# COMPLEX LIE SYMMETRIES FOR DIFFERENTIAL EQUATIONS

*By*

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***Dedicated to***

To My Mother

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SAJID ALI

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## 1.1 Introduction

The history of differential equations (DEs) goes back to the fundamental works of G. W. Leibnitz and I. Newton who developed calculus independently. Newton wrote fundamental laws of nature in terms of DEs that resulted in their first explicit use in the 17th century. They constitute one of the most important disciplines of mathematics and play a fundamental role in mathematical physics. The governing equations of a dynamical system in physical phenomena are DEs, which lead to their diverse use in fields like physics, chemistry, engineering sciences, economics, ecology, biology and many other branches of science.

After the pioneering work of Newton and Leibnitz DEs have been studied as an essential area of mathematics. They are distinguished with respect to their *type, order, homogeneity* and *linearity* and most of all *ordinary* or *partial* differential equations. The order of a DE is the order of the highest derivative. A linear DE is linear in the dependent variable and all its derivatives.

An important, perhaps difficult, aspect of DEs is to find their solutions. It is comparatively easier to deal with ordinary differential equations (ODEs) than partial differential equations (PDEs) for the very basic reason that the solution space of a PDE is infinite dimensional. Another vital issue is with nonlinearity of DEs. There are several methods developed to deal with such an issue. Some ODEs are dealt with by writing them in an appropriate form so that these could be solved analytically easily e.g. separation of variables, homogenous function methods, power series, Frobenius method etc. Generally most of the ODEs encountered are difficult to solve analytically.

S. Lie, who gave another way of dealing with nonlinear DEs, was impressed by Sylow's lecture on the use of group theory to obtain the solvability of algebraic equations. He thought that a unified approach may be constructed that contains all the classical methods for solving DEs known at that time. In 1879 he constructed groups of continuous transformations (named Lie groups after him). He noticed that ODEs can remain invariant under continuous deformations of the dependent and independent variables. This would generalize various methods of change of variables. The observation was very fundamental and led to a new branch of mathematics known as symmetry analysis for solving DEs. *A symmetry group of a system of DEs* is a group of transformations that maps any solution of the system to another solution. Such groups depend upon continuous parameters and consist of point transformations acting on the system's space of independent and dependent variables. Elementary examples of Lie groups include translations, rotations and scalings. Lie gave a complete *complex* classification of non isomorphic algebras admitted by a second-order ODE in [37, 38, 39, 40]. Since then there have been various generalizations of Lie's idea to systems of real ODEs and PDEs [5, 23, 55, 61].

Lie's work systematically relates a miscellany of topics in ODEs including: the integrating factor, separable equations, "homogeneous" equations (in the sense that the functions involved are homogenous functions of some degree), reduction of order and the methods of undetermined coefficients and variation of parameters for linear equations, solution of the Euler equation and the use of the Laplace transform. Lie also indicated that for linear PDEs invariance under a Lie group leads directly to superpositions of solutions in terms of transforms [16].

There are several powerful methods available to solve DEs. Apart from the ODE or PDE divide DEs are classified mainly in two categories, i.e., linear and non linear. One approach to deal with non linear DEs is to find a local transformation that converts it to a linear DE. However, it is difficult to construct such transformations. Lie used the symmetry approach to construct exact solutions and to linearize certain DEs. In all the cases for what exact solutions of a DE can be found, the underlying property is a symmetry of that equation.

Another important branch of mathematics mainly developed in the same century by Riemann was complex analysis. Riemann was interested in the use of complex variables to solve problems that appear in real analysis. In his thesis he stressed on the dependence of a complex valued function on a complex variable, i.e., $u(z)$ which is a complex combination of two real valued functions of two real variables $f(x,y) + \iota g(x,y)$, i.e. by identifying the correspondence of the $xy$ and $fg$-planes [59]. One direct implication of such a construction is that the information of two dependent variables is *encoded* into a single complex dependent variable and of two independent variables into a single complex independent variable with the coupling provided by $\iota$.

In the rest of the thesis we have obtained non trivial results by restricting complex variables in the Lie approach to real variables, which we call complex Lie symmetry (CLS) analysis. It is seen that certain complex ODEs (CODEs) remain invariant under continuous groups of complex transformations of the dependent and independent complex variables known as complex Lie groups. This fact can be used in constructing the complex solutions of CODEs. An important and useful characteristic of CODEs is that we require analyticity instead of $(n + 1)$-times differentiability (for an $n^{th}$ order system of DEs) that helps in many ways. A CODE yields a system of PDEs. However, the fact that complex analyticity implies that the Cauchy-Riemann equations (CREs) hold, incorporates them into the system. We study the *invariance* of such systems of PDEs via the *invariance* of CODEs. Further, if a complex dependent variable is *restricted* to depend on a single real variable, then that CODE (which we call an r-CODE) yields a system of ODEs. The *invariance* of r-CODEs yields the *invariance* of systems of ODEs that explains an unexpected use of complex analysis.

CLS analysis has been mainly used in three ways (a) to study the invariance; (b) linearization; (c) solution of variational problems in CODEs and r-CODEs that yield the *invariance*, *linearization* and *variational problems* of systems of PDEs and ODEs respectively. The remarkable way that complex analysis simplifies complicated expressions and results, which Penrose calls “complex magic” [56], yields a non trivial way to linearize PDEs and their systems.

The outline of the thesis is as follows. The remaining Chapter 1 reviews the basic concepts of real Lie groups of transformations and real Lie algebras used in subsequent chapters. A Lie group of transformations is characterized in terms of its infinitesimal generators, which form a

Lie algebra.

Chapter 2 is concerned with the basic definition of CLSs for CODEs. It also deal with existence and uniqueness of solutions of CODEs. The classification of systems of PDEs with respect to the Lie algebra admitted by a CODE is also given. It also includes the use of complex Lie symmetries to the restricted CODEs that yield analysis of systems of ODEs corresponding to them.

In Chapter 3 we discuss an important method of linearization which is to find a local transformation that maps a non linear DE into a linear DE. We present the Lie approach of finding the linearizing transformation for CODEs and r-CODEs and their use in the linearization of systems of PDEs and systems of ODEs respectively.

Chapter 4 treats another important use of complex Lie symmetries in variational problems by introducing the notion of a complex Lagrangian and its use in complexified Euler-Lagrange equations. The definition of complex Noether symmetries is also presented. The double reduction of order of CODEs and r-CODEs is also mentioned.

We follow the standard procedure of using the principal values of complex-valued functions unless otherwise stated. Further we only classify those CODEs that have analytic coefficients and derivative operators and avoid equations containing $x, y, \bar{z}, |z|^2$ *etc.*

We conclude and indicate directions for future research in the last chapter.

## 1.2 Real manifolds

A *real manifold* is a separable, connected, Hausdorff space with local homeomorphism from open covers on it to $\mathbb{R}^n$. In simple words a manifold locally looks like $\mathbb{R}^n$. Mathematically an $n-$dimensional manifold $M$ is equipped with open charts, $(U_i, \varphi_i)$, such that $\cup U_i = M$, where $U_i,\ i = 1, 2, ..., n$, are open sets in $M$ and the $\varphi_i$ are homeomorphisms from open sets in $M$ to $\mathbb{R}^n$.

We now discuss a few characteristics of manifolds in detail as these are of great relevance in studying symmetries. A *homeomorphism* is a one-to-one, invertible and continuous mapping, i.e., it sends open subsets into open subsets and vice versa. The basic purpose of a homeomorphism is to assign coordinate systems in neighbourhoods of points of a manifold that help in developing calculus on the manifold. Further the choice of coordinate systems on a manifold is not unique. Thus in the theory of manifolds: *all coordinate systems are equally good.* It is also pointed out that this fact is in full harmony with the basic principle of Physics (general covariance): *a physical system must behave in the same way whatever coordinates we use to describe it.*

A space is said to be *separable* if it contains a countable dense subset. A space which has a one-to-one correspondence with the set of natural numbers is said to be countable. Further, a subset is dense if its closure is equal to the set. For example $\mathbb{R}$ is separable because the set of rational numbers, $\mathbb{Q}$, is a countable dense subset in it. More explicitly a separable space is a *continuum,* i.e., it contains all of its accumulation (limit) points.

A space is *disconnected* if it contains two disjoint open subsets such that their union is equal to the space and the closure of one is not contained in the other. For example, $\mathbb{Q}$ is disconnected because it contains two disjoint open subsets,

$$A = \{a \in \mathbb{Q},\ a < \sqrt{2}\} \ \text{ and } \ B = \{b \in \mathbb{Q},\ b > \sqrt{2}\}, \tag{1.1}$$

such that

$$A \cap B = \phi, \quad A \cup B = \mathbb{Q}, \tag{1.2}$$

and further

$$\bar{A} \cap B = \phi, \quad A \cap \bar{B} = \phi. \tag{1.3}$$

A space which does not contain any disconnection is said to be connected. Therefore a connected space contains *no gaps.*

A *Hausdorff* space is one in which distinct points are contained in disjoint neighborhoods. A neighborhood of a point is a set containing an open set containing the point. For example, $\mathbb{R}, \mathbb{R}^2$ etc or $S^1, S^2$ etc. This is an important property in terms of calculus especially in *moving along* the curves on a manifold. Now we take an example of a curved surface, e.g., a sphere $S^2$ in $\mathbb{R}^3$ and determine a coordinate system on that.

**Example 1.1:** A sphere in three dimensions, $S^2$, satisfies

$$x^2 + y^2 + z^2 = a^2, \tag{1.4}$$

where $a$ is the radius of the sphere. There are two coordinate systems that are generally assigned to a sphere: polar and stereographic projection. The polar coordinate system is given by

$$x = a\cos\varphi\sin\psi, \quad y = a\sin\varphi\sin\psi, \quad z = a\cos\psi, \ a \in \mathbb{R}, \tag{1.5}$$

where $\varphi$ and $\psi$ range from $0 \to 2\pi$ and $0 \to \pi$, respectively. The stereographic projection is developed on a sphere by connecting the North Pole $(0, 0, 1)$ (reference point) to a point on the equatorial plane. It may be seen that the line so defined intersects the sphere at some unique point $(X, Y)$. The stereographic coordinates of the point $(X, Y)$ are

$$X = \frac{x}{1-z}, \quad Y = \frac{y}{1-z}. \tag{1.6}$$

We can develop another stereographic coordinate chart, $(U, V)$, by taking South Pole as a reference point so that

$$U = \frac{x}{1+z}, \quad V = \frac{-y}{1+z}. \tag{1.7}$$

It can be proved easily that the transition from one coordinate chart to another is smooth. Thus $S^2$ is a manifold. An important aspect of it is that it may happen that *no coordinate system is usable everywhere at once.*

## 1.3 Point transformations, prolongation and generators

DEs are equations containing the derivatives of one or more dependent variables with respect to one or more independent variables. They are classified according to their **type**, **order** and **linearity**. An *ODE* contains derivatives of one or more dependent variables with respect to a single independent variable. If it contains derivatives of one or more dependent variables relative to more than one independent variables it is a *PDE.* Its order is the highest-order derivative in it. It is *linear* if it can be written in the form

$$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}} + ... + a_1(x)\frac{dy}{dx} + a_0(x)y = g(x), \tag{1.8}$$

where $y$ is the dependent variable and $x$ the independent variable. It is *homogenous* if $g(x) = 0$. An equation that is not linear is said to be *nonlinear*. It is seen that a DE may be simplified by changing the variables,

$$\tilde{x} = p\left(x, y\right), \quad \tilde{y} = q\left(x, y\right), \tag{1.9}$$

where $p$ and $q$ are continuous functions, and (1.9) is known as a point transformation that continuously maps points $(x, y)$ into points $(\tilde{x}, \tilde{y})$. Consider invertible point transformations that depend upon, at least, one arbitrary parameter $\delta$,

$$\tilde{x} = p\left(x, y; \delta\right), \quad \tilde{y} = q\left(x, y; \delta\right), \tag{1.10}$$

where $p$ and $q$ are infinitely differentiable with respect to $x$ and $y$ and analytic in $\delta$. A *one-parameter real Lie group of transformations* is a group $G$ of invertible point transformations of the form (1.10) with an arbitrary parameter $\delta$ [28] if $G$ contains the identity transformation (e.g. $p(\delta)|_{\delta=0} = x, \quad q(\delta)|_{\delta=0} = y$) as well as the inverse of its elements and their composition. The composition is analytic with respect to the arbitrary parameter $\delta$.

The transformations (1.10) give a *symmetry group* of a DE,

$$E\left(x, y, y', y'', ..., y^{(n)}\right) = 0, \tag{1.11}$$

if the equation is form invariant, i.e.

$$E\left(\tilde{x}, \tilde{y}, \tilde{y}', \tilde{y}'', ..., \tilde{y}^{(n)}\right) = 0, \tag{1.12}$$

whenever $E\left(x, y, y', y'', ..., y^{(n)}\right) = 0$ [28, 53, 55, 61]. Here $y' \equiv dy/dx$ etc. A real symmetry group of a DE is also said to be *admitted* by it.

A simple example of a one-parameter group in the real domain is given by the rotations

$$\tilde{x} = x\cos\delta - y\sin\delta, \quad \tilde{y} = x\sin\delta + y\cos\delta. \tag{1.13}$$

We obtain the infinitesimal form of (1.12) by expanding it near zero using Taylor's series as

$$\begin{aligned} \tilde{x} &= x + \xi(x, y)\,\delta + ..., \\ \tilde{y} &= y + \eta(x, y)\,\delta + ..., \end{aligned} \tag{1.14}$$

where the functions $\xi$ and $\eta$ are defined by [61],

$$\xi(x, y) = \frac{\partial \tilde{x}}{\partial \delta}|_{\delta=0}, \quad \eta(x, y) = \frac{\partial \tilde{y}}{\partial \delta}|_{\delta=0}. \tag{1.15}$$

In terms of an operator,

$$\mathbf{X} = \xi(x, y)\frac{\partial}{\partial x} + \eta(x, y)\frac{\partial}{\partial y}, \tag{1.16}$$

(1.14) can be written as

$$\begin{aligned} \tilde{x} &= x + \delta \mathbf{X} x + ..., \\ \tilde{y} &= y + \delta \mathbf{X} y + .... \end{aligned} \tag{1.17}$$

The operator $\mathbf{X}$ is called the *infinitesimal generator* or the *infinitesimal operator* of the transformation as the repeated action of it draws the orbits of the group. It appears as a *symmetry* of a DE provided it satisfies symmetry condition. We will discuss this point later.

**Example 1.2:** Consider the generator

$$\mathbf{X} = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}. \tag{1.18}$$

Then we have $\xi(\tilde{x}, \tilde{y}) = \partial \tilde{x}/\partial \delta = \tilde{x}$ and $\eta(\tilde{x}, \tilde{y}) = \partial \tilde{y}/\partial \delta = \tilde{y}$ so that their solutions with initial values $\tilde{x}(0) = x, \;\; \tilde{y}(0) = y$ is

$$\tilde{x} = e^{\delta} x, \quad \tilde{y} = e^{\delta} y. \tag{1.19}$$

Thus (1.19) is the required group transformation. This is a (special) scaling- or similarity-transformation: all variables are multiplied by the same constant factor.

**Example 1.3:** For small $\delta$, i.e. $\delta << 1$, the rotation (1.13) can be written as

$$\tilde{x} = x - y\delta, \quad \tilde{y} = y + x\delta, \tag{1.20}$$

which gives

$$\xi(x,y) = \frac{\partial \tilde{x}}{\partial \delta}|_{\delta=0} = -y, \quad \eta(x,y) = \frac{\partial \tilde{y}}{\partial \delta}|_{\delta=0} = x. \tag{1.21}$$

Therefore the infinitesimal generator of rotation is

$$\mathbf{X} = -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}. \tag{1.22}$$

This point transformation rotates each point of the plane to another point. Thus we can always obtain the one parameter Lie group of transformations by solving an intial value problem given by the Lie equations, i.e., if we know its infinitesimal generator. This is Lie's first fundamental theorem. The most important thing about infinitesimal generators is that they are linear operators although the transformations can be very complicated. Further this remarkable feature of symmetries indicates the fact that we can use them to remove non linearity in the determining equations for the symmetries.

The generator $\mathbf{X}$ as given by (1.16) explicitly refers to the variables $x$ and $y$. By introducing a new change of variables $u(x,y)$ and $v(x,y)$ the coefficient functions $\xi$ and $\eta$ are changed. For example, if we express the generator (1.18) in coordinates

$$u = y/x \quad \text{and} \quad v = xy, \tag{1.23}$$

we obtain immediately,

$$\mathbf{X}u = 0, \quad \mathbf{X}v = 2v, \tag{1.24}$$

that is,

$$\mathbf{X} = 2v\frac{\partial}{\partial v}. \tag{1.25}$$

As another example the generator (1.22) of a rotation can be expressed as

$$\mathbf{X} = \frac{\partial}{\partial \varphi}, \tag{1.26}$$

known as the polar form of rotation, by introducing the polar coordinates

$$r = \left(x^2 + y^2\right)^{1/2}, \quad \varphi = \arctan y/x. \tag{1.27}$$

There always exist coordinates $r(x,y)$ and $s(x,y)$ for which the symmetry generator (1.16) takes the simple form

$$\mathbf{X} = \frac{\partial}{\partial s} \tag{1.28}$$

and (1.28) is called the normal form or canonical form of the generator **X** [28, 53, 55, 61].

If we want to apply a point transformation (1.10) to an ODE

$$E\left(x, y, y', y'', ..., y^{(n)}\right) = 0, \tag{1.29}$$

we must know how to transform the derivatives $y^{(n)}$, that is how to extend (or prolong) the point transformation to the derivatives. This is done by defining

$$\begin{aligned} \tilde{y}' &= \frac{d\tilde{y}}{d\tilde{x}} = \frac{d\tilde{y}(x,y;\delta)}{d\tilde{x}(x,y;\delta)} = \frac{y'(\partial\tilde{y}/\partial y) + (\partial\tilde{y}/\partial x)}{y'(\partial\tilde{x}/\partial y) + (\partial\tilde{x}/\partial x)} = \tilde{y}'\left(x, y, y'; \delta\right), \\ \tilde{y}'' &= \frac{d\tilde{y}'}{d\tilde{x}} = \tilde{y}''\left(x, y, y', y''; \delta\right) \text{ etc,} \end{aligned} \tag{1.30}$$

that is, the transformed derivatives are the derivatives of, and with respect to, the transformed variables. Thus a nice characteristic of a one-parameter real Lie group of transformations is that it can easily be extended to higher-order derivatives and this fact plays a central role in the theory of symmetry analysis of DEs.

Now for the extensions (or prolongations) of infinitesimal generators, **X**, we write

$$\begin{aligned} \tilde{x} &= x + \delta\xi(x,y) + ... = x + \delta\mathbf{X}x + ..., \\ \tilde{y} &= y + \delta\eta(x,y) + ... = y + \delta\mathbf{X}y + ..., \\ \tilde{y}' &= y' + \delta\eta'\left(x, y, y'\right) + ... = y' + \delta\mathbf{X}y' + ..., \\ \tilde{y}^{(n)} &= y^{(n)} + \delta\eta^{(n)}\left(x, y, y', ..., y^{(n)}\right) + ... = y^{(n)} + \delta\mathbf{X}y^{(n)} + ..., \end{aligned} \tag{1.31}$$

where $\xi$, $\eta$, $\eta', ..., \eta^{(n)}$ are defined by [28, 53, 55, 61]

$$\eta' = \frac{\partial\tilde{y}'}{\partial\delta}|_{\delta=0}, ..., \eta^{(n)} = \frac{\partial\tilde{y}^{(n)}}{\partial\delta}|_{\delta=0}. \tag{1.32}$$

Inserting the expressions (1.31) into (1.32) we obtain

$$\begin{aligned}
\tilde{y}' &= y' + \delta\eta' + ... = \frac{d\tilde{y}}{d\tilde{x}} = \frac{dy + \delta d\eta + ...}{dx + \delta d\xi + ...} \\
&= \frac{y' + \delta\left(d\eta/dx\right) + ...}{1 + \delta\left(d\xi/dx\right) + ...} = y' + \delta\left(\frac{d\eta}{dx} - y'\frac{d\xi}{dx}\right) + ..., \\
\tilde{y}^{(n)} &= y^{(n)} + \delta\eta^{(n)} + ... = \frac{d\tilde{y}^{(n-1)}}{d\tilde{x}} \\
&= y^{(n)} + \delta\left(\frac{d\eta^{(n-1)}}{dx} - y^{(n)}\frac{d\xi}{dx}\right) + ..., \qquad (1.33)
\end{aligned}$$

from which we can see that the $\eta^{(n)}$ [61] are

$$\eta^{(n)} = \mathbf{D}\eta^{(n-1)} - y^{(n)}\mathbf{D}\xi;\ \eta^{(0)} = \eta. \qquad (1.34)$$

Here $\mathbf{D}$ is given by

$$\frac{d}{dx} = \mathbf{D} \equiv \frac{\partial}{\partial x} + y'\frac{\partial}{\partial y} + y''\frac{\partial}{\partial y'} + y'''\frac{\partial}{\partial y''} + .... \qquad (1.35)$$

Therefore the extension (prolongation) of the infinitesimal generator (1.16) up to the $n^{th}$ order is given by

$$\mathbf{X}^{[n]} = \xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y} + \eta'\frac{\partial}{\partial y'} + ... + \eta^{(n)}\frac{\partial}{\partial y^{(n)}}. \qquad (1.36)$$

We drop the index "$[n]$" in future, using the same symbol, $\mathbf{X}$, for the generator and its extension. This sloppiness in notation does not give rise to confusion since in most cases it is clear from the context what is meant. We present the first two values of $\eta^{(n)}$, i.e., $\eta^{(1)} = \eta'$ and $\eta^{(2)} = \eta''$, which are

$$\eta^{(1)} = \eta_x + \left(\eta_y - \xi_x\right)y' - \xi_y y'^2, \qquad (1.37)$$

$$\begin{aligned}
\eta^{(2)} &= \eta_{xx} + \left(2\eta_{xy} - \xi_{xx}\right)y' + \left(\eta_{yy} - 2\xi_{xy}\right)y'^2 - \xi_{yy}y'^3 \\
&\quad + \left(\eta_y - 2\xi_x - 3\xi_y y'\right)y''. \qquad (1.38)
\end{aligned}$$

Here the subscripts denote the partial derivative of the function.

## 1.4 Lie point symmetries of ordinary differential equations

We state an important theorem given in the literature [28, 53, 55, 61] and discuss its applications.

**Theorem:** An ODE (1.20) admits a group of symmetries with generator $\mathbf{X}$ if and only if

$$\mathbf{X}E|_{E=0} \equiv 0 \tag{1.39}$$

holds, i.e., the equation $\mathbf{X}E = 0$ holds on solutions of $E = 0$. Note that (1.39) gives a system of equations called the determining equations.

As an example we find the Lie symmetries of the simple harmonic oscillator linear DE

$$E\left(x, y, y', y''\right) = y'' + y = 0. \tag{1.40}$$

Solving for $\mathbf{X}$ in

$$\mathbf{X}E|_{E=0} = 0 \tag{1.41}$$

gives

$$\mathbf{X}(y'' + y)|_{y''+y=0} = 0. \tag{1.42}$$

This can be written as

$$[\eta^{(2)} + \eta]|_{y''+y=0} = 0. \tag{1.43}$$

Use of the value of $\eta^{(2)}$ from (1.38) and replacment $y''$ by $-y$ give

$$\eta_{xx} + (2\eta_{xy} - \xi_{xx})y' + (\eta_{yy} - 2\xi_{xy})y'^2 - \xi_{yy}y'^3 + (\eta_y - 2\xi_x - 3\xi_y y')(-y) + \eta = 0. \tag{1.44}$$

Equating the coefficients of $y'^0$, $y'^1$, $y'^2$ and $y'^3$ we get

$$y'^0 \quad : \quad \eta_{xx} + (-\eta_y + 2\xi_x)y + \eta = 0, \tag{1.45}$$

$$y'^1 \quad : \quad 2\eta_{xy} - \xi_{xx} + 3\xi_y y = 0, \tag{1.46}$$

$$y'^2 \quad : \quad \eta_{yy} - 2\xi_{xy} = 0, \tag{1.47}$$

$$y'^3 \quad : \quad \xi_{yy} = 0. \tag{1.48}$$

Now (1.48) implies that

$$\xi = a(x)y + b(x). \tag{1.49}$$

Then (1.47) with the use of (1.49) gives

$$\eta = a'(x)y^2 + c(x)y + d(x). \tag{1.50}$$

By invoking (1.49) and (1.50) in (1.46) implies that

$$2(2a''(x)y + c'(x)) - a''(x)y - b''(x) + 3a(x)y = 0. \tag{1.51}$$

Comparing the coefficients of $y^0$ and $y^1$ we get

$$y^0 \quad : \quad 2c'(x) - b''(x) = 0, \tag{1.52}$$

$$y^1 \quad : \quad a''(x) + a(x) = 0. \tag{1.53}$$

The solution of (1.53) can be written as

$$a(x) = A\cos x + B\sin x. \tag{1.54}$$

From (1.45), after substituting (1.49) and (1.50), we get

$$a'''(x)y^2 + c''(x)y + d''(x) + (-2a'(x)y - c(x) + 2a'(x)y + 2b'(x)y + a'(x)y^2 + c(x)y + d(x) = 0. \tag{1.55}$$

Comparison of the coefficients of $y^0$, $y^1$ and $y^2$ yields

$$y^0 \quad : \quad d''(x) + d(x) = 0, \tag{1.56}$$

$$y^1 \quad : \quad c''(x) + 2b'(x) = 0, \tag{1.57}$$

$$y^2 \quad : \quad a'''(x) + a'(x) = 0. \tag{1.58}$$

The solution of (1.56) can be written as

$$d(x) = R\cos x + S\sin x. \tag{1.59}$$

The use of (1.57), after integrating with respect to '$x$', in (1.52) implies that

$$b(x) = P\cos 2x + Q\sin 2x + U. \tag{1.60}$$

Therefore (1.52) gives

$$c(x) = Q\cos 2x - P\sin 2x + V. \tag{1.61}$$

Hence (1.49) and (1.50) take the form

$$\xi = (A\cos x + B\sin x)y + P\cos 2x + Q\sin 2x + U, \tag{1.62}$$

$$\eta = (B\cos x - A\sin x)y^2 + (Q\cos 2x - P\sin 2x + V)y + R\cos x + S\sin x, \tag{1.63}$$

where $A, B, P, Q, R, S, U$ and $V$ are constants. Setting $A = 1$ and all other constants equal to zero yields a generator from the list given below. The same procedure can be followed for the other constants. Thus we have the eight generators

$$\begin{aligned}
\mathbf{X}_1 &= y\cos x\frac{\partial}{\partial x} - y^2\sin x\frac{\partial}{\partial y}, \quad \mathbf{X}_2 = y\sin x\frac{\partial}{\partial x} + y^2\cos x\frac{\partial}{\partial y}, \\
\mathbf{X}_3 &= \cos 2x\frac{\partial}{\partial x} - y\sin 2x\frac{\partial}{\partial y}, \quad \mathbf{X}_4 = \sin 2x\frac{\partial}{\partial x} + y\cos 2x\frac{\partial}{\partial y}, \\
\mathbf{X}_5 &= \cos x\frac{\partial}{\partial y}, \quad \mathbf{X}_6 = \sin x\frac{\partial}{\partial y}, \quad \mathbf{X}_7 = \frac{\partial}{\partial x}, \quad \mathbf{X}_8 = y\frac{\partial}{\partial y}.
\end{aligned} \tag{1.64}$$

## 1.5 Basic properties of real Lie algebras

If the group of real symmetry transformations

$$\tilde{x} = f(x, y; \delta_N), \quad \tilde{y} = g(x, y; \delta_N), \quad N = 1, ..., r, \tag{1.65}$$

depends upon $r-$parameters $\delta_N$, then the general infinitesimal generator of this group is a linear combination

$$\mathbf{X} = b^N \mathbf{X}_N, \quad b^N = \text{const.}, \quad N = 1, ..., r, \tag{1.66}$$

of $r$ linearly independent basic generators $\mathbf{X}_N$, each corresponding to one of the group parameters $\delta_N$, given as

$$\mathbf{X}_N = \xi_N\left(x,y\right)\frac{\partial}{\partial x} + \eta_N\left(x,y\right)\frac{\partial}{\partial y}, \tag{1.67}$$

where

$$\xi_N\left(x,y\right) = \frac{\partial \tilde{x}}{\partial \delta_N}|_{\delta_N=0}, \quad \eta_N\left(x,y\right) = \frac{\partial \tilde{y}}{\partial \delta_N}|_{\delta_N=0}. \tag{1.68}$$

If we change the parameterization of the group, the result will be a linear transformation

$$\mathbf{X}_N = B_N^M \mathbf{X}_M, \;\; B_N^M = \text{const.}, \quad |B_N^M| \neq 0, \tag{1.69}$$

of the basic generators. The main advantage in using the infinistesimal generators instead of the transformations of the finite group is that the generators are linear operators. The superposition law (1.66) is linear and it does not matter in which order generators are added.

The commutator of two generators, $\mathbf{X}_N$ and $\mathbf{X}_M$, is defined by

$$[\mathbf{X}_N, \mathbf{X}_M] = \mathbf{X}_N\mathbf{X}_M - \mathbf{X}_M\mathbf{X}_N, \tag{1.70}$$

which obeys the properties of bilinearity, skew-symmetry and Jacobi's identity [23, 61]:

1. **Bilinearity**

$$[\alpha\mathbf{X}_N + \beta\mathbf{X}_M, \mathbf{X}_P] = \alpha\left[\mathbf{X}_N, \mathbf{X}_P\right] + \beta\left[\mathbf{X}_M, \mathbf{X}_P\right]; \quad \alpha \text{ and } \beta \text{ are scalars.} \tag{1.71}$$

2. **Skew-symmetry**

$$[\mathbf{X}_N, \mathbf{X}_M] = -\left[\mathbf{X}_M, \mathbf{X}_N\right]; \tag{1.72}$$

3. **Jacobi identity**

$$\left[\left[\mathbf{X}_N, \mathbf{X}_M\right], \mathbf{X}_P\right] + \left[\left[\mathbf{X}_M, \mathbf{X}_P\right], \mathbf{X}_N\right] + \left[\left[\mathbf{X}_P, \mathbf{X}_N\right], \mathbf{X}_M\right] = 0. \tag{1.73}$$

All symmetries can be written as a linear combination of the $r$ basic generators and we have

$$[\mathbf{X}_N, \mathbf{X}_M] = C^P_{NM}\mathbf{X}_P \tag{1.74}$$

The constants $C^P_{NM}$ are called the *structure constants* of the group.

For example, if we take $\mathbf{X}_2 = y\sin x\frac{\partial}{\partial x} + y^2\cos x\frac{\partial}{\partial y}$ and $\mathbf{X}_4 = \sin 2x\frac{\partial}{\partial x} + y\cos 2x\frac{\partial}{\partial y}$ as given in (1.64), then

$$\begin{aligned}[\mathbf{X}_2, \mathbf{X}_4] = \mathbf{X}_2\mathbf{X}_4 - \mathbf{X}_4\mathbf{X}_2 &= -y\sin x\frac{\partial}{\partial x} - y^2\cos x\frac{\partial}{\partial y} \\ &= -\mathbf{X}_2. \end{aligned} \tag{1.75}$$

Hence $C^2_{24} = -1 = -C^2_{42}$ and $C^4_{24} = 0 = C^4_{42}$.

A *Lie algebra* is a vector space over $\mathbb{R}$ or $\mathbb{C}$ with a bilinear bracket operator (the commutator) satisfying the properties (1.71), (1.72) and (1.73). Lie's second and third fundamental theorems state that the set of infinitesimal generators $\{\mathbf{X}_N\}$, $N = 1, ...r$, of an $r$-parameter Lie group of transformations (1.65) forms an $r-$dimensional Lie algebra over $\mathbb{R}$ or $\mathbb{C}$.

## 1.6 Symmetries of systems of ordinary differential equations

The system of $p$ ODEs of order $k$,

$$E_i\left(s, \mathbf{x}, \dot{\mathbf{x}}, ..., \mathbf{x}^{(k)}\right) = 0, \quad i = 1, 2, ..., p, \tag{1.76}$$

admits a symmetry algebra with generator,

$$\mathbf{X} = \xi\left(s, \mathbf{x}\right)\frac{\partial}{\partial s} + \eta^{\alpha}\left(s, \mathbf{x}\right)\frac{\partial}{\partial x^{\alpha}}, \tag{1.77}$$

if and only if $\mathbf{X}E_i\mid_{E_i=0} = 0$ holds, where $\mathbf{x}$ is a point in the underlying $m$-dimensional space, $\dot{\mathbf{x}}$ is the first derivative of $\mathbf{x}$, $\mathbf{x}^{(k)}$ is the $k^{th}$-order derivative with respect to $s$ and

$$\mathbf{X} = \xi\left(s, \mathbf{x}\right)\frac{\partial}{\partial s} + \eta^{\alpha}\left(s, \mathbf{x}\right)\frac{\partial}{\partial x^{\alpha}} + \eta^{\alpha}_{,s}\left(s, \mathbf{x}, \dot{\mathbf{x}}\right)\frac{\partial}{\partial \dot{x}^{\alpha}} + ..... + \eta^{\alpha}_{,(k)}\left(s, \mathbf{x}, ...., \mathbf{x}^{(k)}\right)\frac{\partial}{\partial x^{\alpha(k)}}, \tag{1.78}$$

is the extension up to the $k^{th}$ order [28, 61]. The prolongation coefficients are

$$\begin{aligned} \eta^{\alpha}_{,s} &= \frac{d\eta^{\alpha}}{ds} - \dot{x}^{\alpha}\frac{d\xi}{ds}, \\ \eta^{\alpha}_{,(k)} &= \frac{d\eta^{\alpha}_{(k-1)}}{ds} - x^{\alpha(k)}\frac{d\xi}{ds}, \quad k \geq 2. \end{aligned} \tag{1.79}$$

Equation (1.76) generalizes (1.29), which is contained in it as a special case. It serves to determine the components $\eta^{\alpha}_{,(k)}$ of the symmetry generator $\mathbf{X}$ when $E_i$ (the set of DEs) are given.

## 1.7 Symmetries of partial differential equations

In order to obtain the symmetries of PDEs and their systems we need further to extend one parameter Lie point transformations to include all dependent and independent variables. Consider a system of PDEs,

$$K_a(x^i, y^{\alpha}, y^{\alpha}_{,i}, y^{\alpha}_{,ij}, ...) = 0, \tag{1.80}$$

in the $n$ independent variables $x^i$, the $m$ dependent variables $y^{\alpha}(x^i)$ and their derivatives given as

$$y^{\alpha}_{,i} \equiv \frac{\partial y^{\alpha}}{\partial x^i}, \quad y^{\alpha}_{,ij} \equiv \frac{\partial^2 y^{\alpha}}{\partial x^i \partial x^j}, \cdots . \tag{1.81}$$

Our aim is to find the transformation $(\tilde{x}^i, \tilde{y}^{\alpha})$ such that the system (1.80) is form invariant i.e.

$$K_a(\tilde{x}^i, \tilde{y}^{\alpha}, \tilde{y}^{\alpha}_{,i}, \tilde{y}^{\alpha}_{,ij}, ...) = 0. \tag{1.82}$$

For a one-parameter group of transformations

$$\begin{aligned} \tilde{x}^i &= \tilde{x}^i(x^i, y^{\alpha}; \delta), \quad \tilde{y}^{\alpha} = \tilde{y}^{\alpha}(x^i, y^{\alpha}; \delta), \quad \tilde{y}^{\alpha}_{,i} = y^{\alpha}_{,i}(x^i, y^{\alpha}, y^{\alpha}_{,i}, \delta), \\ i &= 1, ..., n, \quad \alpha = 1, ...m, \end{aligned} \tag{1.83}$$

and the use of the Taylor's series

$$\begin{aligned} \tilde{x}^i &= x^i + \xi^i(x^i, y^\alpha)\delta + ..., \quad \xi^i \equiv \frac{\partial \tilde{x}^i}{\partial \delta}|_{\delta=0}, \\ \tilde{y}^\alpha &= y^\alpha + \eta^\alpha(x^i, y^\alpha)\delta + ..., \quad \eta^\alpha \equiv \frac{\partial \tilde{y}^\alpha}{\partial \delta}|_{\delta=0}, \\ \tilde{y}^\alpha_{,i} &= y^\alpha_{,i} + \eta^\alpha_i(x^i, y^\alpha, y^\alpha_{,i})\delta + ..., \quad \eta^\alpha_i \equiv \frac{\partial \tilde{y}^\alpha_{,i}}{\partial \delta}|_{\delta=0} \end{aligned} \tag{1.84}$$

the prolonged symmetry generator becomes

$$\mathbf{X} = \xi^i(x^i, y^\alpha)\frac{\partial}{\partial x^i} + \eta^\alpha(x^i, y^\alpha)\frac{\partial}{\partial y^\alpha} + \eta^\alpha_i(x^i, y^\alpha, y^\alpha_i)\frac{\partial}{\partial y^\alpha_i} + ..., \tag{1.85}$$

where the prolongation coefficients are given as

$$\begin{aligned} \eta^\alpha_i &= \frac{\mathbf{D}\eta^\alpha}{\mathbf{D}x^i} - y^\alpha_j \frac{\mathbf{D}\xi^j}{\mathbf{D}x^i}, \\ \eta^\alpha_{ij} &= \frac{\mathbf{D}\eta^\alpha_i}{\mathbf{D}x^j} - y^\alpha_{ik}\frac{\mathbf{D}\xi^k}{\mathbf{D}x^j}, ..., \end{aligned} \tag{1.86}$$

where

$$\frac{\mathbf{D}}{\mathbf{D}x^i} = \frac{\partial}{\partial x^i} + y^\alpha_{,i}\frac{\partial}{\partial y^\alpha} + y^\alpha_{,ij}\frac{\partial}{\partial y^\alpha_j} + .... \tag{1.87}$$

The symmetry condition is

$$\mathbf{X}K_a = 0, \qquad \mathrm{mod}\, K_a = 0, \tag{1.88}$$

and the DEs $K_a = 0$ have to be used when evaluating $\mathbf{X}K_a = 0$, which is best done by eliminating as many of the highest derivatives as possible.

# Chapter 2

# Symmetry methods for complex ordinary differential equations

This Chapter explores the implications of Lie's complex symmetry analysis by explicitly regarding the complex variables as a pair of real variables with complex structure. A CLS of a CODE (r-CODE) yields two RLSs for the systems of PDEs (ODEs). Further it corresponds to a complex operation that includes the information of two distinct real operations. For instance a rescaling in the complex plane represents rescaling and rotation in the real domain, as should be expected. The complex Lie classification of second-order CODEs (r-CODEs) with respect the complex Lie algebras yields a unique classification of systems of PDEs (ODEs). We then investigate reduction of order in a large class of systems of PDEs and ODEs in a non trivial way by using "complex magic".

## 2.1 Complex manifolds

A complex manifold is a manifold that has a complex structure on it. In comparison to a real manifold, a complex manifold is locally identical to $\mathbb{C}^n$. Since a complex manifold is a manifold, it is separable, connected, Hausdorffness etc. By a complex structure we mean that a complex manifold $M_{\mathbb{C}}$ is completely filled with open charts $(U_j, f_j)$, i.e. $\cup U_j = M_{\mathbb{C}}$, where its $U_j$ $j = 1, 2, ..., n$ are open sets on the manifold and $f_j : U_j \to \mathbb{C}^n$ are holomorphic functions (complex analytic).

It was Riemann who identified the significance of holomorphic functions that provide the basic difference in the understanding of real manifolds in comparison to complex manifolds. Riemann stressed on the geometrical dependence of $f(z)$ by regarding it as the dependence of a plane $\mathcal{A}$ determined by $z = x + \iota y$ on plane $\mathcal{B}$ determined by $f = \varphi + \iota\psi$. It is important to grasp the idea of holomorphic functions. A function $f : D \to \mathbb{C}$, where $D$ is a domain in $\mathbb{C}$, is said to be holomorphic at $c \in D$ if $f$ is complex differentiable at every point of some neighborhood of $c$. This can be translated into the real domain by saying that all first-order partial derivatives of $f$ are continuous and satisfy the CREs at every point of that neighborhood. Therefore, if $z^\mu$ is a trivial coordinate system on a manifold $M_\mathbb{C}$, then

$$\frac{\partial \varphi_j}{\partial x^\mu} = \frac{\partial \psi_j}{\partial y^\mu}, \quad \frac{\partial \psi_j}{\partial y^\mu} = -\frac{\partial \varphi_j}{\partial x^\mu}, \quad \forall \ \ \mu = 1, ..., n, \tag{2.1}$$

where

$$\begin{aligned} f_j &= \varphi_j + \iota\psi_j, \quad \forall \ \ j, \\ z^\mu &= x^\mu + \iota y^\mu. \end{aligned} \tag{2.2}$$

The equations, (2.1), give an analytic structure on the manifold, namely, the existence of CREs at each point of the manifold. If there exists an analytic structure at each point of a real manifold, then it can be complexified, i.e., it can be transformed into a complex manifold. We illustrate this fact by considering the same example of a sphere and the real plane.

**Example 2.1:** In Example 1.1 it was seen that the stereographic coordinates of a point $P(x, y, z) \in S^2 \backslash \{N\}$, where $N$ is the North Pole, projected from the North Pole are

$$(X, Y) = (\frac{x}{1 - z}, \frac{y}{1 - z}), \tag{2.3}$$

while those of a point $P(x, y, z) \in S^2 \backslash \{S\}$, where $S$ is the South Pole, projected from the South Pole are

$$(U, V) = (\frac{x}{1 + z}, \frac{-y}{1 + z}). \tag{2.4}$$

We define complex coordinates

$$Z = X + \iota Y, \quad W = U + \iota V. \tag{2.5}$$

Then $W$ is a holomorphic function of $Z$ in its domain of definition,

$$W = \frac{x - \iota y}{1 + z} = \frac{1 - z}{1 + z}(X - \iota Y) = \frac{X - \iota Y}{X^2 + Y^2} = \frac{1}{Z}. \tag{2.6}$$

Thus $S^2$ is a complex manifold which is identified with the Riemann sphere $\mathbb{C} \cup \{\infty\}$.

Holomorphic functions are of great importance. Their significance is apparent in the fact that these are angle and orientation preserving maps which together amount to *angle preservation with conservation of direction* that according to Riemann: *similarity obtains between the smallest parts of plane $\mathcal{A}$ and their images on plane $\mathcal{B}$.* These properties play a vital role in many physical problems.

## 2.2 First- and second-order CODEs

Consider a general first-order CODE

$$u'(z) = w(z, u(z)), \tag{2.7}$$

where $w$ is holomorphic function and the derivative is with respect to $z$. After substituting

$$\begin{aligned} z &= x + \iota y, u(z) = f(x, y) + \iota g(x, y), \\ w(z, u) &= G(x, y, f, g) + \iota H(x, y, f, g) \end{aligned} \tag{2.8}$$

we obtain the following system of PDEs

$$\begin{aligned} f_x + g_y &= 2G(x, y, f, g), \quad f_x = g_y, \\ g_x - f_y &= 2H(x, y, f, g), \quad f_y = -g_x, \end{aligned} \tag{2.9}$$

where the CREs arise as a consequence of analyticity of $u$. The symmetry conditions for the above system of PDEs are obtained by splitting the complex symmetry condition for CODEs (1.39).

**Examples 2.2:** Consider a complexified Ricatti equation

$$u'(z) + u^2 = 0, \tag{2.10}$$

which has the solution

$$u(z) = \frac{1}{z}. \tag{2.11}$$

We obtain

$$\begin{aligned} f_x + g_y &= 2(g^2 - f^2), \quad f_x = g_y, \\ g_x - f_y &= -4fg, \quad f_y = -g_x \end{aligned} \tag{2.12}$$

by using (2.10). The system above has a solution

$$f = -\frac{x}{x^2 + y^2}, \quad g = \frac{y}{x^2 + y^2}, \tag{2.13}$$

which can be obtained from (2.11). Note that, if we replace (2.10) by

$$u'(z) = \alpha u^2, \tag{2.14}$$

where $\alpha = \alpha_1 + \iota\alpha_2$, it yields the system

$$\begin{aligned} f_x + g_y &= 2\alpha_1(f^2 - g^2) - 4\alpha_2 fg, \quad f_x = g_y, \\ g_x - f_y &= 2\alpha_2(f^2 - g^2) + 4\alpha_1 fg, \quad f_y = -g_x. \end{aligned} \tag{2.15}$$

Note that by simply adding a complex number to a CODE we extend the real system to a general system, i.e., the first equation in the above system includes the quadratic term $fg$ while the second equation contains $f^2 - g^2$ whereas the first equation in (2.12) lacks the $fg$ term and the second equation does not contain $f^2$ and $g^2$ terms. This is an intrinsic property of complex variable theory and plays a central role in dealing with large class of systems of PDEs. Further the solution of the above system can be obtained from the complex solution, $-\alpha/z$, of (2.14), i.e.

$$\begin{aligned} f &= -\frac{\alpha_1 x}{x^2+y^2} - \frac{\alpha_2 y}{x^2+y^2}, \\ g &= -\frac{\alpha_2 x}{x^2+y^2} + \frac{\alpha_1 y}{x^2+y^2}. \end{aligned} \tag{2.16}$$

Similarly a general second-order CODE is given by

$$u''(z) = w(z, u(z), u'), \tag{2.17}$$

where $w$ is a holomorphic function. Further consider

$$w(z, u(z), u') = G(x, y, f, g, h, l) + \iota H(x, y, f, g, h, l), \tag{2.18}$$

where

$$2u' = f_x + g_y + \iota(g_x - f_y) = h + \iota l \tag{2.19}$$

and $G$ and $H$ are arbitrary functions of $x, y, h$ and $l$. The system of PDEs corresponding to (2.17) is

$$\begin{aligned} f_{xx} - f_{yy} + 2g_{xy} &= 4G(x, y, f, g, h, l), \quad f_x = g_y, \\ g_{xx} - g_{yy} - 2f_{xy} &= 4H(x, y, f, g, h, l), \quad f_y = -g_x. \end{aligned} \tag{2.20}$$

Therefore we can deal with properties of the above system by using (2.17) and its symmetry condition (1.39).

**Example 2.3:** The complexified oscillator equation,

$$u'' = -u, \tag{2.21}$$

yields the system

$$\begin{aligned} f_{xx} - f_{yy} + 2g_{xy} &= -4f, \quad f_x = g_y, \\ g_{xx} - g_{yy} - 2f_{xy} &= -4g, \quad f_y = -g_x. \end{aligned} \tag{2.22}$$

The solution of the above system is

$$\begin{aligned} f &= \alpha_1 \cos x \cosh y + \alpha_2 \sin x \sinh y + \beta_1 \sin x \cosh y - \beta_2 \cos x \sinh y, \\ g &= \alpha_2 \cos x \cosh y - \alpha_1 \sin x \sinh y + \beta_1 \cos x \sinh y + \beta_2 \sin x \cosh y, \end{aligned} \tag{2.23}$$

which is obtained from the complex solution $u = \alpha \cos z + \beta \sin z$ of (2.21).

## 2.3 First- and second-order r-CODEs

Consider a first-order CODE

$$u'(z) = w(z, u(z)). \tag{2.24}$$

If we restrict $u$ to depend on a single real variable $x$ instead of the complex variable $z$, then the above equation becomes

$$u'(x) = w(x, u(x)), \tag{2.25}$$

which, upon using

$$u(x) = f(x) + \iota g(x), \quad w = w_1 + \iota w_2, \tag{2.26}$$

becomes

$$\begin{aligned} f' &= w_1(x, f, g), \\ g' &= w_2(x, f, g), \end{aligned} \tag{2.27}$$

where

$$w(x, u(x)) = w_1(x, f, g) + \iota w_2(x, f, g). \tag{2.28}$$

A first-order ODE admits infinitely many symmetries. Similarly a system of two first-order ODEs also admits an infinite-dimensional algebra. Our aim is to extract a subalgebra which is obtained from a complex algebra, i.e. the algebra of conformal symmetries. Similarly a second-order CODE of the form,

$$u''(x) = w(x, u, u'), \tag{2.29}$$

yields

$$\begin{aligned} f'' &= w_1(x, f, g, f', g'), \\ g'' &= w_2(x, f, g, f', g'), \end{aligned} \tag{2.30}$$

where $w$ is an analytic function and $w_1$ and $w_2$ are arbitrary real analytic functions of $x, f, g, f'$ and $g'$. Our aim is to find the symmetries of the above system, (2.30) , by using the symmetries of a general second-order r-CODE of the form (2.29). It must be pointed out that, although the system (2.30) is in a general form, since $w_1$ and $w_2$ are components of $w$, their form depends on the arbitrariness of $w$. This tells us that not every system of ODEs can be transformed into an r-CODE. Further the complex symmetries of (2.29) will only yield the subalgebra of conformal symmetries of system (2.30).

**Example 2.4:** Consider the complexified harmonic oscillator equation

$$u''(x) = -u(x) \tag{2.31}$$

that yields

$$f'' = -f, \quad g'' = -g. \tag{2.32}$$

Notice that the general solution of (2.31) is given as

$$u(x) = \alpha \cos x + \beta \sin x, \tag{2.33}$$

where $\alpha = \alpha_1 + \iota\alpha_2, \beta = \beta_1 + \iota\beta_2$. We can obtain the general solution of (2.32) by substituting the values of $\alpha$ and $\beta$ in (2.33)

$$\begin{aligned} f(x) &= \alpha_1 \cos x + \beta_1 \sin x, \\ g(x) &= \alpha_2 \cos x + \beta_2 \sin x. \end{aligned} \tag{2.34}$$

This reflects the consistency of r-CODEs and their solutions in finding the solutions of systems of ODEs. It is due to a procedure analogous to analytic continuation that in the intervening steps we obtain the solution (2.34) of system (2.32). This observation plays a central role in studying the symmetries of r-CODEs as it yields the symmetries of systems of ODEs. Thus certain systems of ODEs can be treated equally well in the complex domain. It not only helps in understanding the underlying structure of symmetries of r-CODEs but also its use in the compactification of several distinct operations under the complex umbrella.

**Examples 2.5:** Consider the r-CODE

$$(p(x)u')' + q(x)u = 0, \tag{2.35}$$

where

$$p(x) = p_1(x) + \iota p_2(x), \quad q(x) = q_1(x) + \iota q_2(x), \tag{2.36}$$

and each of the functions $p_1(x),\ p_2(x),\ q_1(x)$ and $q_2(x)$ is a continuous real function on the half-line $a \leq x < \infty$. Such DEs have many interpretations and applications. For example, if $p(x) = 1$ and the real and imaginary parts of equation (2.35) are separated, the resulting system of two real equations is

$$\begin{aligned} f_1'' + q_1 f_1 - q_2 f_2 &= 0, \\ f_2'' + q_2 f_1 + q_1 f_2 &= 0, \end{aligned} \tag{2.37}$$

which can be interpreted as equations of motions in the $f_1 f_2-$plane, where $u = f_1 + \iota f_2$. Similarly, if in (2.35) $p(x) = 1$ and $q(x) = \lambda Q(x)$, $Q(x)$ real and positive and $\lambda$ a complex parameter, one obtains

$$\begin{aligned} f_1'' + (\lambda_1 f_1 - \lambda_2 f_2)Q &= 0, \\ f_2'' + (\lambda_2 f_1 + \lambda_1 f_2)Q &= 0, \end{aligned} \tag{2.38}$$

which is used in studying Cauchy's problem for a generalized heat equation. Another important application is obtained by taking a special case of (2.35)

$$[u'/q(x)]' + q(x)u = 0, \tag{2.39}$$

where $q(x)$ is a continuous real function on $a \leq x < \infty$. The fundamental set of solutions consists of functionals $\sin \int_a^x q(t)dt$ and $\cos \int_a^x q(t)dt$.

## 2.4 Complex Lie point transformations

Consider complex parameter Lie point transformations given by

$$\tilde{z} = Z(z, u; \epsilon), \quad \tilde{u} = U(z, u; \epsilon), \tag{2.40}$$

where $\epsilon$ is a complex parameter such that $|\epsilon|^2$ is ignored. The law of composition is holomorphic (complex analytic) and satisfies all the group properties. When one uses Taylor's theorem,

$$\begin{aligned} \tilde{z} &= z + \epsilon \frac{\partial \tilde{z}}{\partial \epsilon}|_{\epsilon=0} + O(|\epsilon|^2) = z + \epsilon\zeta(z, u) + O(|\epsilon|^2), \\ \tilde{u} &= u + \epsilon \frac{\partial \tilde{u}}{\partial \epsilon}|_{\epsilon=0} + O(|\epsilon|^2) = u + \epsilon\chi(z, u) + O(|\epsilon|^2), \end{aligned} \tag{2.41}$$

where

$$\zeta(z, u) = \frac{\partial \tilde{z}}{\partial \epsilon}|_{\epsilon=0}, \quad \chi(z, u) = \frac{\partial \tilde{u}}{\partial \epsilon}|_{\epsilon=0}. \tag{2.42}$$

Then

$$\mathbf{Z} = \zeta \frac{\partial}{\partial z} + \chi \frac{\partial}{\partial u} \tag{2.43}$$

is called an infinitesimal complex generator of the complex Lie point transformations. Our aim is to *extract* two real symmetry generators corresponding to two infinitesimal parameters,

$\epsilon = \epsilon_1 + \iota\epsilon_2$, from a single complex generator $\mathbf{Z}$, i.e.

$$\begin{aligned} 2\mathbf{X} &= \zeta_1\frac{\partial}{\partial x} + \zeta_2\frac{\partial}{\partial y} + \chi_1\frac{\partial}{\partial f} + \chi_2\frac{\partial}{\partial g}, \\ 2\mathbf{Y} &= \zeta_2\frac{\partial}{\partial x} - \zeta_1\frac{\partial}{\partial y} + \chi_2\frac{\partial}{\partial f} - \chi_1\frac{\partial}{\partial g}, \end{aligned} \tag{2.44}$$

where

$$\zeta = \zeta_1 + \iota\zeta_2, \quad \chi = \chi_1 + \iota\chi_2. \tag{2.45}$$

Further, $\zeta$ and $\chi$ are holomorphic functions, i.e.

$$\frac{\partial\zeta_1}{\partial x} = \frac{\partial\zeta_2}{\partial y} \text{ and } \frac{\partial\zeta_1}{\partial y} = -\frac{\partial\zeta_2}{\partial x}, \tag{2.46}$$

hold at each point in the domain of definition of $u$. Therefore all coefficients of a complex Lie symmetry satisfy CREs. The above real generators correspond to the following real infinitesimal transformation

$$\tilde{z} = \tilde{x} + \iota\tilde{y}, \quad \tilde{u} = \tilde{f} + \iota\tilde{g}, \tag{2.47}$$

where

$$\begin{aligned} \tilde{x} &= x + \epsilon_1\zeta_1 + \epsilon_2\zeta_2 + O(\epsilon_1^2, \epsilon_2^2), \\ \tilde{y} &= y + \epsilon_2\zeta_1 - \epsilon_1\zeta_2 + O(\epsilon_1^2, \epsilon_2^2), \\ \tilde{f} &= f + \epsilon_1\chi_1 + \epsilon_2\chi_2 + O(\epsilon_1^2, \epsilon_2^2), \\ \tilde{g} &= g + \epsilon_2\chi_1 - \epsilon_1\chi_2 + O(\epsilon_1^2, \epsilon_2^2). \end{aligned} \tag{2.48}$$

Notice that the above real transformation is a special case of a general real infinitesimal transformation in two real dependent variables of two real independent variables. The coefficients

$\zeta_1, \zeta_2, \chi_1$ and $\chi_2$ can be calculated from (2.42) and are given by

$$\begin{aligned} 2\zeta_1(x,y,f,g) &= \frac{\partial \tilde{x}}{\partial \epsilon_1}|_{\epsilon=0} + \frac{\partial \tilde{y}}{\partial \epsilon_2}|_{\epsilon=0}, \\ 2\zeta_2(x,y,f,g) &= \frac{\partial \tilde{y}}{\partial \epsilon_1}|_{\epsilon=0} - \frac{\partial \tilde{x}}{\partial \epsilon_2}|_{\epsilon=0}, \\ 2\chi_1(x,y,f,g) &= \frac{\partial \tilde{f}}{\partial \epsilon_1}|_{\epsilon=0} + \frac{\partial \tilde{g}}{\partial \epsilon_2}|_{\epsilon=0}, \\ 2\chi_2(x,y,f,g) &= \frac{\partial \tilde{g}}{\partial \epsilon_1}|_{\epsilon=0} - \frac{\partial \tilde{f}}{\partial \epsilon_2}|_{\epsilon=0}. \end{aligned} \tag{2.49}$$

**Example 2.6:** Consider a one complex-parameter transformation of the form

$$\tilde{z} = e^{\epsilon} z, \quad \tilde{u} = e^{\epsilon} u, \quad |\epsilon| < 1. \tag{2.50}$$

Then, with the use of (2.42) the complex generator is given by

$$Z = z\frac{\partial}{\partial z} + u\frac{\partial}{\partial u}. \tag{2.51}$$

We can obtain the real transformations by putting $z = x + \iota y$ and $\epsilon = \epsilon_1 + \iota\epsilon_2$ in (2.50)

$$\begin{aligned} \tilde{z} &= e^{\epsilon_1}(\cos\epsilon_2 + \iota\sin\epsilon_2)(x + \iota y), \\ \tilde{u} &= e^{\epsilon_1}(\cos\epsilon_2 + \iota\sin\epsilon_2)(f + \iota g), \end{aligned} \tag{2.52}$$

which can be written as

$$\begin{aligned} \tilde{z} &= (1+\epsilon_1)\{1 - \frac{\epsilon_2^2}{2} + \iota(\epsilon_2 - \frac{\epsilon_2^3}{6})\}(x + \iota y), \\ \tilde{u} &= (1+\epsilon_1)\{1 - \frac{\epsilon_2^2}{2} + \iota(\epsilon_2 - \frac{\epsilon_2^3}{6})\}(f + \iota g). \end{aligned} \tag{2.53}$$

Since higher orders of $\epsilon_1$ and $\epsilon_2$ are neglected,

$$\begin{aligned} \tilde{z} &= (1+\epsilon_1)(1+\iota\epsilon_2)(x + \iota y), \\ \tilde{u} &= (1+\epsilon_1)(1+\iota\epsilon_2)(f + \iota g), \end{aligned} \tag{2.54}$$

that is

$$\begin{aligned}\tilde{z} &= (1+\epsilon_1)x - \epsilon_2 y + \iota(y + \epsilon_1 y + \epsilon_2 x),\\ \tilde{u} &= (1+\epsilon_1)f - \epsilon_2 g + \iota(g + \epsilon_1 g + \epsilon_2 f).\end{aligned} \tag{2.55}$$

Here the real infinitesimal transformations are

$$\begin{aligned}\tilde{x} &= x + \epsilon_1 x - \epsilon_2 y,\\ \tilde{y} &= y + \epsilon_1 y + \epsilon_2 x,\\ \tilde{f} &= f + \epsilon_1 f - \epsilon_2 g,\\ \tilde{g} &= g + \epsilon_1 g + \epsilon_2 f,\end{aligned} \tag{2.56}$$

and by using (2.49) we get

$$\begin{aligned}\zeta_1(x,y,f,g) &= x, \quad \zeta_2(x,y,f,g) = y,\\ \chi_1(x,y,f,g) &= f, \quad \chi_2(x,y,f,g) = g.\end{aligned} \tag{2.57}$$

Using (2.49) we can obtain the real infinitesimal generators

$$\begin{aligned}2\mathbf{X} &= x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + f\frac{\partial}{\partial f} + g\frac{\partial}{\partial g},\\ 2\mathbf{Y} &= y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y} + g\frac{\partial}{\partial f} - f\frac{\partial}{\partial g}.\end{aligned} \tag{2.58}$$

The physical significance of complex symmetries may be seen by restricting the above example to a single variable, i.e.

$$\tilde{z} = e^{\epsilon} z, \quad |\epsilon| < 1, \tag{2.59}$$

$$\boldsymbol{\zeta}(\tilde{z}) = \frac{d\tilde{z}}{d\epsilon}|_{\epsilon=0} = \tilde{z}. \tag{2.60}$$

Therefore the complex symmetry is

$$\mathbf{Z} = z\frac{\partial}{\partial z} \tag{2.61}$$

and yields

$$2\mathbf{X} = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} \quad \text{and} \quad 2\mathbf{Y} = y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}, \tag{2.62}$$

where $\mathbf{X}$ is the *generator of rescaling* and $\mathbf{Y}$ is the *generator of rotation* about the $z$-axis. Thus a complex symmetry couples two independent real Lie symmetries in an elegant way. This elegance of complex variables is of great significance for many physical applications. The above fact can also be illustrated by considering the product of two complex numbers in their polar forms, namely

$$z_1 z_2 = r_1 r_2 e^{\iota(\theta_1+\theta_2)}. \tag{2.63}$$

Thus the operation of multiplication of two complex numbers describes two independent real operations, i.e. rescaling $r_1 r_2$ and rotation $\theta_1 + \theta_2$. Notice that a simple complex Lie symmetry (2.61) corresponding to infinitesimal transformation (2.59) yields two real Lie symmetries (2.62) corresponding to two real transformations *rescaling* and *rotation*.

## 2.5 Complex Lie symmetries

**Definition**:

Let $M$ be a complex manifold and $G$ be a complex Lie group that acts on $M$. Also consider a system of $m$-complex algebraic equations in $n$-complex variables of the form

$$U_\upsilon(z^\mu) = 0, \ (\upsilon = 1, ..., m, \quad \mu = 1, ..., n), \tag{2.64}$$

where $U_\nu$ are complex analytic functions on $M$. Then a CLS is a symmetry that maps complex solutions of (2.64) into complex solutions. We translate the above equations into real variables by putting

$$z^\mu = x^\mu + \iota y^\mu \tag{2.65}$$

and

$$U_\upsilon = F_\upsilon + \iota G_\upsilon, \tag{2.66}$$

where

$$F_\upsilon = F_\upsilon(x^\mu, y^\mu), \quad G_\upsilon = G_\upsilon(x^\mu, y^\mu). \tag{2.67}$$

Using (2.64) one can write

$$F_{\upsilon}=0,\quad G_{\upsilon}=0, \tag{2.68}$$

which is a system of $2m$ real algebraic equations in $2n$ real variables. Thus each CLS of (2.64) gives two real Lie symmetries for the above system of real algebraic equations. It is obvious that the above system is a special case of a system of $2m$ equations in $2n$ variables since the $U_{\upsilon}$ are holomorphic functions. Thus we have

$$F_{\upsilon,_{x^{\mu}}}=G_{\upsilon,_{y^{\mu}}},\quad F_{\upsilon,_{y^{\mu}}}=-G_{\upsilon,_{x^{\mu}}}, \tag{2.69}$$

It is easily verified that Lie's three fundamental theorems hold in the complex as they do in the real domain [37, 38, 39, 40]. We restate them explicitly for the complex case. Lie's first fundamental theorem simply states that a one-complex dimensional Lie subgroup of $G$ defines an integral complex curve on $G$ of a left- or right-invariant complex vector field determined by specifying a tangent vector at the identity element of $G$ [36]. Lie's second and third theorems reflect the fact that the vector space of all left- or right-invariant complex vector fields has the structure of a complex Lie algebra the commutator of which is the same at all points in $G$ because of the invariance properties of the complex vector fields.

## 2.6 Prolongation and symmetry conditions

Consider an $n^{th}$ order CODE of the form,

$$H(z,u(z),u'(z),u''(z),.....u^{(n)}(z))=0. \tag{2.70}$$

Suppose we can reexpress this equation in the form,

$$u^{(n)}=w(z,u,u',u'',...,u^{(n-1)}), \tag{2.71}$$

where $w$ is a complex analytic function. A complex Lie point transformation describes an infinitesimal motion on a complex manifold. Consider an infinitesimal complex point transformation

$z \longrightarrow z(\epsilon)$. When one uses Taylor's theorem,

$$\tilde{z}(\epsilon) = z + \epsilon\zeta(z) + O(|\epsilon|^2), \quad \tilde{u}(\tilde{z}) = u(z) + \epsilon\chi(z) + O(|\epsilon|^2), \tag{2.72}$$

where

$$\zeta(z) = \partial_\epsilon \tilde{z} \mid_{\epsilon=0}, \quad \chi(u) = \partial_\epsilon \tilde{u} \mid_{\epsilon=0} . \tag{2.73}$$

The infinitesimal generator corresponding to the infinitesimal complex point transformation is

$$\mathbf{Z} = \zeta(z,u)\frac{\partial}{\partial z} + \chi(z,u)\frac{\partial}{\partial u}. \tag{2.74}$$

Here $\mathbf{Z}$ describes a complex vector field that maps each point of a complex manifold onto a tangent vector on the complex manifold. The prolonged symmetry generator in the complex case is similar to the real case, i.e.

$$\mathbf{Z} = \zeta(z)\frac{\partial}{\partial z} + \chi(z)\frac{\partial}{\partial u} + \chi^{[i]}(z)\frac{\partial}{\partial u^{(i)}}, \tag{2.75}$$

where we use the Einstein summation convention that repeated indices are summed over $i = 1, 2, ..., n$. Here

$$\chi^{[n]}(z) = \frac{d\chi^{[n-1]}(z)}{dz} - u^n(z)\frac{d\zeta(z)}{dz}. \tag{2.76}$$

Then the symmetry condition in the complex case is

$$\mathbf{Z}w = \chi^{[n]} \bmod H = 0. \tag{2.77}$$

The above complex symmetry condition yields two real symmetry conditions for the study of symmetries of systems of PDEs.

## 2.7 Classification of systems of PDEs

We have seen that the symmetries of a DE form a Lie algebra. Sometimes we are only able to calculate a subalgebra of symmetries. We can reverse the order, i.e. by the classification of second-order ODEs relative to the algebras they admit. The Lie group classification of a second-order CODE is the same in the complex case as in the real case, i.e. by the calculation of the complex differential invariants [29, 54, 62, 63]. Such a classification of second-order CODEs yields a non trivial classification of systems of PDEs corresponding to them. We present a few examples to illustrate the fact.

**Example 2.7:** A general second-order CODE that admits a trivial complex translational symmetry $\partial/\partial z$ is given as

$$u'' = w(u, u'), \tag{2.78}$$

which is clearly invariant under complex translation $z \to z + c$, where $c$ is a complex number. The complex translation symmetry yields the real translation symmetry generators in the $x$ and $y$ directions

$$\mathbf{X} = \frac{\partial}{\partial x}, \quad \mathbf{Y} = \frac{\partial}{\partial y}. \tag{2.79}$$

The system of PDEs derived from (2.78) is

$$\begin{aligned} f_{xx} - f_{yy} + 2g_{xy} &= 4G(f, g, h, l), \quad f_x = g_y, \\ g_{xx} - g_{yy} - 2f_{xy} &= 4H(f, g, h, l), \quad f_y = -g_x. \end{aligned} \tag{2.80}$$

Notice that the above equations do not depend on $x$ and $y$ explicitly. Thus these admit (2.79). As a special case, if $w$ does not depend upon $u$, then $G$ and $H$ in (2.80) also do not depend upon $f$ and $g$ and the above system also inherits translation symmetries in $f$ and $g$.

**Example 2.8:** We now take a non linear CODE admitting

$$\begin{aligned} \mathbf{Z}_1 &= \partial/\partial z, \quad \mathbf{Z}_2 = \partial/\partial u, \\ \mathbf{Z}_3 &= z\partial/\partial z + (z + u)\partial/\partial u \end{aligned} \tag{2.81}$$

of the form

$$u'' = c \exp(-u'). \tag{2.82}$$

Then the corresponding system of PDEs is

$$\begin{aligned} f_{xx} - f_{yy} + 2g_{xy} &= 4e^{-h}(c_1 \cos(l) + c_2 \sin(l)), \;\; f_x = g_y, \\ g_{xx} - g_{yy} - 2f_{xy} &= 4e^{-h}(c_2 \cos(l) - c_1 \sin(l)), \;\; f_y = -g_x, \end{aligned} \tag{2.83}$$

where $c = c_1 + \iota c_2$. The above system clearly admits trivial translational symmetries in $x, y, f$ and $g$. Further it has

$$\begin{aligned} &x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + (x+f)\frac{\partial}{\partial f} + (y+g)\frac{\partial}{\partial g}, \\ &y\frac{\partial}{\partial y} - x\frac{\partial}{\partial x} + (y+g)\frac{\partial}{\partial f} - (x+f)\frac{\partial}{\partial g}. \end{aligned} \tag{2.84}$$

It seems difficult to find the symmetries (2.84) of the system (2.83) by the usual Lie approach. Notice that the algebraic classification of a non linear CODE yields symmetries of non linear system of PDEs.

**Example 2.9:** The CLSs admitted by a CODE of the form,

$$u''(z) = 0, \tag{2.85}$$

are

$$\begin{aligned} \mathbf{Z}_1 &= \frac{\partial}{\partial z}, \;\; \mathbf{Z}_2 = \frac{\partial}{\partial u}, \;\; \mathbf{Z}_3 = z\frac{\partial}{\partial z}, \;\; \mathbf{Z}_4 = u\frac{\partial}{\partial u}, \;\; \mathbf{Z}_5 = z\frac{\partial}{\partial u}, \\ \mathbf{Z}_6 &= u\frac{\partial}{\partial z}, \;\; \mathbf{Z}_7 = uz\frac{\partial}{\partial u} + z^2\frac{\partial}{\partial z}, \;\; \mathbf{Z}_8 = uz\frac{\partial}{\partial z} + u^2\frac{\partial}{\partial u}. \end{aligned} \tag{2.86}$$

The CLSs have the same form as in the real case. These can be decomposed into their real and imaginary parts to give us symmetries of the following system of PDEs

$$\begin{aligned} f_{xx} - f_{yy} + 2g_{xy} &= 0, \;\; f_x = g_y, \\ g_{xx} - g_{yy} - 2f_{xy} &= 0, \;\; f_y = -g_x, \end{aligned} \tag{2.87}$$

i.e. by placing

$$\mathbf{Z}_j = \mathbf{X}_j + \iota \mathbf{Y}_j, \text{ for all } j = 1, ..., 8. \tag{2.88}$$

$$\mathbf{X}_1 = \frac{\partial}{\partial x}, \quad \mathbf{Y}_1 = \frac{\partial}{\partial y}, \quad \mathbf{X}_2 = \frac{\partial}{\partial f}, \quad \mathbf{Y}_2 = \frac{\partial}{\partial g},$$

$$\mathbf{X}_3 = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}, \quad \mathbf{Y}_3 = y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y},$$

$$\mathbf{X}_4 = f\frac{\partial}{\partial f} + g\frac{\partial}{\partial g}, \quad \mathbf{Y}_4 = g\frac{\partial}{\partial f} - f\frac{\partial}{\partial g},$$

$$\mathbf{X}_5 = x\frac{\partial}{\partial f} + y\frac{\partial}{\partial g}, \quad \mathbf{Y}_5 = y\frac{\partial}{\partial f} - x\frac{\partial}{\partial g},$$

$$\mathbf{X}_6 = f\frac{\partial}{\partial x} + g\frac{\partial}{\partial y}, \quad \mathbf{Y}_6 = g\frac{\partial}{\partial x} - f\frac{\partial}{\partial y},$$

$$\mathbf{X}_7 = (fx - gy)\frac{\partial}{\partial x} + (fy + gx)\frac{\partial}{\partial y} + (f^2 - g^2)\frac{\partial}{\partial f} + 2fg\frac{\partial}{\partial g},$$

$$\mathbf{Y}_7 = (fy + gx)\frac{\partial}{\partial x} - (fx - gy)\frac{\partial}{\partial y} - (f^2 - g^2)\frac{\partial}{\partial g} + 2fg\frac{\partial}{\partial f},$$

$$\mathbf{X}_8 = (fx - gy)\frac{\partial}{\partial f} + (fy + gx)\frac{\partial}{\partial g} + (x^2 - y^2)\frac{\partial}{\partial x} + 2xy\frac{\partial}{\partial y},$$

$$\mathbf{Y}_8 = (fy + gx)\frac{\partial}{\partial f} - (fx - gy)\frac{\partial}{\partial g} - (x^2 - y^2)\frac{\partial}{\partial y} + 2xy\frac{\partial}{\partial x}. \tag{2.89}$$

Since a second-order CODE admits an 8-dimensional maximal algebra (as a real DE [20, 42]), one always gets a 16-dimensional algebra of systems of PDEs (2.87). The complex Lie algebra of an $m$-complex-dimensional complex vector equation,

$$(u^a)'' = 0, \quad u^a = u^a(z), \qquad a = 1, ..., m, \tag{2.90}$$

is isomorphic to $sl(m+2, \mathbb{C})$. The solution of above DE represents a complex line in $m$ dimensional space. One can derive geometrical implications by looking at dimensions of the real and complex algebras. The difference between two algebras can be seen in the Table 1:

Table 1: Comparison of RLS and CLS of a straight line

| $n$ | **Real Algebra Dimension** | **Complex Algebra Real Dimension** | **1st Diff.** | **2nd Diff.** |
|---|---|---|---|---|
| 1 | $\lvert sl(2+2,\mathbb{R})\rvert = 15$ | $\lvert sl(1+2,\mathbb{C})\rvert = 8$<br>16-real sym. | -1 | 6 |
| 2 | $\lvert sl(4+2,\mathbb{R})\rvert = 35$ | $\lvert sl(2+2,\mathbb{C})\rvert = 15$<br>30-real sym. | 5 | 10 |
| 3 | $\lvert sl(6+2,\mathbb{R})\rvert = 63$ | $\lvert sl(3+2,\mathbb{C})\rvert = 24$<br>48-real sym. | 15 | 14 |
| 4 | $\lvert sl(8+2,\mathbb{R})\rvert = 99$ | $\lvert sl(4+2,\mathbb{C})\rvert = 35$<br>70-real sym. | 29 | 18 |
| 5 | $\lvert sl(10+2,\mathbb{R})\rvert = 143$ | $\lvert sl(5+2,\mathbb{C})\rvert = 48$<br>96-real sym. | 47 | ... |
| | $\lvert sl(2n+2,\mathbb{R})\rvert =$<br>$(2n+2)^2-1$ | $\lvert sl(n+2,\mathbb{C})\rvert =$<br>$2[(n+2)^2-1]$ | $2n^2-3$ | $4n+2$ |

Thus one gets

$$\begin{aligned} d_{a+1} &= d_a+4, \qquad a=1,...,n, \\ d_1 &= 6. \end{aligned} \tag{2.91}$$

It may be noticed that a complex line in $\mathbb{C} \equiv \mathbb{R}^2$ has 16 real symmetries whereas a real line in $\mathbb{R}^2$ has only 15 real symmetries. Similarly a complex line in $\mathbb{C}^2 \equiv \mathbb{R}^4$ has only 30 real symmetries whereas a real line in $\mathbb{R}^4$ has 35 real symmetries. Thus in higher dimensions a real line has more real symmetries than a complex line because a single complex symmetry couples two real symmetries.

## 2.8 Reduction in order of systems of PDEs

One of the main uses of symmetries is to construct point transformations that map a DE into a form such that the order of the resulting equation can be reduced. We use a CLS to reduce the order of a CODE in the same way as in the real case.

**Example 2.10:** Consider the complexified harmonic oscillator equation

$$u''(z) = -u(z). \tag{2.92}$$

Then the corresponding PDEs are given by

$$\begin{aligned} f_{xx} - f_{yy} + 2g_{xy} &= -4f, \quad f_x = g_y, \\ g_{xx} - g_{yy} - 2f_{xy} &= -4g, \quad f_y = -g_x. \end{aligned} \tag{2.93}$$

The above system can be mapped to a system which is equivalent to a special case of a Riccatti system in two dimensions. One of the CLSs of CODE (2.92) is given by

$$\mathbf{Z} = u\frac{\partial}{\partial u}, \tag{2.94}$$

which in turn yields the two RLSs of system (2.93)

$$\mathbf{X} = f\frac{\partial}{\partial f} + g\frac{\partial}{\partial g}, \quad \mathbf{Y} = g\frac{\partial}{\partial f} - f\frac{\partial}{\partial g}. \tag{2.95}$$

When one introduces two new complex variables $s$ and $t$ defined by

$$z = s, \quad u = e^t, \tag{2.96}$$

the equation is transformed into

$$t'' + t'^2 + 1 = 0. \tag{2.97}$$

By putting

$$t' = p(s) = a(x, y) + \imath b(x, y) \quad \text{(because } s = z\text{)} \tag{2.98}$$

$$p' = a_x + b_y + \iota(b_x - a_y), \tag{2.99}$$

we get

$$p' + p^2 + 1 = 0, \tag{2.100}$$

which is a CODE of order 1. After using (2.99) we get

$$\begin{aligned} a_x + b_y + a^2 - b^2 + 1 &= 0, \\ b_x - a_y + 2ab &= 0. \end{aligned} \tag{2.101}$$

Notice that this is a special case of Riccatti system of second-order PDEs in two dimensions. Also, the original system (2.93) is equivalent to (2.101). Thus one can use a CLS to reduce the order of a CODE by one which in turn results in the elimination of second-order partial derivatives of two real functions. The above algorithm can be applied to a general linear homogenous second-order CODE of the form

$$u'' + a_1(z)u' + a_2(z)u = 0, \tag{2.102}$$

which also admits (2.94). Therefore its order can be reduced by using the same complex transformation (2.96).

## 2.9 Classification of systems of ODEs

We now present the Lie table for r-CODEs which admit two-dimensional algebras of symmetries in Table 2, thus covering all cases.

Table 2: Lie table for r-CODEs. Here $p = \partial/\partial x$ and $q = \partial/\partial u$.

| Type | Dimension of algebra | Nonzero commutators | Forms for generators | Representative Equation |
|---|---|---|---|---|
| I | 2 | | $\mathbf{X}_1 = p, \quad \mathbf{X}_2 = q$ | $u'' = w(u')$ |
| II | 2 | | $\mathbf{X}_1 = q, \quad \mathbf{X}_2 = xq$ | $u'' = w(x)$ |
| III | 2 | $\mathbf{X}_1$ | $\mathbf{X}_1 = q, \quad \mathbf{X}_2 = xp + uq$ | $xu'' = w(u')$ |
| IV | 2 | $\mathbf{X}_1$ | $\mathbf{X}_1 = q, \quad \mathbf{X}_2 = uq$ | $u'' = w(x)u'$ |

Now we provide an extension of the Lie table for systems of two second-order ODEs. In Table 3 the functions $\alpha_i$, $\sigma$ and $\eta$ are arbitrary functions of their arguments except in III (a) and III (b). In III (a) $\alpha$ and $\beta$ are constrained by the given relation and in III (b) $\sigma(f', g')$ and $\eta(f', g')$ must satisfy analyticity conditions which are forced as a consequence of the representative equation being further subjected to invariance under the operators $\mathbf{X}_1$, $\mathbf{X}_2$ and $\mathbf{Y}_2$.

We notice that Types II, III and IV split up into two cases each. This is due to the fact that the $x$ and $u$ do not have interchangeable roles when they are split as for a system in the extended table there is one independent and two dependent variables. Further we see that II (a) and IV (a) imply linearization. This provides us with an elegant algebraic criterion for linearization for systems directly corresponding to the r-CODE. One can refer integrability properties in the extended table to the Lie Table 3.

Table 3: Lie table for systems of ODEs. Here $p = \partial/\partial x$, $r = \partial/\partial f$ and $s = \partial/\partial g$

| Type | Dimension of algebra | Forms for generators | Nonzero commutators | Representative System Equation |
|---|---|---|---|---|
| I | 3 | $\mathbf{X}_1 = p,\ \mathbf{X}_2 = r,$ $\mathbf{X}_3 = s$ | | $f'' = \sigma(f', g'),\ g'' = \eta(f', g')$ |
| II a | 4 | $\mathbf{X}_1 = r,\ \mathbf{Y}_1 = s,$ $\mathbf{X}_2 = xr,\ \mathbf{Y}_2 = xs$ | | $f'' = \sigma(x),\ g'' = \eta(x)$ |
| II b | 3 | $\mathbf{X}_1 = p,\ \mathbf{X}_2 = fp,$ $\mathbf{Y}_2 = gp$ | | $f'' + \sigma(f,g)f'^3 - 3\sigma(f,g)f'g'^2$ $-3\eta(f,g)f'^2g' + \eta(f,g)g'^3 = 0$ $g'' + 3\sigma(f,g)f'^2g' - \sigma(f,g)g'^3$ $+\eta(f,g)f'^3 - 3\eta(f,g)f'g'^2 = 0$ |
| III a | 4 | $\mathbf{X}_1 = r,\ \mathbf{Y}_1 = s$ $\mathbf{X}_2 = 2xp + fr + gs$ $\mathbf{Y}_2 = gr - fs$ | $[\mathbf{Y}_1, \mathbf{X}_2] = \mathbf{Y}_1,$ $[\mathbf{X}_1, \mathbf{Y}_2] = -\mathbf{Y}_1$ $[\mathbf{X}_1, \mathbf{Y}_2] = \mathbf{X}_1$ | $xf'' = f'\alpha(g'/f')$ $xg'' = f'\beta(g'/f')$ $\beta' = \frac{\alpha f' + \beta g'}{\alpha g' - \beta f'}\alpha'$ |
| III b | 3 | $\mathbf{X}_1 = p,$ $\mathbf{X}_2 = 2xp + fr + gs,$ $\mathbf{Y}_2 = gr - fs$ | $[\mathbf{X}_1, \mathbf{X}_2] = 2\mathbf{X}_1$ | $ff'' - gg'' + \sigma(f', g')(f'^3 - 3f'g'^2)$ $-\eta(f', g')(3f'^2g' - g'^3) = 0$ $gf'' + fg'' + \sigma(f', g')(3f'^2g' - g'^3)$ $+\eta(f', g')(f'^3 - 3f'g'^2) = 0$ |
| IV a | 4 | $\mathbf{X}_1 = r,\ \mathbf{Y}_1 = s$ $\mathbf{X}_2 = fr + gs$ $\mathbf{Y}_2 = gr - fs$ | $[\mathbf{X}_1, \mathbf{X}_2] = \mathbf{X}_1$ $[\mathbf{Y}_1, \mathbf{X}_2] = \mathbf{Y}_1$ $[\mathbf{X}_1, \mathbf{Y}_2] = -\mathbf{Y}_1$ $[\mathbf{Y}_1, \mathbf{Y}_2] = \mathbf{X}_1$ | $f'' = \alpha_1(x)f' - \alpha_2(x)g'$ $g'' = \alpha_1(x)g' + \alpha_2(x)f'$ |
| IV b | 2 | $\mathbf{X}_1 = p,$ $\mathbf{X}_2 = xp$ | $[\mathbf{X}_1, \mathbf{X}_2] = \mathbf{X}_1$ | $f'' + \alpha_1(f,g)f'^2$ $-\alpha_1(f,g)g'^2 - 2\alpha_2(f,g)f'g' = 0$ $g'' + 2\alpha_1(f,g)f'g'$ $+\alpha_2(f,g)(f'^2 - g'^2) = 0$ |

## 2.10 Reduction in order of systems of ODEs

We use a CLS to obtain the reduction of order of the system of ODEs corresponding to r-CODEs. The order of an r-CODE can be reduced by one if it admits a one parameter Lie group of transformations which can be used to construct real transformations that reduce the order of the systems of ODEs.

**Example 2.11:** Consider the complexified harmonic oscillator equation

$$u''(x) = -u(x). \tag{2.103}$$

The corresponding ODEs are given by

$$\begin{aligned} f'' &= -f, \\ g'' &= -g. \end{aligned} \tag{2.104}$$

The above system can be mapped to a system which is equivalent to a special case of the Riccatti system. One of the CLSs of the r-CODE (2.103) is given by

$$\mathbf{Z} = u\frac{\partial}{\partial u}, \tag{2.105}$$

which in turn yields the two RLSs of system (2.104)

$$\mathbf{X} = f\frac{\partial}{\partial f} + g\frac{\partial}{\partial g}, \quad \mathbf{Y} = g\frac{\partial}{\partial f} - f\frac{\partial}{\partial g}. \tag{2.106}$$

When one introduces two new complex variables $s$ and $t$ defined by

$$x = s, \quad u = e^t, \tag{2.107}$$

the equation transforms into

$$t'' + t'^2 - 1 = 0. \tag{2.108}$$

By putting

$$t' = p(s) = a(x) + \iota b(x) \quad (\text{as } s = x) \tag{2.109}$$

$$p' = a' + \iota b', \tag{2.110}$$

we get

$$p' + p^2 - 1 = 0, \tag{2.111}$$

which is a CODE of order 1. Using (2.110) r-CODE (2.111) yields

$$\begin{aligned} a' + a^2 - b^2 - 1 &= 0, \\ b' + 2ab &= 0. \end{aligned} \tag{2.112}$$

This is a special case of a Riccatti system. Further the original system (2.104) is equivalent to (2.112). We can apply the same transformation to reduce the order of

$$u'' + a_1(x)u' + a_2(x)u = 0 \tag{2.113}$$

by using (2.107) .

## Chapter 3

# Linearization of CODEs and r-CODEs

Linearization, i.e. the use of point transformations, to convert nonlinear ODEs to linear form, was introduced by Lie [39]. He provided the linearizability criteria for any scalar second-order RODE to be point transformable to a linear RODE via invertible maps of both independent and dependent variables. He found that the RODE must be at most cubic in the first derivative and that a particular over-determined system of conditions for the coefficients must be satisfied (see e.g., $[37-40]$). Tressè $[65, 66]$ studied the linearization of a second-order RODE by looking at the two relative invariants of the equivalence group of transformations, the vanishing of both of which gives the necessary and sufficient conditions for linearization. These are equivalent to the Lie conditions [45].

In this Chapter we have extended the Lie approach of linearization of RODEs via complex point transformations to the complex domain and obtained corresponding Lie criteria for CODEs. The CODEs can be linearized in the same way as RODEs via complex point transformations. A CODE yields a system of PDEs after decomposing all the complex functions, variables and derivatives into their real and imaginary parts. Further the complex analyticity of $u(z)$ yields CREs which are already linear and thus yield no difference for the linearization of systems of PDEs. The linearization of systems of PDEs corresponding to a CODE follows directly from the linearization of that CODE via real point transformations that are obtained after decomposing the complex point transformation.

## 3.1 Equivalent CODEs

Two CODEs are (locally) equivalent via an invertible complex transformations if one can be transformed into the other by an invertible complex transformation. A CODE is a system of four PDEs, which includes the two linear CREs in two unknown functions of two independent variables. If a CODE is equivalent to another CODE via invertible complex transformations then the system of PDEs corresponding to that CODE is also equivalent to the other system of PDEs. The invertible complex transformation yields two real transformations, which can be used to transform one system of PDEs to another system.

Every first-order CODE,

$$u'(z) = w(z, u), \tag{3.1}$$

can be transformed into the simplest one, $u' = 0$, via a complex point transformation [46]. Similarly every *linear* second-order CODE can always be transformed into its simplest form via invertible complex transformations, specifically into $u''(z) = 0$. For example the complexified Riccati equation,

$$u'(z) + u^2 = 0, \tag{3.2}$$

is transformable to $U' = 0$ by means of

$$Z = z, \quad U = (1/u) - z. \tag{3.3}$$

Also the complexified simple harmonic oscillator equation

$$u''(z) + u = 0, \tag{3.4}$$

can be transformed into $U'' = 0$ via the invertible complex transformation

$$Z = \tan z, \quad U = u \sec z. \tag{3.5}$$

Generally the above argument does not hold true for higher order CODEs. To decompose the above CODEs into corresponding systems of PDEs, we write

$$\begin{aligned} z &= x + \iota y, u(z) = f(x, y) + \iota g(x, y), \qquad (3.6) \\ w &= w_1 + \iota w_2 \qquad (3.7) \end{aligned}$$

and assume that the complex transformation $(z, u) \longrightarrow (Z, U)$ is equivalent to the real transformation $(x, y, f, g) \longrightarrow (X, Y, F, G)$. The system of PDEs corresponding to a first-order CODE is

$$\begin{aligned} f_x + g_y &= 2w_1(x, y, f, g), \\ g_x - f_y &= 2w_2(x, y, f, g), \end{aligned} \qquad (3.8)$$

which can be transformed into

$$F_x + G_y = 0, \quad G_x - F_y = 0, \qquad (3.9)$$

by an invertible real transformation derived from the complex point transformation. Similarly the system of PDEs corresponding to a linear second-order CODE is

$$\begin{aligned} f_{xx} - f_{yy} + 2g_{xy} &= 4w_1(x, y, f, g, h, l), \\ g_{xx} - g_{yy} - 2f_{xy} &= 4w_2(x, y, f, g, h, l), \end{aligned} \qquad (3.10)$$

where both $w_1$ and $w_2$ are two such real functions that do not give rise to a nonlinear system. The above system can be transformed into

$$\begin{aligned} F_{XX} - F_{YY} + 2G_{XY} &= 0, \\ G_{XX} - G_{YY} - 2F_{XY} &= 0, \end{aligned} \qquad (3.11)$$

via an invertible real transformation that is obtained from a complex transformation.

**Examples 3.1:** The complexified Ricatti equation is equivalent to

$$\begin{aligned} f_x + g_y &= 2(-f^2 + g^2), \\ g_x - f_y &= -8fg. \end{aligned} \tag{3.12}$$

If we apply the transformation

$$X = x, \quad Y = y, \tag{3.13}$$

$$F = \frac{f}{f^2 + g^2} - x, \quad G = \frac{-g}{f^2 + g^2} - y, \tag{3.14}$$

we get

$$\begin{aligned} F_X + G_Y &= 0, \\ G_X - F_Y &= 0. \end{aligned} \tag{3.15}$$

Notice that a simple complex transformation for the complexified Riccatti equation yields a non trivial real transformation (3.13) and (3.14) that transforms the system of PDEs (3.12) into its simple analogue (3.15). This remarkable feature of complex variables is of great significance for the linearization of PDEs and their systems.

**Examples 3.2:** The system

$$\begin{aligned} f_{xx} - f_{yy} + 2g_{xy} &= -4f, \\ g_{xx} - g_{yy} - 2f_{xy} &= -4g \end{aligned} \tag{3.16}$$

can be transformed into

$$\begin{aligned} F_{XX} - F_{YY} + 2G_{XY} &= 0, \\ G_{XX} - G_{YY} - 2F_{XY} &= 0 \end{aligned} \tag{3.17}$$

via the real transformations

$$
\begin{aligned}
X &= \frac{(1/2)\sin 2x}{\cos^2 x + \sinh^2 y}, \quad Y = \frac{(1/2)\sinh 2y}{\cos^2 x + \sinh^2 y}, \\
F &= \frac{f\cos x\cosh y - g\sin x\sinh y}{\cos^2 x + \sinh^2 y}, \quad G = \frac{f\sin x\sin y + g\cos x\cosh y}{\cos^2 x + \sinh^2 y}.
\end{aligned} \tag{3.18}
$$

Thus the system of PDEs corresponding to a CODE can easily be transformed into its simplified (linear) form by invoking complex transformations. It would have been very difficult to guess or calculate the real transformations, (3.18), that transform system (3.16) into system (3.17) without this formalism. It is a very nice characteristic of complex variables.

## 3.2 Lie conditions for CODEs and r-CODEs

Lie proved that a second-order scalar ODE which is at most cubic in its first derivative is linearizable provided it satisfies four differential constraints involving two auxiliary variables that Tressé reduced to two differential conditions [65, 66]. We restate the Lie theorem for CODEs by considering a CODE which is at most cubic in its first derivative

$$u''(z) = A(z,u)u'^3 + B(z,u)u'^2 + C(z,u)u' + D(z,u), \tag{3.19}$$

where $A$, $B$, $C$ and $D$ are complex-valued functions, is linearizable according to Lie theorem. Now we compile several known results (see [46]) based on linearization in the complex domain as there is no difference made for the said purpose in Theorem 3.1. In the subsequent section we decompose (3.19) into a system of PDEs by using (2.8) and by taking $h$ and $l$ as real and imaginary parts of $u'$.

**Theorem 3.1.** The following statements are equivalent:

**1.** A scalar second-order r-CODE is linearizable via a complex-like transformation;

**2.** Equation (3.19) has a maximal $8-$dimensional complex Lie algebra;

**3.** The Tressé relative invariants,

$$I_1 = w_{u'u'u'u'},$$

$$I_2 = \frac{d^2}{dz^2}w_{u'u'} - 4\frac{d}{dz}w_{u'u} - 3w_u w_{u'u'} + 6w_{uu} + w_{u'}(4w_{u'u} - \frac{d}{dz}w_{u'u'}), \tag{3.20}$$

both vanish identically for (3.19);

**4.** The coefficients $A$ to $D$ in (3.19) satisfy the condition of Lie type in the complex domain

$$\begin{aligned}
k_z &= kK - AD + \frac{1}{3}C_u - \frac{2}{3}B_z,\\
k_u &= -k^2 - Bk - AK - A_z - AC,\\
K_z &= K^2 + Dk + CW - D_u + BD,\\
K_u &= -Kk + AD + \frac{1}{3}B_z - \frac{2}{3}C_u,
\end{aligned} \tag{3.21}$$

where $k$ and $K$ are auxiliary complex functions;

**5.** The coefficients in (3.19) namely $A$ to $D$ also satisfy the Lie compatibility conditions

$$\begin{aligned}
3A_{zz} + 3A_zC - 3A_uD + C_{uu} - 6AD_u + BC_u - 2BB_z - 2B_{zu} &= 0,\\
6A_zD - 3B_uD + 3AD_z + B_{zz} - 2C_{zu} - 3BD_u + 3D_{uu} + 2CC_u - CB_z &= 0;
\end{aligned} \tag{3.22}$$

**6.** Equation (3.19) has two commuting symmetries, $\mathbf{Z}_1$ and $\mathbf{Z}_2$, with $\mathbf{Z}_1 = \rho(z,u)\mathbf{Z}_2$ for a nonconstant complex function, $\rho$, such that a point transformation, $Z = Z(z,u)$ and $U = U(z,u)$, which brings $\mathbf{Z}_1$ and $\mathbf{Z}_2$ to their canonical forms

$$\mathbf{Z}_1 = \frac{\partial}{\partial U}, \quad \mathbf{Z}_2 = Z\frac{\partial}{\partial U}, \tag{3.23}$$

reduces the equation to the linear form $U'' = W(Z)$;

**7.** Equation (3.19) has two noncommuting symmetries $\mathbf{Z}_1, \mathbf{Z}_2$, in a suitable basis with $[\mathbf{Z}_1, \mathbf{Z}_2] = \mathbf{Z}_1,\ \mathbf{Z}_1 = \rho(z,u)\mathbf{Z}_2$ for a nonconstant complex function, $\rho$, such that a point change of variables $Z = Z(z,u),\ U = U(z,u)$, which brings $\mathbf{Z}_1$ and $\mathbf{Z}_2$ to their canonical forms

$$\mathbf{Z}_1 = \frac{\partial}{\partial U}, \quad \mathbf{Z}_2 = U\frac{\partial}{\partial U}, \tag{3.24}$$

reduces the equation to the linear form $U'' = U'W(Z)$;

**8.** Equation (3.19) has two commuting symmetries, $\mathbf{Z}_1$ and $\mathbf{Z}_2$, with $\mathbf{Z}_1 \neq \rho(z,u)\mathbf{Z}_2$ for a nonconstant complex function, $\rho$, such that a point transformation, $Z = Z(z,u), U = U(z,u)$, which brings $\mathbf{Z}_1$ and $\mathbf{Z}_2$ to their canonical forms

$$\mathbf{Z}_1 = \frac{\partial}{\partial Z}, \quad \mathbf{Z}_2 = \frac{\partial}{\partial U}, \tag{3.25}$$

reduces the equation to one which is at most cubic in the first derivative;

**9.** Equation (3.19) has two noncommuting symmetries $\mathbf{Z}_1$ and $\mathbf{Z}_2$ in a suitable basis with $[\mathbf{Z}_1, \mathbf{Z}_2] = \mathbf{Z}_1$ and $\mathbf{Z}_1 \neq \rho(z,u)\mathbf{Z}_2$ for a nonconstant complex function, $\rho$, such that a point transformation $Z = Z(z,u),\ U = U(z,u)$, which brings $\mathbf{Z}_1$ and $\mathbf{Z}_2$ to their canonical forms

$$\mathbf{Z}_1 = \frac{\partial}{\partial U}, \quad \mathbf{Z}_2 = Z\frac{\partial}{\partial Z} + U\frac{\partial}{\partial U}, \tag{3.26}$$

reduces the equation to

$$ZU'' = aU'^3 + bU'^2 + (1 + \frac{b^2}{3a})U' + \frac{b}{3a} + \frac{b^3}{27a^2}, \tag{3.27}$$

where $a$ $(\neq 0)$ and $b$ are complex constants.

## 3.3 Lie conditions for systems of PDEs

According to Lie's theorem the necessary condition for a second-order CODE to be linearizable is that it is at most cubic in its first-order derivative *implies* the necessary conditions for system of PDEs corresponding to that CODE, i.e. the system must be at most cubic in first derivatives together with certain constraints on coefficients. The real transformations for linearization of a system of nonlinear PDEs can be obtained by decomposing complex transformations that linearize a CODE. The general form of a system of PDEs corresponding to (3.19) is given by

$$\begin{aligned} f_{xx} - f_{yy} + 2g_{xy} &= 4A^1(h^3 - 3hl^2) - 4A^2(3h^2l - l^3) + 4B^1(h^2 - l^2) - \\ &\quad 8B^2hl + 4C^1h - 4C^2l + 4D^1, \\ g_{xx} - g_{yy} - 2f_{xy} &= 4A^1(3h^2l - l^3) + 4A^2(h^3 - 3hl^2) + 4B^2(h^2 - l^2) + \\ &\quad 8B^1hl + 4C^2h + 4C^1l + 4D^2, \end{aligned} \tag{3.28}$$

where all the coefficients $A^i, B^i, C^i$ and $D^i$ are functions of $x, y, f$ and $g$. Notice that the above system is a special class of cubically semi linear non homogenous PDEs. Further the arbitrariness of the functions $A^i, B^i, C^i$ and $D^i$ is restricted by the extent of arbitrariness of the complex functions $A, B, C$ and $D$. For example, if $A$ is linear in $z$ and $u$, i.e.

$$A = \alpha z + \beta u + \gamma, \tag{3.29}$$

then $A^1$ and $A^2$ are of the special form

$$\begin{aligned} A^1 &= \alpha_1 x - \alpha_2 y + \beta_1 f - \beta_2 g + \gamma_1, \\ A^2 &= \alpha_2 x + \alpha_1 y + \beta_2 f + \beta_1 g + \gamma_2. \end{aligned} \tag{3.30}$$

Similarly, if $A$ is quadratic in $z$ or $u$, i.e.

$$A = z^2, \tag{3.31}$$

then

$$\begin{aligned} A^1 &= x^2 - y^2, \\ A^2 &= 2xy. \end{aligned} \tag{3.32}$$

Notice that $A^1$ does not contain the term $xy$ and $A^2$ lacks any of $x^2$ or $y^2$. Therefore $A^1$ and $A^2$ are not in general quadratic in $x$ and $y$ separately. Moreover if we assume $A$ to be

$$A = \alpha z^2, \tag{3.33}$$

then

$$\begin{aligned} A^1 &= \alpha_1(x^2 - y^2) + 2\alpha_2 xy, \\ A^2 &= \alpha_2(x^2 - y^2) + 2\alpha_1 xy, \end{aligned} \tag{3.34}$$

which involves all the quadratic terms, but notice that both $A^1$ and $A^2$ are of the special forms of a general quadratic expression. We stress the fact that we can only deal with those systems of PDEs in the complex domain in which the functions $A^1$ and $A^2$ in (3.28) have these special forms. For other PDEs our complex methods do not apply. This answers the converse question, i.e. "which systems of PDEs can be treated on the complex plane via complex symmetries?" Further we present the following theorem.

**Theorem 3.2.** The following statements are equivalent.

**1.** The above system of PDEs (3.28) is linearizable via real transformations;

**2.** The coefficients in (3.28) satisfy

$$\begin{gathered} 3A^1_{xx} - 3A^1_{yy} + 6A^2_{xy} + 3C^1A^1_x + 3C^1A^2_y - 3A^2_xC^2 + 3C^2A^1_y - 3A^1_fD^1 - 3D^1A^2_g + \\ 3D^2A^2_f - 3D^2A^1_g + 3A^1C^1_x + 3A^1C^2_y - 3A^2C^2_y + 3A^2C^1_x + C^1_{ff} - C^1_{gg} + 2C^2_{fg} - \\ 6A^1D^1_f - 6A^1D^2_g + 6A^2D^2_f - 6A^2D^1_g + B^1C^1_f + B^1C^2_g - B^2C^2_f + B^2C^1_g - \\ 2B^1B^1_x - 2B^1B^2_y + 2B^2B^2_x - 2B^2B^1_y - 2B^1_{xf} - 2B^2_{yf} - 2B^2_{xg} + 2B^1_{yg} = 0, \end{gathered} \tag{3.35}$$

$$\begin{gathered} 3A^2_{xx} - 3A^2_{yy} - 6A^1_{xy} + 3C^2A^1_x + 3C^2A^2_y + 3A^2_xC^1 - 3C^1A^1_y - 3D^2A^1_f - 3D^2A^2_g - \\ 3D^1A^2_f + 3D^1A^1_g + 3A^2C^1_x + 3A^2C^2_y + 3A^1C^2_y - 3A^1C^1_x + C^2_{ff} - C^2_{gg} - 2C^1_{fg} - \\ 6A^2D^1_f - 6A^2D^2_g - 6A^1D^2_f + 6A^1D^1_g + B^2C^1_f + B^2C^2_g + B^1C^2_f - B^1C^1_g - \\ 2B^2B^1_x - 2B^2B^2_y - 2B^1B^2_x + 2B^1B^1_y - 2B^2_{xf} + 2B^1_{yf} + 2B^1_{xg} - 2B^2_{yg} = 0 \end{gathered} \tag{3.36}$$

$$
\begin{aligned}
&6D^1A^1_x + 6D^1A^2_y - 6D^2A^2_x + 6D^2A^1_y - 3D^1B^1_f - 3D^1B^2_g + 3D^2B^2_f - 3D^2B^1_g + \\
&3A^1D^1_x + 3A^1D^2_y - 3A^2D^2_x + 3A^2D^1_y + B^1_{xx} - B^1_{yy} + 2B^2_{xy} - 2C^1_{xf} - 2C^2_{yf} - \\
&2C^2_{xg} + 2C^1_{yg} - 3B^1D^1_f - 3B^1D^2_g + 3B^2D^2_g - 3B^2D^1_g + 3D^1_{ff} - 3D^1_{gg} + 6D^2_{fg} + \\
&2C^1C^1_f + 2C^1C^2_g - 2C^2C^2_f + 2C^2C^1_g - C^1B^1_x - C^1B^2_y + C^2B^2_x - C^2B^1_y = 0, \qquad (3.37)
\end{aligned}
$$

$$
\begin{aligned}
&6D^2A^1_x + 6D^2A^2_y + 6D^1A^2_x - 6D^1A^1_y - 3D^2B^1_f - 3D^2B^2_g - 3D^1B^2_f + 3D^1B^1_g + \\
&3A^2D^1_x + 3A^2D^2_y + 3A^1D^2_x - 3A^1D^1_y + B^2_{xx} - B^2_{yy} - 2B^1_{xy} - 2C^2_{xf} + 2C^1_{yf} + \\
&2C^1_{xg} + 2C^2_{yg} - 3B^2D^1_f - 3B^2D^2_g - 3B^1D^2_f + 3B^1D^1_g + 3D^2_{ff} - 3D^2_{gg} - 6D^1_{fg} + \\
&2C^2C^1_f - 2C^2C^2_g + 2C^1C^2_f - 2C^1C^1_g - C^2B^1_x - C^2B^2_y - C^1B^2_x + C^1B^1_y = 0; \qquad (3.38)
\end{aligned}
$$

**3.** The system of PDEs corresponding to a CODE has four real symmetries, $\mathbf{X}_1,\ \mathbf{Y}_1,\ \mathbf{X}_2$ and $\mathbf{Y}_2$, with

$$
\mathbf{X}_1 = \rho_1\mathbf{X}_2 - \rho_2\mathbf{Y}_2,\ \ \mathbf{Y}_1 = \rho_1\mathbf{Y}_2 + \rho_2\mathbf{X}_2 \qquad (3.39)
$$

for nonconstant $\rho_1$ and $\rho_2$ and they satisfy

$$
[\mathbf{X}_1, \mathbf{X}_2] - [\mathbf{Y}_1, \mathbf{Y}_2] = 0,\ \ [\mathbf{X}_1, \mathbf{Y}_2] + [\mathbf{Y}_1, \mathbf{X}_2] = 0, \qquad (3.40)
$$

such that a point transformation $(x, y, f, g) \longrightarrow (X, Y, F, G)$, which brings $\mathbf{X}_1,\ \mathbf{Y}_1,\ \mathbf{X}_2$ and $\mathbf{Y}_2$ to their canonical form

$$
\mathbf{X}_1 = \frac{\partial}{\partial F},\quad \mathbf{Y}_1 = \frac{\partial}{\partial G},\quad \mathbf{X}_2 = X\frac{\partial}{\partial F} + Y\frac{\partial}{\partial G},\quad \mathbf{Y}_2 = Y\frac{\partial}{\partial F} - X\frac{\partial}{\partial G} \qquad (3.41)
$$

reduces the system (3.28) to the linear form

$$
\begin{aligned}
F_{XX} - F_{YY} + 2G_{XY} &= W_1(X, Y), \\
G_{XX} - G_{YY} - 2F_{XY} &= W_2(X, Y); \qquad (3.42)
\end{aligned}
$$

**4.** The system of PDEs corresponding to a CODE has four real symmetries $\mathbf{X}_1,\ \mathbf{Y}_1,\ \mathbf{X}_2$ and $\mathbf{Y}_2$, with

$$\mathbf{X}_1 = \rho_1\mathbf{X}_2 - \rho_2\mathbf{Y}_2, \quad \mathbf{Y}_1 \ = \rho_1\mathbf{Y}_2 + \rho_2\mathbf{X}_2 \tag{3.43}$$

for nonconstant $\rho_1$ and $\rho_2$ and they satisfy either

$$[\mathbf{X}_1, \mathbf{X}_2] - [\mathbf{Y}_1, \mathbf{Y}_2] \neq 0 \ \text{ or } \ [\mathbf{X}_1, \mathbf{Y}_2] + [\mathbf{Y}_1, \mathbf{X}_2] \neq 0 \tag{3.44}$$

such that a point transformation $(x, y, f, g) \longrightarrow (X, Y, F, G)$, which brings $\mathbf{X}_1,\ \mathbf{Y}_1,\ \mathbf{X}_2$ and $\mathbf{Y}_2$ to their canonical form,

$$\mathbf{X}_1 = \frac{\partial}{\partial F}, \quad \mathbf{Y}_1 = \frac{\partial}{\partial G}, \quad \mathbf{X}_2 = F\frac{\partial}{\partial F} + G\frac{\partial}{\partial G}, \quad \mathbf{Y}_2 = G\frac{\partial}{\partial F} - F\frac{\partial}{\partial G}, \tag{3.45}$$

reduces the system (3.28) to the linear form

$$\begin{aligned} F_{XX} - F_{YY} + 2G_{XY} &= HW_1(X,Y) - LW_2(X,Y), \\ G_{XX} - G_{YY} - 2F_{XY} &= HW_2(X,Y) + LW_1(X,Y), \end{aligned} \tag{3.46}$$

where

$$H = F_X + G_Y, \quad L = G_X - F_Y; \tag{3.47}$$

**5.** The system of PDEs corresponding to a CODE has four real symmetries $\mathbf{X}_1,\ \mathbf{Y}_1,\ \mathbf{X}_2$ and $\mathbf{Y}_2$, with

$$\mathbf{X}_1 \neq \rho_1\mathbf{X}_2 - \rho_2\mathbf{Y}_2, \quad \mathbf{Y}_1 \ \neq \rho_1\mathbf{Y}_2 + \rho_2\mathbf{X}_2 \tag{3.48}$$

for nonconstant $\rho_1$ and $\rho_2$ and they satisfy

$$[\mathbf{X}_1, \mathbf{X}_2] - [\mathbf{Y}_1, \mathbf{Y}_2] = 0, \quad [\mathbf{X}_1, \mathbf{Y}_2] + [\mathbf{Y}_1, \mathbf{X}_2] = 0, \tag{3.49}$$

such that a point transformation $(x, y, f, g) \longrightarrow (X, Y, F, G)$, which brings $\mathbf{X}_1,\ \mathbf{Y}_1,\ \mathbf{X}_2$ and $\mathbf{Y}_2$ to their canonical form

$$\mathbf{X}_1 = \frac{\partial}{\partial X}, \quad \mathbf{Y}_1 = \frac{\partial}{\partial Y}, \quad \mathbf{X}_2 = \frac{\partial}{\partial F}, \quad \mathbf{Y}_2 = \frac{\partial}{\partial G} \tag{3.50}$$

reduces the system (3.28) to the system of PDEs corresponding to that CODE is at most cubic in all its first derivatives;

**6.** The system of PDEs corresponding to a CODE has four real symmetries $\mathbf{X}_1,\ \mathbf{Y}_1,\ \mathbf{X}_2$ and $\mathbf{Y}_2$, with

$$\mathbf{X}_1 \neq \rho_1\mathbf{X}_2 - \rho_2\mathbf{Y}_2, \quad \mathbf{Y}_1 \neq \rho_1\mathbf{Y}_2 + \rho_2\mathbf{X}_2 \tag{3.51}$$

for nonconstant $\rho_1$ and $\rho_2$ and they satisfy either

$$[\mathbf{X}_1, \mathbf{X}_2] - [\mathbf{Y}_1, \mathbf{Y}_2] \neq 0 \ \text{ or } \ [\mathbf{X}_1, \mathbf{Y}_2] + [\mathbf{Y}_1, \mathbf{X}_2] \neq 0 \tag{3.52}$$

such that a point transformation $(x, y, f, g) \longrightarrow (X, Y, F, G)$, which brings $\mathbf{X}_1,\ \mathbf{Y}_1,\ \mathbf{X}_2$ and $\mathbf{Y}_2$ to their canonical form

$$\begin{aligned}\mathbf{X}_1 &= \frac{\partial}{\partial F}, \ \ \mathbf{Y}_1 = \frac{\partial}{\partial G}, \ \ \mathbf{X}_2 = X\frac{\partial}{\partial X} + Y\frac{\partial}{\partial Y} + F\frac{\partial}{\partial F} + G\frac{\partial}{\partial G},\\ \mathbf{Y}_2 &= Y\frac{\partial}{\partial X} - X\frac{\partial}{\partial Y} + G\frac{\partial}{\partial F} - F\frac{\partial}{\partial G},\end{aligned} \tag{3.53}$$

reduces the system (3.28) to the linear form

$$\begin{aligned}&X(F_{XX} - F_{YY} + 2G_{XY}) - Y(G_{XX} - G_{YY} - 2F_{XY}) = a_1(H^3 - 3HL^2)-\\ &a_2(3H^2L - L^3) + b_1(H^2 - L^2) - 2b_2HL + \frac{1}{3(a_1^2 + a_2^2)}\{3(a_1^2 + a_2^2) + (b_1^2 - b_2^2)a_1+\\ &2b_1b_2a_2\}H - \frac{1}{3(a_1^2 + a_2^2)}\{2b_1b_2a_1 - a_2(b_1^2 - b_2^2)\}L + \frac{1}{3(a_1^2 + a_2^2)}(b_1a_1 + b_2a_2)+\\ &\frac{1}{27(a_1^2 + a_2^2)}\{(b_1^3 - 3b_1b_2^2)(a_1^2 - a_2^2) + 2a_1a_2(3b_1^2b_2 - b_2^3),\end{aligned} \tag{3.54}$$

$$\begin{aligned}&Y(F_{XX} - F_{YY} + 2G_{XY}) + X(G_{XX} - G_{YY} - 2F_{XY}) = a_2(H^3 - 3HL^2)+\\ &a_1(3H^2L - L^3) + b_2(H^2 - L^2) + 2b_1HL + \frac{1}{3(a_1^2 + a_2^2)}\{3(a_1^2 + a_2^2) + (b_1^2 - b_2^2)a_1+\\ &2b_1b_2a_2\}L + \frac{1}{3(a_1^2 + a_2^2)}\{2b_1b_2a_1 - a_2(b_1^2 - b_2^2)\}H + \frac{1}{3(a_1^2 + a_2^2)}(b_2a_1 - b_1a_2)+\\ &\frac{1}{27(a_1^2 + a_2^2)}\{(3b_1^2b_2 - b_2^3)(a_1^2 - a_2^2) - 2(b_1^3 - 3b_1b_2^2)a_1a_2\}.\end{aligned} \tag{3.55}$$

The invertible real transformations

$$\begin{aligned}\tilde{X} &= F + \frac{1}{3(a_1^2+a_2^2)}\{(b_1a_1+b_2a_2)X - (b_2a_1-b_1a_2)Y\},\\ \tilde{Y} &= G + \frac{1}{3(a_1^2+a_2^2)}\{(b_1a_1+b_2a_2)Y + (b_2a_1-b_1a_2)X\},\end{aligned} \tag{3.56}$$

$$\begin{aligned}\tilde{F} = \frac{1}{2}(F^2-G^2) + \frac{1}{3(a_1^2+a_2^2)}[\{(b_1a_1+b_2a_2)X-(b_2a_1-b_1a_2)Y\}F-\\ \{(b_1a_1+b_2a_2)Y+(b_2a_1-b_1a_2)X\}G] + \frac{1}{18(a_1^2+a_2^2)^2}[\{(b_1^2-b_2^2)(a_1^2-a_2^2)+\\ 4b_1b_2a_1a_2\}(X^2-Y^2) - 2XY\{(2b_1b_2(a_1^2-a_2^2)-2a_1a_2(b_1^2-b_2^2)\}]+\\ \frac{1}{2(a_1^2+a_2^2)}\{a_1(X^2-Y^2)+2a_2XY\},\end{aligned} \tag{3.57}$$

$$\begin{aligned}\tilde{G} = FG + \frac{1}{3(a_1^2+a_2^2)}[\{(b_1a_1+b_2a_2)X-(b_2a_1-b_1a_2)Y\}G+\\ \{(b_1a_1+b_2a_2)Y+(b_2a_1-b_1a_2)X\}F] + \frac{1}{18(a_1^2+a_2^2)^2}[2XY\{(b_1^2-b_2^2)(a_1^2-a_2^2)+\\ 4b_1b_2a_1a_2\} - (X^2-Y^2)\{2b_1b_2(a_1^2-a_2^2)-2a_1a_2(b_1^2-b_2^2)\}]+\\ \frac{1}{2(a_1^2+a_2^2)}\{2XYa_1-a_2(X^2-Y^2)\}\end{aligned} \tag{3.58}$$

transform (3.54) and (3.55) into the system of linear PDEs

$$\begin{aligned}\tilde{F}_{\tilde{X}\tilde{X}} - \tilde{F}_{\tilde{Y}\tilde{Y}} + 2\tilde{G}_{\tilde{X}\tilde{Y}} &= 0,\\ \tilde{G}_{\tilde{X}\tilde{X}} - \tilde{G}_{\tilde{Y}\tilde{Y}} - 2\tilde{F}_{\tilde{X}\tilde{Y}} &= 0.\end{aligned} \tag{3.59}$$

Now we discuss some illustrative examples.

**Examples 3.3:** Consider a nonlinear CODE of the form (this was discussed in [53] for the real linearization)

$$u'' + 3uu' + u^3 = 0, \tag{3.60}$$

which is linearizable as it satisfies the Lie complex conditions. The set of two non commuting CLSs is

$$\mathbf{Z}_1 = \frac{\partial}{\partial z}, \quad \mathbf{Z}_2 = z\frac{\partial}{\partial z} - u\frac{\partial}{\partial u}. \tag{3.61}$$

Note that $\mathbf{Z}_1 \neq \rho(z,u)\mathbf{Z}_2$. We invoke condition six of Theorem 3.1 to find a linearizing transformation. The complex point transformation that reduces the symmetries (3.61) to their canonical form is

$$Z = \frac{1}{u}, \quad U = z + \frac{1}{u}, \tag{3.62}$$

which reduces (3.60) to

$$ZU'' = -U'^3 + 6U'^2 - 11U' + 6. \tag{3.63}$$

Equation (3.60) linearizes to $\tilde{U}'' = 0$ via the complex transformations which is obtained by placing $a = -1, \ b = 6$ in equation (3.14), that is

$$\tilde{Z} = z - \frac{1}{u}, \quad \tilde{U} = \frac{z^2}{2} - \frac{z}{u}. \tag{3.64}$$

It may be seen that the above transformation is a proper complex transformation. Thus one does not get a contradiction that arises in the linearization of r-CODEs which is discussed in the next section. Further the procedure that seems to be analogous to analytic continuation does not arise here. The system of PDEs corresponding to (3.60) is

$$\begin{aligned} f_{xx} - f_{yy} + 2g_{xy} &= -12(fh - gl) - 4(f^3 - 3fg^2), \\ g_{xx} - g_{yy} - 2f_{xy} &= -12(gh + fl) - 4(3f^2g - g^3), \end{aligned} \tag{3.65}$$

which admits the RLSs

$$\begin{aligned} \mathbf{X}_1 &= \frac{\partial}{\partial x}, \quad \mathbf{Y}_1 = \frac{\partial}{\partial y}, \\ \mathbf{X}_2 &= x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} - f\frac{\partial}{\partial f} - g\frac{\partial}{\partial g}, \\ \mathbf{Y}_2 &= y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y} - g\frac{\partial}{\partial f} + f\frac{\partial}{\partial g}. \end{aligned} \tag{3.66}$$

These generators satisfy (3.51) and (3.52). The transformation,

$$X = \frac{f}{f^2 + g^2}, \quad Y = \frac{-g}{f^2 + g^2}, \tag{3.67}$$

$$F = x + \frac{f}{f^2 + g^2}, \quad G = y - \frac{g}{f^2 + g^2}, \tag{3.68}$$

transforms (3.65) into

$$X(F_{XX} - F_{YY} + 2G_{XY}) - Y(G_{XX} - G_{YY} - 2F_{XY}) = -4H^3 + 12HL^2 - \\ 24H^2 - 44H + 24, \tag{3.69}$$

$$X(G_{XX} - G_{YY} - 2F_{XY}) + Y(F_{XX} - F_{YY} + 2G_{XY}) = 4L^3 - 12H^2L + \\ 48HL - 44L, \tag{3.70}$$

where $H$ and $L$ are given by (3.17). The above system can further be reduced into very simplified linear PDEs by using

$$\tilde{X} = x - \frac{f}{f^2 + g^2}, \quad \tilde{Y} = y + \frac{g}{f^2 + g^2}, \tag{3.71}$$

$$\tilde{F} = \frac{1}{2}(x^2 - y^2) - \frac{1}{f^2 + g^2}(xf + yg), \tag{3.72}$$

$$\tilde{G} = xy - \frac{1}{f^2 + g^2}(yf - xg). \tag{3.73}$$

The linear PDEs are

$$\begin{aligned} \tilde{F}_{\tilde{X}\tilde{X}} - \tilde{F}_{\tilde{Y}\tilde{Y}} + 2\tilde{G}_{\tilde{X}\tilde{Y}} &= 0, \\ \tilde{G}_{\tilde{X}\tilde{X}} - \tilde{G}_{\tilde{Y}\tilde{Y}} - 2\tilde{F}_{\tilde{X}\tilde{Y}} &= 0. \end{aligned} \tag{3.74}$$

**Examples 3.4:** Consider a second-order non linear CODE with an arbitrary function $w(z)$

$$uu'' = u'^2 + w(z)u^2. \tag{3.75}$$

The above CODE admits two CLSs of the form

$$\mathbf{Z}_1 = zu\frac{\partial}{\partial u}, \quad \mathbf{Z}_2 = u\frac{\partial}{\partial u} \tag{3.76}$$

and thus

$$[\mathbf{Z}_1, \mathbf{Z}_2] = 0 \ \text{ and } \ \mathbf{Z}_2 = \frac{1}{z}\mathbf{Z}_1. \tag{3.77}$$

By using the complex transformation

$$Z = \frac{1}{z} \quad \text{and} \quad U = \frac{1}{z}\log u \tag{3.78}$$

(3.75) can be reduced into a linear CODE

$$U'' = \frac{1}{Z^3} w(\frac{1}{Z}). \tag{3.79}$$

The system of PDEs corresponding to (3.75) is

$$\begin{aligned} f(f_{xx} - f_{yy} + 2g_{xy}) - g(g_{xx} - g_{yy} - 2f_{xy}) &= 4h^2 - 4l^2 + 4w_1(f^2 - g^2) - 8fgw_2, \\ f(g_{xx} - g_{yy} - 2f_{xy}) + g(f_{xx} - f_{yy} + 2g_{xy}) &= 8hl + 8w_1 fg + 4w_2(f^2 - g^2). \end{aligned} \tag{3.80}$$

When one introduces the transformation

$$X = \frac{x}{x^2 + y^2}, \quad Y = \frac{-y}{x^2 + y^2}, \tag{3.81}$$

$$F = \frac{1}{x^2 + y^2}[\frac{1}{2}x \ln(f^2 + g^2) + y \tan^{-1}(\frac{g}{f})], \tag{3.82}$$

$$G = \frac{1}{x^2 + y^2}[x \tan^{-1}(\frac{g}{f}) - \frac{y}{2}\ln(f^2 + g^2), \tag{3.83}$$

the system (3.80) is reduced to the linear system of PDEs

$$\begin{aligned} F_{XX} - F_{YY} + 2G_{XY} &= \frac{4}{X^2 + Y^2}[(X^3 - 3XY^2)w_1 - (Y^3 - 3X^2Y)w_2], \\ G_{XX} - G_{YY} - 2F_{XY} &= \frac{4}{X^2 + Y^2}[(X^3 - 3XY^2)w_2 + (Y^3 - 3X^2Y)w_1], \end{aligned} \tag{3.84}$$

where

$$w_1 = w_1(\frac{X}{X^2 + Y^2}, \frac{-Y}{X^2 + Y^2}), \quad w_2 = w_2(\frac{X}{X^2 + Y^2}, \frac{-Y}{X^2 + Y^2}). \tag{3.85}$$

**Examples 3.5:** Consider the nonlinear CODE (cf [45])

$$u'' = \frac{1}{z}(u' + u'^3). \tag{3.86}$$

This equation has CLSs

$$\mathbf{Z}_1 = \frac{1}{z}\frac{\partial}{\partial z}, \quad \mathbf{Z}_2 = \frac{u}{z}\frac{\partial}{\partial z}. \tag{3.87}$$

The complex transformation

$$U = \frac{1}{2}z^2 \text{ and } Z = u, \tag{3.88}$$

transforms (3.86) into a linear CODE

$$U'' + 1 = 0. \tag{3.89}$$

The system of PDEs corresponding to (3.86) is

$$\begin{aligned} f_{xx} - f_{yy} + 2g_{xy} &= \frac{4}{x^2+y^2}[xh(1+h^2-3l^2) + yl(1-l^2+3h^2)], \\ g_{xx} - g_{yy} - 2f_{xy} &= \frac{4}{x^2+y^2}[xl(1-l^2+3h^2) + yh(1+h^2-3l^2)] \end{aligned} \tag{3.90}$$

and admits the symmetries

$$\mathbf{X}_1 = \frac{x}{x^2+y^2}\frac{\partial}{\partial x} - \frac{y}{x^2+y^2}\frac{\partial}{\partial y}, \quad \mathbf{Y}_1 = \frac{-y}{x^2+y^2}\frac{\partial}{\partial x} - \frac{x}{x^2+y^2}\frac{\partial}{\partial y}, \tag{3.91}$$

$$\begin{aligned} \mathbf{X}_2 &= \frac{fx}{x^2+y^2}\frac{\partial}{\partial x} - \frac{fy}{x^2+y^2}\frac{\partial}{\partial y} + \frac{yg}{x^2+y^2}\frac{\partial}{\partial x} - \frac{gx}{x^2+y^2}\frac{\partial}{\partial y}, \\ \mathbf{Y}_2 &= \frac{gx}{x^2+y^2}\frac{\partial}{\partial x} - \frac{gy}{x^2+y^2}\frac{\partial}{\partial y} - \frac{fy}{x^2+y^2}\frac{\partial}{\partial x} - \frac{fx}{x^2+y^2}\frac{\partial}{\partial y}. \end{aligned} \tag{3.92}$$

The above symmetries correspond to the condition five of Theorem 3.2. The real transformation that linearizes system (3.90) is

$$X = f, \quad Y = g, \quad F = \frac{1}{2}(x^2 - y^2), \quad G = xy. \tag{3.93}$$

The linearized system of PDEs is

$$\begin{aligned} F_{XX} - F_{YY} + 2G_{XY} + 4 = 0, \\ G_{XX} - G_{YY} - 2F_{XY} = 0. \end{aligned} \tag{3.94}$$

**Examples 3.6:** Consider the nonlinear CODE

$$u'' = 1 + (u' - z)^2 w(2u - z^2) \tag{3.95}$$

which admits the CLSs

$$\mathbf{Z}_1 = \frac{\partial}{\partial z} + z\frac{\partial}{\partial u},\ \mathbf{Z}_2 = z\frac{\partial}{\partial z} + z^2\frac{\partial}{\partial u}. \tag{3.96}$$

Equation (3.95) reduces to a linear CODE by using the complex transformation

$$Z = 2u - z^2 \ \text{ and } \ U = z. \tag{3.97}$$

It becomes

$$U'' = -\frac{1}{2}U'w(Z). \tag{3.98}$$

The system of PDEs corresponding to (3.95) is

$$\begin{aligned} f_{xx} - f_{yy} + 2g_{xy} &= 4 + 4\{(h-x)^2 - (l-y)^2\}w_1 - 8(h-x)(l-y)w_2, \\ g_{xx} - g_{yy} - 2f_{xy} &= 8(h-x)(l-y)w_1 + 4\{(h-x)^2 - (l-y)^2\}w_2, \end{aligned} \tag{3.99}$$

which admits

$$\mathbf{X}_1 = \frac{\partial}{\partial x} + x\frac{\partial}{\partial f} + y\frac{\partial}{\partial g}, \quad \mathbf{Y}_1 = x\frac{\partial}{\partial g} - y\frac{\partial}{\partial f} - \frac{\partial}{\partial y}, \tag{3.100}$$

$$\begin{aligned} \mathbf{X}_2 &= x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + (x^2 - y^2)\frac{\partial}{\partial f} + 2xy\frac{\partial}{\partial g}, \\ \mathbf{Y}_2 &= y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y} + 2xy\frac{\partial}{\partial f} - (x^2 - y^2)\frac{\partial}{\partial g}, \end{aligned} \tag{3.101}$$

that satisfies condition four of Theorem 3.2. The transformation

$$X = 2f - x^2 + y^2, \quad Y = 2g - 2xy, \tag{3.102}$$

$$F = x, \quad G = y. \tag{3.103}$$

yields system of linear PDEs

$$\begin{aligned} F_{XX} - F_{YY} + 2G_{XY} &= -2(Hw_1 - Lw_2), \\ G_{XX} - G_{YY} - 2F_{XY} &= -2(Hw_2 + Lw_1), \end{aligned} \tag{3.104}$$

where

$$w(z) = w_1 + \iota w_2, \quad H = \frac{1}{2}(F_X + G_Y), \quad L = \frac{1}{2}(G_X - F_Y). \tag{3.105}$$

## 3.4 Linearization of systems of ODEs

Here we are interested in developing the insights obtained by considering ODEs for complex functions of a single real variable. Whereas linear r-CODEs are trivial, nonlinear r-CODEs couple the two components of the complex function to work Penrose's complex magic. An r-CODE can be linearized in the same way as a RODE via invertible complex transformations. It yields two RODEs. The linearization of such systems of RODEs follows directly from the linearization of the corresponding r-CODEs after decomposing the complex transformation into two real point transformations.

## 3.5 Equivalent r-CODEs

Two r-CODEs are (locally) equivalent via an invertible complex transformation if one can be transformed into the other by an invertible complex transformation. An r-CODE is a system of two RODEs in two unknown functions of one variable. If an r-CODE is equivalent to another r-CODE via invertible complex transformations, then the system of RODEs corresponding to that r-CODE is also equivalent to the other system of RODEs. The invertible complex transformation yields two real invertible transformations which can be used to transform one system of RODEs into another system. We illustrate this fact in the following examples. We write $u(x)$ as a complex function $u$ of a single real variable, $x$.

Every first-order r-CODE of the form

$$u'(x) = w(x, u), \tag{3.106}$$

is equivalent to a simple r-CODE, $u' = 0$, via a complex transformation which is the same for RODEs [12]. Further every *linear* second-order r-CODE is also equivalent to its simple analogue, $u''(x) = 0$, via some complex transformation. For example the complex Riccatti equation

$$u'(x) + u^2 = 0, \tag{3.107}$$

is transformable to $U' = 0$ by means of

$$\chi = x, \quad U = (1/u) - x. \tag{3.108}$$

Similarly the complexified simple harmonic oscillator equation,

$$u''(x) + u = 0, \tag{3.109}$$

can be transformed into $U'' = 0$ via an invertible complex transformation,

$$\chi = \tan x \quad \text{and} \quad U = u \sec x, \tag{3.110}$$

in the domain $(-\pi/2, \pi/2)$.

Now we decompose the above r-CODEs into systems of RODEs corresponding to them by

$$u(x) = f(x) + \iota g(x). \tag{3.111}$$

Note that here $f$ and $g$ are functions of a single variable $x$. The system of RODEs corresponding to (3.106) is

$$f' = w_1(x, f, g), \quad g' = w_2(x, f, g), \tag{3.112}$$

where $w_1$ and $w_2$ are the real and imaginary parts of $w$. This system is transformable into

$$\Upsilon' = 0 \quad \text{and} \quad \zeta' = 0 \tag{3.113}$$

by the real invertible transformation derived from the complex transformation, where the complex transformation, $(x, u) \longrightarrow (\chi, U)$, is equivalent to the real transformation $(x, f, g) \longrightarrow$

$(\chi, \Upsilon, \zeta)$. Similarly a general linear second-order r-CODE yields the following system of RODEs

$$\begin{aligned} f'' &= w_1(x, f, g, f', g'), \\ g'' &= w_2(x, f, g, f', g'), \end{aligned} \tag{3.114}$$

where both $w_1$ and $w_2$ are two real functions such that they do not give rise to a nonlinear system. This system can be transformed into

$$\Upsilon'' = 0, \quad \zeta'' = 0. \tag{3.115}$$

Notice that the general solution of (3.109) is

$$u(x) = \alpha \cos x + \beta \sin x, \tag{3.116}$$

where $\alpha$ and $\beta$ are complex constants such that $\alpha = \alpha_1 + \iota\alpha_2$ and $\beta = \beta_1 + \iota\beta_2$, despite the fact that $x$ is a real variable, because the solution is obtained from complex integration. When one uses (3.111), the above solution becomes

$$f(x) = \alpha_1 \cos x + \beta_1 \sin x, \quad g(x) = \alpha_2 \cos x + \beta_2 \sin x, \tag{3.117}$$

which is in fact the solution of the system

$$f'' = -f, \quad g'' = -g, \tag{3.118}$$

corresponding to the r-CODE (3.109) and indicating the consistency of r-CODEs and their solutions. It is important to note that the solutions of r-CODEs yield solutions of systems of ODEs corresponding to them. Now we obtain systems of ODEs corresponding to (3.107) in the following examples.

**Examples 3.7:** The complex Riccatti equation (3.107) yields

$$f' = -f^2 + g^2 \text{ and } g' = -2fg, \tag{3.119}$$

which is a special case in two dimensions of the generalized Riccatti system. Note that what seems to be a trivial complex transformation (3.108) yields the following non trivial real transformation,

$$\chi = x, \quad \Upsilon = \frac{f}{f^2+g^2} - x, \quad \zeta = \frac{-g}{f^2+g^2}, \tag{3.120}$$

that maps system (3.119) into

$$\Upsilon' = 0 \ \text{ and } \ \zeta' = 0. \tag{3.121}$$

**Examples 3.8:** The decoupled linear harmonic oscillator,

$$f'' = -f, \quad g'' = -g, \tag{3.122}$$

can be transformed into

$$\Upsilon'' = 0, \quad \zeta'' = 0 \tag{3.123}$$

via the real transformation

$$\chi = \tan x, \quad \Upsilon = f \sec x, \quad \zeta = g \sec x. \tag{3.124}$$

This transformation could have been easily guessed. However, the complex magic is that it can be *derived* by the use of complex functions of a real variable.

Thus one can use simple complex transformations to map systems of two RODEs into systems of simple RODEs which we state as equivalent systems of RODEs corresponding to some r-CODE.

## 3.6 Lie conditions for complex functions

We state Lie's theorem for r-CODEs that a second-order r-CODE which is at most cubic in its first derivative is linearizable provided it satisfies four differential constraints involving two auxiliary variables that Tressé reduced to two differential conditions [65, 66]. We extend the Lie theorem to r-CODEs by considering an r-CODE of the form

$$u''(x) = A(x,u)u'^3 + B(x,u)u'^2 + C(x,u)u' + D(x,u), \tag{3.125}$$

where $A,\ B,\ C$ and $D$ are complex-valued functions such that

$$A = A^1 + \iota A^2, \quad B = B^1 + \iota B^2, \quad C = C^1 + \iota C^2, \quad D = D^1 + \iota D^2. \tag{3.126}$$

In the subsequent section we obtain the Lie's condition for systems of ODEs corresponding to (3.125).

## 3.7 Lie conditions for systems of two RODEs

The fact that a second-order r-CODE is linearizable if it is at most cubic in its first derivative implies that a system of two RODEs corresponding to that r-CODE which is at most cubic in the first derivative is also linearizable with the relevant conditions on the coefficients. We can easily obtain the real transformations by decomposing each of the complex transformations that linearize an r-CODE. The general form of a system of RODEs corresponding to (3.125) is given by

$$\begin{aligned} f'' &= A_1(f'^3 - 3f'g'^2) - A_2(3f'^2g' - g'^3) + B_1(f'^2 - g'^2) - 2B_2f'g' + C_1f' - C_2g' + D_1, \\ g'' &= A_1(3f'^2g' - g'^3) + A_2(f'^3 - 3f'g'^2) + 2B_1f'g' + B_2(f'^2 - g'^2) + C_2f' + C_1g' + D_2, \end{aligned} \tag{3.127}$$

where the coefficients $A_i, B_i, C_i$ and $D_i$ are function of $x, f, g$. We obtain results for systems of RODEs by decomposing corresponding r-CODEs and state them as Theorem 3.3 with the following notation

$$\begin{aligned} \mathbf{Z}_i &= \mathbf{X}_i + \iota \mathbf{Y}_i, \qquad i = 1, 2, \\ \rho(x) &= \rho_1(x) + \iota\rho_2(x), \quad \chi(x) = \chi_1(x) + \iota\chi_2(x), \end{aligned} \tag{3.128}$$

where $\rho_1(x)$ and $\rho_2(x)$ are nonconstant real functions and the complex transformation $(x, u) \longrightarrow (\chi, U)$, is equivalent to the real transformation $(x, f, g) \longrightarrow (\chi, \Upsilon, \zeta)$.

**Theorem 3.3.** The following statements are equivalent:

**1.** The system of RODEs (3.127) is linearizable via real transformations;

**2.** The coefficients in (3.127) satisfy

$$
\begin{gathered}
3A^1_{xx} + 3C^1A^1_x - 3A^2_xC^2 - 3A^1_fD^1 - 3D^1A^2_g + 3D^2A^2_f - 3D^2A^1_g + 3A^1C^1_x + \\
3A^2C^1_x + C^1_{ff} - C^1_{gg} + 2C^2_{fg} - 6A^1D^1_f - 6A^1D^2_g + 6A^2D^2_f - 6A^2D^1_g + \\
B^1C^1_f + B^1C^2_g - B^2C^2_f + B^2C^1_g - 2B^1B^1_x + 2B^2B^2_x - 2B^1_{xf} - 2B^2_{xg} = 0,
\end{gathered} \tag{3.129}
$$

$$
\begin{gathered}
3A^2_{xx} + 3C^2A^1_x + 3A^2_xC^1 - 3D^2A^1_f - 3D^2A^2_g - 3D^1A^2_f + 3D^1A^1_g + 3A^2C^1_x - \\
3A^1C^1_x + C^2_{ff} - C^2_{gg} - 2C^1_{fg} - 6A^2D^1_f - 6A^2D^2_g - 6A^1D^2_f + 6A^1D^1_g + \\
B^2C^1_f + B^2C^2_g + B^1C^2_f - B^1C^1_g - 2B^2B^1_x - 2B^1B^2_x - 2B^2_{xf} + 2B^1_{xg} = 0,
\end{gathered} \tag{3.130}
$$

$$
\begin{gathered}
6D^1A^1_x - 6D^2A^2_x - 3D^1B^1_f - 3D^1B^2_g + 3D^2B^2_f - 3D^2B^1_g + 3A^1D^1_x - 3A^2D^2_x + \\
B^1_{xx} - 2C^1_{xf}2C^2_{xg} - 3B^1D^1_f - 3B^1D^2_g + 3B^2D^2_g - 3B^2D^1_g + 3D^1_{ff} - \\
3D^1_{gg} + 6D^2_{fg} + 2C^1C^1_f + 2C^1C^2_g - 2C^2C^2_f + 2C^2C^1_g - C^1B^1_x + C^2B^2_x = 0,
\end{gathered} \tag{3.131}
$$

$$
\begin{gathered}
6D^2A^1_x + 6D^1A^2_x - 3D^2B^1_f - 3D^2B^2_g - 3D^1B^2_f + 3D^1B^1_g + 3A^2D^1_x + 3A^1D^2_x + \\
B^2_{xx} - 2C^2_{xf} + 2C^1_{xg} - 3B^2D^1_f - 3B^2D^2_g - 3B^1D^2_f + 3B^1D^1_g + 3D^2_{ff} - \\
3D^2_{gg} - 6D^1_{fg} + 2C^2C^1_f - 2C^2C^2_g + 2C^1C^2_f - 2C^1C^1_g - C^2B^1_x - C^1B^2_x = 0;
\end{gathered} \tag{3.132}
$$

**3.** The system of RODEs corresponding to a r-CODE has four real symmetries $\mathbf{X}_1,\ \mathbf{Y}_1,\ \mathbf{X}_2$ and $\mathbf{Y}_2$ with

$$
\mathbf{X}_1 = \rho_1\mathbf{X}_2 - \rho_2\mathbf{Y}_2, \quad \text{and} \quad \mathbf{Y}_1 = \rho_1\mathbf{Y}_2 + \rho_2\mathbf{X}_2 \tag{3.133}
$$

for nonconstant real functions, $\rho_1(x)$ and $\rho_2(x)$, and they satisfy

$$
[\mathbf{X}_1, \mathbf{X}_2] - [\mathbf{Y}_1, \mathbf{Y}_2] = 0, \quad [\mathbf{X}_1, \mathbf{Y}_2] + [\mathbf{Y}_1, \mathbf{X}_2] = 0, \tag{3.134}
$$

such that a point transformation $\chi = \chi(x, f, g), \Upsilon = \Upsilon(x, f, g)$ and $\zeta = \zeta(x, f, g),$ which brings $\mathbf{X}_1,\ \mathbf{Y}_1,\ \mathbf{X}_2$ and $\mathbf{Y}_2$ to their canonical form

$$\mathbf{X}_1 = \frac{\partial}{\partial \Upsilon}, \quad \mathbf{Y}_1 = -\frac{\partial}{\partial \zeta}, \quad \mathbf{X}_2 = \chi\frac{\partial}{\partial \Upsilon}, \quad \mathbf{Y}_2 = -\chi\frac{\partial}{\partial \zeta}, \tag{3.135}$$

reduces the system (3.127) to the linear form

$$\Upsilon'' = W_1(\chi), \quad \zeta'' = W_2(\chi); \tag{3.136}$$

**4.** The system of RODEs corresponding to an r-CODE has four real symmetries $\mathbf{X}_1,\ \mathbf{Y}_1,\ \mathbf{X}_2$ and $\mathbf{Y}_2$ with

$$\mathbf{X}_1 = \rho_1\mathbf{X}_2 - \rho_2\mathbf{Y}_2, \ \text{ and } \ \mathbf{Y}_1 = \rho_1\mathbf{Y}_2 + \rho_2\mathbf{X}_2 \tag{3.137}$$

for nonconstant $\rho_1$ and $\rho_2$ and they satisfy either

$$[\mathbf{X}_1, \mathbf{X}_2] - [\mathbf{Y}_1, \mathbf{Y}_2] \neq 0 \ \text{ or } \ [\mathbf{X}_1, \mathbf{Y}_2] + [\mathbf{Y}_1, \mathbf{X}_2] \neq 0 \tag{3.138}$$

such that a point transformation $\chi = \chi(x, f, g), \Upsilon = \Upsilon(x, f, g)$ and $\zeta = \zeta(x, f, g),$ which brings $\mathbf{X}_1,\ \mathbf{Y}_1,\ \mathbf{X}_2$ and $\mathbf{Y}_2$ to their canonical form

$$\mathbf{X}_1 = \frac{\partial}{\partial \Upsilon}, \quad \mathbf{Y}_1 = \frac{\partial}{\partial \zeta}, \quad \mathbf{X}_2 = \Upsilon\frac{\partial}{\partial \Upsilon} + \zeta\frac{\partial}{\partial \zeta}, \quad \mathbf{Y}_2 = \zeta\frac{\partial}{\partial \Upsilon} - \Upsilon\frac{\partial}{\partial \zeta}, \tag{3.139}$$

reduces the system (3.127) to the linear form

$$\begin{aligned} \Upsilon'' &= \Upsilon' W_1(\chi) - \zeta' W_2(\chi), \\ \zeta'' &= \Upsilon' W_2(\chi) + \zeta' W_1(\chi); \end{aligned} \tag{3.140}$$

**5.** The system of RODEs corresponding to an r-CODE has four real symmetries $\mathbf{X}_1,\ \mathbf{X}_2$ and $\mathbf{Y}_2$ with

$$\mathbf{X}_1 \neq \rho_1\mathbf{X}_2 - \rho_2\mathbf{Y}_2, \tag{3.141}$$

for nonconstant $\rho_1$ and $\rho_2$ and they satisfy

$$[\mathbf{X}_1, \mathbf{X}_2] = 0, \quad \text{and} \quad [\mathbf{X}_1, \mathbf{Y}_2] = 0, \tag{3.142}$$

such that a point transformation $\chi = \chi(x, f, g), \Upsilon = \Upsilon(x, f, g)$ and $\zeta = \zeta(x, f, g),$ which brings $\mathbf{X}_1,\ \mathbf{X}_2$ and $\mathbf{Y}_2$ to their canonical form

$$\mathbf{X}_1 = \frac{\partial}{\partial \chi}, \quad \mathbf{X}_2 = \frac{\partial}{\partial \Upsilon}, \quad \mathbf{Y}_2 = \frac{\partial}{\partial \zeta}, \tag{3.143}$$

reduces the system (3.127) to the system of RODEs corresponding to r-CODE (3.125) is at most cubic in all its first derivatives;

**6.** The system of RODEs corresponding to a r-CODE has four real symmetries $\mathbf{X}_1,\ \mathbf{Y}_1,\ \mathbf{X}_2$ and $\mathbf{Y}_2$, with

$$\mathbf{X}_1 \neq \rho_1 \mathbf{X}_2 - \rho_2 \mathbf{Y}_2, \quad \mathbf{Y}_1 \neq \rho_1 \mathbf{Y}_2 + \rho_2 \mathbf{X}_2 \tag{3.144}$$

for nonconstant $\rho_1$ and $\rho_2$ and they satisfy either

$$[\mathbf{X}_1, \mathbf{X}_2] - [\mathbf{Y}_1, \mathbf{Y}_2] \neq 0 \quad \text{or} \quad [\mathbf{X}_1, \mathbf{Y}_2] + [\mathbf{Y}_1, \mathbf{X}_2] \neq 0, \tag{3.145}$$

such that a point transformation $\chi = \chi(x, f, g), \Upsilon = \Upsilon(x, f, g)$ and $\zeta = \zeta(x, f, g)$,which brings $\mathbf{X}_1,\ \mathbf{Y}_1,\ \mathbf{X}_2$ and $\mathbf{Y}_2$ to their canonical form

$$\mathbf{X}_1 = \frac{\partial}{\partial \Upsilon}, \quad \mathbf{Y}_1 = \frac{\partial}{\partial \zeta}, \quad \mathbf{X}_2 = \chi\frac{\partial}{\partial \chi} + \Upsilon\frac{\partial}{\partial \Upsilon} + \zeta\frac{\partial}{\partial \zeta},$$

$$\mathbf{Y}_2 = \zeta\frac{\partial}{\partial \Upsilon} - \Upsilon\frac{\partial}{\partial \zeta}, \tag{3.146}$$

reduces the system (3.127) to the linear form

$$\chi\Upsilon'' = a_1(\Upsilon'^3 - 3\Upsilon'\zeta'^2) - a_2(3\Upsilon'^2\zeta' - \zeta'^3) + b_1(\Upsilon'^2 - \zeta'^2) - 2b_2\Upsilon'\zeta' +$$
$$\frac{1}{3(a_1^2+a_2^2)}\{3(a_1^2+a_2^2) + (b_1^2 - b_2^2)a_1 + 2b_1b_2a_2\}\Upsilon' - \frac{1}{3(a_1^2+a_2^2)}\{2b_1b_2a_1 -$$
$$a_2(b_1^2 - b_2^2)\}\zeta' + \frac{1}{3(a_1^2+a_2^2)}(b_1a_1 + b_2a_2) + \frac{1}{27(a_1^2+a_2^2)}\{(b_1^3 - 3b_1b_2^2)(a_1^2 - a_2^2)$$
$$+2a_1a_2(3b_1^2b_2 - b_2^3)\}, \tag{3.147}$$
$$\chi\zeta'' = a_2(\Upsilon'^3 - 3\Upsilon'\zeta'^2) + a_1(3Y'^2\zeta' - \zeta'^3) + b_2(\Upsilon'^2 - \zeta'^2) + 2b_1\Upsilon'\zeta' +$$
$$\frac{1}{3(a_1^2+a_2^2)}\{3(a_1^2+a_2^2) + (b_1^2 - b_2^2)a_1 + 2b_1b_2a_2\}\zeta' + \frac{1}{3(a_1^2+a_2^2)}\{2b_1b_2a_1 -$$
$$a_2(b_1^2 - b_2^2)\}\Upsilon' + \frac{1}{3(a_1^2+a_2^2)}(b_2a_1 - b_1a_2) + \frac{1}{27(a_1^2+a_2^2)}\{(3b_1^2b_2 - b_2^3)(a_1^2 - a_2^2)$$
$$-2(b_1^3 - 3b_1b_2^2)a_1a_2\}.$$

The following invertible real transformation,

$$\tilde{\chi} = \Upsilon + \frac{1}{3(a_1^2+a_2^2)}\{(b_1a_1 + b_2a_2)\chi\}, \tag{3.148}$$
$$\tilde{\Upsilon} = \frac{1}{2}(\Upsilon^2 - \zeta^2) + \frac{1}{3(a_1^2+a_2^2)}[\{(b_1a_1 + b_2a_2)\chi\}\Upsilon - \{(b_2a_1 - b_1a_2)\chi\}\zeta] +$$
$$\frac{1}{18(a_1^2+a_2^2)^2}[\{(b_1^2 - b_2^2)(a_1^2 - a_2^2) + 4b_1b_2a_1a_2\}\chi^2] + \frac{1}{2(a_1^2+a_2^2)}\{a_1\chi^2\}, \tag{3.149}$$

$$\tilde{\zeta} = \Upsilon\zeta + \frac{1}{3(a_1^2+a_2^2)}[\{(b_1a_1 + b_2a_2)\chi\}\zeta + \{(b_2a_1 - b_1a_2)\chi\}F] +$$
$$\frac{1}{18(a_1^2+a_2^2)^2}[-\chi^2\{2b_1b_2(a_1^2 - a_2^2) - 2a_1a_2(b_1^2 - b_2^2)\}] + \frac{1}{2(a_1^2+a_2^2)}\{-a_2\chi^2\}, \tag{3.150}$$

transforms (3.147) into the system of linear RODEs

$$\Upsilon'' = 0, \quad \zeta'' = 0. \tag{3.151}$$

Now we discuss some illustrative examples of linearizable r-CODEs giving linearizable two dimensional systems of RODEs.

**Examples 3.9:** Consider the second-order nonlinear r-CODE with an arbitrary function $w(x)$

$$uu'' = u'^2 + w(x)u^2. \tag{3.152}$$

The above r-CODE admits two CLSs of the form

$$\mathbf{Z}_1 = xu\frac{\partial}{\partial u}, \quad \text{and} \quad \mathbf{Z}_2 = u\frac{\partial}{\partial u}, \tag{3.153}$$

and thus

$$[\mathbf{Z}_1, \mathbf{Z}_2] = 0 \quad \text{and} \quad \mathbf{Z}_2 = \frac{1}{x}\mathbf{Z}_1. \tag{3.154}$$

By using the complex transformation

$$\chi = \frac{1}{x} \quad \text{and} \quad U = \frac{1}{x}\log u, \tag{3.155}$$

(3.152) can be reduced into a linear r-CODE

$$U'' = \frac{1}{\chi^3}w(\frac{1}{\chi}). \tag{3.156}$$

The system of ODEs corresponding to (3.152) is

$$\begin{aligned} ff'' - gg'' &= f'^2 - g'^2 + w_1(f^2 - g^2) - 2fgw_2, \\ fg'' + gf'' &= 2f'g' + 2w_1fg + w_2(f^2 - g^2) \end{aligned} \tag{3.157}$$

with coefficients that satisfy the conditions of Lie type, (3.129) – (3.132). When one invoke the transformation,

$$\chi = \frac{1}{x}, \quad \Upsilon = \frac{1}{2x}\ln(f^2 + g^2), \quad \zeta = \frac{1}{x}\tan^{-1}(\frac{g}{f}), \tag{3.158}$$

the system (3.157) is reduced to the linear system of RODEs

$$\Upsilon'' = \frac{1}{\chi}w_1, \quad \zeta'' = \frac{1}{\chi}w_2, \tag{3.159}$$

where

$$w_1 = w_1(\frac{1}{\chi}), \quad w_2 = w_2(\frac{1}{\chi}). \tag{3.160}$$

**Examples 3.10:** Consider the nonlinear r-CODE (cf [53])

$$u'' + 3uu' + u^3 = 0, \tag{3.161}$$

which is linearizable as it satisfies the Lie conditions. The set of two non commuting CLSs are

$$\mathbf{Z}_1 = \frac{\partial}{\partial x} \quad \text{and} \quad \mathbf{Z}_2 = x\frac{\partial}{\partial x} - u\frac{\partial}{\partial u}. \tag{3.162}$$

Note that $\mathbf{Z}_1 \neq \rho(x,u)\mathbf{Z}_2$. Thus we invoke condition nine of Theorem 3.1 to find a linearizing transformation. The complex point transformation that reduces the symmetries (3.162) to their canonical form is

$$\chi = \frac{1}{u} \quad \text{and} \quad U = x + \frac{1}{u}, \tag{3.163}$$

and (3.161) reduces to

$$\chi U'' = -U'^3 + 6U'^2 - 11U' + 6 \tag{3.164}$$

by means of the transformation (3.163). Equation (3.161) linearizes to $\tilde{U}'' = 0$, via the complex transformations with $a = -1$ and $b = 6$, that is

$$\tilde{\chi} = x - \frac{1}{u} \quad \text{and} \quad \tilde{U} = \frac{x^2}{2} - \frac{x}{u}. \tag{3.165}$$

The transformation (3.165) may seem strange as the variable $x$ is real while $\tilde{\chi}$ is complex. The point is that we are actually working with a complex independent variable *restricted to lie on the real line*. We further require that *at the end* the variable again be restricted to the real line, but in the intervening steps the variable moves off the real line. The procedure is reminiscent of analytic continuation. This odd behavior does not appear for PDEs as everything "lives" in the complex world. In the new coordinates, $(\tilde{\chi}, \tilde{U})$, the solution of (3.161) is

$$\tilde{U} = \alpha\tilde{\chi} + \beta, \tag{3.166}$$

where $\alpha = \alpha_1 + i\alpha_2$ and $\beta = \beta_1 + i\beta_2$ whereas in coordinates $(\chi, u)$ the above equation yields

$$u = \frac{2(x-\alpha)}{x^2 - 2\alpha x - 2\beta}, \tag{3.167}$$

which satisfies r-CODE (3.161). The system of RODEs corresponding to (3.161) is

$$\begin{aligned} f'' &= -3(ff' - gg') - (f^3 - 3fg^2), \\ g'' &= -3(gf' + fg') - (3f^2 g - g^3), \end{aligned} \tag{3.168}$$

with coefficients that satisfy the conditions of Lie type, (3.129) − (3.132). Thus the above system is linearizable. The general solution of system (3.168) is

$$\begin{aligned} f &= \frac{2(x-\alpha_1)(x^2 - 2\alpha_1 x - 2\beta_1) + 4\alpha_2(\alpha_2 x + \beta_2)}{(x^2 - 2\alpha_1 x - 2\beta_1)^2 + (2\alpha_2 x + 2\beta_2)^2}, \\ g &= \frac{4(x-\alpha_1)(\alpha_2 x + \beta_2) - 2\alpha_2(x^2 - 2\alpha_1 x - 2\beta_1)}{(x^2 - 2\alpha_1 x - 2\beta_1)^2 + (2\alpha_2 x + 2\beta_2)^2}, \end{aligned} \tag{3.169}$$

which is obtained from (3.167).

**Examples 3.11:** Consider the nonlinear r-CODE

$$u'' = 1 + (u' - x)^2 w(2u - x^2), \tag{3.170}$$

which admits the CLSs

$$\mathbf{Z}_1 = \frac{\partial}{\partial x} + x\frac{\partial}{\partial u} \quad \text{and} \quad \mathbf{Z}_2 = x\frac{\partial}{\partial x} + x^2\frac{\partial}{\partial u}. \tag{3.171}$$

Equation (3.170) reduces to a linear r-CODE by using the complex transformation,

$$\chi = 2u - x^2 \quad \text{and} \quad U = x, \tag{3.172}$$

to become

$$U'' = -\frac{1}{2}U' w(\chi). \tag{3.173}$$

Again, notice that (3.172) is not the usual complex transformation as it is from (real, complex) to (complex, real) leading to an apparent contradiction. As before it corresponds to analytic continuation. To check consistency we may take $w = 1$, i.e. $w_1 = 1$ and $w_2 = 0$, to obtain

$$u = \alpha + \ln 2 + \frac{x^2}{2} - \log(\beta - x), \tag{3.174}$$

where $\alpha = \alpha_1 + \iota\alpha_2$ and $\beta = \beta_1 + \iota\beta_2$. Note that (3.174) satisfies (3.170). The system of RODEs corresponding to (3.170) is

$$\begin{aligned} f'' &= 1 + \{(f' - x)^2 - g'^2\}w_1 - 2(f' - x)g'w_2, \\ g'' &= 2(f' - x)g'w_1 + \{(f' - x)^2 - g'^2\}w_2 \end{aligned} \tag{3.175}$$

with coefficients satisfying the conditions of Lie type, (3.129) – (3.132), where

$$w_1 = w_1(2f - x^2, 2g) \text{ and } w_2 = w_2(2f - x^2, 2g). \tag{3.176}$$

By placing values of $w_1 = 1, w_2 = 0$, we get the system

$$\begin{aligned} f'' &= 1 + \{(f' - x)^2 - g'^2\}, \\ g'' &= 2(f' - x)g' \end{aligned} \tag{3.177}$$

with general solution

$$\begin{aligned} f &= \alpha_1 - \ln 2 + \frac{x^2}{2} - \frac{1}{2}\log((\beta_1 - x)^2 + \beta_2^2), \\ g &= \alpha_2 - \tan^{-1}(\frac{\beta_2}{\beta_2 - x}). \end{aligned} \tag{3.178}$$

Also, if we take

$$w(2u - x^2) = \frac{1}{2u - x^2}, \tag{3.179}$$

the solution of (3.170) is

$$u = \frac{x^2}{2} + \sqrt{\frac{\beta - x}{2\alpha}}, \tag{3.180}$$

which gives the solution

$$
\begin{aligned}
f &= \frac{x^2}{2} + \frac{1}{\sqrt{2(\alpha_1^2+\alpha_2^2)}}\sqrt{(\alpha_1(\beta_1-x)+\beta_2\alpha_2)^2+(\beta_2\alpha_2-\alpha_2(\beta_1-x))^2}\times \\
&\quad \cos\left[\frac{\beta_2\alpha_2-\alpha_2(\beta_1-x)}{2(\alpha_1(\beta_1-x)+\beta_2\alpha_2)}\right], \\
g &= \frac{1}{\sqrt{2(\alpha_1^2+\alpha_2^2)}}\sqrt{(\alpha_1(\beta_1-x)+\beta_2\alpha_2)^2+(\beta_2\alpha_2-\alpha_2(\beta_1-x))^2}\times \\
&\quad \sin\left[\frac{\beta_2\alpha_2-\alpha_2(\beta_1-x)}{2(\alpha_1(\beta_1-x)+\beta_2\alpha_2)}\right],
\end{aligned} \tag{3.181}
$$

of system

$$
\begin{aligned}
f'' &= 1+\{(f'-x)^2-g'^2\}\frac{2f-x^2}{(2f-x^2)^2+4g^2}+4(f'-x)g'\frac{g}{(2f-x^2)^2+4g^2}, \\
g'' &= 2(f'-x)g'\frac{2f-x^2}{(2f-x^2)^2+4g^2}-2\{(f'-x)^2-g'^2\}\frac{g}{(2f-x^2)^2+4g^2}.
\end{aligned} \tag{3.182}
$$

It is interesting to see that here $w$ is an arbitrary complex function that gives a class of systems of RODEs corresponding to the r-CODE (3.170). Thus the linearization of a general r-CODE like (3.170) encodes the linearization of a large class of systems of RODEs. Furthermore complex solutions like (3.178) and (3.181) directly give us the real solutions of those systems of RODEs and yield non trivial examples of complex magic.

# Chapter 4

# Variational problem for CODEs

Finding a Lagrangian of some DE is one of the important problems of variational calculus, known as "the inverse problem". One of its key uses arises in the application of Noether's theorem [52] that relates the symmetries of the action integral with conservation laws. The construction of a Lagrangian for an arbitrary DE is difficult and the inverse problem is discussed for a class of DEs e.g. $[5-9, 16, 18, 31, 26, 56]$ with the solution of the inverse problem for systems of two ODEs and a class of second-order quasilinear scalar equations. In this Chapter we extend several known results of variational calculus to the complex domain and use them in the construction of Lagrangians for systems of PDEs and ODEs. We also develop several conservation laws for those systems in this chapter.

## 4.1 Complex Lagrangians and complex Euler-Lagrange equations

Consider a second-order CODE of a complex-valued function $u(z)$ of the form

$$u'' = w\ (z, u, u'), \tag{4.1}$$

where $w$ is a complex analytic function of its arguments. The CLSs admitted by the above CODE and the corresponding RLSs of the respective system of PDEs corresponding to the CODE have been discussed in [1]. Assume the above CODE arises from a complex Lagrangian

$L(z,u,u')$, i.e., the above equation is equivalent to the complex EL equation

$$\frac{\partial L}{\partial u}-\frac{d}{dz}(\frac{\partial L}{\partial u'})=0. \tag{4.2}$$

Notice that $L$ is complex scalar, i.e., it yields two real Lagrangians which we call r-Lagrangians and they are not the usual Lagrangians as they do not satisfy the Euler-Lagrange (EL) equations in the real domain. The physical significance of complex Lagrangians is mentioned in the next chapter.

## 4.2 Complex Noether symmetries

**Definition 1.** $\mathbf{Z}$ is said to be a complex Noether point symmetry (NS) corresponding to a c-Lagrangian $L(z,u,u')$ of (4.1) if there exists a complex function $A(z,u)$ such that

$$\mathbf{Z}^{(1)}L+(\frac{d\varsigma}{dz})L=\frac{dA}{dz}, \tag{4.3}$$

where

$$\mathbf{Z}^{(1)}=\varsigma\frac{\partial}{\partial z}+\chi\frac{\partial}{\partial u}+\chi^{(1)}\frac{\partial}{\partial u'},\quad \frac{d}{dz}=\frac{\partial}{\partial z}+u'\frac{\partial}{\partial u}+...\ . \tag{4.4}$$

Now we state Noether's theorem for complex NSs, and theorems for the first integral and solvability by quadratures, without proof as there is no difference made by complexity.

**Theorem 4.1.** *If* $\mathbf{Z}$ *is a Noether point symmetry for a c-Lagrangian* $L(z,u,u')$ *of* (4.1) *then*

$$I=\varsigma L+(\chi-u'\varsigma)L_{u'}-A \tag{4.5}$$

*is a complex first integral of* (4.1) *associated with* $\mathbf{Z}$, i.e., $dI/dz=0$ on solutions of (4.1).

**Theorem 4.2.** *The first integral* $I$ *associated with the complex Noether point symmetry* $\mathbf{Z}$ *satisfies the relation*

$$\mathbf{Z}^{(1)}I=0. \tag{4.6}$$

**Theorem 4.3.** *If for a c-Lagrangian* $L(z,u,u')$ *of* (4.1) *there corresponds a Noether point complex symmetry, then* (4.1) *is solvable by quadrature.*

In the subsequent section we transform the above results into the real domain to obtain EL-like equations, two first integrals and conditions corresponding to (4.3) and (4.5) for a system of PDEs.

## 4.3 Euler-Lagrange-like equations

One obtains the following system of PDEs corresponding to CODE (4.1)

$$\begin{aligned} f_{xx} - f_{yy} + 2g_{xy} &= G(x, y, f, g, h, l), \quad f_x = g_y, \\ g_{xx} - g_{yy} - 2f_{xy} &= H(x, y, f, g, h, l), \quad f_y = -g_x, \end{aligned} \tag{4.7}$$

where we have used (2.8). Since $L$ is a complex scalar, we can write it as

$$L = L_1 + \iota L_2. \tag{4.8}$$

Therefore contrary to the usual Lagrangian approach we have two Lagrangians for system (4.7) that satisfy EL-like equations

$$\begin{aligned} \frac{\partial L_1}{\partial f} + \frac{\partial L_2}{\partial g} - d_x(\frac{\partial L_1}{\partial h} + \frac{\partial L_2}{\partial l}) - d_y(\frac{\partial L_2}{\partial h} - \frac{\partial L_1}{\partial l}) &= 0, \\ \frac{\partial L_2}{\partial f} - \frac{\partial L_1}{\partial g} - d_x(\frac{\partial L_2}{\partial h} - \frac{\partial L_1}{\partial l}) + d_y(\frac{\partial L_1}{\partial h} + \frac{\partial L_2}{\partial l}) &= 0. \end{aligned} \tag{4.9}$$

The above system of DEs couples the two r-Lagrangians $L_1$ and $L_2$. Note that they are not separately Lagrangians as they do not satisfy EL equations separately but only the coupled system (4.9). The corresponding system of conditions relative to (4.3) in the real domain is given by

$$\begin{aligned} 2\mathbf{X}^{(1)}L_1 - 2\mathbf{Y}^{(1)}L_2 + (d_x\varsigma_1 + d_y\varsigma_2)L_1 - (d_x\varsigma_2 - d_y\varsigma_1)L_2 = d_xA_1 + d_yA_2, \\ 2\mathbf{X}^{(1)}L_2 + 2\mathbf{Y}^{(1)}L_1 + (d_x\varsigma_1 + d_y\varsigma_2)L_2 + (d_x\varsigma_2 - d_y\varsigma_1)L_1 = d_xA_2 - d_yA_1, \end{aligned} \tag{4.10}$$

where we have set

$$\zeta = \zeta_1 + \iota\zeta_2 \ \text{ and } \ A = A_1 + \iota A_2. \tag{4.11}$$

The first integral (4.5) results in two real integrals

$$I_1 = \varsigma_1 L_1 - \varsigma_2 L_2 + (\chi_1 - \frac{h}{2}\varsigma_1 - \frac{l}{2}\varsigma_2)(\partial_h L_1 + \partial_l L_2) - (\chi_2 - \frac{h}{2}\varsigma_2 - \frac{l}{2}\varsigma_1)(\partial_h L_2 - \partial_l L_1) - A_1,$$
$$I_2 = \varsigma_1 L_2 + \varsigma_2 L_1 + (\chi_1 - \frac{h}{2}\varsigma_1 - \frac{l}{2}\varsigma_2)(\partial_h L_2 - \partial_l L_1) + (\chi_2 - \frac{h}{2}\varsigma_2 - \frac{l}{2}\varsigma_1)(\partial_h L_1 + \partial_l L_2) - A_2, \quad (4.12)$$

which satisfy

$$\begin{aligned} d_x I_1 + d_y I_2 &= 0, \\ d_y I_1 - d_x I_2 &= 0. \end{aligned} \quad (4.13)$$

In addition, if we set $\chi = \chi_1 + \iota\chi_2$ with $\chi^{(1)} = \chi_1^{(1)} + \iota\chi_2^{(1)}$, then

$$\begin{aligned} 2\mathbf{X} &= \zeta_1\partial_x + \zeta_2\partial_y + \chi_1\partial_f + \chi_2\partial_g + \chi_1^{(1)}\partial_h + \chi_2^{(1)}\partial_l, \\ 2\mathbf{Y} &= \zeta_2\partial_x - \zeta_1\partial_y + \chi_2\partial_f - \chi_1\partial_g + \chi_2^{(1)}\partial_h - \chi_1^{(1)}\partial_l \end{aligned} \quad (4.14)$$

with

$$\begin{aligned} \chi_1^{(1)} &= d_x\chi_1 + d_y\chi_2 - \frac{h}{2}(d_x\zeta_1 + d_y\zeta_2) + \frac{l}{2}(d_x\zeta_2 - d_y\zeta_1), \\ \chi_2^{(1)} &= d_y\chi_1 - d_x\chi_2 - \frac{h}{2}(d_x\zeta_2 - d_y\zeta_1) - \frac{l}{2}(d_x\zeta_1 + d_y\zeta_2). \end{aligned} \quad (4.15)$$

The two first integrals satisfy the coupled equations

$$\begin{aligned} \mathbf{X}^{(1)}I_1 - \mathbf{Y}^{(1)}I_2 &= 0, \\ \mathbf{X}^{(1)}I_2 + \mathbf{Y}^{(1)}I_1 &= 0, \end{aligned} \quad (4.16)$$

where

$$\mathbf{Z}^{(1)} = \mathbf{X}^{(1)} + \iota\mathbf{Y}^{(1)}. \quad (4.17)$$

Now we present some illustrative examples.

**Examples 4.1:** Consider the complexified free particle equation

$$u'' = 0, \quad (4.18)$$

which admits a c-Lagrangian of the form

$$L = \frac{1}{2} u'^{\,2}. \tag{4.19}$$

The CODE (4.18) admits an 8−dimensional complex Lie algebra [1]. It has 5 CLSs which are also the NSs with respect to the c-Lagrangian (4.19) (for simplicity we do not write the first extension)

$$\mathbf{Z}_1 = \frac{\partial}{\partial z}, \quad \mathbf{Z}_2 = \frac{\partial}{\partial u}, \quad \mathbf{Z}_3 = 2z\frac{\partial}{\partial z} + u\frac{\partial}{\partial u}, \tag{4.20}$$

$$\mathbf{Z}_4 = z^2\frac{\partial}{\partial z} + zu\frac{\partial}{\partial u}, \quad \mathbf{Z}_5 = z\frac{\partial}{\partial u}. \tag{4.21}$$

One can reduce the order of (4.18) by two using any one of the above symmetries. For example for the NS $\mathbf{Z}_2$ the first integral is

$$I = u' = a \tag{4.20}$$

by the use of (4.5). From (4.6) $\mathbf{Z}_2$ is also a symmetry of the Lagrangian (4.19). The above equation (4.20) yields the solution

$$u = az + b, \tag{4.21}$$

where $a$ and $b$ are complex constants. Similarly, if we use the NS $\mathbf{Z}_3$, we obtain

$$I = uu' - zu'^2 = c. \tag{4.22}$$

Since $\mathbf{Z}_3$ satisfies (4.6) , it is also a symmetry of (4.22). It can be transformed into separable form by introducing

$$w = uz^{-\frac{1}{2}}, \tag{4.23}$$

which is

$$w^2 - 4z^2w'^2 = C. \tag{4.24}$$

Thus an NS reduces a second-order CODE to quadratures. The solution in the new coordinates $(z, w)$ is

$$w(z) = \alpha z^{\frac{1}{2}} + \beta z^{-\frac{1}{2}}, \tag{4.25}$$

where $\alpha$ and $\beta$ are complex constants. The system of PDEs corresponding to the CODE (4.18) is

$$\begin{aligned} f_{xx} - f_{yy} + 2g_{xy} &= 0, \quad f_x = g_y, \\ g_{xx} - g_{yy} - 2f_{xy} &= 0, \quad f_y = -g_x. \end{aligned} \tag{4.26}$$

The respective r-Lagrangians, $L_1$ and $L_2$, are

$$\begin{aligned} L_1 &= \frac{1}{8}(h^2 - l^2), \\ L_2 &= \frac{1}{4}hl. \end{aligned} \tag{4.27}$$

One obtains the respective system of PDEs (4.26) by replacing the above r-Lagrangians in (4.9). In order to reduce the order we decompose $\mathbf{Z}_2$ into its real parts,

$$\mathbf{X}_2 = \frac{\partial}{\partial f} \quad \text{and} \quad \mathbf{Y}_2 = \frac{\partial}{\partial g}, \tag{4.28}$$

which satisfy (4.10). From (4.12), setting $I_1 = c_1$ and $I_2 = c_2$, the two conserved quantities are $(A_1 = A_2 = 0)$

$$I_1 = h = 2c_1 \quad \text{and} \quad I_2 = l = 2c_2. \tag{4.29}$$

Using the CREs one obtains the solution of the system (4.26). Similarly, $\mathbf{Z}_3$ yields $(A_1 = A_2 = 0)$

$$\mathbf{X}_3 = 2x\frac{\partial}{\partial x} + 2y\frac{\partial}{\partial y} + f\frac{\partial}{\partial f} + g\frac{\partial}{\partial g} \quad \text{and} \quad \mathbf{Y}_3 = 2y\frac{\partial}{\partial x} - 2x\frac{\partial}{\partial y} + g\frac{\partial}{\partial f} - f\frac{\partial}{\partial g} \tag{4.30}$$

which satisfy (4.10). The conserved quantities from (4.12), setting $I_1 = c_1$ and $I_2 = c_2$, are $(A_1 = A_2 = 0)$

$$\begin{aligned} I_1 &= fh - gl - \frac{1}{2}x(h^2 - l^2) + yhl = 2c_1, \\ I_2 &= fl + gh - \frac{1}{2}y(h^2 - l^2) - xhl = 2c_2. \end{aligned} \tag{4.31}$$

By introducing new coordinates we have

$$\begin{aligned} F &= r^{-\frac{1}{2}}(f\cos\theta/2 + g\sin\theta/2), \\ G &= r^{-\frac{1}{2}}(g\cos\theta/2 - f\sin\theta/2), \end{aligned} \tag{4.32}$$

where

$$r^2 = x^2 + y^2, \quad \theta = \tan^{-1}(y/x), \tag{4.33}$$

we obtain the following system

$$\begin{aligned} F^2 - G^2 - 4r^2[(H^2 - J^2)\cos 2\theta - 2HJ\sin 2\theta] &= C_1, \\ 2FG - 4r^2[(H^2 - J^2)\sin 2\theta + 2HJ\cos 2\theta] &= C_2, \end{aligned} \tag{4.34}$$

where

$$H = \frac{1}{2}(F_x + G_y), \quad J = \frac{1}{2}(G_x - F_y) \tag{4.35}$$

and

$$\partial_x = \cos\theta\partial_r - \frac{1}{r}\sin\theta\partial_\theta, \quad \partial_y = \sin\theta\partial_r + \frac{1}{r}\partial_\theta\,, \tag{4.36}$$

so that $F_x = \cos\theta F_r - (1/r)\sin\theta F_\theta$ etc. in (3.40). The solution of the system (4.34) with $\alpha = \alpha_1 + \iota\alpha_2$ and $\beta = \beta_1 + \iota\beta_2$ is

$$\begin{aligned} F &= r^{\frac{1}{2}}(\alpha_1\cos\theta/2 - \alpha_2\sin\theta/2) + r^{-\frac{1}{2}}(\beta_1\cos\theta/2 + \beta_2\sin\theta/2), \\ G &= r^{\frac{1}{2}}(\alpha_1\sin\theta/2 + \alpha_2\cos\theta/2) + r^{-\frac{1}{2}}(-\beta_1\sin\theta/2 + \beta_2\cos\theta/2). \end{aligned} \tag{4.37}$$

Notice that the above solution cannot be obtained directly without using complex transformations. It also demonstrates the fact that a simple solution may look complicated in some coordinate system.

**Examples 4.2:** The complexified oscillator equation is given by

$$u'' = -u. \tag{4.38}$$

The CLS of this CODE have been studied in [7]. An obvious c-Lagrangian admitted by such an equation is

$$L = \frac{1}{2}u'^2 - \frac{1}{2}u^2. \tag{4.39}$$

A simple NS of (4.38) with respect to the c-Lagrangian (4.39) is

$$\mathbf{Z} = \frac{\partial}{\partial z}, \tag{4.40}$$

which gives the first integral

$$I = u^2 + u'^2 = 2c. \tag{4.41}$$

The solution of the above CODE (4.41) is

$$u = \alpha \cos z + \beta \sin z, \tag{4.42}$$

where $\alpha$ and $\beta$ are complex constants. The system of PDEs corresponding to (4.41) is

$$\begin{aligned} f_{xx} - f_{yy} + 2g_{xy} &= -4f, \quad f_x = g_y, \\ g_{xx} - g_{yy} - 2f_{xy} &= -4g, \quad f_y = -g_x. \end{aligned} \tag{4.43}$$

The respective r-Lagrangians, $L_1$ and $L_2$, are

$$\begin{aligned} L_1 &= \frac{1}{8}(h^2 - l^2) - \frac{1}{2}(f^2 - g^2), \\ L_2 &= \frac{1}{4}hl - fg. \end{aligned} \tag{4.44}$$

The obvious NSs of system (4.43) for the r-Lagrangians (4.44) are

$$\mathbf{X} = \frac{\partial}{\partial x}, \quad \mathbf{Y} = \frac{\partial}{\partial y}, \tag{4.45}$$

which satisfy (4.9). From (4.12) two conserved quantities, $I_1 = c_1$ and $I_2 = c_2$, are ($A_1 = A_2 = 0$)

$$\begin{aligned} I_1 &= f^2 - g^2 + \frac{1}{4}(h^2 - l^2) = -2c_1, \\ I_2 &= fg + \frac{1}{4}hl = -c_2, \end{aligned} \tag{4.46}$$

which on use of the CREs, yields the solution of (4.43) as

$$\begin{aligned} f &= \alpha_1 \cos x \cosh y + \alpha_2 \sin x \sinh y + \beta_1 \sin x \cosh y - \beta_2 \cos x \sinh y, \\ g &= \alpha_2 \cos x \cosh y - \alpha_1 \sin x \sinh y + \beta_1 \cos x \sinh y + \beta_2 \sin x \cosh y. \end{aligned} \tag{4.47}$$

This is easier to obtain by utilising (4.42).

**Examples 4.3:** Consider the complexified Emden-Fowler equation

$$u'' + \frac{2}{z}u' = 3u^5. \tag{4.48}$$

It admits a c-Lagrangian of the form

$$L = \frac{1}{2}z^2 u'^2 + \frac{1}{2}z^2 u^6. \tag{4.49}$$

The complex NS of (4.48) with respect to the c-Lagrangian (4.49) is

$$\mathbf{Z} = 2z\frac{\partial}{\partial z} - u\frac{\partial}{\partial u}. \tag{4.50}$$

The first integral is

$$I = -z^3 u'^2 - z^2 uu' + z^3 u^6 \tag{4.51}$$

by use of (4.5). Hence the reduced equation is

$$z^3 u'^2 + z^2 uu' - z^3 u^6 = c, \tag{4.52}$$

which can be put into a separable form on introducing

$$w = uz^{1/2}, \tag{4.53}$$

which is

$$z^2 w'^2 - \frac{1}{4} w^2 - w^6 = C. \tag{4.54}$$

This first-order equation (4.54) is variables separable and thus can be reduced to quadrature. The corresponding system of PDEs of the CODE (4.48) is

$$\begin{aligned}
f_{xx} - f_{yy} + 2g_{xy} + \frac{4x}{x^2+y^2}h + \frac{4y}{x^2+y^2}l &= 12(f^5 - 8f^3g^2 + 5fg^4), \quad f_x = g_y, \\
g_{xx} - g_{yy} - 2f_{xy} + \frac{4x}{x^2+y^2}l - \frac{4y}{x^2+y^2}h &= 12(g^5 - 8f^2g^3 + 5f^4g), \quad f_y = -g_x. \quad (4.55)
\end{aligned}$$

The c-Lagrangian $L$ (4.49) yields the two r-Lagrangians

$$\begin{aligned}
L_1 &= \frac{1}{8}(x^2-y^2)(h^2-l^2) - \frac{1}{2}xyhl + \frac{1}{2}(x^2-y^2)(f^2-g^2)(f^4+g^4-14f^2g^2) \\
&\quad -2xyfg(3f^4+3g^4-10f^2g), \\
L_2 &= \frac{1}{4}(x^2-y^2)hl + \frac{1}{4}xy(h^2-l^2) + (x^2-y^2)fg(3f^4+3g^4-10f^2g^2) \\
&\quad +xy(f^2-g^2)(f^4+g^4-14f^2g^2). \quad (4.56)
\end{aligned}$$

It turns out that the system of PDEs (4.55) is obtained on inserting $L_1$ and $L_2$ in the EL-like equations (4.9). The two real NSs corresponding to the system of PDEs (4.55) for the r-Lagrangians, $L_1$ and $L_2$, are

$$\mathbf{X} = 2x\frac{\partial}{\partial x} + 2y\frac{\partial}{\partial y} - f\frac{\partial}{\partial f} - g\frac{\partial}{\partial g}, \tag{4.57}$$

$$\mathbf{Y} = 2y\frac{\partial}{\partial x} - 2x\frac{\partial}{\partial y} + f\frac{\partial}{\partial g} - g\frac{\partial}{\partial f}. \tag{4.58}$$

The conserved quantities are given by ($A_1 = A_2 = 0$)

$$
\begin{aligned}
I_1 = & \frac{1}{4}x(x^2-3y^2)(h^2-l^2) - \frac{1}{2}y(3x^2-y^2)hl + \frac{1}{2}h(f(x^2-y^2)-2xyg) - \\
& \frac{1}{2}l(g(x^2-y^2)+2xyf) + x(x^2-3y^2)(f^2-g^2)(f^4+g^4-14f^2g^2) \\
& -2yfg(3x^2-y^2)(3f^4+3g^4-10f^2g^2), \qquad (4.59) \\
I_2 = & \frac{1}{4}y(3x^2-y^2)(h^2-l^2) + \frac{1}{2}xhl(x^2-3y^2) + \frac{1}{2}h(g(x^2-y^2)+2xyf) + \\
& \frac{1}{2}l(f(x^2-y^2)-2xyg) + 2x(x^2-3y^2)fg(3f^4+3g^4-10f^2g^2) \\
& +y(3x^2-y^2)(f^2-g^2)(f^4+g^4-14f^2g^2) \qquad (4.60)
\end{aligned}
$$

as these satisfy the coupled system (4.12). We introduce

$$
F = r^{1/2}(f\cos\theta/2 - g\sin\theta/2),
$$

$$
G = r^{1/2}(g\cos\theta/2 + f\sin\theta/2), \qquad (4.61)
$$

where $r$ and $\theta$ are defined by (4.33). One obtains the system

$$
\begin{aligned}
r^2(H^2-J^2)\cos 2\theta - 2HJ\sin 2\theta - \tfrac{1}{4}(F^2-G^2) - (F^2-G^2)^3 + 12F^2G^2(F^2-G^2) = C_1 \\
r^2(H^2-J^2)\sin 2\theta + 2HJ\cos 2\theta - \tfrac{1}{2}FG + 8F^3G^3 - 6FG(F^2-G^2)^2 = C_2, \qquad (4.62)
\end{aligned}
$$

where $H$ and $J$ are given by (4.35). The solution of (4.62) can easily be deduced from the solution of (4.48) by using separation of variables. Then one can revert to the original variables via the transformation (4.61).

## 4.4 Variational problem for systems of ODEs

In this section we use EL equations for r-CODEs and then utilize symmetries to write the first integrals for systems of ODEs. The Lagrangian $L(x, u, u')$ of an r-CODE is a complex Lagrangian restricted to real line. It yields two real 'Lagrangians' $L_1(x, f, g, f', g')$ and $L_2(x, f, g, f', g')$ which satisfy the system of EL like equations, (4.9), restricted to real line. Further we present the double reduction in systems of ODEs.

**Example 4.4**: Consider the complexified free particle equation

$$u'' = 0, \tag{4.63}$$

which admits a c-Lagrangian of the form

$$L = \frac{1}{2}u'^{\,2}. \tag{4.64}$$

The r-CODE (4.63) admits an 8-dimensional complex Lie algebra [1]. It has 5 CLSs which are also the NSs with respect to the c-Lagrangian (4.64) (for simplicity we do not write the first extension)

$$\begin{aligned} \mathbf{Z}_1 &= \frac{\partial}{\partial x}, \quad \mathbf{Z}_2 = \frac{\partial}{\partial u}, \quad \mathbf{Z}_3 = 2x\frac{\partial}{\partial x} + u\frac{\partial}{\partial u}, \\ \mathbf{Z}_4 &= x^2\frac{\partial}{\partial x} + xu\frac{\partial}{\partial u}, \quad \mathbf{Z}_5 = x\frac{\partial}{\partial u}. \end{aligned} \tag{4.65}$$

We use the NS $\mathbf{Z}_2$ to reduce the order of (4.65). The first integral is

$$I = u' = a, \tag{4.66}$$

by the use of (4.5). From (4.6) $\mathbf{Z}_2$ is also a symmetry of the c-Lagrangian (4.19). The above equation (4.66) yields the solution

$$u = ax + b, \tag{4.67}$$

where $a$ and $b$ are complex constants. Similarly, if we use the NS $\mathbf{Z}_3$, we obtain

$$I = uu' - xu'^2 = c. \tag{4.68}$$

Since $\mathbf{Z}_3$ satisfies (4.6) , it is also a symmetry of the above equation (4.68). It can be transformed into separable form by introducing

$$w = ux^{-\frac{1}{2}}, \tag{4.69}$$

which is

$$w^2 - 4x^2w'^2 = C. \tag{4.70}$$

Thus a NS reduces a second-order r-CODE to quadratures. The solution in new coordinates $(z, w)$ is

$$w(x) = \alpha x^{\frac{1}{2}} + \beta x^{-\frac{1}{2}}, \tag{4.71}$$

where $\alpha$ and $\beta$ are complex constants. The system of ODEs corresponding to the r-CODE (4.63) is

$$f'' = 0, \quad g'' = 0. \tag{4.72}$$

The respective r-Lagrangians, $L_1$ and $L_2$, are

$$\begin{aligned} L_1 &= \frac{1}{2}(f'^2 - g'^2), \\ L_2 &= f'g'. \end{aligned} \tag{4.73}$$

One obtains the respective system of ODEs (4.72) by replacing the above r-Lagrangians in (4.9). In order to reduce the order we decompose $\mathbf{Z}_2$ into its real parts

$$\mathbf{X}_2 = \frac{\partial}{\partial f} \quad \text{and} \quad \mathbf{Y}_2 = \frac{\partial}{\partial g}, \tag{4.74}$$

which satisfy (4.10). From (4.12), setting $I_1 = c_1$ and $I_2 = c_2$, the two conserved quantities are $(A_1 = A_2 = 0)$

$$I_1 = h = 2c_1 \quad \text{and} \quad I_2 = l = 2c_2. \tag{4.75}$$

Therefore we can apply CLS analysis to solve variational problems for systems of ODEs. It may be noticed that two copies of the real Lagrangians are contained in a single complex Lagrangian.

# Chapter 5

# Conclusion and future directions

CLS analysis of CODEs leads us to calculate RLSs of real PDEs corresponding to those CODEs. It is seen that finding symmetries of PDEs and their systems is sometimes not only difficult but highly non trivial [44]. We can specify the symmetries of only those systems of PDEs that are obtained from CODEs and their systems, which therefore satisfy CREs. Further we provide an approach that helps not only to classify but also to find symmetries of certain systems of PDEs. A better understanding of this approach is expected to provide a means to solve the inverse problem, i.e. by doing the symmetry analysis of an arbitrary system of PDEs in the complex framework. Certainly the complex approach is a fruitful choice for a non trivial generalization of the classical Lie approach. Since the CREs directly imply the existence of Laplace equations (for both $f$ and $g$ in our notation), one can look for the symmetries of systems of Laplace equations.

We know that PDEs and their systems can have infinitely many symmetries. In our CLS analysis we have calculated finite-dimensional subalgebras of special classes of systems of PDEs. It would be worth while to classify such systems in more detail. The classification of systems of PDEs with respect to the algebra is an open problem. It is also important to point out that the arbitrariness of the functions $G$ and $H$ in system (2.9) and (2.20) is constrained by the analyticity of $w$ in (2.7) and (2.17) respectively i.e. they satisfy CREs. Thus we only deal with a special class of systems of PDEs. This is what enables us to obtain results that would be surprising in a general context.

CLS analysis was also applied to the invariance of r-CODEs and systems of corresponding

ODEs. The idea is to restrict CODEs on the real line. This yields systems of ODEs. The solution of such systems of ODEs is obtained from the complex solution of corresponding r-CODEs by again restricting them to the real line. It was found that the symmetries of certain systems of ODEs can be obtained from CLSs of r-CODEs. The real transformations that are obtained from complex transformation can be used to map a system of ODEs into another system of ODEs. It is important to notice the invariance of the systems of ODEs under the complex transformations corresponding to CLSs because we restricted complex functions to a single real variable which looks an apparent contradiction. This is the analogue of analytic continuation for ODEs. Another important use of CLS analysis in systems of ODEs is the extraction of their solutions from complex solutions of r-CODEs.

Generally, a first-order linear ODE admits infinitely many symmetries and a second-order DE can have a symmetry algebra of dimension 0, 1, 2, 3 or 8 [37, 42]. Also a linear $n^{th}$-order ($n \geq 3$) equation does not possess a maximal symmetry algebra of dimension $n+3$ [41]. It will be interesting to explore and extend these results to CODEs and r-CODEs and then use these results in systems of real PDEs and systems of ODEs, respectively. The classification of CODEs and r-CODEs by their complex algebras needs also to be explored [35]. Further it is suggested that several known results for a system of two second-order RODEs admits $5, 6, 7, 8$ or 15 RLSs and the maximal symmetry algebra is $sl(4, \mathbb{R})$ for the simplest system [65] can be extended to systems of two CODEs and two r-CODEs which in turn yield the symmetry analysis of systems of eight PDEs and four ODEs respectively. Therefore CLS analysis can be used to extend several classical results. It is very difficult to deal with large systems of PDEs and ODEs especially in the context of their symmetries by the usual classical Lie approach. It may also be pointed out that we only obtain an even number of PDEs (ODEs) from CODEs (r-CODEs), thus indicating another limitation of CLS analysis.

The geometrical aspects of the CLS analysis can be exploited by taking a complexified free-particle equation. The projective transformation that maps a complex line into a complex line contains 8 complex parameters. It remains to be rigorously proved that the maximal symmetry algebra of a CODE is 8-dimensional. In order to prove this fact geometrically one might need to use the stereographic projection on the Riemann sphere [59]. Notice that the simplest system of ODEs,

$$f'' = 0, \quad g'' = 0,$$

admits 15 RLSs [65] whereas through CLS analysis we only obtain a subalgebra. An explicit understanding of these results is also required.

One can use a CLS to reduce the order of a CODE that corresponds not only to the reduction of higher-order derivatives but also the number of variables in the corresponding system of PDEs. The CLS of *systems of CODEs* [19] and their use in reducing the higher-orders from corresponding PDEs will give us non trivial results. Similarly, reduction of order in r-CODEs yields the reduction of order in the systems of ODEs.

Finding exact solutions of nonlinear DEs is difficult. Linearization of DEs is a method to convert the original nonlinear DE to a linear DE so that exact solutions can be constructed directly. This requires the existence and construction of such transformations (point, tangent, contact or complex) that transforms nonlinear DEs into linear DEs. For scalar ODEs Lie provided nontrivial ways of constructing point transformations via symmetries. However, it is difficult to extend the Lie conditions for systems of ODEs and for PDEs (see e.g.,[$46-50, 68$]). Finding transformations that map systems of nonlinear ODEs and PDEs into linear systems is not only tedious but also highly nontrivial.

In this work we have looked at linearizability criteria for second-order CODEs. We obtained analogous Lie conditions for CODEs and applied them to linearization of systems of nonlinear PDEs. The corresponding statement for a system of two PDEs (in fact four, i.e. by incorporating linear CREs) associated to a second-order CODE was presented. Examples were given for the linearization of systems of PDEs including the construction of the linearizing point transformations. It is important to point that we only obtain those systems of PDEs that correspond to CODEs. For example system (3.28) is a special case of a general system of second-order PDEs in two dimensions. The question arises: "How can we obtain more general systems of second-order PDEs from CODEs?" It is suggested that the Lie criteria for linearization of two CODEs can be found in a similar way, which can then be used to linearize four nonlinear PDEs corresponding to a system of two CODEs.

We have seen that linearization theorems can be carried over from the real case to the complex case with remarkable results for the linearization of systems of PDEs. Thus we were able to linearize those systems of nonlinear PDEs that correspond to some CODEs. There are several possible ways of using CLS analysis. It would be of great interest to extend the geometric proof of linearization for RODEs [25, 48] to CODEs. The analysis can also be used for classification of those systems of nonlinear PDEs that correspond to CODEs. An extension of the Lie linearization results for systems of quadratically and cubically semilinear CODEs in $[46-50]$ in the complex domain may yield useful and highly nontrivial results for systems of PDEs. Furthermore results of linearization for third- and fourth-order CODEs may be obtained by using results for third- and fourth-order RODEs in $[46-50]$.

We have also used CLS analysis to construct linearizability criteria for certain systems of nonlinear ODEs by considering r-CODEs. The linearization of r-CODEs directly gives linearizability criteria for systems of ODEs corresponding to those r-CODEs [3]. The study of linearization of RODEs is significant as it plays an important role in the reduction of equations to simple form, from which one can construct solutions. It also includes finding the transformations that map nonlinear RODEs into linear RODEs. Linearization of systems of nonlinear RODEs is far from trivial as the Lie conditions are very complicated and involve a system of more than ten equations to be solved simultaneously $[46-50]$. The system (3.127) is a special case of a general system of two second-order cubically semilinear ODEs which is linearizable. It remains to answer the following question "Can any general system of two second-order nonlinear ODEs be linearized via CLS analysis restricted to single real variable?"

An extensive classification of those systems of RODEs that correspond to r-CODEs would be of great worth. It is important to point out the fact that the system $(3.129)-(3.132)$ provides a necessary and sufficient condition for system (3.127) to be linearizable. Lie conditions for a general system of second-order quadratically semi linear ODEs were obtained in [47]. Whether system (3.127) constitute a special case of system of second-order cubically semi linear ODEs in [48] and whether conditions $(3.129)-(3.132)$ can be derived from conditions in [48] remains to be explored.

It is suggested that Lie criteria for linearization of two r-CODEs can be found in a similar way, which can then be used to linearize four nonlinear RODEs corresponding to a system of two r-CODEs. It is well known that a system of two second-order RODEs admits $5, 6, 7, 8$ or 15 RLSs and the maximal symmetry algebra is $sl(4, \mathbb{R})$ for all the simplest system [65].

Thus it is conjectured that the system of two second-order r-CODEs admits $10, 12, 14, 16$ or $30$ RLSs corresponding to $5, 6, 7, 8$ or $15$ CLSs. The RLSs $10, 12, 14, 16$ or $30$ are basically the symmetries of four second-order RODEs that correspond to two second-order r-CODEs. Further the simplest systems of two second-order r-CODEs admits a maximal symmetry algebra $sl(4, \mathbb{C})$. The earlier procedures $[46-50]$ already became unwieldy for systems of four RODEs. It seems likely that the system of two r-CODEs may provide a convenient way to obtain results for systems of four RODEs.

It may also be proposed that one can construct criteria of Lie type for third-order CODEs (r-CODEs) which will help in the linearization of systems of two nonlinear PDEs (RODEs) of third-order. Previous results for third- and fourth-order RODE $[48-50]$ might be extendible by this method to systems of CODEs and PDEs.

We have also applied the CLS analysis to the variational problems [2], which yields conservation laws for systems of linear and nonlinear PDEs. Suppose a CODE can be obtained from a variational principle, i.e., it admits a complex Lagrangian the EL equation which is the given CODE. We know a Lagrangian is scalar quantity. It represents the residual energy in the case of a simple harmonic oscillator. An importan question arises here: "What is the physical meaning of a complex Lagrangian?" A complex Lagrangian consists of two parts "What are those real 'Lagrangians'? and what DEs admit those 'Lagrangians'?" We have only been able to answer this question partially in this thesis, but a deeper understanding of these mathematical operators is required. Now symmetries of CODEs are CLSs and of Lagrangians are NSs. NSs play an essential role in finding conservation laws of dynamical equations. We know that complex NSs form a subalgebra of the algebra of CLSs. It would be interesting to classify subalgebra of NSs for systems of PDEs corresponding to CODEs. The results for the double reduction of a system of PDEs corresponding to a CODE have also been explored. This can be further extended to r-CODEs to solve inverse problems, i.e. to find Lagrangians of systems of ODEs and used complex variables in developing their conservation laws. It must also be mentioned we can also obtain the double-double reduction of systems of PDEs and ODEs. It can be done by taking those CODEs (or r-CODEs) that admit two-dimensional complex algebra.

It is important to note that complex variables play a vital role in quantum theory. Therefore CLS analysis can be used to explore the physical significance of complex symmetries, complex

Lagrangians and complex conservation laws. The Hamiltonian specifies the energy levels and time evolution of a quantum theory. A standard axiom of quantum mechanics requires that Hamiltonians be Hermitian because hermiticity guarantees that the energy spectrum be real and that time evolution be unitary (probability-preserving). Recently Bender [10, 11, 12, 13] introduced complex Lagrangians to use non-hermitian Hamiltonians for the generalization of quantum theory. They describe an alternative formulation of quantum mechanics in which the mathematical axiom of Hermiticity (transpose + complex conjugate) is replaced by the physically transparent condition of space-time reflection (PT) symmetry, thereby, requiring only parity-time symmetry in comparison to charge-parity-time (CPT) symmetry as in usual quantum theory. One might think that quantum theory based on non hermitian Hamiltonians is not unitary. However, if the Hamiltonian respects PT symmetry, then it is possible to develop un noticed physical symmetries of Hamiltonians to construct an inner product the norm of which is positive definite. This construction is general and works for PT-symmetric Hamiltonians. The dynamics is described by unitary time evolution same as in standard quantum mechanics. Many possible observable and experimental consequences can be drawn by extending quantum mechanics to the complex domain in atomic physics, particle physics and solid state physics. There are infinitely many new Hamiltonians that might be constructed and fit experimental data. Could non hermitian, PT-symmetric Hamiltonians be used to describe experimentally observable phenomena? Non hermitian Hamiltonians have already been used to describe interacting systems. For example in 1959 Wu showed that the ground state of a Bose system of hard spheres is described by a non Hermitian Hamiltonian [66]. Wu found that the ground-state energy of this system is real and conjectured that all of the energy levels were real. In 1992 Hollowood showed that even though the Hamiltonian of a complex Toda lattice is non-hermitian, the energy levels are real [22]. Non hermitian Hamiltonians of the form, $H = p^2 + \iota x^3$, also arise in various models in Reggeon field theory that exhibit real positive spectra [17]. In each of these cases the fact that a non-hermitian Hamiltonian has a real spectrum appeared mysterious at the time, but now the explanation is simple: In each of these cases it is easy to show that the non hermitian Hamiltonian is PT-symmetric, that is, the Hamiltonian in each case is constructed so that the position operator $x$ or the field operator $\varphi$ is always multiplied by $\iota$. An *experimental signal* of a complex Hamiltonian might be found in the context of condensed matter physics. Consider the complex crystal lattice the potential of which is given by $V(x) = i \sin x$. While the Hamiltonian, $H = p^2 + i \sin x$, is not Hermitian, it is PT-symmetric and all of the energy bands are real. However, at the edge of the bands the wave function of a particle in such a lattice is always bosonic ($2\pi$-periodic) and, unlike the case of ordinary crystal lattices, the wave function is never fermionic ($4\pi$-periodic) [19]. Direct observation of such a band structure would give unambiguous evidence of a PT-symmetric Hamiltonian. Thus it may be seen that complex variables play an essential role in the understanding of physical processes. One could apply CLS analysis to simplify problems in PT-quantum theory. Further several experimental arrangments can be set up with the help of symmetries depending upon initial and boundary conditions. For example a rotational symmetry in DEs describe spherical symmetry. We hope that the use of CLSs lead to the deeper insights into the new quantum theory might be achieved.

Another important use of PT-symmetric Hamiltonians arises in cosmology. The model is based on a classical complex Lagrangian of a complex scalar field which has real solutions of the classical equations of motion. Further the potential is taken to be complex and PT symmetric in the usual quantum mechanical sense, thereby providing a cosmological model that describes an evolution from the big bang to the big rip involving the transitition from normal matter to phantom matter, crossing smoothly the phantom divide line. The interest of their appoach [10] is to describe an interaction between two important fields. Now there are three main ways in which CLS analysis can be applied to this approach: (a) in finding the relationship between the symmetries of a complex Lagrangian and non hermitian Hamiltonians as these are connected with each other by Hamilton's equations; (b) in looking at the insights that can be obtained from the physical significance of non-hermitition Hamiltonians in quantum cosmology; and (c) in developing conservation laws (first integrals) corresponding to those equations of motions.

It is also hoped that Penrose's complex magic can be further extended to "*hypercomplex magic*" by introducing hypercomplex variables (e.g. quaternions, octonions, Clifford or Grassmanian variables) that may result in hypercomplex Lie symmetries. Then these can be used to reduce the order, linearize and solve large classes of systems of PDEs. Further restricted hypercomplex variables are hoped to reveal more classes of systems of PDEs which may be done by projecting down to lower dimensions keeping analyticity of hypercomplex functions.

In this way the process of analytic continuation can be generalized to hypercomplex variables. It is hoped that the non commutative behavior of hypercomplex variables may yield a non trivial generalization of Lie's work. An important observation in this regard comes from the fact that certain algebras are subalgebras of Clifford and Grassmanian algebras. Furthermore use of the hypercomplex Lie symmetries in geometric calculus may result in non trivial physical implications.

Twistor theory [57, 58] was developed by Roger Penrose. He realized that using the space-time continuum picture to describe physical processes is inadequate not only at the Planck scale of $10^{-33}$ cm [57, 58] but also at the much larger scales of elementary particles, or perhaps atoms, where the quantum effects become important. The aim was to construct a mathematical tool that contain variables or operators suitable to write fundamental equations of relativity and quantum field theory in a unified framework. He observed that the proper orthochronous Lorentz group, $so(1, 3)$, is two-to-one isomorphic to the complex group, $sl(2, \mathbb{C})$, i.e., corresponding to each element of $so(1, 3)$ there exists two unique elements in $sl(2, \mathbb{C})$. Each element of $sl(2, \mathbb{C})$ is known as a spinor. This thus laid down the spinor formulation of relativity. He believes that space-time is created from quantum processes themselves at the subatomic level. The mathematical tool in field theories is not suitable for the new formulation since the field equations are based on well-behaved functions varying smoothly in space-time. Thus his mathematical tool is geometry instead of differential equations. However, space-time descriptions of the normal kind have been used at the atomic or particle level for a long time with extraordinary accuracy. Thus this new geometrical picture must, at that level, be mathematically equivalent to the normal space-time picture in the sense that some kind of mathematical transformation must exist between the two pictures. Therefore, he constucted variables and operators in the complex domain which were named twistors and used their holomorphicity to write fundamental equations of motions. The twistor space is defined by four complex dimensions. Since a complex number consists of two independent parts (such as Z = X + iY), it should contain more information than the "conventional space-time" with four real dimensions. A twistor $\mathbb{Z}$ is a point in this twistor space. In order to make use of complex symmetries it is required that we extend them to higher dimensions. It is hoped that we can not only simplify twistor equations by their symmetries but may also be able to construct conservation laws contained in them.

CLS analysis may be used to extend physical theories into the complex domain. The complex domain is huge compared with the real domain and therefore there are many exciting new theories to explore. The obvious potential problem with extending a real theory into complex space is that one may lose some of the characteristics that a valid physical theory must possess. Thus it is necessary that this complex extension be tightly constrained. In Bender's work [10, 11, 12, 13] the essential physical axioms of quantum theory are retained for non hermitian Hamiltonian theory if the complex extension is done in such a way as to preserve PT symmetry. The complex theories that can be constructed are often far more elaborate and diverse than theories that are restricted to the real domain. Upon entering the complex world we have found a gold mine of new physical theories that have strange and fascinating properties. We have just begun to study the vast new panorama that has opened up and we can hardly begin to guess what new kinds of phenomena have yet to be discovered.

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