Multiple Soliton, Lump and its Interaction Solutions For Nonlinear evolution equations

by

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Dedicated

to

My Beloved Parents

and

Honorable Teachers

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Abstract

Within this thesis, a comprehensive study is conducted on the behavior of kink-solitons, lumps, and their interaction solutions when subjected to periodic and kink waves of some nonlinear evolution equations. Among these NLEEs, new extended (3+1)-dimensional B-type Kadomtsev-Petviashvili equation and generalized (2+1)-dimensional Soliton equation are given. Through the utilization of the simplified Hirota's bilinear method, kink-soliton solutions specially one-kink, two-kink and three-kink solutions are obtained. By the aid of direct method based on Hirota's bilinear form lump and lump interaction solutions are obtained which includes lump interaction with stripe solutions, lump-periodic solution, breathers solutions and solitary wave solutions. Extreme value points of lump solutions are obtained to describe maximum and minimum points that give insights about the motion, amplitudes and velocities of these solutions. Physical attributes of obtained results are demonstrated by 3D plots, contour maps, density graphs, and 2D plots. It is also studied that physical dynamics changed by changing the values of parameters involved in test functions.

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Chapter 1

Introduction

Mathematical equations play a vital part in the characterization and understanding of physical phenomena. They provide a framework for quantitatively describing the nature, properties, and relationships of natural phenomena. Mathematical representation of these phenomena allows scientists to quantify and analyze diversity. By formulating these physical phenomena into mathematical equations, scientists and researchers can analyze and model the behavior of complex systems, and unravel the underlying principles and laws that govern them.

Differential equations are significant in comprehending and modeling various phenomena in the field of natural sciences. These equations are important to explain how quantities change and interact with each other over time, making them valuable in domains such as biology, engineering, physics and economics. Among differential equations, there are two frequently employed categories known as ordinary differential equations (ODEs) and partial differential equations (PDEs). These equations provide a mathematical framework to capture the dynamics and behaviors of natural systems.

Nonlinear evolution equations (NLEEs) are PDEs which outline the temporal progression of a physical system based on provided initial information. Evolution equations are derived from a variety of fields in applied and engineering sciences. For example, nonlinear Schrödinger equations [1] from quantum mechanics, Navier-Stokes equations [2, 3] from fluid mechanics, nonlinear reaction-diffusion equations [4, 5] from heat transfers and biological sciences, and Korteweg–de Vries equation [6] from water-waves theory, to name just a few, serving as notable illustrations within this mathematical field. Unlike linear equations, the solution of NLEEs is not always straightforward and often requires advanced mathematical techniques. These equations arise in numerous fields and are essential for studying phenomena with nonlinear characteristics. NLEEs capture intricate relationships and nonlinear interactions among variables, enabling a deeper understanding of the underlying dynamics.

Exact solutions of NLEEs hold significant relevance within the domain of nonlinear sciences. They serve multiple purposes, such as validating numerical results obtained through computational simulations and shedding light on the fundamental laws governing the system. Finding the exact solutions of NLEEs is a challenging task, and their discovery often requires significant mathematical sophistication and sophisticated computational techniques. It is worth noting that exact solutions have the potential to act as a prototypical model for testing the validity and precision of various numerical and analytic approaches, even if they lack a clear physical description. Within the field of nonlinear sciences, the ability to formulate precise solutions for nonlinear PDEs is a fundamental component, and this discipline has advanced significantly in recent decades because of advances in computer technology.

NLEEs are classified into two broad categories, integrable and non-integrable equations. Integrable equations possess special solutions known as solitons. Solitons are distinctive wave-like structures that maintain their shape and propagate without dispersion. These solitons often have fascinating forms, such as the bell-shaped sech (hyperbolic secant) solution or the kink-shaped tanh (hyperbolic tangent) solution. Solitons find applications in various fields, including optics, fluid dynamics, and nonlinear wave phenomena.

The advancement of computer technology and the availability of symbolic computation softwares have greatly contributed to the exploration and discovery of exact solutions to nonlinear PDEs. With these tools, researchers can explore very complex areas of mathematics and discover novel solutions that were previously inaccessible. The combination of mathematical insight and mathematical ability has led to rapid progress in the understanding of nonlinear phenomena and their mathematical representation.

1.1 Preliminaries

A travelling wave is one where the medium moves in the wave's propagation direction. A travelling wave is also related with maintaining a consistent velocity during its propagation. Such waves have been recorded in a variety of fields, including combustion, which can come from a chemical process. Mathematical biology illustrates the transmission of impulses in nerve fibers through the depiction of traveling waves. Also, conservation laws are related to fluid dynamics difficulties. Traveling waves are also used to describe shock characteristics. However, before moving forward, some key terms must be understood.

1.1.1 Travelling Wave Solutions

A permanent form solution travelling at a constant speed is known as a travelling wave solution. Typically, in order to obtain traveling wave solutions, the nonlinear evolution equations are transformed into corresponding ordinary differential equations. Travelling waves arise in the study of NLEEs where waves are represented by the form

$$\phi(x,t) = f(x - ct), \tag{1.1}$$

where $\phi(x,t)$ represents a disturbance moving in the negative or positive x direction if $c < 0$ or $c > 0$ respectively. Here x and t are spatial and temporal variables.

(a)

(b)

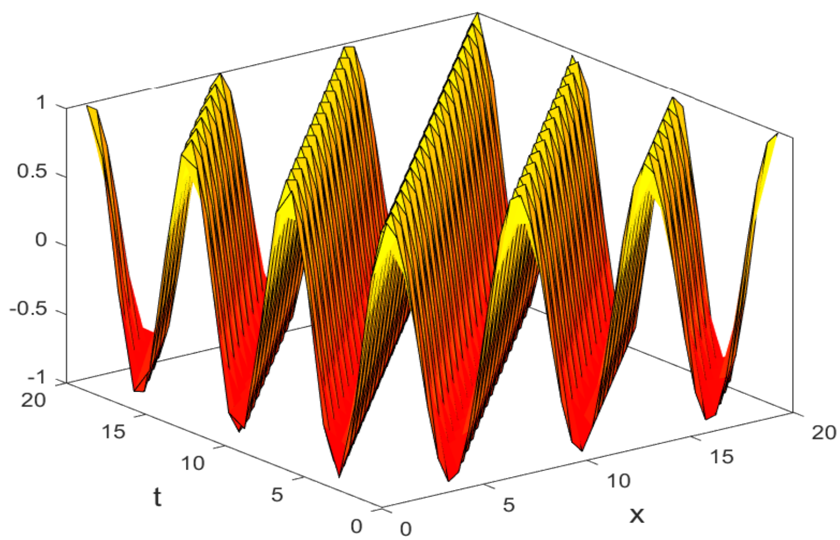
Figure 1.1: Travelling waves in water

1.1.2 Solitary Waves

Solitary waves are traveling waves [8] that exhibit a confined nature, characterized by unvarying speeds and shapes, ultimately tending towards zero as they propagate over infinite distances. As the solitary wave propagates from $\Theta = -\infty$ to $\Theta = \infty$, its transition between the two asymptotic states is confined within a localized region along the Θ variable, where $\Theta = x - wt$, and w is wave speed.

1.1.3 Periodic Solutions

Periodic solutions, such as $\cos(x - wt)$, are periodic travelling wave solutions. Periodic solutions can be found using the conventional wave equation $\phi_{tt} = \phi_{xx}$.

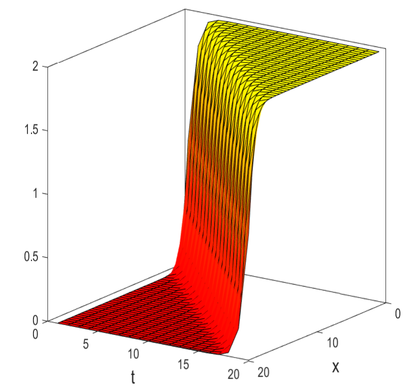


(a)

Figure 1.2: Graphical representation of a periodic solution $\phi(x,t) = \cos(x - wt), -2\pi \leq x, t \leq 2\pi$.

1.1.4 Kink Waves

Kink waves undergo a transition from one asymptotic state to the next, exhibiting either an upward or downward motion. The value of the kink solution becomes constant at infinity. The dissipative Burger equation $\phi_t + \phi_{xx} = v\phi_{xx}$ give the kink wave solutions, where v is the viscosity coefficient.



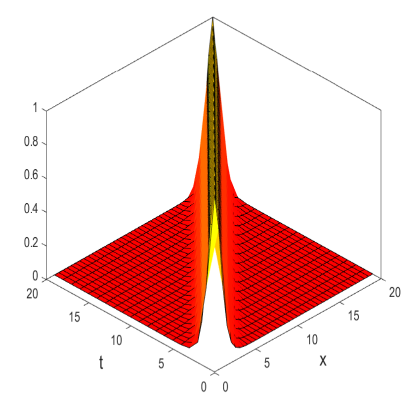
(a)

Figure 1.3: Visualization of a kink solution $\phi(x,t) = 1 - \tanh(x - wt), -20 \leqslant x, t \leq 20$.

1.2 Solitons

A Soliton is travelling wave solution [9] of NLEEs having following properties:

- A wave of permanent form should be visible in the solution.
- The soliton retains its distinctiveness as it interacts with other solitons.
- The solution exhibits localization, indicating that it either decays exponentially towards zero, as observed in the solitons of the KdV equation, or convergence to a constant value at infinity, as seen in the solitons of the Sine-Gordon equation.



(a)

Figure 1.4: Graphical representation of a soliton solution $\phi(x,t) = \text{sech}^2(x - t), -2 \leqslant x, t \leq 2$.

1.2.1 Discovery of Solitons

The initial sighting of solitary waves took place in Edinburgh in 1834 by John Scott Russell, a renowned Scottish engineer. He noticed a massive swelling wave of water moving through the canal with a consistent shape. Quoting Russell directly, "I was observing the motion of a boat which was rapidly drawn along a narrow channel by a pair of horses, when the boat suddenly stopped not so the mass of water in the channel which it had put in motion; it accumulated round the prow of the vessel in a state of violent agitation, then suddenly leaving it behind, rolled forward with great velocity, assuming the form of a large solitary elevation, a rounded, smooth and well-defined heap of water, which continued its course along the channel apparently without change of form or diminution of speed. I followed it on horseback, and overtook it still rolling on at a rate of some eight or nine miles an hour, preserving its original figure some thirty feet long and a foot to a foot and a half in height. Its height gradually diminished, and after a chase of one or two miles I lost it in the windings of the channel. Such in the month of August 1834, was my first chance interview with that singular and beautiful phenomenon which I have called wave of translation" [11]. Scott Rusell's findings contradicted Daniel Bernoulli's and Isaac Newton's hydrodynamic theories. George Gabriel Stokes and George Biddell Airy declined to acknowledge Scott Russell's experimental findings due to discrepancies with existing water wave theories. It was not until the 1870s that Lord Rayleigh and Joseph Boussinesq disclosed the solutions to their discrepancies. However, in 1895, Diederik Korteweg and Gustav de Vries published the Korteweg-de Vries equation, which not only addressed those discrepancies but also provided solutions for solitary waves [12]. Using a finite difference approach, Norman Zabusky of Bell Labs and Martin Kruskal of Princeton University identified soliton in the Kortewegde Vries equation (KdV equation) in 1965.

1.3 Application of Solitons

Solitons are used in many fields of pure and applied mathematics, including differential equations, Lie groups, Lie algebras, differential and algebraic geometry, and so on.

1.3.1 In Plasma Physics

In plasma physics, solitons are of great significance when it comes to elucidating wave characteristics. Solitons, which are essential for describing wave behavior in plasmas, become particularly relevant in the context of nonlinear plasma waves, including ion-acoustic waves [13].

1.3.2 In Water waves

Solitons have a vital part to play in the examination of tsunami waves, particularly in regions characterized by inhomogeneities. Tsunamis are long-period waves generated by large-scale disturbances, such as earthquakes or underwater landslides. As these waves propagate over varying bathymetry, they can undergo transformations and exhibit solitonic characteristics [14].

1.3.3 In Fiber Optics

In fibre optics applications, solitons have been extensively tested. Solitons have been used successfully in the transmission of digital signals over large distances [15].

1.3.4 In Biology

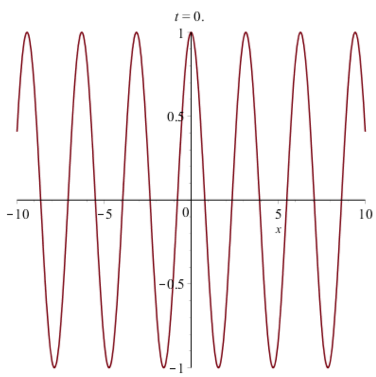
The utilization of soliton theory has been employed to elucidate the phenomenon of low frequency collective motion observed in proteins and DNA, as well as signal and energy propagation in biomembranes, the nervous system, and these phenomena [16–18].

1.3.5 In Nuclear Physics

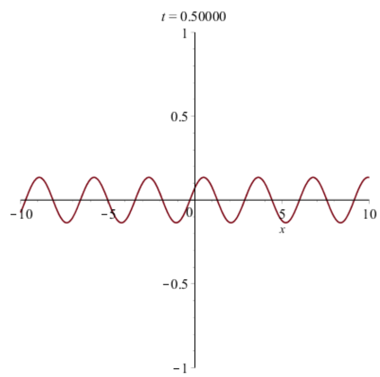
Under certain temperature and energy conditions, the soliton structure is predicted to encompass the entirety of the nuclear wave function. The presence of these conditions is believed to be present within the innermost regions of certain stars, when nuclei show no reactivity and simply move past each other unchanged, keeping their soliton waves as a result of the collision [19].

1.4 Dispersion and Dissipation

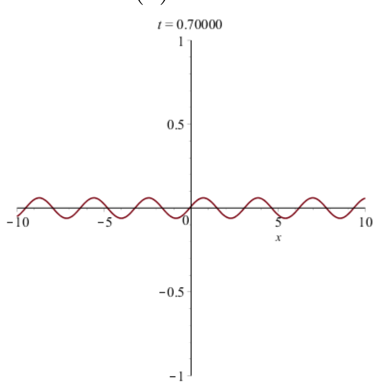
The correlation between a wave's frequency and its phase velocity in optics is known as dispersion. The phase velocities of waves of different frequency will differ. Wave dispersion refers to the change in phase velocity of water waves as the wave number undergoes variation. Dispersive medium are those that have these characteristics. The phenomenon of dispersion can be exemplified by a rainbow, where light is dispersed and separated into various wavelengths, giving rise to different colors. A dissipative wave refers to a wave that experiences a gradual reduction in its amplitude as a result of energy dissipation. The solution of dissipative equation shows that wave propagate with unity speed. In dispersive media, velocity varies with wave number and in dissipative media wave propagate at a unity speed. Some 2D graphs are shown to see dynamic behaviour of dissipation term in PDEs.



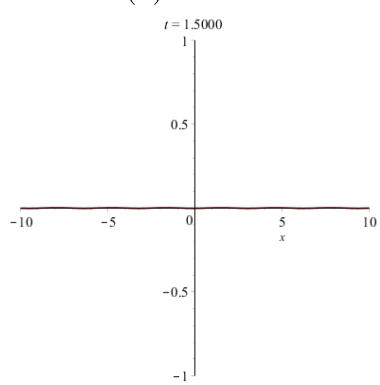
(a) $t = 0$



(b) $t = 0.5$



(c) $t = 0.7$



(d) $t = 1.5$

Figure 1.5: 2D Graphs of solution of dissipative equation

1.5 Literature Review

The study of partial differential equations, particularly nonlinear evolution equations, has proven invaluable for scientists and researchers in understanding various natural phenomena. By analyzing and solving NLEEs, researchers have been able to uncover intricate patterns, uncover hidden relationships, and predict the behavior of physical systems under different conditions. One crucial aspect of studying these equations is the discovery and analysis of exact solutions. Exact solutions provide explicit mathematical expressions that describe the precise evolution of the system being studied. These solutions serve as essential tools for validating numerical methods, testing theoretical conjectures, and gaining insights into the underlying dynamics.

Over the years, numerous methods and a range of strategies have been introduced and refined for the purpose of identifying exact solutions to nonlinear evolution equations. These equations often involve intricate nonlinearities that make their solutions challenging to obtain analytically. However, the quest for exact solutions has motivated researchers to devise innovative approaches, paving the way for significant advancements in the field.

The revelation of solitary waves served as a catalyst for scientists to engage in a significant amount of research in order to investigate this phenomenon. Korteweg and deVries, in 1895, formulated a nonlinear PDE, famously referred to as the KdV equation, for describing the elevation of a shallow water surface with solitary waves [20]. In the realm of studying plasma waves, the KdV equation proved useful in analyzing their behavior within dispersive mediums. A study conducted by Zabusky and Kruskal in 1965 [21] focused on investigating the interaction between solitary waves and the recurrence of initial states. B.B. Kadomtsev and V.I. Petviashvili studied stability of solitary waves in weakly dispersive media in 1970 [22]. In 1971, R. Hirota examined the multiple collisions of soliton solutions of KdV equation [23]. In 1973, R. Hirota achieved a significant breakthrough by obtaining exact N-soliton solutions for a nonlinear wave equation [24]. In 1976, dynamics of KdV equation investigated by Caudrey, Dodd, and Gibbon [25]. M. Ito achieved a breakthrough in 1980 by expanding the scope of research on nonlinear evolution equations. This progress specifically focused on extending the KdV (mKdV) type equations to higher orders [26]. In 1987 J. Hietarinta Investigated KdV-type Bilinear equations satisfying Hirota's three-soliton condition I [27] and also mKdV-type bilinear equations satisfying Hirota's three-soliton condition II [28]. Solitons in NLEEs and inverse scattering were proven by Ablowitz and Clarkson [29] in 1991. Also in 1991 Hereman, W. and W. Zhuang given a MACSYMA Program for the Hirota Method [30]. In 1993, P. Rosenau and J.M. Hyman studied Compactons: Solitons with finite wavelengths [31]. In 1994, Hereman, W. and W. Zhuang provided a symbolic computation of solitons via Hirota's Bilinear Method [32]. In 1994, G. Adomian using Decomposition method solved Frontier problems of Physics [33]. In 1996, Wang and Q. Zhou utilized the homogeneous balancing approach to successfully solve precise nonlinear equations in the field of mathematical physics [34]. W. Hereman and A. Nuseir simplified the Hirota's bilinear method by Symbolic methods to construct exact solutions of nonlinear PDEs in 1997 [35]. In 1999, W. Hong and Y. D. Jung made significant progress by discovering analytic solutions and utilizing auto-Backlund transformation for the variable-coefficient Korteweg-de Vries (KdV) problem [36]. In 2000, E. Fan introduced the extended tanh-function approach [37] and explored its applications to various nonlinear equations. Furthermore, in 2002, Z. Feng employed the first integral approach [38] to investigate the Burgers-KdV problem.

The Hirota's bilinear method and its simplified version considered as the most appropriate methods for finding the N-soliton solutions of NLEEs. Based on this method researchers explored and obtained soliton solutions of a large number of NLEEs. Grammaticos B., Ramani A. and Hietarinta J. in 1994 [54] introduced multilinear operators, which generalize Hirota's bilinear D operator, based on the principle of gauge invariance of the τ functions. In 1996 Johnson, R. S. obtained soliton solution of two dimensional Boussinesq equation for water waves using this method [55]. Hu X.-B. and Tam H.-W. applied bilinear method to integrable differential equations [56] in 2001.

In last two decades, Hirota's bilinear method applied by many researcher for obtaining the soliton solutions of NLEEs and systems which advanced the soliton theory. Wazwaz, A.M., Tian, B., Zhang, S., Liu, W.J., Gao, Y.T., Seadawy, A.R., Zhang, H.Q., Zayed, E.M.E., Lou, S.Y. and Masood, W. played a vital role for the advancement of theory of solitons through Hirota's blinear method. Wazwaz's noteable work include Multiple-soliton solutions for the KP equation [57], Multiple-front solutions for the Burgers equation and

the coupled Burgers equations [58], multiple-soliton solutions of the Sawada-Kotera-Ito seventh-order equation [59], multiple soliton solutions and multiple singular soliton solutions of (3+1)-dimensional evolution equation [60], Multiple-soliton solutions of the perturbed KdV equation [61], Multiple soliton solutions for the Whitham-Broer-Kaup model [62], Multiple soliton solutions of generalized Kaup-Boussinesq equation [63], The collision dynamics between double-hump solitons in two mode optical fibers [64] and Lump, periodic and travelling wave solutions to the (2+1)-dimensional pKP-BKP model [65].
In recent years, the investigation of lump solutions has gained significant interest in the field of solving Nonlinear Evolution Equations (NLEE). Lump solutions, which take the form of rational functions, exhibit polynomial decay in all spatial directions. A lump solution for the Kadomtsev-Petviashvili I (KPI) equation was initially described mathematically in 1977 [39]. Ablowitz and Satsuma introduced the long wave limit method, based on exponential functions, in 1978 [40, 41]. Imai devised the Grammian determinant technique by employing binary Darboux transformations [42]. Ma presented an alternative approach known as the direct method, which utilizes a specific class of quadratic polynomial functions [43, 44]. Dai et al., on the other hand, introduced the heteroclinic breather limit method, employing trigonometric functions as key elements [45, 46], alongside various other contributions. A method that has been gaining attention is the long wave limit approach, mainly because of its practicality in addressing lump solutions and N-solitons [47–49]. Recently, the study of solitons related to lump solutions, such as lump-type solutions and rogue waves, especially their interactions with solitons, has gained considerable prominence within the domain of nonlinear natural science [50–53].

Chapter 2

Methodologies

2.1 The Hirota's Bilinear Method

In 1971, Hirota published a paper introducing the Hirota direct method. This novel approach revolutionized the process of finding N-soliton solutions for nonlinear evolutionary equations by providing a direct method. Hirota's approach stands as a highly effective direct methodology in the realm of mathematical physics and soliton theory, renowned for its ability to generate exact solutions across a broad spectrum of nonlinear partial differential equations. Any equation that can be formulated in bilinear form can be effectively addressed using this technique, once the bilinear form is acquired, the methodology transforms into an algorithmic process, relying solely on calculus and algebra. However, the computations grow significantly longer and more complex, especially when dealing with high-order partial differential equations or those containing highly nonlinear terms. Furthermore, the intricacy of the computations significantly escalates depending on the desired type of soliton solution. While single soliton solutions can be easily calculated, even manually, two- and three-soliton solutions require considerable effort. Four-soliton solutions push the limits of what can be achieved with symbolic programming. Nevertheless, once the structure of the two- and three-soliton solutions is discerned, it provides insights into the form of higher-order soliton solutions.

Essentially, Hirota's method necessitates three fundamental steps:

(i) The implementation of a clever transformation of the dependent variable. (ii) The introduction of a novel differential operator. (iii) Employing a perturbation expansion approach for the effective resolution of the resultant bilinear equation.

2.1.1 The Auxiliary Function

Inspired by the N-soliton solution of the Korteweg-de Vries (KdV) equation, which was established through methods like the Inverse Scattering Transform, as well as the transformation of the Burgers' equation known as the Cole-Hopf transformation which transforms it into the linear heat equation, Hirota introduced a novel function called the auxiliary function in 1971. This introduction was driven by the intention to uncover a similar form of solution and enhance the understanding of nonlinear equations. The function f is often denoted as the "tau function" in many books. This name is attributed to its dual significance within Hirota's method. On one hand, the tau function is instrumental in the construction of soliton solutions. But also, Hirota's method relies heavily on its crucial contribution within the theoretical framework.

2.1.2 The Hirota's Derivative Operator

Hirota introduced the D-operator as a binary form that takes a pair of functions as input and produces a new function as output. This operator possesses several advantageous properties that facilitate the analysis of differential equations. Notably, it enables the discovery of analytic soliton solutions for these equations. Over time, the D-operator has garnered significant attention and has been subject to further extensive research. In the following section, we will outline and elaborate upon its definition and important properties.

Definition

Hirota introduced the differential operator, that operates on a pair of functions (τ, ξ) of a real variable x, by

$$D_x(\tau \cdot \xi) = \left(\frac{\partial}{\partial x} - \frac{\partial}{\partial x'}\right) \tau(x)\xi(x')\Big|_{x'=x}. \tag{2.1}$$

To enable multiple iterations of the operator and while employing it to different variables (such as x and t in this definition, but it can be extended to any number of real variables), Hirota defined

$$D_x^m D_t^n(\tau \cdot \xi) = \left(\frac{\partial}{\partial x} - \frac{\partial}{\partial x'}\right)^m \left(\frac{\partial}{\partial t} - \frac{\partial}{\partial t'}\right)^n \tau(x,t)\xi(x',t')\Big|_{x'=x,t'=t}. \tag{2.2}$$

To provide a better grasp of the D-operator and develop some intuitive insights, let's consider a few quick results.

$$\begin{aligned} D_x(\tau \cdot \xi) &= \tau_x \xi - \tau \xi_x \ , \\ D_x^3(\tau \cdot \xi) &= \tau_{xxx}\xi - 3\tau_{xx}\xi_x + 3\tau_x\xi_{xx} - \tau\xi_{xxx} \ , \\ D_x D_t(\tau \cdot \xi) &= \tau_{xt}\xi - \tau_x\xi_t - \tau_t\xi_x + \tau\xi_{xt} \ . \end{aligned}$$

In fact, we can observe the formula given below for expanding the n^{th} power of the operator, formed by expanding powers of Eq. (2.1) through binomial expansion.

$$D_x^n(\tau \cdot \xi) = \sum_{k=0}^{n} (-1)^k \binom{n}{k} \partial_x^{n-k} \partial_{x'}^{k} \tau(x)\xi(x'). \tag{2.3}$$

A notable comparison can be made with the conventional product rule for differentiation, the primary factor that sets them apart is the alternating sign, denoted by $(-1)^k$, which precedes every term that is generated.

The final defining property of the D-operator, as implied by its name, is its bilinearity. This implies that it exhibits linearity independently for each argument within the input tuple. For instance, considering functions m, n, p, q, and scalar δ, we have the following relationship:

$$\begin{aligned} D_x(m + \delta n) \cdot (p + \delta q) &= D_x(m \cdot p + m \cdot \delta q + \delta n \cdot p + \delta n \cdot \delta q) \\ &= D_x(m \cdot p) + \delta D_x(m \cdot q + n \cdot p) + \delta^2 D_x(n \cdot q). \end{aligned}$$

Properties

Now, let's explore some notable properties of the operator that we can utilize when solving bilinear equations.

Consider a general polynomial P in $D = (D_t, D_x, D_y, \ldots)$. Additionally, we introduce the notation $\partial = (\partial_t, \partial_x, \partial_y, \ldots)$. In this context, we introduce a variety of valuable properties:

$$P(D)\tau \cdot 1 = P(\partial)\tau, \tag{2.4}$$

$$P(D)1 \cdot \tau = P(-\partial)\tau, \tag{2.5}$$

$$D_x^n((\tau \cdot \xi) = (-1)^n D_x^n(\xi \cdot \tau), \tag{2.6}$$

$$D_x^{2n+1}(\tau \cdot \tau) = 0, \ n \in \mathbb{N}. \tag{2.7}$$

The Eqs. (2.4)-(2.6) can be derived directly from the definition when applied to the specified tuples of functions and constants. Additionally, Eq. (2.7) is obtained by directly applying Eq. (2.6) to the tuple $(\tau \cdot \tau)$. If we now introduce the exponential function given by e^{Θ_i}, with $\Theta_i = w_i t + p_i + q_i y + ... + \Theta_i^0$ and where Θ_i^0 is constant, we can present an other result

$$P(D)\exp(\Theta_1) \cdot \exp(\Theta_2) = P(w_1 - w_2, p_1 - p_2, q_1 - q_2, \ldots)\exp(\Theta_1 + \Theta_2). \tag{2.8}$$

In fact, this particular property could potentially serve as the fundamental reason behind the existence of soliton solutions in these types of equations.

2.1.3 Finding soliton solutions for bilinear equation

Now our approach involves utilizing the operator in solving some differential equation written in bilinear form. Assuming that the bilinear equation can be represented as follows

$$P(D)\tau \cdot \tau = 0, \tag{2.9}$$

where P and D defined as in section 2.1.2 and with the additional restriction that

$$P(0) = 0. \tag{2.10}$$

The aforementioned restriction is essential for discovering soliton solutions [4], and its significance becomes evident through the following reasoning.
Next, we will make use of the standard perturbation method. This involves representing the function τ as a formal power series in a small parameter. This expansion yields the following expression

$$\tau = 1 + \epsilon\tau_1 + \epsilon^2\tau_2 + \epsilon^3\tau_3 + \ldots \tag{2.11}$$

By substituting this expansion into Eq. (2.9) and employing the bilinearity of the D-operator, we obtain

$$P(D)(1 \cdot 1 + 1 \cdot \epsilon\tau_1 + \epsilon\tau_1 \cdot 1 + \epsilon\tau_1 \cdot \epsilon\tau_1 + 1 \cdot \epsilon^2\tau_1 + \epsilon^2\tau_1 \cdot 1 + \ldots) = 0, \tag{2.12}$$

A set of equations obtained by collecting the terms that correspond to every exponent of the expansion parameter, given as

$$\epsilon^0 \;: P(D)(1 \cdot 1) = 0, \tag{2.13}$$

$$\epsilon^1 \;: P(D)(1 \cdot \tau_1 + \tau_1 \cdot 1) = 0, \tag{2.14}$$

$$\epsilon^2 \;: P(D)(1 \cdot \tau_2 + \tau_1 \cdot \tau_1 + \tau_2 \cdot 1) = 0, \tag{2.15}$$

$$\epsilon^3 \;: P(D)(1 \cdot \tau_3 + \tau_1 \cdot \tau_2 + \tau_2 \cdot \tau_1 + \tau_3 \cdot 1) = 0. \tag{2.16}$$

Initially, let us observe that Eq. (2.13) is inherently valid due to Eq. (2.10), as $\frac{\partial}{\partial x}(1) = 0$ irrespective of the specific variable we differentiate with respect to. Subsequently, we simplify Eq. (2.14)

$$P(D)(1 \cdot \tau_1 + \tau_1 \cdot 1) = 2P(D)(\tau_1 \cdot 1) = 2P(\partial)\tau_1 = 0 \iff P(\partial)\tau_1 = 0. \tag{2.17}$$

The first step involves utilizing Eqs. (2.6) and (2.7), while the subsequent stage involves applying Eq. (2.4). It becomes evident that every term containing an odd number of D-operators gets eliminated, while each term with an even number is multiplied by two through the application of identities (2.6) and (2.7). Consequently, Eq. (2.17) retains a linear PDE with unchanging coefficients, which, as customary, is solved using an exponential expression. Now let us proceed by assuming the solution $\tau_1 = \exp(\Theta)$ with $\Theta_i = w_i t + p_i + q_i y + ... + \Theta_i^0$ and where Θ_i^0 is constant. Subsequently, we obtain

$$P(w, p, q, \ldots) \exp(\Theta) = 0 \iff P(w, p, q, \ldots) = 0. \tag{2.18}$$

Therefore, when the coefficients of the variables in the exponent, represented by Θ, fulfill the polynomial equation, the presumed solution will align with a valid solution. The equation $P(w, p, q, \ldots) = 0$ represents the dispersion relation of this solution.

It should be emphasized that our main emphasis is on solutions that decay exponentially, as they have a fundamental significance in the emergence of solitons. This is in contrast to plane wave solutions, which are typically examined in classical wave equations such as the one mentioned earlier. For further insights on this matter, additional details can be located in Hirota's work [23].

Next, we examine Eq. (2.15) and apply bilinearity, followed by the identical procedure used in Eq. (2.17) for the "mirrored" terms involving τ_2. This allows us to derive the resulting expression

$$2P(\partial)\tau_2 = -P(D)(\tau_1 \cdot \tau_1). \tag{2.19}$$

Observing the situation, it becomes apparent that our proposed approach for τ_1, combined with Eqs. (2.7) and (2.9), effectively nullifies the right-hand side. Consequently, what remains is

$$P(\partial)\tau_2 = 0. \tag{2.20}$$

It is apparent that we can make the choice of setting $\tau_2 = 0$ and still achieve a valid solution.

For Eq. (2.16) this allows us to swiftly reduce the expression into

$$P(\partial)\tau_3 = 0. \tag{2.21}$$

Choosing $\tau_3 = 0$ remains a valid option once more.

Subsequently, the identical process is replicated for every subsequent exponent of ϵ, leading us to the conclusion that the perturbation series terminates when $\tau_n = 0$ for all $n > 1$. Consequently, a solution is obtained

$$\tau = 1 + \epsilon\tau_1 = 1 + \exp(\Theta). \tag{2.22}$$

The perturbation coefficient has been incorporated into the constant Θ^0 in the exponent during the final step. As a result, we have obtained the one-soliton solution.

To acquire the solution for two solitons, as well as the more general solution for N solitons, we exploit the property that by virtue of being a linear differential equation, Eq. (2.17) abides by the superposition principle. To attain the two-soliton solution, we employ the following ansatz

$$\tau_1 = \exp(\Theta_1) + \exp(\Theta_2) \tag{2.23}$$

Both Θ_1 and Θ_2 individually fulfill the dispersion relation (2.18). Now, let's examine the impact of this assumption on (2.14). Similarly to before, we obtain the expression (2.17). However, unlike the previous case of the one-soliton, the right-hand side of Eq. (2.19) doesn't equate to zero. Instead, a different outcome arises

$$\begin{aligned}
&P(D)(\tau_1 \cdot \tau_1) \\
&= P(D)(\exp(\Theta_1) + \exp(\Theta_2)) \cdot (\exp(\Theta_1) + \exp(\Theta_2)) \\
&= (\exp(\Theta_1) \cdot \exp(\Theta_1) + \exp(\Theta_1) \cdot \exp(\Theta_2) + \exp(\Theta_2) \cdot \exp(\Theta_1) + \exp(\Theta_2) \cdot \exp(\Theta_2)) \\
&= 2P(D)(\exp(\Theta_1) \cdot \exp(\Theta_2)).
\end{aligned}$$

The presence of the second equals sign is solely due to the bilinearity, while the third equals sign arises from the subtraction of equation (2.6) from equation (2.8). This subtraction results in the cancellation or doubling of terms based on whether the derivative is of even or odd order, leading to a coefficient of 2. Additionally, equation (2.8) eliminates pairs of identical exponents in the form of $\exp(\Theta_1) \cdot \exp(\Theta_2)$. We then proceed to employ (2.8) once more, yielding the subsequent outcome.

$$2P(D)\exp(\Theta_1) \cdot \exp(\Theta_2) = 2P(w_1 - w_2, p_1 - p_2, q_1 - q_2, \ldots)\exp(\Theta_1 + \Theta_2). \tag{2.24}$$

Therefore, we are directed to select the solution

$$\tau_2 = a_{12}\exp(\Theta_1 + \Theta_2), \tag{2.25}$$

where the value of a_{12} is determined by

$$a_{12} = -\frac{P(w_1 - w_2, p_1 - p_2, q_1 - q_2, \ldots)}{P(w_1 + w_2, p_1 + p_2, q_1 + q_2, \ldots)}. \tag{2.26}$$

Upon examining Eq. (2.19), it becomes clear that the numerator of a_{12} cancels out the coefficient in (2.24), while the denominator cancels out the outcome of the dispersion relation of the solution τ_2 on the left-hand side, resembling the process by which we obtained (2.18).

Moving forward, we continue with Eq. (2.16), employing a similar approach to how we handled (2.15) by dividing it into two separate expressions. This leads us to acquire

$$P(D)(1 \cdot \tau_1 + \tau_1 \cdot 1) = P(D)(\tau_1 \cdot \tau_2 + \tau_2 \cdot \tau_1). \tag{2.27}$$

After applying the same reduction method used to obtain Eq. (2.17), the left side is diminished. Now, let's examine the right side of the equation. We substitute our initial assumption for τ_1 and substitute the obtained value for τ_2, yielding the following expression

$$
\begin{aligned}
P(D)([\exp(\Theta_2) + \exp(\Theta_2)] \cdot a_{12}\exp(\Theta_1+\Theta_2) + a_{12}\exp(\Theta_1+\Theta_2)\cdot[\exp(\Theta_2)+\exp(\Theta_2)]) \\
&= 2a_{12}P(D)([\exp(\Theta_1)+\exp(\Theta_2)]\cdot\exp(\Theta_1+\Theta_2) \\
&= 2a_{12}(P(D)\exp(\Theta_1)\cdot\exp(\Theta_1+\Theta_2) + P(D)\exp(\Theta_2)\cdot\exp(\Theta_1+\Theta_2)).
\end{aligned}
$$

We finally use the rule (2.7) to get

$$
2P(w_1,p_1,q_1,\ldots)\exp(2\Theta_1+\Theta_2) + 2P(w_2,p_2,q_2,\ldots)\exp(\Theta_1+2\Theta_2). \tag{2.28}
$$

Drawing from our prior experiences, our intention is for the entire expression to amount to zero. As a result, we can set τ_3 to 0 and disregard the expansion of the sum. As a matter of fact, it is already 0. We selected the function τ_1 in such a way that each individual term adheres to the polynomial P, allowing for the superposition of valid solutions of (2.17). Given that any valid solution of (2.17) also meets the requirements of the dispersion relation (2.18), we can infer that the right side of (2.27) yields a value of zero. Consequently, we find ourselves in the same situation as before and require τ_3 to fulfill

$$
P(\partial)\tau_3 = 0, \tag{2.29}
$$

Consequently, we can easily select $\tau_3 = 0$, leading to a clear choice. Utilizing analogous reasoning as mentioned earlier, we have effectively truncated the series and obtained a solution

$$
\tau = 1 + \exp(\Theta_1) + \exp(\Theta_2) + a_{12}\exp(\Theta_1+\Theta_2). \tag{2.30}
$$

The constants of the exponents have assimilated the expansion coefficients, as mentioned previously. Consequently, the outcome is the two-soliton solution.

The argument mentioned earlier can be reiterated, and by making use of the D-operator's properties at specific instances, the overall solution consisting of N-soliton can be derived. The crucial step involves the selection of a specific number of terms in τ_1, which determines the value of N in the solution. This numerical representation corresponds to the quantity of individual solitons that will diminish gradually as time passes. Hirota offers a concise representation for the solution of N solitons, which is restated below for reference. However, due to its intricate nature, the expression holds limited computational significance.

$$\tau = \sum_{\nu=0,1} \exp(\sum_{i<j}^{(N)} \nu_i \nu_j A_{ij} + \sum_{i=1}^{N} \nu_i \Theta_i), \tag{2.31}$$

where first $\sum$ implies that combinations of $\nu_i = 0, 1$ for all $1 \leq i \leq N$ and $\sum_{i<j}^{N}$ represents a summation for all possible pairs (i, j) such that $1 \leq i, j \leq N$ and $i < j$. For example if we set $N = 3$, the resulting solution is the 3-soliton solution, which takes the form of

$$\begin{aligned}\tau = 1 &+ \exp(\Theta_1) + \exp(\Theta_2) + \exp(\Theta_3)\\ &+ \exp(A_{12} + \exp(\Theta_1) + \exp(\Theta_2)) + \exp(A_{13} + \exp(\Theta_1) + \exp(\Theta_3))\\ &+ \exp(A_{23} + \exp(\Theta_2) + \exp(\Theta_3)) + \exp(A_{12} + A_{13} + A_{23} + \Theta_1 + \Theta_2 + \Theta_3).\end{aligned}$$

Also note that the relationship between the coefficients can be described as $a_{ij} = \exp(A_{ij})$. We have demonstrated that, regardless of the equation's structure, characterized by $P(D)\tau \cdot \tau = 0$ and subject to the vacuum restriction mentioned earlier, it will possess N-soliton solutions precisely as described earlier. The discrepancies among these solutions, governed by the foundational equation P, the given result will depend on the coefficients and the dispersion relations they fulfill. You might be questioning the significance of this if you haven't encountered an equation in the aforementioned form before. Thus, the question arises: What methods can be employed to reveal or find this elusive bilinear form?

2.1.4 How to find Bilinear form

Mathematicians in various fields frequently encounter differential equations that are rarely provided in the required format for discovering N-soliton solutions. Determining which equations can be transformed into this form often requires a certain level of artistry rather than a straightforward approach. Nonetheless, this section provides insights into identifying certain characteristics. The objective is to rephrase an equation using D-operators. The initial step involves a change of variables.

Dependent variable transformations

In the following discussion, we will address two commonly used variable transformations employed to find bilinear forms. It is important to note that there are various variations of these transformations, and it is necessary to exercise some caution in selecting the appropriate one. In order to ascertain the validity of these transformations, we will require certain results

$$\exp(\delta\partial_x)p(x) = p(x+\delta), \tag{2.32}$$

$$\exp(\delta D_x)p(x)\cdot q(x) = p(x+\delta)q(x-\delta x), \tag{2.33}$$

$$2\cosh(\delta\partial_x)p(x) = p(x+\delta)+p(x-\delta), \tag{2.34}$$

$$\cosh(\delta D_x)p(x)\cdot p(x) = p(x+\delta)p(x-\delta). \tag{2.35}$$

All of these are driven by examining the Taylor series expansion centered at the given point x of $p(x+\delta)$ given by

$$p(x+\delta) = \sum_{n=0}^{\infty}\frac{(\partial_x^n)(x)}{n!}((x+\delta)-x)^n = \sum_{n=0}^{\infty}\frac{(\partial_x)^n}{n!}p(x).$$

The Eq. (2.32) follows directly from the observation that changing the sign of each term with an odd number of derivatives in a Taylor expansion is reversed when the exponent changes sign. Similarly, the second function in the input of the D-operator also exhibits the same pattern, resulting in a translation in the opposite (negative) direction, as shown for q in Eq. (2.33). From these observations, the remaining results can be derived straightforwardly. Now, let's examine the two transformations.

(i) The rational transform

Starting with a given solution u of a nonlinear partial equation, we can perform a rational transformation to convert it into a corresponding rational function

$$\phi = \frac{p}{q}, \tag{2.36}$$

where u, p, and q be functions of the same real variables. In the case where the nonlinear terms in the equation can be formulated as a polynomial function involving u and its derivatives, the transformation of variables will yield a homogeneous expression in terms of p and q. This homogeneous expression can be further separated into a set of quadratic forms. The motivation behind this transformation is supported by the following identity.

$$\exp(\delta\partial_x)\frac{p}{q} = \frac{\exp(\delta D_x)p\cdot q}{\cosh(\delta D_x)q\cdot q}, \tag{2.37}$$

The proof can be obtained by directly applying the results stated in Eqs. (2.32), (2.33), and (2.35). By expanding both sides of equation (2.37) with respect to the parameter δ and organizing the terms based on each order of δ^n, this might require engaging in algebraic manipulations to manipulate the expansions, we can derive the desired outcome

$$\frac{\partial}{\partial x}\frac{p}{q} = \frac{D_x p \cdot q}{q^2}, \tag{2.38}$$

$$\frac{\partial^2}{\partial x^2}\frac{p}{q} = \frac{D_x^2 p \cdot q}{q^2} - \frac{p}{q}\frac{D_x^2 q \cdot q}{q^2}, \tag{2.39}$$

$$\frac{\partial^3}{\partial x^3}\frac{p}{q} = \frac{D_x p \cdot q}{q^3} - 3\frac{D_x p \cdot q}{q^2}\frac{D_x^2 q \cdot q}{q^2}. \tag{2.40}$$

and so on. After applying these relations to any equation, it is common for the equation to be decoupled, resulting in the introduction of an arbitrary decoupling function λ into a system of bilinear equations [4, 3]. Let's proceed to the next transformation.

(ii) **The logarithmic transformation**

The logarithmic transformation, which is even more prevalent than the previous transformation, is expressed as follows

$$\phi = 2\alpha(\log \tau)_{xx}, \tag{2.41}$$

given that ϕ and f are functions of real variables, and α is a free parameter, there exists another motivating identity for this transformation as

$$2\alpha \cosh\left(\delta\frac{\partial}{\partial x}\right)\log\tau(x) = \alpha\log[\cosh(\delta D_x)\tau(x)\cdot\tau(x)]. \tag{2.42}$$

Proved by using (2.34) and (2.34), as well as standard results about the logarithm as follows

$$\begin{aligned} LHS &= \log\tau(x+\delta) + \log\tau(x-\delta) \\ &= \log[\tau(x+\delta)\tau(z-\delta)] = RHS. \end{aligned} \tag{2.43}$$

Expanding both sides with respect to δ and organizing the terms according to their powers of δ yields the following relationships.

$$2\alpha\frac{\partial^2}{\partial x^2}\log\tau = \alpha\frac{D_x^2\tau\cdot\tau}{\tau^2}, \tag{2.44}$$

$$2\alpha\frac{\partial^2}{\partial x\partial t}\log\tau = \alpha\frac{D_xD_t\tau\cdot\tau}{\tau^2}, \tag{2.45}$$

$$2\alpha\frac{\partial^4}{\partial x^4}\log\tau = \left[\frac{D_x^2\tau\cdot\tau}{\tau^2} - 3\left(\frac{D_x^2\tau\cdot\tau}{\tau^2}\right)^2\right]. \tag{2.46}$$

and so on.

2.1.5 Description of Simplified Hirota Method

In their work, Hereman et al. presented a simplified variation of Hirota's method, which allows for the direct derivation of exact solitons without the utilization of bilinear forms. This method involves solving a perturbation scheme utilizing symbolic manipulation software. The subsequent discussion provides a summary of the primary steps involved in this simplified version of Hirota's method.

To provide an example of the methodology, we take the commonly observed KdV equation

$$\phi_t + 6\phi\phi_x + \phi_{xxx} = 0, \tag{2.47}$$

where $\phi = \phi(x,t)$ and subscripts denote derivatives.

Step 1: A change of dependent variable

$$\phi(x,t) = \frac{2\partial^2\tau(x,t)}{\partial x^2} = 2\frac{(\tau\tau_{xx} - \tau_x^2)}{\tau^2}, \tag{2.48}$$

enables us to express (2.47) as a quadratic equation in terms of τ and its derivatives

$$\tau[\tau_{xt} + \tau_{3x}] - \tau_x\tau_t - 4\tau_x\tau_{3x} + 3\tau_{2x}^2 = 0. \tag{2.49}$$

To adapt Hirota's technique to equations where the bilinear form is not readily available, we intentionally leave out Hirota's bilinear operators [9] and instead represent Eq. (2.49) in the following manner

$$\tau L(\tau) + N(\tau,\tau) = 0, \tag{2.50}$$

where L denote the linear differential operator

$$L(\tau) = \frac{\partial^2 \tau}{\partial x \partial t} + \frac{\partial^4 \tau}{\partial x^4}, \tag{2.51}$$

and the nonlinear differential operator N is defined as

$$N(\tau, \xi) = -\tau_x \xi_t - 4\tau_x \xi_{3x} + 3\tau_{2x}\xi_{2x}. \tag{2.52}$$

Step 2: We seek solution of type

$$\tau(x,t) = 1 + \sum_{n=1}^{\infty} \epsilon^n \tau_n(x,t), \tag{2.53}$$

within this framework, ϵ plays the role of a parameter employed for maintaining records. Progressing as outlined in Hirota's methodology, we substitute (2.53) into (2.50), and equate to zero the different powers of ϵ:

$$\epsilon^1 \ : L\tau_1 = 0, \tag{2.54}$$

$$\epsilon^2 \ : L\tau_2 = -N(\tau_1, \tau_1) = 0, \tag{2.55}$$

$$\epsilon^3 \ : L\tau_3 = -\tau_1 L\tau_2 - N(\tau_1, \tau_2) - N(\tau_2, \tau_1) = 0, \tag{2.56}$$

$$\vdots$$

$$\epsilon^n \ : L\tau_n = -\sum_{j=1}^{n-1} [\tau_j L\tau_{n-j} + N(\tau_j, \tau_{n-j})] = 0. \tag{2.57}$$

Step 3: The N-soliton solution is obtained from

$$\tau_1 = \sum_{i=1}^{N} \exp(\Theta_i), \tag{2.58}$$

where

$$\Theta_i = k_i x - c_i t, \tag{2.59}$$

is called phase shift and c_i, k_i are called the wave speed and wave number respectively. Substituting

$$\phi(x,t) = \exp(k_i x - c_i t), \tag{2.60}$$

into the linear terms of the KdV Eq. (2.47) gives the dispersion relation

$$c_i = k_i^3. \tag{2.61}$$

We therefore find

$$\Theta_i = k_i x - k_i^3 t. \tag{2.62}$$

By using $N = 1$ in Eq. (2.58) we obtained

$$\tau_1 = \exp(\Theta_1) = \exp(k_1 x - k_1^3 t). \tag{2.63}$$

One-soliton solution

For finding one-soliton solution, we set $n = 1$ in Eq. (2.53)

$$\tau = 1 + \tau_1 = 1 + \exp(k_1 x - k_1^3 t), \tag{2.64}$$

where we used $\epsilon = 1$ and now substituting above equation into (2.48) gives one-soliton solution for KdV equation

$$\phi(x,t) = \frac{2k_1^2 \exp(k_1(x - k_1^2 t))}{1 + \exp(k_1(x - k_1^2 t))^2}. \tag{2.65}$$

Two-soliton solution

In order to find the two-soliton solutions, we begin by substituting $N = 2$ into Eq. (2.58). This gives the following result

$$\tau_1 = \exp(\Theta_1) + \exp(\Theta_1), \tag{2.66}$$

In order to obtain τ_2, the right-hand side of (2.55) has to be calculated

$$-N(\tau_1, \tau_1) = -2(k_1 - k_2)^2 (m_2 k_1^2 + 2k_1 m_1 k_2 + 2k_1 m_2 k_2 + m_1 k_2^2) \exp(\Theta_1 + \Theta_2). \tag{2.67}$$

Obviously, τ_2 must be of the form

$$\tau_2 = a_{12} \exp(\Theta_1 + \Theta_2), \tag{2.68}$$

where a_{12} is a constant to be determined. Computation of the left-hand side of Eq. (2.55) yields

$$L\tau_2 = -2a_{12}(k_1 + k_2)^2 (m_2 k_1^2 + 2k_1 m_1 k_2 + 2k_1 m_2 k_2 + m_1 k_2^2) \exp(\Theta_1 + \Theta_2). \tag{2.69}$$

Equating Eq. (2.67) and Eq. (2.69) gives

$$a_{12} = \frac{(k_1 - k_2)^2}{(k_1 + k_2)^2}. \tag{2.70}$$

Therefore Eq. (2.68) becomes

$$\tau_2 = \frac{(k_1 - k_2)^2}{(k_1 + k_2)^2} \exp(\Theta_1 + \Theta_2), \tag{2.71}$$

Now for finding two-soliton solution we set $n = 2$ in Eq. (2.53) gives

$$\tau(x,t) = 1 + \epsilon\tau_1 + \epsilon^2\tau_2 = 1 + \exp(\Theta_1) + \exp(\Theta_1) + \frac{(k_1 - k_2)^2}{(k_1 + k_2)^2} \exp(\Theta_1 + \Theta_2), \tag{2.72}$$

for simplicity we choose $\epsilon = 1$ which is arbitrary constant. Now by using Eq. (2.72) into Eq. (2.48) we get two-soliton solution for KdV equation.

Three-soliton solution

We further go on to search for three-soliton solution. To this end, substituting $N = 3$ in Eq. (2.58) yields

$$\tau_1 = \exp(\Theta_1) + \exp(\Theta_2) + \exp(\Theta_3), \tag{2.73}$$

where $\Theta_i = k_i x - k_i^3 t$ for $i = 1, 2, 3$. Inserting (2.73) into Eq. (2.55), calculating and equating left and right sides we get

$$\tau_2 = a_{12} \exp(\Theta_1 + \Theta_2) + a_{13} \exp(\Theta_1 + \Theta_3) + a_{23} \exp(\Theta_2 + \Theta_3), \tag{2.74}$$

with

$$a_{12} = \frac{(k_1 - k_2)^2}{(k_1 + k_2)^2}, a_{13} = \frac{(k_1 - k_3)^2}{(k_1 + k_3)^2}, a_{23} = \frac{(k_2 - k_3)^2}{(k_2 + k_3)^2}. \tag{2.75}$$

Proceeding in a similar way with (2.56) we get

$$\tau_3 = b_{123} \exp(\Theta_1 + \Theta_2 + \Theta_3), \tag{2.76}$$

with

$$b_{123} = a_{12}a_{13}a_{23}. \tag{2.77}$$

Thus for finding three-soliton solution we set $N = 3$ in Eq. (2.53) gives

$$\tau(x,t) = 1 + \epsilon\tau_1 + \epsilon^2\tau_2 + \epsilon^3\tau_3. \tag{2.78}$$

Substituting Eq. (2.78) along with Eqs. (2.73), (2.74) and (2.76) into Eq. (2.48) gives three-soliton solution for KdV equation.

2.2 Description of Direct Method using Hirota Bilinear form for Lump and its Interaction Solutions

In order to demonstrate the essential stages of a direct approach for acquiring lump solitons, we examine a generic formulation of a nonlinear evolution equation in higher dimensions as

$$F(\phi, \phi_t, \phi_x, \phi_y, \phi_z, \phi_{xx}, \phi_{yy}, \phi_{zz}, \ldots) = 0, \tag{2.79}$$

where $\phi = \phi(x, y, z, t)$, F is a polynomial about u and its derivatives.

The fundamental concept of the direct method can be articulated as follows (its application demonstrated on certain equations [43, 44]).

Step 1: We undergo a transformation, considering it as

$$\phi = T(\tau), \tag{2.80}$$

where τ is a new unknown (auxiliary) function. Then, the equation adopts the subsequent Hirota bilinear formulation

$$G(D_t, D_x, D_y, D_z, \ldots)(\tau, \tau) = 0, \tag{2.81}$$

where the D-operators are Hirota's bilinear operators [24].

Step 2: To find the lump solution of Eq. (2.79), we make the assumption that the equation possesses a lump solution expressed in the following form

$$\tau = g^2 + h^2 + m, \tag{2.82}$$

$$g = a_1x + a_2y + a_3z + a_4t + \ldots + a_n, \quad h = b_1x + b_2y + b_3z + b_4t + \ldots + b_n,$$

where $a_i, b_i (1 \leq i \leq n)$ and m represent real parameters to be found at a later stage. By putting Eq. (2.82) into Eq. (2.81) and equating the coefficients of all variables for each polynomial term, then equating these coefficients to zero. This process yields a system of algebraic equations involving parameters.

Step 3: By utilizing symbolic computation [43], we can solve the system of algebraic equations presented in step 2 and determine the values of $a_i, b_i (1 \leq i \leq n)$ and m. By substituting these identified values of parameters into Eq. (2.82) and then into Eq. (2.80), we can acquire numerous precise lump solutions for Eq. (2.79).

2.2.1 Test Functions for Lump Interaction Solutions

Let $n \in N$, and we introduce n wave variables as

$$\begin{aligned} g &= a_1x + a_2y + a_3z + a_4t + \ldots + a_n, \\ h &= b_1x + b_2y + b_3z + b_4t + \ldots + b_n, \\ l &= c_1x + c_2y + c_3z + c_4t + \ldots + c_n, \end{aligned} \tag{2.83}$$

where $a_i, b_i, c_i (1 \leq i \leq n)$ and m are constant parameters.

Lump Interaction with Multi-stripe Solutions

To examine lump interaction with mulit-stripe solutions we employ the combination of two quadratic functions and N exponential function as

$$\tau = g^2 + h^2 + m + \sum_{i=1}^{N} e^{l_i}, \tag{2.84}$$

Lump Interaction with Periodic Waves

The test function for examining the interaction phenomenon among lump and periodic waves is given as

$$\tau = g^2 + h^2 + m + \cos(l), \tag{2.85}$$

Breathers wave solution

To obtain breathers waves solution we consider the test function of the following form

$$\tau = e^{-k_1 l} + c_1 e^{k_1 l} + c_2 \cos(k_2 g), \tag{2.86}$$

Solitary wave solution

To obtain breathers waves solution we consider the test function of the following form

$$\tau = q_1 e^{l} + q_2 \tan(g) + q_3 \tanh(h), \tag{2.87}$$

where k_1, k_2, q_1, q_2 and q_3 are also constant parameters.

Chapter 3

Lump and Soliton solutions of an Extended (3+1)-dimensional B-type Kadomtsev-Petviashvili equation

In 1970, kadomstsev-petviashivili (KP) equation [22] was proposed as a two dimensional extension of Kortweg-de Vries equation. As like other NLEEs, KP equation also possesses soliton solutions which have great importance in many domains of nonlinear phenomenon such as optics, plasma physics and water waves. For instance, KP equation is used to study ion acoustic waves in magnetized dusty plasma and in the study of water waves, at the bottom of ocean, tsunami waves travelling in the in-homogeneous zone are described by KP equation. In the last couple of decades, KP equation caught eyes of many researchers which resulted into many extensions of KP equation.

In this chapter, by using Hirota's Bilinear method, multi-soliton, lump and lump interaction solutions with soliton and periodic waves for an extended (3+1)-dimensional B-type Kadomtsev-Petviashvili (eBKP) equation are obtained which is one of the extension of KP equation. The (3+1)-dimensional eBKP is given as follows

$$\phi_{tz} + \phi_{ty} - \phi_{xxxy} - 3(\phi_x \phi_y)_x + 3\phi_{xz} = 0, \tag{3.1}$$

where $\phi = \phi(x, y, t, z)$.

This chapter is designed as: In Sect. 3.1, we obtain bilinear form of Eq. (3.1). Sect. 3.2 contains 1, 2 and 3-soliton solutions of Eq. (3.1) while in Sect. 3.3, lump solutions are presented. In Section 3.4, lump interaction solutions with periodic and soliton waves are presented.

3.1 Bilinear Form of eBKP

By dependent variable transformation

$$\phi = 2(\ln \tau)_x, \tag{3.2}$$

where ϕ and τ are functions of x, y, z, and t. By Eq. (3.2), Eq. (3.1) transforms into Hirota Bilinear form [23] as follows:

$$(3D_xD_z + D_yD_t + D_zD_t - D_x^3D_y)(\tau \cdot \tau) = 0, \tag{3.3}$$

where D_z, D_t, D_x and D_y are derivative operators given as follows:

$$D_x^pD_t^q(\tau \cdot \xi) = (\partial_x - \partial_{x'})^p(\partial_t - \partial_{t'})^q\tau(x,t)\xi(x^{'}, t^{'})|_{x'=x,t'=t}, \tag{3.4}$$

where $p, q \geq 0$ and $p + q \geq 1$. So,

$$D_x^3D_y = -2(\tau_{xxxy}\tau + \tau_{xxx}\tau_y + 3\tau_{xxy}\tau_x - 3\tau_{xx}\tau_{xy})(\tau \cdot \tau) = 0, \tag{3.5}$$

$$D_xD_z = 2(\tau_{xz}\tau - \tau_x\tau_z), \tag{3.6}$$

$$D_yD_t = 2(\tau_{yt}\tau - \tau_y\tau_t), \tag{3.7}$$

$$D_zD_t = 2(\tau_{zt}\tau - \tau_z\tau_t). \tag{3.8}$$

By using Eqs. (3.5),(3.6),(3.7) and (3.8), Eq.(3.3) can be written as

$$3\tau_{xz}\tau - 3\tau_x\tau_z + \tau_{yt}\tau - \tau_y\tau_t + \tau_{zt}\tau - \tau_z\tau_t - \tau_{xxxy}\tau - \tau_{xxx}\tau_y - 3\tau_{xxy}\tau_x + 3\tau_{xx}\tau_{xy} = 0. \tag{3.9}$$

The above equation will be used for finding Lump solutions later.

3.2 Kink-Solitons Solutions

3.2.1 1-kink soliton Solution

To find the dispersion relation, substitute

$$\phi = e^{\Theta_i} = e^{k_ix + r_iy + s_iz - c_it}, \tag{3.10}$$

into the linear part of Eq. (3.1), which gives

$$c_i = -\frac{k_i(k_i^2 r_i - 3s_i)}{r_i + s_i}. \tag{3.11}$$

Hence, the dispersion variable takes the form

$$\Theta_i = k_i x + r_i y + s_i z + \frac{k_i(k_i^2 r_i - 3s_i)}{r_i + s_i} t. \tag{3.12}$$

For 1-kink soliton solution

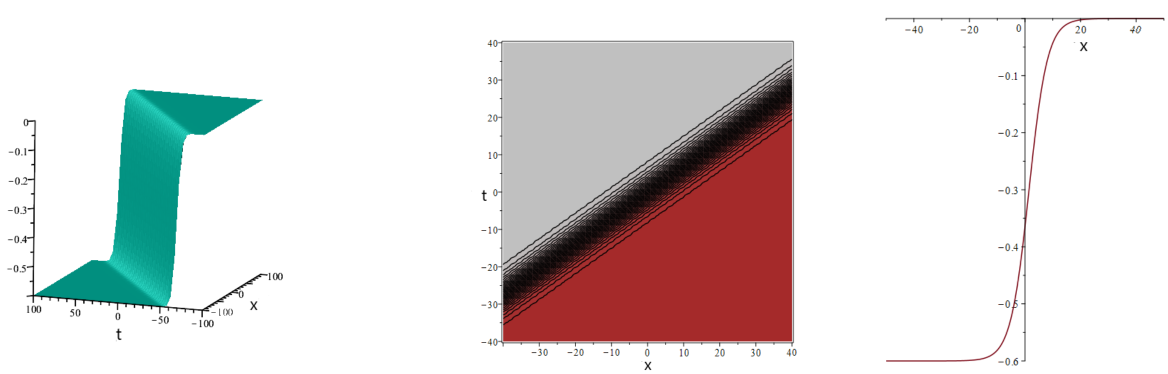
$$\tau_1 = e^{\Theta_1}, \tag{3.13}$$

where $\Theta_1 = k_1 x + r_1 y + s_1 z + \frac{k_1(k_1^2 r_1 - 3s_1)}{r_1 + s_1} t$, therefore auxiliary function τ takes the form as follows:

$$\tau = 1 + e^{\Theta_1} = 1 + e^{k_1 x + r_1 y + s_1 z + \frac{k_1(k_1^2 r_1 - 3s_1)}{r_1 + s_1} t}. \tag{3.14}$$

Hence, using $\phi = 2(\ln \tau)_x$, 1-kink soliton is given as

$$\phi(x, y, z, t) = \frac{2k_1 e^{k_1 x + r_1 y + s_1 z + \frac{k_1(k_1^2 r_1 - 3s_1)}{r_1 + s_1} t}}{1 + e^{k_1 x + r_1 y + s_1 z + \frac{k_1(k_1^2 r_1 - 3s_1)}{r_1 + s_1} t}}. \tag{3.15}$$



(a) (b) (c)

Figure 3.1: (a) 1-kink soliton solution at $y = 0, z = 0$ with $k_1 = -0.3$, $r_1 = 0.5$ and $s_1 = 0.5$ (b) contour plot (c) 2D representation.

3.2.2 2-kink soliton Solution

For 2-kink soliton, let

$$\tau = 1 + \tau_1 + \tau_2, \tag{3.16}$$

where $\tau_1 = e^{\Theta_1} + e^{\Theta_2}$ and $\tau_2 = d_{12}e^{\Theta_1+\Theta_2}$ also $\Theta_1 = k_1x + r_1y + s_1z + \frac{k_1(k_1^2r_1-3s_1)}{r_1+s_1}t$ and $\Theta_2 = k_2x + r_2y + s_2z + \frac{k_2(k_2^2r_2-3s_2)}{r_2+s_2}t$, here d_{12} is called phase shift. So, Eq. (3.16) takes the form

$$\tau = 1 + e^{\Theta_1} + e^{\Theta_2} + d_{12}e^{\Theta_1+\Theta_2}, \tag{3.17}$$

Phase shift d_{12} can be calculated by substituting directly Eq. (3.17) under dependent variable transformation $\phi = 2(\ln\tau)_x$ into Eq. (3.1) which gives d_{12} for general case

$$d_{ij} = \frac{L}{M}, \qquad 1 \leq i \leq j \leq N, \tag{3.18}$$

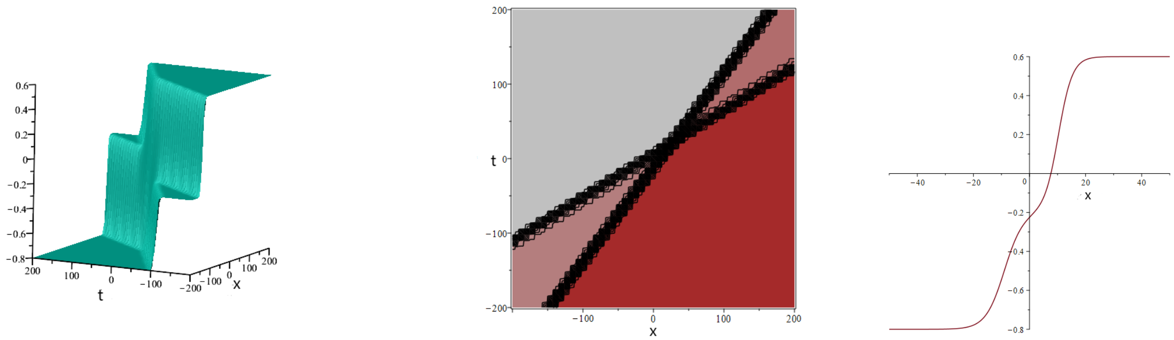
where

$$\begin{aligned}
L = {}& k_i^3r_ir_js_j + k_i^3r_is_j^2 - k_i^3r_j^2s_i - k_i^3r_js_is_j - 3k_i^2k_jr_i^2r_j - 3k_i^2k_jr_i^2s_j + 3k_i^2k_jr_ir_j^2 \\
&- 3k_i^2k_jr_ir_js_i + 3k_i^2k_jr_ir_js_j - 3k_i^2k_jr_is_is_j + 3k_i^2k_jr_j^2s_i + 3k_i^2k_jr_js_is_j + 3k_ik_j^2r_i^2r_j \\
&+ 3k_ik_j^2r_i^2s_j - 3k_ik_j^2r_ir_j^2 + 3k_ik_j^2r_ir_js_i - 3k_ik_j^2r_ir_js_j + 3k_ik_j^2r_is_is_j - 3k_ik_j^2r_j^2s_i \\
&- 3k_ik_j^2r_js_is_j - k_j^3r_i^2s_j + k_j^3r_ir_js_i - k_j^3r_is_is_j + k_j^3r_js_i^2 + 3k_ir_ir_js_j + 3k_ir_is_j^2 \\
&- 3k_1r_j^2s_i - 3k_ir_js_is_j - 3k_jr_i^2s_j + 3k_jr_ir_js_i - 3k_jr_is_is_j + 3k_jr_js_i^2,
\end{aligned} \tag{3.19}$$

and

$$\begin{aligned}
M = {}& k_i^3r_ir_js_j + k_i^3r_is_j^2 - k_i^3r_j^2s_i - k_i^3r_js_is_j - 3k_i^2k_jr_i^2r_j - 3k_i^2k_jr_i^2s_j - 3k_i^2k_jr_ir_j^2 \\
&- 3k_i^2k_jr_ir_js_i - 3k_i^2k_jr_ir_js_j - 3k_i^2k_jr_is_is_j - 3k_i^2k_jr_j^2s_i - 3k_i^2k_jr_js_is_j - 3k_ik_j^2r_i^2r_j \\
&- 3k_ik_j^2r_i^2s_j - 3k_ik_j^2r_ir_j^2 - 3k_ik_j^2r_ir_js_i - 3k_ik_j^2r_ir_js_j - 3k_ik_j^2r_is_is_j - 3k_ik_j^2r_j^2s_i \\
&- 3k_ik_j^2r_js_is_j - k_j^3r_i^2s_j + k_j^3r_ir_js_i - k_j^3r_is_is_j + k_j^3r_js_i^2 + 3k_ir_ir_js_j + 3k_ir_is_j^2 \\
&- 3k_ir_j^2s_i - 3k_ir_js_is_j - 3k_jr_i^2s_j + 3k_jr_ir_js_i - 3k_jr_is_is_j + 3k_jr_js_i^2,
\end{aligned} \tag{3.20}$$

Hence 2-kink soliton solution can be determined by using Eq. (3.17) with dependent variable transformation $\phi = 2(\ln\tau)_x$.



(a) (b) (c)

Figure 3.2: (a) 2-kink soliton solution at $y = 5, z = 1$ with $k_1 = -0.4$, $r_1 = s_2 = 0.7$, $s_1 = k_2 = 0.5$ and $r_2 = 0.5$ (b) contour plot (c) 2D representation.

3.2.3 3-kink soliton Solution

To retrieve 3-kink soliton, let

$$\tau = 1 + \tau_1 + \tau_2 + \tau_3, \tag{3.21}$$

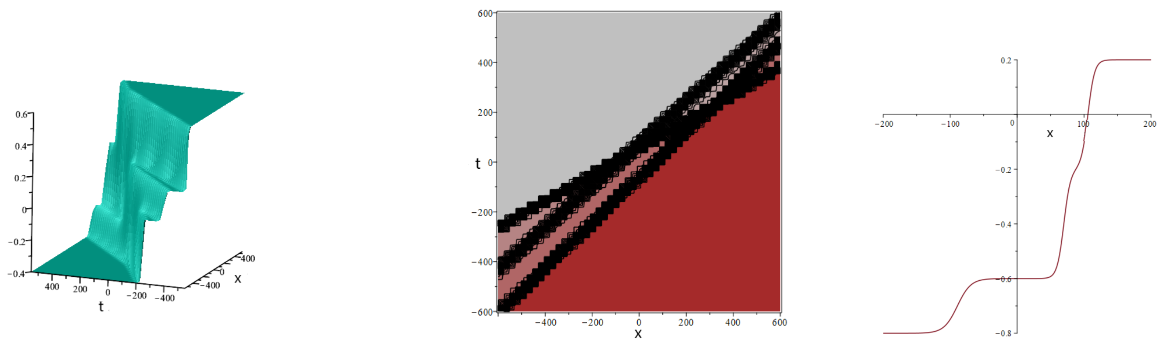
where $\tau_1 = e^{\Theta_1} + e^{\Theta_2} + e^{\Theta_3}$, $\tau_2 = d_{12}e^{\Theta_1+\Theta_2} + d_{13}e^{\Theta_1+\Theta_3} + d_{23}e^{\Theta_2+\Theta_3}$ and $\tau_3 = b_{123}e^{\Theta_1+\Theta_2+\Theta_3}$. With dispersion variables $\Theta_1 = k_1x + r_1y + s_1z + \frac{k_1(k_1^2r_1-3s_1)}{r_1+s_1}t$, $\Theta_2 = k_2x + r_2y + s_2z + \frac{k_2(k_2^2r_2-3s_2)}{r_2+s_2}t$ and $\Theta_3 = k_3x + r_3y + s_3z + \frac{k_3(k_3^2r_3-3s_3)}{r_3+s_3}t$,
where b_{123} in τ_3 is given as

$$b_{123} = d_{12} \cdot d_{13} \cdot d_{23}. \tag{3.22}$$

Therefore Eq. (3.21) takes the form

$$\begin{aligned}\tau &= 1 + e^{\Theta_1} + e^{\Theta_2} + e^{\Theta_3} + d_{12}e^{\Theta_1+\Theta_2} + d_{13}e^{\Theta_1+\Theta_3} \\ &+ d_{23}e^{\Theta_2+\Theta_3} + b_{123}e^{\Theta_1+\Theta_2+\Theta_3}.\end{aligned} \tag{3.23}$$

To find 3-kink soliton solution, substitute Eq. (3.23) into the formula $\phi = 2(\ln \tau)_x$. In the same manners, we can find kink-soliton solutions for $N \geq 4$.



(a) (b) (c)

Figure 3.3: (a) 3-kink soliton solution at $y = 3, z = 41$ with $k_1 = -0.2,\ k_2 = 0.1,\ r_1 = s_3 = 0.5,\ s_1 = r_2 = r_3 = 0.5$ and $s_2 = k_3 = 0.5$ (b) contour plot (c) 2D representation.

3.3 Lump Solutions

3.3.1 Lump solution consisting of two quadratic functions

Suppose τ to be a positive quadratic function of the subsequent form [44]

$$\tau = \tau_1^2 + \tau_2^2 + m_{11}, \tag{3.24}$$

with

$$\tau_1 = m_1x + m_2y + m_3z + m_4t + m_5,\ \ \tau_2 = m_6x + m_7y + m_8z + m_9t + m_{10}, \tag{3.25}$$

where m_i, $1 \leq i \leq 11$ are constants which are to be found.

By the aid of computing mathematical tool Maple we will obtain the values of constants m_i using (3.24) in bilinear equation (3.9), which are as follows

$$m_2 = -\frac{1}{m_4^2 + m_9^2}(3m_1m_3m_4 + 3m_1m_8m_9 + m_3m_4^2 + 3m_3m_6m_9 + m_3m_9^2 - 3m_4m_6m_8), \tag{3.26}$$

$$m_7 = \frac{1}{m_4^2 + m_9^2}(3m_1m_3m_9 - 3m_1m_4m_8 - 3m_3m_4m_6 - m_4^2m_8 - 3m_6m_8m_9 - m_8m_9^2), \tag{3.27}$$

$$\begin{aligned} m_{11} = &-\frac{1}{(m_3m_9 - m_4m_8)(m_1m_9 - m_4m_6)}(3m_1^4m_3m_4 + 3m_1^4m_8m_9 + m_1^3m_3m_4^2 + m_1^3m_3m_9^2 \\ &+ 6m_1^2m_3m_4m_6^2 + m_1^2m_4^2m_6m_8 + 6m_1^2m_6^2m_8m_9 + m_1^2m_6m_8m_9^2 + m_1m_3m_4^2m_6^2 + m_1m_3m_6^2m_9^2 \\ &+ 3m_3m_4m_6^4 + m_4^2m_6^3m_8 + 3m_6^4m_8m_9 + m_6^3m_8m_9^2), \end{aligned} \tag{3.28}$$

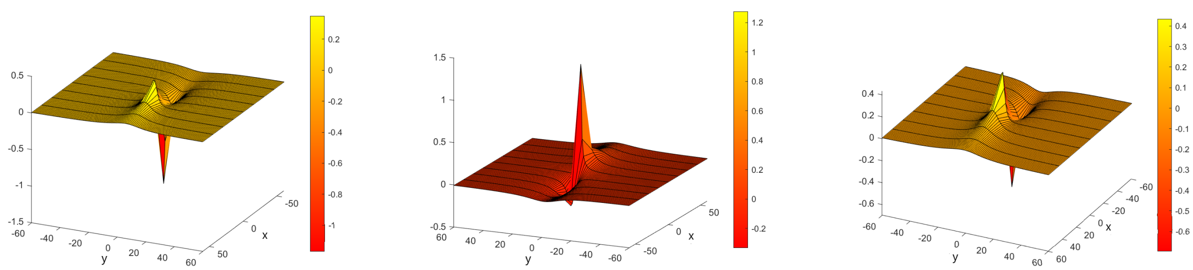
The other constants are all free parameters satisfying the conditions, $m_3m_9 - m_4m_8 \neq 0$, $m_1m_9 - m_4m_6 \neq 0$ and $m_4^2 + m_9^2 \neq 0$, $m_3^2 + m_8^2 \neq 0$, and $m_1^2 + m_6^2 \neq 0$.
By substituting constants m_i's into (3.24), we get the required solution τ to the bilinear equation (3.9). Using $\phi = 2(\ln \tau)_x$, the lump solution for Eq. (3.1) is given as follows:

$$\phi = \frac{4m_1\tau_1 + 2m_6\tau_2}{\tau}, \tag{3.29}$$

where $\tau = \tau_1^2 + \tau_2^2 + m_{11}$,

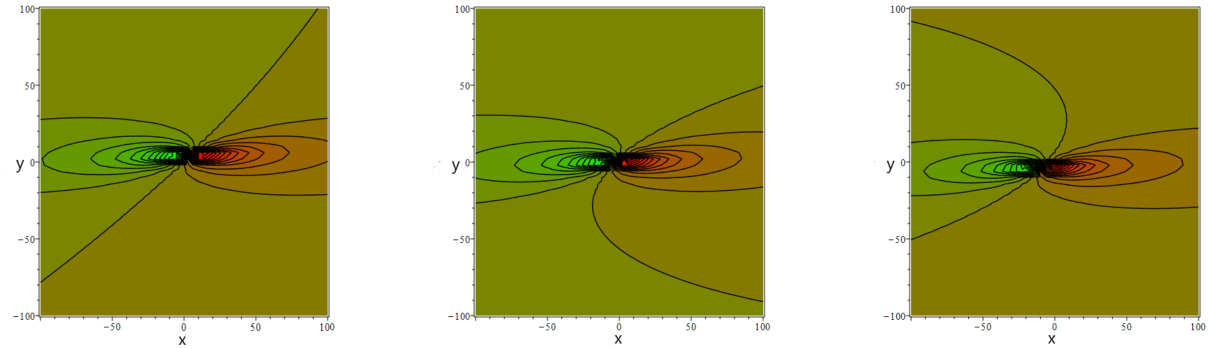
$$\tau_1 = m_1x - \frac{3m_1m_3m_4 + 3m_1m_8m_9 + m_3m_4^2 + 3m_3m_6m_9 + m_3m_9^2 - 3m_4m_6m_8}{m_4^2 + m_9^2} y + m_3z + m_4t + m_5, \tag{3.30}$$

$$\tau_2 = m_6x + \frac{3m_1m_3m_9 - 3m_1m_4m_8 - 3m_3m_4m_6 - m_4^2m_8 - 3m_6m_8m_9 - m_8m_9^2}{m_4^2 + m_9^2} y + m_8z + m_9t + m_{10}\,. \tag{3.31}$$



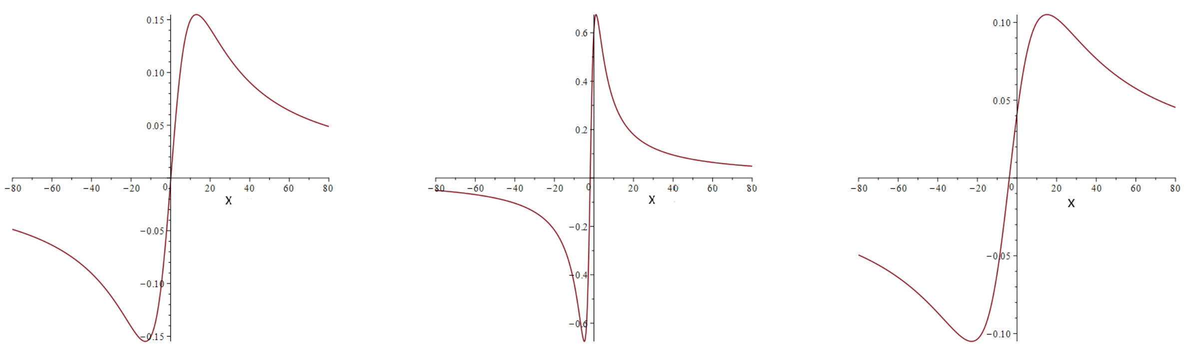
(a) (b) (c)

Figure 3.4: 3D graphs of Lump solutions Eq. (3.29) with $z = 1$ and $m_1 = 0.5, m_3 = m_8 = 1.5, m_4 = -1.5, m_5 = m_6 = m_9 = m_{10} = 1$ (a) $t = -10$ (b) $t = 0$ (c) $t = 10$.



(a) (b) (c)

Figure 3.5: Contour graphs of Lump solutions Eq. (3.29) with $z = 1$ and $m_1 = 0.5, m_3 = m_8 = 1.5$,
$m_4 = -1.5, m_5 = m_6 = m_9 = m_{10} = 1$ (a) $t = -10$ (b) $t = 0$ (c) $t = 10$.



(a) (b) (c)

Figure 3.6: 2D graphs of Lump solutions Eq. (3.29) with $z = 1, y = 0$, $m_1 = 0.5, m_3 = m_8 = 1.5, m_4 = -1.5, m_5 = m_6 = m_9 = m_{10} = 1$ (a) $t = -10$ (b) $t = 0$ (c) $t = 10$.

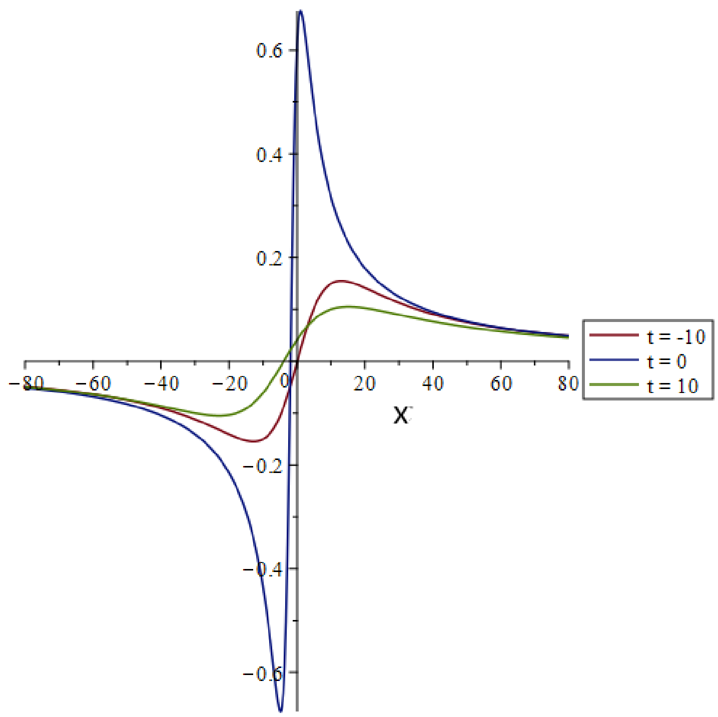


Figure 3.7: Combined 2D representation of lump solutions at $t = -10, t = 0$ and $t = -10$.

With arbitrary selection of parameters lump solution (3.29) is shown in figure 3.4. Applying multi-variable calculus, minimum point (extreme value point) of lump is obtained at $\left(-\frac{82}{11}, -\frac{169}{44}\right)$ with the amplitude $-\frac{1}{93750000}$. It travels along the $y = -\frac{169}{44}$ line with a velocity of $-\frac{82}{11}$. In simpler terms, this type of motion is referred to as uniform linear motion in the field of physics.

3.3.2 Lump solution consisting of three quadratic functions

To find lump solution consisting of three quadratic functions we suppose τ of the following form [68]

$$\tau = \tau_1^2 + \tau_2^2 + \tau_3^2 + m_{16}, \tag{3.32}$$

with

$$\begin{aligned} \tau_1 &= m_1 x + m_2 y + m_3 z + m_4 t + m_5, \\ \tau_2 &= m_6 x + m_7 y + m_8 z + m_9 t + m_{10}, \\ \tau_3 &= m_{11} x + m_{12} y + m_{13} z + m_{14} t + m_{15}. \end{aligned} \tag{3.33}$$

where m_j, $1 \leq j \leq 16$ are constants which are to be found.

By substituting Eq. (3.32) into the Eq. (3.9) and collecting coefficients of the all powers of variables to 0, we get system of equations and solving it with the help of computing tool Maple we obtained following relations of constants m_j's as

$$\begin{aligned}
m_2 &= \frac{m_8(3m_6^2 + m_9m_6 + 3m_{11}^2)}{m_9m_6}, \\
m_3 &= -\frac{m_8(3m_1^2 + 3m_6^2 + m_9m_6 + 3m_{11}^2)}{m_9m_6}, \\
m_4 &= -\frac{3m_1^2 + 3m_6^2 + m_9m_6 + 3m_{11}^2}{m_1}, \\
m_7 &= -\frac{m_8(m_9 + 3m_6)}{m_9}, \\
m_{12} &= -\frac{3m_8m_{11}}{m_9}, \\
m_{13} &= m_{14} = 0.
\end{aligned} \tag{3.34}$$

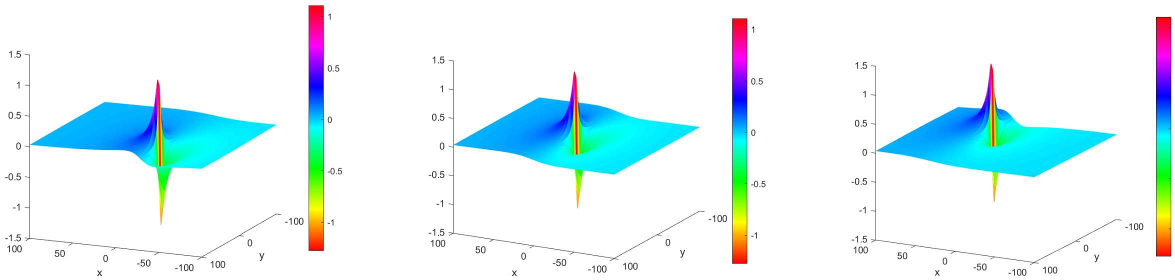
where $m_1, m_5, m_6, m_8, m_9, m_{10}, m_{11}, m_{15}$ and m_{16} are all free parameters. The auxiliary function τ is analytical provided that $m_9m_6 \neq 0, m_9 \neq 0$, $m_1 \neq 0$ and $m_{16} > 0$.

By substituting Eq. (3.32) along with Eq. (3.33) and Eq. (3.34) into the Eq. (3.2) we will get lump solution consisting of three quadratic functions to Eq. (3.1) as follows

$$\phi(x, y, z, t) = \frac{4(m_1\tau_1 + m_6\tau_2 + m_{11}\tau_3)}{\tau_1^2 + \tau_2^2 + \tau_3^2 + m_{16}}, \tag{3.35}$$

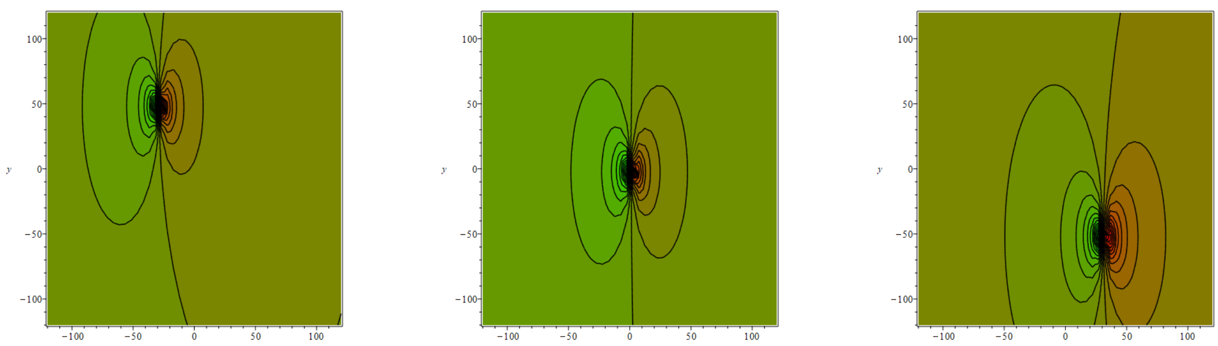
with

$$\begin{aligned}
\tau_1 = &\ m_1x + \frac{m_8(3m_6^2 + m_9m_6 + 3m_{11}^2)}{m_9m_6}y - \frac{m_8(3m_1^2 + 3m_6^2 + m_9m_6 + 3m_{11}^2)}{m_9m_6}z \\
&- \frac{3m_1^2 + 3m_6^2 + m_9m_6 + 3m_{11}^2}{m_1}t + m_5, \\
\tau_2 = &\ m_6x - \frac{m_8(m_9 + 3m_6)}{m_9}y + m_8z + m_9t + m_{10}, \\
\tau_3 = &\ m_{11}x - \frac{3m_8m_{11}}{m_9}y + m_{15}.
\end{aligned} \tag{3.36}$$



(a) (b) (c)

Figure 3.8: 3D graphs of Lump solutions Eq. (3.35) with $z = 1$ and $m_1 = 1.5, m_5 = -4, m_6 = 0.5, m_9 = -0.5, m_8 = 0.1, m_{10} = 2, m_{11} = -0.8, m_{13} = m_{14} = 0, m_{15} = m_{16} = 1$ (a) $t = -10$ (b) $t = 0$ (c) $t = 10$.



(a) (b) (c)

Figure 3.9: Contour graphs of Lump solutions Eq. (3.35) with $z = 1$ and $m_1 = 1.5, m_5 = -4, m_6 = 0.5, m_9 = -0.5, m_8 = 0.1, m_{10} = 2, m_{11} = -0.8, m_{13} = m_{14} = 0, m_{15} = m_{16} = 1$ (a) $t = -10$ (b) $t = 0$ (c) $t = 10$.

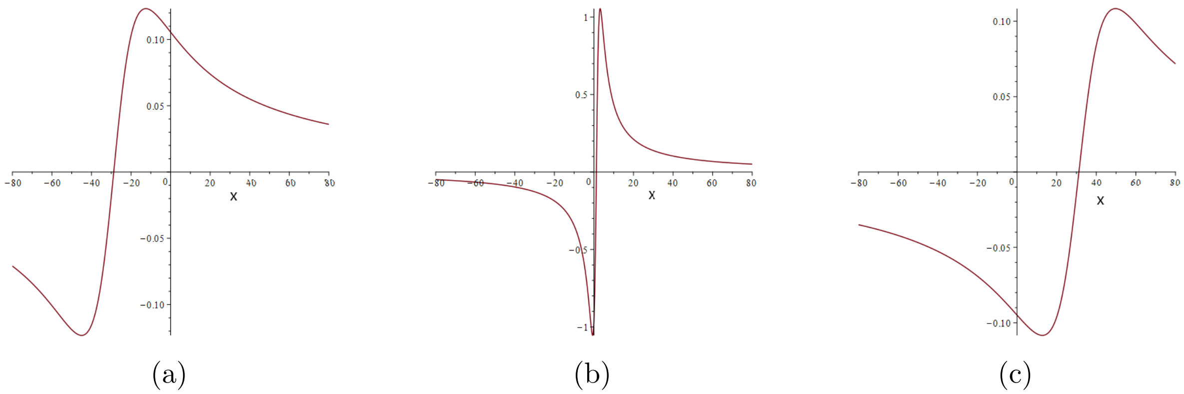


Figure 3.10: 2D graphs of Lump solutions Eq. (3.35) with $z = 1, y = 1$ and $m_1 = 1.5, m_5 = -4, m_6 = 0.5, m_9 = -0.5, m_8 = 0.1, m_{10} = 2, m_{11} = -0.8, m_{13} = m_{14} = 0, m_{15} = m_{16} = 1$ (a) $t = -10$ (b) $t = 0$ (c) $t = 10$.

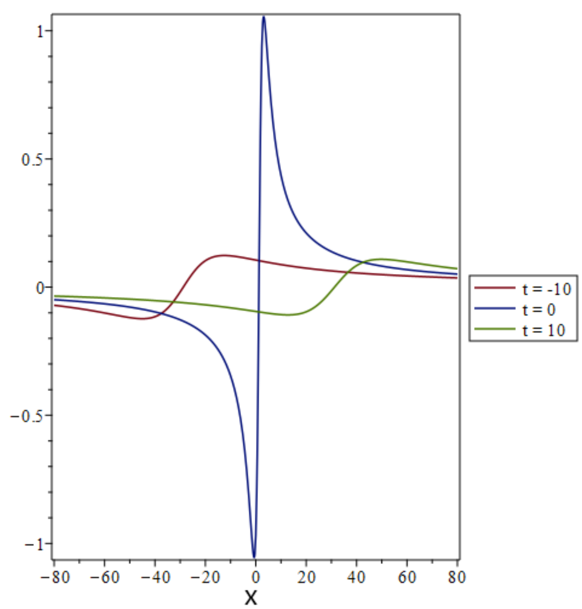


Figure 3.11: Combined 2D representation of lump solutions at $t = -10, t = 0$ and $t = -10$.

In figure 3.8, lump solution (3.35) is shown with the arbitrary selection of parameters. Using multi-variable calculus, minimum point (extreme value point) of lump is obtained at $\left(-\frac{317123}{10461}, -\frac{302734}{5687}\right)$ with the amplitude $-\frac{8543}{132843}$. It travels along the y $=-\frac{302734}{5687}$ line with a velocity of $-\frac{317123}{10461}$. In simpler terms, this type of motion is referred to as uniform linear motion in the field of physics.

As compared to the lump obtained in previous section 5.1, this lump solution also has uniform linear motion, so both lump solutions have similar motion but extreme points and amplitudes are different in both cases.

3.4 Lump interaction solutions with Periodic and kink waves

3.4.1 Lump interaction with periodic waves

To examine interaction among a lump and periodic waves we consider auxiliary function τ of the following form

$$\tau = \tau_1^2 + \tau_2^2 + \cos(l) + m_{16}, \tag{3.37}$$

with

$$\begin{aligned} \tau_1 &= m_1 x + m_2 y + m_3 z + m_4 t + m_5, \\ \tau_2 &= m_6 x + m_7 y + m_8 z + m_9 t + m_{10}, \\ l &= m_{11} x + m_{12} y + m_{13} z + m_{14} t + m_{15}. \end{aligned} \tag{3.38}$$

where m_j, $1 \leq j \leq 16$ are constants which are to be found.

By putting Eq. (3.37) into Eq. (3.9) and equating coefficients of the all powers of variables to 0, we get system of equation which is solved with help of Maple and obtained following relations of constants as

$$\begin{aligned} m_2 &= \frac{m_9 m_8 (m_4 + 3m_1)}{m_4^2}, \\ m_3 &= -\frac{m_9 m_8}{m_4}, \\ m_6 &= \frac{m_9 m_1}{m_4}, \\ m_7 &= -\frac{m_8 (m_4 + 3m_1)}{m_4}, \\ m_{12} &= -\frac{m_{13} (m_4 + 3m_1)}{m_4}, \\ m_{11} &= m_{14} = 0. \end{aligned} \tag{3.39}$$

where $m_1, m_4, m_5, m_8, m_9, m_{10}, m_{13}, m_{15}$ and m_{16} are all free parameters. The auxiliary function τ is analytical provided that $m_4 \neq 0$ and $m_{16} > 0$.

By substituting Eq. (3.37) along with Eq. (3.38) and Eq. (3.38) into the Eq. (3.2) we will get interaction solution between a lump and periodic waves to Eq. (3.1) as follows

$$\phi(x, y, z, t) = \frac{4m_1(m_4\tau_1 + m_9\tau_2)}{m_4(\tau_1^2 + \tau_2^2 + \cos(l) + m_{16})}. \tag{3.40}$$

where

$$\begin{aligned} \tau_1 &= m_1x + \frac{m_9m_8(m_4 + 3m_1)}{m_4^2}y - \frac{m_9m_8}{m_4}z + m_4t + m_5, \\ \tau_2 &= \frac{m_9m_1}{m_4}x - \frac{m_8(m_4 + 3m_1)}{m_4} + m_8z + m_9t + m_{10}, \\ l &= -\frac{m_{13}(m_4 + 3m_1)}{m_4}y + m_{13}z + m_{15}. \end{aligned} \tag{3.41}$$

In Figure 3.12, profile of interaction between a lump and periodic waves presented with $z = 0.5, t = 10$ and $m_1 = 1.5, m_4 = m_5 = -4, m_6 = 0.5, m_8 = 0.1, m_9 = -0.5, m_{13} = -1, m_{11} = m_{14} = 0, m_{10} = m_{15} = m_{16} = 1$.

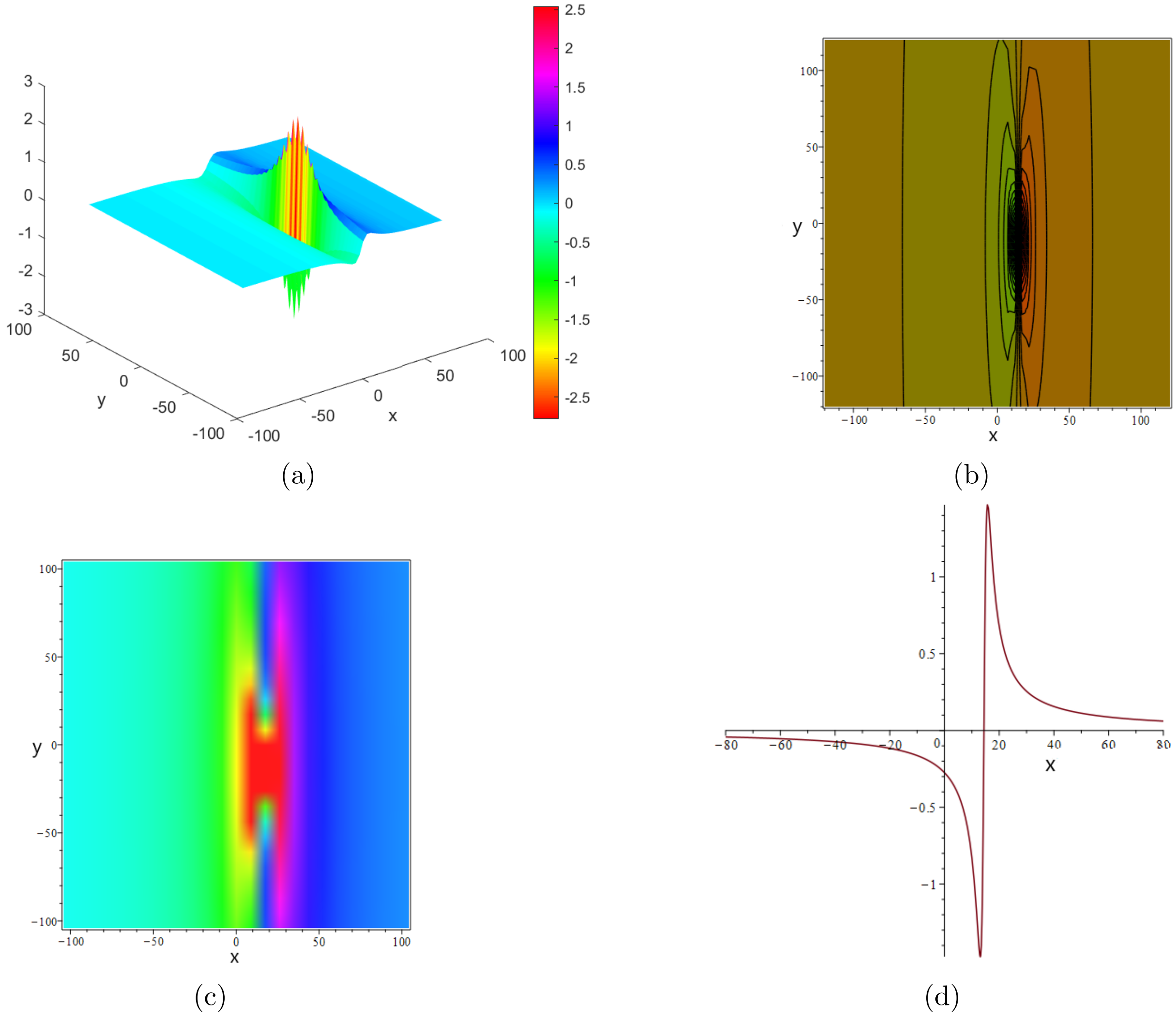


Figure 3.12: Visuals of Interaction among a lump and periodic waves in Eq. (47) with $z = 0.5, t = 10$ (a) Three-dimensional representation (b) contour plot (c) density plot, and (d) 2D profile with $y = 1$.

3.4.2 Lump interaction with kink waves

To find interaction solution between a lump and kink waves we suppose auxiliary function τ of the following form

$$\tau = \tau_1^2 + \tau_2^2 + m_{15}e^{l} + m_{16}, \tag{3.42}$$

with

$$\begin{aligned}
&\tau_1 = m_1 x + m_2 y + m_3 z + m_4 t + m_5, \\
&\tau_2 = m_6 x + m_7 y + m_8 z + m_9 t + m_{10}, \\
&l = m_{11} x + m_{12} y + m_{13} z + m_{14} t.
\end{aligned} \tag{3.43}$$

where m_j, $1 \leq j \leq 16$ are constants which are to be found.

By substituting Eq. (3.42) into the Eq. (3.9) and collecting coefficients of the all powers of variables to 0, we get system of equations and solving it with the help of computing tool Maple we obtained following relations of constants m_j's as

$$\begin{aligned}
&m_2 = \frac{m_1 m_8 m_6}{m_1^2 + m_6^2}, \\
&m_7 = -\frac{m_8 m_1^2}{m_1^2 + m_6^2}, \\
&m_9 = -\frac{3(m_1^2 + m_6^2)}{m_6}, \\
&m_{13} = \frac{m_8 m_{11}(m_{11}^2 + 3) m_6}{3(m_1^2 + m_6^2)}, \\
&m_{14} = -3 m_{11}, \\
&m_3 = m_4 = m_{12} = 0.
\end{aligned} \tag{3.44}$$

where $m_1, m_5, m_6, m_8, m_{10}, m_{11}, m_{15}$ and m_{16} are free parameters. The auxiliary function τ is analytical provided that $m_1 \neq 0$,$m_6 \neq 0$ and $m_{16} > 0$.
By substituting Eq. (3.42) along with Eq. (3.43) and Eq. (3.44) into Eq. (3.2) we get lump interaction solution with kink waves for Eq. (3.1) as follows

$$\phi(x, y, z, t) = \frac{2(2 m_1 \tau_1 + 2 m_6 \tau_2 + m_{11} m_{15} e^l)}{\tau_1^2 + \tau_2^2 + m_{15} e^l + 1}. \tag{3.45}$$

where

$$\begin{aligned}
&\tau_1 = m_1 x + \frac{m_1 m_8 m_6}{m_1^2 + m_6^2} y + m_5, \\
&\tau_2 = m_6 x - \frac{m_8 m_1^2}{m_1^2 + m_6^2} y + m_8 z - \frac{3(m_1^2 + m_6^2)}{m_6} t + m_{10}, \\
&l = (x - 3t) m_{11} + \frac{m_8 m_{11}(m_{11}^2 + 3) m_6}{3(m_1^2 + m_6^2)} z.
\end{aligned} \tag{3.46}$$

In Figure 3.13, profile of interaction between a lump and kink waves presented with $z = 1, t = 0$ and $m_1 = 0.5, m_3 = m_4 = m_{12} = 0, m_5 = m_6 = 1, m_8 = -0.5, m_{10} = 5, m_{14} = -3m_{11}, m_{11} = m_{15} = m_{16} = 1$.

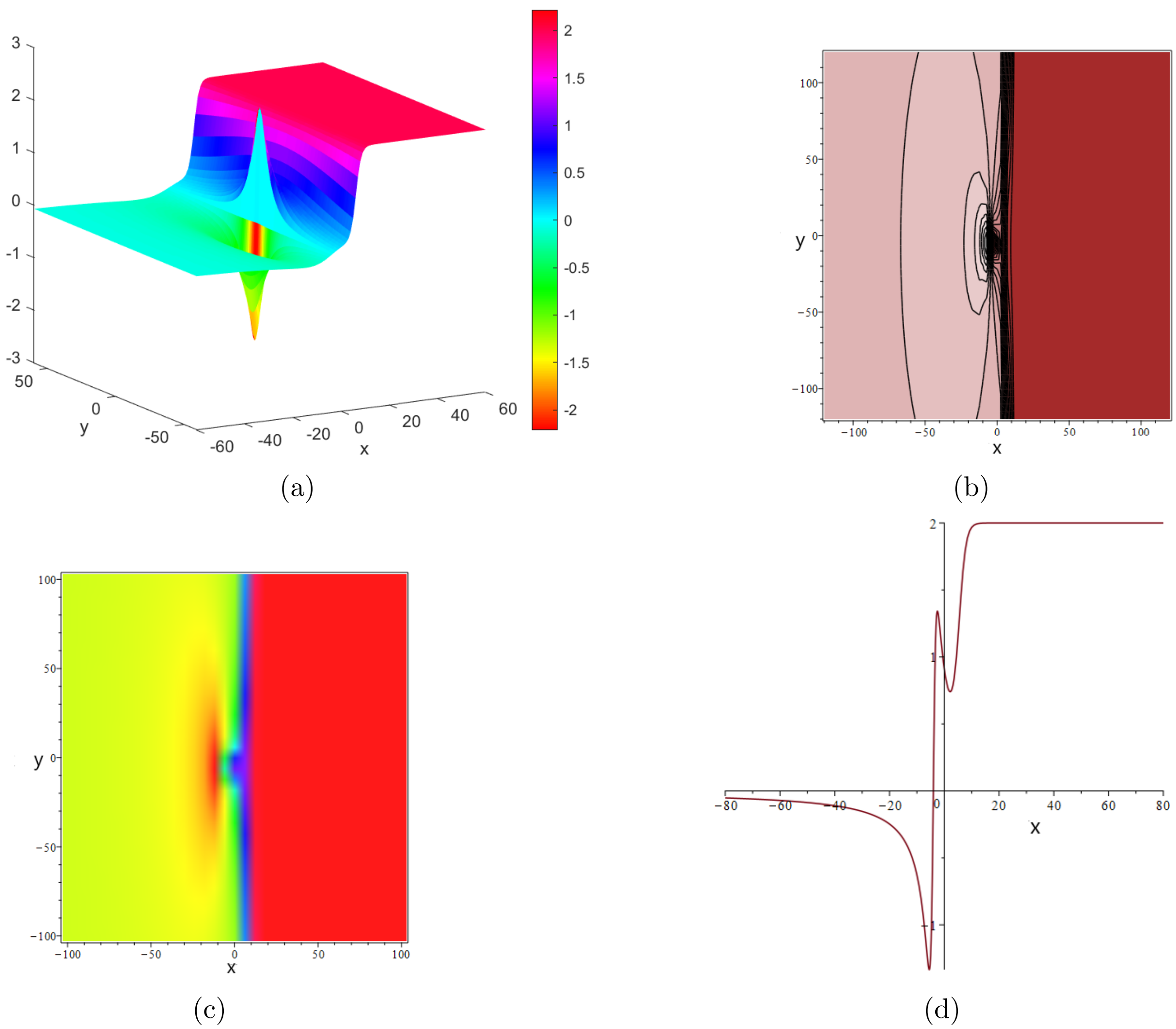


Figure 3.13: Visuals of Interaction between a lump and kink waves in Eq. (52) with $z = 1, t = 0$ (a) Three-dimensional representation (b) contour plot (c) density plot, and (d) 2D profile with $y = 1$.

Chapter 4

Breathers wave, Solitary wave, Lump and lump Interaction solutions for generalized (2+1)-dimensional Soliton equation

In this chapter we will focus on generalized (2+1)-dimensional soliton equation [66, 67] which is given as

$$\phi_t + \alpha(\phi_{xxx} + 6\phi\phi_x) + \beta\partial_x^{-1}\phi_{yy} + \gamma(\phi_{xxy} + 3\phi\phi_y + 3\phi_x\partial_x^{-1}\phi_y) = 0, \tag{4.1}$$

where $\phi = \phi(x, y, t)$ and ∂_x^{-1} is the anti-derivative with respect to x. In [66, 67] C. Li et al. investigated the integrability of Eq. (4.1) and Wang D. et al. by using Bell polynomials obtained Hirota Bilinear form of Eq. (4.1) and explored soliton solutions. Under the dependent variable transformation

$$\phi = 2(\ln\tau)_{xx}, \tag{4.2}$$

where $\phi = \phi(x, y, t)$, the Hirota's bilinear form of Eq. (4.1) [67] is given as follows

$$\left(D_t D_x + \alpha D_x^4 + \beta D_y^2 + \gamma D_x^3 D_y\right)\tau\cdot\tau = 0. \tag{4.3}$$

where $\tau = \tau(x, y, t)$ and D_x, D_y and D_t are Hirota bilinear operators, and the Hirota bilinear operator is defined as

$$D_{w_1}^{n_1}\cdots D_{w_l}^{n_l}(\tau\cdot\xi) = \left(\partial_{w_1} - \partial_{w_1'}\right)^{n_1}\cdots\left(\partial_{w_l} - \partial_{w_l'}\right)^{n_l}\tau\left(w_1,\cdots,w_l\right)\xi\left(w_1',\cdots,w_l'\right)\big|_{w_1'=w_1,\cdots,w_l'=w_l}, \tag{4.4}$$

in which $n_1 + n_2 + ... + n_l \geq 0$. By using Eq. (4.4), Eq. (4.3) can be written as

$$\tau\tau_{xt}-\tau_x\tau_t+\alpha(\tau\tau_{xxxx}-4\tau_{xxx}\tau_x+3\tau_{xx}^2)+\beta(\tau\tau_{yy}-\tau_y^2)+\gamma(\tau\tau_{xxxy}-\tau_{xxx}\tau_y-3\tau_{xxy}\tau_x+3\tau_{xx}\tau_{xy}) = 0. \tag{4.5}$$

We will utilize Eq. (4.5) to find lump and its interaction solutions also breathers and solitary waves solutions.

4.1 Lump Solution to the generalized (2+1)-dimensional Soliton Equation

To search for lump solution we take τ as of the following form with two quadratic functions

$$\begin{aligned}
\tau &= d_1^2 + d_2^2 + q_9, \\
d_1 &= q_1x + q_2y + q_3t + q_4, \\
d_2 &= q_5x + q_6y + q_7t + q_8, ,
\end{aligned} \tag{4.6}$$

where $q_j (1 \leq j \leq 9)$ are all constant parameters yet to be known.
By inserting Eq. (4.6) into Eq. (4.5) and equating the coefficients of each power of variables to zero, a system of equations is attained and with the help of computing mathematical tool Maple we derived following relations of parameters :

Case 1

$$\begin{aligned}
q_3 &= -\frac{\beta(q_1q_2^2 - q_1q_6^2 + 2q_2q_5q_6)}{q_1^2 + q_5^2}, q_7 = -\frac{\beta(2q_1q_2q_6 - q_2^2q_5 + q_5q_6^2)}{q_1^2 + q_5^2}, \\
q_9 &= -\frac{3\left(\alpha q_1^6 + 3\alpha q_1^4q_5^2 + 3\alpha q_1^2q_5^4 + \alpha q_5^6 + \gamma q_1^5q_2 + \gamma q_1^4q_5q_6 + 2\gamma q_1^3q_2q_5^2 + 2\gamma q_1^2q_5^3q_6 + \gamma q_1q_2q_5^4 + \gamma q_5^5q_6\right)}{\beta(q_1q_6 - q_2q_5)^2},
\end{aligned} \tag{4.7}$$

where q_1, q_2, q_4, q_5, q_6 and q_8 are free parameters with $\beta \neq 0, q_1^2+q_5^2 \neq 0$, and $q_1q_6-q_2q_5 \neq 0$.
On substituting Eq. (4.7) into Eq. (4.6) and then into Eq. (4.2), we will get lump solution to Eq. (4.1) as

$$\phi = \frac{2\left(2q_1^2 + 2q_5^2\right)}{\tau} - \frac{2\left(2q_1d_1 + 2q_5d_2\right)^2}{\tau^2}, \tag{4.8}$$

where

$$\tau = d_1^2 + d_2^2 + q_9,$$
$$d_1 = q_1x + q_2y - \frac{q_1q_2^2 - q_1q_6^2 + 2q_2q_5q_6}{q_1^2 + q_5^2}\beta t + q_4,$$
$$d_2 = q_5x + q_6y - \frac{2q_1q_2q_6 - q_2^2q_5 + q_5q_6^2}{q_1^2 + q_5^2}\beta t + q_8.$$

Lump solution in Eq. (4.8) is shown in Fig. 1 with suitable choice of parameters.

It is apparent from the Fig. 4.1, the solution comprising the lump exhibits a prominent peak and two minor depressions that gradually fade away in all spatial dimensions. To analyze the properties of Eq. (4.8), one can set $t = 0$ and treat ϕ as a function of x and y, let

$$\frac{\partial\phi(x,y)}{\partial x} = 0, \quad \frac{\partial\phi(x,y)}{\partial y} = 0, \tag{4.9}$$

we can obtain three extreme points to ϕ as

$$\begin{aligned}
(x_1, y_1) = \Bigg(&-\frac{\beta tq_1q_2^2q_6 + \beta tq_1q_6^3 - \beta tq_2^3q_5 - \beta tq_2q_5q_6^2 - q_1^2q_2q_8 + q_1^2q_4q_6 - q_2q_5^2q_8 + q_4q_5^2q_6}{(q_1^2+q_5^2)(q_1q_6 - q_2q_5)},\\
&\frac{2\beta tq_1^2q_2q_6 - 2\beta tq_1q_2^2q_5 + 2\beta tq_1q_5q_6^2 - 2\beta tq_2q_5^2q_6 - q_1^3q_8 + q_1^2q_4q_5 - q_1q_5^2q_8 + q_4q_5^3}{q_1^3q_6 - q_1^2q_2q_5 + q_1q_5^2q_6 - q_2q_5^3}\Bigg),\\
(x_2, y_2) = \Bigg(&\frac{-p}{\beta(q_1q_6 - q_2q_5)(q_1^2+q_5^2)},\\
&\frac{2\beta tq_1^2q_2q_6 - 2\beta tq_1q_2^2q_5 + 2\beta tq_1q_5q_6^2 - 2\beta tq_2q_5^2q_6 - q_1^3q_8 + q_1^2q_4q_5 - q_1q_5^2q_8 + q_4q_5^3}{q_1^3q_6 - q_1^2q_2q_5 + q_1q_5^2q_6 - q_2q_5^3}\Bigg),\\
(x_3, y_3) = \Bigg(&\frac{q}{\beta(q_1q_6 - q_2q_5)(q_1^2+q_5^2)},\\
&\frac{2\beta tq_1^2q_2q_6 - 2\beta tq_1q_2^2q_5 + 2\beta tq_1q_5q_6^2 - 2\beta tq_2q_5^2q_6 - q_1^3q_8 + q_1^2q_4q_5 - q_1q_5^2q_8 + q_4q_5^3}{q_1^3q_6 - q_1^2q_2q_5 + q_1q_5^2q_6 - q_2q_5^3}\Bigg).
\end{aligned} \tag{4.10}$$

where

$$p = \left(\beta^2 t q_1 q_2^2 q_6 + \beta^2 t q_1 q_6^3 - \beta^2 t q_2^3 q_5 - \beta^2 t q_2 q_5 q_6^2 - \beta q_1^2 q_2 q_8 + \beta q_1^2 q_4 q_6 - \beta q_2 q_5^2 q_8 + \beta q_4 q_5^2 q_6\right)$$
$$+ 3\Big(- \alpha\beta q_1^8 - 4\alpha\beta q_1^6 q_5^2 - 6\alpha\beta q_1^4 q_5^4 - 4\alpha\beta q_1^2 q_5^6 - \alpha\beta q_5^8 - \beta\gamma q_1^7 q_2 - \beta\gamma q_1^6 q_5 q_6 - 3\beta\gamma q_1^5 q_2 q_5^2$$
$$- 3\beta\gamma q_1^4 q_5^3 q_6 - 3\beta\gamma q_1^3 q_2 q_5^4 - 3\beta\gamma q_1^2 q_5^5 q_6 - \beta\gamma q_1 q_2 q_5^6 - \beta\gamma q_5^7 q_6\Big)^{\frac{1}{2}},$$
$$q = \left(-\beta^2 t q_1 q_2^2 q_6 - \beta^2 t q_1 q_6^3 + \beta^2 t q_2^3 q_5 + \beta^2 t q_2 q_5 q_6^2 + \beta q_1^2 q_2 q_8 - \beta q_1^2 q_4 q_6 + \beta q_2 q_5^2 q_8 - \beta q_4 q_5^2 q_6\right)$$
$$+ 3\Big(- \alpha\beta q_1^8 - 4\alpha\beta q_1^6 q_5^2 - 6\alpha\beta q_1^4 q_5^4 - 4\alpha\beta q_1^2 q_5^6 - \alpha\beta q_5^8 - \beta\gamma q_1^7 q_2 - \beta\gamma q_1^6 q_5 q_6 - 3\beta\gamma q_1^5 q_2 q_5^2$$
$$- 3\beta\gamma q_1^4 q_5^3 q_6 - 3\beta\gamma q_1^3 q_2 q_5^4 - 3\beta\gamma q_1^2 q_5^5 q_6 - \beta\gamma q_1 q_2 q_5^6 - \beta\gamma q_5^7 q_6\Big)^{\frac{1}{2}}.$$

Our next step involves determining the Hessian matrix and calculating the value of ϕ_{xx} at the extreme point (x_1, y_1).

$$\begin{aligned}
\Delta &= \begin{vmatrix} \frac{\partial^2\phi(x,y)}{\partial x^2} & \frac{\partial^2\phi(x,y)}{\partial x\partial y} \\ \frac{\partial^2\phi(x,y)}{\partial x\partial y} & \frac{\partial^2\phi(x,y)}{\partial y^2} \end{vmatrix}_{(x_1,y_1)} \\
&= \frac{64\beta^4(q_1q_6 - q_2q_5)^8(3q_1^2q_2^2 + q_1^2q_6^2 + 4q_1q_2q_5q_6 + q_2^2q_5^2 + 3q_5^2q_6^2)(q_1^2 - q_1q_2 + q_5^2 - q_5q_6)}{27(q_1^2 + q_5^2)^7(\alpha q_1^2 + \alpha q_5^2 + \gamma q_1 q_2 + \gamma q_5 q_6)^4} \\
&= \frac{22346}{49537} > 0, \\
\phi_{xx}(x_1, y_1) &= -\frac{8\beta^2(q_1q_6 - q_2q_5)^4}{3(q_1^2 + q_5^2)^2(\alpha q_1^2 + \alpha q_5^2 + \gamma q_1q_2 + \gamma q_5q_6)^2} = -\frac{17764}{62979} < 0,
\end{aligned} \tag{4.11}$$

also the Hessian matrix and ϕ_{xx} at (x_2, y_2), (x_3, y_3) is

$$\begin{aligned}
\Delta &= \begin{vmatrix} \frac{\partial^2\phi(x,y)}{\partial x^2} & \frac{\partial^2\phi(x,y)}{\partial x\partial y} \\ \frac{\partial^2\phi(x,y)}{\partial x\partial y} & \frac{\partial^2\phi(x,y)}{\partial y^2} \end{vmatrix}_{(x_r,y_r)} \\
&= \frac{(q_1^2 - q_1q_2 + q_5^2 - q_5q_6)(3q_1^2q_2^2 + 4q_1^2q_6^2 - 2q_1q_2q_5q_6 + 4q_2^2q_5^2 + 3q_5^2q_6^2)(q_1q_6 - q_2q_5)^8\beta^4}{432(q_1^2 + q_5^2)^7(\alpha q_1^2 + \alpha q_5^2 + \gamma q_1q_2 + \gamma q_5q_6)^4} \\
&= \frac{538}{516241} > 0, \\
\phi_{xx}(x_r, y_r) &= \frac{(q_1q_6 - q_2q_5)^4\beta^2}{12(q_1^2 + q_5^2)^2(\gamma q_1q_2 + \gamma q_5q_6 + \alpha q_1^2 + \alpha q_5^2)^2} = \frac{3228}{366217} > 0, \qquad r = 2, 3.
\end{aligned} \tag{4.12}$$

Therefore, at the extreme point (x_1, y_1), the maximum value of the lump solution is achieved. This means that the peak height is

$$H = \phi_{max} = \phi\left(\frac{10}{19}, \frac{-9}{10}\right) = \frac{13732}{31667}.$$

Furthermore, at the coordinates (x_2, y_2) and (x_3, y_3), the lump solution exhibits its minimum value which is basically is the depth of troughs given as respectively

$$D = \phi_{min} = \phi\left(\frac{80306}{16963}, \frac{-9}{10}\right) = -\frac{3610}{77179},$$
$$D = \phi_{min} = \phi\left(\frac{158825}{27446}, \frac{-9}{10}\right) = -\frac{4318}{79661}.$$

From Fig. 4.1, it can be observed that the lump solution exhibits a prominent maximum along with two points of lowest magnitude. This observation provides valuable insights into the role of wave phenomena exhibiting nonlinearity in various domains, including physics and oceanography.

Case 2

$$\begin{aligned} q_3 &= -\frac{\beta q_1 \alpha^2}{\gamma^2}, q_1 = q_1, q_2 = -\frac{q_1 \alpha}{\gamma}, q_4 = q_4, \\ q_7 &= \frac{q_6 \alpha \beta}{\gamma}, q_5 = -\frac{\gamma q_6}{\alpha}, q_6 = q_6, q_8 = q_8, q_9 = q_9, \end{aligned} \tag{4.13}$$

where q_1, q_4, q_6, q_8 and q_9 are free parameters and $\gamma \neq 0, \alpha \neq 0$. On substituting Eq. (4.13) into (4.6) and then into Eq. (4.2), we will get lump solution to Eq. (4.1) as

$$\phi = \frac{2\left(2q_1^2 + \frac{2\gamma^2 q_6^2}{\alpha^2}\right)}{\tau} - \frac{2\left(2d_1 q_1 - 2d_2 \frac{\gamma q_6}{\alpha}\right)^2}{\tau^2}, \tag{4.14}$$

where

$$\begin{aligned} \tau &= d_1^2 + d_2^2 + q_9, \\ d_1 &= -\frac{\beta q_1 \alpha^2}{\gamma^2} t + q_1 x - q_1 \frac{\alpha}{\gamma} y + q_4, \\ d_2 &= \frac{q_6 \alpha \beta}{\gamma} t - \frac{\gamma q_6}{\alpha} x + q_6 y + q_8. \end{aligned}$$

Lump solution in Eq. (4.14) is shown in Fig. 4.2 with suitable choice of parameters.
In that case we can also obtain three extreme points, one represent a tall peak and two represent shallow troughs.

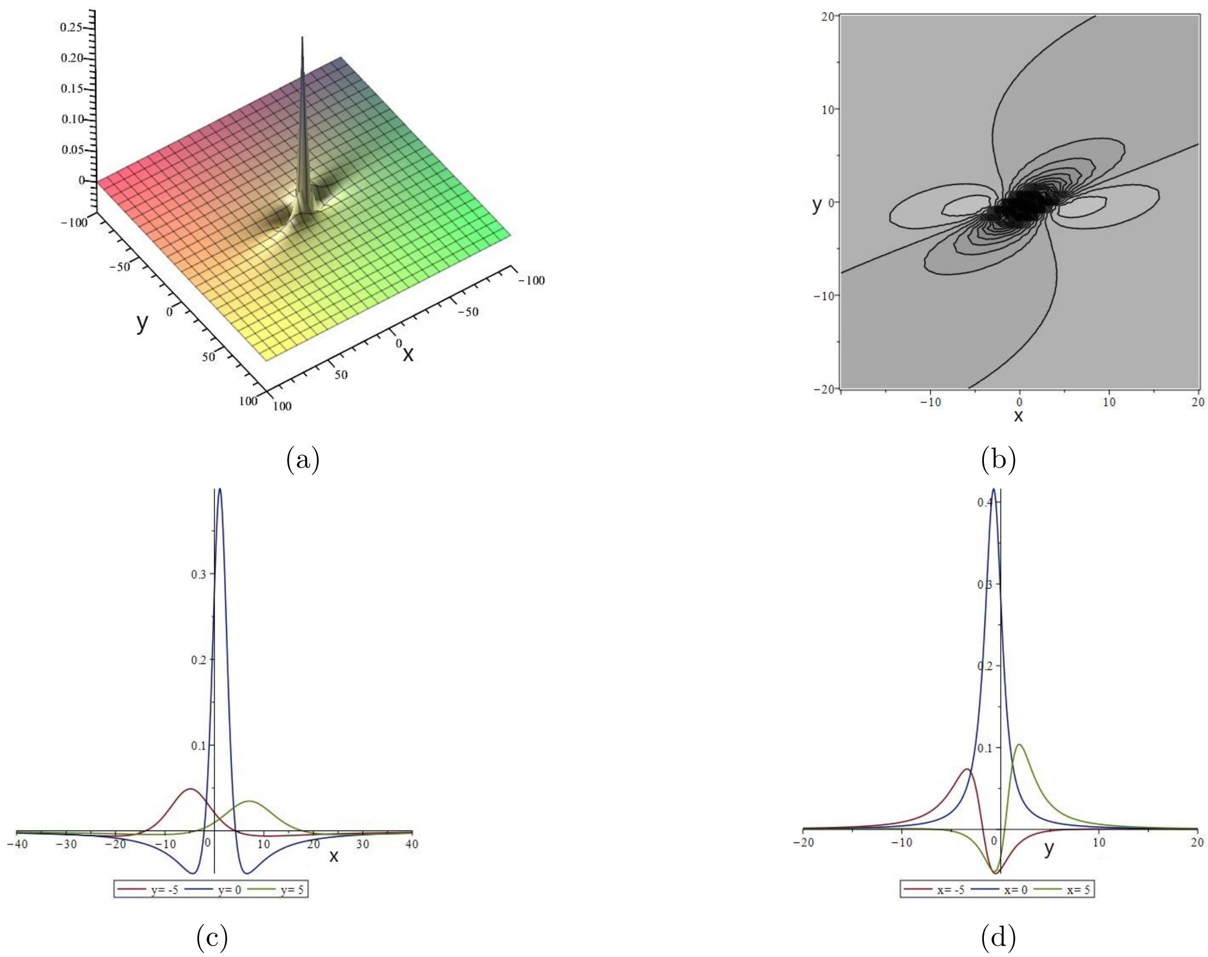
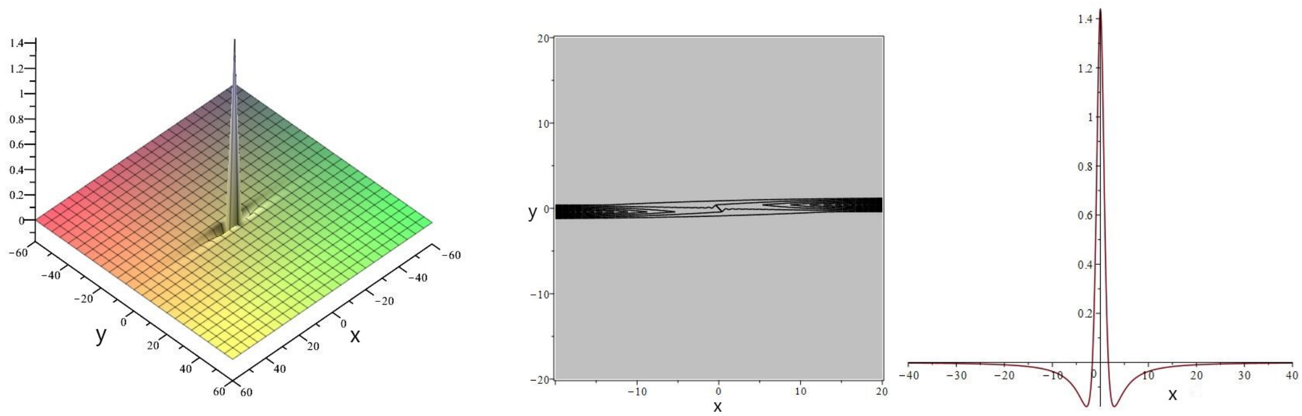


Figure 4.1: Visuals of lump solution Eq. (4.14) at $q_1 = -0.1, q_2 = 2, q_4 = 1, q_5 = -1, q_6 = 1, q_8 = 1, q_9 = 1, \alpha = 0.1, \beta = 0.1, \gamma = 1$ and $t = 0$ (a) three-dimensional view (b) contour map (c) 2D profile at $y = -5, y = 0, y = 5$, and (d) 2D profile at $x = -5, x = 0, x = 5$.



(a) (b) (c)

Figure 4.2: Graphical representation of lump solution Eq. (4.14) at $q_1 = 0.6,\ q_4 = 0,\ q_6 = 0.1,\ q_8 = 0,\ q_9 = 1,\ \alpha = 1.5,\ \beta = 1,\ \gamma = 0.02$, and $t = 0$. (a) Three-dimensional graph (b) contour map, and (c) 2D profile with $y = 0$.

4.2 Interaction between lump and stripe solutions

4.2.1 Lump and 1-stripe

To find interaction solution among a lump and 1-stripe we consider a function τ with the following form

$$
\begin{aligned}
\tau &= d_1^2 + d_2^2 + e^l + q_9, \\
d_1 &= q_1 x + q_2 y + q_3 t + q_4, \\
d_2 &= q_5 x + q_6 y + q_7 t + q_8, \\
l &= b_1 x + b_2 y + b_3 t + b_4,
\end{aligned}
\tag{4.15}
$$

where $q_j (1 \le j \le 9)$ and $b_k (1 \le k \le 4)$ are all constant parameters yet to be known. By putting Eq. (4.15) into Eq. (4.5) and equating the coefficients of all powers of variables to zero, a set of equations is attained and solving this system we obtain following relations of parameters :

$$q_1 = q_1, q_2 = -\frac{\alpha q_1}{\gamma}, q_3 = -\frac{q_1\alpha^2\beta}{\gamma^2}, q_4 = q_4, q_5 = q_5, q_6 = -\frac{\alpha q_5}{\gamma},$$
$$q_7 = -\frac{\beta q_5\alpha^2}{\gamma^2}, q_8 = q_8, q_9 = q_9, b_1 = b_1, b_2 = -\frac{b_1\alpha}{\gamma}, b_3 = -\frac{\beta b_1\alpha^2}{\gamma^2}, b_4 = b_4, \tag{4.16}$$

where $q_1, q_4, q_5, q_8, q_9, b_1$ and b_4 are all free parameters and $\gamma \neq 0$. By inserting Eq. (4.16) into Eq. (4.15) and then into Eq. (4.2) we get lump interaction solution with 1-stripe solution to the Eq. (4.1) as

$$\phi = \frac{2\left(2q_1^2 + 2q_5^2 + b_1^2 e^l\right)}{\tau} - \frac{2\left(2d_1q_1 + 2d_2q_5 + b_1e^l\right)^2}{\tau^2}, \tag{4.17}$$

where

$$d_1 = q_1x - \frac{\alpha q_1}{\gamma}y - \frac{q_1\alpha^2\beta}{\gamma^2}t + q_4,$$
$$d_2 = q_5x - \frac{\alpha q_5}{\gamma}y - \frac{\beta q_5\alpha^2}{\gamma^2}t + q_8,$$
$$l = b_1x - \frac{\alpha b_1}{\gamma}y - \frac{\beta b_1\alpha^2}{\gamma^2}t + b_4.$$

By choosing the values of parameters and substituting all these values in Eq. (4.17) we get lump interaction solution with 1-stripe shown in Fig. 4.3.

4.2.2 Lump and 2-stripe

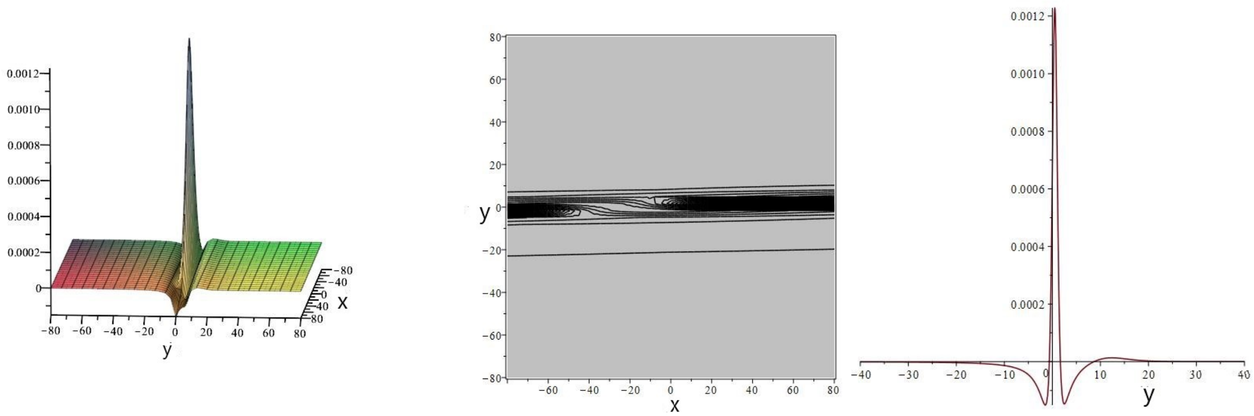
For finding lump interaction with 2-stripe we consider function τ of the following structure

$$\tau = d_1^2 + d_2^2 + e^l + e^k + q_9, \tag{4.18}$$

with

$$d_1 = q_1x + q_2y + q_3t + q_4,$$
$$d_2 = q_5x + q_6y + q_7t + q_8,$$
$$l = b_1x + b_2y + b_3t + b_4,$$
$$k = b_5x + b_6y + b_7t + b_8,$$

where $q_j (1 \leq j \leq 9)$ and $b_k (1 \leq k \leq 8)$ are all constant parameters yet to be known.



(a) (b) (c)

Figure 4.3: Graphical representation of lump interaction solution with 1-stripe Eq. (4.17) at $q_1 = 0.01, q_9 = 1, q_5 = 1, q_8 = 1, q_4 = 1, b_1 = 1, b_4 = 1, \alpha = 0.1, \beta = 1, \gamma = 0.01$ and $t = 0$. (a) Three-dimensional representation (b) contour map, and (c) 2D profile with $y = 0$.

By inserting Eq. (4.18) into Eq. (4.5) and comparing the coefficients of the variables raised to different powers, a set of equations is obtained. Through the solving of this set of equations, we are able to deduce the following parameter relationships

$$\begin{aligned} q_3 &= -\frac{q_1\alpha^2\beta}{\gamma^2},\ q_2 = -\frac{\alpha q_1}{\gamma},\ q_7 = -\frac{\beta q_5\alpha^2}{\gamma^2},\ q_6 = -\frac{\alpha q_5}{\gamma}, \\ b_2 &= -\frac{\alpha b_1}{\gamma},\ b_3 = -\frac{\beta\alpha^2 b_1}{\gamma^2},\ b_6 = -\frac{\alpha b_5}{\gamma},\ b_7 = -\frac{\beta\alpha^2 b_5}{\gamma^2}. \end{aligned} \tag{4.19}$$

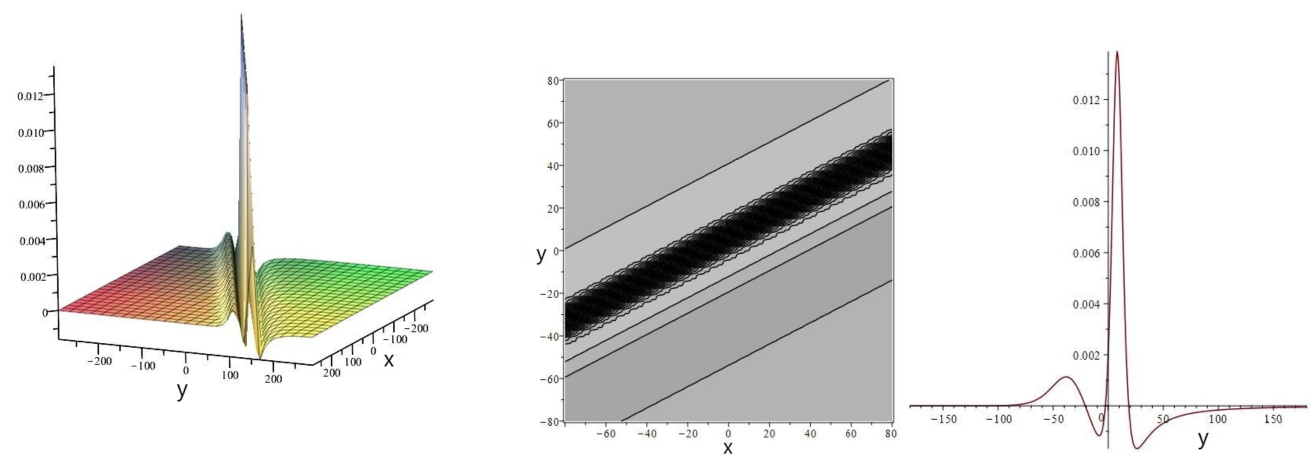
where $q_1, q_4, q_5, q_8, q_9, b_1$ and b_3 are all free parameters and $\gamma \neq 0$. By putting Eq. (4.19) into Eq. (4.18) and then into Eq. (4.2) we get lump and interaction wih 2-stripe solution to the Eq. (4.1) as

$$\phi = \frac{2(2q_1^2 + 2q_5^2 + b_1^2e^l + b_5^2e^k)}{\tau} - \frac{2(2d_1q_1 + 2d_2q_5 + b_1e^l + b_5e^k)^2}{\tau^2}, \tag{4.20}$$

where

$$d_1 = q_1 x - \frac{\alpha q_1}{\gamma} y - \frac{q_1 \alpha^2 \beta}{\gamma^2} t + q_4,$$
$$d_1 = q_5 x - \frac{\alpha q_5}{\gamma} y - \frac{\beta q_5 \alpha^2}{\gamma^2} t + q_8,$$
$$l = b_1 x - \frac{\alpha b_1}{\gamma} y - \frac{\beta b_1 \alpha^2}{\gamma^2} t + b_4,$$
$$k = b_5 x - \frac{\alpha b_5}{\gamma} y - \frac{\beta b_5 \alpha^2}{\gamma^2} t + b_8.$$

By choosing the values of parameters involved in Eq. (4.20) we get lump interaction with 2-stripe solution which is shown in Fig. 4.4.



(a) (b) (c)

Figure 4.4: Graphical representation of lump interaction solution with 2-stripe Eq. (4.20) at $q_1 = 0.05, q_4 = 1, q_5 = 0.1, q_8 = 1, q_9 = 1, b_1 = 0.01, b_4 = 1, b_5 = 0.08, b_8 = 0, \alpha = 1, \beta = 1, \gamma = 0.5$ and $t = 0$. (a) Three-dimensional graph (b) contour map, and (c) 2D profile with $y = 0$.

4.3 Interaction between lump and Periodic waves

4.3.1 Lump interaction with Periodic waves

The test function for examining the interaction phenomenon among lump and periodic waves is given as

$$\tau = d_1^2 + d_2^2 + \cos l + q_9, \tag{4.21}$$

where

$$d_1 = q_1 x + q_2 y + q_3 t + q_4,$$
$$d_2 = q_5 x + q_6 y + q_7 t + q_8,$$
$$l = b_1 x + b_2 y + b_3 t + b_4,$$

where $q_j (1 \le j \le 9)$ and $b_k (1 \le k \le 4)$ are all constant parameters yet to be known.

By inserting Eq. (4.21) into Eq. (4.5) and comparing the coefficients of the variables raised to different powers, a system of equations is obtained. On solving, we are able to deduce the following parameter relationships

$$\begin{aligned} &q_1 = q_1, q_2 = -\frac{\alpha q_1}{\gamma}, q_3 = -\frac{q_1 \alpha^2 \beta}{\gamma^2}, q_4 = q_4, q_5 = q_5, q_6 = -\frac{\alpha q_5}{\gamma}, \\ &q_7 = -\frac{\beta q_5 \alpha^2}{\gamma^2}, q_8 = q_8, q_9 = q_9, b_1 = b_1, b_2 = -\frac{b_1 \alpha}{\gamma}, b_3 = -\frac{\beta b_1 \alpha^2}{\gamma^2}, b_4 = b_4, \end{aligned} \tag{4.22}$$

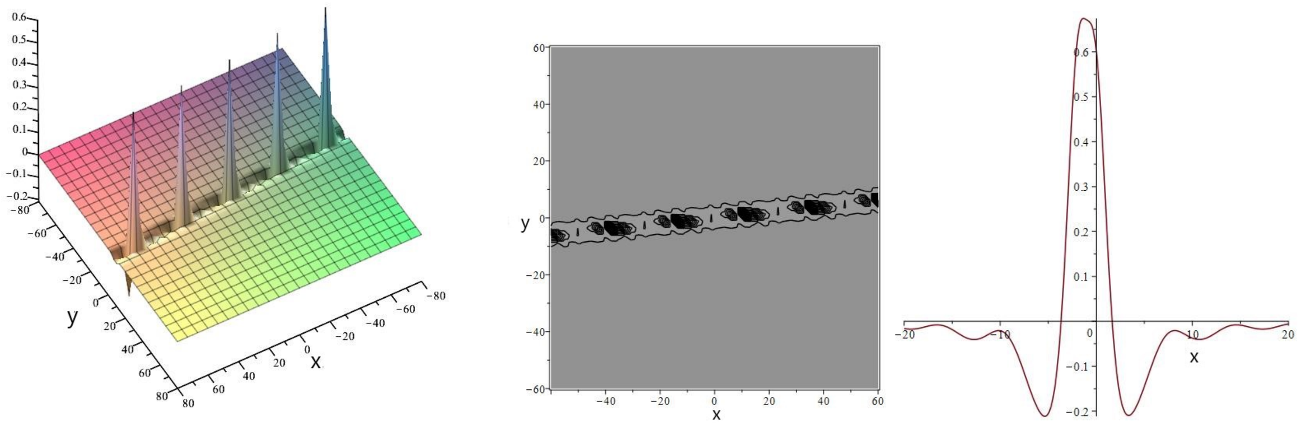
where $q_1, q_4, q_5, q_8, q_9, b_1$ and b_4 are all free parameters and $\gamma \neq 0$. By inserting Eq. (4.22) into Eq. (4.21) and then into Eq. (4.2) we get lump interaction solution with periodic waves to the Eq. (4.1) as

$$\phi = \frac{2\left(2q_1^2 + 2q_5^2 - b_1^2 \cos(l)\right)}{\tau} - \frac{2\left(2d_1 q_1 + 2d_2 q_5 + b_1 \sin(l)\right)^2}{\tau^2}, \tag{4.23}$$

where

$$d_1 = q_1 x - \frac{\alpha q_1}{\gamma} y - \frac{q_1 \alpha^2 \beta}{\gamma^2} t + q_4,$$
$$d_1 = q_5 x - \frac{\alpha q_5}{\gamma} y - \frac{\beta q_5 \alpha^2}{\gamma^2} t + q_8,$$
$$l = b_1 x - \frac{\alpha b_1}{\gamma} y - \frac{\beta b_1 \alpha^2}{\gamma^2} t + b_4.$$

By selecting the values of parameters and substituting in Eq. (4.23) we get lump interaction solution with periodic waves shown in Fig. 4.5.



(a) (b) (c)

Figure 4.5: Graphical visualization of lump interaction solution with periodic wave Eq. (4.23) at $q_1 = 0.01, q_4 = 1, q_5 = 1, q_8 = 1, q_9 = 1, b_1 = 1, b_4 = 1, \alpha = 0.1, \beta = 1, \gamma = 0.01$ and $t = 0$. (a) Three-dimensional view (b) contour map, and (c) 2D profile with $y = 0$.

4.3.2 Interaction between a lump, periodic wave and 1-stripe

To investigate the interaction phenomenon between a lump, periodic waves and 1-stripe we take test function of the following form

$$\tau = d_1^2 + d_2^2 + \cos l + e^k + q_9, \tag{4.24}$$

with

$$d_1 = q_1 x + q_2 y + q_3 t + q_4,$$
$$d_2 = q_5 x + q_6 y + q_7 t + q_8,$$
$$l = b_1 x + b_2 y + b_3 t + b_4,$$
$$k = b_5 x + b_6 y + b_7 t + b_8,$$

where $q_j (1 \leq j \leq 9)$ and $b_k (1 \leq k \leq 8)$ are all constant parameters yet to be known.

By inserting Eq. (4.24) into Eq. (4.5) and comparing the coefficients of the variables raised to different powers, a system of equations is obtained. On solving, we are able to deduce the following parameter relationships

$$q_3=-\frac{q_1\alpha^2\beta}{\gamma^2},\ q_2=-\frac{\alpha q_1}{\gamma},\ q_7=-\frac{\beta q_5\alpha^2}{\gamma^2},\ q_6=-\frac{\alpha q_5}{\gamma},$$
$$b_2=-\frac{\alpha b_1}{\gamma},\ b_3=-\frac{\beta\alpha^2 b_1}{\gamma^2},\ b_6=-\frac{\alpha b_5}{\gamma},\ b_7=-\frac{\beta\alpha^2 b_5}{\gamma^2}. \tag{4.25}$$

where $q_1, q_4, q_5, q_8, q_9, b_1$ and b_3 are all free parameters and $\gamma \neq 0$. By inserting Eq. (4.25) into Eq. (4.24) and then into Eq. (4.2) we get lump and interaction wih 2-stripe solution to the Eq. (4.1) as

$$\phi=\frac{2\left(2q_1^2+2q_5^2-b_1^2\cos(l)+b_5^2e^k\right)}{\tau}-\frac{2\left(2d_1q_1+2d_2q_5+b_1\sin(l)+b_5e^k\right)^2}{\tau^2}, \tag{4.26}$$

where

$$d_1=q_1x-\frac{\alpha q_1}{\gamma}y-\frac{q_1\alpha^2\beta}{\gamma^2}t+q_4,$$
$$d_1=q_5x-\frac{\alpha q_5}{\gamma}y-\frac{\beta q_5\alpha^2}{\gamma^2}t+q_8,$$
$$l=b_1x-\frac{\alpha b_1}{\gamma}y-\frac{\beta b_1\alpha^2}{\gamma^2}t+b_4,$$
$$k=b_5x-\frac{\alpha b_5}{\gamma}y-\frac{\beta b_5\alpha^2}{\gamma^2}t+b_8.$$

By choosing the values of parameters we obtain the interaction phenomenon between lump, periodic waves and 1-stripe solution which is shown in Fig. 4.6.

4.4 Breather wave solution

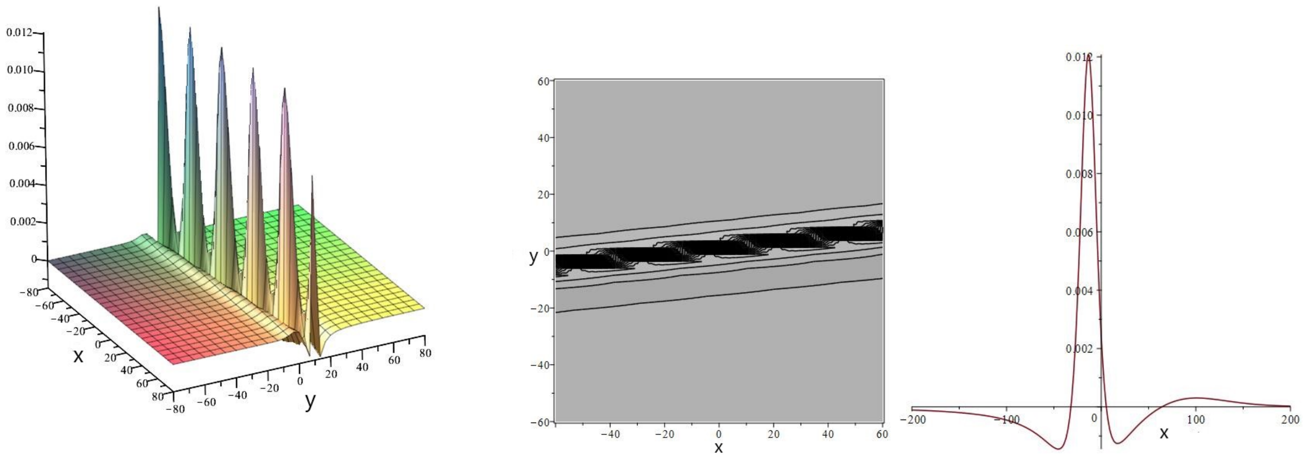
Within this part, we examine breather solution to Eq. (4.1) by considering auxiliary function to Eq. (4.5) of the following form

$$\tau=e^{-k_1l}+c_1e^{k_1l}+c_2\cos(k_2m), \tag{4.27}$$

with

$$l=q_1x+q_2y+q_3t+q_4,$$
$$m=b_1x+b_2y+b_3t+b_4,$$

where c_1, c_2, k_1, k_2 and $q_i, b_i (1\le i\le 4)$ are constant parameters which are to be determined.



(a) (b) (c)

Figure 4.6: Visuals of lump interaction solution with stripe and periodic wave Eq. (4.26) at $q_1 = 0.01,\ q_4 = 1,\ q_5 = 0.1,\ q_8 = 1,\ q_9 = 1,\ b_1 = 0.01, b_4 = 1,\ b_5 = 0.05,\ b_8 = 1,\ \alpha = 1,\ \beta = 1,\ \gamma = 0.1$ and $t = 0$. (a) Three-dimensional view (b) contour map, and (c) 2D profile at $y = 0$.

By putting Eq. (4.27) into Eq. (4.5) and equating the coefficients of all powers of the variables by setting equal to zero, an algebraic system of equations is obtained and then by solving the system following relationships of parameters are derived

$$\begin{aligned} &q_3 = -\frac{\beta b_2 k_2 \alpha}{k_1 \gamma},\ q_1 = \frac{\gamma b_2 k_2}{\alpha k_1},\ q_2 = -\frac{b_2 k_2}{k_1},\ q_4 = q_4,\ b_1 = -\frac{\gamma b_2}{\alpha},\ b_4 = b_4, \\ &b_2 = b_2,\ b_3 = \frac{\beta b_2 \alpha}{\gamma},\ b_5 = b_5,\ c_1 = c_1,\ c_2 = c_2,\ k_1 = k_1,\ k_2 = k_2, \end{aligned} \tag{4.28}$$

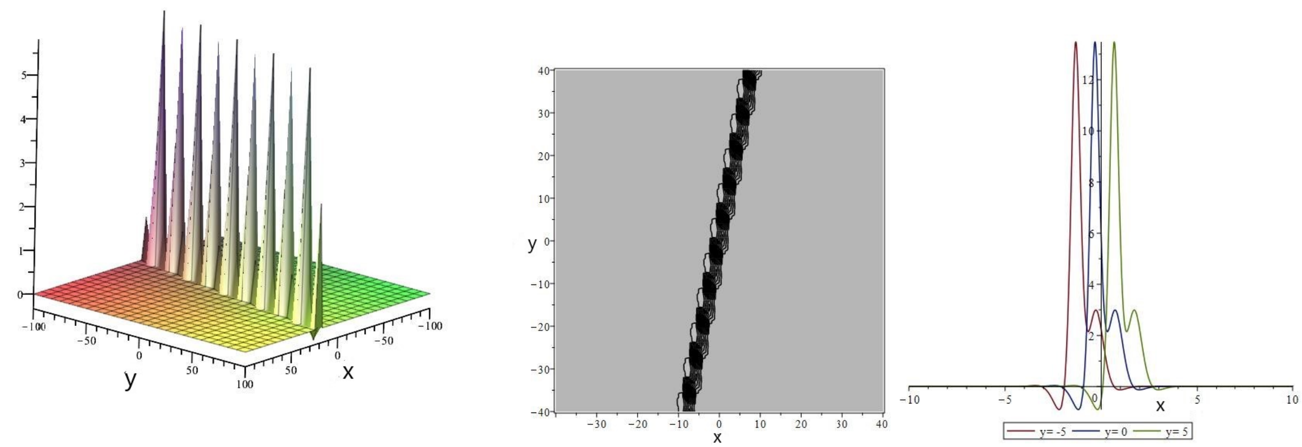
where $q_4, b_4, b_2, b_5, c_1, c_2, k_1$ and k_2 are all free parameters with $k_1 \neq 0, \alpha \neq 0, \gamma \neq 0$. By inserting Eq. (4.28) into Eq. (4.27) and then into Eq. (4.2) we get breather wave solution to the Eq. (4.1) of the followig form

$$\begin{aligned} \phi = &\ \frac{2(\gamma^2 b_2^2 k_2^2 e^{-k_1 l} + c_1 \gamma^2 b_2^2 k_2^2 e^{k_1 l} - c_2 \gamma^2 b_2^2 k_2^2 \cos(k_2 m))}{\alpha^2 \tau} \\ &- \frac{2(-\gamma b_2 k_2 e - k_1 l + c_1 \gamma b_2 k_2 e k_1 l + c_2 \gamma b_2 k_2 \sin(k_2 m))^2}{\alpha^2 \tau^2}, \end{aligned} \tag{4.29}$$

where

$$l = \frac{\gamma b_2 k_2}{\alpha k_1}x - \frac{b_2 k_2}{k_1}y - \frac{b_2 k_2 \alpha\beta}{k_1\gamma}t + q_4,$$
$$m = -\frac{\gamma b_2}{\alpha}x + b_2 y + \frac{b_2\alpha\beta}{\gamma}t + b_4,$$

By choosing the values of parameters and substituting into Eq. (4.29) we obtain breather waves solution which is can be seen in Fig. 4.7.



(a) (b) (c)

Figure 4.7: Graphical representation of breather wave solution Eq. (4.29) at $k_1 = 1, b_2 = 0.5, c_1 = 1, c_2 = 1, k_2 = 1, q_4 = 0, b_4 = 1, \alpha = 0.1, \beta = 1, \gamma = 0.5$ and $t = 0$. (a) three-dimensional view (b) contour map, and (c) 2D profile with $y = -5, y = 0, y = 5$.

4.5 Solitary wave solution

For finding solitary wave solution we take test function as

$$\tau = q_4 e^l + q_8 \tan(m) + q_{12}\tanh(n), \tag{4.30}$$

where

$$l = q_1 x + q_2 y + q_3 t,$$
$$m = q_5 x + q_6 y + q_7 t,$$
$$n = q_9 x + q_{10} y + q_{11} t,$$

where $q_i (1 \leq i \leq 12)$ are constant parameters which are to be determined.

By putting Eq. (4.30) into Eq. (4.5) and collecting the coefficients of all powers of the variables by setting equal to zero, an algebraic system of equations is obtained and then by solving the system following relations of parameters are derived

$$\begin{aligned} &q_3 = -\frac{q_1\alpha^2\beta}{\gamma^2},\ q_1 = q_1,\ q_2 = -\frac{\alpha q_1}{\gamma},\ q_4 = q_4,\ q_7 = -\frac{\beta\alpha^2 q_5}{\gamma^2},\ q_5 = q_5, \\ &q_6 = -\frac{\alpha q_5}{\gamma},\ q_8 = q_8,\ q_9 = q_9,\ q_{11} = -\frac{\beta\alpha^2 q_9}{\gamma^2},\ q_{10} = -\frac{\alpha q_9}{\gamma},\ q_{12} = q_{12}, \end{aligned} \tag{4.31}$$

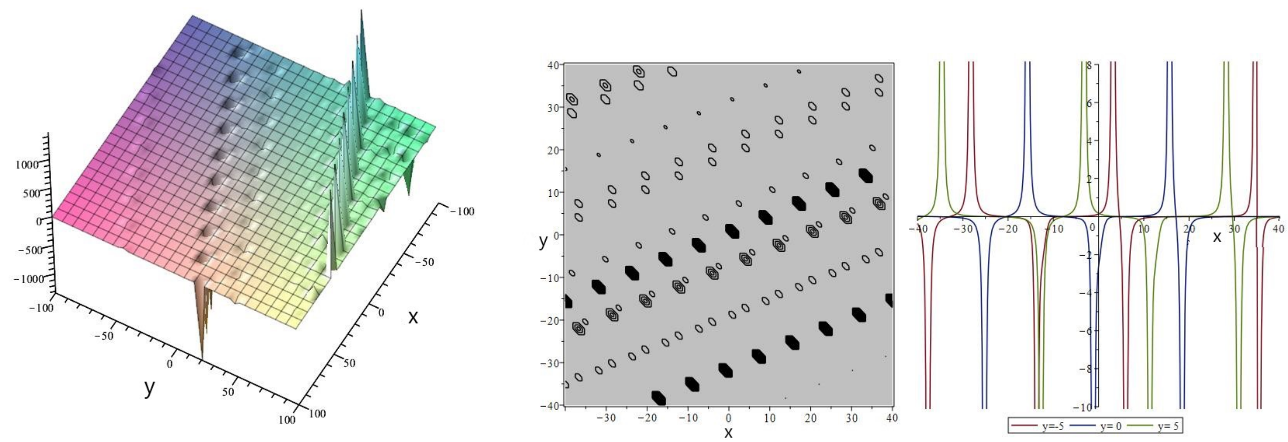
where q_1, q_4, q_5, q_8, q_9 and k_{12} are all free parameters with $\gamma \neq 0$. By inserting Eq. (4.31) into Eq. (4.30) and then into Eq. (4.2) we get solitary wave solution to the Eq. (4.1) of the followig form

$$\begin{aligned} \phi = &\frac{2\left(q_4 q_1^2 e^l + 2q_8 q_5^2 \tan(m)(1 - \tan(m)^2) - 2q_{12} q_9^2 \tanh(n)(1 + \tanh(n)^2)\right)}{\tau} \\ &- \frac{2\left(q_4 q_1 e^l + q_8 q_5 (1 - \tan(m)^2) + q_{12} q_9 (1 + \tanh(n)^2)\right)^2}{\tau^2}, \end{aligned} \tag{4.32}$$

where

$$\begin{aligned} l &= q_1 x - \frac{\alpha q_1}{\gamma} y - \frac{q_1\alpha^2\beta}{\gamma^2} t, \\ m &= q_5 x - \frac{\alpha q_5}{\gamma} y - \frac{q_5\alpha^2\beta}{\gamma^2} t, \\ n &= q_9 x - \frac{\alpha q_9}{\gamma} y - \frac{q_9\alpha^2\beta}{\gamma^2} t. \end{aligned}$$

By letting the values of parameters and substituting in Eq. (4.32) we get solitary waves solution shown in Fig. 4.8.



(a) (b) (c)

Figure 4.8: Graphical representation of solitary wave solution Eq. (4.32) at $q_1 = 0.05,\ q_4 = 1,\ q_5 = 0.1,\ q_8 = 1,\ q_9 = 1,\ b_1 = 0.01,\ b_4 = 1, b_5 = 0.08,\ b_8 = 0,\ q_{12} = 1,\ \alpha = 1,\ \beta = 1,\ \gamma = 0.4\ t = 0$. (a) three-dimensional view (b) contour map, and (c) 2D profile with $y = -5, y = 0, y = 5$.

Chapter 5

Summary and Conclusion

In this thesis, we used Hirota's Bilinear method more precisely its simplified version and direct approach based on Hirota bilinear form for finding soliton solution for two NLLEs. The brief description of Hirota method is given in Chapter 2. In Chapter 3, we first obtained Hirota bilinear form using dependent variable transformation. By using this bilinear form we obtained kink solution such as 1-kink, 2-kink and 3-kink soliton solution. Later on, a lump solution containing two quadratic functions and three quadratic functions is obtained. To examine their motion, the extreme value points are found using multivariable calculus, and their amplitude and velocity are observed, which demonstrate uniform motion for both solutions differing in amplitude and minimum points. At the end of chapter 3, we obtained lump interaction solutions with periodic and kink waves. The obtained results are presented in 3D, 2D, and contour plots. In Chapter 4, firstly we obtained lump solution which shows uniform motion under the random selection of parameters then we obtained extreme value points which has given one maximum point $\left(\frac{10}{19}, \frac{-9}{10}\right)$ shows peak height of lump which has value $\frac{13732}{31667}$. We also obtained two minimum points $\left(\frac{80306}{16963}, \frac{-9}{10}\right)$ and $\left(\frac{158825}{27446}, \frac{-9}{19}\right)$ which shows trough values of lump having depth values $-\frac{3610}{77179}$ and $-\frac{4318}{79661}$ respectively. Secondly, we obtianed lump interaction solutions with 1-stripe and 2-stripe. Finally, breathers waves and solitary waves solutions obtained. It is seen obtained results depends on the parameters involved in finding solutions. 3D, 2D and contour plots visualizes the motions of all obtained results.

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