Some Algebraic Invariants of Edge Ideals of Corona Product of Multi Triangular Snake and Ouroboros Snake Graphs

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I devote this thesis to my beloved parents who have been believing in me, supporting and encouraging me throughout the research.

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All praise be to the creator of this universe for blessing me with the ability to understand the field of mathematics, which is woven into the very fabric of this universe.

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Abstract

In this thesis, we discuss some fundamental concept of Abstract Algebra like rings and modules. The main concern of this dissertation is with the algebraic and geometric invariants such as depth, Stanley depth and regularity. We also discuss some known results related to these invariants. Furthermore, we compute exact values and bounds for depth, Stanley depth and regularity of cyclic modules associated with corona product of multi triangular snake and ouroboros snake graphs with any graph $\mathcal{Q}$.

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Introduction

Depth, Stanley depth, Projective dimension, and Castelnuovo-Mumford regularity are four significant and interconnected invariants in the fields of commutative algebra and algebraic geometry. Richard P. Stanley is known for his work to develop a relationship in Algebra and Geometry. In 1982, Stanley proposed a conjecture [31]. According to Stanley conjecture, Stanley depth of a module is atleast the depth of a module. Later on it was proved by Duval et al. [11] in year 2015 that Stanley's conjecture generally does not hold for Λ/I type modules, where S is defined as a ring of polynomials and I is a monomial ideal. Yet, finding classes which still satisfy the Stanley's inequality is a challenging task. Number of papers on Stanley depth and depth have been written up till now, and still more work is being done on these invariants and the conjecture relating them.

Castelnuovo-Mumford regularity [23] was introduced in the early 1980s by Eisenbud, Goto, and Ooishi as a mathematical concept that corresponds to the idea of regularity. One of the important aspects of Castelnuovo-Mumford regularity is that it can also be defined and used to determine the vanishing of local cohomology modules and estimates the complexity of the minimal free resolution [4, 33]. This triple nature of Castelnuovo-Mumford regularity is commonly discussed in the context of graded rings over base fields, but it holds true in general, as demonstrated in this thesis. Furthermore, this thesis computes the precise values of Depth and Stanley depth for the Cyclic Modules associated with the Corona Product of certain graphs. For some recent results related to the said invariants we refer the readers to [16, 18, 19, 27].

The first chapter provides an overview, definitions and findings for algebraic structures ring, module and graph theory along with the relevant examples and results. In second chapter, the fundamental theory of depth, Stanley depth and Stanley decomposition of ideals and modules have been discussed. Different results related to regularity and projective dimension are also stated in this chapter. In third chapter, the exact values of depth, Stanley depth and projective dimension of cyclic modules associated with corona product of multi-triangular and ouroboros snake graph with any graph $\mathcal{Q}$ are computed by using short exact sequences. In fourth chapter, the exact values for regularity of cyclic modules associated with corona product of multi-triangular and ouroboros snake graph with any graph $\mathcal{Q}$ are computed.

Chapter 1

Preliminaries

1.1 Ring Theory

In 1914, Fraenkel gave the definition of ring. Initially the concept of ring was studied in 1800, when Emmy Noether introduced the general concept of commutative rings. Also throughout this thesis rings would be considered as commutative rings. Later, the concept of non-commutative rings was also addressed as generalised concept of commutative rings. Polynomial rings, fields, ring of integers are examples of commutative rings. In algebra, the algebraic structures are dealt under the flag of ring theory, which have defined operations of multiplication and addition. For further details we refer the readers to [10].

Definition 1.1.1. A non empty set Λ with well defined operations
" $+$ " and " $\times$ " forms a ring if it satisfies these axioms:

- Λ is abelian with respect to addition "$+$".
- Associative law holds with respect to multiplication " $\times$ " in Λ.
- Distributive laws (left and right) holds in Λ.

If a ring Λ is commutative w.r.t multiplication, then it is called a commutative ring. The ring Λ is known to have an identity $1 \in \Lambda$ if $\forall\ \lambda \in \Lambda$

$$\lambda \times 1 = 1 \times \lambda = \lambda.$$

Definition 1.1.2. A subring $\mathcal{S}$ is a subgroup of the ring Λ that is non-empty and satisfies closure under multiplication.

Definition 1.1.3. A subring $\mathcal{L}$ of a ring Λ is called an ideal if it satisfies the following axioms:

- $\mathcal{L}$ is a commutative subgroup of Λ w.r.t addition.
- If $\lambda \in \mathcal{L}$ and $\phi \in \Lambda$, then $\lambda\phi \in \mathcal{L}$.

The ideal $\{0\}$ is considered to be the trivial ideal.

Definition 1.1.4. Let $\mathcal{J}$ be a proper ideal of ring Λ. If there exist an ideal $\mathcal{L}$ such that if $\mathcal{J} \subset \mathcal{L} \subset \Lambda$, then $\mathcal{J} = \mathcal{L}$ or $\mathcal{L} = \Lambda$, then $\mathcal{L}$ is known as maximal ideal.

Definition 1.1.5. Local ring is a ring Λ with unique maximal ideal.

Definition 1.1.6. For a proper ideal $\mathcal{L}$, a quotient ring $\Lambda/\mathcal{L}$ can be formed, which consists of cosets $\lambda + \mathcal{L}$, where $\lambda \in \Lambda$, and the product of cosets is defined as:

$$(\lambda_1 + \mathcal{L})(\lambda_2 + \mathcal{L}) = \lambda_1\lambda_2 + \mathcal{L}.$$

Definition 1.1.7. Consider an ideal $\mathcal{L}$ of a ring Λ. Then $(0 : \mathcal{L})$ is an ideal known as the annihilator of $\mathcal{L}$ represented as $\text{Ann}(\mathcal{L})$ defined as

$$\text{Ann}(\mathcal{L}) = \{\lambda \in \Lambda : \lambda\mathcal{L} = 0\}.$$

Definition 1.1.8. An ideal $\mathcal{L}$ of Λ is primary ideal if $\lambda_1\lambda_2 \in \mathcal{L}$, for $\lambda_1, \lambda_2 \in \Lambda$, then either $\lambda_1 \in \mathcal{L}$ or $\lambda_2^n \in \mathcal{L}$ for some $n \geq 1$.

Definition 1.1.9. Consider the ring $\Lambda = \mathcal{F}[\lambda_1, \ldots, \lambda_n]$, where $\mathcal{F}$ is a field. In this ring, the monomials form a natural $\mathcal{F}$-basis for Λ. Let $\mathbf{u} = (u_1, \ldots, u_n) \in \mathbb{R}^n$ be a vector with non-negative components $u_j \geq 0$. A monomial is defined as a product of the form $\lambda_1^{u_1} \ldots \lambda_n^{u_n}$, where each u_j is a non-negative integer. We can express a monomial w as $w = \lambda^{\mathbf{u}}$, where $\mathbf{u} = (u_1, \ldots, u_n) \in \mathbb{Z}_+^n$, and the multiplication of monomials satisfies the rule

$$\lambda^{\mathbf{u_1}}\lambda^{\mathbf{u_2}} = \lambda^{\mathbf{u_1}+\mathbf{u_2}}.$$

Definition 1.1.10. A monomial $\lambda^{\mathbf{b}}$ is said to be squarefree if $\mathbf{b}$ has components 0 and 1. An ideal with a generating set containing only squarefree monomials is known as squarefree monomial ideal.

Let Λ be a ring. Set of polynomials in one or more than one indeterminants form a polynomial ring with co-efficents in $\digamma$.

Definition 1.1.11. A polynomial over a field $\digamma$ with the variable λ is an expression of the form $\Lambda(x) = a_0 + a_1\lambda + a_2\lambda^2 + \cdots + a_n\lambda^n$, where $a_o, a_1, \ldots, a_n \in \digamma$ and $n \geq 0$. The polynomial Ring over $\digamma$ is

$\Lambda[x] = \{a_0 + a_1\lambda + a_2\lambda^2 + \cdots + a_n\lambda^n\colon\ a_o, a_1, \ldots, a_n \in \digamma, n \geq 0\ \}.$

Definition 1.1.12. The polynomial ring in the variables $\lambda_1, \lambda_2, \ldots, \lambda_n$ and coefficients belonging to Λ (commutative with identity) is defined inductively

$$\Lambda[\lambda_1, \lambda_2, \ldots, \lambda_n] = \Lambda[\lambda_1, \lambda_2, \ldots, \lambda_{n-1}][\lambda_n].$$

A ring homomorphism is a mapping between two rings that preserves their respective additive and multiplicative structures.

Definition 1.1.13. Consider two rings Λ_1 and Λ_2.

A ring homomorphism is a map $\mathcal{M} : \Lambda_1 \to \Lambda_2$ which satisfies the following axioms for all $\lambda_1, \lambda_2 \in \Lambda_1$

- $\mathcal{M}(\lambda_1 + \lambda_2) = \mathcal{M}(\lambda_1) + \mathcal{M}(\lambda_2)$,
- $\mathcal{M}(\lambda_1\lambda_2) = \mathcal{M}(\lambda_1)\mathcal{M}(\lambda_2)$,

A homomorphism which is both injective and surjective is known as ring isomorphism.

1.2 Module Theory

Module is an algebraic object with a ring attached to it, and actually the ring acts on module. Emmy Noether made important contributions to Abstract Algebra, and so she also introduced the use of modules. In this section, various properties of modules are discussed along with the examples and some important results of Module Theory.

Definition 1.2.1. Assume a ring Λ that is commutative, an Λ-module $\mathcal{D}$ is a commutative group w.r.t addition, along with a scalar multiplication map $\cdot : \Lambda \times \mathcal{D} \to \mathcal{D}$, defined as $\cdot((\lambda, \phi)) = \lambda\phi$, which holds the following axioms:

(1) $\lambda(\phi_1 + \phi_2) = \lambda\phi_1 + \lambda\phi_2$ for all $\lambda \in \Lambda$ and $\phi_1, \phi_2 \in \mathcal{D}$,

(2) $(\lambda_1 + \lambda_2)\phi = \lambda_1\phi + \lambda_2\phi$ for all $\lambda_1, \lambda_2 \in \Lambda$ and $\phi \in \mathcal{D}$,

(3) $(\lambda_1\lambda_2)\phi = \lambda_1(\lambda_2\phi)$ for all $\lambda_1, \lambda_2 \in \Lambda$ and $\phi \in \mathcal{D}$.

If the ring Λ has unity then we impose one additional axiom that is

(4) $1\phi = \phi$ for all $\phi \in \mathcal{D}$.

Definition 1.2.2. A submodule $\mathcal{P}$ of a Λ-module $\mathcal{D}$ is a subgroup of $\mathcal{D}$ under addition such that $\lambda\phi \in \Lambda$ for all $\lambda \in \Lambda$ and $\phi \in \mathcal{D}$.

Definition 1.2.3. *(Submodule Criterion)*

For a Λ-module $\mathcal{D}$, $\mathcal{P} \subseteq \mathcal{D}$ then $\mathcal{P}$ is said to be a submodule of $\mathcal{D}$ if and only if

1. $\mathcal{P} \neq \emptyset$,

2. $p_1 + \lambda p_2 \in \mathcal{P}, \qquad \forall\, \lambda \in \Lambda$ and $p_1, p_2 \in \mathcal{P}$.

Definition 1.2.4. For any poset P with respect to $\leq$, the following statements are equivalent

1. Any increasing sequence $\lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_r \leq \ldots$ in P is stationary, that is there exist $r \in \mathbb{N}$ for which $\lambda_s = \lambda_r$, for all $s \geq r$.

2. Any $\emptyset \neq H \subset P$ possesses a maximal element.

If P be the set of submodules of $\mathcal{D}$ which is ordered w.r.t the relation $\subseteq$ then statement 1 is known as ascending chain condition and statement 2 is known as the maximal condition.

Definition 1.2.5. Consider a commutative ring Λ, a Λ -module $\mathcal{D}$ is known as Noetherian if each ascending chain of Λ-submodules of $\mathcal{D}$ is stationary. A ring Λ is Noetherian if Λ is Noetherian as a Λ -module.

Definition 1.2.6. Let Λ be a commutative ring. Consider a sequence on Λ-modules

$$\dots \longrightarrow \Upsilon_{k-1} \xrightarrow{h_k} \Upsilon_k \xrightarrow{h_{k+1}} \Upsilon_{k+1} \xrightarrow{h_{k+2}} \dots$$

it is exact at Υ_k if $\mathrm{Im}(h_k) = \ker(h_{k+1})$. If the sequence is exact at each Υ_k, then it is called exact sequence.

Definition 1.2.7. The sequence

$$0 \longrightarrow \Upsilon_1 \xrightarrow{h} \Upsilon \xrightarrow{g} \Upsilon_2 \longrightarrow 0$$

is considered to be short exact sequence if and only if h is one to one, g is onto and $\mathrm{Im}(h) = \ker(g)$.

Definition 1.2.8 (Regular Element)**.** Let Λ be a ring and $\mathcal{D}$ be a module over Λ. An element $0 \neq \lambda \in \Lambda$ is considered a regular element on $\mathcal{D}$ if, for any $\phi \in \mathcal{D}$, the condition $\lambda\phi = 0$ implies that $\phi = 0$.

Definition 1.2.9 (Regular Sequence)**.** A sequence of elements $\lambda_1, \lambda_2, \dots, \lambda_n$ in Λ is referred as $\mathcal{D}$-regular where $\mathcal{D}$ is a Λ-module, if it fulfills the following conditions:

1. λ_i is regular on $\mathcal{D}/(\lambda_1, \lambda_2, \dots, \lambda_{i-1})\,\mathcal{D}$ for all $i = 1, \dots, n$;

2. $\mathcal{D} \neq (\lambda_1, \lambda_2, \dots, \lambda_i)\mathcal{D}$.

1.3 Graded Rings and Graded Modules

Consider a commutative semigroup (w.r.t addition) H. An H-graded ring is such type of a ring Λ alongside a decomposition

$$\Lambda = \bigoplus_{u \in H} \Lambda_u \text{ (as a group)},$$

such that $\Lambda_u \Lambda_v \subset \Lambda_{u+v}\ \forall\ u, v \in H$.
Then for $\lambda \in \Lambda$, we can write a unique expression

$$\lambda = \sum_{u \in H} \lambda_u,$$

where $\lambda_u \in \Lambda_u$ and almost all $\lambda_u = 0$. The element λ_u is called the u^{th} homogeneous component and if $\lambda = \lambda_u$, then λ is homogeneous of degree u. $\Lambda[\lambda]$ and $\Lambda[\lambda, \phi]$ are $\mathbb{Z}$-graded rings as

- $\Lambda[\lambda] = \Lambda \oplus \Lambda\lambda \oplus \Lambda\lambda^2 \oplus \Lambda\lambda^3 \oplus \Lambda\lambda^4 \oplus \Lambda\lambda^5 \oplus \cdots$.
- $\Lambda[\lambda, \phi] = \Lambda \oplus (\Lambda\lambda + \Lambda\phi) \oplus (\Lambda\lambda^2 + \Lambda\lambda\phi + \Lambda\phi^2) \oplus (\Lambda\lambda^3 + \Lambda\lambda^2\phi + \Lambda\lambda\phi^2 + \Lambda\phi^3) \oplus \cdots$.

For an H-graded ring Λ and Λ-module $\mathcal{D}$

$$\mathcal{D} = \bigoplus_{u \in H} \mathcal{D}_u \text{ (as a group)},$$

with $\Lambda_u \mathcal{D}_v \subset \mathcal{D}_{u+v}$ for all $u, v \in H$, then $\mathcal{D}$ is said to be an H-graded module. A non zero element of $\mathcal{D}_u$ is called a homogeneous element of degree u.

For a polynomial ring Λ defined over the field $\mathcal{F}$, suppose $\mathbf{b} \in \mathbb{Z}^n$, then $h \in \Lambda$ is said to be homogeneous of degree $\mathbf{b}$ when h has the form $\beta\lambda^{\mathbf{b}}$, where $\beta \in \mathcal{F}$. Also Λ is $\mathbb{Z}^n$-graded with graded components:

$$\Lambda_{\mathbf{b}} = \begin{cases} \mathcal{F}\lambda^{\mathbf{b}}, & \text{if } \mathbf{b} \in \mathbb{Z}^n_+; \\ 0\,, & \text{otherwise.} \end{cases}$$

A Λ-module $\mathcal{D}$ is $\mathbb{Z}^n$-graded if $\mathcal{D} = \bigoplus_{\mathbf{b} \in \mathbb{Z}^n} \mathcal{D}_{\mathbf{b}}$ and $\Lambda_{\mathbf{b_1}} \mathcal{D}_{\mathbf{b_2}} \subset \mathcal{D}_{\mathbf{b_1}+\mathbf{b_2}}$ for all $\mathbf{b_1}, \mathbf{b_2} \in \mathbb{Z}^n$.

Definition 1.3.1. A graded ideal in the polynomial ring is defined as an ideal $\mathcal{L}$ that is generated by homogeneous elements.

Example 1. Let $S = \mathcal{F}[\lambda_1, \lambda_2, \lambda_3]$ be a polynomial ring, then ideal

(1) $\mathcal{L} = (\lambda_1^2\lambda_2 - \lambda_1\lambda_2^2, \lambda_3^2\lambda_1 - \lambda_1^2\lambda_3, \lambda_2^2\lambda_3)$ is a graded ideal.

(2) $\mathcal{J} = (\lambda_1^3\lambda_2 - \lambda_1\lambda_2^3 + 1, \lambda_1\lambda_3^3 - \lambda_2^3\lambda_3)$ is not a graded ideal.

1.4 Graph Theory

In 18th century, Euler solved the Konigsberg's bridge problem which lead to new branch of mathematics called Graph Theory. Graph Theory is considered as a field of modern mathematics. Anderson et al. [2] proposed the idea of associating a graph to a commutative ring in 1991, which is now commonly used in research. In this section, we discuss some fundamentals of Graph Theory. We also discuss different types of graph which we will use in next chapters. For further details we refer the readers to [37].

1.4.1 Basic Definitions

A graph W is an ordered pair $W = (V(W), E(W))$, where $V(W)$ can be referred as vertex set and $E(W)$ is referred as an edge set. Each edge consists of two vertices which are its endpoints. If e_1 is an edge whose end points are same then e_1 is a loop. If e_2 and e_3 are the edges with exactly the same set of endpoints then e_2 and e_3 are multiple edges. If edges e_2 and e_3 have a common endpoint then they are adjacent edges. Two vertices joined by an edge is known as *adjacent* vertices. Also a graph which has no loops and multiple edges is a *simple* graph. In this thesis, we are considering simple graphs. Number of edges adjacent to a vertex is known as its *degree*. If $V(W_v) = \{\lambda_1, \ldots, \lambda_v\}$ and $E(W_v) = \emptyset$, then a graph W_v is a *null* graph on v vertices. A *bipartite* graph is a graph whose vertex set can be split up into two disjoint independent sets called *partite* sets. A simple graph P_r is a *path* if its vertices can be ordered in such a way that two vertices have an edge between them if and only if they are consecutive in the list. A graph consisting of r vertices ($r \geq 3$) is known as a *cycle* if we join first and last vertices of path graph by an edge. Deleting one edge from a cycle forms a path. A path and cycle on r vertices are represented by P_r and C_r, respectively. When every pair of vertices is connected by an edge, a graph is said to be *complete* for $r \geq 1$ (unless $r = 1$, in which case $E(K_1)$ is empty). Let $s \geq 2$, an $s-star$ denoted by Λ_s is a graph on $s + 1$ vertices, having one internal vertex of degree s and all other vertices having degree 1.

Definition 1.4.1. A vertex λ in a connected graph is a cut vertex whose deletion together with incident edges disconnects the graph.

Definition 1.4.2. If there is no cut vertex in a maximal connected subgraph of W, then it is called a block. W is itself a block if it is connected and has no cut vertex.

Definition 1.4.3. For a set $M \subseteq V(W)$, an induced subgraph of W is a graph $H = (M, E(W'))$ such that $E(W) = \{\{\lambda_i, \lambda_j\} \in E(W) : \lambda_i, \lambda_j \in M\}$.

Definition 1.4.4. Let $W = (V(W), E(W))$ be a graph. The squarefree monomial ideal of W is the edge ideal associated with Λ, such that

$$I(W) = (\lambda_i\lambda_j | \{\lambda_i, \lambda_j\} \in E(W)).$$

Definition 1.4.5. A triangular snake graph Γ_r [35], is a connected graph where all the blocks are triangles, and the block cut vertex graph forms a path. In other words, Γ_r represents a triangular snake graph with r blocks.

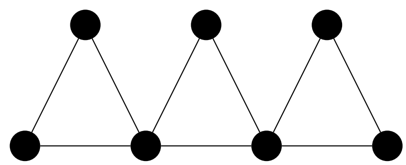


Figure 1.1: Γ_3

Definition 1.4.6. For any values where $r \geq 1$ and $s \geq 2$, the graph $\Gamma_{r,s}$ represents a multi triangular snake. This graph consists of r blocks, and each block contains r triangles, all sharing a common edge. For any values of $r, s \geq 1$, the graph $\Gamma_{r,s}$ is referred to as an s-triangular snake. Specifically, when $s = 1$, $\Gamma_{r,s}$ becomes Γ_r, which is a triangular snake and when $s \geq 2$, $\Gamma_{r,s}$ is called a multi-triangular snake.

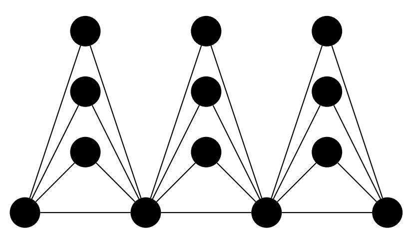


Figure 1.2: $\Gamma_{3,3}$

Definition 1.4.7. If the vertices λ_1 and λ_2 in a graph W are replaced by one new vertex λ, that is every edge that was adjacent to either λ_1 or λ_2, or both, is now adjacent to λ. .

Definition 1.4.8. If we merge vertices λ_1 and λ_{r+1} in the $\Gamma_{r,s}$ graph, we get a new graph denoted $\Omega_{r,s}$, which is called an s-triangular ouroboros snake. In particular, if $s = 1$, then we call $\Omega_{r,1}$ a triangular ouroboros snake, and if $s \geq 2$ then we call $\Omega_{r,s}$ a multi triangular ouroboros snake [29].

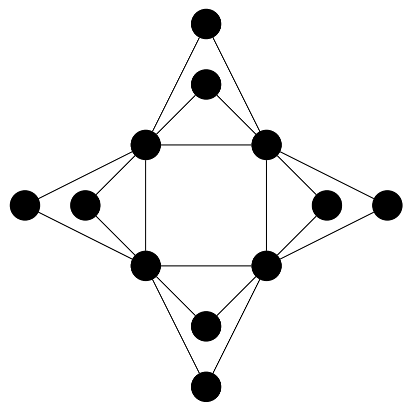
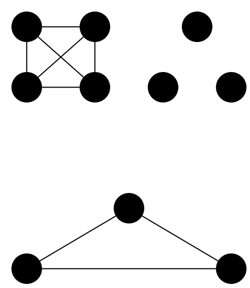


Figure 1.3: $\mathcal{O}_{4,2}$

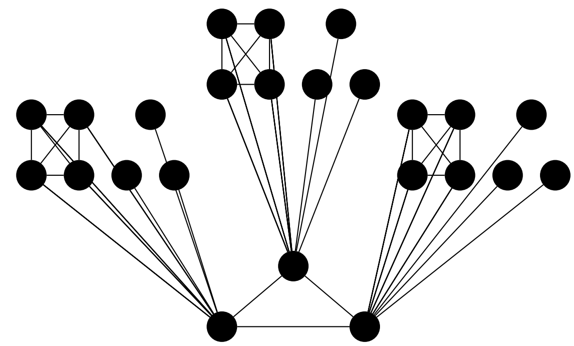
1.4.2 Graph Operations

Definition 1.4.9. Let $W_1 = (V, E)$ and $W_2 = (V', E')$ are two simple graphs. The union of these two graphs is a simple graph having edge set $E \cup E'$ and vertex set $V \cup V'$. The union of W_1 and W_2 is denoted by $W_1 \cup W_2$.

Definition 1.4.10. The Corona Product [13] of two graphs, W_1 and W_2, is obtained by choosing one copy of W_1 and $|V(W_1)|$ copies of W_2. Each vertex of W_1 in this product is connected to every vertex in the corresponding copy of W_2. This connection is established for the j^{th} vertex of W_1 with each vertex in the j^{th} copy of W_2, where $1 \leq j \leq |V(W_1)|$.



(a) Graphs C_3 and $\mathcal{Q}$



(b) $C_3 \odot \mathcal{Q}$

Figure 1.4: Corona product of two graphs

Chapter 2

Depth, Stanley Depth and Regularity

In this chapter, we provide a summary of the currently known values and limitations regarding the depth and Stanley depth of monomial ideals in polynomial rings, as well as their quotients.

2.1 Depth and Stanley Depth

Definition 2.1.1. Consider a Λ-module $\mathcal{D}$. A zero divisor of a module $\mathcal{D}$ is an element $0 \neq \lambda \in \Lambda$ such that $\lambda\phi = 0$, where $0 \neq \phi \in \mathcal{D}$.

Definition 2.1.2. Consider $\mathcal{D}$, a finitely generated Λ-module, and let $\mathcal{L}$ be unique maximal ideal of local Noetherian ring Λ. Then depth of $\mathcal{D}$ is the common length of all maximal $\mathcal{D}$-sequences in $\mathcal{L}$, represented by depth$(\mathcal{D})$.

Definition 2.1.3. Let $\Lambda := \digamma[\lambda_1, \lambda_2, \ldots, \lambda_n]$ be a ring of polynomials and consider $\mathbb{Z}^n$-graded Λ-module $\mathcal{D}$ over field $\digamma$. Suppose $\lambda \in \mathcal{D}$ and also consider $H \subset \{\lambda_1, \ldots, \lambda_n\}$, then $\lambda Z[H]$ represents the Z-subspace of $\mathcal{D}$, whose generating set comprises of elements (homogeneous in degree) of the form λh, where h is a monomial in $Z[H]$. When $\lambda Z[H]$ is a free $Z[H]$-module, it is referred to as a Stanley space with a dimension of $|H|$. The Stanley decomposition of $\mathcal{D}$ is then determined as:

$$M : \mathcal{D} = \bigoplus_{r=1}^{n} \lambda_p Z[H_p]$$

$$\text{sdepth}(A) = \min\{|H_p|: p = 1, 2, \ldots, n\}.$$

Also

$$\text{sdepth}(\mathcal{D}) = \max\{\text{sdepth}(A) : \ A \text{ is a Stanley decomposition of} \mathcal{D}\}.$$

2.1.1 Stanley's Conjecture

In 1982, Stanley gave a conjecture about an upper bound for the depth of a $\mathbb{Z}^n$-graded Λ-modules.

$$\text{depth}(\mathcal{D}) \leq \text{sdepth}(\mathcal{D}).$$

It has been extremely significant as it gave a comparison of two very different invariants of modules. For a ring of polynomials Λ in n number of variables, consider $\mathcal{L} \subset \Lambda$ be the monomial ideal, then for $n \leq 3$, $n = 4$ and $n = 5$ the conjecture for $\Lambda/\mathcal{L}$ is proved by Apel [3], Anwar and Popescu [2, 25], respectively. Also, when $\mathcal{L}$ is an intersection of three monomial prime ideals, or three monomial primary ideals or four monomial prime ideals of Λ , the conjecture holds for $\mathcal{L}$. But in 2016, Duval et al. [11] proved that Stanley's coniecture is generally false, by giving a counter example for the module of type $\Lambda/\mathcal{L}$.

2.1.2 Method to Compute Stanley Depth

In 2009, Herzog et al. gave a method of computing the lower bound for Stanley depth of monomial ideals in finite number of steps by using posets. Assume $\mathcal{L}$ be a squarefree monomial ideal generated by $e_1, \ldots, e_m$. The characteristic poset of $\mathcal{L}$ w.r.t $g = (1, \ldots, 1)$, written as $\mathcal{P}_{\mathcal{L}}^{(1,\ldots,1)}$ is defined as

$$\mathcal{P}_{\mathcal{L}}^{(1,\ldots,1)} = \{\beta \subset [n] \mid \beta \text{ contains } \operatorname{supp}(e_j) \text{ for some } j\},$$

where $\operatorname{supp}(e_j) = \{i : \lambda_i | e_j\} \subseteq [n] := \{1, \ldots, n\}$. For each $\rho, \phi \in \mathcal{P}_{\mathcal{L}}^{(1,\ldots,1)}$ where $\rho \subseteq \phi$, and

$$[\rho\,,\,\phi] = \{\beta \in \mathcal{P}_{\mathcal{L}}^{(1,\ldots,1)} : \rho \subseteq \beta \subseteq \phi\}.$$

Let $\mathcal{P} : \mathcal{P}_{\mathcal{L}}^{(1,\ldots,1)} = \cup_{j=1}^{l}[\beta_j \,,\, \eta_j]$ be a partition of $\mathcal{P}_{\mathcal{L}}^{(1,\ldots,1)}$, and for every j, suppose $s(j) \in \{0,1\}^n$ is the tuple with $\text{supp}(\lambda^{s(j)}) = \beta_j$, then the Stanley decomposition $\mathcal{D}(\mathcal{P})$ of $\mathcal{L}$ is given by

$$\mathcal{D}(\mathcal{P}) \,:\, \mathcal{L} = \bigoplus_{j=1}^{r} \lambda^{s(j)} \digamma[\{\lambda_l \,|\, l \in \eta_j\}].$$

Clearly, $\text{sdepth}\, \mathcal{D}(\mathcal{P}) = \min\{|\eta_1|, \ldots, |\eta_r|\}$ and

$$\text{sdepth}(\mathcal{L}) = \max\{\text{sdepth}\, \mathcal{D}(\mathcal{P}) \mid \mathcal{P} \quad \text{is a partition of } \mathcal{P}_{\mathcal{L}}^{(1,\ldots,1)}\}.$$

Example 2.1.4. Consider $\mathcal{L} = (\lambda_1\lambda_4, \lambda_1\lambda_2, \lambda_2\lambda_4, \lambda_1\lambda_3) \subset \digamma[\lambda_1, \lambda_2, \lambda_3, \lambda_4]$ be a square-free monomial ideal and $\mathcal{J} = 0$. Set $\alpha_1 = (1,0,0,1)$, $\alpha_2 = (1,1,0,0)$, $\alpha_3 = (0,1,0,1)$ and $\alpha_4 = (1,0,1,0)$. Thus $\mathcal{L}$ is generated by $\lambda^{\alpha_1}, \lambda^{\alpha_2}, \lambda^{\alpha_3}, \lambda^{\alpha_4}$ and choose $g = (1,1,1,1)$. The poset $r = P_{\mathcal{L}/\mathcal{J}}^{g}$ is given by

$$\begin{aligned} t = \{&(1,1,0,0),(1,0,1,0),(1,0,0,1),(0,1,0,1),(1,1,1,0),(1,1,0,1),(1,0,1,1),\\ &(0,1,1,1),(1,1,1,1)\}. \end{aligned}$$

Partitions of P are given by

$$\begin{aligned} \mathcal{P}_1 :\ & [(1,1,0,0),(1,1,0,0)] \bigcup [(1,0,1,0),(1,0,1,0)] \bigcup [(0,1,0,1),(0,1,0,1)] \bigcup \\ & [(1,0,0,1),(1,0,0,1)] \bigcup [(1,1,1,0),(1,1,1,0)] \bigcup [(1,1,0,1),(1,1,0,1)] \bigcup \\ & [(1,0,1,1),(1,0,1,1)] \bigcup [(0,1,1,1),(0,1,1,1)] \bigcup [(1,1,1,1),(1,1,1,1)]. \end{aligned}$$

$$\begin{aligned} \mathcal{P}_2 :\ & [(1,1,0,0),(1,1,1,0)] \bigcup [(1,0,0,1),(1,1,0,1)] \bigcup [(1,0,1,0),(1,0,1,1)] \bigcup \\ & [(0,1,0,1),(0,1,1,1)] \bigcup [(1,1,1,1),(1,1,1,1)]. \end{aligned}$$

and the corresponding Stanley decomposition is

$$\begin{aligned} \mathcal{D}(\mathcal{P}_1) \ :=\ & \lambda_1\lambda_2\digamma[\lambda_1,\lambda_2] \oplus \lambda_1\lambda_3\digamma[\lambda_1,\lambda_3] \oplus \lambda_1\lambda_4\digamma[\lambda_1,\lambda_4] \oplus \lambda_2\lambda_4\digamma[\lambda_2,\lambda_4] \oplus \\ & \lambda_2\lambda_3\lambda_4\digamma[\lambda_2,\lambda_3,\lambda_4] \oplus \lambda_1\lambda_2\lambda_4\digamma[\lambda_1,\lambda_2,\lambda_4] \oplus \lambda_1\lambda_3\lambda_4\digamma[\lambda_1,\lambda_3,\lambda_4] \oplus \\ & \lambda_1\lambda_2\lambda_3\digamma[\lambda_1,\lambda_2,\lambda_3] \oplus \lambda_1\lambda_2\lambda_3\lambda_4\digamma[\lambda_1,\lambda_2,\lambda_3,\lambda_4]. \end{aligned}$$

$$\begin{aligned}\mathcal{D}(\mathcal{P}_2) \ := \ & \lambda_1\lambda_3 F[\lambda_1,\lambda_3,\lambda_4] \oplus \lambda_1\lambda_4 F[\lambda_1,\lambda_2,\lambda_4] \oplus \lambda_1\lambda_2 F[\lambda_1,\lambda_2,\lambda_3] \oplus \lambda_2\lambda_4 F[\lambda_2,\lambda_3,\lambda_4] \oplus \\ & \lambda_1\lambda_2\lambda_3\lambda_4 F[\lambda_1,\lambda_2,\lambda_3,\lambda_4].\end{aligned}$$

Then

$$\begin{aligned}\text{sdepth}(\mathcal{L}) \ & \geq \ \max\{\text{sdepth}(\mathcal{D}(\mathcal{P}_1))\,,\,\text{sdepth}(\mathcal{D}(\mathcal{P}_2))\} \\ & \geq \ \max\{2,3\} \\ & \geq \ 3.\end{aligned}$$

Example 2.1.5. For $\Lambda = F[\lambda_1,\lambda_2,\lambda_3,\lambda_4,\lambda_5]$, consider $\mathcal{L} = (\lambda_1\lambda_5, \lambda_2\lambda_3\lambda_4, \lambda_1\lambda_2, \lambda_1\lambda_4)$. Then choose $g = (1,1,1,1,1)$ and the poset $P = P^g_{\Lambda/\mathcal{Q}}$ is given by

$$\begin{aligned}P = \{ & (0,0,0,0,0),(1,0,0,0,0),(0,1,0,0,0),(0,0,1,0,0),(0,0,0,1,0),(0,0,0,0,1), \\ & (1,0,1,0,0),(0,1,1,0,0),(0,1,0,1,0),(0,1,0,0,1),(0,0,1,1,0),(0,0,1,0,1), \\ & (0,0,0,1,1),(0,1,1,0,1),(0,1,0,1,1),(0,0,1,1,1)\}.\end{aligned}$$

Partitions of P are given by

$$\begin{aligned}\mathcal{P}_1 : \ & [(0,0,0,0,0),(0,0,1,1,1)] \bigcup [(1,0,0,0,0),(1,0,0,0,0)] \bigcup \\ & [(0,1,0,0,0),(0,1,0,0,0)] \bigcup [(0,0,1,0,0),(0,0,1,0,0)] \bigcup \\ & [(0,0,0,1,0),(0,0,0,1,0)] \bigcup [(0,0,0,0,1),(0,0,0,0,1)] \bigcup \\ & [(1,0,1,0,0),(1,0,1,0,0)] \bigcup [(0,1,1,0,0),(0,1,1,0,0)] \bigcup \\ & [(0,1,0,1,0),(0,1,0,1,0)] \bigcup [(0,1,0,0,1),(0,1,0,0,1)] \bigcup \\ & [(0,0,1,1,0),(0,0,1,1,0)] \bigcup [(0,0,1,0,1),(0,0,1,0,1)] \bigcup \\ & [(0,0,0,1,1),(0,0,0,1,1)] \bigcup [(0,1,1,0,1),(0,1,1,0,1)] \bigcup \\ & [(0,1,0,1,1),(0,1,0,1,1)].\end{aligned}$$

$$\mathcal{P}_2 : [(0,0,0,0,0),(1,0,1,0,0)] \bigcup [(0,1,0,0,0),(0,1,1,0,0)] \bigcup$$
$$[(0,0,0,1,0),(0,1,0,1,0)] \bigcup [(0,0,0,0,1),(0,1,0,0,1)] \bigcup$$
$$[(0,0,0,1,1),(0,1,0,1,1)] \bigcup [(0,0,1,0,1),(0,1,1,0,1)] \bigcup$$
$$[(0,0,1,1,0),(0,0,1,1,1)].$$

and the corresponding Stanley decomposition is

$$\begin{aligned}\mathcal{D}(\mathcal{P}_1) := {} & F[\lambda_3,\lambda_4,\lambda_5] \oplus \lambda_1 F[\lambda_1] \oplus \lambda_2 F[\lambda_2] \oplus \lambda_3 F[\lambda_3] \oplus \lambda_4 F[\lambda_4] \oplus \lambda_5 F[\lambda_5] \oplus \\ & \lambda_1\lambda_3 F[\lambda_1,\lambda_3] \oplus \lambda_2\lambda_3 F[\lambda_2,\lambda_3] \oplus \lambda_2\lambda_4 F[\lambda_2,\lambda_4] \oplus \lambda_2\lambda_5 F[\lambda_2,\lambda_5] \oplus \\ & \lambda_3\lambda_4 F[\lambda_3,\lambda_4] \oplus \lambda_3\lambda_5 F[\lambda_3,\lambda_5] \oplus \lambda_4\lambda_5 F[\lambda_4,\lambda_5] \oplus \lambda_2\lambda_3\lambda_5 F[\lambda_2,\lambda_3,\lambda_5] \oplus \\ & \lambda_2\lambda_4\lambda_5 F[\lambda_2,\lambda_4,\lambda_5].\end{aligned}$$

$$\begin{aligned}\mathcal{D}(\mathcal{P}_2) := {} & F[\lambda_1,\lambda_3] \oplus \lambda_2 F[\lambda_2,\lambda_3] \oplus \lambda_4 F[\lambda_2,\lambda_4] \oplus \lambda_5 F[\lambda_2,\lambda_5] \oplus \lambda_4\lambda_5 F[\lambda_2,\lambda_4,\lambda_5] \oplus \\ & \lambda_3\lambda_5 F[\lambda_2,\lambda_3,\lambda_5] \oplus \lambda_3\lambda_4 F[\lambda_3,\lambda_4,\lambda_5].\end{aligned}$$

Then

$$\begin{aligned}\operatorname{sdepth}(\Lambda/\mathcal{L}) & \geq \max\{\operatorname{sdepth}(\mathcal{D}(\mathcal{P}_1)), \operatorname{sdepth}(\mathcal{D}(\mathcal{P}_2))\} \\ & \geq \max\{1,2\} \\ & \geq 2.\end{aligned}$$

2.1.3 Some Known Results for Depth and Stanley Depth

Following are the results in which the classes of monomial ideals including monomial edge ideals and residue class rings of monomial ideals are considered for computing their depth as well as Stanley depth.

Lemma 2.1.6 ([14])**.** *(Depth Lemma) Let Υ_1, Υ_2 and Υ_3 be $\mathbb{Z}^n$-graded Λ-modules and*

$$0 \longrightarrow \Upsilon_1 \longrightarrow \Upsilon_2 \longrightarrow \Upsilon_3 \longrightarrow 0$$

be a short exact sequence. Then

(a) $\operatorname{depth}(\Upsilon_2) \geq \min\{\operatorname{depth}(\Upsilon_3), \operatorname{depth}(\Upsilon_1)\}$.

(b) $\operatorname{depth}(\Upsilon_1) \geq \min\{\operatorname{depth}(\Upsilon_2), \operatorname{depth}(\Upsilon_3)+1\}$.

(c) $\operatorname{depth}(\Upsilon_3) \geq \min\{\operatorname{depth}(\Upsilon_1)-1, \operatorname{depth}(\Upsilon_2)\}$.

Proposition 2.1.7 ([27, Lemma 2.2]). *Consider a short exact sequence* $0 \longrightarrow \Upsilon_1 \longrightarrow \Upsilon_2 \longrightarrow \Upsilon_3 \longrightarrow 0$ *of* $\mathbb{Z}^n$*-graded* Λ*-modules. In this case, we have the inequality*

$$\operatorname{sdepth}(\Upsilon_2) \geq \min\{\operatorname{sdepth}(\Upsilon_1), \operatorname{sdepth}(\Upsilon_3)\}.$$

Lemma 2.1.8 ([14, Lemma 3.6]). *Let* $\mathcal{L}$ *and* $\mathcal{J}$ *be two monomial ideals with* $\mathcal{J} \subset \mathcal{L}$*, suppose* $\Lambda' = \Lambda[\lambda_{n+1}]$*. Then*

(a) $\operatorname{depth}(\mathcal{L}\Lambda'/\mathcal{J}\Lambda') = \operatorname{depth}(\mathcal{L}\Lambda/\mathcal{J}\Lambda)+1$.

(b) $\operatorname{sdepth}(\mathcal{L}\Lambda'/\mathcal{J}\Lambda') = \operatorname{sdepth}(\mathcal{L}\Lambda/\mathcal{J}\Lambda)+1$.

Lemma 2.1.9 ([27, Theorem 3.1]). *Consider two monomial ideals such that* $\mathcal{L} \subset \Lambda' = \digamma[\lambda_1, \ldots, \lambda_u]$ *and* $\mathcal{J} \subset \Lambda'' = \digamma[\lambda_{u+1}, \ldots, \lambda_n]$ *with* $1 \leq u < n$*. If* $\Lambda = \Lambda' \otimes_K \Lambda''$*, then*

(a) $\operatorname{depth}_\Lambda(\Lambda'/\mathcal{L} \otimes_K \Lambda''/\mathcal{J}) = \operatorname{depth}_\Lambda(\Lambda/(\mathcal{L}\Lambda+\mathcal{J}\Lambda)) = \operatorname{depth}_{\Lambda'}(\Lambda^{'}/\mathcal{L})+\operatorname{depth}_{\Lambda''}(\Lambda^{''}/\mathcal{J})$.

(b) $\operatorname{sdepth}_\Lambda(\Lambda'/\mathcal{L} \otimes_K \Lambda''/\mathcal{J}) = \operatorname{sdepth}_\Lambda(\Lambda/(\mathcal{L}\Lambda+\mathcal{J}\Lambda)) \geq \operatorname{sdepth}_{\Lambda'}(\Lambda^{'}/\mathcal{L})+\operatorname{sdepth}_{\Lambda''}(\Lambda^{''}/\mathcal{J})$.

Corollary 2.1.10 ([27, Corollary 1.3]). *Assume a proper monomial ideal* $\mathcal{L}$ *of* Λ *and* $\lambda \notin \mathcal{L}$*. Then*

$$\operatorname{depth}(\Lambda/(\mathcal{L} : \lambda)) \geq \operatorname{depth}(\Lambda/\mathcal{L}).$$

Proposition 2.1.11 ([38, Proposition 2.7]). *Assume a proper monomial ideal* $\mathcal{L}$ *of* Λ *and* $\lambda \notin \mathcal{L}$*. Then*

$$\operatorname{sdepth}(\Lambda/(\mathcal{L} : \lambda)) \geq \operatorname{sdepth}(\Lambda/\mathcal{L}).$$

Lemma 2.1.12 ([30, Lemma 2.2]). *Consider a null graph* W_1*. Let* $\Lambda = \digamma[V(W_1 \odot \mathcal{Q})]$ *and* $I(W_1 \odot \mathcal{Q})$ *be the edge ideal. Then*

$$\operatorname{depth}(\Lambda/I(W_1 \odot \mathcal{Q})) = \operatorname{sdepth}(\Lambda/I(W_1 \odot \mathcal{Q})) = 1.$$

Lemma 2.1.13 ([1, Theorem 2.6]). *Let* $\Lambda = \digamma[V(\Lambda_s)]$ *and* $I(\Lambda_s)$ *be an edge ideal of* s*-star. Then*

$$\operatorname{depth}(\Lambda/I(\Lambda_s)) = \operatorname{sdepth}(\Lambda/I(\Lambda_s)) = 1.$$

Theorem 2.1.14 ([30, Theorem 2.17]). *Let $\Lambda = \digamma[V(\Lambda_s \odot \mathcal{Q})]$. Then*

(a) $\operatorname{depth}(\Lambda/I(\Lambda_s \odot \mathcal{Q})) = m + \operatorname{depth}(\digamma[V(\mathcal{Q})]/I(\mathcal{Q})) + |i(\mathcal{Q})|$.

(b) $\operatorname{sdepth}(\Lambda/I(\Lambda_s \odot \mathcal{Q})) \geq m + \operatorname{sdepth}(\digamma[V(\mathcal{Q})]/I(\mathcal{Q})) + |i(\mathcal{Q})|$.

Lemma 2.1.15 ([21, Lemma 2.8]). *Let $\mathcal{L} = I(P_r)$ be an edge ideal of a path graph on r vertices and $r \geq 2$. Then*

(a) $\operatorname{depth}(\Lambda/\mathcal{L}) = \lceil \frac{r}{3} \rceil$.

(b) $\operatorname{sdepth}(\Lambda/\mathcal{L}) = \lceil \frac{r}{3} \rceil$.

Theorem 2.1.16 ([30, Theorem 2.5]). *Let $r \geq 1$ and $\Lambda = \digamma[V(P_r \odot \mathcal{Q})]$. Then*

(a) $\operatorname{depth}(\Lambda/I(P_r \odot \mathcal{Q})) = \lceil \frac{r}{2} \rceil + \lceil \frac{r-1}{2} \rceil (\operatorname{depth}(\digamma[V(\mathcal{Q})]/I(\mathcal{Q})) + |i(\mathcal{Q})|)$.

(b) $\operatorname{sdepth}(\Lambda/I(P_r \odot \mathcal{Q})) \geq \lceil \frac{r}{2} \rceil + \lceil \frac{r-1}{2} \rceil (\operatorname{sdepth}(\digamma[V(\mathcal{Q})]/I(\mathcal{Q})) + |i(\mathcal{Q})|)$.

To be more specific, if $\mathcal{Q}$ is a null graph, then

$$\operatorname{sdepth}(\Lambda/I(P_r \odot \mathcal{Q})) = \lceil \frac{r}{2} \rceil + \lceil \frac{r-1}{2} \rceil |V(\mathcal{Q})|.$$

Proposition 2.1.17 ([9, Proposition 1.3]). *Let $\mathcal{L} = I(C_r)$ be an edge ideal of ring Λ associated with a cycle graph. Then*

(a) $\operatorname{depth}(\Lambda/\mathcal{L}) = \lceil \frac{r-1}{3} \rceil$.

(b) $\operatorname{sdepth}(\Lambda/\mathcal{L}) \geq \lceil \frac{r-1}{3} \rceil$.

Theorem 2.1.18 ([30, Theorem 2.9]). *If $r \geq 3$ and $\Lambda = \digamma[V(C_r \odot \mathcal{Q})]$, then*

(a) $\operatorname{depth}(\Lambda/I(C_r \odot \mathcal{Q})) = \lceil \frac{r-1}{2} \rceil + \lceil \frac{r}{2} \rceil (\operatorname{depth}(\digamma[V(\mathcal{Q})]/I(\mathcal{Q})) + |i(\mathcal{Q})|)$.

(b) $\operatorname{sdepth}(\Lambda/I(C_r \odot \mathcal{Q})) \geq \lceil \frac{r-1}{2} \rceil + \lceil \frac{r}{2} \rceil (\operatorname{sdepth}(\digamma[V(\mathcal{Q})]/I(\mathcal{Q})) + |i(\mathcal{Q})|)$.

To be more specific, if $\mathcal{Q}$ is a null graph, then

$$\operatorname{sdepth}(\Lambda/I(C_r \odot \mathcal{Q})) = \lceil \frac{r-1}{2} \rceil + \lceil \frac{r}{2} \rceil |V(\mathcal{Q})|.$$

Lemma 2.1.19 ([6, Theorems 1.3.3]). *If Λ is a Noetherian local ring that is also a commutative ring, and $\mathcal{D}$ is a non-zero finite Λ-module of finite projective dimension, then*

$$\operatorname{pdim}(\mathcal{D}) + \operatorname{depth}(\mathcal{D}) = \operatorname{depth}(\Lambda).$$

2.2 Regularity

Consider a field $\digamma$ and a polynomial ring in n variables over $\digamma$ denoted by $\Lambda := \digamma[\lambda_1, \dots, \lambda_n]$. Suppose we have a finitely generated $\mathbb{Z}$-graded Λ-module $\mathcal{D}$, and it possesses a minimal free resolution.

$$0 \longrightarrow \bigoplus_{k\in\mathbb{Z}} \Lambda(-k)^{\phi_{l,k}(\mathcal{D})} \longrightarrow \bigoplus_{k\in\mathbb{Z}} \Lambda(-k)^{\phi_{l-1,k}(\mathcal{D})} \longrightarrow \dots \longrightarrow \bigoplus_{k\in\mathbb{Z}} \Lambda(-k)^{\phi_{0,k}(\mathcal{D})} \longrightarrow \mathcal{D} \longrightarrow 0,$$

then the regularity and projective dimension of C can be determined by $\operatorname{reg}(\mathcal{D}) = \max\{k - l : \phi_{l,k}(\mathcal{D}) \notin 0\}$ and $\operatorname{pdim}(\mathcal{D}) = \max\{l : \phi_{l,k}(\mathcal{D}) \notin 0\}$, respectively. The role of regularity as a key indicator of module's complexity is significant and serves as an important invariant in the field of Commutative Algebra. Numerous researchers have examined the values and limitations of regularity and projective dimension of edge ideals.

2.2.1 Some Known Results for Regularity

Theorem 2.2.1 ([7, Theorems 4.7]). *Assume a monomial ideal $\mathcal{L}$ with λ as a variable of Λ. Then*

(a) $\operatorname{reg}(\Lambda/\mathcal{L}) = \operatorname{reg}\Lambda/(\mathcal{L} : \lambda) + 1,$ *if* $\operatorname{reg}(\Lambda/\mathcal{L} : \lambda) > \operatorname{reg}\Lambda/(\mathcal{L}, \lambda),$

(b) $\operatorname{reg}(\Lambda/\mathcal{L}) \in \{\operatorname{reg}\Lambda/(\mathcal{L}, \lambda) + 1, \operatorname{reg}\Lambda/(\mathcal{L}, \lambda)\},$ *if* $\operatorname{reg}(\Lambda/\mathcal{L} : \lambda) = \operatorname{reg}\Lambda/(\mathcal{L}, \lambda)$ *and*

(c) $\operatorname{reg}(\Lambda/\mathcal{L}) = \operatorname{reg}(\lambda/\mathcal{L}, \lambda)$ *if* $\operatorname{reg}(\lambda/\mathcal{L} : \lambda) < \operatorname{reg}\Lambda/(\mathcal{L}, \lambda).$

Lemma 2.2.2 ([36, Lemma 8]). *Consider two disjoint graphs $\mathcal{V}_1$ and $\mathcal{V}_2$, and let $\mathcal{A} = \mathcal{V}_1 \cup \mathcal{V}_2$. Then*

$$\operatorname{reg}(\digamma[V(\mathcal{A})]/I(\mathcal{A})) = \operatorname{reg}(\digamma[V(\mathcal{A})]/I(\mathcal{V}_1)) + \operatorname{reg}(\digamma[V(\mathcal{A})]/I(\mathcal{V}_2)).$$

Lemma 2.2.3 ([22, Lemma 3.6]). *Let $\mathcal{J} \subset \Lambda = \digamma[\lambda_1, \dots, \lambda_n]$ be a monomial ideal and $\Lambda' = \Lambda \bigotimes_K \digamma[\lambda_{n+1}, \dots, \lambda_{n+s}]$. Then*

$$\operatorname{reg}(\Lambda'/\mathcal{J}) = \operatorname{reg}(\Lambda/\mathcal{J}),$$

Theorem 2.2.4 ([30, Lemma 2.26])**.** *Let $s \geq 1$. If $\Lambda = \mathcal{F}[V(\Lambda_s)]$ and $I(\Lambda_s)$ be an edge ideal of an s-star, then* $\operatorname{reg}(\Lambda/I(\Lambda_s)) = 1$.

Lemma 2.2.5 ([30, Lemma 2.26])**.** *Consider a ring $\Lambda = \mathcal{F}[V(W_1 \odot \mathcal{Q})]$. Then*

$$\operatorname{reg}(\Lambda/I(W_1 \odot \mathcal{Q})) = \begin{cases} 1\,, & \textit{if } \mathcal{Q} \textit{ is a null graph;} \\ \operatorname{reg}(\mathcal{F}[V(\mathcal{Q})]/I(\mathcal{Q}))\,, & \textit{otherwise.} \end{cases}$$

Theorem 2.2.6 ([30, Theorem 2.34])**.** *Consider a ring Λ such that $\Lambda = \mathcal{F}[V(\Lambda_s \odot \mathcal{Q})]$ where Λ_s is an s-star on $s+1$ vertices. Then*

$$\operatorname{reg}(\Lambda/I(\Lambda_s \odot \mathcal{Q})) = \begin{cases} s\,, & \textit{if } \mathcal{Q} \textit{ is a null graph;} \\ (s+1).reg(\mathcal{F}[V(\mathcal{Q})]/I(\mathcal{Q}))\,, & \textit{otherwise.} \end{cases}$$

Chapter 3

Depth, Stanley Depth and Projective Dimension of Cyclic Modules Associated with Corona Product Of Certain Graphs

In this chapter, we compute Stanley depth, depth and projective dimension of edge ideal associated with corona product of multi-triangular graph $\Gamma_{r,s}$ with any graph $\mathcal{Q}$. The process of computing exact values of these invariants involves the use of short exact sequences and some important results of the Stanley depth, depth and projective dimension.

3.1 Depth, Stanley Depth and Projective Dimension of Cyclic Module Associated with $\Gamma_{r,s} \odot \mathcal{Q}$

Consider a graph $\mathcal{Q}$ with w vertices such that $|V(\mathcal{Q})|= w$. Let $r, s \geq 1$. Then corona product of multi triangular snake graph $\Gamma_{r,s}$ with any graph $\mathcal{Q}$ denoted by $\Gamma_{r,s} \odot \mathcal{Q}$ is obtained by connecting all vertices of $\mathcal{Q}$ to every vertex of $\Gamma_{r,s}$. If $s = 1$ then $\Gamma_{r,s} = \Gamma_{r,1} = \Gamma_r$ a triangular snake graph. Clearly $|V(\Gamma_{r,s} \odot \mathcal{Q})|= |V(\Gamma_{r,s})|(|V(\mathcal{Q})|+1)$ and $E(\Gamma_{r,s} \odot \mathcal{Q}) = |E(\Gamma_{r,s})|+(|V(\Gamma_{r,s})||E(\mathcal{Q})|) + (|V(\Gamma_{r,s})||E(\mathcal{Q})|)$. The vertices of the $\Gamma_{r,s} \odot \mathcal{Q}$ are labelled by using the following sets of variables $\{\mu_1, \mu_2, \ldots, \mu_{r+1}\}$,

$\{\mu_{r1}, \mu_{r2}, \ldots, \mu_{rw}\}$,$\{\sigma_{r1}, \sigma_{r2}, \ldots, \sigma_{rs}\}$ and $\{\sigma_{rs1}, \sigma_{rs2}, \ldots, \sigma_{rsw}\}$. For example see figure 3.1. Also we denote the edge ideal of polynomial ring $\Lambda_{r,s} = F[V(\Gamma_{r,s} \odot \mathcal{Q})]$ by $I_{r,s}$.

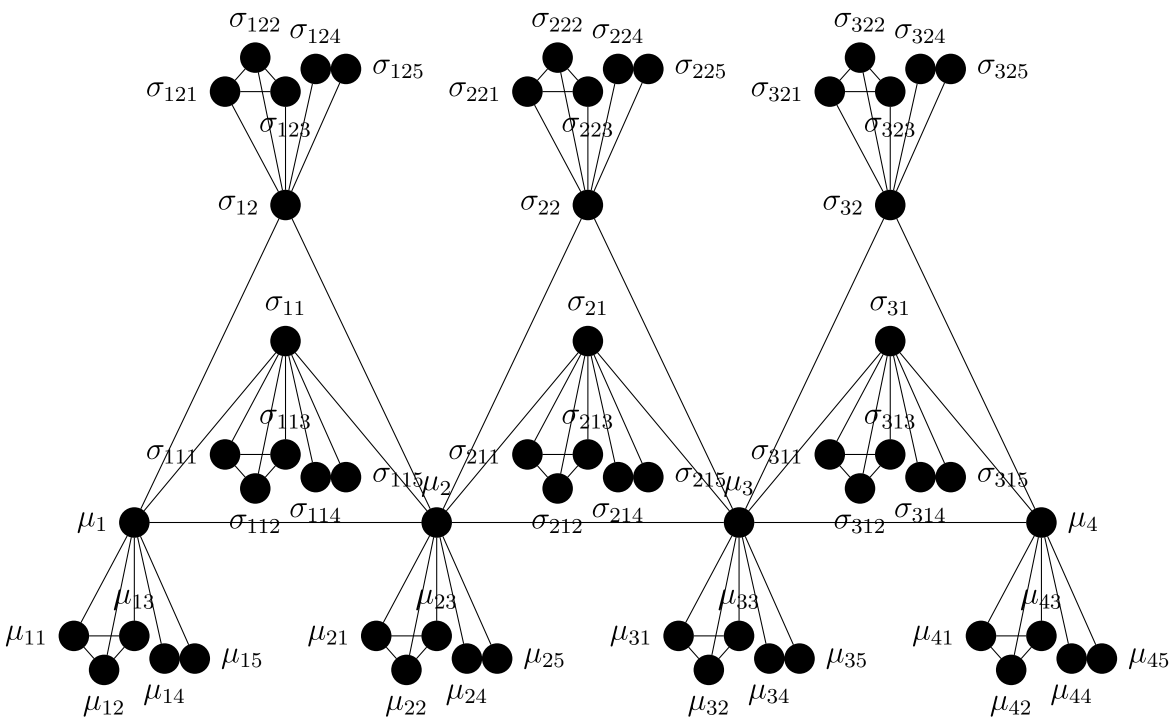


Figure 3.1: $\Gamma_{3,2} \odot \mathcal{Q}$

Let us consider a supergraph $\Gamma'_{r,s} \odot \mathcal{Q}$ of the graph $\Gamma_{r,s} \odot \mathcal{Q}$. The vertex and edge sets of $\Gamma'_{r,s} \odot \mathcal{Q}$ are $|V(\Gamma'_{r,s} \odot \mathcal{Q})| = |V(\Gamma'_{r,s})|(|V(\mathcal{Q})|+1) = V(T_{r,s}) \bigcup \{\sigma_{(r+1)1}, \sigma_{(r+1)2}, \ldots, \sigma_{(r+1)s}\}$ and $E(\Gamma'_{r,s} \odot \mathcal{Q}) = |E(\Gamma'_{r,s})| + (|V(\Gamma'_{r,s})||V(\mathcal{Q})|) + (|V(\Gamma'_{r,s})||E(\mathcal{Q})|) = E(T_{r,s}) \bigcup \{\{\mu_{r+1}, \sigma_{(r+1)k}\} : 1 \leq k \leq s\}$. For example of graph $\Gamma'_{r,s} \odot \mathcal{Q}$ see figure 3.2. We denote the edge ideal of graph $\Gamma'_{r,s} \odot \mathcal{Q}$ with $I'_{r,s}$, which is monomial ideal of the polynomial ring $\Lambda'_{r,s} = F[V(\Gamma'_{r,s} \odot \mathcal{Q})]$.

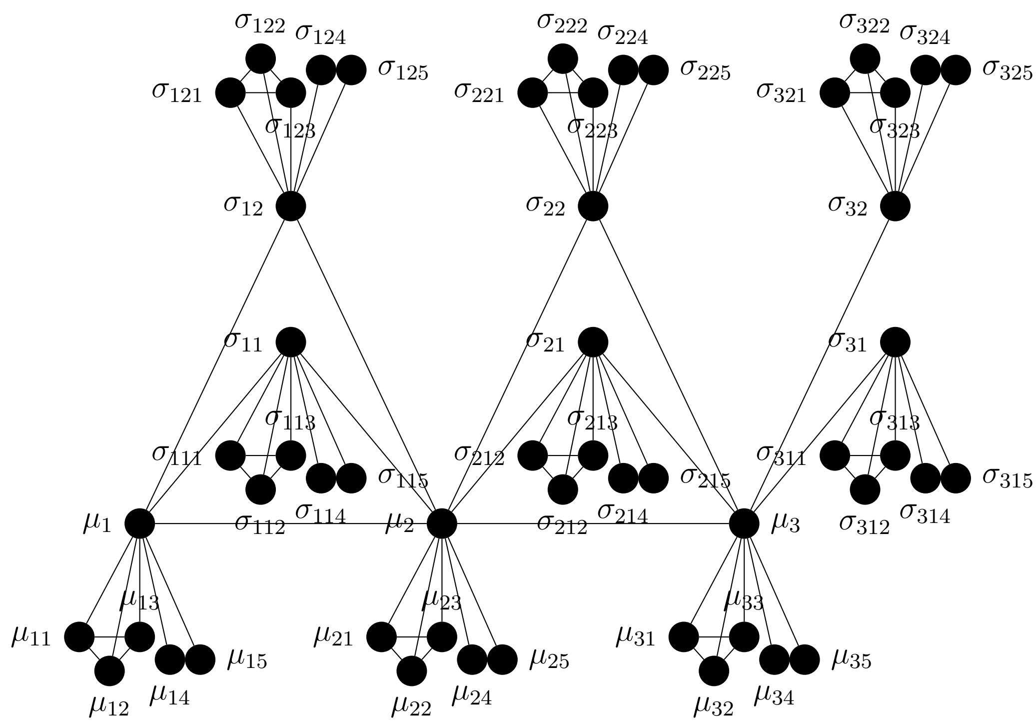


Figure 3.2: $\Gamma'_{2,2} \odot \mathcal{Q}$

Clearly $\Gamma_{r,s}$ is a connected graph. It is important to note that the graph $\Gamma_{r,s} \odot \mathcal{Q}$ does not contain any isolated vertices, regardless of whether $\mathcal{Q}$ itself has isolated vertices. Therefore, we allow the presence of isolated vertices in the graph $\mathcal{Q}$. The presence of isolated vertices within $\mathcal{Q}$ is of great importance in our findings.

Remark 3.1.1. Let $\mathcal{Q}$ be a graph, and we denote the set of isolated vertices in $\mathcal{Q}$ as $i(\mathcal{Q})$. If we define $C := V(\mathcal{Q}) \setminus i(\mathcal{Q})$, we use $\mathcal{Q}'$ to represent the induced subgraph of $\mathcal{Q}$ on the vertex set C. It is clear that $|V(\mathcal{Q})| = |i(\mathcal{Q})| + |C|$ and $I(\mathcal{Q}) = (q_i q_j, q_k : \{q_i, q_j\} \in E(\mathcal{Q}'))$ and $q_k \in i(\mathcal{Q}))$. Also $\digamma[V(\mathcal{Q})]/I(\mathcal{Q}) \cong \digamma[V(\mathcal{Q}')]/I(\mathcal{Q}')$.

Lemma 3.1.2. *Let $r, s \geq 1$. If $\Lambda'_{r,s} = \digamma[V(\Gamma'_{r,s} \odot \mathcal{Q})]$ and $I'_{r,s} = I(\Gamma'_{r,s} \odot \mathcal{Q})$, then*

(1) $\operatorname{depth}(\Lambda'_{r,s}/I'_{r,s}) = (r+1)(\operatorname{depth}(\digamma[V(\mathcal{Q})]/I(\mathcal{Q})) + |i(\mathcal{Q})| + s)$.

(2) $\operatorname{sdepth}(\Lambda'_{r,s}/I'_{r,s}) = (r+1)(\operatorname{sdepth}(\digamma[V(\mathcal{Q})]/I(\mathcal{Q})) + |i(\mathcal{Q})| + s)$.

Proof. We will prove this result by induction on r. Following is the sequence which is short exact

$$0 \longrightarrow \Lambda'_{r,s}/(I'_{r,s} : \mu_{r+1}) \xrightarrow{\cdot \mu_{r+1}} \Lambda'_{r,s}/I'_{r,s} \longrightarrow \Lambda'_{r,s}/(I'_{r,s}, \mu_{r+1}) \longrightarrow 0.$$

By Depth Lemma, we have

$$\operatorname{depth}(\Lambda'_{r,s}/I'_{r,s}) \geq \min\{\operatorname{depth}(\Lambda'_{r,s}/(I'_{r,s} : \mu_{r+1})), \operatorname{depth}(\Lambda'_{r,s}/(I'_{r,s}, \mu_{r+1}))\}.$$

If $r = 1$, then we have the following isomorphism:

$$\Lambda'_{1,s}/(I'_{1,s} : \mu_2) \cong \bigotimes_{k=1}^{2s+1} \mathcal{F}[V(\mathcal{Q})]/I(\mathcal{Q}) \bigotimes_{k=1}^{2s+1} \mathcal{F}[i(\mathcal{Q})] \bigotimes_{K} \mathcal{F}[\mu_2].$$

Then by Lemma 2.1.9, we have

$$\begin{aligned}\operatorname{depth}(\Lambda'_{1,s}/(I'_{1,s} : \mu_2)) = &\sum_{k=1}^{2s+1} \operatorname{depth}(\mathcal{F}[V(\mathcal{Q})]/I(\mathcal{Q})) + \sum_{k=1}^{2s+1} \operatorname{depth}(\mathcal{F}[i(\mathcal{Q})]) \\ &+ \operatorname{depth}(\mathcal{F}[\mu_2]),\end{aligned}$$

which implies that

$$\operatorname{depth}(\Lambda'_{1,s}/(I'_{1,s} : \mu_2)) = (2s+1)(\operatorname{depth}(\mathcal{F}[V(\mathcal{Q})]/I(\mathcal{Q})) + |i(Q)|) + 1.$$

Similarly

$$\begin{aligned}\Lambda'_{1,s}/(I'_{1,s}, \mu_2) \cong &\mathcal{F}[V(\Lambda_s \odot \mathcal{Q})]/I(\Lambda_s \odot \mathcal{Q}) \bigotimes_{K} \mathcal{F}[V(\mathcal{Q})]/I(\mathcal{Q}) \\ &\bigotimes_{k=1}^{s} \mathcal{F}[V(W_1 \odot \mathcal{Q})]/I(W_1 \odot \mathcal{Q}) \bigotimes_{K} \mathcal{F}[i(\mathcal{Q})].\end{aligned}$$

Therefore again by Lemma 2.1.9

$$\begin{aligned}\operatorname{depth}(\Lambda'_{1,s}/(I'_{1,s},\mu_2)) =& \operatorname{depth}(\digamma[V(\Lambda_s\odot\mathcal{Q})]/I(\Lambda_s\odot\mathcal{Q})) + \operatorname{depth}(\digamma[V(\mathcal{Q})]/I(\mathcal{Q}))\\ &+\sum_{k=1}^{s}\operatorname{depth}(\digamma[V(W_1\odot\mathcal{Q})]/I(W_1\odot\mathcal{Q})) + \operatorname{depth}(\digamma[i(\mathcal{Q})]).\end{aligned}$$

Thus by Theorem 2.1.14

$$\operatorname{depth}(\Lambda'_{1,s}/(I'_{1,s},\mu_2)) = 2(\operatorname{depth}(\digamma[V(\mathcal{Q})]/I(\mathcal{Q})) + |i(\mathcal{Q})|+s).$$

It follows by Lemma 2.1.6

$$\operatorname{depth}(\Lambda'_{1,s}/I'_{1,s}) \geq 2(\operatorname{depth}(\digamma[V(\mathcal{Q})]/I(\mathcal{Q})) + |i(\mathcal{Q})|+s). \tag{3.1}$$

Now since $\sigma_{21}\sigma_{22}\dots\sigma_{2s}\notin I'_{1,s}$, we have

$$\begin{aligned}\Lambda'_{1,s}/(I'_{1,s}:\sigma_{21}\sigma_{22}\dots\sigma_{2s}) \cong& \digamma[V(\Lambda_s\odot\mathcal{Q})]/I(\Lambda_s\odot\mathcal{Q})\bigotimes_K\digamma[i(\mathcal{Q})]\\ &\bigotimes_K\digamma[V(\mathcal{Q})]/I(\mathcal{Q})\bigotimes_K\digamma[\sigma_{21},\sigma_{22},\dots,\sigma_{2s}],\end{aligned}$$

by using Lemma 2.1.9, we get

$$\begin{aligned}\operatorname{depth}(\Lambda'_{1,s}/(I'_{1,s}:\sigma_{21}\sigma_{22}\dots\sigma_{2s})) =& \operatorname{depth}(\digamma[V(\Lambda_s\odot\mathcal{Q})]/I(\Lambda_s\odot\mathcal{Q}))\\ &+\operatorname{depth}(\digamma[V(\mathcal{Q})]/I(\mathcal{Q})) + \operatorname{depth}(\digamma[i(\mathcal{Q})])\\ &+\operatorname{depth}(\digamma[\sigma_{21},\sigma_{22},\dots,\sigma_{2s}]).\end{aligned}$$

Thus by Theorem 2.1.14

$$\operatorname{depth}(\Lambda'_{1,s}/(I'_{1,s}:\sigma_{21}\sigma_{22}\dots\sigma_{2s})) = 2(\operatorname{depth}(\digamma[V(\mathcal{Q})]/I(\mathcal{Q})) + |i(\mathcal{Q})|+s).$$

By using Corollary 2.1.10, we have

$$\operatorname{depth}(\Lambda'_{1,s}/I'_{1,s}) \leq 2(\operatorname{depth}(\digamma[V(\mathcal{Q})]/I(\mathcal{Q})) + |i(\mathcal{Q})|+s). \tag{3.2}$$

Therefore by Eq (3.1) and Eq (3.2) we have the following result

$$\operatorname{depth}(\Lambda'_{1,s}/I'_{1,s}) = 2(\operatorname{depth}(\digamma[V(\mathcal{Q})]/I(\mathcal{Q})) + |i(\mathcal{Q})| + s).$$

If $r = 2$, then using the similar arguments and case $r = 1$, one can easily prove that $\operatorname{depth}(\Lambda'_{2,s}/I'_{2,s}) = 3(\operatorname{depth}(\digamma[V(\mathcal{Q})]/I(\mathcal{Q})) + |i(\mathcal{Q})| + s)$.
Let $r \geq 3$. Then we have

$$\Lambda'_{r,s}/(I'_{r,s} : \mu_{r+1}) \cong \Lambda'_{(r-2),s}/I'_{(r-2),s} \bigotimes_{k=1}^{2s+1} \digamma[V(\mathcal{Q})]/I(\mathcal{Q}) \bigotimes_{k=1}^{2s+1} \digamma[i(\mathcal{Q})] \bigotimes_{K} \digamma[\mu_{r+1}].$$

Then by Lemma 2.1.9

$$\begin{aligned}\operatorname{depth}(\Lambda'_{r,s}/(I'_{r,s} : \mu_{r+1})) = &\operatorname{depth}(\Lambda'_{(r-2),s}/I'_{(r-2),s}) + \sum_{k=1}^{2s+1} \operatorname{depth}(\digamma[V(\mathcal{Q})]/I(\mathcal{Q})) \\ &+ \sum_{k=1}^{2s+1} \operatorname{depth}(\digamma[i(\mathcal{Q})]) + \operatorname{depth}(\digamma[\mu_{r+1}]).\end{aligned}$$

By induction, we have

$$\operatorname{depth}(\Lambda'_{r,s}/(I'_{r,s} : \mu_{r+1})) = (r + 2s)(\operatorname{depth}(\digamma[V(\mathcal{Q})]/I(\mathcal{Q})) + |i(Q)|) + rs - s + 1.$$

Also

$$\begin{aligned}\Lambda'_{r,s}/(I'_{r,s}, \mu_{r+1}) \cong &\Lambda'_{(r-1),s}/I'_{(r-1),s} \bigotimes_{K} \digamma[V(\mathcal{Q})]/I(\mathcal{Q}) \bigotimes_{k=1}^{s} \digamma[V(W_1 \odot \mathcal{Q})]/I(W_1 \odot \mathcal{Q}) \\ &\bigotimes_{K} \digamma[i(\mathcal{Q})].\end{aligned}$$

Then by Lemma 2.1.9

$$\begin{aligned}\operatorname{depth}(\Lambda'_{r,s}/(I'_{r,s}, \mu_{r+1})) = &\operatorname{depth}(\Lambda'_{(r-1),s}/I'_{(r-1),s}) + \operatorname{depth}(\digamma[V(\mathcal{Q})]/I(\mathcal{Q})) \\ &+ \sum_{k=1}^{s} \operatorname{depth}(\digamma[V(W_1 \odot \mathcal{Q})]/I(W_1 \odot \mathcal{Q})) + \operatorname{depth}(\digamma[i(\mathcal{Q})]).\end{aligned}$$

By induction and Lemma 2.1.12, we get

$$\operatorname{depth}(\Lambda'_{r,s}/(I'_{r,s},\mu_{r+1})) = (r+1)(\operatorname{depth}(\mathcal{F}[(\mathcal{Q})]/I(\mathcal{Q})) + |i(\mathcal{Q})| + s).$$

Again by Lemma 2.1.6

$$\operatorname{depth}(\Lambda'_{r,s}/I'_{r,s}) \geq (r+1)(\operatorname{depth}(\mathcal{F}[V(\mathcal{Q})]/I(\mathcal{Q})) + |i(\mathcal{Q})| + s). \tag{3.3}$$

Now since $\sigma_{(r+1)1}\sigma_{(r+1)2}\ldots\sigma_{(r+1)s} \notin I'_{r,s}$, we have

$$\begin{aligned}\Lambda'_{r,s}/(I'_{r,s} : \sigma_{(r+1)1}\sigma_{(r+1)2}\ldots\sigma_{(r+1)s}) \cong{}& \Lambda'_{(r-1),s}/I'_{(r-1),s} \bigotimes_K \mathcal{F}[i(\mathcal{Q})] \bigotimes_K \mathcal{F}[V(\mathcal{Q})]/I(\mathcal{Q}) \\ & \bigotimes_K \mathcal{F}[\sigma_{(r+1)1}, \sigma_{(r+1)2}, \ldots, \sigma_{(r+1)s}].\end{aligned}$$

Thus by Lemma 2.1.9

$$\begin{aligned}\operatorname{depth}(\Lambda'_{r,s}/(I'_{r,s} : \sigma_{(r+1)1}\sigma_{(r+1)2}\ldots\sigma_{(r+1)s})) ={}& \operatorname{depth}(\Lambda'_{(r-1),s}/I'_{(r-1),s}) + \operatorname{depth}(\mathcal{F}[i(\mathcal{Q})] \\ & + \operatorname{depth}(\mathcal{F}[V(\mathcal{Q})]/I(\mathcal{Q})) \\ & + \operatorname{depth}(\mathcal{F}[\sigma_{(r+1)1}, \sigma_{(r+1)2}, \ldots, \sigma_{(r+1)s}]).\end{aligned}$$

By induction $\operatorname{depth}(\Lambda'_{r,s}/(I'_{r,s} : \sigma_{(r+1)1}\sigma_{(r+1)2}\ldots\sigma_{(r+1)s})) = (r+1)(\operatorname{depth}(\mathcal{F}[V(\mathcal{Q})]/I(\mathcal{Q})) + |i(\mathcal{Q})| + s)$. Again by Corollary 2.1.10, we have

$$\operatorname{depth}(\Lambda'_{r,s}/I'_{r,s}) \leq (r+1)(\operatorname{depth}(\mathcal{F}[V(\mathcal{Q})]/I(\mathcal{Q})) + |i(\mathcal{Q})| + s). \tag{3.4}$$

Hence by Eq (3.3) and (3.4), we get the desired result

$$\operatorname{depth}(\Lambda'_{r,s}/(I'_{r,s})) = (r+1)(\operatorname{depth}(\mathcal{F}[V(\mathcal{Q})]/I(\mathcal{Q})) + |i(\mathcal{Q})| + s).$$

Now we prove the result for the Stanley depth. We obtain the lower bound for the Stanley depth by induction, Proposition 2.1.7, Lemma 2.1.8(b), and Lemma 2.1.9(b). For upper bound we use Proposition 2.1.11 instead of Corollary 2.1.10.

□

Corollary 3.1.3. *Let r,s* ≥ 1. *Then*

$$\operatorname{pdim}(\Lambda'_{r,s}/I'_{r,s}) = (r+1)(s+1)(|V(\mathcal{Q})|) - (r+1)\operatorname{depth}(\digamma[V(\mathcal{Q})]/I(\mathcal{Q})) + r + 1.$$

Proof. The result for projective dimension follows by using Lemma 2.1.19, that is

$$\operatorname{pdim}(\Lambda'_{r,s}/I'_{r,s}) + \operatorname{depth}(\Lambda'_{r,s}/I'_{r,s}) = \operatorname{depth}(\Lambda'_{r,s}),$$

which implies that

$$\operatorname{pdim}(\Lambda'_{r,s}/I'_{r,s}) = \operatorname{depth}(\Lambda'_{r,s}) - \operatorname{depth}(\Lambda'_{r,s}/I'_{r,s}).$$

Hence by Lemma 3.1.2, we have

$$\operatorname{pdim}(\Lambda'_{r,s}/I'_{r,s}) = (r+1)(s+1)(|V(\mathcal{Q})|) - (r+1)\operatorname{depth}(\digamma[V(\mathcal{Q})]/I(\mathcal{Q}) + r + 1.$$

□

Theorem 3.1.4. *Let* $r, s \geq 1$. *If* $\Lambda_{r,s} = \digamma[V(\Gamma_{r,s} \odot \mathcal{Q})]$ *and* $I_{r,s} = I(\Gamma_{r,s} \odot \mathcal{Q})$, *then*

(1) $\operatorname{depth}(\Lambda_{r,s}/I_{r,s}) = (r+1)(\operatorname{depth}(\digamma[V(\mathcal{Q})]/I(\mathcal{Q})) + |i(\mathcal{Q})|) + rs.$

(2) $\operatorname{sdepth}(\Lambda_{r,s}/I_{r,s}) = (r+1)(\operatorname{sdepth}(\digamma[V(\mathcal{Q})]/I(\mathcal{Q})) + |i(\mathcal{Q})|) + rs.$

Proof. First we demonstrate the result for depth. Consider the sequence which is short exact

$$0 \longrightarrow \Lambda_{r,s}/(I_{r,s} : \mu_{r+1}) \xrightarrow{\cdot \mu_{r+1}} \Lambda_{r,s}/I_{r,s} \longrightarrow \Lambda_{r,s}/(I_{r,s}, \mu_{r+1}) \longrightarrow 0.$$

By applying Depth Lemma, we get

$$\operatorname{depth}(\Lambda_{r,s}/I_{r,s}) \geq \min\{\operatorname{depth}(\Lambda_{r,s}/(I_{r,s} : \mu_{r+1})), \operatorname{depth}(\Lambda_{r,s}/(I_{r,s}, \mu_{r+1}))\}.$$

If $r = 1$, then we have the following isomorphism:

$$\Lambda_{1,s}/(I_{1,s} : \mu_2) \cong \bigotimes_{k=1}^{s+1} \digamma[V(\mathcal{Q})]/I(\mathcal{Q}) \bigotimes_{k=1}^{s+1} \digamma[i(\mathcal{Q})] \bigotimes_{K} \digamma[\mu_2].$$

Then by Lemma 2.1.9

$$\begin{aligned}\operatorname{depth}(\Lambda_{1,s}/(I_{1,s}:\mu_2)) =& \sum_{k=1}^{s+1}\operatorname{depth}(\mathcal{F}[V(\mathcal{Q})]/I(\mathcal{Q})) + \sum_{k=1}^{s+1}\operatorname{depth}(\mathcal{F}[i(\mathcal{Q})]) \\ &+ \operatorname{depth}(\mathcal{F}[\mu_2]),\end{aligned}$$

implies that

$$\operatorname{depth}(\Lambda_{1,s}/(I_{1,s})) : \mu_2) = (s+1)(\operatorname{depth}(\mathcal{F}[V(\mathcal{Q})]/I(\mathcal{Q})) + |i(\mathcal{Q})|) + 1.$$

Similarly

$$\Lambda_{1,s}/(I_{1,s},\mu_2) \cong \mathcal{F}[V(\Lambda_s \odot \mathcal{Q})]/I(\Lambda_s \odot \mathcal{Q}) \bigotimes_K \mathcal{F}[V(\mathcal{Q})]/I(\mathcal{Q}) \bigotimes_K \mathcal{F}[i(\mathcal{Q})].$$

Therefore again by Lemma 2.1.9, we have

$$\begin{aligned}\operatorname{depth}(\Lambda_{1,s}/(I_{1,s},\mu_2)) =& \operatorname{depth}(\mathcal{F}[V(\Lambda_s \odot \mathcal{Q})]/I(\Lambda_s \odot \mathcal{Q})) + \operatorname{depth}(\mathcal{F}[V(\mathcal{Q})]/I(\mathcal{Q})) \\ &+ \operatorname{depth}(\mathcal{F}[i(\mathcal{Q})]).\end{aligned}$$

Thus by Theorem 2.1.14

$$\operatorname{depth}(\Lambda_{1,s}/(I_{1,s},\mu_2)) = 2(\operatorname{depth}(\mathcal{F}[V(\mathcal{Q})]/I(\mathcal{Q})) + |i(\mathcal{Q})|) + s.$$

It follows by Lemma 2.1.6

$$\operatorname{depth}(\Lambda_{1,s}/I_{1,s}) \geq 2(\operatorname{depth}(\mathcal{F}[V(\mathcal{Q})]/I(\mathcal{Q})) + |i(\mathcal{Q})|) + s. \tag{3.5}$$

Now since $\sigma_{21}\sigma_{22}\ldots\sigma_{2s} \notin I_{1,s}$, we have

$$\Lambda_{1,s}/(I_{1,s}:\sigma_{21}\sigma_{22}\ldots\sigma_{2s}) \cong \bigotimes_{k=1}^{2}\mathcal{F}[V(\mathcal{Q})]/I(\mathcal{Q}) \bigotimes_{k=1}^{2}\mathcal{F}[i(\mathcal{Q})] \bigotimes_K \mathcal{F}[\sigma_{21},\sigma_{22},\ldots,\sigma_{2s}].$$

By using Lemma 2.1.9

$$\operatorname{depth}(\Lambda_{1,s}/(I_{1,s}:\sigma_{21}\sigma_{22}\dots\sigma_{2s})) = \sum_{i=1}^{2}\operatorname{depth}(\digamma[V(\mathcal{Q})]/I(\mathcal{Q})) + \sum_{i=1}^{2}\operatorname{depth}(\digamma[i(\mathcal{Q})]) + \operatorname{depth}(\digamma[\sigma_{21},\sigma_{22},\dots,\sigma_{2s}]).$$

Thus we have

$$\operatorname{depth}(\Lambda_{1,s}/(I_{1,s}:\sigma_{21}\sigma_{22}\dots\sigma_{2s})) = 2(\operatorname{depth}(\digamma[V(\mathcal{Q})]/I(\mathcal{Q})) + |i(\mathcal{Q})|) + s.$$

By using Corollary 2.1.10, we get

$$\operatorname{depth}(\Lambda_{1,s}/I_{1,s}) \le 2(\operatorname{depth}(\digamma[V(\mathcal{Q})]/I(\mathcal{Q})) + |i(\mathcal{Q})|) + s. \tag{3.6}$$

Therefore by Eq (3.5) and (3.6) we have

$$\operatorname{depth}(\Lambda_{1,s}/I_{1,s}) = 2(\operatorname{depth}(\digamma[V(\mathcal{Q})]/I(\mathcal{Q})) + |i(\mathcal{Q})|) + s.$$

If $r = 2$, then using the similar arguments and case $r = 1$, one can easily prove that $\operatorname{depth}(\Lambda_{2,s}/I_{2,s}) = 3(\operatorname{depth}(\digamma[V(\mathcal{Q})]/I(\mathcal{Q})) + |i(\mathcal{Q})|) + s$. Let $r \ge 3$. Then we have

$$\Lambda_{r,s}/(I_{r,s}:\mu_{r+1}) \cong \Lambda'_{(r-2),s}/I'_{(r-2),s} \bigotimes_{k=1}^{s+1} \digamma[V(\mathcal{Q})]/I(\mathcal{Q}) \bigotimes_{k=1}^{s+1} \digamma[i(\mathcal{Q})] \bigotimes_{K} \digamma[\mu_{r+1}].$$

Then by Lemma 2.1.9

$$\operatorname{depth}(\Lambda_{r,s}/(I_{r,s}:\mu_{r+1})) = \operatorname{depth}(\Lambda'_{(r-2),s}/I'_{(r-2),s}) + \sum_{k=1}^{s+1}\operatorname{depth}(\digamma[V(\mathcal{Q})]/I(\mathcal{Q})) + \sum_{k=1}^{s+1}\operatorname{depth}(\digamma[i(\mathcal{Q})]) + \operatorname{depth}(\digamma[\mu_{r+1}]).$$

By Lemma 3.1.2

$$\operatorname{depth}(\Lambda_{r,s}/(I_{r,s}:\mu_{r+1})) = (r+s)(\operatorname{depth}(\digamma[V(\mathcal{Q})]/I(\mathcal{Q})) + |i(Q)|) + rs - s + 1.$$

Also

$$\Lambda_{r,s}/(I_{r,s},\mu_{r+1}) \cong \Lambda'_{(r-1),s}/I'_{(r-1),s} \bigotimes_K \digamma[V(\mathcal{Q})]/I(\mathcal{Q}) \bigotimes_K \digamma[i(\mathcal{Q})].$$

Then again by Lemma 2.1.9, we get

$$\begin{aligned}\operatorname{depth}(\Lambda_{r,s}/(I_{r,s},\mu_{r+1})) = & \operatorname{depth}(\Lambda'_{(r-1),s}/I'_{(r-1),s}) + \operatorname{depth}(\digamma[V(\mathcal{Q})]/I(\mathcal{Q})) \\ & + \operatorname{depth}(\digamma[i(\mathcal{Q})]).\end{aligned}$$

By Lemma 3.1.2

$$\operatorname{depth}(\Lambda_{r,s}/(I_{r,s},\mu_{r+1})) = (r+1)(\operatorname{depth}(\digamma[V(\mathcal{Q})]/I(\mathcal{Q})) + |i\mathcal{Q}|) + rs.$$

Again by Lemma 2.1.6

$$\operatorname{depth}(\Lambda_{r,s}/I_{r,s}) \geq (r+1)(\operatorname{depth}(\digamma[\mathcal{Q}]/I(\mathcal{Q})) + |i(\mathcal{Q})|) + rs. \tag{3.7}$$

Now since $\sigma_{r1}\sigma_{r2}\ldots\sigma_{rs} \notin I_{r,s}$, we have

$$\begin{aligned}\Lambda_{r,s}/(I_{r,s} : \sigma_{r1}\sigma_{r2}\ldots\sigma_{rs}) \cong & \Lambda'_{(r-1),s}/I'_{(r-1),s} \bigotimes_{k=1}^{2} \digamma[V(\mathcal{Q})]/I(\mathcal{Q}) \\ & \bigotimes_{k=1}^{2} \digamma[i(\mathcal{Q})] \bigotimes_K \digamma[\sigma_{r1},\sigma_{r2},\ldots,\sigma_{rs}].\end{aligned}$$

Thus by Lemma 2.1.9

$$\begin{aligned}\operatorname{depth}(\Lambda_{r,s}/(I_{r,s} : \sigma_{r1}\sigma_{r2}\ldots\sigma_{rs})) = & \operatorname{depth}(\Lambda'_{(r-1),s}/I'_{(r-1),s}) \\ & + \sum_{k=1}^{2} \operatorname{depth}(\digamma[V(\mathcal{Q})]/I(\mathcal{Q})) \\ & + \sum_{k=1}^{2} \operatorname{depth}(\digamma[i(\mathcal{Q})] \\ & + \operatorname{depth}(\digamma[\sigma_{r1},\sigma_{r2},\ldots,\sigma_{rs}]).\end{aligned}$$

By Lemma 3.1.2

$$\begin{aligned}\operatorname{depth}(\Lambda_{r,s}/(I_{r,s}:\sigma_{r1}\sigma_{r2}\dots\sigma_{rs})) =&(r+1)(\operatorname{depth}(\mathcal{F}[V(\mathcal{Q})]/I(\mathcal{Q}))\\ &+|i(\mathcal{Q})|)+rs.\end{aligned}$$

Again by Corollary 2.1.10, we have

$$\operatorname{depth}(\Lambda_{r,s}/I_{r,s}) \leq (r+1)(\operatorname{depth}(\mathcal{F}[V(\mathcal{Q})]/I(\mathcal{Q}))+|i(\mathcal{Q})|)+rs. \tag{3.8}$$

Hence by Eq (3.7) and Eq (3.8) we get the desired result

$$\operatorname{depth}(\Lambda_{r,s}/I_{r,s}) = (r+1)(\operatorname{depth}(\mathcal{F}[V(\mathcal{Q})]/I(\mathcal{Q}))+|i(\mathcal{Q})|)+rs.$$

Now we prove the result for the Stanley depth. We obtain the lower bound for the Stanley depth by induction, Proposition 2.1.7, Lemma 2.1.8(b), and Lemma 2.1.9(b). For upper bound we use Proposition 2.1.11 instead of Corollary 2.1.10. □

Corollary 3.1.5. *Let $r,s \geq 1$. Then*

$$\operatorname{pdim}(\Lambda_{r,s}/I_{r,s}) = (rs+r+1)(|V(\mathcal{Q})|) - (r+1)\operatorname{depth}(\mathcal{F}[V(\mathcal{Q})]/I(\mathcal{Q})) + r + 1.$$

Proof. The result for projective dimension follows by using Lemma 2.1.19, that is

$$\operatorname{pdim}(\Lambda_{r,s}/I_{r,s}) + \operatorname{depth}(\Lambda_{r,s}/I_{r,s}) = \operatorname{depth}(\Lambda_{r,s}),$$

which implies that

$$\operatorname{pdim}(\Lambda_{r,s}/I_{r,s}) = \operatorname{depth}(\Lambda_{r,s}) - \operatorname{depth}(\Lambda_{r,s}/I_{r,s}).$$

Hence by Theorem 3.1.4, we get

$$\operatorname{pdim}(\Lambda_{r,s}/I_{r,s}) = (r+1)(s+1)(|V(\mathcal{Q})|) - (r+1)\operatorname{depth}(\mathcal{F}[V(\mathcal{Q})]/I(\mathcal{Q}) + r + 1.$$

□

3.2 Depth, Stanley Depth and Projective Dimension of Cyclic Module Associated with $\Omega_{r,s} \odot \mathcal{Q}$

In this section, we compute Stanley depth, depth and projective dimension of edge ideal associated with corona product of multi triangular ouroboros snake graph $\mathcal{O}_{r,s}$ with any graph $\mathcal{Q}$.

Let $r, s \geq 1$. Then corona product of multi triangular ouroboros snake graph $\mathcal{O}_{r,s}$ with any graph $\mathcal{Q}$ denoted by $\mathcal{O}_{r,s} \odot \mathcal{Q}$ is obtained by connecting all vertices of $\mathcal{Q}$ to every vertex of $\mathcal{O}_{r,s}$. For example if $s = 1$ then $\mathcal{O}_{r,s} = \mathcal{O}_{r,1} = \mathcal{O}_r$ a triangular ouroboros snake graph. Clearly $|V(\mathcal{O}_{r,s} \odot \mathcal{Q})| = |V(\mathcal{O}_{r,s})|(|V(\mathcal{Q})|+1)$ and $E(\mathcal{O}_{r,s} \odot \mathcal{Q}) = |E(\mathcal{O}_{r,s})|+(|V(\mathcal{O}_{r,s})||E(\mathcal{Q})|)+(|V(\mathcal{O}_{r,s})||V(\mathcal{Q})|)$. The vertices of the $\Gamma_{r,s} \odot \mathcal{Q}$ are labelled by using the following sets of variables $\{\mu_1, \mu_2, \ldots, \mu_{r+1}\}$, $\{\mu_{r1}, \mu_{r2}, \ldots, \mu_{rw}\}$,$\{\sigma_{r1}, \sigma_{r2}, \ldots, \sigma_{rs}\}$ and $\{\sigma_{rs1}, \sigma_{rs2}, \ldots, \sigma_{rsw}\}$. For example see figure 3.4. Also we denote the square free monomial ideal of polynomial ring $\Omega_{r,s} = \mathcal{F}[V(\mathcal{O}_{r,s} \odot \mathcal{Q})]$ by $J_{r,s}$.

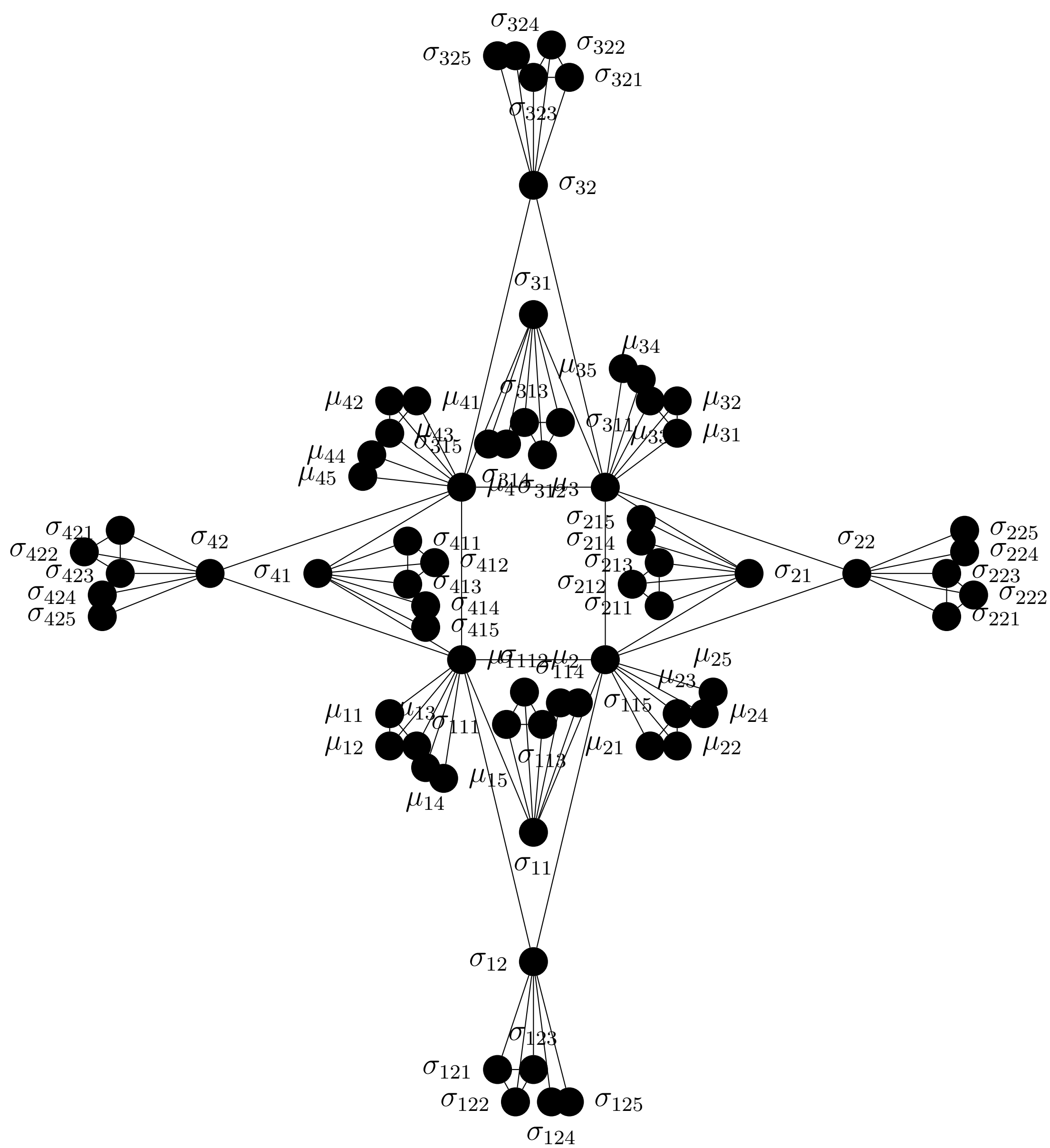


Figure 3.3: $\Omega_{4,2} \odot \mathcal{Q}$

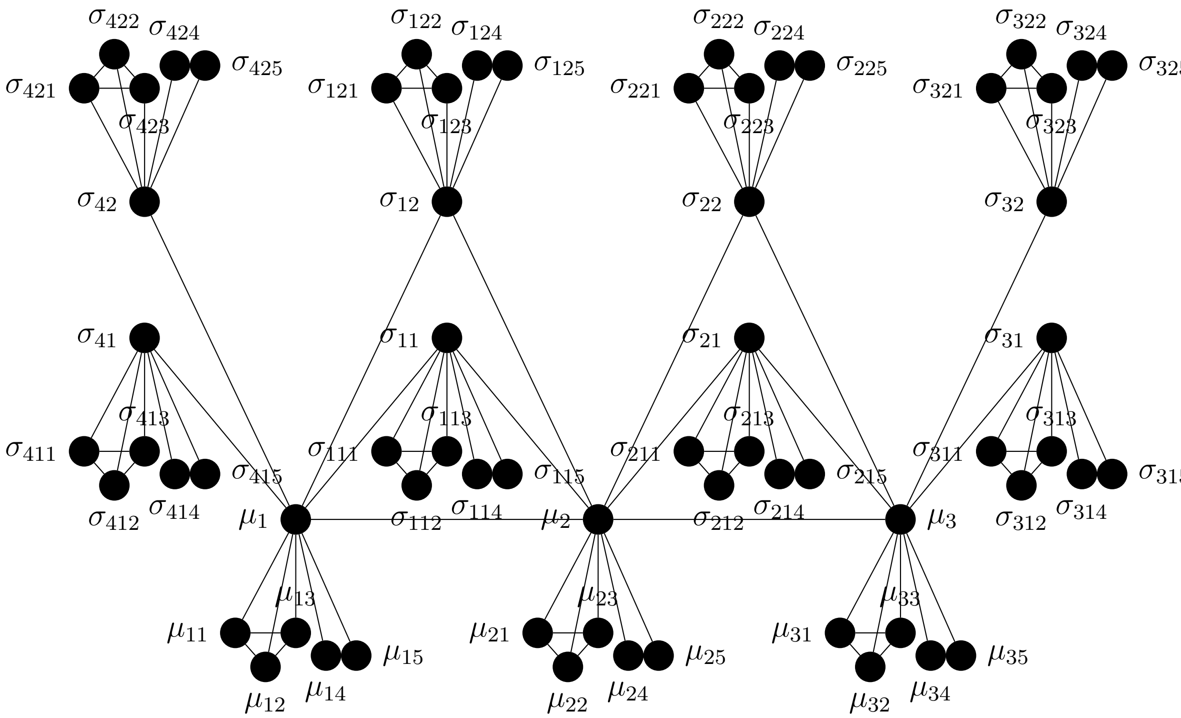


Figure 3.4: $\Gamma''_{2,s} \odot \mathcal{Q}$

Let us consider a super graph $\Gamma''_{r,s} \odot \mathcal{Q}$ of the graph $\Gamma'_{r,s} \odot \mathcal{Q}$. The vertex and edge sets of $\Gamma''_{r,s} \odot \mathcal{Q}$ are $|V(\Gamma''_{r,s} \odot \mathcal{Q})|= |V(\Gamma''_{r,s})|(|V(\mathcal{Q})|+1) = V(T'_{r,s}) \bigcup\{\sigma_{(r+2)1}, \sigma_{(r+2)2}, \dots, \sigma_{(r+2)s}\}$ and $E(\Gamma''_{r,s}\odot\mathcal{Q}) = |E(\Gamma''_{r,s})|+(|V(\Gamma''_{r,s})||V(\mathcal{Q})|)+(|V(\Gamma'''_{r,s})||E(\mathcal{Q})|) = E(T'_{r,s}) \bigcup\{\{\mu_1, \sigma_{(r+2)k}\} : 1 \leq k \leq s\}$. Clearly $T''_{r,s} \odot \mathcal{Q}$. For example of graph $\Gamma''_{r,s} \odot \mathcal{Q}$ see figure 3.4. We denote the edge ideal of graph $\Gamma''_{r,s} \odot \mathcal{Q}$ with $J''_{r,s}$ which is monomial ideal of the polynomial ring $\Lambda''_{r,s} = \mathcal{F}[V(\Gamma''_{r,s} \odot \mathcal{Q})]$.

Lemma 3.2.1. *Let $r, s \geq 1$. If $\Lambda''_{r,s} = \mathcal{F}[V(\Gamma''_{r,s} \odot \mathcal{Q})]$ and $J''_{r,s} = I(\Gamma''_{r,s} \odot \mathcal{Q})$, then*

(1) $\operatorname{depth}(\Lambda''_{r,s}/J''_{r,s}) = (r+1)(\operatorname{depth}(\mathcal{F}[V(\mathcal{Q})]/I(\mathcal{Q})) + |i(\mathcal{Q})|) + (r+2)s.$

(2) $\operatorname{sdepth}(\Lambda''_{r,s}/J''_{r,s}) = (r+1)(\operatorname{sdepth}(\mathcal{F}[V(\mathcal{Q})]/I(\mathcal{Q})) + |i(\mathcal{Q})|) + (r+2)s.$

Proof. First we demonstrate the result for depth by induction on r. Consider the sequence which is short exact

$$0 \longrightarrow \Lambda''_{r,s}/(J''_{r,s} : \mu_{r+1}) \xrightarrow{\cdot\mu_{r+1}} \Lambda''_{r,s}/J''_{r,s} \longrightarrow \Lambda''_{r,s}/(J''_{r,s}, \mu_{r+1}) \longrightarrow 0$$

by applying Depth Lemma, we get

$$\operatorname{depth}(\Lambda''_{r,s}/J''_{r,s}) \geq \min\{\operatorname{depth}(\Lambda''_{r,s}/(J''_{r,s} : \mu_{r+1})), \operatorname{depth}(\Lambda''_{r,s}/(J''_{r,s}, \mu_{r+1}))\}.$$

If $r = 1$, then we have the following isomorphism:

$$\Lambda''_{1,s}/(J''_{1,s} : \mu_2) \cong \bigotimes_{k=1}^{2s+1} \digamma[V(\mathcal{Q})]/I(\mathcal{Q}) \bigotimes_{k=1}^{2s+1} \digamma[i(\mathcal{Q})] \bigotimes_{k=1}^{s} \digamma[V(W_1 \odot \mathcal{Q})]/I(W_1 \odot \mathcal{Q}) \bigotimes_{K} \digamma[\mu_2].$$

Then by Lemma 2.1.9

$$\operatorname{depth}(\Lambda''_{1,s}/(J''_{1,s} : \mu_2)) = \sum_{k=1}^{2s+1} \operatorname{depth}(\digamma[V(\mathcal{Q})]/I(\mathcal{Q})) + \sum_{k=1}^{2s+1} \operatorname{depth}(\digamma[i(\mathcal{Q})]) + \sum_{k=1}^{s} \operatorname{depth}(\digamma[V(W_1 \odot \mathcal{Q})]/I(W_1 \odot \mathcal{Q})) + \operatorname{depth}(\digamma[\mu_2]),$$

which implies by Lemma 2.1.12

$$\operatorname{depth}(\Lambda''_{1,s}/(J''_{1,s} : \mu_2)) = (2s+1)(\operatorname{depth}(\digamma[V(\mathcal{Q})]/I(\mathcal{Q})) + |i(Q)|) + s + 1.$$

Similarly

$$\Lambda''_{1,s}/(J''_{1,s}, \mu_2) \cong \digamma[V(\Lambda_{2s} \odot \mathcal{Q})]/I(\Lambda_{2s} \odot \mathcal{Q}) \bigotimes_{K} \digamma[V(\mathcal{Q})]/I(\mathcal{Q}) \bigotimes_{K} \digamma[i(\mathcal{Q})] \bigotimes_{k=1}^{s} \digamma[V(W_1 \odot \mathcal{Q})]/I(W_1 \odot \mathcal{Q}).$$

Therefore again by Lemma 2.1.9, we have

$$\begin{aligned}\operatorname{depth}(\Lambda''_{1,s}/(J''_{1,s},\mu_2)) =& \operatorname{depth}(F[V(\Lambda_{2s}\odot \mathcal{Q})]/I(\Lambda_{2s}\odot \mathcal{Q})) + \operatorname{depth}(F[V(\mathcal{Q})]/I(\mathcal{Q}))\\ &+ \operatorname{depth}(F[i(\mathcal{Q})]) + \sum_{k=1}^{s}\operatorname{depth}(F[V(W_1\odot \mathcal{Q})]/I(W_1\odot \mathcal{Q})).\end{aligned}$$

Thus we have by Theorem 2.1.14

$$\operatorname{depth}(\Lambda''_{1,s}/(J''_{1,s},\mu_2)) = 2(\operatorname{depth}(F[V(\mathcal{Q})]/I(\mathcal{Q})) + |i(\mathcal{Q})|) + 3s.$$

It follows by Lemma 2.1.6

$$\operatorname{depth}(\Lambda''_{1,s}/J''_{1,s}) \geq 2(\operatorname{depth}(F[V(\mathcal{Q})]/I(\mathcal{Q})) + |i(\mathcal{Q})|) + 3s. \tag{3.9}$$

Now since $\sigma_{21}\sigma_{22}\dots\sigma_{2s} \notin J''_{1,s}$, we have

$$\begin{aligned}\Lambda''_{1,s}/(J''_{1,s}:\sigma_{21}\sigma_{22}\dots\sigma_{2s}) \cong& F[V(\Lambda_{2s}\odot \mathcal{Q})]/I(\Lambda_{2s}\odot \mathcal{Q}) \bigotimes_K F[V(\mathcal{Q})]/I(\mathcal{Q})\\ &\bigotimes_K F[i(\mathcal{Q})] \bigotimes_K F[\sigma_{21},\sigma_{22},\dots,\sigma_{2s}].\end{aligned}$$

By using Lemma 2.1.9

$$\begin{aligned}\operatorname{depth}(\Lambda''_{1,s}/(J''_{1,s}:\sigma_{21}\sigma_{22}\dots\sigma_{2s})) =& \operatorname{depth}(F[V(\Lambda_{2s}\odot \mathcal{Q})]/I(\Lambda_{2s}\odot \mathcal{Q}))\\ &+ \operatorname{depth}(F[V(\mathcal{Q})]/I(\mathcal{Q}))\\ &+ \operatorname{depth}(F[i(\mathcal{Q})]) + \operatorname{depth}(F[\sigma_{21},\sigma_{22},\dots,\sigma_{2s}]).\end{aligned}$$

Thus we have by Theorem 2.1.14

$$\operatorname{depth}(\Lambda''_{1,s}/(J''_{1,s}:\sigma_{21}\sigma_{22}\dots\sigma_{2s})) = 2(\operatorname{depth}(F[V(\mathcal{Q})]/I(\mathcal{Q})) + |i(\mathcal{Q})|) + 3s.$$

By using Corollary 2.1.10, we get

$$\operatorname{depth}(\Lambda''_{1,s}/J''_{1,s}) \leq 2(\operatorname{depth}(F[V(\mathcal{Q})]/I(\mathcal{Q})) + |i(\mathcal{Q})|) + 3s. \tag{3.10}$$

Therefore by Eq (3.9) and Eq (3.10) we get the following result

$$\operatorname{depth}(\Lambda''_{1,s}/(J''_{1,s}) = 2(\operatorname{depth}(\digamma[V(\mathcal{Q})]/I(\mathcal{Q})) + |i(\mathcal{Q})|) + 3s.$$

If $r = 2$, then using the similar arguments and case $r = 1$, one can easily prove that $\operatorname{depth}(\Lambda''_{2,s}/J''_{2,s}) = 3(\operatorname{depth}(\digamma[V(\mathcal{Q})]/I(\mathcal{Q})) + |i(\mathcal{Q})|) + 4s$. Let $r \geq 3$, thus we have

$$\begin{aligned}\Lambda''_{r,s}/(J''_{r,s} : \mu_{r+1}) \cong & \Lambda''_{(r-2),s}/J''_{(r-2),s} \bigotimes_{k=1}^{2s+1} \digamma[V(\mathcal{Q})]/I(\mathcal{Q}) \bigotimes_{k=1}^{2s+1} \digamma[i(\mathcal{Q})] \\ & \bigotimes_{K} \digamma[\mu_{r+1}].\end{aligned}$$

Then by Lemma 2.1.9

$$\begin{aligned}\operatorname{depth}(\Lambda''_{r,s}/(J''_{r,s} : \mu_{r+1})) = & \operatorname{depth}(\Lambda''_{(r-2),s}/J''_{(r-2),s}) + \sum_{k=1}^{2s+1} \operatorname{depth}(\digamma[V(\mathcal{Q})]/I(\mathcal{Q})) \\ & + \sum_{k=1}^{2s+1} \operatorname{depth}(\digamma[i(\mathcal{Q})]) + \operatorname{depth}(\digamma[\mu_{r+1}]).\end{aligned}$$

Now by induction

$$\operatorname{depth}(\Lambda''_{r,s}/(J''_{r,s} : \mu_{r+1})) = (r + 2s)(\operatorname{depth}(\digamma[V(\mathcal{Q})]/I(\mathcal{Q})) + |i(Q)|) + rs + 1.$$

Also

$$\begin{aligned}\Lambda''_{r,s}/(J''_{r,s}, \mu_{r+1}) \cong & \Lambda''_{(r-1),s}/J''_{(r-1),s} \bigotimes_{K} \digamma[V(\mathcal{Q})]/I(\mathcal{Q}) \bigotimes_{K} \digamma[i(\mathcal{Q})] \\ & \bigotimes_{k=1}^{s} \digamma[V(W_1 \odot \mathcal{Q})]/I(W_1 \odot \mathcal{Q}).\end{aligned}$$

Then again by Lemma 2.1.9

$$\begin{aligned}\operatorname{depth}(\Lambda''_{r,s}/(J''_{r,s}, \mu_{r+1})) = & \operatorname{depth}(\Lambda''_{(r-1),s}/J''_{(r-1),s}) + \operatorname{depth}(\digamma[V(\mathcal{Q})]/I(\mathcal{Q})) \\ & + \operatorname{depth}(\digamma[i(\mathcal{Q})]) + \sum_{k=1}^{s} \operatorname{depth}(\digamma[V(W_1 \odot \mathcal{Q})]/I(W_1 \odot \mathcal{Q})).\end{aligned}$$

By induction, we get

$$\operatorname{depth}(\Lambda''_{r,s}/(J''_{r,s},\mu_{r+1})) = (r+1)(\operatorname{depth}(\mathcal{F}[V(\mathcal{Q})]/I(\mathcal{Q})) + |i(\mathcal{Q})|) + (r+2)s.$$

Again by Lemma 2.1.6

$$\operatorname{depth}(\Lambda''_{r,s}/J''_{r,s}) \geq (r+1)(\operatorname{depth}(\mathcal{F}[V(\mathcal{Q})]/I(\mathcal{Q})) + |i(\mathcal{Q})|) + (r+2)s. \tag{3.11}$$

Now since $\sigma_{(r+1)1}\sigma_{(r+1)2}\dots\sigma_{(r+1)s} \notin J''_{r,s}$, we have

$$\begin{aligned}\Lambda''_{r,s}/(J''_{r,s}:\sigma_{(r+1)1}\sigma_{(r+1)2}\dots\sigma_{(r+1)s}) \cong &\Lambda''_{(r-1),s}/J''_{(r-1),s}\bigotimes_K \mathcal{F}[V(\mathcal{Q})]/I(\mathcal{Q})\bigotimes_K \mathcal{F}[i(\mathcal{Q})]\\ &\bigotimes_K \mathcal{F}[\sigma_{(r+1)1},\sigma_{(r+1)2},\dots,\sigma_{(r+1)s}].\end{aligned}$$

Thus by Lemma 2.1.9, we get

$$\begin{aligned}\operatorname{depth}(\Lambda''_{r,s}/(J''_{r,s}:\sigma_{(r+1)1}\sigma_{(r+1)2}\dots\sigma_{(r+1)s})) = &\operatorname{depth}(\Lambda''_{(r-1),s}/J''_{(r-1),s}\odot W)\\ &+\operatorname{depth}(\mathcal{F}[V(\mathcal{Q})]/I(\mathcal{Q}))\\ &+\operatorname{depth}(\mathcal{F}[i(\mathcal{Q})]\\ &+\operatorname{depth}(\mathcal{F}[\sigma_{(r+1)1},\sigma_{(r+1)2},\dots,\sigma_{(r+1)s}]).\end{aligned}$$

Then by induction

$$\begin{aligned}\operatorname{depth}(\Lambda''_{r,s}/(J''_{r,s}:\sigma_{(r+1)1}\sigma_{(r+1)2}\dots\sigma_{(r+1)s})) = &(r+1)(\operatorname{depth}(\mathcal{F}[V(\mathcal{Q})]/I(\mathcal{Q})) + |i(\mathcal{Q})|)\\ &+(r+2)s.\end{aligned}$$

Again by Corollary 2.1.10,

$$\operatorname{depth}(\Lambda''_{r,s}/J''_{r,s}) \leq (r+1)(\operatorname{depth}(\mathcal{F}[V(\mathcal{Q})]/I(\mathcal{Q})) + |i(\mathcal{Q})|) + (r+2)s. \tag{3.12}$$

Hence we have he desired result by Eq (3.11) and Eq (3.12) such that

$$\operatorname{depth}(\Lambda''_{r,s}/J''_{r,s}) = (r+1)(\operatorname{depth}(\mathcal{F}[V(\mathcal{Q})]/I(\mathcal{Q})) + |i(\mathcal{Q})|) + (r+2)s.$$

Now we prove the result for the Stanley depth. We obtain the lower bound for the Stanley depth by induction, Proposition 2.1.7, Lemma 2.1.8(b), and Lemma 2.1.9(b). For upper bound we use Proposition 2.1.11 instead of Corollary 2.1.10.

□

Corollary 3.2.2. *Let r,s* ≥ 1. *Then*

$$\text{pdim}(\Lambda''_{r,s}/J''_{r,s}) = (r+1)(s+1)(|V(\mathcal{Q})|) - (r+1)\,\text{depth}(\digamma[V(\mathcal{Q})]/I(\mathcal{Q})) + r + 1.$$

Proof. The result for projective dimension follows by using Lemma 2.1.19, that is

$$\text{pdim}(\Lambda''_{r,s}/J''_{r,s}) + \text{depth}(\Lambda''_{r,s}/J''_{r,s}) = \text{depth}(\Lambda''_{r,s}),$$

which implies that $\text{pdim}(\Lambda''_{r,s}/J''_{r,s}) = \text{depth}(\Lambda''_{r,s}) - \text{depth}(\Lambda''_{r,s}/J''_{r,s})$. Hence by Lemma 3.2.1, we have

$$\text{pdim}(\Lambda''_{r,s}/J''_{r,s}) = (r+1)(s+1)(|V(\mathcal{Q})|) - (r+1)\,\text{depth}(\digamma[V(\mathcal{Q})]/I(\mathcal{Q})) + r + 1.$$

□

Theorem 3.2.3. *Let* $r \geq 3, s \geq 1$. *If* $\Omega_{r,s} = \digamma[V(O_{r,s} \odot \mathcal{Q})]$ *and* $J_{r,s} = I(O_{r,s} \odot \mathcal{Q})$, *then*

(1) $\text{depth}(\Omega_{r,s}/J_{r,s}) = r(\text{depth}(\digamma[V(\mathcal{Q})]/I(\mathcal{Q})) + |i(\mathcal{Q})| + s)$.

(2) $\text{sdepth}(\Omega_{r,s}/J_{r,s}) = r(\text{sdepth}(\digamma[V(\mathcal{Q})]/I(\mathcal{Q})) + |i(\mathcal{Q})| + s)$.

Proof. First we demonstrate the result for depth. Consider the sequence which is short exact

$$0 \longrightarrow \Omega_{r,s}/(J_{r,s} : \mu_r) \xrightarrow{\cdot \mu_r} \Omega_{r,s}/J_{r,s} \longrightarrow \Omega_{r,s}/(J_{r,s}, \mu_r) \longrightarrow 0.$$

By applying Depth Lemma

$$\text{depth}(\Omega_{r,s}/J_{r,s}) \geq \min\{\text{depth}(\Omega_{r,s}/(J_{r,s} : \mu_{r+1})), \text{depth}(\Omega_{r,s}/(J_{r,s}, \mu_{r+1}))\}.$$

If $r = 3$, then we have

$$\Omega_{3,s}/(J_{3,s} : \mu_3) \cong \bigotimes_{k=1}^{2s+2} \digamma[V(\mathcal{Q})]/I(\mathcal{Q}) \bigotimes_{k=1}^{2s+2} \digamma[i(\mathcal{Q})] \bigotimes_{k=1}^{s} \digamma[V(W_1 \odot \mathcal{Q})]/I(W_1 \odot \mathcal{Q}) \bigotimes_{K} \digamma[\mu_3].$$

Then by Lemma 2.1.9

$$\begin{aligned}\operatorname{depth}(\Omega_{3,s}/(J_{3,s} : \mu_3)) &= \sum_{k=1}^{2s+2} \operatorname{depth}(\digamma[V(\mathcal{Q})]/I(\mathcal{Q})) + \sum_{k=1}^{2s+2} \operatorname{depth}(\digamma[i(\mathcal{Q})]) \\ &+ \sum_{k=1}^{s} \operatorname{depth}(\digamma[V(W_1 \odot \mathcal{Q})]/I(W_1 \odot \mathcal{Q})) + \operatorname{depth}(\digamma[\mu_3]),\end{aligned}$$

which implies by Lemma 2.1.12

$$\operatorname{depth}(\Omega_{3,s}/(J_{3,s} : \mu_3)) = (2s+2)(\operatorname{depth}(\digamma[V(\mathcal{Q})]/I(\mathcal{Q})) + i(Q)|) + s + 1.$$

Similarly

$$\Omega_{3,s}/(J_{3,s}, \mu_3) \cong \Lambda''_{1,s}/J''_{1,s} \bigotimes_{K} \digamma[V(\mathcal{Q})]/I(\mathcal{Q}) \bigotimes_{K} \digamma[i(\mathcal{Q})].$$

Therefore again by Lemma 2.1.9

$$\operatorname{depth}(\Omega_{3,s}/(J_{3,s}, \mu_3)) = \operatorname{depth}(\Lambda''_{1,s}/J''_{1,s}) + \operatorname{depth}(\digamma[V(\mathcal{Q})]/I(\mathcal{Q})) + \operatorname{depth}(\digamma[i(\mathcal{Q})]).$$

Thus by Lemma 3.2.1, we have

$$\operatorname{depth}(\Omega_{3,s}/(J_{3,s}, \mu_3)) = 3(\operatorname{depth}(\digamma[V(\mathcal{Q})]/I(\mathcal{Q})) + |i(\mathcal{Q})| + s).$$

It follows by Lemma 2.1.6

$$\operatorname{depth}(\Omega_{3,s}/J_{3,s}) \geq 3(\operatorname{depth}(\digamma[V(\mathcal{Q})]/I(\mathcal{Q})) + |i(\mathcal{Q})| + s). \tag{3.13}$$

Now since $\sigma_{31}\sigma_{32}\ldots\sigma_{3s} \notin J_{3,s}$, we have

$$\Omega_{3,s}/(J_{3,s} : \sigma_{31}\sigma_{32}\ldots\sigma_{3s}) \cong \mathcal{F}[V(\Lambda_{2s} \odot \mathcal{Q})]/I(\Lambda_{2s} \odot \mathcal{Q}) \bigotimes_{k=1}^{2} \mathcal{F}[i(\mathcal{Q})] \bigotimes_{k=1}^{2} \mathcal{F}[V(\mathcal{Q})]/I(\mathcal{Q}) \bigotimes_{K} \mathcal{F}[\sigma_{31}, \sigma_{32}, \ldots, \sigma_{3s}].$$

By using Lemma 2.1.9

$$\begin{aligned} \operatorname{depth}(\Omega_{3,s}/(J_{3,s} : \sigma_{31}\sigma_{32}\ldots\sigma_{3s})) = &\operatorname{depth}(\mathcal{F}[V(\Lambda_{2s} \odot \mathcal{Q})]/I(\Lambda_{2s} \odot \mathcal{Q})) \\ &+ \sum_{i=1}^{2} \operatorname{depth}(\mathcal{F}[i(\mathcal{Q})]) + \sum_{i=1}^{2} \operatorname{depth}(\mathcal{F}[V(\mathcal{Q})]/I(\mathcal{Q})) \\ &+ \operatorname{depth}(\mathcal{F}[\sigma_{31}, \sigma_{32}, \ldots, \sigma_{3s}]). \end{aligned}$$

Thus by Theorem 2.1.14, we have

$$\operatorname{depth}(\Omega_{3,s}/(J_{3,s} : \sigma_{31}\sigma_{32}\ldots\sigma_{3s})) = 3(\operatorname{depth}(\mathcal{F}[V(\mathcal{Q})]/I(\mathcal{Q})) + |i(\mathcal{Q})| + s).$$

By using Corollary 2.1.10

$$\operatorname{depth}(\Omega_{3,s}/J_{3,s}) \leq 3(\operatorname{depth}(\mathcal{F}[V(\mathcal{Q})]/I(\mathcal{Q})) + |i(\mathcal{Q})| + s). \quad (3.14)$$

Therefore by Eq (3.13) and (3.14), we have the following result

$$\operatorname{depth}(\Omega_{3,s}/J_{3,s}) = 3(\operatorname{depth}(\mathcal{F}[V(\mathcal{Q})]/I(\mathcal{Q})) + |i(\mathcal{Q})| + s).$$

If $r = 4$, then using the similar arguments and case $r = 3$, one can easily prove that $\operatorname{depth}(\Omega_{4,s}/J_{4,s}) = 4(\operatorname{depth}(\mathcal{F}[V(\mathcal{Q})]/I(\mathcal{Q})) + |i(\mathcal{Q})| + s)$. Let $r \geq 5$. Then we have

$$\Omega_{r,s}/(J_{r,s} : \mu_r) \cong \Lambda''_{(r-4),s}/J''_{(r-4),s} \bigotimes_{k=1}^{2s+2} \mathcal{F}[V(\mathcal{Q})]/I(\mathcal{Q}) \bigotimes_{k=1}^{2s+2} \mathcal{F}[i(\mathcal{Q})] \bigotimes_{K} \mathcal{F}[\mu_r].$$

Then by Lemma 2.1.9

$$\begin{aligned}\operatorname{depth}(\Omega_{r,s}/(J_{r,s}:\mu_r)) =& \operatorname{depth}(\Lambda''_{(r-4),s}/J''_{(r-4),s}) + \sum_{k=1}^{2s+2}\operatorname{depth}(\mathcal{F}[V(\mathcal{Q})]/I(\mathcal{Q})) \\ &+ \sum_{k=1}^{2s+2}\operatorname{depth}(\mathcal{F}[i(\mathcal{Q})]) + \operatorname{depth}(\mathcal{F}[\mu_r]).\end{aligned}$$

By Lemma 3.2.1

$$\operatorname{depth}(\Omega_{r,s}/(J_{r,s}:\mu_r)) = (r+2s-1)(\operatorname{depth}(\mathcal{F}[V(\mathcal{Q})]/I(\mathcal{Q})) + |i(Q)|) + (r-2)s+1.$$

Also

$$\Omega_{r,s}/(J_{r,s},\mu_r) \cong \Lambda''_{(r-2),s}/J''_{(r-2),s} \bigotimes_K \mathcal{F}[V(\mathcal{Q})]/I(\mathcal{Q}) \bigotimes_K \mathcal{F}[i(\mathcal{Q})].$$

Then again by Lemma 2.1.9, we get

$$\begin{aligned}\operatorname{depth}(\Omega_{r,s}/(J_{r,s},\mu_r)) =& \operatorname{depth}(\Lambda''_{(r-2),s}/J''_{(r-2),s}) + \operatorname{depth}(\mathcal{F}[V(\mathcal{Q})]/I(\mathcal{Q})) \\ &+ \operatorname{depth}(\mathcal{F}[i(\mathcal{Q})]).\end{aligned}$$

By Lemma 3.2.1

$$\operatorname{depth}(\Omega_{r,s}/(J_{r,s},\mu_r)) = r(\operatorname{depth}(\mathcal{F}[V(\mathcal{Q})]/I(\mathcal{Q})) + |i(\mathcal{Q})| + s).$$

Again by Lemma 2.1.6

$$\operatorname{depth}(\Omega_{r,s}/J_{r,s},\mu_r) \geq r(\operatorname{depth}(\mathcal{F}[V(\mathcal{Q})]/I(\mathcal{Q})) + |i(\mathcal{Q})| + s). \tag{3.15}$$

Now since $\sigma_{r1}\sigma_{r2}\ldots\sigma_{rs} \notin J_{r,s}$, we have

$$\begin{aligned}\Omega_{r,s}/(J_{r,s}:\sigma_{r1}\sigma_{r2}\ldots\sigma_{rs}) \cong& \Lambda''_{(r-3),s}/J''_{(r-3),s} \bigotimes_{k=1}^{2} \mathcal{F}[i(\mathcal{Q})] \bigotimes_{k=1}^{2} \mathcal{F}[V(\mathcal{Q})]/I(\mathcal{Q}) \\ &\bigotimes_K \mathcal{F}[\sigma_{r1},\sigma_{r2},\ldots,\sigma_{rs}].\end{aligned}$$

Thus by Lemma 2.1.9

$$\begin{aligned}\operatorname{depth}(\Omega_{r,s}/(J_{r,s}:\sigma_{r1}\sigma_{r2}\dots\sigma_{rs})) = &\operatorname{depth}(\Lambda''_{(r-3),s}/J''_{(r-3),s}) + \sum_{k=1}^{2}\operatorname{depth}(\digamma[i(\mathcal{Q})]\\ &+\sum_{k=1}^{2}\operatorname{depth}(\digamma[V(\mathcal{Q})]/I(\mathcal{Q}))\\ &+\operatorname{depth}(\digamma[\sigma_{r1},\sigma_{r2},\dots,\sigma_{rs}]).\end{aligned}$$

Again by Lemma 3.2.1

$$\operatorname{depth}(\Omega_{r,s}/(J_{r,s}:\sigma_{r1}\sigma_{r2}\dots\sigma_{rs})) = r(\operatorname{depth}(\digamma[V(\mathcal{Q})]/I(\mathcal{Q})) + |i(\mathcal{Q})|+s).$$

Again by Corollary 2.1.10

$$\operatorname{depth}(\Omega_{r,s}/J_{r,s}) \leq r(\operatorname{depth}(\digamma[V(\mathcal{Q})]/I(\mathcal{Q})) + |i(\mathcal{Q})|+s). \tag{3.16}$$

Hence we have the desired result by Eq (3.15) and Eq (3.16) that is

$$\operatorname{depth}(\Omega_{r,s}/J_{r,s}) = r(\operatorname{depth}(\digamma[V(\mathcal{Q})]/I(\mathcal{Q})) + |i(\mathcal{Q})|+s).$$

Now we prove the result for the Stanley depth. We obtain the lower bound for the Stanley depth by induction, Proposition 2.1.7, Lemma 2.1.8(b), and Lemma 2.1.9(b). For upper bound we use Proposition 2.1.11 instead of Corollary 2.1.10.

□

Corollary 3.2.4. *Let $r \geq 3, s \geq 1$. Then*

$$\operatorname{pdim}(\Omega_{r,s}/J_{r,s}) = (rs+r+1)(|V(W)|) - (r+1)\operatorname{depth}(\digamma[V(\mathcal{Q})]/I(\mathcal{Q})) + r + 1.$$

Proof. The result for projective dimension follows by using Lemma 2.1.19, that is

$$\operatorname{pdim}(\Omega_{r,s}/J_{r,s}) + \operatorname{depth}(\Omega_{r,s}/J_{r,s}) = \operatorname{depth}(\Omega_{r,s}),$$

which implies that

$$\mathrm{pdim}(\Omega_{r,s}/J_{r,s}) = \mathrm{depth}(\Omega_{r,s}) - \mathrm{depth}(\Omega_{r,s}/J_{r,s}).$$

Hence by theorem 3.2.3, we have

$$\mathrm{pdim}(\Omega_{r,s}/J_{r,s}) = (r+1)(s+1)(|V(\mathcal{Q})|) - (r+1)\,\mathrm{depth}(\mathcal{F}[V(\mathcal{Q})]/I(\mathcal{Q}) + r + 1.$$

□

3.3 Krull Dimension and Cohen-Macaulay Graphs

In this section, we establish a formula for the residue class rings of edge ideals associated with the corona product of two graphs if the Krull dimension of one graph is known. In addition, we use the depth values provided in the Sections 3.1 and 3.2 to characterize distinct Cohen-Macaulay graphs.

Definition 3.3.1. A finite (i.e. finitely generated) Λ-module $\mathcal{D} \neq 0$ is a Cohen-Macaulay module for a commutative Noetherian local ring if $\mathrm{depth}(\mathcal{D}) = \dim(\mathcal{D})$ (in general, $\mathrm{depth}(\mathcal{D}) \leq \dim(\mathcal{D})$). However, because Λ is a module on itself, any Cohen-Macaulay module that is also a Λ-module is referred to as a Cohen-Macaulay ring. A maximal Cohen-Macaulay module is defined as one with the property $\mathrm{depth}(\mathcal{D}) = \mathrm{depth}(\Lambda)$.

Definition 3.3.2. In commutative algebra, the supremum of all chains of prime ideals is known as the Krull dimension of a commutative ring R, after Wolfgang Krull. The Krull dimension does not have to be finite for a Noetherian ring. In general, the deviation of a submodule's poset can be used to calculate the Krull dimension for modules spanning potentially non-commutative rings.

Theorem 3.3.3. *Let* $\amalg$ *and* $\mathcal{Q}$ *represent any two graphs, and* $\Lambda = \mathcal{F}[V(\amalg \odot \mathcal{Q})]$. *Then*

$$\dim(\Lambda/I(\amalg \odot \mathcal{Q})) = |V(\amalg)| \cdot (\dim(\mathcal{F}[V(\mathcal{Q})]/I(\mathcal{Q})) + |i(\mathcal{Q})|).$$

Lemma 3.3.4. *Let $\mathcal{Q}$ be a non-trivial connected graph and $\Lambda = \mathcal{F}[V(\mathcal{Q})]$. Then* $\dim(\Lambda/I(\mathcal{Q})) = 1$ *if and only if $\mathcal{Q}$ is complete graph.*

Theorem 3.3.5. *Let $\mathcal{Q}$ be any graph, $\amalg \in \{\Gamma'_{r,s}, \Gamma_{r,s}, \Gamma''_{r,s}, \Omega_{r,s}\}$ and $\Lambda = \mathcal{F}[V(\amalg \odot \mathcal{Q})]$. Then $\Lambda/I(\amalg \odot \mathcal{Q})$ is Cohen-Macaulay if and only if $\mathcal{Q}$ is a complete graph.*

Proof. By Lemma 3.3.4, a non-trivial connected graph $\mathcal{Q}$ is complete if and only if $\dim(\mathcal{F}[V(\mathcal{Q})]/I(\mathcal{Q})) = \text{depth}(\mathcal{F}[V(\mathcal{Q})]/I(\mathcal{Q})) = 1$. We will discuss all the cases one by one as follows:

(1) Let $\amalg = \Gamma_{r,s}$. By Theorem 3.1.4 and Theorem 3.3.5, the module $\Lambda/I(\Gamma_{r,s} \odot \mathcal{Q})$ is Cohen-Macaulay if and only if $\text{depth}(\Lambda/I(\Gamma_{r,s} \odot \mathcal{Q})) = \dim(\Lambda/I(\Gamma_{r,s} \odot \mathcal{Q}))$ if and only if

$$(r+1)(\text{depth}(\mathcal{F}[V(\mathcal{Q})]/I(\mathcal{Q})) + |i(\mathcal{Q})|) + rs = (rs + r + 1)\cdot \big(\dim(\mathcal{F}[V(\mathcal{Q})]/I(\mathcal{Q})) + |i(\mathcal{Q})|\big)$$

if and only if $|i(\mathcal{Q})| = 1$ and $\text{depth}(\mathcal{F}[V(\mathcal{Q})]/I(\mathcal{Q})) = 0 = \dim(\mathcal{F}[V(\mathcal{Q})]/I(\mathcal{Q}))$ or $|i(\mathcal{Q})| = 0$ and $\dim(\mathcal{F}[V(\mathcal{Q})]/I(\mathcal{Q})) = \text{depth}(\mathcal{F}[V(\mathcal{Q})]/I(\mathcal{Q})) = 1$ if and only if $\mathcal{Q}$ is a complete graph.

(2) Let $\amalg = \Omega_{r,s}$. By Theorem 3.2.3 and Theorem 3.3.5, the module $\Lambda/I(\Omega_{r,s} \odot \mathcal{Q})$ is Cohen-Macaulay if and only if $\text{depth}(\Lambda/I(\Omega_{r,s} \odot \mathcal{Q})) = \dim(\Lambda/I(\Omega_{r,s} \odot \mathcal{Q}))$ if and only if

$$r(\text{depth}(\mathcal{F}[V(\mathcal{Q})]/I(\mathcal{Q})) + |i(\mathcal{Q})| + s) = r(s+1)\cdot \big(\dim(\mathcal{F}[V(\mathcal{Q})]/I(\mathcal{Q})) + |i(\mathcal{Q})|\big)$$

if and only if $|i(\mathcal{Q})| = 1$ and $\text{depth}(\mathcal{F}[V(\mathcal{Q})]/I(\mathcal{Q})) = 0 = \dim(\mathcal{F}[V(\mathcal{Q})]/I(\mathcal{Q}))$ or $|i(\mathcal{Q})| = 0$ and $\dim(\mathcal{F}[V(\mathcal{Q})]/I(\mathcal{Q})) = \text{depth}(\mathcal{F}[V(\mathcal{Q})]/I(\mathcal{Q})) = 1$ if and only if $\mathcal{Q}$ is a complete graph.

□

Chapter 4

Regularity of the Cyclic Modules Associated with Corona Product of Certain Graphs

In this chapter, we compute regularity of edge ideal associated with corona product of multi triangular snake and ouroboros snake graph with any graph $\mathcal{Q}$. We also use some important results of regularity to find our results.

4.1 Regularity of Cyclic Module Associated with $\Gamma_{r,s} \odot \mathcal{Q}$

Lemma 4.1.1. *Let $r, s \geq 1$. If $\Lambda'_{r,s} = \digamma[V(\Gamma'_{r,s} \odot \mathcal{Q})]$ and $I'_{r,s} = I(\Gamma'_{r,s} \odot \mathcal{Q})$, then*

$$\operatorname{reg}(\Lambda'_{r,s}/I'_{r,s}) = (r+1)(s+1)\operatorname{reg}(\digamma[V(\mathcal{Q})]/I(\mathcal{Q})).$$

Proof. Now, using induction on r, we demonstrate the result for regularity. If $r = 1$, then we have

$$\operatorname{reg}(\Lambda'_{1,s}/(I'_{1,s} : \mu_2)) = \sum_{k=1}^{2s+1} \operatorname{reg}(\digamma[V(\mathcal{Q})]/I(\mathcal{Q})) = (2s+1)\operatorname{reg}(\digamma[V(\mathcal{Q})]/I(\mathcal{Q})).$$

Also by applying Lemma 2.2.2 and Lemma 2.2.3, we get

$$\begin{aligned}\operatorname{reg}(\Lambda'_{1,s}/(I'_{1,s},\mu_2)) =& \operatorname{reg}(\mathcal{F}[V(\Lambda_s \odot \mathcal{Q})]/I(\Lambda_s \odot \mathcal{Q})) + \operatorname{reg}(\mathcal{F}[V(\mathcal{Q})]/I(\mathcal{Q})) \\ &+ \sum_{k=1}^{s} \operatorname{reg}(\mathcal{F}[V(W_1 \odot \mathcal{Q})]/I(W_1 \odot \mathcal{Q})).\end{aligned}$$

Then by using Theorem 2.2.5 and Theorem 2.2.6, we get

$$\operatorname{reg}(\Lambda'_{1,s}/(I'_{1,s},\mu_2)) = 2(s+1)\operatorname{reg}(\mathcal{F}[V(\mathcal{Q})]/I(\mathcal{Q})).$$

The required result follows by Theorem 2.2.1(c), such that

$$\operatorname{reg}(\Lambda'_{1,s}/I'_{1,s}) = 2(s+1)\operatorname{reg}(\mathcal{F}[V(\mathcal{Q})]/I(\mathcal{Q})).$$

If $r = 2$, then by using the similar arguments and case $r = 1$, we get the desired result, that is

$$\operatorname{reg}(\Lambda'_{2,s}/(I'_{2,s}) = 3(s+1)\operatorname{reg}(\mathcal{F}[V(\mathcal{Q})]/I(\mathcal{Q})).$$

Now let $r \geq 3$. By Lemma 2.2.2 and Lemma 2.2.3, we have

$$\operatorname{reg}(\Lambda'_{r,s}/(I'_{r,s} : \mu_{r+1})) = \operatorname{reg}(\Lambda'_{(r-2),s}/I'_{(r-2),s}) + \sum_{k=1}^{2s+1} \operatorname{reg}(\mathcal{F}[V(\mathcal{Q})]/I(\mathcal{Q})).$$

So by induction, we have $\operatorname{reg}(\Lambda'_{r,s}/(I'_{r,s} : \mu_{r+1})) = (rs+r+s)(\operatorname{reg}(\mathcal{F}[V(\mathcal{Q})]/I(\mathcal{Q}))$. Again applying Lemma 2.2.2 and Lemma 2.2.3, we get

$$\begin{aligned}\operatorname{reg}(\Lambda'_{r,s}/(I'_{r,s},\mu_{r+1})) =& \operatorname{reg}(\Lambda'_{(r-1),s}/I'_{(r-1),s}) + \sum_{k=1}^{s} \operatorname{reg}(\mathcal{F}[V(W_1 \odot \mathcal{Q})]/I(W_1 \odot \mathcal{Q})) \\ &+ \operatorname{reg}(\mathcal{F}[V(\mathcal{Q})]/I(\mathcal{Q})).\end{aligned}$$

Again by induction and Theorem 2.2.5, we get

$$\operatorname{reg}(\Lambda'_{r,s}/(I'_{r,s},\mu_{r+1})) = (r+1)(s+1)\operatorname{reg}(\mathcal{F}[V(\mathcal{Q})]/I(\mathcal{Q})).$$

Hence by Theorem 2.2.1(c) we have

$$\operatorname{reg}(\Lambda'_{r,s}/I'_{r,s}) = (r+1)(s+1)\operatorname{reg}(\digamma[V(\mathcal{Q})]/I(\mathcal{Q})).$$

□

Theorem 4.1.2. *Let $r, s \geq 1$. If $\Lambda_{r,s} = \digamma[V(\Gamma_{r,s} \odot \mathcal{Q})]$ and $I_{r,s} = I(\Gamma_{r,s} \odot \mathcal{Q})$, then*

$$\operatorname{reg}(\Lambda_{r,s}/I_{r,s}) = (r(s+1)+1)\operatorname{reg}(\digamma[V(\mathcal{Q})]/I(\mathcal{Q})).$$

Proof. Now, we will be using Lemma 3.1.2 to demonstrate the result for regularity. If $r = 1$, then we have

$$\operatorname{reg}(\Lambda_{1,s}/(I_{1,s} : \mu_2)) = \sum_{k=1}^{s+1}\operatorname{reg}(\digamma[V(\mathcal{Q})]/I(\mathcal{Q})) = (s+1)\operatorname{reg}(\digamma[V(\mathcal{Q})]/I(\mathcal{Q})).$$

Also by applying Lemma 2.2.2 and Lemma 2.2.3, we get

$$\operatorname{reg}(\Lambda_{1,s}/(I_{1,s}, \mu_2)) = \operatorname{reg}(\digamma[V(\Lambda_s \odot \mathcal{Q})]/I(\Lambda_s \odot \mathcal{Q})) + \operatorname{reg}(\digamma[V(\mathcal{Q})]/I(\mathcal{Q})).$$

Then by using Theorem 2.2.6, we get

$$\operatorname{reg}(\Lambda_{1,s}/(I_{1,s}, \mu_2)) = (s+2)\operatorname{reg}(\digamma[V(\mathcal{Q})]/I(\mathcal{Q})).$$

The required result follows by Theorem 2.2.1(c), such that

$$\operatorname{reg}(\Lambda_{1,s}/I_{1,s}) = (s+2)\operatorname{reg}(\digamma[V(\mathcal{Q})]/I(\mathcal{Q})).$$

If $r = 2$, then by using the similar arguments and case $r = 1$, we get the desired result, that is

$$\operatorname{reg}(\Lambda_{2,s}/(I_{2,s}) = (2s+3)\operatorname{reg}(\digamma[V(\mathcal{Q})]/I(\mathcal{Q})).$$

Now let $r \geq 3$, Again by Lemma 2.2.3, we have

$$\operatorname{reg}(\Lambda_{r,s}/(I_{r,s} : \mu_{r+1})) = \operatorname{reg}(\Lambda'_{(r-2),s}/I'_{(r-2),s}) + \sum_{k=1}^{s+1}\operatorname{reg}(\digamma[V(\mathcal{Q})]/I(\mathcal{Q})).$$

So by Lemma 4.1.1, we have

$$\operatorname{reg}(\Lambda_{r,s}/(I_{r,s} : \mu_{r+1})) = (rs + r)(\operatorname{reg}(\digamma[V(\mathcal{Q})]/I(\mathcal{Q})).$$

Now applying Lemma 2.2.2 and Lemma 2.2.3, we get

$$\operatorname{reg}(\Lambda_{r,s}/(I_{r,s}, \mu_{r+1})) = \operatorname{reg}(\Lambda'_{(r-1),s}/I'_{(r-1),s}) + \operatorname{reg}(\digamma[V(\mathcal{Q})]/I(\mathcal{Q})).$$

Again by Lemma 4.1.1, we get

$$\operatorname{reg}(\Lambda_{r,s}/(I_{r,s}, \mu_{r+1})) = (r(s + 1) + 1) \operatorname{reg}(\digamma[V(\mathcal{Q})]/I(\mathcal{Q})).$$

Hence by Theorem 2.2.1(c) we have

$$\operatorname{reg}(\Lambda_{r,s}/I_{r,s}) = (r(s + 1) + 1) \operatorname{reg}(\digamma[V(\mathcal{Q})]/I(\mathcal{Q})).$$

□

4.2 Regularity of Cyclic Module Associated with $\Omega_{r,s}\odot \mathcal{Q}$

Lemma 4.2.1. *Let $r, s \geq 1$. If $\Lambda''_{r,s} = \digamma[V(\Gamma''_{r,s} \odot \mathcal{Q})]$ and $J''_{r,s} = I(\Gamma''_{r,s} \odot \mathcal{Q})$, then*

$$\operatorname{reg}(\Lambda''_{r,s}/J''_{r,s}) = ((r + 2)s + r + 1) \operatorname{reg}(\digamma[V(\mathcal{Q})]/I(\mathcal{Q})).$$

Proof. Now, using induction on r, we demonstrate the result for regularity.

If $r = 1$, then we have

$$\operatorname{reg}(\Lambda''_{1,s}/(J''_{1,s} : \mu_2)) = \sum_{k=1}^{s} \digamma[V(W_1 \odot \mathcal{Q})]/I(W_1 \odot \mathcal{Q}) + \sum_{k=1}^{2s+1} \operatorname{reg}(\digamma[V(\mathcal{Q})]/I(\mathcal{Q})).$$

Then by Theorem 2.2.5

$$\operatorname{reg}(\Lambda''_{1,s}/(J''_{1,s} : \mu_2) = (3s + 1) \operatorname{reg}(\digamma[V(\mathcal{Q})]/I(\mathcal{Q})).$$

Also by applying Lemma 2.2.2 and Lemma 2.2.3, we get

$$\begin{aligned}\operatorname{reg}(\Lambda''_{1,s}/(J''_{1,s},\mu_2)) =& \operatorname{reg}(\digamma[V(\Lambda_{2s}\odot\mathcal{Q})]/I(\Lambda_{2s}\odot\mathcal{Q}))\\ &+\sum_{k=1}^{s}\operatorname{reg}(\digamma[V(W_1\odot\mathcal{Q})]/I(W_1\odot\mathcal{Q})+\operatorname{reg}(\digamma[V(\mathcal{Q})]/I(\mathcal{Q})).\end{aligned}$$

Then by using Theorem 2.2.5 and Theorem 2.2.6

$$\operatorname{reg}(\Lambda''_{1,s}/(J''_{1,s},\mu_2)) = (3s+2)\operatorname{reg}(\digamma[V(\mathcal{Q})]/I(\mathcal{Q})).$$

The required result follows by Theorem 2.2.1(c), such that

$$\operatorname{reg}(\Lambda''_{1,s}/I''_{1,s}) = (3s+2)\operatorname{reg}(\digamma[V(\mathcal{Q})]/I(\mathcal{Q})).$$

If $r=2$, then by using the similar arguments and case $r=1$, we get the desired result, that is

$$\operatorname{reg}(\Lambda''_{2,s}/J''_{2,s}) = (4s+3)\operatorname{reg}(\digamma[V(\mathcal{Q})]/I(\mathcal{Q})).$$

Now let $r\geq 3$, Again by Lemma 2.2.3

$$\operatorname{reg}(\Lambda''_{r,s}/(J''_{r,s}:\mu_{r+1})) = \operatorname{reg}(\Lambda''_{(r-2),s}/J''_{(r-2),s}) + \sum_{k=1}^{2s+1}\operatorname{reg}(\digamma[V(\mathcal{Q})]/I(\mathcal{Q})).$$

So by induction, we have

$$\operatorname{reg}(\Lambda''_{r,s}/(J''_{r,s}:\mu_{r+1})) = (rs+r+s-1)(\operatorname{reg}(\digamma[V(\mathcal{Q})]/I(\mathcal{Q})).$$

Now applying Lemma 2.2.2 and Lemma 2.2.3

$$\begin{aligned}\operatorname{reg}(\Lambda''_{r,s}/(J''_{r,s},\mu_{r+1})) =& \operatorname{reg}(\Lambda''_{(r-1),s}/J''_{(r-1),s}) + \sum_{k=1}^{s}\operatorname{reg}(\digamma[V(W_1\odot\mathcal{Q})]/I(W_1\odot\mathcal{Q}))\\ &+\operatorname{reg}(\digamma[V(\mathcal{Q})]/I(\mathcal{Q})).\end{aligned}$$

Again by induction and Theorem 2.2.5

$$\operatorname{reg}(\Lambda''_{r,s}/(J''_{r,s},\mu_{r+1})) = ((r+2)s+r+1)\operatorname{reg}(\digamma[V(\mathcal{Q})]/I(\mathcal{Q})).$$

Hence by Theorem 2.2.1(c) we have

$$\operatorname{reg}(\Lambda''_{r,s}/J''_{r,s}) = ((r+2)s + r + 1)\operatorname{reg}(\digamma[V(\mathcal{Q})]/I(\mathcal{Q})).$$

□

Theorem 4.2.2. *Let $r \geq 3, s \geq 1$. If $\Omega_{r,s} = \digamma[V(O_{r,s} \odot \mathcal{Q})]$ and $J_{r,s} = I(O_{r,s} \odot \mathcal{Q})$, then*

$$\operatorname{reg}(\Omega_{r,s}/J_{r,s}) = r(s+1)\operatorname{reg}(\digamma[V(\mathcal{Q})]/I(\mathcal{Q})).$$

Proof. Now, using Lemma 4.2.1, we demonstrate the result for regularity. If $r = 3$, then we have

$$\operatorname{reg}(\Omega_{1,s}/(J_{1,s} : \mu_3)) = \sum_{k=1}^{2s+2}\operatorname{reg}(\digamma[V(\mathcal{Q})]/I(\mathcal{Q})) + \sum_{k=1}^{s}\operatorname{reg}(\digamma[V(W_1 \odot \mathcal{Q})]/I(W_1 \odot \mathcal{Q})).$$

Then by Theorem 2.2.5, we get

$$\operatorname{reg}(\Omega_{3,s}/(J_{3,s} : \mu_3)) = (3s+2)\operatorname{reg}(\digamma[V(\mathcal{Q})]/I(\mathcal{Q})).$$

Also by applying Lemma 2.2.2 and Lemma 2.2.3, we get

$$\operatorname{reg}(\Omega_{3,s}/(J_{3,s}, \mu_3)) = \Lambda''_{1,s}/J''_{1,s} + \operatorname{reg}(\digamma[V(\mathcal{Q})]/I(\mathcal{Q})).$$

Then by using Lemma 4.2.1, we get

$$\operatorname{reg}(\Omega_{3,s}/(J_{3,s}, \mu_3)) = 3(s+1)\operatorname{reg}(\digamma[V(\mathcal{Q})]/I(\mathcal{Q})).$$

The required result follows by Theorem 2.2.1(c), such that

$$\operatorname{reg}(\Omega_{3,s}/J_{3,s}) = 3(s+1)\operatorname{reg}(\digamma[V(\mathcal{Q})]/I(\mathcal{Q})).$$

If $r = 4$, then by using the similar arguments and case $r = 3$, we get the desired result

$$\operatorname{reg}(\Omega_{4,s}/J_{4,s}) = 4(s+1)\operatorname{reg}(\digamma[V(\mathcal{Q})]/I(\mathcal{Q})).$$

Now let $r \geq 5$. Again by Lemma 2.2.3, we have

$$\operatorname{reg}(\Omega_{r,s}/(J_{r,s} : \mu_r)) = \operatorname{reg}(\Lambda''_{(r-4),s}/J''_{(r-4),s}) + \sum_{k=1}^{2s+2} \operatorname{reg}(\mathcal{F}[V(\mathcal{Q})]/I(\mathcal{Q})).$$

So by Lemma 4.2.1, we have

$$\operatorname{reg}(\Omega_{r,s}/(J_{r,s} : \mu_r)) = (rs + r - 1)(\operatorname{reg}(\mathcal{F}[V(\mathcal{Q})]/I(\mathcal{Q})).$$

Now applying Lemma 2.2.2 and Lemma 2.2.3, we get

$$\operatorname{reg}(\Omega_{r,s}/(J_{r,s}, \mu_r)) = \operatorname{reg}(\Lambda''_{(r-1),s}/J''_{(r-1),s}) + \operatorname{reg}(\mathcal{F}[V(\mathcal{Q})]/I(\mathcal{Q})).$$

By using Lemma 4.2.1, we have $\operatorname{reg}(\Omega_{r,s}/(J_{r,s}, \mu_r)) = r(s+1)\operatorname{reg}(\mathcal{F}[V(\mathcal{Q})]/I(\mathcal{Q}))$. Hence by Theorem 2.2.1(c), we have the desired result

$$\operatorname{reg}(\Omega_{r,s}/J_{r,s}) = r(s+1)\operatorname{reg}(\mathcal{F}[V(\mathcal{Q})]/I(\mathcal{Q})).$$

□

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