Depth and Stanley depth of the edge ideals of some classes of graphs

By

Zahid Iqbal

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Thesis Supervisor: Dr. Muhammad Ishaq

Department of Mathematics

School of Natural Sciences

National University of Sciences and Technology

H-12, Islamabad, Pakistan

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Student Name: **Mr. Zahid Iqbal** Signature: Zahid Iqbal

Examination Committee:

a)	**External Examiner:**		Signature
	Name	Dr. Tariq Shah	
	Designation	Professor	
	Official Address	Department of Mathematics, Quaid-i-Azam University, Islamabad.	

b)	**External Examiner:**		Signature
	Name	Dr. Sarfraz Ahmad	
	Designation	Associate Professor	
	Official Address	Department of Mathematics COMSATS University Islamabad, Lahore Campus.	

c)	**Internal Examiner:**		Signature
	Name	Dr. Matloob Anwar	
	Designation	Associate Professor	
	Official Address	NUST-SNS	

Supervisor Name: **Dr. Muhammad Ishaq** Signature: ____________

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Dedicated
to my
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Acknowledgment

To begin with the name of ALLAH the most benevolent and merciful, the creator of the universe, He Who has knowledge of the most secret part of everything, the most merciful, Who showered upon me His blessings throughout the thick and thin of my life and Who blessed me the courage, good health, company and support of good teachers and friends to conceptualize, develop and complete my research. I offer my humblest, sincerest and millions Duroods to the Holy Prophet Hazrat Muhammad (peace be upon him) who is forever a symbol of guidance for humanity.

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Abstract

In 1982, Stanley suggested the prominent conjecture in which he estimated a combinatorial upper bound for the depth of any finitely generated multigraded module over a polynomial ring. The estimated invariant is now named as the Stanley depth. The Stanley conjecture is attractive in the sense that it compares a homological invariant with a combinatorial invariant of the module. In 2015, Duval et al. constructed a counterexample for Stanley's conjecture. However, there still looks to be a profound and attractive relationship between these two invariants, which is yet to be understood. Furthermore, it is still fascinating to confirm Stanley's inequality for some classes of modules and as result a lower bound for the Stanley depth can be achieved. The study of Stanley depth for modules is a complex problem. Herzog, Vladoiu and Zheng gave a combinatorial method to find Stanley depth. However, it is too difficult to calculate Stanley depth by their method because this is based on hard combinatorial techniques. The aim of this thesis is to provide the values and bounds of Stanley depth and depth of the edge ideals and quotient rings of the edge ideals associated with some classes of graphs. Furthermore, thesis gives a positive answer to Stanley's inequality for quotient rings of the edge ideals related to some classes of graphs. In addition, a positive answer is also given to the Conjecture of Herzog for the edge ideals associated with some classes of graphs.

List of publications

(1) Z. Iqbal, M. Ishaq, M. Aamir, *Depth and Stanley depth of edge ideals of square paths and square cycles*, Comm. Algebra, **46**(3) (2018) 1188-1198.

(2) Z. Iqbal, M. Ishaq, *Depth and Stanley depth of the edge ideals of the powers of paths and cycles*, An. St. Univ. Ovidius Constanta, **27**(3) (2019) 113-135.

(3) Z. Iqbal, M. Ishaq, Muhammad Ahsan Binyamin, *Depth and Stanley depth of the edge ideals of the strong product of some graphs*, to be appear in Hacettepe Journal of Mathematics and Statistics.
https://dergipark.org.tr/en/pub/hujms/issue/42398/638033.

(4) Z. Iqbal, M. Ishaq, *Depth and Stanley depth of edge ideals associated to some line graphs,* AIMS Mathematics, **4**(3) (2019) 686-698.

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Chapter 1

Introduction

1.1 Research Background

In the last few years, combinatorial commutative algebra manifested as a novel and energetic part of mathematics. The work of Melvin Hochster invited the interest of a great number of researchers due to that the modern commutative algebra has become one of the flourishing lines of research. Richard P. Stanley is famous for his appreciable participation to combinatorics and its connection to geometry and algebra, specifically in the theory of simplicial complexes. In 1975, Richard Stanley [74] provided a positive answer to the upper bound conjecture for spheres. This provides new directions in the field of commutative algebra, as it proved that commutative algebra furnishes fundamental approaches in the algebraic study of combinatorics on convex polytopes and simplicial complexes. Stanley was the pioneer to apply the ideas and strategies from commutative algebra in an organized manner to analyse the simplicial complexes by considering of the Hilbert function of Stanley-Reisner rings. Since then the subject of squarefree monomial ideals has become an eye-catching field of research in commutative algebra. Partitionable and Cohen-Macaulay complexes both have prominent places in other complexes, and they play pivotal roles in combinatorics. Stanley presented the main conjecture linking these two notions as follows; are all Cohen-Macaulay simplicial complexes partitionable? In 1982, Stanley defined a notion, which is now named as the Stanley depth of a graded module over a graded commutative ring. By a conjecture of Stanley, this geometric invariant connects to an algebraic invariant depth.

1.2 Problem Statement

Let $S = K[z_1, \ldots, z_n]$ is a polynomial ring in n indeterminates with coefficients in the field K. Let Q be a finitely generated $\mathbb{Z}^n$-graded S-module. The Stanley depth of Q (usually represented by $\mathrm{sdepth}(Q)$), is a combinatorial invariant of Q which was first studied by Apel in [4] and the study of Stanley depth attracted the many researchers, for instance; see [1, 2, 7, 10, 13, 22–25, 33, 36, 62]. The interested reader is referred to [29, 68] for a brief introduction to this subject. The Stanley depth can be described in the form of specific combinatorial decompositions, which are named as Stanley decompositions. These Stanley decompositions have some applications in both pure and applied mathematics. Stanley decompositions may be used to express finitely generated graded algebras, for instance, rings of invariants under some group action [77]. For some other applications in the systems of differential equations; see [58, 59, 72]. Stanley conjectured in [75] that $\mathrm{sdepth}(Q) \geq \mathrm{depth}(Q)$ for any $\mathbb{Z}^n$-graded S-module Q. After 33 years, this conjecture was disproved by Duval and his co-authors in [21] as it was expected due to the different nature of these two invariants. However, there still looks to be a deep and fascinating relationship between depth and Stanley depth, which is yet to be exactly understood. The relation between Stanley depth and some other invariants has already been established; see [34, 36, 65, 78].

In [36], Herzog and his co-authors proved that the Stanley depth of Q can be calculated in a limited number of steps if $Q = B/A$, where $A \subset B \subset S$ are monomial ideals. However, practically it is too hard to find Stanley depth by using this method; see, for instance [7, 15, 17, 53, 55]. The interested reader is encouraged to consult [40,51,66,67,70] for some computing techniques of Stanley depth for some classes of modules. For some parallel results for the Stanley depth and depth, the interested reader may consult to [30, 35-38, 45-47, 66]. For the monomial ideal $A \subset S$, it is well known that $\text{depth}(A) = \text{depth}(S/A) + 1$, this means that once you know about $\text{depth}(S/A)$ then you also know about $\text{depth}(A)$ and vice versa. Whereas for Stanley depth this is not the case, examples are existed where $\text{sdepth}(A) > \text{sdepth}(S/A)$ but until now, no example is known where $\text{sdepth}(A) < \text{sdepth}(S/A)$. Looking at the behaviour of $\text{sdepth}(S/A)$ and $\text{sdepth}(A)$ it seems that the latter inequality is false. For the monomial ideal $A \subset S$, Asia asked the following question.

Question 1.2.1. *[70] Does the following inequality holds*

$$\text{sdepth}(A) \geq \text{sdepth}(S/A) + 1?$$

A weaker form of the above inequality is given as a conjecture by Herzog as follows:

Conjecture 1.2.1 ([29, Conjecture 64])**.** *Let $A \subset S$ be a monomial ideal then*

$$\text{sdepth}(A) \geq \text{sdepth}(S/A).$$

The aforementioned conjecture has been proved in some special cases by Popescu and Qureshi in [66] and Asia in [70]. Recently, Keller and Young in [55], proved it for any squarefree monomial ideal of $S = K[z_1, \ldots, z_7]$.

1.3 Research Objectives

The main objectives of this research are stated in the following:

(1) To determine the values and bounds of Stanley depth and depth of the edge ideals and quotient rings of the edge ideals associated with some classes of graphs.

(2) To give a positive answer to the Conjecture 1.2.1 for the edge ideals associated with some classes of graphs.

(3) To show that Stanley's inequality holds for the quotient rings of the edge ideals associated with some classes of graphs.

1.4 Scope of the Research

This research focuses on the values and bounds of Stanley depth and depth of the edge ideals and quotient rings of the edge ideals associated with some classes of graphs. A positive answer is provided to the Conjecture 1.2.1 for these classes of edge ideals. Furthermore, this research shows that Stanley's inequality holds for the quotient rings of the edge ideals associated with some classes of graphs.

1.5 Significance of the Research

In 2015, Duval et al. [21] proved that Stanley conjecture is false for modules of the type S/I in general. However, there still looks to be a deep and interesting relationship between depth and Stanley depth, which is yet to be exactly understood. Furthermore, it is still interesting to prove Stanley's inequality for some classes of modules because in this case, a lower bound for the Stanley depth can be obtained. The study of Stanley depth for modules is a hard problem because, until now, there is no efficient method to compute Stanley depth. Let $I \subset J \subset S$ be monomial ideals, Herzog et al. in [36] gave a combinatorial method to compute Stanley depth of J/I. However, it is too hard to compute Stanley depth by their method because the method is based on hard combinatorial techniques.

This research provides the values and bounds of Stanley depth and depth of the edge ideals and quotient rings of the edge ideals associated with some classes of graphs. Furthermore, a positive answer is given to the Conjecture 1.2.1 for the edge ideals associated with some classes of graphs. Furthermore, this research shows that Stanley's inequality holds for the quotient rings of the edge ideals associated with some classes of graphs.

1.6 Research Methodology

In this research, by using the combinatorial/homological techniques, values and bounds of Stanley depth and depth of the edge ideals and quotient rings of the edge ideals associated with some classes of graphs are provided. In most of the results, for initial cases, the computer algebra system CoCoA [71] is used. For the remaining cases, the mathematical induction is used.

1.7 Organization of the Thesis

This thesis is structured as follows.

Chapter 1 describes the research background, research objectives, research methodology, the scope of the research, significance of the research as well as thesis organization.

Chapter 2 contains some required definitions of graph theory. In chapter 3, some necessary material of commutative algebra is given. Furthermore, some known results and computational technique related to Stanley depth of multigraded S-modules are discussed in this chapter.

In chapter 4, precise values of Stanley depth and depth for the quotient ring of the edge ideal related to a square path on n vertices are given. For $n \equiv 0, 3, 4 (\text{mod } 5)$, exact values of Stanley depth and depth for the quotient ring of the edge ideal related to a square cycle on n vertices are provided. In the remaining cases, tight bounds are established. Furthermore, the conjecture of Herzog presented in [29] is proved for the edge ideals of square paths and square cycles. These results are published in [42].

Chapter 5 provides the generalized results of the fourth chapter. Precise values are provided for the Stanley depth and depth of the quotient ring of the edge ideal related with the t^{th} power of a path on n vertices, where $t \geq 3$. If $n \equiv 0, t+1, t+2, \dots, 2t (\text{mod}(2t+1))$, then values are given for the Stanley depth and depth of the quotient ring of the edge ideal associated to the t^{th} power of a cycle on n vertices and tight bounds otherwise. Also, lower bounds for the Stanley depth of the edge ideals associated with the t^{th} power of a path and a cycle are established. These bounds are good enough that a positive answer is given to the conjecture of Herzog for these ideals. These results are elaborated also in [44].

In chapter 6, some upper and lower bounds for Stanley depth and depth of edge ideals associated to line graphs of the ladder and circular ladder graphs are established. Furthermore, some bounds for the dimension of the quotient rings of the edge ideals related to these graphs are also given. These results are taken from [43].

In chapter 7, results are discussed that are related to Stanley depth and depth of the edge ideals and quotient rings of the edge ideals associated with classes of graphs obtained by taking the strong product of two graphs. In this chapter, the strong product of two graphs is considered, when either both graphs are arbitrary paths or one is an arbitrary path and the other is an arbitrary cycle. Exact formulae for values of Stanley depth and depth for some subclasses are given. Also some sharp upper bounds for Stanley depth and depth in the general cases are established. These results are discussed in [45].

The final chapter provides a recap of the entire thesis and sums up the whole idea. The future perspectives of the research, and the conclusion are presented in Chapter 8.

Figure 1.1 gives the organization of the whole thesis.

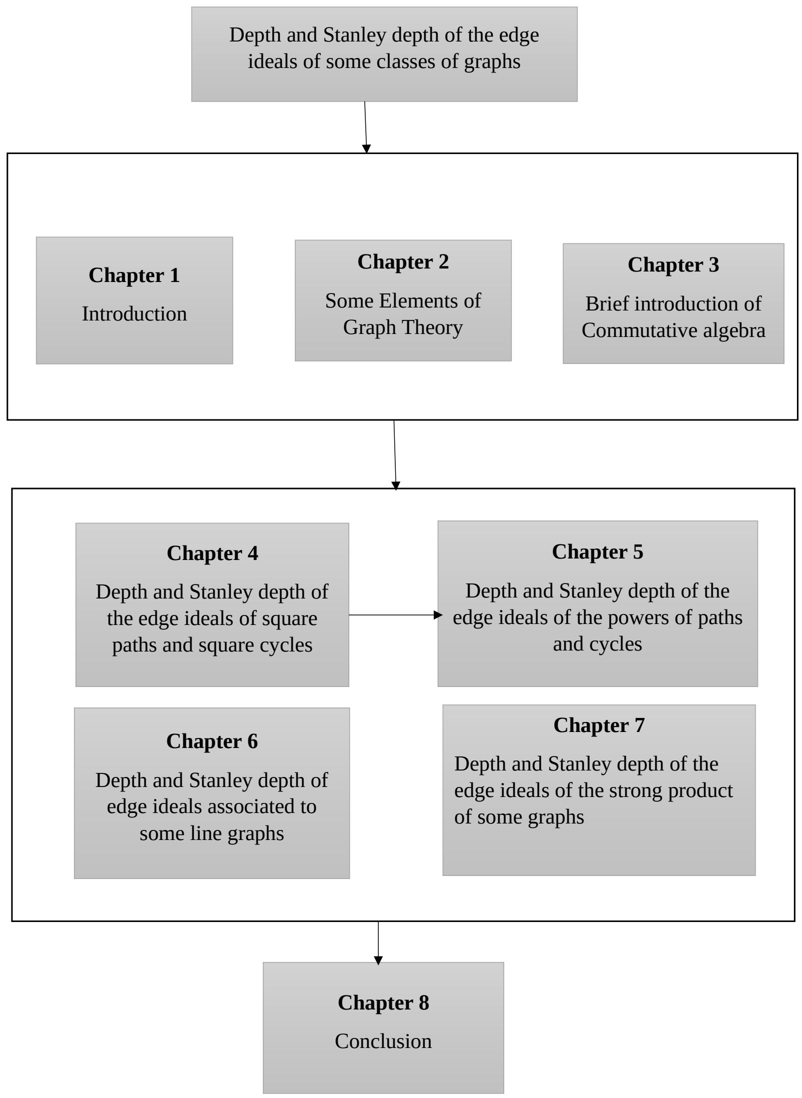

Figure 1.1: Organization of the whole thesis.

Chapter 2

Some Elements of Graph Theory

This chapter describes some basic notions, terminologies and definitions of graph theory that are used in the rest of this dissertation. For further details, the interested reader is referred to [8, 11, 28].

2.1 Basic Notions

Definition 2.1.1. *[8]* A graph U is an ordered pair $U = (V(U), E(U))$, where $V(U)$ is a non-empty set and $E(U)$ is a set of two-element subsets of $V(U)$. The elements of $V(U)$ are the vertices of U and the elements of $E(U)$ are the edges of U.

The order and size of a graph U are defined as the cardinalities of $V(U)$ and $E(U)$, respectively. The vertex set $V(U)$ of a graph U is always non-empty. A graph with a finite set of vertices is said to be a finite graph, on the other hand, it is called an infinite graph. Two or more edges that attach the same pair of vertices are known as multiple edges, and a loop is an edge that joins a vertex with itself. A graph U is said to be a simple graph if it has no loop and multiple edges. Throughout this work, all the graphs are undirected, simple and finite.

Definition 2.1.2. *[11]* If $v, w \in V(U)$, then $e = vw$ is an edge between v and w, and they are said to be end vertices of e.

Definition 2.1.3. *[11]* The degree of a vertex v is the number of edges that are incident with it.

Definition 2.1.4. *[11]* For a vertex u in a graph U, the neighborhood of u is commonly expressed and defined as follows:

$$N_U(u) = \{v \ : \ \{v, u\} \in E(U)\},$$

that is, the set consisting of all neighbors of u.

Definition 2.1.5. *[8]* A graph $U = (V(U), E(U))$ is said to be a subgraph of a graph $T = (V(T), E(T))$, if $V(U) \subseteq V(T)$ and $E(U) \subseteq E(T)$, and in this case T is called supergraph of U.

Definition 2.1.6. *[11]* A vertex cover of a graph U is a subset B of $V(U)$ such that for every edge $e \in E(U)$, $e \cap B \neq \emptyset$ and B is minimal with respect to this property, that is for any proper subset B' of B, then there exists an edge $f \in E(U)$ with $f \cap B' = \emptyset$.

2.2 Some Common types of Graphs

Definition 2.2.1. *[8]* A graph U is known as the r-regular if for every vertex $v \in U$, the set $N_U(v)$ has the cardinality r.

Definition 2.2.2. *[11]* If U is a graph with $V(U) = \{a_1, a_2, \ldots, a_n\}$, then U is said to be a path if $E(U) = \{\{a_i, a_{i+1}\} : i \in [n-1]\}$.

Definition 2.2.3. *[8]* If U is a graph with $V(U) = \{w_1, \ldots, w_n\}$, then U is called a cycle if $E(U) = \{\{w_i, w_{i+1}\} : i \in [n-1]\} \bigcup \{\{w_1, w_n\}\}$.

Definition 2.2.4. *[11]* An n-vertex graph is called a complete graph denoted by K_n, if each pair of vertices of K_n is connected through an edge. The complete graph K_1 is a trivial and simple graph with exactly one vertex.

Definition 2.2.5. *[8]* An n-vertex graph U is called a bipartite graph if $V(U)$ can be classified into disjoint sets U_1 and U_2 such that each edge of U has one end point in U_1 and other end point in U_2.

A complete bipartite graph $K_{n,m}$ is a bipartite graph with bipartition (U_1, U_2) where $|U_1| = n$ and $|U_2| = m$ such that each vertex of U_1 is joined with each vertex of U_2 through an edge.

2.3 Connected and Disconnected Graphs

Definition 2.3.1. *[8]* A graph U is said to be a connected graph, if for every pair of vertices $a, b \in V(U)$, there exists a a-b path in U. If the graph U does not satisfy this property, then it is called a disconnected graph.

Definition 2.3.2. *[11]* For $a, b \in V(U)$ of a graph U, the length of a shortest path from a to b is called the distance between them, and it is symbolized by $\mathrm{d}_U(a, b)$. If there exists no such path between them, then $d_U(a, b) = \infty$.

Definition 2.3.3. *[11]* The diameter of a connected graph U is denoted and define as follows:

$$\mathrm{diam}(U) := \max\{\mathrm{d}_U(a, b) : a, b \in V(U)\}.$$

2.4 Powers of Graphs

Definition 2.4.1. *[8]* For a simple n-vertex graph U, the square of a graph U is the graph U^2 on the vertex set $V(U)$, and every two vertices, which are at distance 2 or less in U, they are connected by an edge in the graph U^2.

Definition 2.4.2. *[11]* Let $n \geq 3$ and P_n be a path on $[n] = \{1, 2, \ldots, n\}$ vertices, then the edge set of *square of a path* (or *square path*) is

$$E(P_n^2) = \big\{\{k, k+1\}, \{k, k+2\} : k \in [n-2]\big\} \cup \big\{\{n-1, n\}\big\}.$$

See Fig. 2.1 for examples of P_n^2.

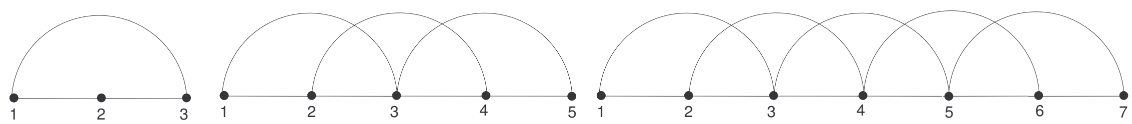

Figure 2.1: From left to right; square paths P_n^2 with $n = 3$, $n = 5$ and $n = 7$, respectively.

Definition 2.4.3. *[11]* Let $n \geq 5$ and C_n be a cycle on $[n]$ vertices, then the edge set of *square of cycle* (or *square cycle*) is

$$E(C_n^2) = \big\{\{k, k+1\}, \{k, k+2\} : k \in [n-2]\big\} \cup \big\{\{n-1, 1\}, \{n, 1\}, \{n-1, n\}, \{n, 2\}\big\}.$$

See Fig. 2.2 for examples of C_n^2.

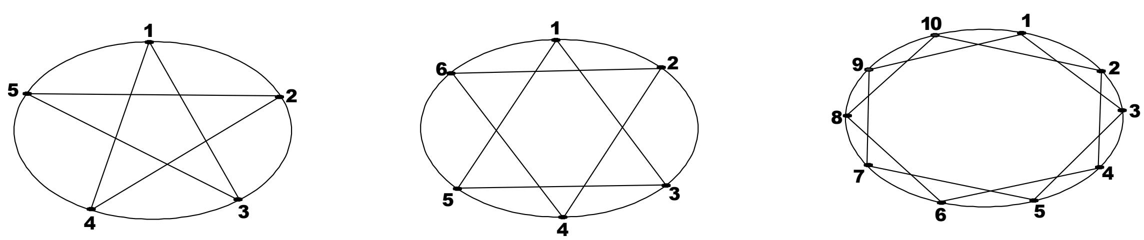

Figure 2.2: From left to right; square cycles C_n^2 with $n = 5$, $n = 6$ and $n = 10$, respectively.

Definition 2.4.4. *[11]* Let U be a simple graph. For $t \in \mathbb{Z}^+$, the t^{th} power of a graph U is another graph U^t on the vertex set $V(U)$, such that two vertices are connected through an edge in U^t when in U, the distance between them is at most t.

Definition 2.4.5. *[11]* For $n \geq 2$, the t^{th} power of a path, denoted by P_n^t, is a graph such that $\forall\ 1 \leq j < k \leq n$, $\{j,k\} \in E(P_n^t)$ iff $0 < k-j \leq t$. If $n \leq t+1$, then P_n^t is a complete graph on $[n]$ vertices. If $n \geq t+2$, then

$$E(P_n^t) = \cup_{j=1}^{n-t}\big\{\{j,j+1\},\dots,\{j,j+t\}\big\}\cup$$
$$\cup_{k=n-t+1}^{n-1}\big\{\{k,k+1\},\dots,\{k,n\}\big\}.$$

Definition 2.4.6. *[11]* For $n \geq 3$, the t^{th} power of a cycle, denoted by C_n^t, is a graph such that $\forall\ 1 \leq j,k \leq n$, $\{j,k\} \in E(C_n^t)$ iff $|k-j| \leq t$ or $|k-j| \geq n-t$. If $n \leq 2t+1$, then C_n^t is a complete graph on $[n]$ vertices. If $n \geq 2t+2$, then

$$E(C_n^t) = E(P_n^t)\cup\cup_{l=1}^{t}\big\{\{l,l+n-t\},\{l,l+n-t+1\},\{l,l+n-t+2\},\dots,\{l,n\}\big\}.$$

For examples of powers of paths and cycles see Figures 2.3 and 2.4.

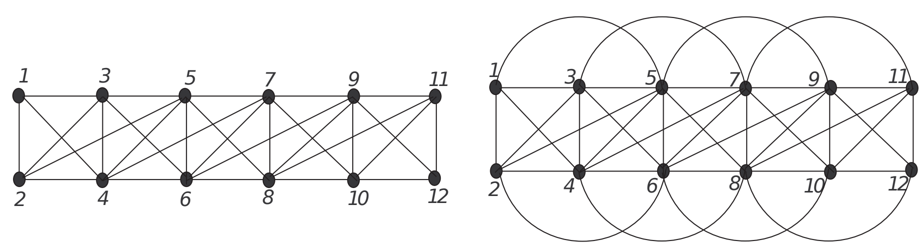

Figure 2.3: From left to right, P_{12}^3 and P_{12}^4 respectively.

2.5 Two Standard Graph Products

Definition 2.5.1. *[27]* The Cartesian product of two graphs U_1 and U_2 is usually denoted by $U_1 \square U_2$. This graph has vertex set $V(U_1) \times V(U_2)$, and for $(v_1,u_1),(v_2,u_2) \in V(U_1 \square U_2)$, $(v_1,u_1)(v_2,u_2) \in E(U_1 \square U_2)$, if either

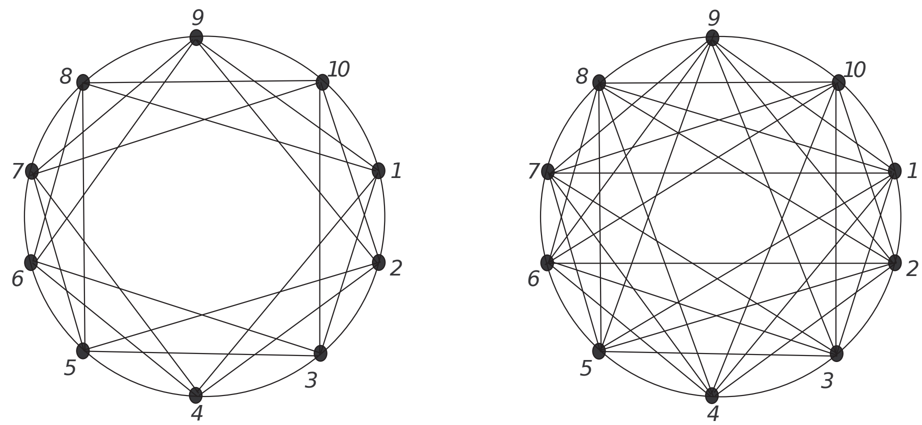

Figure 2.4: From left to right, C_{10}^3 and C_{10}^4 respectively.

- $\{v_1, v_2\} \in E(U_1)$ and $u_1 = u_2$ or
- $v_1 = v_2$ and $\{u_1, u_2\} \in E(U_2)$.

If $n \geq 2$, then the Cartesian product $P_2 \square P_n$ of two paths P_2 and P_n is called the ladder graph which is denoted as $\mathcal{L}_n$ that is $\mathcal{L}_n := P_2 \square P_n$. For $n \geq 3$, the Cartesian product of P_2 and C_n is the circular ladder graph which is represented by $\mathcal{CL}_n$ that is $\mathcal{CL}_n := P_2 \square C_n$.

Definition 2.5.2. *[8]* For a given graph U, the line graph of U is usually denoted by $L(U)$. It is a graph with vertex set $V(L(U)) = E(U)$ and two vertices in $L(U)$ are adjacent iff the corresponding edges in U share a vertex.

For examples of the ladder, circular ladder graphs and their corresponding line graphs see Figures 2.5 and 2.6.

Definition 2.5.3 ([27])**.** The strong product $U_1 \boxtimes U_2$ of graphs U_1 and U_2 is a graph with $V(U_1 \boxtimes U_2) = V(U_1) \times V(U_2)$, and for $(v_1, u_1), (v_2, u_2) \in V(U_1 \boxtimes U_2)$, $(v_1, u_1)(v_2, u_2) \in E(U_1 \boxtimes U_2)$, whenever

- $\{v_1, v_2\} \in E(U_1)$ and $u_1 = u_2$ or

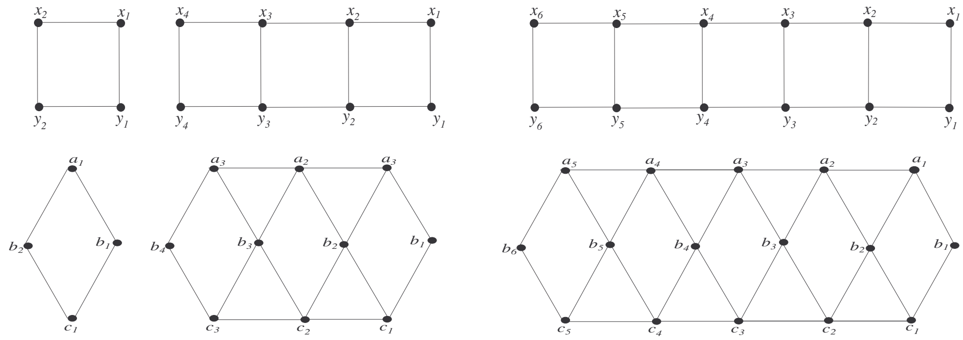

Figure 2.5: $\mathcal{L}_2$, $\mathcal{L}_4$, $\mathcal{L}_6$ and their line graphs $L(\mathcal{L}_2)$, $L(\mathcal{L}_4)$, $L(\mathcal{L}_6)$ respectively.

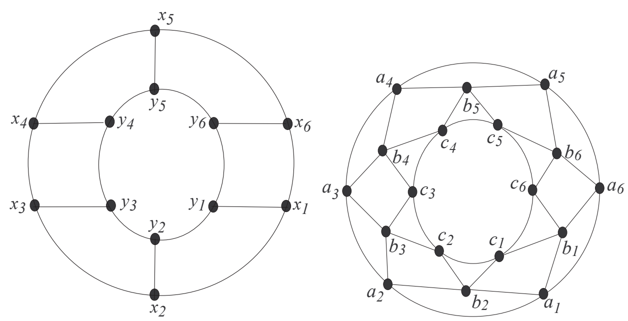

Figure 2.6: From left to right, $\mathcal{CL}_6$ and $L(\mathcal{CL}_6)$.

- $v_1 = v_2$ and $\{u_1, u_2\} \in E(U_2)$ or
- $\{v_1, v_2\} \in E(U_1)$ and $\{u_1, u_2\} \in E(U_2)$.

For $n \geq 2$, let $\mathcal{P}_{n,m} := P_n \boxtimes P_m \cong P_m \boxtimes P_n$, and for $m \geq 1$ and $n \geq 3$, let $\mathcal{C}_{n,m} := C_n \boxtimes P_m \cong P_m \boxtimes C_n$. For examples of $\mathcal{P}_{n,m}$ and $\mathcal{C}_{n,m}$, see Figures 2.7 and 2.8.

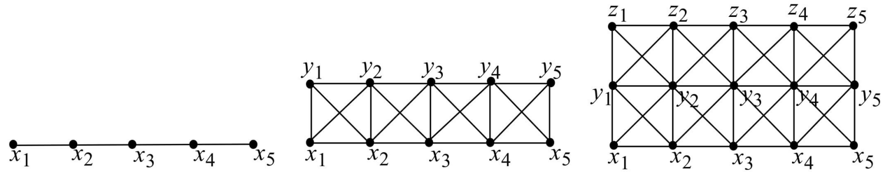

Figure 2.7: From left to right; $\mathcal{P}_{5,1}$, $\mathcal{P}_{5,2}$ and $\mathcal{P}_{5,3}$.

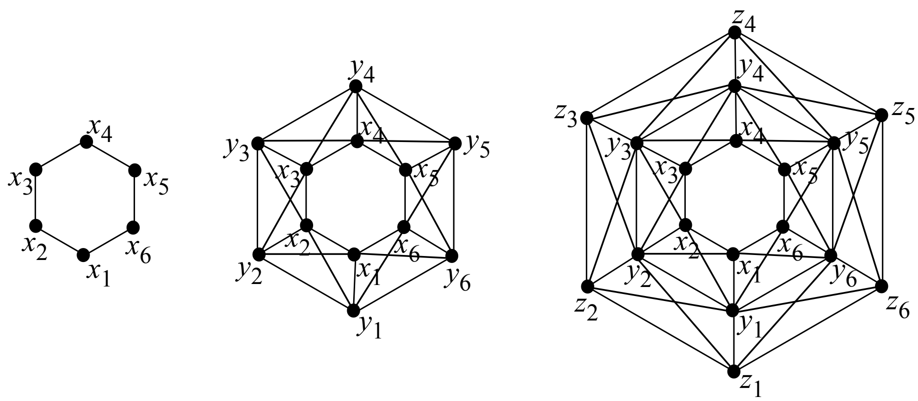

Figure 2.8: From left to right; $\mathcal{C}_{6,1}$, $\mathcal{C}_{6,2}$ and $\mathcal{C}_{6,3}$.

2.6 Conclusion

In this chapter, some basic notions, terminologies and definitions from graph theory are presented. Furthermore, the concept of power of a graph is discussed. The two well-known graph operations are explained at the end of the chapter.

Chapter 3

Brief introduction of Commutative algebra

This chapter is divided into eight sections. In the first five sections, some necessary material of commutative algebra is given. A method for computing the Stanley depth of a special type of module is given in section 6.

Some known results of depth and Stanley depth are presented in the last section. For further details, the interested reader is referred to [6, 19, 30, 79].

3.1 Ring theory

Definition 3.1.1. *[19]* A ring is a set $\mathfrak{T}$ with two operations $+$ and $\times$ such that $(\mathfrak{T}, +)$ is an abelian group, the multiplication is associative and left and right distributes laws hold in $\mathfrak{T}$.

Definition 3.1.2. *[19]* The ring $\mathfrak{T}$ is commutative if for all $a, c \in \mathfrak{T}$, $ac = ca$.

Definition 3.1.3. *[19]* The ring $\mathfrak{T}$ is said to be a ring with unity if there is an element $1 \in \mathfrak{T}$ with $1 \times z = z \times 1 = z$ for all $z \in \mathfrak{T}$.

Throughout this dissertation, all rings are considered to be commutative with unity.

Definition 3.1.4. *[19]* Let $\mathfrak{T}$ be a ring. Then the set of all polynomials in a variable z with coefficients in $\mathfrak{T}$ form a ring under usual addition and multiplication of polynomials, this ring is denoted by $\mathfrak{T}[z]$. The polynomial ring in the variables $z_1, z_2, \dots, z_n$ with coefficients in $\mathfrak{T}$, represented by $\mathfrak{T}[z_1, \dots, z_n]$, is defined inductively by $\mathfrak{T}[z_1, \dots, z_n] = \mathfrak{T}[z_1, \dots, z_{n-1}][z_n]$.

Definition 3.1.5. *[19]* A subset A of a ring $\mathfrak{T}$ is called an ideal of $\mathfrak{T}$ if:

- A is an additive subgroup of $\mathfrak{T}$ and,
- $\forall\, x \in A$ and $\forall\, r \in \mathfrak{T}$, $rx \in A$.

Definition 3.1.6. *[19]* An ideal A of $\mathfrak{T}$ is called a primary ideal if $xy \in A$ then $x \in A$ or $y^a \in A$ for some $a \geq 1$.

Definition 3.1.7. *[19]* An ideal A of $\mathfrak{T}$ is called a prime ideal if $xy \in A$ then $x \in A$ or $y \in A$. Let $A_0, A_1, \dots, A_k$ be the prime ideals of $\mathfrak{T}$. A chain of prime ideals $A_0 \subsetneq A_1 \subsetneq \cdots \subsetneq A_k$ is said to be a chain of length k.

Definition 3.1.8. *[19]* The Krull dimension of a ring $\mathfrak{T}$, usually represented by $\dim(\mathfrak{T})$, is the length of the longest chain of its prime ideals.

Definition 3.1.9. *[19]* An ideal $\mathfrak{m}$ in a ring $\mathfrak{T}$ is said to be a maximal ideal if I is an ideal such that $\mathfrak{m} \subseteq I \subseteq \mathfrak{T}$, then $\mathfrak{m} = I$ or $I = \mathfrak{T}$.

Definition 3.1.10. *[19]* A ring $\mathfrak{T}$ is a local ring if it has a unique maximal ideal.

Definition 3.1.11. *[19]* A ring $\mathfrak{T}$ is a Noetherian ring if it satisfies the ascending chain condition on its ideals, that is for any chain of ideals:

$$A_1 \subset A_2 \subset \cdots \subset A_{k+1} \subset \dots$$

there exists $k \in \mathbb{Z}^+$ such as

$$A_k = A_{k+1} = \dots.$$

3.2 Module theory

Definition 3.2.1. *[19]* Let $\mathfrak{T}$ be a ring, a $\mathfrak{T}$-module Q is an abelian group under addition and an action of $\mathfrak{T}$ on Q (that is a map $\mathfrak{T} \times Q \longrightarrow Q$) represented by az for all $a \in \mathfrak{T}$ and $z \in Q$ that satisfies the following axioms:

1. $a(y+z) = ay + az$, $\forall\ a \in \mathfrak{T}$ and $y, z \in Q$,

2. $(ab)z = a(bz)$, $\forall\ a, b \in \mathfrak{T}$ and $z \in Q$,

3. $(a+b)z = az + bz$, $\forall\ a, b \in \mathfrak{T}$ and $z \in Q$,

4. $1z = z$, $\forall\ z \in Q$.

Definition 3.2.2. *[19]* Let Q is a $\mathfrak{T}$-module, a $\mathfrak{T}$-submodule A of Q is an additive subgroup of Q such that $ra \in A$ for all $r \in \mathfrak{T}$, $a \in A$.

Definition 3.2.3. *[19]* Let Q is a $\mathfrak{T}$-module and $Q_1, Q_2, \dots, Q_n$ be submodules of Q. The sum of $Q_1, Q_2, \dots, Q_n$ is denoted by $Q_1 + Q_2 + \dots + Q_n$ and it is the set of all finite sums of elements of Q_i as follows:

$$Q_1 + Q_2 + \dots + Q_n = \{b_1 + b_2 + \dots + b_n \mid b_i \in Q_i \ \ \forall\ i\}.$$

Definition 3.2.4. *[19]* Let Q is a $\mathfrak{T}$-module and $B \subset Q$. For $i \in \mathbb{Z}^+$, $r_1, r_2, \dots, r_i \in \mathfrak{T}$, $b_1, b_2, \dots, b_i \in B$, then $\mathfrak{T}B = \{r_1b_1 + r_2b_2 + \dots + r_ib_i\}$. If $B = \{b_1, b_2, \dots, b_m\}$, then $\mathfrak{T}B$ can be written as $\mathfrak{T}B = \mathfrak{T}b_1 + \mathfrak{T}b_2 + \dots + \mathfrak{T}b_m$.

Definition 3.2.5. *[19]* A submodule A of Q is called finitely generated if there is some finite subset B of Q such that $A = \mathfrak{T}B$.

Definition 3.2.6. A $\mathfrak{T}$-module Q is said to be Noetherian if each of its submodule is finitely generated.

Definition 3.2.7. *[19]* A submodule A of Q is called cyclic if there exists an element $m \in Q$ such that $A = \mathfrak{T}m$, that is, if A can be generated by a single element:

$$A = \mathfrak{T}m = \{rm \mid r \in \mathfrak{T}\}.$$

Definition 3.2.8. *[19]* Let $Q_1, Q_2, \ldots, Q_m$ be a collection of $\mathfrak{T}$-modules. The collection of i-tuples $(n_1, n_2, \ldots, n_i)$ where $n_k \in Q_k$ with addition and action of $\mathfrak{T}$ defined componentwise is said to be the direct product of these modules.

Remark 3.2.1. *[19]* The direct product of a collection of $\mathfrak{T}$-modules is again a $\mathfrak{T}$-module. The direct product of $Q_1, Q_2, \ldots, Q_m$ is also referred as the direct sum of $Q_1, Q_2, \ldots, Q_m$.

Definition 3.2.9. *[19]* A $\mathfrak{T}$-module Q is called free module on the subset A of Q, if for any non-zero element $m \in Q$, there exist unique non-zero elements $x_1, x_2, \ldots, x_n$ of $\mathfrak{T}$ and unique elements $a_1, a_2, \ldots, a_n$ in A such that $m = x_1a_1 + x_2a_2 + \cdots + x_na_n$ for some $n \in \mathbb{Z}^+$. Then A is said to be basis or set of free generators of Q.

Definition 3.2.10. *[19]* Let A, B and C be $\mathfrak{T}$-modules over ring $\mathfrak{T}$. Then the pair of $\mathfrak{T}$-homomorphisms $A \xrightarrow{\alpha} B \xrightarrow{\beta} C$ is called exact at B if image$(\alpha) = \ker(\beta)$.

Definition 3.2.11. *[19]* A sequence $\ldots \longrightarrow A_{m-1} \longrightarrow A_m \longrightarrow A_{m+1} \ldots$ of $\mathfrak{T}$-modules and $\mathfrak{T}$-homomorphisms is called an exact sequence if it is exact at every A_m between a pair of homomorphisms.

Proposition 3.2.1 ([19, Proposition 22])**.** *Let X, Y and Z be $\mathfrak{T}$-modules over ring $\mathfrak{T}$. Then*

1. *The sequence* $0 \longrightarrow X \xrightarrow{\pi} Y$ *is called exact at* X *if and only if* π *is injective.*

2. *The sequence* $Y \xrightarrow{\psi} Z \longrightarrow 0$ *is called exact at* Z *if and only if* ψ *is surjective.*

Corollary 3.2.1 ([19, Corollary 23]). *The sequence* $0 \longrightarrow X \xrightarrow{\pi} Y \xrightarrow{\psi} Z \longrightarrow 0$ *is exact if and only if* π *is injective,* ψ *is surjective and* $\text{image}(\pi) = \ker(\psi)$.

Definition 3.2.12. *[19]* The exact sequence $0 \longrightarrow X \xrightarrow{\pi} Y \xrightarrow{\psi} Z \longrightarrow 0$ is said to be a short exact sequence.

Definition 3.2.13. *[30]* Let $\mathfrak{T}$ be a ring, Q be a $\mathfrak{T}$-module and A be a proper subset of Q. The annihilator of A is usually represented by $\text{Ann}_{\mathfrak{T}}(A)$. It consists of all elements r in $\mathfrak{T}$ such that for each a in A, $ra = 0$. In set notation, it can be written as follows:

$$\text{Ann}_{\mathfrak{T}}(A) = \{r \in \mathfrak{T} : \forall\ a \in A, ra = 0\}.$$

Definition 3.2.14. *[30]* Let Q be a $\mathfrak{T}$-module, a monomial prime ideal P of $\mathfrak{T}$ is called an associated prime of Q if $P = \text{Ann}(m)$ for a non-zero element m of Q. The set of associated primes of Q is denoted by $\text{Ass}_{\mathfrak{T}}(Q)$.

3.3 Monomial ideals

Let $S = K[z_1, \dots, z_n]$, and $\mathbb{N}^n$ represents the set of vectors $\beta = (b_1, \dots, b_n)$ with each $b_j \geq 0$ is an integer. Any arbitrary product of the type $u = z_1^{b_1} z_2^{b_2} \dots z_n^{b_n}$ with $b_j \in \mathbb{N}$ is said to be a monomial, and it can be written as $u = z^{\beta}$. If $\mathcal{B}$ denotes the set of monomials of S then $\mathcal{B}$ is a K-basis of S. That is, if $g \in S$ then g can be written as a linear combination of elements of $\mathcal{B}$ over K as follows:

$$g = \sum_{u \in \mathcal{B}} u b_u,$$

where $b_u \in K$. The sets $\text{supp}(g) = \{u \in \mathcal{B} : b_u \neq 0\}$, and $\text{supp}(u) = \{j : z_j | u\}$ represent the support of g and u respectively.

Definition 3.3.1. *[30]* Let $A \subset S$ be an ideal, then A is a monomial ideal if it is generated by the set of monomials.

Theorem 3.3.1. *[30] If $\mathcal{L}$ is the set of monomials of the monomial ideal A, then $\mathcal{L}$ is a K-basis of A.*

Proposition 3.3.1. *[30] Let $A \subset S$ be a monomial ideal, then A has a unique minimal set of monomial generators.*

The unique minimal set of the monomial ideal A is denoted by $\mathcal{G}(A)$.

Definition 3.3.2. *[30]* A monomial a is said to be a squarefree if $a = z_{j_1} z_{j_2} \dots z_{j_m}$ for some $1 \leq j_1 < j_2 < \cdots < j_m \leq n$.

Definition 3.3.3. *[30]* Any monomial ideal is said to be a squarefree monomial ideal if it is generated by squarefree monomials.

Definition 3.3.4. *[30]* Let $U := (V(U), E(U))$ be a graph with vertex set $V(U) := \{x_1, x_2, \dots, x_n\}$ and edge set $E(U)$. The edge ideal $I(U)$ associated to U is the squarefree monomial ideal, that is $I(U) = (x_i x_j : \{x_i, x_j\} \in E(U))$.

Remark 3.3.1. The intersection, product, and sum of monomial ideals are again monomial ideals. Furthermore, for the monomial ideals A_1 and A_2 of S, it follows that $\mathcal{G}(A_1 + A_2) \subset \mathcal{G}(A_1) \cup \mathcal{G}(A_2)$, $\mathcal{G}(A_1 A_2) \subset \mathcal{G}(A_1)\mathcal{G}(A)$ and $A_1 \cap A_2 = (\operatorname{lcm}(a, b) : a \in \mathcal{G}(A_1), b \in \mathcal{G}(A_2))$, where $\operatorname{lcm}(a, b)$ represents the least common multiple of a and b.

Proposition 3.3.2. *[30] The colon ideal $(A_1 : A_2)$ of two monomial ideals A_1 and A_2 of S, is again a monomial ideal and $(A_1 : A_2) = \bigcap_{e \in \mathcal{G}(A_2)} (A_1 : (e))$. Moreover, $\{d / \gcd(e, d) : d \in \mathcal{G}(A_1)\}$ is a set of generators of $(A_1 : (e))$.*

Example 3.3.1. Let $S = K[z_1, z_2, z_3, z_4]$, $A_1 = \{z_2 z_3^2, z_1 z_2^2, z_1^2, z_3 z_4^2\}$ and $A_2 = \{z_2 z_1^3, z_2 z_3, z_4 z_3\}$ are monomial ideals of S. Then

$$A_1 A_2 = \{z_1^2 z_3 z_4, z_1^2 z_2 z_3, z_3^2 z_4^3, z_1 z_2^2 z_3 z_4, z_2 z_3^3 z_4, z_2 z_3^2 z_4^2, z_1 z_2^3 z_3, z_2^2 z_3^3, z_1^5 z_2, z_1^4 z_2^3\},$$

$$A_1 + A_2 = \{z_3z_4, z_2z_3, z_1^2, z_1z_2^2\},$$

$$A_1 \cap A_2 = \{z_2z_3^2, z_3z_4^2, z_1^3z_2, z_1^2z_2z_3, z_1z_2^2z_3, z_1^2z_3z_4\},$$

$$(A_1 : A_2) = \{z_2z_3, z_1^2, z_4^2, z_1z_2z_4, z_1z_2^2, z_3z_4\}.$$

3.4 Graded rings and graded modules

Definition 3.4.1. *[30]* Let $(H, +)$ be an abelian semigroup. An H-graded ring is a ring $\mathfrak{T}$ together with a decomposition

$$\mathfrak{T} = \bigoplus_{a \in H} \mathfrak{T}_a \text{ (as a group)},$$

such that $\mathfrak{T}_a\mathfrak{T}_b \subset \mathfrak{T}_{a+b}$ for all $a, b \in H$. Then for $r \in \mathfrak{T}$, a unique expression can be written as follows:

$$r = \sum_{a \in H} r_a,$$

where $r_a \in \mathfrak{T}_a$ and almost all $r_a = 0$. The element r_a is called the a^{th} homogeneous component and if $r = r_a$, then r is homogeneous of degree a.

Definition 3.4.2. *[30]* Let $(H, +)$ be an abelian semigroup, $\mathfrak{T}$ is an H-graded ring and Q is a $\mathfrak{T}$-module with

$$Q = \bigoplus_{a \in H} Q_a \text{ (as a group)}$$

such that $\mathfrak{T}_aQ_b \subset Q_{a+b}$ $\forall a, b \in H$, then Q is said to be an H-graded module. A non-zero element of Q_a is called a homogenous element of degree a.

Example 3.4.1. Let $\beta = (b_1, b_2, \ldots, b_n) \in \mathbb{Z}^n$ and $z^\beta = z_1^{b_1}z_2^{b_2}\ldots z_n^{b_n}$. Then $g \in S := K[z_1, \ldots, z_n]$ is said to be a homogeneous element of degree β if $g = cz^\beta$ and $c \in K$. Here S is clearly a $\mathbb{Z}^n$-graded ring that is $S = \bigoplus_{\beta \in \mathbb{Z}^n} S_\beta$, where

$$S_\beta = \begin{cases} Kz^\beta, & \text{if } \beta \in \mathbb{Z}_+^n; \\ 0, & \text{otherwise.} \end{cases}$$

An S-module Q is said to be a $\mathbb{Z}^n$-graded if $Q = \bigoplus_{\beta \in \mathbb{Z}^n} Q_\beta$ and $S_\beta Q_\gamma \subset Q_{\beta+\gamma}$ for all $\beta, \gamma \in \mathbb{Z}^n$.

3.5 Depth

In this section, some definitions related to depth are provided.

Definition 3.5.1. *[19]* Let $\mathfrak{T}$ be a ring and Q be a $\mathfrak{T}$ module. A zero divisor of a module Q is an element x of $\mathfrak{T}$ such that $xq = 0$ for some non-zero q in Q.

Definition 3.5.2. *[30]* Let Q be a module over a ring $\mathfrak{T}$, $a \in \mathfrak{T}$ is said to be Q-regular if for any $m \in Q$, $am = 0$ implies $m = 0$. In other words, a is not a zero-divisor on Q or the multiplication by a on Q is an injective map.

Example 3.5.1. (1) Let $S = K[z_1, z_2]$, and $Q = S$. As S is an integral domain, thus every monomial of S is S-regular element.

(2) Let $S = K[z_1, z_2, z_3]$ be the polynomial ring, and $Q = S/(z_2^2 z_3)$. Then z_1^3 is Q-regular element, as z_1^3 is not a zero divisor of Q.

Definition 3.5.3. *[30]* Let Q be a module over a ring $\mathfrak{T}$ and $z = z_1, \ldots, z_n$ be a sequence of elements in $\mathfrak{T}$, it is said to be a Q-regular sequence, if it satisfies the following conditions:

- z_i is $Q/(z_1, \ldots, z_{i-1})Q$ regular for any i;
- $Q \neq (z)Q$.

Example 3.5.2. Let $S = K[z_1, \ldots, z_n]$. Then $z_1, \ldots, z_n$ is a regular sequence on S.

Example 3.5.3. Let $S = K[z_1, z_2, z_3, z_4]$, and $Q = S/J$, where $J = (z_2^2, z_3 z_4) \subset S$ is a monomial ideal. Then $z = z_3 + z_4$, z_1^2 is a Q-regular sequence, as $z_3 + z_4$ is not a zero divisor in Q, and $z_1^2 \notin (z_2, z_3, z_4)$.

Theorem 3.5.1. *[9] Let $\mathfrak{T}$ be a Noetherian ring, Q is a finitely generated $\mathfrak{T}$-module and $J \subset \mathfrak{T}$ such that $JQ \neq Q$. Then all maximal Q-sequences in J have the same length n, that is given by*

$$n = \min\{i : \operatorname{Ext}^i_{\mathfrak{T}}(\mathfrak{T}/J, Q) \neq 0\}.$$

Definition 3.5.4. *[30]* Let $\mathfrak{T}$ be a local Noetherian ring with unique maximal ideal $\mathfrak{m}$, and Q a finitely generated $\mathfrak{T}$-module. The common length of all the maximal Q-sequences in $\mathfrak{m}$ is said to be the depth of Q and it is usually represented by depth(Q).

Example 3.5.4. Let $S = K[z_1, \ldots, z_n]$. Then $z = z_1, \ldots, z_n$ is a regular sequence on S. Since $S/(z) \cong K$, thus depth$(S) = n$.

Example 3.5.5. Let $S = K[z_1, z_2, z_3, z_4]$ be the polynomial ring and $Q = S/J$, where $J = (z_2^2, z_3z_4) \subset S$ is a monomial ideal. Then $z = z_3 + z_4, z_1^2$ is a maximal Q-sequence.

3.6 Stanley depth

Let $S = K[z_1, \ldots, z_n]$ be a polynomial ring over a field K, and A is a finitely generated $\mathbb{Z}^n$-graded S-module. Let $a \in A$ be a homogeneous element and $X \subset \{z_1, z_2, \ldots, z_n\}$. $aK[X]$ represents as the K-subspace of A generated by all elements ab where b is a monomial in $K[X]$. The $\mathbb{Z}^n$-graded K-subspace $aK[X]$ of A is said to be a Stanley space of dimension $|X|$, if $aK[X]$ is a free $K[X]$-module. A Stanley decomposition of A is a presentation of K-vector space A as a finite direct sum of Stanley spaces

$$\mathcal{H} : A = \bigoplus_{i=1}^{r} a_i K[X_i].$$

The number $\text{sdepth}(\mathcal{H}) = \min\{|X_i| : i = 1, \dots, r\}$ is the Stanley depth of $\mathcal{H}$. Let $\text{sdepth}(A) = \max\{\text{sdepth}(\mathcal{H}) : \mathcal{H} \text{ is a Stanley decomposition of } A\}$, then $\text{sdepth}(A)$ is called the Stanley depth of A. Herzog et. al initiated a method [36], in order to find the Stanley depth of a module of the form B/A where $A \subset B \subset S$ are monomial ideals. By using this method, examples are given in the next chapters. Let $Q = B/A$, $A \subset B$ are monomial ideals of S. Let " $\leq$ " be the natural partial order on $\mathbb{N}^n$ given by $b \leq c$ if $b(k) \leq c(k)$ for all $k \in [n]$ and $z^b = z_1^{b(1)} \dots z_n^{b(n)}$ for $b \in \mathbb{N}^n$. Suppose that B is generated by the monomials $z^{b_1}, \dots, z^{b_r}$ and A by the monomials $z^{c_1}, \dots, z^{c_s}$, $b_k, c_l \in \mathbb{N}^n$. Choose $a \in \mathbb{N}^n$ such that $b_k \leq a, c_l \leq a$ for all k, l. Let $P^a_{B/A}$ be the sub-poset of $\mathbb{N}^n$ given by all $d \in \mathbb{N}^n$ with $d \leq a$ and such that $b_k \leq d$ for some k and $d \ngeq c_l$ for all l. $P^a_{B/A}$ is said to be the characteristic poset of B/A with respect to a. Given a finite poset P and $b, c \in P$, $[b, c] = \{d \in P : b \leq d \leq c\}$ is an interval. A partition of P is a disjoint union $\beta : P = \bigcup_{k=1}^{r} [b_k, c_k]$ of intervals, for $d \in P$, set $Z_d = \{z_l : d(l) = a(l)\}$ and let $\psi : P \longrightarrow \mathbb{N}$ be the map given by $d \longrightarrow |Z_d|$.

Theorem 3.6.1. *[36]*

1. *Let $\beta : P^a_{B/A} = \bigcup_{k=1}^{r} [d_k, e_k]$ be a partition of $P^a_{B/A}$. Then*

$$\mathcal{H}(\beta) : B/A = \bigoplus_{k=1}^{r} (\bigoplus_{d} y^d [Z_{e_k}])$$

   is a Stanley decomposition of B/A, where the inner direct sum is taken over all $d \in [d_k, e_k]$ for which $d(l) = d_k(l)$ for all l with $z_l \in Z_{e_k}$. Moreover, $\text{sdepth}(\mathcal{H}(\beta)) = \min\{\psi(e_k) : k \in [r]\}$.

2. *Let $\mathcal{H}$ be a Stanley decomposition of B/A. Then there exists a partition β of $P^a_{B/A}$ such that* $\text{sdepth}(\mathcal{H}(\beta)) \geq \text{sdepth}(\mathcal{H})$

In particular, Stanley depth of B/A can be calculated in the limited number of steps.

Example 3.6.1. Let $B = (z_2z_4, z_1z_2, z_3z_4, z_1z_3) \subset K[z_1, z_2, z_3, z_4]$ and $A = 0$. Set $b_1 = (0,1,0,1)$, $b_2 = (1,1,0,0)$, $b_3 = (0,0,1,1)$ and $b_4 = (1,0,1,0)$. Thus B is generated by $z^{b_1}, z^{b_2}, z^{b_3}, z^{b_4}$ and a can be chosen as $a = (1,1,1,1)$. The poset $P = P^a_{B/A}$ is given by

$P = \{(1,0,1,0),(1,0,0,1),(0,1,1,0),(0,1,0,1),(1,1,1,0),(1,1,0,1),(1,0,1,1),$
$(0,1,1,1),(1,1,1,1)\}$

A partition $\mathcal{P}$ of P is given by

$$[(0,1,0,1),(1,1,0,1)]\bigcup[(1,1,0,0),(1,1,1,0)]\bigcup[(0,0,1,1),(1,0,1,1)]$$
$$\bigcup[(1,0,1,0),(0,1,1,1)]\bigcup[(1,1,1,1),(1,1,1,1)].$$

From finitely many Stanley decompositions, some are as follows:

$$\beta_1 : B = z_2z_4K[z_1,z_2,z_4]\oplus z_2z_1K[z_1,z_2,z_3]\oplus z_3z_4K[z_3,z_4,z_2]\oplus z_3z_1K[z_3,z_1,z_4]\oplus$$
$$z_2z_3z_4z_1K[z_2,z_3,z_4,z_1].$$

$$\beta_2 : B = z_2z_4K[z_2,z_4,z_3]\oplus z_1z_2K[z_1,z_4,z_2]\oplus z_3z_4K[z_3,z_4,z_1]\oplus$$
$$z_1z_3K[z_1,z_3,z_2]\oplus z_1z_2z_3z_4K[z_1,z_3,z_2,z_4].$$

$$\beta_3 : B = z_2z_4K[z_4,z_2]\oplus z_1z_2K[z_1,z_3,z_2]\oplus z_3z_4K[z_2,z_3,z_4]\oplus$$
$$z_1z_3K[z_3,z_1,z_4]\oplus z_1z_2z_4K[z_1,z_4,z_2]\oplus z_1z_2z_3z_4K[z_1,z_3,z_4,z_2].$$

$$\beta_4 : B = z_2z_4K[z_2,z_4,z_3]\oplus z_1z_2K[z_2,z_1]\oplus z_3z_4K[z_1,z_4,z_3]\oplus$$
$$z_2z_4z_1K[z_2,z_1,z_4]\oplus z_3z_1K[z_1,z_3,z_2]\oplus z_1z_2z_3z_4K[z_1,z_3,z_2,z_4].$$

$$\beta_5 : B = z_2z_4K[z_2,z_1,z_4]\oplus z_1z_2K[z_1,z_3,z_2]\oplus z_3z_4K[z_3,z_4]\oplus$$
$$z_2z_3z_4K[z_2,z_4,z_3]\oplus z_1z_3K[z_1,z_4,z_3]\oplus z_1z_2z_3z_4K[z_1,z_4,z_3,z_2].$$

$$\beta_6 : B = z_2z_4K[z_1, z_4, z_2] \oplus z_1z_2K[z_1, z_3, z_2] \oplus z_3z_4K[z_4, z_3, z_2] \oplus$$
$$z_1z_3K[z_1, z_3] \oplus z_1z_3z_4K[z_4, z_3, z_1] \oplus z_1z_2z_3z_4K[z_1, z_4, z_3, z_2].$$

The Stanley depth of B is

$$\operatorname{sdepth}(B) = \max\{\operatorname{sdepth}\beta_i : \beta_i \text{ is a Stanley decomposition of } B\} \geq 3.$$

As $\operatorname{sdepth}(B) = 4$ if and only if B is a principal ideal. Since B is not a principal ideal, thus $\operatorname{sdepth}(B) = 3$.

3.7 Some known results for depth and Stanley depth

A monomial ideal $B \subset S = K[z_1, z_2, \ldots, z_n]$ is said to be a Stanley ideal if Stanley's inequality is true for S/B.

Theorem 3.7.1. *[5] Any monomial ideal of $S = K[z_1, z_2, z_3]$ is a Stanley ideal.*

Theorem 3.7.2. *[3] Let $B \subset S$ be a monomial ideal. If $n = 4$, then B is a Stanley ideal.*

Theorem 3.7.3. *[63] For any monomial ideal B of $S = K[z_1, z_2, z_3, z_4, z_5]$, B is a Stanley ideal.*

Theorem 3.7.4. *[62] If $B = \bigcap_{i=1}^{3} B_i$, where each B_i is a monomial prime ideal of S, then B is a Stanley ideal.*

Theorem 3.7.5. *[80] If $B = \bigcap_{i=1}^{3} B_i$, where each B_i is a monomial primary ideal of S, then B is a Stanley ideal.*

Theorem 3.7.6. *[64] If $B = \bigcap_{i=1}^{4} B_i$, where each B_i is a monomial prime ideal of S, then B is a Stanley ideal.*

Let $B \subset A \subset S$ be monomial ideals. Herzog et al. [36] calculated the Stanley depth of $\mathfrak{m} := (z_1, \ldots, z_n) \subset S$ for $n \leq 9$ by applying their method, and show that $\operatorname{sdepth}(\mathfrak{m}) = \lceil \frac{n}{2} \rceil$. They conjectured that this result holds for any n. Using combinatorial techniques, Biro et al. [7] provided a positive answer to this conjecture. A subset B of S is said to be a complete intersection monomial ideal of S if for any two monomials v_1 and v_2 of $\mathcal{G}(B)$, $\operatorname{supp}(v_1) \bigcap \operatorname{supp}(v_2) = \emptyset$. For a complete intersection monomial ideal B of S, some results are as follows.

Proposition 3.7.1 ([36, Proposition 3.8]). *Let* $|\mathcal{G}(B)| = 3$. *Then* $\operatorname{sdepth}(B) = n - 1$.

Theorem 3.7.7 ([73, Theorem 2.3]). *Let* $|\mathcal{G}(B)| = r$. *Then* $\operatorname{sdepth}(B) = n - \lfloor \frac{r}{2} \rfloor$.

Shen proposed the upcoming question for the squarefree monomial ideal T of S.

Question 3.7.1. *[73] Let* $|\mathcal{G}(T)| = r$. *Is it true that* $\operatorname{sdepth}(T) \geq n - \lfloor \frac{r}{2} \rfloor$?

Keller and Young gave a positive answer to the above question in [54].

Theorem 3.7.8. *[54] Let* $|\mathcal{G}(T)| = r$. *Then* $\operatorname{sdepth}(T) \geq n - \lfloor \frac{r}{2} \rfloor$.

Let B ba an ideal of S, then the ideal

$$\sqrt{B} := \{r \in S : r^a \in B \text{ for some } a \in \mathbb{Z}^+\}$$

is said to be a radical ideal of B. It is a famous result that $\operatorname{depth}(S/B) \leq \operatorname{depth}(S/\sqrt{B})$ and equivalently $\operatorname{depth}(B) \leq \operatorname{depth}(\sqrt{B})$ [35]. Apel showed that the first inequality holds also for Stanley depth.

Theorem 3.7.9. *[5]* $\operatorname{sdepth}(S/B) \leq \operatorname{sdepth}(S/\sqrt{B})$.

Ishaq extended the above result, which is shown in [46] as follows:

Theorem 3.7.10. *Let* $B \subset A \subset S = K[z_1, \ldots, z_n]$ *be monomial ideals. Then*

$$\operatorname{sdepth}(\sqrt{A}/\sqrt{B}) \geq \operatorname{sdepth}(A/B).$$

He also gave the following interesting upper bound.

Theorem 3.7.11. *[47] Let $A \subset S$ be a monomial ideal. Then*

$$\operatorname{sdepth}(A) \leq \min\{P_i, i = 1, \ldots, s\},$$

where $P_i \in Ass_S(S/A)$.

Now some results related to the edge ideals of graphs are presented here. Ishaq and Qureshi [51] gave the upper bounds of Stanley depth for the edge ideals of a k-uniform complete bipartite hypergraph and k-partite complete graph.

Corollary 3.7.1 ([51, Corollary 2.9]). *If B is an edge ideal of complete k-partite graph then Stanley's inequality is true for B.*

Let $I(P_n)$ and $I(C_n)$ be the edge ideals of the path and cyclic graphs respectively. By applying Depth Lemma, Morey showed the following result.

Lemma 3.7.1 ([57, Lemma 2.8]). *Let $n \geq 2$, then* $\operatorname{depth}(S/I(P_n)) = \lceil \frac{n}{3} \rceil$.

Stefan proved a similar result for Stanley depth.

Lemma 3.7.2 ([76, Lemma 4]). *Let $n \geq 2$, then* $\operatorname{sdepth}(S/I(P_n)) = \lceil \frac{n}{3} \rceil$.

Theorem 3.7.12 ([67, Theorem 2.7]). *Let H be a forest with k connected components $H_1, \ldots, H_k$ and let $I(H)$ be its edge ideal in S. Let e_i is the diameter of H_i, $1 \leq i \leq k$, and $e = \max\{e_i \mid 1 \leq i \leq k\}$. Then for every $r \geq 1$*

$$\operatorname{sdepth}(S/(I(H))^r) \geq \max\{\lceil \frac{e - r + 2}{3} \rceil + k - 1, k\}.$$

Cimpoeas proved the following results for the edge ideals of the cyclic graph.

Proposition 3.7.2 ([16, Proposition 1.3]). *Let $n \geq 3$, then*

$$\operatorname{depth}(S/I(C_n)) = \lceil \frac{n-1}{3} \rceil.$$

Theorem 3.7.13 ([16, Theorem 1.9]). *Let $n \geq 3$, then*

(1) $\operatorname{sdepth}(S/I(C_n)) = \lceil \frac{n-1}{3} \rceil$, *if* $n \equiv 0, 2 \pmod 3$.

(2) $\operatorname{sdepth}(S/I(C_n)) \leq \lceil \frac{n}{3} \rceil$, *if* $n \equiv 1 \pmod 3$.

Fouli and Morey gave the following lower bound in terms of the diameter of a graph U.

Theorem 3.7.14 ([26, Theorems 3.1 and 4.18]). *Let U be a connected graph and $A = I(U)$ be the edge ideal of U. If $d = \operatorname{diam}(U)$, then*

$$\operatorname{depth}(S/A), \operatorname{sdepth}(S/A) \geq \lceil \frac{d+1}{3} \rceil.$$

For more results about these invariants that are related to the powers of edge ideals, the interested reader is referred to [20, 26, 57, 81]. Some results that are referred several times in the rest of the dissertation are as follows.

Lemma 3.7.3 (Depth Lemma). *[79, Lemma 1.3.9] Let $0 \longrightarrow A_1 \longrightarrow A_2 \longrightarrow A_3 \longrightarrow 0$ be the short exact sequence, where A_1, A_2 and A_3 are the finitely generated $\mathbb{Z}^n$-graded S-modules. Then*

1. $\operatorname{depth}(A_2) \geq \min\{\operatorname{depth}(A_3), \operatorname{depth}(A_1)\}$.

2. $\operatorname{depth}(A_3) \geq \min\{\operatorname{depth}(A_1) - 1, \operatorname{depth}(A_2)\}$.

3. $\operatorname{depth}(A_1) \geq \min\{\operatorname{depth}(A_2), \operatorname{depth}(A_3) + 1\}$.

Lemma 3.7.4 ([70, Lemma 2.2]). *Let $0 \longrightarrow T_1 \longrightarrow T_2 \longrightarrow T_3 \longrightarrow 0$ be the short exact sequence, where T_1, T_2 and T_3 are $\mathbb{Z}^n$-graded S-modules. Then*

$$\operatorname{sdepth}(T_2) \geq \min\{\operatorname{sdepth}(T_1), \operatorname{sdepth}(T_3)\}.$$

Lemma 3.7.5 ([36, Lemma 3.6]). *Let $T \subset S = K[z_1, \ldots, z_n]$ is a monomial ideal, and $\bar{S} = S[z_{n+1}, z_{n+2}, \ldots, z_{n+r}]$ is a polynomial ring. Then*

$$\operatorname{depth}(\bar{S}/T\bar{S}) = \operatorname{depth}(S/TS) + r \quad \textit{and} \quad \operatorname{sdepth}(\bar{S}/T\bar{S}) = \operatorname{sdepth}(S/TS) + r.$$

Theorem 3.7.15 ([60, Theorem 2.3]). *Let $T \subset S$ be a monomial ideal of S and $|\mathcal{G}(T)| = r$. Then*

$$\operatorname{sdepth}(T) \geq \min\left\{1, n - \lfloor\frac{r}{2}\rfloor\right\}.$$

Corollary 3.7.2 ([70, Corollary 1.3]). *Let $T \subset S$ be a monomial ideal. Then* $\operatorname{depth}(S/(T:v)) \geq \operatorname{depth}(S/T) \quad \forall\ v \notin T.$

Proposition 3.7.3 ([14, Proposition 2.7]). *Let $T \subset S$ be a monomial ideal. Then*

$\operatorname{sdepth}(S/(T:v)) \geq \operatorname{sdepth}(S/T) \quad \forall\ v \notin T.$

3.8 Conclusion

In this chapter, some basic notions, terminologies and definitions from commutative algebra are presented.The fundamental material on monomial ideals is summarized in it. Moreover,the basic literature related to Stanley depth and depth is given in this chapter. Furthermore, a computational technique related to Stanley depth of multigraded S-modules is presented in this chapter.

Chapter 4

Depth and Stanley depth of the edge ideals of square paths and square cycles

This chapter consists of two sections. In the first section, the exact value of depth and Stanley depth for $S/I(P_n^2)$ is computed. These invariants are also computed for $S/I(C_n^2)$, when $n \equiv 0, 3, 4 (\mathrm{mod}\ 5)$, and tight bounds are given in other cases. In the next section, a lower bound is provided for the Stanley depth of $I(P_n^2)$, and this is good enough that a positive answer to the Conjecture 1.2.1 as well as to the Question 1.2.1 for $I(P_n^2)$ can be given. Also, a positive answer is given to the Conjecture 1.2.1 for $I(C_n^2)$ at the end of this chapter.

Throughout this chapter, set $S_m := K[a_1, a_2, \dots, a_m]$. The edge ideal $I(P_n^2)$ associated with the graph P_n^2 is the squarefree monomial ideal of polynomial ring S_m and the generating set contains monomials of the type $a_i a_j$, which corresponds to the edges in the graph P_n^2. For examples of P_n^2, see Fig. 2.1. Edge ideal of P_n^2

is given by

$$I(P_n^2) = (a_1a_2, a_1a_3, a_2a_3, a_2a_4, \dots, a_ia_{i+1}, a_ia_{i+2}, \dots, a_{n-4}a_{n-3}, a_{n-4}a_{n-2},$$
$$a_{n-3}a_{n-2}, a_{n-3}a_{n-1}, a_{n-2}a_{n-1}, a_{n-2}a_n, a_{n-1}a_n).$$

Similarly, the edge ideal of C_n^2 is given by

$$I(C_n^2) = (a_1a_2, a_1a_3, a_2a_3, a_2a_4, \dots, a_ia_{i+1}, a_ia_{i+2}, \dots, a_{n-4}a_{n-3}, a_{n-4}a_{n-2},$$
$$a_{n-3}a_{n-2}, a_{n-3}a_{n-1}, a_{n-2}a_{n-1}, a_{n-2}a_n, a_{n-1}a_n, a_{n-1}a_1, a_na_1, a_na_2).$$

See Fig. 2.2 for examples of C_n^2.

4.1 Depth and Stanley depth of cyclic modules associated to square paths and square cycles

In this section, the values of depth and Stanley depth are computed for the cyclic module $S/I(P_n^2)$, and these results are used in computations of depth and Stanley depth of $S/I(C_n^2)$.

Theorem 4.1.1. *Let* $n \geq 3$. *Then* $\operatorname{depth}(S/I(P_n^2)) = \lceil \frac{n}{5} \rceil$.

Proof. To prove it, the induction hypothesis is used. For $3 \leq n \leq 7$, the result is true. Now consider $n \geq 8$. For this, assume the short exact sequence as follows:

$$0 \longrightarrow S/(I(P_n^2) : a_{n-2}) \xrightarrow{\cdot a_{n-2}} S/I(P_n^2) \longrightarrow S/(I(P_n^2), a_{n-2}) \longrightarrow 0, \qquad (4.1)$$

by Depth Lemma

$$\operatorname{depth}(S/I(P_n^2)) \geq \min\{\operatorname{depth}(S/(I(P_n^2) : a_{n-2})), \operatorname{depth}(S/(I(P_n^2), a_{n-2})\}.$$

$$\begin{aligned}(I(P_n^2):a_{n-2}) &= (a_1a_2, a_1a_3, a_2a_3, a_2a_4, \dots, a_ia_{i+1}, a_ia_{i+2}, \dots, a_{n-6}a_{n-5}, a_{n-4},\\ &\qquad a_{n-3}, a_{n-1}, a_n)\\ &= (I(P_{n-5}^2),\ a_{n-4}, a_{n-3}, a_{n-1}, a_n).\end{aligned}$$

This further implies that $S/(I(P_n^2):a_{n-2}) \cong (S_{n-5}/I(P_{n-5}^2))[a_{n-2}]$. The induction hypothesis and Lemma 3.7.5 give

$$\operatorname{depth}(S/(I(P_n^2):a_{n-2}) = \lceil\frac{n-5}{5}\rceil + 1 = \lceil\frac{n}{5}\rceil.$$

Now let

$$T := (I(P_n^2), a_{n-2}) = (I(P_{n-3}^2),\ a_{n-3}a_{n-1}, a_{n-1}a_n, a_{n-2}).$$

For this, assume the short exact sequence as follows:

$$0 \longrightarrow S/(T:a_{n-1}) \xrightarrow{\cdot a_{n-1}} S/T \longrightarrow S/(T, a_{n-1}) \longrightarrow 0. \tag{4.2}$$

By Depth Lemma, it follows that

$$\operatorname{depth}(S/T) \geq \min\{\operatorname{depth}(S/(T:a_{n-1})), \operatorname{depth}(S/(T, a_{n-1}))\}.$$

Here

$$(T:a_{n-1}) = (I(P_{n-4}^2),\ a_{n-3}, a_{n-2}, a_n) \text{ and } (T, a_{n-1}) = (I(P_{n-3}^2),\ a_{n-2}, a_{n-1}).$$

That gives

$$S/(T:a_{n-1}) \cong (S_{n-4}/I(P_{n-4}^2))[a_{n-1}] \text{ and } S/(T, a_{n-1}) \cong (S_{n-3}/I(P_{n-3}^2))[a_n].$$

Thus inductive hypothesis on n and Lemma 3.7.5 yield

$$\operatorname{depth}(S/(T:a_{n-1})) = \lceil\frac{n-4}{5}\rceil + 1 \text{ and } \operatorname{depth}(S/(T, a_{n-1})) = \lceil\frac{n-3}{5}\rceil + 1.$$

Since $\operatorname{depth}(S/(T, a_{n-1})) \geq \operatorname{depth}(S/(T:a_{n-1}))$, then by Depth Lemma

$$\operatorname{depth}(S/(I(P_n^2), a_{n-2})) = \operatorname{depth}(S/T) = \lceil\frac{n-4}{5}\rceil + 1.$$

Also as $\operatorname{depth}(S/(I(P_n^2), a_{n-2})) \geq \operatorname{depth}(S/(I(P_n^2) : a_{n-2}))$, again by Depth Lemma

$$\operatorname{depth}(S/I(P_n^2)) = \lceil \frac{n}{5} \rceil.$$

□

Proposition 4.1.1. *Let $n \geq 3$. Then* $\operatorname{sdepth}(S/I(P_n^2)) \geq \lceil \frac{n}{5} \rceil$.

Proof. If $3 \leq n \leq 7$, then by using the method given in [36], the Stanley decompositions of the desired Stanley depth are as follows.

$$\text{If } n = 3, \quad \text{then} \qquad S/I(P_3^2) = K[a_1] \oplus a_2K[a_2] \oplus a_3K[a_3].$$

$$\text{If } n = 4, \quad \text{then} \qquad S/I(P_4^2) = K[a_1, a_4] \oplus a_2K[a_2] \oplus a_3K[a_3].$$

$$\text{If } n = 5, \quad \text{then} \qquad S/I(P_5^2) = K[a_1, a_4] \oplus a_2K[a_2, a_5] \oplus a_3K[a_3] \oplus a_5K[a_1, a_5].$$

$$\text{If } n = 6, \quad \text{then } S/I(P_6^2) = K[a_1, a_4] \oplus a_2K[a_2, a_5] \oplus a_3K[a_3, a_6] \oplus a_5K[a_1, a_5] \oplus a_6K[a_1, a_6] \oplus a_2a_6K[a_2, a_6].$$

$$\text{If } n = 7, \quad \text{then } S/I(P_7^2) = K[a_1, a_4] \oplus a_2K[a_2, a_5] \oplus a_3K[a_3, a_6] \oplus a_5K[a_1, a_5] \oplus a_6K[a_1, a_6] \oplus a_7K[a_1, a_7] \oplus a_2a_6K[a_2, a_6] \oplus a_2a_7K[a_2, a_7] \oplus a_3a_7K[a_3, a_7] \oplus a_4a_7K[a_1, a_4, a_7].$$

It remains to be shown that the result holds for $n \geq 8$. For this, the proposition can be proved by taking the similar steps as in Theorem 4.1.1. By applying Lemma 3.7.4 instead of Depth Lemma on the exact sequences (4.1) and (4.2), then the required result holds by inductive hypothesis. □

The following elementary lemma is helpful in finding an upper bound for the Stanley depth.

Lemma 4.1.1. *Let I is a squarefree monomial ideal of S, if* $\operatorname{supp}(I) = [n]$, *and $v := a_{i_1}a_{i_2}\cdots a_{i_q} \in S/I$ such that $a_jv \in I$ for all $j \in [n]\backslash \operatorname{supp}(v)$. Then* $\operatorname{sdepth}(S/I) \leq q$.

Proof. Suppose that $\operatorname{sdepth}(S/I) > q$, then by [31, Lemma 4.1], a squarefree Stanley decomposition of S/I, is as follows

$$S/I = \bigoplus_{r=1}^{s} u_r K[Z_r],$$

where $|Z_r| > q$, u_r's are squarefree monomials in S/I and $\operatorname{supp}(u_r) \subset Z_r \ \forall\ r$. Since $v \in S/I$, thus $v \in u_{r'}K[Z_{r'}]$ for some r', and by supposition $|Z_{r'}| > q$, therefore $\operatorname{supp}(v) \subsetneq Z_{r'}$. Let $a_l \in Z_{r'}\backslash \operatorname{supp}(v)$ then $a_lv \in I$ and hence $u_{r'}K[Z_{r'}]$ is not a free $K[Z_{r'}]$ module, a contradiction, thus $\operatorname{sdepth}(S/I) \leq q$. □

Theorem 4.1.2. *Let $n \geq 3$. Then* $\operatorname{sdepth}(S/I(P_n^2)) = \lceil \frac{n}{5} \rceil$.

Proof. Using Proposition 4.1.1, it only needs to prove that $\operatorname{sdepth}(S/I(P_n^2)) \leq \lceil \frac{n}{5} \rceil$. If $n = 3$ or 4, then the result can be easily verified. For $n \geq 5$, assume the following cases:

Case 1. Let $n = 5k$, where $k \geq 1$. Since $v = a_3a_8a_{13}\cdots a_{5k-2} \in S\backslash I(P_n^2)$, but $a_lv \in I(P_n^2)$ for all $l \in [n]\backslash \operatorname{supp}(v)$, thus by using Lemma 4.1.1, $\operatorname{sdepth}(S/I(P_n^2)) \leq k = \lceil \frac{n}{5} \rceil$.

Case 2. Let $n = 5k + t \geq 7$, where $t \in \{1, 2, 3\}$ and $k \geq 1$. Since $v = a_3a_8a_{13}\cdots a_{5k-2}a_{5k+t} \in S\backslash I(P_n^2)$, but $a_lv \in I(P_n^2)$ for all $l \in [n]\backslash \operatorname{supp}(v)$, therefore by Lemma 4.1.1, $\operatorname{sdepth}(S/I(P_n^2)) \leq k + 1 = \lceil \frac{n}{5} \rceil$.

Case 3. Let $n = 5k + 4 \geq 9$, where $k \geq 1$. Since $v = a_3a_8a_{13}\cdots a_{5k+3} \in S\backslash I(P_n^2)$, but $a_lv \in I(P_n^2)$ for all $l \in [n]\backslash \operatorname{supp}(v)$, thus again by Lemma 4.1.1, $\operatorname{sdepth}(S/I(P_n^2)) \leq k + 1 = \lceil \frac{n}{5} \rceil$. □

From Theorems 4.1.1 and 4.1.2, the next result can be easily deduced.

Corollary 4.1.1. *The Stanley's inequality holds for $I(P_n^2)$.*

Theorem 4.1.3. *Let $n \geq 5$. Then* $\operatorname{depth}(S/I(C_n^2)) \geq \lceil \frac{n-2}{5} \rceil$.

Proof. The result can be easily verified for $5 \leq n \leq 7$. For $n \geq 8$, assume the short exact sequence as follows:

$$0 \longrightarrow S/(I(C_n^2) : a_{n-2}) \xrightarrow{\cdot a_{n-2}} S/I(C_n^2) \longrightarrow S/(I(C_n^2), a_{n-2}) \longrightarrow 0.$$

Then by Depth Lemma

$$\operatorname{depth}(S/I(C_n^2)) \geq \min\{\operatorname{depth}(S/(I(C_n^2) : a_{n-2})), \operatorname{depth}(S/(I(C_n^2), a_{n-2}))\}.$$

As $S/(I(C_n^2) : a_{n-2}) \cong S_{n-5}/I(P_{n-5}^2)[a_{n-2}]$, then by Proposition 4.1.1 and Lemma 3.7.5, it follows that

$$\operatorname{depth}(S/(I(C_n^2) : a_{n-2})) = \lceil \frac{n-5}{5} \rceil + 1 = \lceil \frac{n}{5} \rceil.$$

Let

$$J' := (I(C_n^2), a_{n-2}) = (a_1a_2, a_1a_3, a_2a_3, a_2a_4, \ldots, a_ia_{i+1}, a_ia_{i+2}, \ldots, a_{n-5}a_{n-4},$$
$$a_{n-5}a_{n-3}, a_{n-4}a_{n-3}, a_{n-3}a_{n-1}, a_{n-1}a_n, a_{n-1}a_1, a_na_1, a_na_2, a_{n-2}).$$

Now assume the following exact sequence

$$0 \longrightarrow S/(J' : a_{n-1}) \xrightarrow{\cdot a_{n-1}} S/J' \longrightarrow S/(J', a_{n-1}) \longrightarrow 0.$$

By using Depth Lemma,

$$\operatorname{depth}(S/J') \geq \min\{\operatorname{depth}(S/(J' : a_{n-1})), \operatorname{depth}(S/(J', a_{n-1}))\}.$$

Here

$$(J' : a_{n-1}) = (a_2a_3, a_2a_4, \ldots, a_ia_{i+1}, a_ia_{i+2}, \ldots, a_{n-5}a_{n-4}, a_{n-3}, a_n, a_1, a_{n-2}).$$

After renumbering the variables, it follows that

$$S/(J' : a_{n-1}) \cong S_{n-5}/I(P_{n-5}^2)[a_{n-1}].$$

Therefore by Theorem 4.1.2 and Lemma 3.7.5, it results that

$$\operatorname{depth}(S/(J' : a_{n-1})) = \lceil \frac{n-5}{5} \rceil + 1 = \lceil \frac{n}{5} \rceil.$$

Also

$$(J', a_{n-1}) = (a_n a_1, a_n a_2, a_1 a_2, a_1 a_3, a_2 a_3, a_2 a_4, \dots, a_i a_{i+1}, a_i a_{i+2}, \dots, a_{n-5} a_{n-4},$$
$$a_{n-5} a_{n-3}, a_{n-4} a_{n-3}, a_{n-1}, a_{n-2}).$$

After renumbering the variables

$$S/(J', a_{n-1}) \cong S_{n-2}/I(P_{n-2}^2).$$

By Theorem 4.1.2, it follows that

$$\operatorname{depth}(S/(J', a_{n-1})) = \lceil \frac{n-2}{5} \rceil.$$

Thus $\operatorname{depth}(S/I(C_n^2)) \geq \lceil \frac{n-2}{5} \rceil$.

□

Corollary 4.1.2. *Let* $n \geq 5$, *if* $n \equiv 1, 2 \pmod 5$, *then*

$$\lceil \frac{n}{5} \rceil - 1 \leq \operatorname{depth}(S/I(C_n^2)) \leq \lceil \frac{n}{5} \rceil.$$

Otherwise,

$$\operatorname{depth}(S/I(C_n^2)) = \lceil \frac{n}{5} \rceil.$$

Proof. The result can be easily verified if $5 \leq n \leq 7$. For $n \geq 8$, by Theorem 4.1.3 it only needs to show that $\operatorname{depth}(S/I(C_n^2)) \leq \lceil \frac{n}{5} \rceil$. Since $a_{n-2} \notin I(C_n^2)$, therefore by Corollary 3.7.2, it follows that

$$\operatorname{depth}(S/I(C_n^2)) \leq \operatorname{depth}(S/(I(C_n^2) : a_{n-2})).$$

Since $S/(I(C_n^2) : a_{n-2}) \cong S_{n-5}/I(P_{n-5}^2)[a_{n-2}]$, thus by Lemma 3.7.5 and Theorem 4.1.1, $\operatorname{depth}(S_{n-5}/I(P_{n-5}^2)[a_{n-2}]) = \lceil \frac{n}{5} \rceil$ and the required result follows. □

Theorem 4.1.4. *Let $n \geq 5$. Then* $\operatorname{sdepth}(S/I(C_n^2)) \geq \lceil \frac{n-2}{5} \rceil$.

Proof. If $5 \leq n \leq 7$, then by using the method given in [36], the Stanley decompositions of the desired Stanley depth are as follows.

$$\text{If } n=5, \text{ then} \quad S/I(C_5^2) = K[a_1] \oplus a_2K[a_2] \oplus a_3K[a_3] \oplus a_4K[a_4] \oplus a_5K[a_5].$$

$$\text{If } n=6, \text{ then} \quad S/I(C_6^2) = K[a_1,a_4] \oplus a_2K[a_2,a_5] \oplus a_3K[a_3,a_6] \oplus a_5K[a_5] \oplus a_6K[a_6].$$

$$\text{If } n=7, \text{ then} \quad S/I(C_7^2) = K[a_1,a_4] \oplus a_2K[a_2,a_5] \oplus a_3K[a_3,a_6] \oplus a_5K[a_1,a_5] \oplus a_6K[a_2,a_6] \oplus a_7K[a_3,a_7] \oplus a_4a_7K[a_4,a_7].$$

Now consider $n \geq 8$. For this, the theorem can be proved in similar way as the proof of Theorem 4.1.3. By applying Lemma 3.7.4 instead of Depth Lemma and Theorem 4.1.2 instead of Theorem 4.1.1, the required result holds. □

Corollary 4.1.3. *Let $n \geq 5$, if $n \equiv 1,2 \pmod 5$, then*

$$\lceil \frac{n}{5} \rceil - 1 \leq \operatorname{sdepth}(S/I(C_n^2)) \leq \lceil \frac{n}{5} \rceil.$$

Otherwise,

$$\operatorname{sdepth}(S/I(C_n^2)) = \lceil \frac{n}{5} \rceil.$$

Proof. By Theorem 4.1.4, if $n \equiv 1,2 (\bmod 5)$, then $\operatorname{sdepth}(S/I(C_n^2)) \geq \lceil \frac{n}{5} \rceil - 1$ and otherwise $\operatorname{sdepth}(S/I(C_n^2)) \geq \lceil \frac{n}{5} \rceil$, thus it is only need to show that $\operatorname{sdepth}(S/I(C_n^2)) \leq \lceil \frac{n}{5} \rceil$. To do this, assume the three cases as follows:

Case 1. Let $n = 5k$, where $k \geq 1$. Consider the monomial v such that

$$v = a_1a_6a_{11} \cdots a_{5(k-2)+1}a_{5(k-1)+1},$$

then clearly $v \in S\backslash I(C_n^2)$, but $a_l v \in I(C_n^2)$ for all $l \in [n]\backslash \operatorname{supp}(v)$, thus by Lemma 4.1.1, $\operatorname{sdepth}(S/I(C_n^2)) \leq k = \lceil \frac{n}{5} \rceil$.

Case 2. Let $n = 5k + t$, where $t \in \{1, 2, 3\}$ and $k \geq 1$. Consider the monomial

$$v = a_1 a_6 a_{11} \cdots a_{5(k-1)+1} a_{5k-1},$$

then clearly $v \in S\backslash I(C_n^2)$, but $a_l v \in I(C_n^2)$ for all $l \in [n]\backslash \operatorname{supp}(v)$, therefore by Lemma 4.1.1, $\operatorname{sdepth}(S/I(C_n^2)) \leq k + 1 = \lceil \frac{n}{5} \rceil$.

Case 3. Let $n = 5k+4$, where $k \geq 1$. Consider the monomial $v = a_1 a_6 \cdots a_{5(k-1)+1} a_{5k+1}$, then clearly $v \in S\backslash I(C_n^2)$, but $a_l v \in I(C_n^2)$ for all $l \in [n]\backslash \operatorname{supp}(v)$, thus by Lemma 4.1.1, $\operatorname{sdepth}(S/I(C_n^2)) \leq k + 1 = \lceil \frac{n}{5} \rceil$. □

From Corollaries 4.1.2 and 4.1.3, the following result can be easily deduced.

Corollary 4.1.4. *The Stanley's inequality holds for $I(C_n^2)$.*

4.2 Stanley depth of edge ideals of square paths and square cycles

The main goal of this section is to show that the Conjecture 1.2.1 is true for the edge ideals of square paths and square cycles. Moreover, a positive answer to the Question 1.2.1 for edge ideal of an arbitrary square path, and for some classes of edge ideals of square cycle is given.

Theorem 4.2.1. *Let $n \geq 3$. Then* $\operatorname{sdepth}(I(P_n^2)) \geq \lceil \frac{n}{5} \rceil + 1$.

Proof. To prove it, the induction hypothesis is used. It is easy to see that the result is true for $3 \leq n \leq 7$. Now consider $n \geq 8$. For this, since $a_{n-2} \notin I(P_n^2)$, thus it follows that

$$I(P_n^2) = I(P_n^2) \cap S' \bigoplus a_{n-2}\big(I(P_n^2) : a_{n-2}\big)S,$$

where $S' = K[a_1, \dots, a_{n-3}, a_{n-1}, a_n]$, $I(P_n^2) \cap S' = \big(\mathcal{G}(I(P_{n-3}^2)), a_{n-3}a_{n-1}, a_{n-1}a_n\big)$, and $\big(I(P_n^2) : a_{n-2}\big) = \big(\mathcal{G}(I(P_{n-5}^2)), a_{n-4}, a_{n-3}, a_{n-1}, a_n\big)S$.

$$I(P_n^2) \cap S' = \big(\mathcal{G}(I(P_{n-3}^2)), a_{n-3}a_{n-1}, a_{n-1}a_n\big) =$$
$$\big(\mathcal{G}(I(P_{n-3}^2)), a_{n-3}a_{n-1}, a_{n-1}a_n\big) \cap S'' \bigoplus a_{n-1}\big((\mathcal{G}(I(P_{n-3}^2)), a_{n-3}a_{n-1}, a_{n-1}a_n) : a_{n-1}\big)S',$$

where $S'' = K[a_1, \dots, a_{n-3}, a_n]$, $\big(\mathcal{G}(I(P_{n-3}^2)), a_{n-3}a_{n-1}, a_{n-1}a_n\big)S'' = \big(\mathcal{G}(I(P_{n-3}^2))\big)S''$, and $\big((\mathcal{G}(I(P_{n-3}^2)), a_{n-3}a_{n-1}, a_{n-1}a_n) : a_{n-1}\big) = \big(\mathcal{G}(I(P_{n-4}^2)), a_{n-3}, a_n\big)S'$. Thus

$$\operatorname{sdepth}\big(I(P_n^2)\big) \geq \min\Big\{\operatorname{sdepth}\big((\mathcal{G}(I(P_{n-5}^2)), a_{n-4}, a_{n-3}, a_{n-1}, a_n)S\big),$$
$$\operatorname{sdepth}\big((\mathcal{G}(I(P_{n-3}^2))S''\big), \operatorname{sdepth}\big((\mathcal{G}(I(P_{n-4}^2)), a_{n-3}, a_n)S'\big)\Big\}.$$

By Lemma 3.7.5, it follows that

$$\operatorname{sdepth}\big((\mathcal{G}(I(P_{n-5}^2)), a_{n-4}, a_{n-3}, a_{n-1}, a_n)S\big)$$
$$= \operatorname{sdepth}\big((\mathcal{G}(I(P_{n-5}^2)), a_{n-4}, a_{n-3}, a_{n-1}, a_n)S'\big) + 1,$$

and by [14, Theorem 1.3], it results that

$$\operatorname{sdepth}\big((\mathcal{G}(I(P_{n-5}^2)), a_{n-4}, a_{n-3}, a_{n-1}, a_n)S'\big) \geq \min\Big\{\operatorname{sdepth}\big(I(P_{n-5}^2)S_{n-5}\big) + 4,$$
$$\operatorname{sdepth}\big((a_{n-4}, a_{n-3}, a_{n-1}, a_n)\bar{S}\big) + \operatorname{sdepth}\big(S_{n-5}/I(P_{n-5}^2)S_{n-5}\big)\Big\},$$

where $\bar{S} = K[a_{n-4}, a_{n-3}, a_{n-1}, a_n]$. Now by induction on n, by [7, Theorem 2.2] and Theorem 4.1.3, it follows that

$$\operatorname{sdepth}\big((\mathcal{G}(I(P_{n-5}^2)), a_{n-4}, a_{n-3}, a_{n-1}, a_n)\big)S' \geq \min\big\{\lceil\tfrac{n-5}{5}\rceil + 1 + 4, 2 + \lceil\tfrac{n-5}{5}\rceil\big\}$$
$$= \lceil\frac{n}{5}\rceil + 1.$$

Therefore $\operatorname{sdepth}\big((\mathcal{G}(I(P_{n-5}^2)), a_{n-4}, a_{n-3}, a_{n-1}, a_n)S\big) > \lceil\frac{n}{5}\rceil + 1$. Now by inductive hypothesis and Lemma 3.7.5, it follows that

$$\operatorname{sdepth}\big((\mathcal{G}(I(P_{n-3}^2)), a_{n-3}a_{n-1}, a_{n-1}a_n)S''\big) = \operatorname{sdepth}\big((\mathcal{G}(I(P_{n-3}^2)))S''\big)$$
$$\geq \lceil\frac{n-3}{5}\rceil + 1 + 1.$$

By [47, Lemma 2.11], the following inequality holds

$$\text{sdepth}\left((\mathcal{G}((I(P_{n-4}^2)), a_{n-3}, a_n)S'\right) \geq \text{sdepth}\left(I(P_{n-4}^2)S_{n-4}[a_{n-1}]\right),$$

by Lemma 3.7.5 and induction on n, it follows that

$$\text{sdepth}\left(((\mathcal{G}(I(P_{n-3}^2)), a_{n-3}a_{n-1}, a_{n-1}a_n) : a_{n-1})S'\right) =$$

$$\text{sdepth}\left((\mathcal{G}(I(P_{n-4}^2)), a_{n-3}, a_n)S'\right) \geq \text{sdepth}\left(I(P_{n-4}^2)S_{n-4}[a_{n-1}]\right) \geq \lceil\frac{n-4}{5}\rceil+1+1.$$

Thus

$$\text{sdepth}(I(P_n^2)) \geq \lceil\frac{n}{5}\rceil + 1.$$

□

Corollary 4.2.1. *Let* $n \geq 3$. *Then* $\text{sdepth}(I(P_n^2)) \geq \text{sdepth}(S/I(P_n^2)) + 1$.

Proposition 4.2.1. *Let* $n \geq 5$. *Then* $\text{sdepth}(I(C_n^2)/I(P_n^2)) \geq \lceil\frac{n+3}{5}\rceil$.

Proof. If $5 \leq n \leq 9$, then by using the method given in [36], the Stanley decompositions of desired Stanley depth are as follows:
If $n = 5$ or 6, then

$$I(C_n^2)/I(P_n^2) = a_1a_{n-1}K[a_1, a_{n-1}] \oplus a_1a_nK[a_1, a_n] \oplus a_2a_nK[a_2, a_n].$$

If $n = 7$, then

$$I(C_7^2)/I(P_7^2) = a_1a_6K[a_1, a_6] \oplus a_1a_7K[a_1, a_7] \oplus a_2a_7K[a_2, a_7] \oplus a_1a_4a_7K[a_1, a_4, a_7].$$

If $n = 8$, then

$$I(C_8^2)/I(P_8^2) = a_1a_7K[a_1, a_4, a_7] \oplus a_1a_8K[a_1, a_4, a_8] \oplus a_2a_8K[a_2, a_5, a_8] \oplus a_1a_5a_8K[a_1, a_5, a_8].$$

If $n=9$, then

$$I(C_9^2)/I(P_9^2) = a_1a_8K[a_1,a_4,a_8] \oplus a_1a_9K[a_1,a_4,a_9]$$
$$\oplus a_2a_9K[a_2,a_5,a_9] \oplus a_1a_5a_8K[a_1,a_5,a_8] \oplus a_1a_5a_9K[a_1,a_5,a_9]$$
$$\oplus a_1a_6a_9K[a_1,a_6,a_9] \oplus a_2a_5a_8K[a_2,a_5,a_8] \oplus a_2a_6a_9K[a_2,a_6,a_9].$$

It remains to be shown that the result holds for $n \geq 10$. For this, consider the following K-vector space isomorphism:

$$I(C_n^2)/I(P_n^2) \cong$$
$$a_1a_n\frac{K[a_4,a_5,\dots,a_{n-3}]}{(a_4a_5,a_4a_6,\dots,a_{n-4}a_{n-3})}[a_1,a_n] \oplus a_2a_n\frac{K[a_5,a_6,\dots,a_{n-3}]}{(a_5a_6,a_5a_7,\dots,a_{n-4}a_{n-3})}[a_2,a_n]$$
$$\oplus\, a_1a_{n-1}\frac{K[a_4,a_5,\dots,a_{n-4}]}{(a_4a_5,a_4a_6,\dots,a_{n-5}a_{n-4})}[a_1,a_{n-1}].$$

Indeed, if $u \in I(C_n^2)$ and $u \notin I(P_n^2)$, then $(a_1a_n)|u$ or $(a_2a_n)|u$ or $(a_1a_{n-1})|u$. If $(a_1a_n)|u$, then $u = x_1^{\gamma_1}x_n^{\delta_1}v_1$, $v_1 \in K[a_4,a_5,\dots,a_{n-3}]$, since $v_1 \notin I(P_n^2)$, it follows that $v_1 \notin (a_4a_5,a_4a_6,\dots,a_{n-4}a_{n-3})$. Now if $(a_2a_n)|u$, then $u = x_2^{\gamma_2}x_n^{\delta_2}v_2$, $v_2 \in K[a_5,a_6,\dots,a_{n-3}]$, since $v_2 \notin I(P_n^2)$, it follows that $v_2 \notin (a_5a_6,\dots,a_{n-4}a_{n-3})$. Finally, if $(a_1a_{n-1})|u$, then $u = x_1^{\gamma_3}x_{n-1}^{\delta_3}v_3$, $v_3 \in K[a_4,\dots,a_{n-4}]$, since $v_3 \notin I(P_n^2)$, it follows that $v_3 \notin (a_4a_5,a_4a_6,\dots,a_{n-5}a_{n-4})$. Thus by Theorem 4.1.2 and Lemma 3.7.5, it follows that

$$\operatorname{sdepth}(I(C_n^2)/I(P_n^2)) \geq \min\{\lceil\frac{n+3}{5}\rceil, \lceil\frac{n+3}{5}\rceil, \lceil\frac{n+4}{5}\rceil\} = \lceil\frac{n+3}{5}\rceil.$$

□

Theorem 4.2.2. *Let* $n \geq 5$. *Then* $\operatorname{sdepth}(I(C_n^2)) \geq \lceil\frac{n-2}{5}\rceil + 1$.

Proof. Assume the exact sequence as follows:

$$0 \longrightarrow I(P_n^2) \longrightarrow I(C_n^2) \longrightarrow I(C_n^2)/I(P_n^2) \longrightarrow 0,$$

then by using Lemma 3.7.4,

$$\operatorname{sdepth}(I(C_n^2)) \geq \min\{\operatorname{sdepth}(I(P_n^2)), \operatorname{sdepth}(I(C_n^2)/I(P_n^2))\}.$$

By Theorem 4.2.1, it follows that

$$\operatorname{sdepth}(I(P_n^2)) \geq \lceil \frac{n}{5} \rceil + 1,$$

and by using Proposition 4.2.1, it concludes that

$$\operatorname{sdepth}(I(C_n^2)/I(P_n^2)) \geq \lceil \frac{n+3}{5} \rceil = \lceil \frac{n-2}{5} \rceil + 1,$$

this finishes the proof. □

Corollary 4.2.2. *Let* $n \geq 5$, *if* $n \equiv 1, 2 \pmod 5$, *then*

$$\operatorname{sdepth}(I(C_n^2)) \geq \operatorname{sdepth}(S/I(C_n^2)).$$

Otherwise,

$$\operatorname{sdepth}(I(C_n^2)) \geq \operatorname{sdepth}(S/I(C_n^2)) + 1.$$

4.3 Conclusion

In this chapter, it is proved that $\lceil \frac{n}{5} \rceil$ is the exact value of both Stanley depth and depth for the quotient ring of the edge ideal related to a square path on n vertices. For $n \equiv 0, 3, 4 (\bmod\ 5)$, it is confirmed that $\lceil \frac{n}{5} \rceil$ is the value of Stanley depth and depth for the quotient ring of the edge ideal related to a square cycle on n vertices. In the remaining cases, tight bounds are provided. Furthermore, a positive answer is given to the Conjecture 1.2.1 for the edge ideals of square paths and square cycles.

Chapter 5

Depth and Stanley depth of the edge ideals of the powers of paths and cycles

In the first section of this chapter, the values of depth and Stanley depth of $S/I(P_n^t)$ are computed, where $I(P_n^t)$ denotes the edge ideal of the t^{th} power of a path P_n on n vertices. Then it is proved that

$$\operatorname{depth}(S/I(P_n^t)) = \operatorname{sdepth}(S/I(P_n^t)) = \lceil \frac{n}{2t+1} \rceil.$$

Let $I(C_n^t)$ represents the edge ideal of the t^{th} power of a cycle C_n on n vertices. In second section, some lower bounds for depth and Stanley depth of $S/I(C_n^t)$ are given. If $n \geq 2t+2$, then by Corollaries 5.2.2 and 5.2.3, it is proved that if $n \equiv 0, t+1, \ldots, 2t(\operatorname{mod}(2t+1))$ then $\operatorname{depth}(S/I(C_n^t)) = \operatorname{sdepth}(S/I(C_n^t)) = \lceil \frac{n}{2t+1} \rceil$. Otherwise,

$$\lceil \frac{n}{2t+1} \rceil - 1 \leq \operatorname{depth}(S/I(C_n^t)), \operatorname{sdepth}(S/I(C_n^t)) \leq \lceil \frac{n}{2t+1} \rceil.$$

Last section is devoted to Conjecture 1.2.1 for $I(P_n^t)$ and $I(C_n^t)$. By Theorem 5.3.1, it follows that

$$\text{sdepth}(I(P_n^t)) \geq \lceil \frac{n}{2t+1} \rceil + 1,$$

which shows that $I(P_n^t)$ satisfies Conjecture 1.2.1. For $n \geq 2t+1$, Proposition 5.3.1 gives a lower bound for $I(C_n^t)/I(P_n^t)$ that is

$$\text{sdepth}(I(C_n^t)/I(P_n^t)) \geq \lceil \frac{n+t+1}{2t+1} \rceil.$$

Corollary 5.3.1 of this chapter proves that $I(C_n^t)$ satisfies Conjecture 1.2.1.

Throughout this chapter, set $S_m := K[a_1, a_2, \ldots, a_m]$. Let $I \subset S$ be an ideal. Then it is represented by I instead of IS. Thus every ideal is considered to be an ideal of S unless otherwise stated. Let I and J be monomial ideals of S, then for $I+J$, it is denoted by (I, J). In the whole chapter, the vertices of the graph U are lebeled by $1, 2, 3, \ldots, n$. The set of vertices of U is represented by $[n] := \{1, 2, \ldots, n\}$ and its edge set by $E(U)$. Assume that all graphs and their powers are simple graphs and all graphs have at least two vertices and a non-empty edge set.

If $n \leq t+1$, then $I(P_n^t)$ is a squarefree Veronese ideal of degree 2. If $n \geq t+2$, then

$$\mathcal{G}(I(P_n^t)) = \cup_{i=1}^{n-t}\{a_ia_{i+1}, a_ia_{i+2}, \ldots, a_ia_{i+t}\} \cup \cup_{j=n-t+1}^{n-1} \{a_ja_{j+1}, a_ja_{j+2}, \ldots, a_ja_n\}.$$

If $n \leq 2t+1$, then $I(C_n^t)$ is a squarefree Veronese ideal of degree 2. If $n \geq 2t+2$, then

$$\mathcal{G}(I(C_n^t)) = \mathcal{G}(I(P_n^t)) \cup \cup_{l=1}^{t}\{a_la_{l+n-t}, a_la_{l+n-t+1}, \ldots, a_la_n\}.$$

Let U be a graph and $i \in [n]$, then $N_U(a_i) := \{a_j : a_ix_j \in \mathcal{G}\big(I(U)\big)\}$, where $j \in [n] \backslash \{i\}$. For $t \geq 2$, $0 \leq i \leq t-1$ and $n \geq 2t+2$, let A_{n-t-1}, A_{n-t+i}, B_{n-t+i}

and D_{n-t+i} denote the monomial prime ideals of S such that $A_{n-t-1} = (0)$, $A_{n-t+i} := (a_{n-t}, a_{n-t+1}, \ldots, a_{n-t+i})$,

$$B_{n-t+i} := \big(a_j : a_j \in N_{P_n^t}(a_{n-t+i})\big)$$
$$= (a_{n-2t+i}, a_{n-2t+i+1}, \ldots, a_{n-t+i-1}, a_{n-t+i+1}, \ldots, a_n),$$

and $D_{n-t+i} := \big(a_j : a_j \in N_{C_n^t}(a_{n-t+i})\big)$. Thus if $i = 0$, then

$$D_{n-t+i} = (a_{n-2t}, a_{n-2t+1}, \ldots, a_{n-t-1}, a_{n-t+1}, \ldots, a_n)$$

and if $1 \leq i \leq t-1$, then

$$D_{n-t+i} = (a_{n-2t+i}, a_{n-2t+i+1}, \ldots, a_{n-t+i-1}, a_{n-t+i+1}, \ldots, a_n, a_1, \ldots, a_i).$$

These monomial prime ideals and the following function play important role in the proof of main theorems of this chapter. For $t \geq 2$ and $2t+2 \leq n \leq 3t+1$, define a function

$$f : \{n-t, n-t+1, \ldots, n-t+i, \ldots, n-1\} \longrightarrow \mathbb{Z}^+ \cup \{0\}, \text{ by}$$

$$f(n-t+i) = \begin{cases} t, & \text{if } n-2t-1+i \geq t+1; \\ n-2t-2+i, & \text{if } 2 \leq n-2t-1+i < t+1. \end{cases}$$

5.1 Depth and Stanley of cyclic modules associated to the edge ideals of the powers of a path

In the start of this section, some results are proved, that are used to complete the main task of this section.

Lemma 5.1.1. *Let $a \geq 2$ be an integer, $\{E_i : 1 \leq i \leq a\}$ and $\{G_i : 0 \leq i \leq a\}$ be the some families of $\mathbb{Z}^n$-graded S-modules with the following short exact sequences:*

$$0 \longrightarrow E_1 \longrightarrow G_0 \longrightarrow G_1 \longrightarrow 0 \tag{1}$$

$$0 \longrightarrow E_2 \longrightarrow G_1 \longrightarrow G_2 \longrightarrow 0 \tag{2}$$

$$\vdots$$

$$0 \longrightarrow E_{a-1} \longrightarrow G_{a-2} \longrightarrow G_{a-1} \longrightarrow 0 \tag{$a-1$}$$

$$0 \longrightarrow E_a \longrightarrow G_{a-1} \longrightarrow G_a \longrightarrow 0 \tag{a}$$

and $\operatorname{depth}(G_a) \geq \operatorname{depth}(E_a)$, $\operatorname{depth}(E_i) \geq \operatorname{depth}(E_{i-1})$ *for all* $2 \leq i \leq a$. *Then* $\operatorname{depth}(G_0) = \operatorname{depth}(E_1)$.

Proof. By assumption, it follows that $\operatorname{depth}(G_a) \geq \operatorname{depth}(E_a)$, using Depth Lemma on (a), $\operatorname{depth}(G_{a-1}) = \operatorname{depth}(E_a)$. Also, by assumption

$$\operatorname{depth}(G_{a-1}) = \operatorname{depth}(E_a) \geq \operatorname{depth}(E_{a-1}).$$

By applying Depth Lemma on $(a-1)$, $\operatorname{depth}(G_{a-2}) = \operatorname{depth}(E_{a-1})$. By repeating the same steps on all exact sequences one by one from bottom to top gives that $\operatorname{depth}(G_{i-1}) = \operatorname{depth}(E_i)$ for all i. Thus if $i = 1$, then $\operatorname{depth}(G_0) = \operatorname{depth}(E_1)$. $\square$

Lemma 5.1.2. *Let $t \geq 2$ and $n \geq 2t + 2$. Then*

$$S/(I(P_n^t), A_{n-1}) \cong S_{n-t-1}/I(P_{n-t-1}^t)[a_n].$$

Proof. Since,

$$\mathcal{G}(I(P_n^t)) = \cup_{i=1}^{n-t}\{a_i a_{i+1}, a_i a_{i+2}, \ldots, a_i a_{i+t}\} \cup \\ \cup_{i=n-t+1}^{n-1}\{a_i a_{i+1}, a_i a_{i+2}, \ldots, a_i a_n\},$$

which further implies that

$$I(P_n^t) + A_{n-1} = A_{n-1} +$$
$$\Big[\sum_{i=n-2t}^{n-t}(a_ia_{i+1}, a_ia_{i+2}, \ldots, a_ia_{i+t}) + \sum_{i=1}^{n-2t-1}(a_ia_{i+1}, a_ia_{i+2}, \ldots, a_ia_{i+t}) +$$
$$\sum_{i=n-t+1}^{n-1}(a_ia_{i+1}, a_ia_{i+2}, \ldots, a_ia_n)\Big] = \sum_{i=1}^{n-2t-1}\big(a_ia_{i+1}, a_ia_{i+2}, \ldots, a_ia_{i+t}\big) +$$
$$\sum_{i=n-2t}^{n-t-2}(a_ia_{i+1}, a_ia_{i+2}, \ldots, a_ia_{n-t-1}) + A_{n-1} = I(P_{n-t-1}^t) + A_{n-1}.$$

Thus the required result follows. □

Lemma 5.1.3. *Let $t \geq 2$, $0 \leq i \leq t-1$ and $n \geq 3t+2$. Then*

$$S/(I(P_n^t) : a_{n-t+i}) \cong S_{n-2t-1+i}/I(P_{n-2t-1+i}^t)[a_{n-t+i}].$$

Proof. It suffices to show that $(I(P_n^t) : a_{n-t+i}) = (I(P_{n-2t-1+i}^t), B_{n-t+i})$. Clearly

$$I(P_{n-2t-1+i}^t) \subset I(P_n^t) \subset (I(P_n^t) : a_{n-t+i}).$$

Let $u \in B_{n-t+i}$, by definition of $I(P_n^t)$, $ux_{n-t+i} \in I(P_n^t)$ i.e. $u \in (I(P_n^t) : a_{n-t+i})$. Thus $B_{n-t+i} \subset (I(P_n^t) : a_{n-t+i})$ and it further implies that $\big(I(P_{n-2t-1+i}^t), B_{n-t+i}\big) \subset (I(P_n^t) : a_{n-t+i})$. Now let w be a monomial generator of $(I(P_n^t) : a_{n-t+i})$, then $w = \frac{v}{gcd(v, a_{n-t+i})}$, where $v \in \mathcal{G}(I(P_n^t))$. If $\mathrm{supp}(v) \cap \mathcal{G}(B_{n-t+i}) \neq \emptyset$, then it follows that $w \in \mathcal{G}(B_{n-t+i})$ and if $\mathrm{supp}(v) \cap \mathcal{G}(B_{n-t+i}) = \emptyset$, then $w \in \mathcal{G}(I(P_n^t)) \cap K[a_1, a_2, \ldots, a_{n-2t-1+i}] = \mathcal{G}(I(P_{n-2t-1+i}^t))$. □

Lemma 5.1.4. *Let $n \geq 3t+2$ and $0 \leq i \leq t-1$, then*

$$S/((I(P_n^t), A_{n-t+(i-1)}) : a_{n-t+i}) \cong S_{n-2t-1+i}/I(P_{n-2t-1+i}^t)[a_{n-t+i}].$$

Proof. As $\big((I(P_n^t), A_{n-t+(i-1)}) : a_{n-t+i}\big) = \big((I(P_n^t) : a_{n-t+i}), A_{n-t+(i-1)}\big)$. Now using the proof of Lemma 5.1.3, it follows that

$$\big((I(P_n^t) : a_{n-t+i}), A_{n-t+(i-1)}\big) =$$
$$\big(I(P_{n-2t-1+i}^t), B_{n-t+i}, A_{n-t+(i-1)}\big) = \big(I(P_{n-2t-1+i}^t), B_{n-t+i}\big),$$

as $A_{n-t+(i-1)} \subset B_{n-t+i}$. Thus the required result follows by Lemma 5.1.3. $\square$

Remark 5.1.1. Let $m \geq 2$ and $I(P_m^{m-1}) \subset S_m = K[a_1, a_2, \dots, a_m]$ be the edge ideal of the $(m-1)^{th}$ power of path P_m. Then $I(P_m^{m-1})$ is a squarefree Veronese ideal in variables $a_1, a_2, \dots, a_m$ and it has degree 2. Thus by [30, Corollary 10.3.7] and Theorem 5.1.2, it follows that

$$\operatorname{depth}(S_m/I(P_m^{m-1})) = \operatorname{sdepth}(S_m/I(P_m^{m-1})) = 1.$$

Remark 5.1.2. Let $t \geq 2$ and $2t+2 \leq n \leq 3t+1$, then it is easy to see that

(1) If $n = 2t+2$, then

$$S/(I(P_n^t) : a_{n-t}) = S/(a_2, \dots, a_{n-t-1}, a_{n-t+1}, \dots, a_n) \cong K[a_1, a_{n-t}].$$

(2) If $0 \leq i \leq t-1$ and $n > 2t+2$, then

$$\begin{aligned} S/(I(P_n^t) : a_{n-t+i}) &= S/((I(P_n^t), A_{n-t+(i-1)}) : a_{n-t+i}) \\ &\cong S_{n-2t-1+i}/I(P_{n-2t-1+i}^{f(n-t+i)})[a_{n-t+i}] \\ &= \begin{cases} S_{n-2t-1+i}/I(P_{n-2t-1+i}^{t})[a_{n-t+i}], & \text{if } n-2t-1+i \geq t+1; \\ S_{n-2t-1+i}/I(P_{n-2t-1+i}^{n-2t-2+i})[a_{n-t+i}], & \text{otherwise.} \end{cases} \end{aligned}$$

Theorem 5.1.1. *Let $n \geq 2$. Then* $\operatorname{depth}(S/I(P_n^t)) = \lceil \frac{n}{2t+1} \rceil$.

Proof. (a) If $n \leq t+1$, then $I(P_n^t)$ is a squarefree Veronese ideal thus by Remark 5.1.1, $\operatorname{depth}(S/I(P_n^t)) = 1 = \lceil \frac{n}{2t+1} \rceil$.

(b) For $n \geq t+2$, assume the following cases:

(1) If $t = 1$, then by [57, Lemma 2.8],

$$\operatorname{depth}(S/I(P_n^1)) = \lceil \frac{n}{3} \rceil = \lceil \frac{n}{2t+1} \rceil.$$

(2) If $t \geq 2$ and $t+2 \leq n \leq 2t+1$, then $\operatorname{depth}(S/I(P_n^t)) \geq 1$ as $\mathfrak{m} \notin \operatorname{Ass}(S/I(P_n^t))$. Since $a_{t+1} \notin I(P_n^t)$ and $a_s a_{t+1} \in \mathcal{G}(I(P_n^t))$ for all $s \in \{1, \ldots t, t+2, \ldots, n\}$, that gives

$$(I(P_n^t) : a_{t+1}) = (a_1, \ldots a_t, a_{t+2}, \ldots, a_n).$$

By Corollary 3.7.2, it follows that

$$\begin{aligned}\operatorname{depth}(S/I(P_n^t)) &\leq \operatorname{depth}(S/(I(P_n^t) : a_{t+1})) \\ &= \operatorname{depth}(S/(a_1, \ldots a_t, a_{t+2}, \ldots, a_n)) = 1.\end{aligned}$$

Thus $\operatorname{depth}(S/I(P_n^t)) = 1 = \lceil \frac{n}{2t+1} \rceil$.

(3) For $t \geq 2$, $2t+2 \leq n \leq 3t+1$ and $0 \leq i \leq t-1$, assume the family of short exact sequences:

$$0 \longrightarrow S/((I(P_n^t), A_{n-t+(i-1)}) : a_{n-t+i}) \xrightarrow{\cdot a_{n-t+i}} S/(I(P_n^t), A_{n-t+(i-1)}) \longrightarrow S/(I(P_n^t), A_{n-t+i}) \longrightarrow 0$$

By Lemma 5.1.2, $S/(I(P_n^t), A_{n-1}) \cong S_{n-t-1}/I(P_{n-t-1}^t)[a_n]$. Since the case $2t+2 \leq n \leq 3t+1$ is considered here, which implies that $t+1 \leq n-t-1 \leq 2t$. If $n-t-1 = t+1$, then $S_{n-t-1}/I(P_{n-t-1}^t) = S_{t+1}/I(P_{t+1}^t)$, by Remark 5.1.1 and Lemma 3.7.5, $\operatorname{depth}\big(S/(I(P_n^t), A_{n-1})\big) = 2$. If $t+1 < n-t-1 \leq 2t$, then by case(b)(2), $\operatorname{depth}(S_{n-t-1}/I(P_{n-t-1}^t)) = 1$. By Lemma 3.7.5, $\operatorname{depth}\big(S/(I(P_n^t), A_{n-1})\big) = 2$. Now to prove that $\operatorname{depth}\big(S/(I(P_n^t) : a_{n-t})\big) = 2$, consider the following cases:
If $n = 2t+2$, then by Remark 5.1.2,

$$S/(I(P_n^t) : a_{n-t}) = S/(a_2, a_3, \ldots, a_{n-t-1}, a_{n-t+1}, \ldots, a_n) \cong K[a_1, a_{n-t}],$$

and thus $\operatorname{depth}\big(S/(I(P_n^t):a_{n-t})\big)=2$. If $n>2t+2$, by Remark 5.1.2 it follows that

$$S/(I(P_n^t):a_{n-t})\cong S_{n-2t-1}/I(P_{n-2t-1}^{n-2t-2})[a_{n-t}],$$

where $2\le n-2t-1\le t$. Thus by Remark 5.1.1 and Lemma 3.7.5, $\operatorname{depth}\big(S/(I(P_n^t):a_{n-t})\big)=2$. Now for $1\le i\le t-1$, by Remark 5.1.2, it follows that

$$\begin{aligned} S/((I(P_n^t),A_{n-t+(i-1)}):a_{n-t+i}) &= S/(I(P_n^t):a_{n-t+i}) \\ &\cong S_{n-2t-1+i}/I(P_{n-2t-1+i}^{f(n-t+i)})[a_{n-t+i}]. \end{aligned}$$

Let $T:=S_{n-2t-1+i}/I(P_{n-2t-1+i}^{f(n-t+i)})[a_{n-t+i}]$. Assume the following three cases:

(i) If $t+1=n-2t-1+i$, then $T=S_{t+1}/I(P_{t+1}^t)[a_{n-t+i}]$, thus by case(a) and Lemma 3.7.5, $\operatorname{depth}(T)=2$.

(ii) For $t+1<n-2t-1+i$, $T=S_{n-2t-1+i}/I(P_{n-2t-1+i}^t)[a_{n-t+i}]$. Since $t+2\le n-2t-1+i\le 2t-1$, thus by case(b)(2) and Lemma 3.7.5, $\operatorname{depth}(T)=2$.

(iii) If $2\le n-2t-1+i<t+1$, then $T=S_{n-2t-1+i}/I(P_{n-2t-1+i}^{n-2t-2+i})[a_{n-t+i}]$, by Remark 5.1.1 and Lemma 3.7.5, $\operatorname{depth}(T)=2$.

Thus by Lemma 5.1.1, $\operatorname{depth}(S/I(P_n^t))=2$.

(4) For $t\ge 2$, $n\ge 3t+2$ and $0\le i\le t-1$, assume the family of short

exact sequences

$$0 \longrightarrow S/((I(P_n^t), A_{n-t+(i-1)}) : a_{n-t+i}) \xrightarrow{\cdot a_{n-t+i}}$$
$$S/(I(P_n^t), A_{n-t+(i-1)}) \longrightarrow S/(I(P_n^t), A_{n-t+i}) \longrightarrow 0$$

By Lemma 5.1.2, $S/(I(P_n^t), A_{n-1})) \cong S_{n-t-1}/I(P_{n-t-1}^t)[a_n]$. The induction hypothesis and Lemma 3.7.5 yield that $\operatorname{depth}(S/(I(P_n^t), A_{n-1})) = \lceil \frac{n-t-1}{2t+1} \rceil + 1$. By Lemma 5.1.4, it follows that

$$S/((I(P_n^t), A_{n-t+(i-1)}) : a_{n-t+i}) \cong S_{n-2t-1+i}/I(P_{n-2t-1+i}^t)[a_{n-t+i}].$$

Thus by induction on n and Lemma 3.7.5, it results that

$$\operatorname{depth}(S/((I(P_n^t), A_{n-t+(i-1)}) : a_{n-t+i})) = \lceil \frac{n-2t-1+i}{2t+1} \rceil + 1.$$

Here

$$\operatorname{depth}(S/(I(P_n^t), A_{n-1})) = \lceil \tfrac{n-t-1}{2t+1} \rceil + 1 \geq$$
$$\lceil \frac{n-t-2}{2t+1} \rceil + 1 = \operatorname{depth}(S/(I(P_n^t), A_{n-2}) : a_{n-1})),$$

and for all $1 \leq i \leq t-1$,

$$\operatorname{depth}(S/((I(P_n^t), A_{n-t+(i-1)}) : a_{n-t+i})) = \lceil \tfrac{n-2t-1+i}{2t+1} \rceil + 1 \geq$$
$$\lceil \frac{n-2t-2+i}{2t+1} \rceil + 1 = \operatorname{depth}(S/((I(P_n^t), A_{n-t+(i-2)}) : a_{n-t+(i-1)})).$$

Thus by Lemma 5.1.1, it follows that

$$\operatorname{depth}(S/I(P_n^t)) = \lceil \frac{n-2t-1}{2t+1} \rceil + 1 = \lceil \frac{n}{2t+1} \rceil.$$

□

Let $e \in [n]$ and $I_{n,e} := (v \in S$ be a square free monomial : $\deg(v) = e)$. Then Cimpoeas proved the following theorem for $I_{n,e}$.

Theorem 5.1.2 ([15, Theorem 1.1]). *(1)* $\operatorname{sdepth}(S/I_{n,e}) = e - 1$.

(2) $e \leq \operatorname{sdepth}(I_{n,e}) \leq \frac{n-e}{e+1} + e$.

Theorem 5.1.3 ([12, Theorem 1.4]). *Let T be a $\mathbb{Z}^n$-graded S-module and* $\operatorname{sdepth}(T) = 0$*, then it follows that* $\operatorname{depth}(T) = 0$*. On the other hands, if* $\operatorname{depth}(T) = 0$ *and* $\dim_K(T_b) = 1$ *for any $b \in \mathbb{Z}^n$, then* $\operatorname{sdepth}(T) = 0$*.*

Lemma 5.1.5 ([76, Lemma 4]). *For $n \geq 2$,* $\operatorname{sdepth}(S/I(P_n^1)) = \lceil \frac{n}{3} \rceil$*.*

Example 5.1.1. *For $n \geq 2$, and $n \leq 2t + 1$,* $\operatorname{sdepth}(S/I(P_n^t)) = 1$*.*

Proof. If $n \leq t + 1$, then by Theorem 5.1.2 $\operatorname{sdepth}(S/I(P_n^t)) = 1$. Now if $t + 2 \leq n \leq 2t + 1$, then $\operatorname{depth}(S/I(P_n^t)) \geq 1$ as $\mathfrak{m} \notin \operatorname{Ass}(S/I(P_n^t))$, thus by Theorem 5.1.3, $\operatorname{sdepth}(S/I(P_n^t)) \geq 1$. Since $a_{t+1} \notin I(P_n^t)$ and $a_i a_{t+1} \in \mathcal{G}(I(P_n^t))$ for all $i \in \{1, \ldots t, t+2, \ldots, n\}$, therefore $(I(P_n^t) : a_{t+1}) = (a_1, \ldots a_t, a_{t+2}, \ldots, a_n)$. Thus by Proposition 3.7.3, it follows that

$$\operatorname{sdepth}(S/I(P_n^t)) \leq \operatorname{sdepth}(S/(I(P_n^t) : a_{t+1})) = \operatorname{sdepth}(S/(a_1, \ldots a_t, a_{t+2}, \ldots, a_n)) = 1.$$

□

Proposition 5.1.1. *For $t \geq 2$ and $n \geq 2t + 2$,*

$$\operatorname{sdepth}(S/I(P_n^t)) \geq \lceil \frac{n}{2t+1} \rceil.$$

Proof. (1) If $2t + 2 \leq n \leq 3t + 1$, then by applying Lemma 3.7.4 on the exact sequences in case(b)(3) of Theorem 5.1.1, $\operatorname{sdepth}(S/I(P_n^t)) \geq 2 = \lceil \frac{n}{2t+1} \rceil$.

(2) If $n \geq 3t + 2$, then the proof is identical to the proof of Theorem 5.1.1. By applying Lemma 3.7.4 on the exact sequences in case(b)(4) of Theorem

5.1.1, it follows that

$$\operatorname{sdepth}(S/I(P_n^t)) \geq \min\Big\{\operatorname{sdepth}(S/(I(P_n^t), A_{n-1})),$$
$$\min_{i=0}^{t-1}\{\operatorname{sdepth}(S/((I(P_n^t), A_{n-t+(i-1)}) : a_{n-t+i}))\}\Big\} \geq \lceil\frac{n}{2t+1}\rceil.$$

□

Theorem 5.1.4. *Let* $n \geq 2$, *then* $\operatorname{sdepth}(S/I(P_n^t)) = \lceil\frac{n}{2t+1}\rceil$.

Proof. The result follows by Lemma 5.1.5 for $t = 1$. Let $t \geq 2$. If $n \leq 2t+1$, then by Example 5.1.1 the required result holds. If $n \geq 2t+2$, then by Proposition 5.1.1 it follows that

$$\operatorname{sdepth}(S/I(P_n^t)) \geq \lceil\frac{n}{2t+1}\rceil.$$

It suffices to show that $\operatorname{sdepth}(S/I(P_n^t)) \leq \lceil\frac{n}{2t+1}\rceil$, for this, consider the following cases:

(1) If $n = (2t+1)l$, where $l \geq 1$, then

$$v = a_{t+1}a_{3t+2}a_{5t+3}\cdots a_{(2t+1)l-t} \in S\backslash I(P_n^t),$$

but $a_{f_1}v \in I(P_n^t)$ for all $f_1 \in [n]\backslash \operatorname{supp}(v)$, thus by Lemma 4.1.1,

$$\operatorname{sdepth}(S/I(P_n^t)) \leq l = \lceil\frac{n}{2t+1}\rceil.$$

(2) If $n = (2t+1)l + r$, where $r \in \{1, 2, 3, \ldots, t+1\}$ and $l \geq 1$, then it follows that

$$v = a_{t+1}a_{3t+2}a_{5t+3}\cdots a_{(2t+1)l-t}a_{(2t+1)l+r} \in S\backslash I(P_n^t),$$

and $a_{f_2}v \in I(P_n^t)$ for all $f_2 \in [n]\backslash \operatorname{supp}(v)$, so by Lemma 4.1.1,

$$\operatorname{sdepth}(S/I(P_n^t)) \leq l+1 = \lceil\frac{n}{2t+1}\rceil.$$

(3) If $n=(2t+1)l+s$, where $s\in\{t+2,t+3,\dots,2t\}$ and $l\geq 1$, since

$$v=a_{t+1}a_{3t+2}a_{5t+3}\cdots a_{(2t+1)l+t+1}\in S\backslash I(P_n^t),$$

but $a_{f_3}v\in I(P_n^t)$ for all $f_3\in[n]\backslash\operatorname{supp}(v)$, by Lemma 4.1.1, it follows that

$$\operatorname{sdepth}(S/I(P_n^t))\leq l+1=\lceil\frac{n}{2t+1}\rceil.$$

□

From Theorems 5.1.1 and 5.1.4, the next result can be derived.

Corollary 5.1.1. *The Stanley's inequality holds for $I(P_n^t)$.*

5.2 Depth and Stanley depth of cyclic modules associated to the edge ideals of the powers of a cycle

Here, some values and bounds are computed for depth and Stanley depth of cyclic modules associated to the edge ideals of powers of a cycle. In the start of this section, some results are proved, that are used to complete the main task of this section.

Lemma 5.2.1. *Let $t\geq 2$ and $n\geq 3t+2$, then $S/(I(C_n^t),A_{n-1})\cong S_{n-t}/I(P_{n-t}^t)$.*

Proof. Since,

$$\mathcal{G}(I(C_n^t))=\mathcal{G}(I(P_n^t))\cup\cup_{l=1}^{t-1}\{a_la_{l+n-t},a_la_{l+n-t+1},\dots,a_la_{n-1}\}\cup\{a_1a_n,\dots,a_ta_n\},$$

that gives

$$I(C_n^t)+A_{n-1}=$$
$$I(P_n^t)+\sum_{l=1}^{t-1}(a_la_{l+n-t},a_la_{l+n-t+1},\dots,a_la_{n-1})+(a_1a_n,a_2a_n,\dots,a_ta_n)+A_{n-1}.$$

Thus by the proof of Lemma 5.1.2, it follows that $I(P_n^t)+A_{n-1} = I(P_{n-t-1}^t)+A_{n-1}$. As

$$\sum_{l=1}^{t-1}(a_l a_{l+n-t}, a_l a_{l+n-t+1}, \ldots, a_l a_{n-1}) + A_{n-1} = A_{n-1}.$$

Therefore $S/(I(C_n^t), A_{n-1}) = S/(I(P_{n-t-1}^t), A_{n-1}, (a_1 a_n, a_2 a_n, \ldots, a_t a_n))$

$$\cong K[a_1, a_2, \ldots, a_{n-t-1}, a_n]/\big(I(P_{n-t-1}^t), (a_1 a_n, a_2 a_n, \ldots, a_t a_n)\big).$$

After renumbering the variables, it results that

$$K[a_1, \ldots, a_{n-t-1}, a_n]/\big(I(P_{n-t-1}^t), (a_1 a_n, a_2 a_n, \ldots, a_t a_n)\big) \cong S_{n-t}/I(P_{n-t}^t).$$

□

Lemma 5.2.2. *Let $t \geq 2$ and $n \geq 3t+2$ and $0 \leq i \leq t-1$, then*

$$S/(I(C_n^t) : a_{n-t+i}) \cong S_{n-2t-1}/I(P_{n-2t-1}^t)[a_{n-t+i}].$$

Proof. Let w be a monomial generator of $(I(C_n^t) : a_{n-t+i})$. Then $w = \frac{v}{gcd(v, a_{n-t+i})}$, where $v \in \mathcal{G}(I(C_n^t))$. If $\text{supp}(v) \cap \mathcal{G}(D_{n-t+i}) \neq \emptyset$, then $w \in \mathcal{G}(D_{n-t+i})$ and if $\text{supp}(v) \cap \mathcal{G}(D_{n-t+i}) = \emptyset$ then $w \in E := \mathcal{G}(I(C_n^t)) \cap K[a_{i+1}, a_{i+2}, \ldots, a_{n-2t-1+i}]$. Thus $(I(C_n^t) : a_{n-t+i}) \subset E + D_{n-t+i}$. On the other hand, $(I(C_n^t) : a_{n-t+i}) = E + D_{n-t+i}$, which further implies that $S/(I(C_n^t) : a_{n-t+i}) = S/(E + D_{n-t+i})$. After renumbering the variables, it follows that

$$S/(I(C_n^t) : a_{n-t+i}) = S/(E, D_{n-t+i}) \cong S_{n-2t-1}/I(P_{n-2t-1}^t)[a_{n-t+i}].$$

□

Lemma 5.2.3. *Let $t \geq 2$, $n \geq 3t+2$ and $0 \leq i \leq t-1$. Then*

$$S/\big((I(C_n^t), A_{n-t+(i-1)}) : a_{n-t+i}\big) \cong S_{n-2t-1}/I(P_{n-2t-1}^t)[a_{n-t+i}].$$

Proof. As $\big((I(C_n^t), A_{n-t+(i-1)}) : a_{n-t+i}\big) = \big((I(C_n^t) : a_{n-t+i}), A_{n-t+(i-1)}\big)$. By using the same arguments as in the proof of Lemma 5.2.2, it follows that

$$\big((I(C_n^t) : a_{n-t+i}), A_{n-t+(i-1)}\big) = \big(E, D_{n-t+i}, A_{n-t+(i-1)}\big) = \big(E, D_{n-t+i}\big)$$

as $A_{n-t+(i-1)} \subset D_{n-t+i}$. Thus the required result follows by Lemma 5.2.2. □

Corollary 5.2.1 ([30, Corollary 10.3.7])**.** *Let* $2 \leq e < n$. *Then*

$$\operatorname{depth}(S/I_{n,e}^k) = \max\{0, n - k(n-e) - 1\}.$$

Theorem 5.2.1. *Let* $n \geq 3$, *then*

$$\begin{aligned} \operatorname{depth}(S/I(C_n^t)) &= 1, \quad \textit{if } \; n \leq 2t+1; \\ \operatorname{depth}(S/I(C_n^t)) &\geq \lceil \frac{n-t}{2t+1} \rceil, \quad \textit{if } \; n \geq 2t+2. \end{aligned}$$

Proof. (a) If $n \leq 2t+1$, then $I(C_n^t)$ is a squarefree Veronese ideal of degree 2. Thus by Corollary 5.2.1, $\operatorname{depth}(S/I(C_n^t)) = 1$.

(b) For $n \geq 2t+2$, consider the following cases:

(1) If $t = 1$, then by [16, Proposition 1.3], $\operatorname{depth}(S/I(C_n^1)) = \lceil \frac{n-1}{3} \rceil$.

(2) If $t \geq 2$ and $2t+2 \leq n \leq 3t+1$, then $\operatorname{depth}(S/I(C_n^t)) \geq 1 = \lceil \frac{n-t}{2t+1} \rceil$ as $\mathfrak{m} \notin \operatorname{Ass}(S/I(C_n^t))$.

(3) For $t \geq 2$, $n \geq 3t+2$ and $0 \leq i \leq t-1$, assume the family of short exact sequences as follows:

$$0 \longrightarrow S/((I(C_n^t), A_{n-t+(i-1)}) : a_{n-t+i}) \xrightarrow{\cdot a_{n-t+i}} S/(I(C_n^t), A_{n-t+(i-1)}) \longrightarrow S/(I(C_n^t), A_{n-t+i}) \longrightarrow 0$$

By Lemma 5.2.1, $S/(I(C_n^t), A_{n-1})) \cong S_{n-t}/I(P_{n-t}^t)$. Now by Lemma 5.2.3, it follows that

$$S/((I(C_n^t), A_{n-t+(i-1)}) : a_{n-t+i}) \cong S_{n-2t-1}/I(P_{n-2t-1}^t)[a_{n-t+i}].$$

By Theorem 5.1.1 and Lemma 3.7.5, it results that

$$\operatorname{depth}(S/((I(C_n^t), A_{n-t+(i-1)}) : a_{n-t+i})) = \lceil \frac{n-2t-1}{2t+1} \rceil + 1 = \lceil \frac{n}{2t+1} \rceil.$$

Again by Theorem 5.1.1, it follows that $\operatorname{depth}(S/(I(C_n^t), A_{n-1})) = \lceil \frac{n-t}{2t+1} \rceil$. Thus by applying Lemma 3.7.3(1) on the family of short exact sequences, the desired result $\operatorname{depth}(S/I(C_n^t)) \geq \lceil \frac{n-t}{2t+1} \rceil$ holds.

□

Corollary 5.2.2. *Let $n \geq 3$, If $n \geq 2t+2$, then*

$$\operatorname{depth}(S/I(C_n^t)) = \lceil \frac{n}{2t+1} \rceil, \qquad \text{if } n \equiv 0, t+1, \ldots, 2t \pmod{2t+1};$$
$$\lceil \frac{n}{2t+1} \rceil - 1 \leq \operatorname{depth}(S/I(C_n^t)) \leq \lceil \frac{n}{2t+1} \rceil, \quad \text{if } n \equiv 1, \ldots, t \pmod{2t+1}.$$

Proof. By Theorem 5.2.1, it is suffices to prove $\operatorname{depth}(S/I(C_n^t)) \leq \lceil \frac{n}{2t+1} \rceil$, for $t \geq 2$ and $n \geq 2t+2$. Since $a_{n-t} \notin I(C_n^t)$, thus by Corollary 3.7.2, $\operatorname{depth}(S/I(C_n^t)) \leq \operatorname{depth}(S/(I(C_n^t) : a_{n-t}))$. Now, consider two cases:

(1) Let $2t+2 \leq n \leq 3t+1$, then $S/(I(C_n^t) : a_{n-t}) = S/(I(P_n^t) : a_{n-t})$, so by the proof of Theorem 5.1.1 it follows that $\operatorname{depth}(S/(I(P_n^t) : a_{n-t})) = 2 = \lceil \frac{n}{2t+1} \rceil$. Therefore

$$\operatorname{depth}(S/I(C_n^t)) \leq \operatorname{depth}(S/(I(C_n^t) : a_{n-t})) = 2 = \lceil \frac{n}{2t+1} \rceil.$$

(2) Let $n \geq 3t+2$, then by Lemma 5.2.2,

$$S/(I(C_n^t) : a_{n-t}) \cong S_{n-2t-1}/I(P_{n-2t-1}^t)[a_{n-t}].$$

By Lemma 3.7.5 and Theorem 5.1.1, $\operatorname{depth}(S_{n-2t-1}/I(P_{n-2t-1}^t)[a_{n-t}]) = \lceil \frac{n}{2t+1} \rceil$. Thus $\operatorname{depth}(S/I(C_n^t)) \leq \operatorname{depth}(S/(I(C_n^t) : a_{n-t})) = \lceil \frac{n}{2t+1} \rceil$.

□

Theorem 5.2.2. *Let $n \geq 3$, then*

$$\begin{aligned}
\operatorname{sdepth}(S/I(C_n^t)) &= 1, && \text{if } n \leq 2t+1;\\
\operatorname{sdepth}(S/I(C_n^t)) &\geq \lceil \frac{n-t}{2t+1} \rceil, && \text{if } n \geq 2t+2.
\end{aligned}$$

Proof. (a) If $n \leq 2t+1$, then $\operatorname{sdepth}(S/I(C_n^t)) = 1$ by Theorem 5.1.2.

(b) For $n \geq 2t+2$, consider the following cases:

(1) If $t = 1$, then by [16, Proposition 1.8] $\operatorname{sdepth}(S/I(C_n^1)) \geq \lceil \frac{n-1}{3} \rceil$.

(2) If $t \geq 2$ and $2t+2 \leq n \leq 3t+1$, then $\operatorname{depth}(S/I(C_n^t)) \geq 1$ as $\mathfrak{m} \notin \operatorname{Ass}(S/I(C_n^t))$, thus by Theorem 5.1.3, $\operatorname{sdepth}(S/I(C_n^t)) \geq 1 = \lceil \frac{n-t}{2t+1} \rceil$.

(3) For $t \geq 2$, $n \geq 3t+2$ and $0 \leq i \leq t-1$, consider the family of short exact sequences

$$0 \longrightarrow S/((I(C_n^t), A_{n-t+(i-1)}) : a_{n-t+i}) \xrightarrow{\cdot a_{n-t+i}} S/(I(C_n^t), A_{n-t+(i-1)}) \longrightarrow S/(I(C_n^t), A_{n-t+i}) \longrightarrow 0.$$

By Lemma 5.2.1, $S/(I(C_n^t), A_{n-1})) \cong S_{n-t}/I(P_{n-t}^t)$. Now by Lemma 5.2.3, it follows that

$$S/((I(C_n^t), A_{n-t+(i-1)}) : a_{n-t+i}) \cong S_{n-2t-1}/I(P_{n-2t-1}^t)[a_{n-t+i}].$$

By Theorem 5.1.4 and Lemma 3.7.5, it follows that

$$\operatorname{sdepth}(S/((I(C_n^t), A_{n-t+(i-1)}) : a_{n-t+i})) = \lceil \frac{n-2t-1}{2t+1} \rceil + 1 = \lceil \frac{n}{2t+1} \rceil.$$

Again by Theorem 5.1.4, $\operatorname{sdepth}(S/(I(C_n^t), A_{n-1})) = \lceil \frac{n-t}{2t+1} \rceil$. By applying Lemma 3.7.4 on the above family of short exact sequences, the required result $\operatorname{sdepth}(S/I(C_n^t)) \geq \lceil \frac{n-t}{2t+1} \rceil$ holds.

□

Corollary 5.2.3. *Let* $n \geq 3$*, if* $n \geq 2t+2$*, then*

$$\operatorname{sdepth}(S/I(C_n^t)) = \lceil \frac{n}{2t+1} \rceil, \qquad \textit{if} \;\; n \equiv 0, t+1, \ldots, 2t \,(\operatorname{mod}(2t+1));$$
$$\lceil \frac{n}{2t+1} \rceil - 1 \leq \operatorname{sdepth}(S/I(C_n^t)) \leq \lceil \frac{n}{2t+1} \rceil, \;\; \textit{if} \;\; n \equiv 1, \ldots, t \,(\operatorname{mod}(2t+1)).$$

Proof. When $t = 1$, then by [16, Theorem 1.9], $\operatorname{sdepth}(S/I(C_n^t)) \leq \lceil \frac{n}{3} \rceil$. By Theorem 5.2.2, it is suffices to show that $\operatorname{sdepth}(S/I(C_n^t)) \leq \lceil \frac{n}{2t+1} \rceil$ for $t \geq 2$ and $n \geq 2t+2$. Since $a_{n-t} \notin I(C_n^t)$, thus by Proposition 3.7.3 it follows that

$$\operatorname{sdepth}(S/I(C_n^t)) \leq \operatorname{sdepth}(S/(I(C_n^t) : a_{n-t})).$$

Now, consider two cases:

(1) Let $2t+2 \leq n \leq 3t+1$, then $S/(I(C_n^t) : a_{n-t}) = S/(I(P_n^t) : a_{n-t})$ so by the proof of Theorem 5.1.4, $\operatorname{sdepth}(S/(I(P_n^t) : a_{n-t})) = 2 = \lceil \frac{n}{2t+1} \rceil$. Therefore

$$\operatorname{sdepth}(S/I(C_n^t)) \leq \operatorname{sdepth}(S/(I(C_n^t) : a_{n-t})) = 2 = \lceil \frac{n}{2t+1} \rceil.$$

(2) Let $n \geq 3t+2$, then by Lemma 5.2.2

$$S/(I(C_n^t) : a_{n-t}) \cong S_{n-2t-1}/I(P_{n-2t-1}^t)[a_{n-t}].$$

By Lemma 3.7.5 and Theorem 5.1.4, $\operatorname{sdepth}(S_{n-2t-1}/I(P_{n-2t-1}^t)[a_{n-t}]) = \lceil \frac{n}{2t+1} \rceil$. Thus $\operatorname{sdepth}(S/I(C_n^t)) \leq \operatorname{sdepth}(S/(I(C_n^t) : a_{n-t})) = \lceil \frac{n}{2t+1} \rceil$.

□

From Corollaries 5.2.2 and 5.2.3, the next result can be derived.

Corollary 5.2.4. *The Stanley's inequality holds for* $I(C_n^t)$.

5.3 Lower bounds for Stanley depth of edge ideals of t^{th} powers of paths and cycles and a conjecture of Herzog

This section consists of some lower bounds for $\text{sdepth}(I(P_n^t))$ and $\text{sdepth}(I(C_n^t))$. These bounds are good enough to prove that the Conjecture 1.2.1 is true for $I(P_n^t)$ and $I(C_n^t)$. Let $0 \leq i \leq t-1$,

$$R_{n-t+i} := K[\{a_1, a_2, \dots, a_n\} \backslash \{a_{n-t}, a_{n-t+1}, \dots, a_{n-t+i}\}]$$

be a subring of S and

$$B'_{n-t+i} := (a_j : a_j \in N_{P_n^t}(a_{n-t+i}) \backslash \{a_{n-t}, a_{n-t+1}, \dots, a_{n-t+(i-1)}\}).$$

be a monomial prime ideal of S. Let $I \subset Z = K[a_{i_1}, a_{i_1}, \dots, a_{i_r}]$ be a monomial ideal and $Z' := Z[a_{i_r+1}]$. Then one can write $IZ' = I[a_{i_r+1}]$. Now, recall a useful remark of Cimpoeas.

Remark 5.3.1. *[14, Remark 1.7]* Let A be a monomial ideal of S, and $A' = (A, a_{n+1}, \dots, a_{n+m})$ be a monomial ideal of $S' = S[a_{n+1}, a_{n+2}, \dots, a_{n+m}]$. Then

$$\text{sdepth}_{S'}(A') \geq \min\{\text{sdepth}_S(A) + m, \text{sdepth}_S(S/A) + \lceil \frac{m}{2} \rceil\}.$$

Theorem 5.3.1. *For $n \geq 2$,* $\text{sdepth}(I(P_n^t)) \geq \lceil \frac{n}{2t+1} \rceil + 1$.

Proof. (a) If $n \leq 2t+1$, then as the minimal generators of $I(P_n^t)$ have degree 2, by [53, Lemma 2.1] it follows that $\text{sdepth}(I(P_n^t)) \geq 2 = \lceil \frac{n}{2t+1} \rceil + 1$.

(b) For $n \geq 2t+2$, if $t = 1$, then by [60, Theorem 2.3], $\text{sdepth}(I(P_n^1)) \geq n - \lfloor \frac{n-1}{2} \rfloor = \lceil \frac{n-1}{2} \rceil + 1 \geq \lceil \frac{n}{3} \rceil + 1$. Now for $t \geq 2$, by using induction on n

and consider the decomposition of $I(P_n^t)$ as a vector space in the following way:

$$I(P_n^t) = I(P_n^t) \cap R_{n-t} \oplus a_{n-t}(I(P_n^t) : a_{n-t})S.$$

Similarly, one can decompose $I(P_n^t) \cap R_{n-t}$ as follows:

$$I(P_n^t) \cap R_{n-t} = I(P_n^t) \cap R_{n-t+1} \oplus a_{n-t+1}(I(P_n^t) \cap R_{n-t} : a_{n-t+1})R_{n-t}.$$

Continuing in the same way for $1 \leq i \leq t-1$, it follows that

$$\begin{aligned} I(P_n^t) \cap R_{n-t+i} = I(P_n^t) \cap R_{n-t+(i+1)} \oplus \\ a_{n-t+(i+1)}(I(P_n^t) \cap R_{n-t+i} : a_{n-t+(i+1)})R_{n-t+i}. \end{aligned}$$

Finally, the following decomposition of $I(P_n^t)$ can be obtained.

$$\begin{aligned} I(P_n^t) &= I(P_n^t) \cap R_{n-1} \oplus \\ &\oplus_{i=1}^{t-1} a_{n-t+i}(I(P_n^t) \cap R_{n-t+(i-1)} : a_{n-t+i})R_{n-t+i} \oplus a_{n-t}(I(P_n^t) : a_{n-t})S. \end{aligned}$$

Therefore

$$\begin{aligned} \operatorname{sdepth}(I(P_n^t)) \geq \min \Big\{ & \operatorname{sdepth}(I(P_n^t) \cap R_{n-1}), \operatorname{sdepth}((I(P_n^t) : a_{n-t})S), \\ & \min_{i=1}^{t-1}\{\operatorname{sdepth}((I(P_n^t) \cap R_{n-t+(i-1)} : a_{n-t+i})R_{n-t+i})\}\Big\}. \end{aligned}$$

As $I(P_n^t) \cap R_{n-1} = \mathcal{G}(I(P_{n-t-1}^t))[a_n]$, thus the induction hypothesis and Lemma 3.7.5 yield $\operatorname{sdepth}(I(P_n^t) \cap R_{n-1}) \geq \lceil \frac{n-t-1}{2t+1} \rceil + 1 + 1 \geq \lceil \frac{n}{2t+1} \rceil + 1$. Now, it needs to show that $\operatorname{sdepth}((I(P_n^t) : a_{n-t})S) \geq \lceil \frac{n}{2t+1} \rceil + 1$ and

$$\operatorname{sdepth}((I(P_n^t) \cap R_{n-t+(i-1)} : a_{n-t+i})R_{n-t+i}) \geq \lceil \frac{n}{2t+1} \rceil + 1.$$

For this, consider the following cases:

(1) Let $2t+2 \leq n \leq 3t+1$. If $n = 2t+2$, then $(I(P_n^t) : a_{n-t})S = (a_2, \ldots, a_{n-t-1}, a_{n-t+1}, \ldots, a_n)S$, thus by [7, Theorem 2.2] and Lemma 3.7.5, it follows that

$$\operatorname{sdepth}((I(P_n^t) : a_{n-t})S) = \lceil \frac{n-2}{2} \rceil + 2 \geq \lceil \frac{n}{2t+1} \rceil + 1.$$

If $2t+3 \leq n \leq 3t+1$, then by Remark 5.1.2, it results that

$$(I(P_n^t) : a_{n-t})S = (\mathcal{G}(I(P_{n-2t-1}^{f(n-t)})), B_{n-t})[a_{n-t}].$$

Since $\operatorname{sdepth}(I(P_{n-2t-1}^{f(n-t)})) + |\mathcal{G}(B_{n-t})| \geq 2$, by Remark 5.1.1, it follows that

$$\operatorname{sdepth}(S_{n-2t-1}/I(P_{n-2t-1}^{f(n-t)})) + \lceil \frac{|\mathcal{G}(B_{n-t})|}{2} \rceil \geq 2,$$

then by Remark 5.3.1, $\operatorname{sdepth}(\mathcal{G}(I(P_{n-2t-1}^{f(n-t)})), B_{n-t}) \geq 2$, and by Lemma 3.7.5, $\operatorname{sdepth}((I(P_n^t) : a_{n-t})S) \geq 3 = \lceil \frac{n}{2t+1} \rceil + 1$. Now since

$$(I(P_n^t) \cap R_{n-t+(i-1)} : a_{n-t+i})R_{n-t+i}) = (\mathcal{G}(I(P_{n-2t-1+i}^{f(n-t+i)})), B'_{n-t+i})[a_{n-t+i}].$$

So by the same arguments, it follows that

$$\operatorname{sdepth}((I(P_n^t) \cap R_{n-t+(i-1)} : a_{n-t+i})R_{n-t+i}) \geq 3 = \lceil \frac{n}{2t+1} \rceil + 1.$$

(2) If $n \geq 3t+2$, then by the proof of Lemma 5.1.3, it follows that

$$(I(P_n^t) : a_{n-t})S = (\mathcal{G}(I(P_{n-2t-1}^t)), B_{n-t})[a_{n-t}]$$

and

$$(I(P_n^t) \cap R_{n-t+(i-1)} : a_{n-t+i})R_{n-t+i} = (\mathcal{G}(I(P_{n-2t-1+i}^t)), B'_{n-t+i})[a_{n-t+i}].$$

By Remark 5.3.1, it follows that

$$\operatorname{sdepth}(\mathcal{G}(I(P^t_{n-2t-1})), B_{n-t}) \geq \min\Big\{\operatorname{sdepth}(\mathcal{G}(I(P^t_{n-2t-1})))+ |\mathcal{G}(B_{n-t})|, \operatorname{sdepth}(S_{n-2t-1}/I(P^t_{n-2t-1})) + \lceil\frac{|\mathcal{G}(B_{n-t})|}{2}\rceil\Big\}.$$

The induction hypothesis gives $\operatorname{sdepth}(\mathcal{G}(I(P^t_{n-2t-1}))) \geq \lceil\frac{n-2t-1}{2t+1}\rceil+1 = \lceil\frac{n}{2t+1}\rceil$, and by Theorem 5.1.4, $\operatorname{sdepth}(S_{n-2t-1}/I(P^t_{n-2t-1})) = \lceil\frac{n}{2t+1}\rceil-1$. Therefore $\operatorname{sdepth}(\mathcal{G}(I(P^t_{n-2t-1})), B_{n-t}) \geq \lceil\frac{n}{2t+1}\rceil + 1$. Thus by Lemma 3.7.5, $\operatorname{sdepth}((I(P^t_n) : a_{n-t})S) > \lceil\frac{n}{2t+1}\rceil + 1$.

Now using Remark 5.3.1 again, the following inequality holds.

$$\operatorname{sdepth}(\mathcal{G}(I(P^t_{n-2t-1+i})), B'_{n-t+i}) \geq \min\Big\{\operatorname{sdepth}(\mathcal{G}(I(P^t_{n-2t-1+i}))) + |\mathcal{G}(B'_{n-t+i})|, \operatorname{sdepth}(S_{n-2t-1+i}/I(P^t_{n-2t-1+i})) + \lceil\frac{|\mathcal{G}(B'_{n-t+i})|}{2}\rceil\Big\}.$$

The induction hypothesis yield $\operatorname{sdepth}(\mathcal{G}(I(P^t_{n-2t-1+i}))) \geq \lceil\frac{n-2t-1+i}{2t+1}\rceil+1$, and by Theorem 5.1.4, it follows that $\operatorname{sdepth}(S_{n-2t-1+i}/I(P^t_{n-2t-1+i})) = \lceil\frac{n-2t-1+i}{2t+1}\rceil$. Therefore

$$\operatorname{sdepth}(\mathcal{G}(I(P^t_{n-2t-1+i})), B'_{n-t+i}) \geq \lceil\frac{n-2t-1+i}{2t+1}\rceil + 1.$$

Thus by Lemma 3.7.5

$$\operatorname{sdepth}((I(P^t_n) \cap R_{n-t+(i-1)} : a_{n-t+i})R_{n-t+i}) \geq \lceil\frac{n}{2t+1}\rceil + 1.$$

This finishes the proof.

□

Proposition 5.3.1. *Let* $n \geq 2t+1$, *then* $\operatorname{sdepth}(I(C^t_n)/I(P^t_n)) \geq \lceil\frac{n+t+1}{2t+1}\rceil$.

Proof. If $t = 1$, then by [16, Proposition 1.10], the required result holds. Now assume that $t \geq 2$ and consider the following cases:

(1). If $2t+1 \leq n \leq 3t+1$, then as $I(C_n^t)$ is a monomial ideal generated by degree 2, by [36, Theorem 2.1], $\operatorname{sdepth}(I(C_n^t)/I(P_n^t)) \geq 2 = \lceil \frac{n+t+1}{2t+1} \rceil$.

(2). If $3t+2 \leq n \leq 4t+1$, then by using the method given in [36], there exist Stanley decompositions of desired Stanley depth. Let $s \in \{1, 2, \dots t\}$, $j_s \in \{1, 2, \dots, t+1-s\}$ and

$$L := \oplus_{s=1}^{t} \Big(\oplus_{j_s=1}^{t+1-s} a_{j_s} a_{n+1-s} K[a_{j_s}, a_{j_s+t+1}, a_{n+1-s}] \Big).$$

It is easy to observe that $L \subset I(C_n^t) \backslash I(P_n^t)$. Now let $u_i \in I(C_n^t) \backslash I(P_n^t)$ be a squarefree monomial such that $u_i \notin L$ then clearly $\deg(u_i) \geq 3$. Since

$$I(C_n^t)/I(P_n^t) \cong L \oplus_{u_i} u_i K[\operatorname{supp}(u_i)],$$

thus $\operatorname{sdepth}(I(C_n^t)/I(P_n^t)) \geq 3 = \lceil \frac{n+t+1}{2t+1} \rceil$ as required.

(3). If $n \geq 4t+2$, then consider the following K-vector space isomorphism:

$$
\begin{aligned}
&I(C_n^t)/I(P_n^t) \cong \\
&\oplus_{j_1=1}^{t} a_{j_1} a_n \frac{K[a_{j_1+t+1}, a_{j_1+t+2}, \dots, a_{n-t-1}]}{(a_{j_1+t+1}a_{j_1+t+2}, a_{j_1+t+1}a_{j_1+t+3}, \dots, a_{n-t-2}a_{n-t-1})}[a_{j_1}, a_n] \oplus \\
&\oplus_{j_2=1}^{t-1} a_{j_2} a_{n-1} \frac{K[a_{j_2+t+1}, a_{j_2+t+2}, \dots, a_{n-t-2}]}{(a_{j_2+t+1}a_{j_2+t+2}, a_{j_2+t+1}a_{j_1+t+3}, \dots, a_{n-t-3}a_{n-t-2})}[a_{j_2}, a_{n-1}] \oplus \\
&\quad \vdots \\
&\oplus_{j_{t-1}=1}^{2} a_{j_{t-1}} a_{n-t+2} \frac{K[a_{j_{t-1}+t+1}, a_{j_{t-1}+t+2}, \dots, a_{n-2t+1}]}{(a_{j_{t-1}+t+1}a_{j_{t-1}+t+2}, \dots, a_{n-2t}a_{n-2t+1})}[a_{j_{t-1}}, a_{n-t+2}] \oplus \\
&a_1 a_{n-(t-1)} \frac{K[a_{t+2}, a_{t+3}, \dots, a_{n-2t}]}{(a_{t+2}a_{t+3}, a_{t+2}a_{t+4}, \dots, a_{n-2t-1}a_{n-2t})}[a_1, a_{n-(t-1)}].
\end{aligned}
$$

Thus

$$
\begin{aligned}
&I(C_n^t)/I(P_n^t) \cong \\
&\oplus_{s=1}^{t} \Big(\oplus_{j_s=1}^{t+1-s} a_{j_s} a_{n+1-s} \big(S_{j_s+t+1, n-s-t} / \big(\mathcal{G}(I(P_n^t)) \cap S_{j_s+t+1, n-s-t}\big)[a_{j_s}, a_{n+1-s}] \Big),
\end{aligned}
$$

where $S_{j_s+t+1,n-s-t} = K[a_{j_s+t+1}, a_{j_s+t+2}, \ldots, a_{n-s-t}]$. Indeed, if $u \in I(C_n^t)$ such that $u \notin I(P_n^t)$ then $(a_{j_s}a_{n+1-s})|u$ for only one pair of s and j_s. If $(a_{j_s}a_{n+1-s})|u$ then $u = a_{j_s}^{\gamma}a_{n+1-s}^{\delta}v$ and $v \in S_{j_s+t+1,n-s-t}$. Since $v \notin I(P_n^t)$, it follows that $v \notin \mathcal{G}(I(P_n^t)) \cap S_{j_s+t+1,n-s-t}$. Clearly

$$S_{j_s+t+1,n-s-t}/\mathcal{G}(I(P_n^t)) \cap S_{j_s+t+1,n-s-t} \cong S_{n-(j_s+2t+s)}/I(P_{n-(j_s+2t+s)}^t).$$

Thus by Theorem 5.1.4 and Lemma 3.7.5, it follows that

$$\operatorname{sdepth}(I(C_n^t)/I(P_n^t)) \geq \min_{s=1}^{t}\{\lceil\frac{n-(j_s+s+2t)}{2t+1}\rceil + 2\}.$$

It is noted that $\max\{j_s + s\} = t + 1$. Therefore

$$\operatorname{sdepth}(I(C_n^t)/I(P_n^t)) \geq \lceil\frac{n-(3t+1)}{2t+1}\rceil + 2 = \lceil\frac{n+t+1}{2t+1}\rceil.$$

□

Theorem 5.3.2. *Let $n \geq 3$, then*

$$\operatorname{sdepth}(I(C_n^t)) \geq 2, \qquad \text{if } n \leq 2t+1;$$
$$\operatorname{sdepth}(I(C_n^t)) \geq \lceil\frac{n-t}{2t+1}\rceil + 1, \qquad \text{if } n \geq 2t+2.$$

Proof. (a) If $n \leq 2t+1$, then as the minimal generators of $I(C_n^t)$ have degree 2, by [53, Lemma 2.1], $\operatorname{sdepth}(I(C_n^t)) \geq 2$.

(b) If $n \geq 2t+2$, then assume the short exact sequence as follows:

$$0 \longrightarrow I(P_n^t) \longrightarrow I(C_n^t) \longrightarrow I(C_n^t)/I(P_n^t) \longrightarrow 0,$$

by Lemma 3.7.4,

$$\operatorname{sdepth}(I(C_n^t)) \geq \min\{\operatorname{sdepth}(I(P_n^t)), \operatorname{sdepth}(I(C_n^t)/I(P_n^t))\}.$$

By Theorem 5.3.1, $\operatorname{sdepth}(I(P_n^t)) \geq \lceil\frac{n}{2t+1}\rceil + 1$, and by Proposition 5.3.1, it follows that $\operatorname{sdepth}(I(C_n^t)/I(P_n^t)) \geq \lceil\frac{n+t+1}{2t+1}\rceil = \lceil\frac{n-t}{2t+1}\rceil + 1$.

□

Corollary 5.3.1. *Let $n \geq 3$, if $n \leq 2t+1$, then*

$$\operatorname{sdepth}(I(C_n^t)) \geq 2 = \operatorname{sdepth}(S/I(C_n^t)) + 1.$$

If $n \geq 2t+2$, then

$$\operatorname{sdepth}(I(C_n^t)) \geq \operatorname{sdepth}(S/I(C_n^t)), \qquad \text{if} \quad n \equiv 1, \ldots, t \,(\operatorname{mod}(2t+1));$$
$$\operatorname{sdepth}(I(C_n^t)) \geq \operatorname{sdepth}(S/I(C_n^t)) + 1, \quad \text{if} \quad n \equiv 0, t+1, \ldots, 2t \,(\operatorname{mod}(2t+1)).$$

Proof. Proof follows by Corollary 5.2.3, Theorem 5.2.2 and Theorem 5.3.2. □

5.4 Conclusion

In this chapter, it is verified that $\lceil \frac{n}{2t+1} \rceil$ is the precise value of both Stanley depth and depth for the quotient ring of the edge ideal related to the t^{th} power of a path on n vertices. For $n \equiv 0, t+1, t+2, \ldots, 2t(\operatorname{mod}(2t+1))$, it is proved that $\lceil \frac{n}{2t+1} \rceil$ is the value of Stanley depth and depth for the quotient ring of the edge ideal related to a square cycle on n vertices. In the remaining cases, tight bounds are established. Furthermore, lower bounds for the Stanley depth of the edge ideals associated with the t^{th} power of a path and a cycle are given. These bounds are good enough that a positive answer is given to the Conjecture 1.2.1 for these ideals.

Chapter 6

Depth and Stanley depth of edge ideals associated to some line graphs

In this chapter, bounds for the Stanley depth and depth of cyclic modules associated to line graphs of the ladder graph $\mathcal{L}_n$ and circular ladder graph $\mathcal{CL}_n$ are given. Also, some bounds for Krull dimension of these cyclic modules are established.

6.1 Results and discussions

Here, the edge ideals of the line graphs of $\mathcal{L}_n$ and $\mathcal{CL}_n$ are denoted by I_n and J_n respectively. Also, the vertices of the line graphs of $\mathcal{L}_n$ and $\mathcal{CL}_n$ are labeled by using three sets of variables $\{a_1, a_2, \dots, a_n\}$, $\{b_1, b_2, \dots, b_n\}$ and $\{c_1, c_2, \dots, c_n\}$, see Figures 2.5 and 2.6. Let $S_n := K[a_1, a_2, \dots, a_{n-1}, b_1, b_2, \dots, b_n, c_1, c_2, \dots, c_{n-1}]$ and $\overline{S}_n = S_n[a_n, c_n]$ be the polynomials rings in these variables over the field K. Then I_n and J_n are squarefree monomial ideals of S_n and $\overline{S}_n$ respectively. With the labeling as shown in Figures 2.5 and 2.6, the minimal generating sets for I_n and J_n can be written as follows:

$$\mathcal{G}(I_n) = \bigcup_{i=1}^{n-1}\{a_ib_i, b_ic_i, a_ib_{i+1}, b_{i+1}c_i\} \bigcup \bigcup_{i=1}^{n-2}\{a_ia_{i+1}, c_ic_{i+1}\},$$

$$\mathcal{G}(J_n) = \mathcal{G}(I_n) \bigcup \{a_1a_n, c_1c_n, a_{n-1}a_n, c_{n-1}c_n, b_1a_n, b_1c_n, a_nb_n, b_nc_n\}.$$

Lemma 6.1.1. *If* $2 \leq n \leq 4$ *then* $\operatorname{depth}(S_n/I_n) = \operatorname{sdepth}(S_n/I_n) = n-1$.

Proof. If $n = 2$, then $\mathcal{G}(I_2) = \{a_1b_1, b_1c_1, a_1b_2, b_2c_1\}$, which is a minimal generating set of the edge ideal of C_4. Thus by Proposition 3.7.2 it follows that $\operatorname{depth}(S_2/I_2) = 1$. If $n = 3$, then $\mathcal{G}(I_3) = \mathcal{G}(I_2) \bigcup \{a_2b_2, b_2c_2, a_2b_3, b_3c_2, a_1a_2, c_1c_2\}$. Consider the following short exact sequence

$$0 \longrightarrow S_3/(I_3 : b_2) \xrightarrow{\cdot b_2} S_3/I_3 \longrightarrow S_3/(I_3, b_2) \longrightarrow 0. \tag{6.1}$$

Here $(I_3 : b_2) = (a_1, a_2, c_1, c_2)$, that gives $S_3/(I_3 : b_2) \cong K[b_1, b_2, b_3]$, thus $\operatorname{depth}(S_3/(I_3 : b_2)) = 3$. Also $(I_3, b_2) = (a_1b_1, b_1c_1, c_1c_2, c_2b_3, b_3a_2, a_2a_1, b_2)$, that implies $S_3/(I_3, b_2) \cong K[a_1, a_2, b_1, b_3, c_1, c_2]/(a_1b_1, b_1c_1, c_1c_2, c_2b_3, b_3a_2, a_2a_1) \cong K[a_1, a_2, b_1, b_3, c_1, c_2$ by Proposition 3.7.2, $\operatorname{depth}(S_3/(I_3, b_2)) = 2$. By applying Depth lemma on the short exact sequence (6.1), it follows that $\operatorname{depth}(S_3/I_3) \geq 2$. For the upper bound, since $b_3 \notin I_3$, by Corollary 3.7.2, $\operatorname{depth}(S_3/I_3) \leq \operatorname{depth}(S_3/(I_3 : b_3))$. As $(I_3 : b_3) = (a_2, c_2, I_2)$, thus $S_3/(I_3 : b_3) \cong S_2/I_2[b_3]$, by Lemma 3.7.5, it follows that $\operatorname{depth}(S_3/(I_3 : b_3)) \leq \operatorname{depth}(S_2/I_2) + 1 = 1 + 1 = 2$. Hence $\operatorname{depth}(S_3/I_3) = 2$. If $n = 4$, then $\mathcal{G}(I_4) = \mathcal{G}(I_3) \bigcup \{a_3b_3, b_3c_3, a_3b_4, b_4c_3, a_2a_3, c_2c_3\}$. Consider the following short exact sequence

$$0 \longrightarrow S_4/(I_4 : b_3) \xrightarrow{\cdot b_3} S_4/I_4 \longrightarrow S_4/(I_4, b_3) \longrightarrow 0. \tag{6.2}$$

Here $(I_4 : b_3) = (I_2, a_2, c_2, a_3, c_3)$, that gives $S_4/(I_4 : b_3) \cong S_2/I_2[b_3, b_4]$, thus Lemma 3.7.5 yields $\operatorname{depth}(S_4/(I_4 : b_3)) = \operatorname{depth}(S_2/I_2) + 2 = 1 + 2 = 3$. Let

$T := (I_4, b_3) = (I_2, a_2b_2, b_2c_2, a_1a_2, a_2a_3, c_1c_2, c_2c_3, a_3b_4, c_3b_4, b_3)$. Again assume the short exact sequence as follows:

$$0 \longrightarrow S_4/(T : b_2) \xrightarrow{\cdot b_2} S_4/T \longrightarrow S_4/(T, b_2) \longrightarrow 0. \tag{6.3}$$

Here $(T : b_2) = (a_1, a_2, c_1, c_2, b_3, a_3b_4, c_3b_4)$, that gives

$$S_4/(T : b_2) \cong K[a_3, b_4, c_3]/(a_3b_4, c_3b_4)[b_1, b_2],$$

by Lemmas 3.7.5 and 3.7.1, $\text{depth}(S_4/(T : b_2)) = 1 + 2 = 3$.
Also $(T, b_2) = (a_1b_1, b_1c_1, c_1c_2, c_2c_3, c_3b_4, b_4a_3, a_3a_2, a_2a_1, b_2, b_3)$, which implies that

$$\begin{aligned} S_4/(T, b_2) &\cong K[a_1, a_2, a_3, c_1, c_2, c_3, b_1, b_4]/(a_1b_1, b_1c_1, c_1c_2, c_2c_3, c_3b_4, b_4a_3, a_3a_2, a_2a_1) \\ &\cong K[a_1, a_2, a_3, c_1, c_2, c_3, b_1, b_4]/I(C_8) \end{aligned}$$

thus Proposition 3.7.2 gives that $\text{depth}(S_4/(T, b_2)) = 3$. By applying [42, Lemma 3.1] on the exact sequences (6.2), and (6.3), it follows that $\text{depth}(S_4/I_4) = 3$.
For Stanley depth, if $n = 2$, then by Theorem 3.7.13, $\text{sdepth}(S_2/I_2) \leq 1$. Also, by using the method in [36], there exists Stanley decomposition of desired Stanley depth as follows.

$$S_2/I_2 = K[a_1] \oplus b_1K[b_1, b_2] \oplus c_1K[a_1, c_1] \oplus b_2K[b_2].$$

Thus $\text{sdepth}(S_2/I_2) = 1$. If $n = 3$, then by applying Lemmas 3.7.3, 3.7.2, and Theorem 3.7.13 on the exact sequences (6.1), it follows that $\text{sdepth}(S_3/I_3) \geq 2$. For upper bound, since $b_3 \notin I_3$, by Proposition 3.7.3, $\text{sdepth}(S_3/I_3) \leq \text{sdepth}(S_3/(I_3 : b_3))$. As $(I_3 : b_3) = (a_2, c_2, I_2)$, thus $S_3/(I_3 : b_3) \cong S_2/I_2[b_3]$, by Lemma 3.7.5, it follows that $\text{sdepth}(S_3/(I_3 : b_3)) \leq \text{sdepth}(S_2/I_2) + 1 = 1 + 1 = 2$. Hence $\text{sdepth}(S_3/I_3) = 2$. If $n = 4$, by using Lemmas 3.7.3, 3.7.2, and Theorem 3.7.13 on the exact sequences (6.2) and (6.3), it follows that $\text{sdepth}(S_4/I_4) \geq 3$. For upper bound, since $b_4 \notin I_4$, by Proposition 3.7.3, it follows that $\text{sdepth}(S_4/I_4) \leq \text{sdepth}(S_4/(I_4 : b_4))$. As $(I_4 : b_4) = (a_3, c_3, I_3)$, thus $S_4/(I_4 : b_4) \cong S_3/I_3[b_4]$, by Lemma 3.7.5, it follows that $\text{sdepth}(S_4/(I_4 : b_4)) \leq \text{sdepth}(S_3/I_3)+1 = 2+1 = 3$. Hence $\text{sdepth}(S_4/I_4) = 3$. This completes the proof. □

Let $1 \leq k \leq n-1$ and $A_k := K[a_{n-1}, a_{n-2}, \dots, a_{n-k}], C_k := K[c_{n-1}, c_{n-2}, \dots, c_{n-k}]$, $D_k := A_k \otimes_K C_k$ and $\overline{D}_k := D_k \otimes_K K[b_1]$ be the subrings of S_n. Let $B_0 := (0)$, $B_j := (b_n, b_{n-1}, \dots, b_{n-j+1})$, for $1 \leq j \leq n$, and for $3 \leq j \leq n-2$, $\mathcal{P}_{j-1} := (a_{n-j+1}a_{n-j+2}, a_{n-j+2}a_{n-j+3}, \dots, a_{n-2}a_{n-1})$ and $\overline{\mathcal{P}}_{j-1} = (c_{n-j+1}c_{n-j+2}, \dots, c_{n-2}c_{n-1})$ are the squarefree monomial ideals of S_n. In the following result, some bounds for depth and Stanley depth of S_n/I_n are given.

Theorem 6.1.1. *For $n \geq 2$,*

$$\lceil \frac{n}{2} \rceil \leq \text{depth}(S_n/I_n), \text{sdepth}(S_n/I_n) \leq n-1$$

.

Proof. If $2 \leq n \leq 4$, then the result follows by Lemma 6.1.1. For $n \geq 5$, first to prove $\lceil \frac{n}{2} \rceil \leq \text{depth}(S_n/I_n) \leq n-1$, the induction on n is used. For $0 \leq j \leq n-2$, assume the following family of short exact sequences

$$0 \longrightarrow S_n/((I_n, B_0) : b_n) \xrightarrow{\cdot b_n} S_n/(I_n, B_0) \longrightarrow S_n/(I_n, B_1) \longrightarrow 0 \qquad (E_1)$$

$$0 \longrightarrow S_n/((I_n, B_1) : b_{n-1}) \xrightarrow{\cdot b_{n-1}} S_n/(I_n, B_1) \longrightarrow S_n/(I_n, B_2) \longrightarrow 0 \qquad (E_2)$$

$$0 \longrightarrow S_n/((I_n, B_2) : b_{n-2}) \xrightarrow{\cdot b_{n-2}} S_n/(I_n, B_2) \longrightarrow S_n/(I_n, B_3) \longrightarrow 0 \qquad (E_3)$$

$$\vdots$$

$$0 \longrightarrow S_n/((I_n, B_j) : b_{n-j}) \xrightarrow{\cdot b_{n-j}} S_n/(I_n, B_j) \longrightarrow S_n/(I_n, B_{j+1}) \longrightarrow 0 \qquad (E_{j+1})$$

$$\vdots$$

$$0 \longrightarrow S_n/((I_n, B_{n-2}) : b_2) \xrightarrow{\cdot b_2} S_n/(I_n, B_{n-2}) \longrightarrow S_n/(I_n, B_{n-1}) \longrightarrow 0 \qquad (E_{n-1})$$

(1) If $j = 0$, then $(I_n : b_n) = (I_{n-1}, a_{n-1}, c_{n-1})$, that gives $S_n/(I_n : b_n) \cong S_{n-1}/I_{n-1}[b_n]$, the induction hypothesis and Lemma 3.7.5 give

$$\operatorname{depth}(S_n/(I_n : b_n)) \geq \lceil \frac{n-1}{2} \rceil + 1 = \lceil \frac{n+1}{2} \rceil.$$

(2) If $j = 1$, then $((I_n, B_1) : b_{n-1}) = (I_{n-2}, a_{n-1}, a_{n-2}, c_{n-1}, c_{n-2}, B_1)$, that further implies that $S_n/((I_n, B_1) : b_{n-1}) \cong S_{n-2}/I_{n-2}[b_{n-1}]$, applying induction hypothesis and Lemma 3.7.5, it follows that $\operatorname{depth}(S_n/((I_n, B_1) : b_{n-1})) \geq \lceil \frac{n-2}{2} \rceil + 1 = \lceil \frac{n}{2} \rceil$.

(3) If $j = 2$, then $((I_n, B_2) : b_{n-2}) = (I_{n-3}, a_{n-2}, a_{n-3}, c_{n-2}, c_{n-3}, B_2)$, that further implies that $S_n/((I_n, B_2) : b_{n-2}) \cong S_{n-3}/I_{n-3}[a_{n-1}, b_{n-2}, c_{n-1}]$, the induction hypothesis and Lemma 3.7.5 give that $\operatorname{depth}(S_n/((I_n, B_2) : b_{n-2})) \geq \lceil \frac{n-3}{2} \rceil + 3 = \lceil \frac{n+1}{2} \rceil + 1$.

(4) If $3 \leq j \leq n-3$, then

$((I_n, B_j) : b_{n-j}) = \Big(I_{n-(j+1)}, (a_{n-j+1}a_{n-j+2}, a_{n-j+2}a_{n-j+3}, \dots, a_{n-2}a_{n-1}),$ and

$(c_{n-j+1}c_{n-j+2}, c_{n-j+2}c_{n-j+3}, \dots, c_{n-2}c_{n-1}), a_{n-j}, a_{n-(j+1)}, c_{n-j}, c_{n-(j+1)}, B_j\Big),$

that further implies

$$S_n/((I_n, B_j) : b_{n-j}) \cong (S_{n-(j+1)}/I_{n-(j+1)}) \otimes_K (A_{j-1}/\mathcal{P}_{j-1}) \otimes_K (C_{j-1}/\overline{\mathcal{P}}_{j-1}) \otimes_K K[b_{n-j}].$$

By applying [79, Theorem 2.2.21], it follows that

$$\operatorname{depth}(S_n/((I_n, B_j) : b_{n-j})) = \operatorname{depth}(S_{n-(j+1)}/I_{n-(j+1)}) + \operatorname{depth}(A_{j-1}/\mathcal{P}_{j-1}) + \operatorname{depth}(C_{j-1}/\overline{\mathcal{P}}_{j-1}) + 1.$$

By Lemma 3.7.1, $\operatorname{depth}(A_{j-1}/\mathcal{P}_{j-1}) = \lceil \frac{j-1}{3} \rceil = \operatorname{depth}(C_{j-1}/\overline{\mathcal{P}}_{j-1})$ and by induction on n, $\operatorname{depth}(S_{n-(j+1)}/I_{n-(j+1)}) \geq \lceil \frac{n-(j+1)}{2} \rceil$. Thus the following inequality holds.

$$\operatorname{depth}(S_n/((I_n, B_j) : b_{n-j})) \geq \lceil \frac{n-(j+1)}{2} \rceil + \lceil \frac{j-1}{3} \rceil + \lceil \frac{j-1}{3} \rceil + 1.$$

(5) If $j = n-2$, then $((I_n, B_{n-2}) : b_2) = \Big((a_3a_4, a_4a_5, \ldots, a_{n-2}a_{n-1}), a_{n-j}, a_{n-(j+1)}, c_{n-j}, c_{n-(j+1)}, (c_3c_4, \ldots, c_{n-2}c_{n-1}), B_j\Big)$, thus $S_n/((I_n, B_{n-2}) : b_2) \cong (A_{n-3}/\mathcal{P}_{n-3}) \otimes_K (C_{n-3}/\overline{\mathcal{P}}_{n-3}) \otimes_K K[b_1, b_2]$, by [79, Theorem 2.2.21], it follows that

$$\operatorname{depth}(S_n/((I_n, B_{n-2}) : b_2)) = \operatorname{depth}(A_{n-3}/\mathcal{P}_{n-3}) + \operatorname{depth}(C_{n-3}/\overline{\mathcal{P}}_{n-3}) + 2.$$

By Lemma 3.7.1, $\operatorname{depth}(A_{n-3}/\mathcal{P}_{n-3}) = \lceil \frac{n-3}{3} \rceil = \operatorname{depth}(C_{n-3}/\overline{\mathcal{P}}_{n-3})$. Thus $\operatorname{depth}(S_n/((I_n, B_{n-2}) : b_2)) = \lceil \frac{n-3}{3} \rceil + \lceil \frac{n-3}{3} \rceil + 2$ as desired.

Also $(I_n, B_{n-1}) = (a_{n-1}a_{n-2}, \ldots, a_2a_1, a_1b_1, b_1c_1, c_1c_2, c_2c_3, \ldots, c_{n-2}c_{n-1}, B_{n-1})$, that gives $S_n/(I_n, B_{n-1}) \cong \overline{D}_{n-1}/I(\mathcal{P}_{2n-1})$. Thus by Lemma 3.7.1, it follows that $\operatorname{depth}(S_n/(I_n, B_{n-1})) = \lceil \frac{2n-1}{3} \rceil$. By applying Depth Lemma on the above family of short exact sequences, the required lower bound for depth is holds. Now with the help of induction hypothesis on n, the inequality holds $\operatorname{depth}(S_n/I_n) \leq n-1$. For $n \geq 5$, as $b_n \notin I_n$, by Corollary 3.7.2, it follows that $\operatorname{depth}(S_n/I_n) \leq \operatorname{depth}(S_n/(I_n : b_n))$. Since $S_n/(I_n : b_n) \cong S_{n-1}/I_{n-1}[b_n]$, the induction hypothesis and Lemma 3.7.5 yield

$$\operatorname{depth}(S_n/(I_n : b_n)) \leq n - 1 - 1 + 1 = n - 1.$$

Now, it remains to show the result for Stanley depth. The required lower bound can be obtained by applying Lemmas 3.7.3, 3.7.2, and [70, Theorem 3.1] instead of Depth Lemma, Lemma 3.7.1, and [79, Theorem 2.2.21] respectively on above family of short exact sequences. Finally, the inequality $\operatorname{sdepth}(S_n/I_n) \leq n-1$ is proved by using induction on n. For $n \geq 5$, as $b_n \notin I_n$, from Proposition 3.7.3, $\operatorname{sdepth}(S_n/I_n) \leq \operatorname{sdepth}(S_n/(I_n : b_n))$. As $S_n/(I_n : b_n) \cong S_{n-1}/I_{n-1}[b_n]$, by induction and Lemma 3.7.5, it follows that $\operatorname{sdepth}(S_n/(I_n : b_n)) \leq n-1-1+1 = n-1$. This finishes the proof. □

Remark 6.1.1. Clearly $\operatorname{diam}(L(\mathcal{L}_n)) = n$, then by Theorems 3.7.14 it follows that

$$\operatorname{depth}(S_n/I_n), \operatorname{sdepth}(S_n/I_n) \geq \lceil \frac{n+1}{3} \rceil.$$

Theorem 6.1.1 shows $\operatorname{depth}(S_n/I_n), \operatorname{sdepth}(S_n/I_n) \geq \lceil \frac{n}{2} \rceil$. Thus in Theorem 6.1.1, there is a better lower bound for these classes of edge ideals.

In order to compute bounds for depth and Stanley depth of the cyclic module $\overline{S}_n/J_n$, consider two supergraphs U_n and V_n of $L(\mathcal{L}_n)$. The vertex and edge sets of U_n are $V(U_n) = V(L(\mathcal{L}_n)) \cup \{c_n\}$ and $E(U_n) = E(L(\mathcal{L}_n)) \cup \{c_{n-1}c_n, b_nc_n\}$ respectively. The vertex and edge sets of V_n are $V(V_n) = V(U_n) \cup \{c_{n+1}\}$ and $E(V_n) = E(U_n) \cup \{c_1c_{n+1}, b_1c_{n+1}\}$ respectively. For examples of U_n and V_n, see Figure 6.1. Denote the edge ideals of U_n and V_n with I_n^* and I_n^{**} respectively. The minimal generating sets of I_n^* and I_n^{**} are $\mathcal{G}(I_n^*) = \mathcal{G}(I_n) \bigcup \{c_{n-1}c_n, b_nc_n\}$ and $\mathcal{G}(I_n^{**}) = \mathcal{G}(I_n^*) \bigcup \{c_1c_{n+1}, b_1c_{n+1}\}$. The bounds for depth and Stanley depth of the cyclic modules S_n^*/I_n^* and S_n^{**}/I_n^{**} are provided in the next results, where $S_n^* = S_n[c_n]$ and $S_n^{**} = S_n[c_n, c_{n+1}]$.

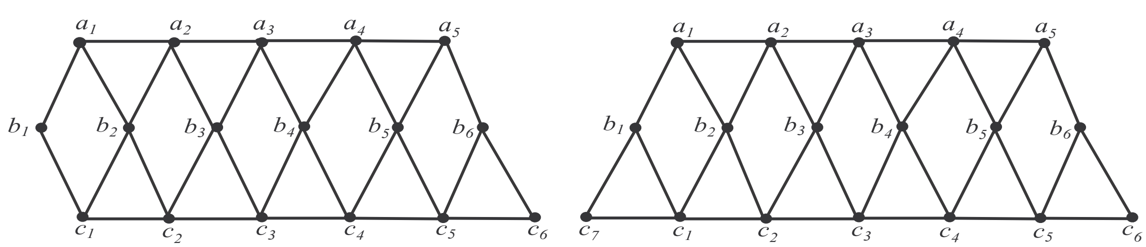

Figure 6.1: From left to right, supergraphs U_6 and V_6 of $L(\mathcal{L}_6)$ respectively.

Proposition 6.1.1. *Let $n \geq 2$. Then $\lceil \frac{n}{2} \rceil \leq \operatorname{depth}(S_n^*/I_n^*), \operatorname{sdepth}(S_n^*/I_n^*) \leq n$.*

Proof. If $n = 2$, then by using CoCoA, it follows that $\operatorname{depth}(S_n^*/I_n^*) = \operatorname{sdepth}(S_n^*/I_n^*) = 2$. For $n \geq 3$, the inequality $\lceil \frac{n}{2} \rceil \leq \operatorname{depth}(S_n^*/I_n^*)$ is proved by using induction on n. To do this, assume the following exact sequence

$$0 \longrightarrow S_n^*/(I_n^* : c_n) \xrightarrow{\cdot c_n} S_n^*/I_n^* \longrightarrow S_n^*/(I_n^*, c_n) \longrightarrow 0. \tag{6.4}$$

Here $(I_n^* : c_n) =$

$$(\bigcup_{i=1}^{n-2}\{a_ib_i, b_ic_i, a_ib_{i+1}, b_{i+1}c_i\}\bigcup\bigcup_{i=1}^{n-3}\{a_ia_{i+1}, c_ic_{i+1}\}, a_{n-1}a_{n-2}, a_{n-1}b_{n-1}, b_n, c_{n-1}),$$

that gives $S_n^*/(I_n^* : c_n) \cong S_{n-1}^*/I_{n-1}^*[c_n]$. By using induction and Lemma 3.7.5, the following inequality holds.

$$\operatorname{depth}(S_n^*/(I_n^* : c_n)) \geq \lceil\frac{n-1}{2}\rceil + 1 = \lceil\frac{n+1}{2}\rceil.$$

As $(I_n^*, c_n) = (\bigcup_{i=1}^{n-1}\{a_ib_i, b_ic_i, a_ib_{i+1}, b_{i+1}c_i\}\bigcup\bigcup_{i=1}^{n-2}\{a_ia_{i+1}, c_ic_{i+1}\}, c_n) = (I_n, c_n)$, that gives $S_n^*/(I_n^*, c_n) \cong S_n/I_n$. By Theorem 6.1.1, it follows that $\operatorname{depth}(S_n^*/(I_n^*, c_n)) \geq \lceil\frac{n}{2}\rceil$. Therefore by applying Depth Lemma on the sequence (7.3), it follows that $\operatorname{depth}(S_n^*/I_n^*) \geq \lceil\frac{n}{2}\rceil$. Now, the inequality $\operatorname{depth}(S_n^*/I_n^*) \leq n$ is also proved by using induction on n. For $n \geq 3$, as $c_n \notin I_n^*$, from Corollary 3.7.2, $\operatorname{depth}(S_n^*/I_n^*) \leq \operatorname{depth}(S_n^*/(I_n^* : c_n))$. Since $S_n^*/(I_n^* : c_n) \cong S_{n-1}^*/I_{n-1}^*[c_n]$, by induction and Lemma 3.7.5, $\operatorname{depth}(S_n^*/I_n^*) \leq n - 1 + 1 = n$. It remains to show the result for Stanley depth. For $n \geq 3$, by using induction hypothesis and Lemma 3.7.3 on the exact sequence (7.3), the inequality $\operatorname{sdepth}(S_n^*/I_n^*) \geq \lceil\frac{n}{2}\rceil$ holds. For upper bound of Stanley depth, one can repeat the proof for depth by using Proposition 3.7.3 instead of Corollary 3.7.2. □

Proposition 6.1.2. *If $n \geq 2$, then*

$$\lceil\frac{n}{2}\rceil \leq \operatorname{depth}(S_n^{**}/I_n^{**}), \operatorname{sdepth}(S_n^{**}/I_n^{**}) \leq n + 1.$$

Proof. If $n = 2$, then by using CoCoA, $\operatorname{depth}(S_n^{**}/I_n^{**}) = \operatorname{sdepth}(S_n^{**}/I_n^{**}) = 2$, and for $n = 3$, $\operatorname{depth}(S_n^{**}/I_n^{**}) = \operatorname{sdepth}(S_n^{**}/I_n^{**}) = 3$. For $n \geq 4$, to prove $\operatorname{depth}(S_n^{**}/I_n^{**}) \geq \lceil\frac{n}{2}\rceil$, the induction on n is used. Assume the following short exact sequence

$$0 \longrightarrow S_n^{**}/(I_n^{**} : c_n) \xrightarrow{\cdot c_n} S_n^{**}/I_n^{**} \longrightarrow S_n^{**}/(I_n^{**}, c_n) \longrightarrow 0. \tag{6.5}$$

As $(I_n^{**}, c_n) = (\bigcup_{i=1}^{n-1}\{a_ib_i, b_ic_i, a_ib_{i+1}, b_{i+1}c_i\} \bigcup \bigcup_{i=1}^{n-2}\{a_ia_{i+1}, c_ic_{i+1}\}, c_1c_{n+1}, b_1c_{n+1}, c_n)$,

that gives $S_n^{**}/(I_n^{**}, c_n) \cong S_n^*/I_n^*$. Therefore by Proposition 6.1.1, it follows that

$$\operatorname{depth}(S_n^{**}/(I_n^{**}, c_n)) \geq \lceil \frac{n}{2} \rceil.$$

Let $T = (I_n^{**} : c_n) = (\bigcup_{i=1}^{n-2}\{a_ib_i, b_ic_i, a_ib_{i+1}, b_{i+1}c_i\} \bigcup \bigcup_{i=1}^{n-3}\{a_ia_{i+1}, c_ic_{i+1}\}, a_{n-1}a_{n-2}, b_n$
$, a_{n-1}b_{n-1}, c_1c_{n+1}, b_1c_{n+1}, c_{n-1}) = (I_{n-1}^*, a_{n-1}a_{n-2}, a_{n-1}b_{n-1}, b_n, c_{n-1})$.

Now consider another short exact sequence as follows:

$$0 \longrightarrow S_n^{**}/(T : a_{n-1}) \xrightarrow{\cdot a_{n-1}} S_n^{**}/T \longrightarrow S_n^{**}/(T, a_{n-1}) \longrightarrow 0, \qquad (6.6)$$

$$(T : a_{n-1}) = (\bigcup_{i=1}^{n-3}\{a_ib_i, b_ic_i, a_ib_{i+1}, b_{i+1}c_i\} \bigcup \bigcup_{i=1}^{n-4}\{a_ia_{i+1}, c_ic_{i+1}\}, b_{n-2}c_{n-2}, c_{n-2}c_{n-3},$$
$$b_{n-1}, c_1c_{n+1}, b_1c_{n+1}, b_n, c_{n-1}, a_{n-2}) = (I_{n-2}^{**}, b_n, c_{n-1}, a_{n-2}, b_{n-1}),$$

that gives $S_n^{**}/(T : a_{n-1}) \cong S_{n-2}^{**}/I_{n-2}^{**}[a_{n-1}, c_n]$. Thus induction on n and Lemma 3.7.5 give that $\operatorname{depth}(S_n^{**}/(T : a_{n-1})) \geq \lceil \frac{n-2}{2} \rceil + 2 = \lceil \frac{n}{2} \rceil + 1$. As $(T, a_{n-1}) = (I_{n-1}^*, a_{n-1}, b_n, c_{n-1})$, which implies $S_n^{**}/(T, a_{n-1}) \cong S_{n-1}^*/I_{n-1}^*$. By Proposition 6.1.1 and Lemma 3.7.5, $\operatorname{depth}(S_n^{**}/(T, a_{n-1})) \geq \lceil \frac{n-1}{2} \rceil + 1 = \lceil \frac{n+1}{2} \rceil$. Therefore by applying Depth Lemma on the exact sequences (7.5) and (7.6), it follows that $\operatorname{depth}(S_n^{**}/I_n^{**}) \geq \lceil \frac{n}{2} \rceil$. Now to prove $\operatorname{depth}(S_n^{**}/I_n^{**}) \leq n+1$, the induction hypothesis is used. For $n \geq 4$, as $a_{n-1}c_n \notin I_n^{**}$, from Corollary 3.7.2, it implies that

$$\operatorname{depth}(S_n^{**}/I_n^{**}) \leq \operatorname{depth}(S_n^{**}/(I_n^{**} : a_{n-1}c_n)).$$

Since $S_n^{**}/(I_n^{**} : a_{n-1}c_n) \cong S_{n-2}^{**}/I_{n-2}^{**}[a_{n-1}, c_n]$, by using induction hypothesis and Lemma 3.7.5, $\operatorname{depth}(S_n^{**}/I_n^{**}) \leq n-2+1+2 = n+1$. It remains to prove the result for Stanley depth. For $n \geq 4$, by using induction on n, and by applying Lemma

3.7.3 on the exact sequences (7.5) and (7.6), the inequality $\operatorname{sdepth}(S_n^{**}/I_n^{**}) \geq \lceil \frac{n}{2} \rceil$ holds. Similarly, one can obtain the required upper bound for Stanley depth by using Proposition 3.7.3 instead of Corollary 3.7.2. □

Theorem 6.1.2. *Let $n \geq 3$. Then $\lceil \frac{n}{2} \rceil \leq \operatorname{depth}(\overline{S}_n/J_n) \leq n-1$, and $\lceil \frac{n}{2} \rceil \leq \operatorname{sdepth}(\overline{S}_n/J_n) \leq n$.*

Proof. For $3 \leq n \leq 4$, by using CoCoA, (for Stanley depth, SdepthLib:coc [71] is used), $\operatorname{depth}(\overline{S}_3/J_3) = \operatorname{sdepth}(\overline{S}_3/J_3) = 2$, $\operatorname{depth}(\overline{S}_4/J_4) = \operatorname{sdepth}(\overline{S}_4/J_4) = 3$. Now to show that $\operatorname{depth}(\overline{S}_n/J_n) \geq \lceil \frac{n}{2} \rceil$ for $n \geq 5$. Assume the short exact sequence as follows:

$$0 \longrightarrow \overline{S}_n/(J_n : a_n) \xrightarrow{\cdot a_n} \overline{S}_n/J_n \longrightarrow \overline{S}_n/(J_n, a_n) \longrightarrow \overline{S}_n/(J_n, a_n) \longrightarrow 0. \quad (6.7)$$

Let $U = (J_n, a_n) = (\bigcup_{i=1}^{n-1}\{a_ib_i, b_ic_i, a_ib_{i+1}, b_{i+1}c_i\} \bigcup \bigcup_{i=1}^{n-2}\{a_ia_{i+1}, c_ic_{i+1}\}, c_1c_n, c_{n-1}c_n, b_1c_n, b_nc_n, a_n)$.

Now assume another short exact sequence

$$0 \longrightarrow \overline{S}_n/(U : c_n) \xrightarrow{\cdot c_n} \overline{S}_n/U \longrightarrow \overline{S}_n/(U, c_n) \longrightarrow 0. \quad (6.8)$$

As $(U, c_n) = (\bigcup_{i=1}^{n-1}\{a_ib_i, b_ic_i, a_ib_{i+1}, b_{i+1}c_i\} \bigcup \bigcup_{i=1}^{n-2}\{a_ia_{i+1}, c_ic_{i+1}\}, c_n, a_n)$,

that implies $\overline{S}_n/(U, c_n) \cong S_n/I_n$. Thus Theorem 6.1.1 gives that $\operatorname{depth}(\overline{S}_n/(U, c_n)) \geq \lceil \frac{n}{2} \rceil$.

Also $(U : c_n) = (\bigcup_{i=2}^{n-2}\{a_ib_i, b_ic_i, a_ib_{i+1}, b_{i+1}c_i\} \bigcup \bigcup_{i=2}^{n-3}\{a_ia_{i+1}, c_ic_{i+1}\}, a_1a_2, a_1b_2, a_{n-1}a_{n-2}, a_n, a_{n-1}b_{n-1}, b_1, b_n, c_1, c_{n-1}) = (I_{n-2}^{**}, a_n, b_1, b_n, c_1, c_{n-1})$, that gives $\overline{S}_n/(U : c_n) \cong S^{**}_{n-2}/I^{**}_{n-2}[c_n]$. Thus by Proposition 6.1.2 and Lemma 3.7.5 the following inequality holds.

$$\operatorname{depth}(\overline{S}_n/(U : c_n)) \geq \lceil \frac{n-2}{2} \rceil + 1 = \lceil \frac{n}{2} \rceil.$$

Let $V = (J_n : a_n) = (\bigcup_{i=2}^{n-2}\{a_ib_i, b_ic_i, a_ib_{i+1}, b_{i+1}c_i\} \bigcup \bigcup_{i=2}^{n-3}\{a_ia_{i+1}, c_ic_{i+1}\}, c_1b_2, c_1c_2,$
$c_{n-1}c_{n-2}, c_{n-1}b_{n-1}, c_1c_n, c_nc_{n-1}, a_1, a_{n-1}, b_1, b_n).$

Now assume the short exact sequence as follows:

$$0 \longrightarrow \overline{S}_n/(V : c_n) \xrightarrow{\cdot c_n} \overline{S}_n/V \longrightarrow \overline{S}_n/(V, c_n) \longrightarrow 0, \tag{6.9}$$

$$\begin{aligned}(V : c_n) &= (\bigcup_{i=2}^{n-2}\{a_ib_i, b_ic_i, a_ib_{i+1}, b_{i+1}c_i\} \bigcup \bigcup_{i=2}^{n-3}\{a_ia_{i+1}, c_ic_{i+1}\}, c_1, c_{n-1}, a_1, b_1, b_n, a_{n-1}) \\ &= (I_{n-2}, c_1, c_{n-1}, a_1, b_1, b_n, a_{n-1}),\end{aligned}$$

that gives $\overline{S}_n/(V : c_n) \cong S_{n-2}/I_{n-2}[a_n, c_n]$. Thus by Theorem 6.1.1 and Lemma 3.7.5, it follows that

$$\operatorname{depth}(\overline{S}_n/(V : c_n)) \geq \lceil \frac{n-2}{2} \rceil + 2 = \lceil \frac{n}{2} \rceil + 1.$$

As $(V, c_n) = (\bigcup_{i=2}^{n-2}\{a_ib_i, b_ic_i, a_ib_{i+1}, b_{i+1}c_i\} \bigcup \bigcup_{i=2}^{n-3}\{a_ia_{i+1}, c_ic_{i+1}\}, c_1b_2, c_1c_2, c_{n-1}c_{n-2},$
$c_{n-1}b_{n-1}, a_1, a_{n-1}, b_1, b_n, c_n),$

that gives $\overline{S}_n/(V, c_n) \cong S^{**}_{n-2}/I^{**}_{n-2}$. By Proposition 6.1.2 and Lemma 3.7.5, $\operatorname{depth}(\overline{S}_n/(V, c_n)) \geq \lceil \frac{n-2}{2} \rceil + 1 = \lceil \frac{n}{2} \rceil$. Therefore by applying Depth Lemma on the exact sequences (6.7), (6.8) and (6.9), it follows that $\operatorname{depth}(\overline{S}_n/J_n) \geq \lceil \frac{n}{2} \rceil$. Now to prove $\operatorname{depth}(\overline{S}_n/J_n) \leq n-1$. For $n \geq 5$, as $a_nc_n \notin J_n$, from Corollary 3.7.2, it follows that $\operatorname{depth}(\overline{S}_n/J_n) \leq \operatorname{depth}(\overline{S}_n/(J_n : a_nc_n))$. Since $\overline{S}_n/(J_n :$

$a_n c_n) \cong S_{n-2}/I_{n-2}[a_n, c_n]$, by Theorem 6.1.1 and Lemma 3.7.5, it follows that $\text{depth}(\overline{S}_n/J_n) \leq n-2-1+2 = n-1$.

It remains to show the result for Stanley depth. For $n \geq 5$, by applying Lemma 3.7.3 on the exact sequences (6.7), (6.8) and (6.9), it follows that $\text{sdepth}(\overline{S}_n/J_n) \geq \lceil \frac{n}{2} \rceil$. Similarly, one can obtain the required upper bound for Stanley depth by using Proposition 3.7.3 instead of Corollary 3.7.2. □

Remark 6.1.2. It is clear that $\text{diam}(L(\mathcal{CL}_n)) = \lceil \frac{n+1}{2} \rceil$, then by Theorems 3.7.14, $\text{depth}(\overline{S}_n/J_n), \text{sdepth}(\overline{S}_n/J_n) \geq \lceil \frac{n+2}{6} \rceil$. Theorem 6.1.2 shows that $\text{depth}(\overline{S}_n/J_n), \text{sdepth}(\overline{S}_n/J_n) \geq \lceil \frac{n}{2} \rceil$. Thus in Theorem 6.1.2, there is a better lower bound for these classes of edge ideals.

Proposition 6.1.3. *If* $n \geq 2$, *then* $\dim(S_n/I_n) \geq n$.

Proof. Let $E = \{a_1, a_2, \ldots, a_{n-1}, c_1, c_2, \ldots, c_{n-1}\}$ be a subset of vertex set $V(L(\mathcal{L}_n))$. The set E is a vertex cover because it covers all the edges. Now by removing a_i for some $1 \leq i \leq n-1$ from set E then the resulting set is not a vertex cover because the edges $a_i b_i$ and $a_i b_{i+1}$ are not covered. Similarly, by removing c_i for some $1 \leq i \leq n-1$ from set E then the resulting set is not a vertex cover because the edges $c_i b_i$ and $c_i b_{i+1}$ are not covered. This shows that the set E forms a minimal vertex cover of I_n. Thus it follows that $\text{ht}(I_n) \leq 2n-2$. Since S_n is a polynomial ring of dimension $3n-2$, which implies that $\dim(S_n/I_n) \geq 3n-2-(2n-2) = n$. □

Proposition 6.1.4. *Let* $n \geq 3$. *Then* $\dim(\overline{S}_n/J_n) \geq n$.

Proof. As in the Proposition 6.1.3, one can show in a similar way that the set $F = \{a_1, a_2, \ldots, a_n, c_1, c_2, \ldots, c_n\}$ forms a minimal vertex cover of J_n, therefore $\text{ht}(J_n) \leq 2n$. As $\overline{S}_n$ is a polynomial ring of dimension $3n$, thus $\dim(\overline{S}_n/J_n) \geq n$. □

Remark 6.1.3. By Theorem 6.1.1 and 6.1.2, $\operatorname{depth}(S_n/I_n), \operatorname{depth}(\overline{S}_n/J_n) \leq n-1$, and by Proposition 6.1.3 and 6.1.4, it follows that $\dim(S_n/I_n), \dim(\overline{S}_n/J_n) \geq n$. Thus graphs $L(\mathcal{L}_n)$ and $L(\mathcal{CL}_n)$ are not Cohen-Macaulay.

6.2 Conclusion

In this chapter, some upper and lower bounds are established for Stanley depth and depth of edge ideals associated to line graphs of the ladder $L(\mathcal{L}_n)$ and circular ladder graphs $L(\mathcal{CL}_n)$. Furthermore, some bounds are given for the dimension of the quotient rings of the edge ideals related to these graphs. These bounds are good enough that one can concluded that the graphs $L(\mathcal{L}_n)$ and $L(\mathcal{CL}_n)$ are not Cohen-Macaulay.

Chapter 7

Depth and Stanley depth of the edge ideals of the strong product of some graphs

In this chapter, some results are provided that are related to Stanley depth and depth of the edge ideals and quotient ring of the edge ideals associated to classes of graphs $\mathcal{H} := \{P_n \boxtimes P_m : n, m \geq 1\}$ and $\mathcal{K} := \{C_n \boxtimes P_m : n \geq 3, m \geq 1\}$. In first section, Stanley depth and depth of the quotient ring of edge ideals associated to some sub-classes of $\mathcal{H}$ and $\mathcal{K}$ are presented. In section 2 of this chapter, the Conjecture 1.2.1 for the edge ideals related to some subclasses of $\mathcal{H}$ and $\mathcal{K}$ is proved. In the last section, sharp upper bounds are given for Stanley depth and depth of the quotient ring of the edge ideals associated to $\mathcal{H}$ and $\mathcal{K}$. Let P_1 denotes the null graph on one vertex that is $V(P_1) := \{x_1\}$ and $E(P_1) := \emptyset$. Let $\mathcal{P}_{n,m} := P_n \boxtimes P_m \cong P_m \boxtimes P_n$, if $m = n = 1$, then $\mathcal{P}_{1,1} \cong P_1$, this trivial case is excluded. For $n \geq 3$ and $m \geq 1$, $\mathcal{C}_{n,m} := C_n \boxtimes P_m \cong P_m \boxtimes C_n$.

Remark 7.0.1. $|V(\mathcal{P}_{n,m})| = nm$, $|V(\mathcal{C}_{n,m})| = nm$, $|E(\mathcal{P}_{n,m})| = (m-1) + 4(n-1)(m-1) + (n-1)$, and $|E(\mathcal{C}_{n,m})| = 3(m-1) + 1 + |E(\mathcal{P}_{n,m})|$.

Since both graphs $\mathcal{P}_{n,m}$ and $\mathcal{C}_{n,m}$ are on nm vertices, for the sake of convenience, the vertices of $\mathcal{P}_{n,m}$ and $\mathcal{C}_{n,m}$ are classified by using m sets of variables $\{x_{1j}, x_{2j}, \dots, x_{nj}\}$ where $1 \leq j \leq m$. Let $S_{n,m} := K[\cup_{j=1}^{m}\{x_{1j}, x_{2j}, \dots, x_{nj}\}]$.

Remark 7.0.2. 1. For $m, n \in \mathbb{Z}^+$ such that m and n are not equal to 1 simultaneously, $\mathcal{G}(I(\mathcal{P}_{n,m}))$ is given as:

$$\mathcal{G}(I(\mathcal{P}_{n,m})) = \cup_{i=1}^{n-1}\Big\{ \cup_{j=1}^{m-1} \{x_{ij}x_{i(j+1)}, x_{ij}x_{(i+1)(j+1)}, x_{ij}x_{(i+1)j}, x_{(i+1)j}x_{i(j+1)}, x_{nj}x_{n(j+1)}\Big\}, x_{im}x_{(i+1)m}\Big\}.$$

2. For $n \geq 3$, $m \geq 1$, $\mathcal{G}(I(\mathcal{C}_{n,m}))$ is as follows:

$$\mathcal{G}(I(\mathcal{C}_{n,m})) = \mathcal{G}(I(\mathcal{P}_{n,m})) \cup \Big\{ \cup_{j=1}^{m-1} \{x_{1j}x_{n(j+1)}, x_{1j}x_{nj}, x_{1(j+1)}x_{nj}\Big\}, x_{1m}x_{nm}\Big\}.$$

3. $\mathcal{P}_{n,1} \cong P_n$ and $\mathcal{C}_{n,1} \cong C_n$.

4. For $n, m \geq 1$, $\mathcal{P}_{n,m} \cong \mathcal{P}_{m,n}$, so without loss of generality the strong product of two paths can be represented as $\mathcal{P}_{n,m}$ with $m \leq n$. Thus in some proofs by induction on n, whenever the case is reduced where $\mathcal{P}_{n',m}$ with $n' < m$, in that case after a suitable relabeling of vertices, $\mathcal{P}_{n',m} \cong \mathcal{P}_{m,n'}$. Therefore, one can simply replace $I(\mathcal{P}_{n',m})$ by $I(\mathcal{P}_{m,n'})$ and $S_{n',m}/I(\mathcal{P}_{n',m})$ by $S_{m,n'}/I(\mathcal{P}_{m,n'})$.

7.1 Results of cyclic modules associated to $\mathcal{P}_{n,m}$ and $\mathcal{C}_{n,m}$ when $1 \leq m \leq 3$

Here, the results related to depth and Stanley depth of the cyclic modules $S_{n,m}/I(\mathcal{P}_{n,m})$ and $S_{n,m}/I(\mathcal{C}_{n,m})$ are presented, when $m = 1, 2, 3$. Let $n \geq 2$ and $1 \leq i \leq n$, for convenience, set $x_i := x_{i1}$, $y_i := x_{i2}$ and $z_i := x_{i3}$, see Figures 2.7 and 2.8. Let $S_{n,1} := K[x_1, x_2, \dots, x_n]$, $S_{n,2} := K[x_1, x_2, \dots x_n, y_1, y_2, \dots, y_n]$ and

$S_{n,3} := K[x_1, x_2, \dots x_n, y_1,$
$y_2, \dots, y_n, z_1, z_2, \dots, z_n]$. Clearly $\mathcal{P}_{n,1} \cong P_n$ and $\mathcal{C}_{n,1} \cong C_n$, the minimal generating sets for the edge ideals of $\mathcal{P}_{n,2}$, $\mathcal{P}_{n,3}$, $\mathcal{C}_{n,2}$ and $\mathcal{C}_{n,3}$ are given as:

$$\mathcal{G}(I(\mathcal{P}_{n,2})) = \cup_{i=1}^{n-1}\{x_iy_i, x_iy_{i+1}, x_ix_{i+1}, x_{i+1}y_i, y_iy_{i+1}\} \cup \{x_ny_n\},$$

$$\mathcal{G}(I(\mathcal{P}_{n,3})) = \cup_{i=1}^{n-1}\{x_iy_i, x_iy_{i+1}, x_ix_{i+1}, x_{i+1}y_i, y_iy_{i+1}, y_iz_i, y_iz_{i+1}, y_{i+1}z_i, z_iz_{i+1}\} \cup\{x_ny_n, y_nz_n\},$$

$$\mathcal{G}(I(\mathcal{C}_{n,2})) = \mathcal{G}(I(\mathcal{P}_{n,2})) \cup \{x_1y_n, x_1x_n, y_1x_n, y_1y_n\} \text{ and}$$

$$\mathcal{G}(I(\mathcal{C}_{n,3})) = \mathcal{G}(I(\mathcal{P}_{n,3})) \cup \{x_1y_n, x_1x_n, y_1x_n, y_1y_n, y_1z_n, z_1y_n, z_1z_n\}.$$

Remark 7.1.1. Note that for $n \geq 2$, $S_{n,1}/I(\mathcal{P}_{n,1}) \cong S/I(P_n)$, thus by [57, Lemma 2.8] and [76, Lemma 4], it follows that $\operatorname{depth}(S_{n,1}/I(\mathcal{P}_{n,1})) = \operatorname{sdepth}(S_{n,1}/I(\mathcal{P}_{n,1})) = \lceil \frac{n}{3} \rceil$. If $n \geq 3$, then $S_{n,1}/I(\mathcal{C}_{n,1}) \cong S/I(C_n)$, and by [16, Propositions 1.3 and 1.8], it results that $\operatorname{depth}(S_{n,1}/I(\mathcal{C}_{n,1})) = \lceil \frac{n-1}{3} \rceil \leq \operatorname{sdepth}(S_{n,1}/I(\mathcal{C}_{n,1})) \leq \lceil \frac{n}{3} \rceil$.

Lemma 7.1.1. *Let $n \geq 1$. Then*

$$\operatorname{depth}(S_{n,2}/I(\mathcal{P}_{n,2})) = \operatorname{sdepth}(S_{n,2}/I(\mathcal{P}_{n,2})) = \lceil \frac{n}{3} \rceil.$$

Proof. If $n = 1$, then by Remark 7.1.1, the required result holds. Let $n \geq 2$, to prove the result for depth. As $\operatorname{diam}(\mathcal{P}_{n,2}) = n - 1$, thus by Theorem 3.7.14, $\operatorname{depth}(S_{n,2}/I(\mathcal{P}_{n,2})) \geq \lceil \frac{n}{3} \rceil$. If $n = 2$ or $n = 3$, then the reverse inequality is trivially holds. For $n \geq 4$, to show the inequality, the induction on n is used. Since $y_{n-1} \notin I(\mathcal{P}_{n,2})$, thus by Corollary 3.7.2, it follows that

$$\operatorname{depth}(S_{n,2}/I(\mathcal{P}_{n,2})) \leq \operatorname{depth}(S_{n,2}/(I(\mathcal{P}_{n,2}) : y_{n-1})).$$

As $S_{n,2}/(I(\mathcal{P}_{n,2}) : y_{n-1}) \cong S_{n-3,2}/I(\mathcal{P}_{n-3,2})[y_{n-1}]$, therefore by induction and Lemma 3.7.5, it follows that $\operatorname{depth}(S_{n,2}/(I(\mathcal{P}_{n,2}) : y_{n-1})) \leq \lceil \frac{n-3}{3} \rceil + 1 = \lceil \frac{n}{3} \rceil$.

The proof of Stanley depth is similar by using Theorem 3.7.14 and Proposition 3.7.3. □

Lemma 7.1.2. *Let* $n \geq 1$. *Then*

$$\operatorname{depth}(S_{n,3}/I(\mathcal{P}_{n,3})) = \operatorname{sdepth}(S_{n,3}/I(\mathcal{P}_{n,3})) = \lceil \frac{n}{3} \rceil.$$

Proof. If $n = 1$, then the result follows by Remark 7.1.1. If $n = 2$, then $S_{2,3}/I(P_{2,3}) \cong S_{3,2}/I(P_{3,2})$ so the result holds by Lemma 7.1.1. For $n \geq 3$, first, it proves that the result holds for depth. As $\operatorname{diam}(\mathcal{P}_{n,3}) = n - 1$, then by Theorem 3.7.14, $\operatorname{depth}(S_{n,3}/I(\mathcal{P}_{n,3})) \geq \lceil \frac{n}{3} \rceil$. Now, for $n = 3$, the reverse inequality $\operatorname{depth}(S_{n,3}/I(\mathcal{P}_{n,3})) \leq \lceil \frac{n}{3} \rceil$ is trivially holds. If $n = 3$, then the required inequality is trivial. For $n \geq 4$, the inequality is proved by using induction on n. As $y_2 \notin I(\mathcal{P}_{n,3})$, thus by Corollary 3.7.2, it follows that

$$\operatorname{depth}(S_{n,3}/I(\mathcal{P}_{n,3})) \leq \operatorname{depth}(S_{n,3}/(I(\mathcal{P}_{n,3}) : y_2)).$$

Since $S_{n,3}/(I(\mathcal{P}_{n,3}) : y_2) \cong S_{n-3,3}/I(\mathcal{P}_{n-3,3})[y_2]$. Therefore the induction hypothesis and Lemma 3.7.5 give that $\operatorname{depth}(S_{n,3}/(I(\mathcal{P}_{n,3}) : y_2)) \leq \lceil \frac{n-3}{3} \rceil + 1 = \lceil \frac{n}{3} \rceil$. It remains to be shown that $\operatorname{sdepth}(S_{n,3}/I(\mathcal{P}_{n,3})) = \lceil \frac{n}{3} \rceil$. For this, it is similar by using Theorem 3.7.14 and Proposition 3.7.3. □

Theorem 7.1.1. *Let* $n \geq 3$. *Then*

$$\operatorname{sdepth}(S_{n,2}/I(\mathcal{C}_{n,2})) \geq \operatorname{depth}(S_{n,2}/I(\mathcal{C}_{n,2})) = \lceil \frac{n-1}{3} \rceil.$$

Proof. It is proved first that $\operatorname{depth}(S_{n,2}/I(\mathcal{C}_{n,2})) = \lceil \frac{n-1}{3} \rceil$. For $n = 3$, or $n = 4$, the result is trivial. Let $n \geq 5$, assume the short exact sequence as follows:

$$0 \longrightarrow S_{n,2}/(I(\mathcal{C}_{n,2}) : x_n) \xrightarrow{\cdot x_n} S_{n,2}/I(\mathcal{C}_{n,2}) \longrightarrow S_{n,2}/(I(\mathcal{C}_{n,2}), x_n) \longrightarrow 0, \quad (7.1)$$

by Depth Lemma

$$\operatorname{depth}(S_{n,2}/I(\mathcal{C}_{n,2})) \geq \min\{\operatorname{depth}(S_{n,2}/(I(\mathcal{C}_{n,2}) : x_n)), \operatorname{depth}(S_{n,2}/(I(\mathcal{C}_{n,2}), x_n))\}.$$

Here

$$(I(\mathcal{C}_{n,2}):x_n)=\big(\cup_{i=2}^{n-3}\{x_iy_i,x_iy_{i+1},x_ix_{i+1},x_{i+1}y_i,y_iy_{i+1}\},x_{n-2}y_{n-2},x_1,y_1,x_{n-1},y_{n-1},y_n\big).$$

After renumbering the variables, $S_{n,2}/(I(\mathcal{C}_{n,2}):x_n)\cong S_{n-3,2}/I(\mathcal{P}_{n-3,2})[x_n]$. Thus by Lemmas 7.1.1 and 3.7.5, it follows that $\operatorname{depth}(S_{n,2}/(I(\mathcal{C}_{n,2}):x_n))=\lceil\frac{n-3}{3}\rceil+1=\lceil\frac{n}{3}\rceil$. Let

$$J=(I(\mathcal{C}_{n,2}),x_n)=\big(\cup_{i=1}^{n-2}\{x_iy_i,x_iy_{i+1},x_ix_{i+1},x_{i+1}y_i,y_iy_{i+1}\},x_{n-1}y_{n-1},x_n,x_{n-1}y_n,y_{n-1}y_n,y_1y_n,x_1y_n\big)=(I(\mathcal{P}_{n-1,2}),x_n,x_{n-1}y_n,y_{n-1}y_n,y_1y_n,x_1y_n).$$

Assume the following exact sequence:

$$0\longrightarrow S_{n,2}/(J:y_n)\xrightarrow{\cdot y_n}S_{n,2}/J\longrightarrow S_{n,2}/(J,y_n)\longrightarrow 0, \tag{7.2}$$

by Depth Lemma

$$\operatorname{depth}(S_{n,2}/J)\geq\min\{\operatorname{depth}(S_{n,2}/(J:y_n)),\operatorname{depth}(S_{n,2}/(J,y_n))\}.$$

As $(J,y_n)=(I(\mathcal{P}_{n-1,2}),x_n,y_n)$ and $S_{n,2}/(J,y_n)\cong S_{n-1,2}/I(\mathcal{P}_{n-1,2})$. Therefore by Lemma 7.1.1, it follows that $\operatorname{depth}(S_{n,2}/(J,y_n))=\lceil\frac{n-1}{3}\rceil$. Also

$$(J:y_n)=\big(\cup_{i=2}^{n-3}\{x_iy_i,x_iy_{i+1},x_ix_{i+1},x_{i+1}y_i,y_iy_{i+1}\},x_{n-2}y_{n-2},x_1,y_1,y_{n-1},x_n,x_{n-1}\big).$$

After renumbering the variables, $S_{n,2}/(J:y_n)\cong S_{n-3,2}/I(\mathcal{P}_{n-3,2})[y_n]$. Therefore by Lemmas 7.1.1 and 3.7.5, it follows that $\operatorname{depth}(S_{n,2}/(I(\mathcal{C}_{n,2}):y_n))=\lceil\frac{n-3}{3}\rceil+1=\lceil\frac{n}{3}\rceil$. If $n\equiv 2(\operatorname{mod}3)$ or $n\equiv 0(\operatorname{mod}3)$ then $\lceil\frac{n-1}{3}\rceil=\lceil\frac{n}{3}\rceil$. By applying Depth Lemma on exact sequences (7.1) and (7.2), it results that $\operatorname{depth}(S_{n,2}/I(\mathcal{C}_{n,2}))=\lceil\frac{n-1}{3}\rceil$ as required. Now for $n\equiv 1(\operatorname{mod}3)$, assume that $n\geq 7$, then consider the following $S_{n,2}$-module isomorphism:

$$(I(\mathcal{C}_{n,2}):x_n)/I(\mathcal{C}_{n,2}) \cong x_1\frac{K[x_3,\dots,x_{n-1},y_3,\dots,,y_{n-1}]}{(\bigcup_{i=3}^{n-2}\{x_iy_i,x_iy_{i+1},x_ix_{i+1},x_{i+1}y_i,y_iy_{i+1}\},x_{n-1}y_{n-1})}[x_1]$$
$$\oplus\, y_1\frac{K[x_3,\dots,x_{n-1},y_3,\dots,,y_{n-1}]}{(\bigcup_{i=3}^{n-2}\{x_iy_i,x_iy_{i+1},x_ix_{i+1},x_{i+1}y_i,y_iy_{i+1}\},x_{n-1}y_{n-1})}[y_1]$$
$$\oplus\, y_n\frac{K[x_2,\dots,x_{n-2},y_2,\dots,,y_{n-2}]}{(\bigcup_{i=2}^{n-3}\{x_iy_i,x_iy_{i+1},x_ix_{i+1},x_{i+1}y_i,y_iy_{i+1}\},x_{n-2}y_{n-2})}[y_n]$$
$$\oplus\, x_{n-1}\frac{K[x_2,\dots,x_{n-3},y_2,\dots,y_{n-3}]}{(\bigcup_{i=2}^{n-4}\{x_iy_i,x_iy_{i+1},x_ix_{i+1},x_{i+1}y_i,y_iy_{i+1}\},x_{n-3}y_{n-3})}[x_{n-1}]$$
$$\oplus\, y_{n-1}\frac{K[x_2,\dots,x_{n-3},y_2,\dots,y_{n-3}]}{(\bigcup_{i=2}^{n-4}\{x_iy_i,x_iy_{i+1},x_ix_{i+1},x_{i+1}y_i,y_iy_{i+1}\},x_{n-3}y_{n-3})}[y_{n-1}].$$

It is easy to see that the first three summands are isomorphic to $S_{n-3,2}/I(\mathcal{P}_{n-3,2})[x_n]$ and last two summands are isomorphic to $S_{n-4,2}/I(\mathcal{P}_{n-4,2})[x_n]$. Thus by Lemmas 7.1.1 and 3.7.5, it follows that

$$\operatorname{depth}(I(\mathcal{C}_{n,2}):x_n)/I(\mathcal{C}_{n,2})) = \min\{\lceil\frac{n-3}{3}\rceil+1,\lceil\frac{n-4}{3}\rceil+1\} = \lceil\frac{n-1}{3}\rceil.$$

Applying Depth Lemma on the next sequence, the required result holds.

$$0\longrightarrow (I(\mathcal{C}_{n,2}):x_n)/I(\mathcal{C}_{n,2}) \xrightarrow{\cdot x_n} S_{n,2}/I(\mathcal{C}_{n,2}) \longrightarrow S_{n,2}/(I(\mathcal{C}_{n,2}):x_n) \longrightarrow 0.$$

For Stanley depth, the required result follows by applying Lemma 3.7.4 on the exact sequences (7.1) and (7.2). $\square$

Corollary 7.1.1. *Let* $n\geq 3$. *Then* $\lceil\frac{n-1}{3}\rceil \leq \operatorname{sdepth}(S_{n,2}/I(\mathcal{C}_{n,2})) \leq \lceil\frac{n}{3}\rceil$.

Proof. As $I(\mathcal{C}_{3,2})$ is a squarefree Veronese ideal, so by using [15, Theorem 1.1], $\operatorname{sdepth}(S_{n,2}/I(\mathcal{C}_{n,2})) = 1$. If $n\geq 4$, then by Proposition 3.7.3, it follows that $\operatorname{sdepth}(S_{n,2}/I(\mathcal{C}_{n,2})) \leq \operatorname{sdepth}(S_{n,2}/(I(\mathcal{C}_{n,2}):x_n))$. Since $S_{n,2}/(I(\mathcal{C}_{n,2}):x_n) \cong S_{n-3,2}/I(\mathcal{P}_{n-3,2})[x_n]$. Using Lemmas 7.1.1 and 3.7.5, it follows that $\operatorname{sdepth}(S_{n,2}/(I(\mathcal{C}_{n,2}):x_n)) = \lceil\frac{n-3}{3}\rceil+1 = \lceil\frac{n}{3}\rceil$. $\square$

For $n\geq 2$, define a supergraph of $\mathcal{P}_{n,3}$ denoted by $\mathcal{P}^*_{n,3}$ with the set of vertices $V(\mathcal{P}^*_{n,3}) := V(\mathcal{P}_{n,3})\cup\{z_{n+1}\}$ and edge set $E(\mathcal{P}^*_{n,3}) := E(\mathcal{P}_{n,3})\cup\{z_nz_{n+1},y_nz_{n+1}\}$.

Also, define a supergraph of $\mathcal{P}^*_{n,3}$ represented by $\mathcal{P}^{**}_{n,3}$ with the set of vertices $V(\mathcal{P}^{**}_{n,3}) := V(\mathcal{P}^*_{n,3}) \cup \{z_{n+2}\}$ and edge set $E(\mathcal{P}^{**}_{n,3}) := E(\mathcal{P}^*_{n,3}) \cup \{z_1 z_{n+2}, y_1 z_{n+2}\}$. For examples of $\mathcal{P}^*_{n,m}$ and $\mathcal{P}^{**}_{n,m}$ see Fig. 7.1. Let $S^*_{n,3} := S_{n,3}[z_{n+1}]$ and $S^{**}_{n,3} := S_{n,3}[z_{n+1}, z_{n+2}]$, then in the following lemmas, the results related to the depth and Stanley depth are presented.

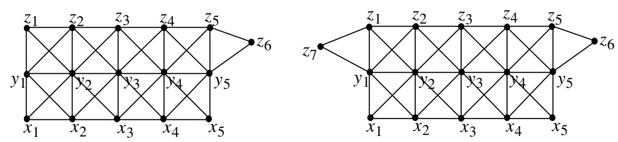

Figure 7.1: From left to right; $\mathcal{P}^*_{5,3}$ and $\mathcal{P}^{**}_{5,3}$.

Lemma 7.1.3. *Let $n \geq 2$. Then*

$$\operatorname{depth}(S^*_{n,3}/I(\mathcal{P}^*_{n,3})) = \operatorname{sdepth}(S^*_{n,3}/I(\mathcal{P}^*_{n,3})) = \lceil \frac{n+1}{3} \rceil.$$

Proof. To prove the result for depth, as $\operatorname{diam}(\mathcal{P}^*_{n,3}) = n$, then by Theorem 3.7.14, $\operatorname{depth}(S^*_{n,3}/I(\mathcal{P}^*_{n,3})) \geq \lceil \frac{n+1}{3} \rceil$. For the reverse inequality, if $n = 2$, then the result is trivial. For $n \geq 3$, as $y_n \notin I(\mathcal{P}^*_{n,3})$, so by Corollary 3.7.2, it follows that $\operatorname{depth}(S^*_{n,3}/I(\mathcal{P}^*_{n,3})) \leq \operatorname{depth}(S^*_{n,3}/(I(\mathcal{P}^*_{n,3}) : y_n))$. Since $S^*_{n,3}/(I(\mathcal{P}^*_{n,3}) : y_n) \cong (S_{n-2,3}/I(\mathcal{P}_{n-2,3}))[y_n]$, by Lemmas 7.1.2 and 3.7.5, it follows that $\operatorname{depth}(S^*_{n,3}/(I(\mathcal{P}^*_{n,3}) : y_n)) = \lceil \frac{n-2}{3} \rceil + 1 = \lceil \frac{n+1}{3} \rceil$. Thus $\operatorname{depth}(S^*_{n,3}/I(\mathcal{P}^*_{n,3})) \leq \lceil \frac{n+1}{3} \rceil$. The proof for Stanley depth is similar by using Proposition 3.7.3 and Theorem 3.7.14. □

Lemma 7.1.4. *Let $n \geq 2$. Then*

$$\operatorname{depth}(S^{**}_{n,3}/I(\mathcal{P}^{**}_{n,3})) = \operatorname{sdepth}(S^{**}_{n,3}/I(\mathcal{P}^{**}_{n,3})) = \lceil \frac{n+2}{3} \rceil.$$

Proof. Clearly $\operatorname{diam}(\mathcal{P}^{**}_{n,3}) = n+1$, then by Theorem 3.7.14, $\operatorname{depth}(S^{**}_{n,3}/I(\mathcal{P}^{**}_{n,3})) \geq \lceil \frac{n+2}{3} \rceil$. The reverse inequality is true when $n = 2$ or $n = 3$. For $n \geq 4$, as $y_n \notin I(\mathcal{P}^{**}_{n,3})$ so by Corollary 3.7.2, $\operatorname{depth}(S^{**}_{n,3}/I(\mathcal{P}^{**}_{n,3})) \leq \operatorname{depth}(S^{**}_{n,3}/(I(\mathcal{P}^{**}_{n,3}) : y_n))$. Since $S^{**}_{n,3}/(I(\mathcal{P}^{**}_{n,3}) : y_n) \cong (S^*_{n-2,3}/I(\mathcal{P}^*_{n-2,3}))[y_n]$, by Lemmas 7.1.3 and

3.7.5, it follows that $\operatorname{depth}(S^*_{n,3}/I(\mathcal{P}^*_{n,3}) : y_n) = \lceil \frac{n-2+1}{3} \rceil + 1 = \lceil \frac{n+2}{3} \rceil$. Thus $\operatorname{depth}(S^{**}_{n,3}/I(\mathcal{P}^{**}_{n,3})) \leq \lceil \frac{n+2}{3} \rceil$. Similarly, one can show the result for Stanley depth by using Proposition 3.7.3 and Theorem 3.7.14. □

Theorem 7.1.2. *Let $n \geq 3$ and $n \equiv 0, 2 \pmod 3$. Then*

$$\operatorname{depth}(S_{n,3}/I(\mathcal{C}_{n,3})) = \operatorname{sdepth}(S_{n,3}/I(\mathcal{C}_{n,3})) = \lceil \frac{n-1}{3} \rceil,$$

and if $n \equiv 1 \pmod 3$, then $\lceil \frac{n-1}{3} \rceil \leq \operatorname{depth}(S_{n,3}/I(\mathcal{C}_{n,3})), \operatorname{sdepth}(S_{n,3}/I(\mathcal{C}_{n,3})) \leq \lceil \frac{n}{3} \rceil$.

Proof. To show the result for depth, for $n = 3$ or $n = 4$, the result is clear. For $n \geq 5$, assume the short exact sequence as follows:

$$0 \longrightarrow S_{n,3}/(I(\mathcal{C}_{n,3}) : x_n) \xrightarrow{\cdot x_n} S_{n,3}/I(\mathcal{C}_{n,3}) \longrightarrow S_{n,3}/(I(\mathcal{C}_{n,3}), x_n) \longrightarrow 0, \quad (7.3)$$

Let $A := (I(\mathcal{C}_{n,3}) : x_n) = \big(\cup_{i=2}^{n-3} \{x_iy_i, x_ix_{i+1}, x_iy_{i+1}, x_{i+1}y_i, y_iy_{i+1}, y_iz_i, y_iz_{i+1}, y_{i+1}z_i, z_iz_{i+1}\}, x_{n-2}y_{n-2}, y_{n-2}z_{n-2}, x_1, x_{n-1}, y_1, y_{n-1}, y_n, z_nz_{n-1}, z_{n-1}z_{n-2}, y_{n-2}z_{n-1}, z_nz_1, z_1z_2, y_2z_1 \big)$,

and consider the following exact sequence

$$0 \longrightarrow S_{n,3}/(A : z_n) \xrightarrow{\cdot z_n} S_{n,3}/A \longrightarrow S_{n,3}/(A, z_n) \longrightarrow 0, \quad (7.4)$$

Here

$$(A, z_n) = \big(\cup_{i=2}^{n-3} \{x_iy_i, x_ix_{i+1}, x_iy_{i+1}, x_{i+1}y_i, y_iy_{i+1}, y_iz_i, y_iz_{i+1}, y_{i+1}z_i, z_iz_{i+1}\}, x_{n-2}y_{n-2}, y_{n-2}z_{n-2}, x_1, x_{n-1}, y_1, y_{n-1}, y_n, z_n, z_{n-1}z_{n-2}, y_{n-2}z_{n-1}, z_1z_2, y_2z_1 \big).$$

After renumbering the variables, $S_{n,3}/(A, z_n) \cong (S^{**}_{n-3,3}/I(\mathcal{P}^{**}_{n-3,3}))[x_n]$. Thus by Lemmas 7.1.4 and 3.7.5, it follows that $\operatorname{depth}(S_{n,3}/(A, z_n)) = \lceil \frac{n-3+2}{3} \rceil + 1 =$

$\lceil \frac{n-1}{3} \rceil + 1$. Also

$$(A : z_n) = \big(\cup_{i=2}^{n-3} \{x_iy_i, x_iy_{i+1}, x_ix_{i+1}, x_{i+1}y_i, y_iy_{i+1}, y_iz_i, y_iz_{i+1}, y_{i+1}z_i, z_iz_{i+1}\}, x_{n-2}y_{n-2}, y_{n-2}z_{n-2}, x_1, x_{n-1}, z_1, z_{n-1}, y_1, y_{n-1}, y_n \big).$$

After renumbering the variables, $S_{n,3}/(A : z_n) \cong (S_{n-3,3}/I(\mathcal{P}_{n-3,3}))[x_n, z_n]$. Thus by Lemmas 7.1.2 and 3.7.5, it follows that $\text{depth}(S_{n,3}/(A : z_n)) = \lceil \frac{n-3}{3} \rceil + 2 = \lceil \frac{n}{3} \rceil + 1$. Now let

$$\overline{A} := (I(\mathcal{C}_{n,3}), x_n) = \big(\cup_{i=1}^{n-2} \{x_iy_i, x_iy_{i+1}, x_ix_{i+1}, x_{i+1}y_i, y_iy_{i+1}, y_iz_i, y_iz_{i+1}, y_{i+1}z_i, z_iz_{i+1}\}, x_{n-1}y_{n-1}, y_{n-1}z_{n-1}, x_n, x_{n-1}y_n, y_{n-1}y_n, y_nz_{n-1}, y_{n-1}z_n, z_{n-1}z_n, y_nz_n, y_1y_n, x_1y_n, y_1z_n, y_nz_1, z_1z_n \big) = (I(\mathcal{P}_{n-1,3}), x_n, x_{n-1}y_n, y_{n-1}y_n, y_nz_{n-1}, y_{n-1}z_n, z_{n-1}z_n, y_nz_n, y_1y_n, x_1y_n, y_1z_n, y_nz_1, z_1z_n),$$

and the following exact sequence

$$0 \longrightarrow S_{n,3}/(\overline{A} : y_n) \xrightarrow{\cdot y_n} S_{n,3}/\overline{A} \longrightarrow S_{n,3}/(\overline{A}, y_n) \longrightarrow 0, \tag{7.5}$$

As $(\overline{A} : y_n) = \big(\cup_{i=2}^{n-3}\{x_iy_i, x_iy_{i+1}, x_ix_{i+1}, x_{i+1}y_i, y_iy_{i+1}, y_iz_i, y_iz_{i+1}, y_{i+1}z_i, z_iz_{i+1}\}, x_{n-2}y_{n-2}, y_{n-2}z_{n-2}, x_n, x_1, y_1, z_1, x_{n-1}, y_{n-1}, z_{n-1}, z_n\big)$.

After renumbering the variables, $S_{n,3}/(\overline{A} : y_n) \cong S_{n-3,3}/I(\mathcal{P}_{n-3,3})[y_n]$. Therefore by Lemmas 7.1.2 and 3.7.5, it follows that $\text{depth}(S_{n,3}/(\overline{A} : y_n)) = \lceil \frac{n-3}{3} \rceil + 1 = \lceil \frac{n}{3} \rceil$. Now let

$$\widehat{A} := (\overline{A}, y_n) = (I(\mathcal{P}_{n-1,3}), x_n, y_n, y_{n-1}z_n, z_{n-1}z_n, y_1z_n, z_1z_n),$$

and the following short exact sequence

$$0 \longrightarrow S_{n,3}/(\widehat{A} : z_n) \xrightarrow{\cdot z_n} S_{n,3}/\widehat{A} \longrightarrow S_{n,3}/(\widehat{A}, z_n) \longrightarrow 0, \tag{7.6}$$

thus $S_{n,3}/(\widehat{A}, z_n) \cong S_{n-1,3}/I(\mathcal{P}_{n-1,3})$. Therefore by Lemma 7.1.2, it results that

$$\operatorname{depth}(S_{n,3}/(\widehat{A}, z_n)) = \lceil \frac{n-1}{3} \rceil.$$

Also

$$(\widehat{A} : z_n) = \big(\cup_{i=2}^{n-3} \{x_iy_i, x_iy_{i+1}, x_ix_{i+1}, x_{i+1}y_i, y_iy_{i+1}, y_iz_i, y_iz_{i+1}, y_{i+1}z_i, z_iz_{i+1}\},$$
$$x_{n-2}y_{n-2}, y_{n-2}z_{n-2}, z_1, y_1, z_{n-1}, y_{n-1}, y_n, x_n, x_{n-1}x_{n-2}, x_{n-1}y_{n-2}, x_1x_2, x_1y_2\big).$$

After renumbering the variables, $S_{n,3}/(\widehat{A} : z_n) \cong (S^{**}_{n-3,3}/I(\mathcal{P}^{**}_{n-3,3}))[z_n]$. Thus by Lemmas 7.1.4 and 3.7.5, it follows that $\operatorname{depth}(S_{n,3}/(\widehat{A} : z_n)) = \lceil \frac{n-3+2}{3} \rceil + 1 = \lceil \frac{n-1}{3} \rceil + 1$. By applying Depth Lemma on the exact sequences (7.3), (7.4), (7.5) and (7.6) it follows that $\operatorname{depth}(S_{n,3}/I(\mathcal{C}_{n,3})) \geq \lceil \frac{n-1}{3} \rceil$. For upper bound, by Corollary 3.7.2, $\operatorname{depth}(S_{n,3}/I(\mathcal{C}_{n,3})) \leq \operatorname{depth}(S_{n,3}/(I(\mathcal{C}_{n,3}) : y_n))$. Since $(S_{n,3}/(I(\mathcal{C}_{n,3}) : y_n)) \cong (S_{n-3,3}/(I(\mathcal{P}_{n-3,3}))[y_n]$, then Lemmas 7.1.2 and 3.7.5 give that $\operatorname{depth}(S_{n,3}/I(\mathcal{C}_{n,3})) \leq \lceil \frac{n}{3} \rceil$, if $n \equiv 2 (\bmod 3)$ or $n \equiv 0 (\bmod 3)$ then $\lceil \frac{n-1}{3} \rceil = \lceil \frac{n}{3} \rceil$. If $n \equiv 1 (\bmod 3)$, then $\lceil \frac{n-1}{3} \rceil \leq \operatorname{depth}(S_{n,3}/I(\mathcal{C}_{n,3})) \leq \lceil \frac{n}{3} \rceil$. The proof for Stanley depth is similar by using Lemma 3.7.4 and Proposition 3.7.3. □

Example 7.1.1. One can expect that $\operatorname{depth}(S_{n,3}/I(\mathcal{C}_{n,3})) = \lceil \frac{n-1}{3} \rceil$ as in [16, Proposition 1.3] and Theorem 7.1.1. But examples show that in the essential case for $n \equiv 1 (\bmod 3)$, the upper bound in Theorem 7.1.2 is reached. For instance, if $n = 4$, then $\operatorname{depth}(S_{4,3}/I(\mathcal{C}_{4,3})) = 2 = \lceil \frac{4}{3} \rceil$.

7.2 Lower bounds for Stanley depth of $I(\mathcal{P}_{n,m})$ and $I(\mathcal{C}_{n,m})$ when $1 \leq m \leq 3$

In this section, some lower bounds are given for Stanley depth of $I(\mathcal{P}_{n,m})$ and $I(\mathcal{C}_{n,m})$, when $m \leq 3$. These bounds together with the results of previous section allow to give a positive answer to the conjecture 1.2.1. The following elementary lemma is helpful in proving the main results of this section.

Lemma 7.2.1. *Let A_1 and A_2 be two disjoint sets of variables, $I_1 \subset K[A_1]$ and $I_2 \subset K[A_2]$ be squarefree monomial ideals such that* $\operatorname{sdepth}_{K[A_1]}(I_1) > \operatorname{sdepth}(K[A_1]/I_1)$. *Then*

$$\operatorname{sdepth}_{K[A_1\cup A_2]}(I_1 + I_2) \geq \operatorname{sdepth}(K[A_1]/I_1) + \operatorname{sdepth}_{K[A_2]}(I_2).$$

Proof. Proof follows by [14, Theorem 1.3]. □

Remark 7.2.1. Since $I(\mathcal{P}_{n,1}) \cong I(P_n)$, thus by [60, Theorem 2.3] and [67, Prposition 2.1], it follows that $\operatorname{sdepth}(I(\mathcal{P}_{n,1})) > \operatorname{sdepth}(S_{n,1}/I(\mathcal{P}_{n,1})) = \lceil \frac{n}{3} \rceil$.

Theorem 7.2.1. *Let $n \geq 1$. Then* $\operatorname{sdepth}(I(\mathcal{P}_{n,2})) > \operatorname{sdepth}(S_{n,2}/I(\mathcal{P}_{n,2})) = \lceil \frac{n}{3} \rceil$.

Proof. Let $1 \leq t \leq n$, then by Lemma 7.1.1, $\operatorname{sdepth}(S_{t,2}/I(\mathcal{P}_{t,2})) = \lceil \frac{t}{3} \rceil$. The Lemma 7.1.1 is used in the proof without referring it again and again. By the same lemma it is enough to show that $\operatorname{sdepth}(I(\mathcal{P}_{n,2})) > \lceil \frac{n}{3} \rceil$. To do it, the induction on n is used. If $n = 1$, then by Remark 7.2.1, the required result holds. If $n = 2, 3$, then by [53, Lemma 2.1], it follows that $\operatorname{sdepth}(I(\mathcal{P}_{n,2})) > \lceil \frac{n}{3} \rceil$. Now assume that $n \geq 4$. Since $x_{n-1} \notin I(\mathcal{P}_{n,2})$, thus

$$I(\mathcal{P}_{n,2}) = I(\mathcal{P}_{n,2}) \cap S' \oplus x_{n-1}\big(I(\mathcal{P}_{n,2}) : x_{n-1}\big)S_{n,2},$$

where $S' = K[x_1, x_2, \ldots, x_{n-2}, x_n, y_1, y_2, \ldots, y_n]$. Now

$$I(\mathcal{P}_{n,2}) \cap S' = \big(\mathcal{G}(I(\mathcal{P}_{n-2,2})), x_{n-2}y_{n-1}, y_{n-2}y_{n-1}, x_ny_n, y_{n-1}x_n, y_{n-1}y_n\big) \text{ and}$$

$$\big(I(\mathcal{P}_{n,2}) : x_{n-1}\big)S_{n,2} = \big(\mathcal{G}(I(\mathcal{P}_{n-3,2})), x_{n-2}, x_n, y_{n-2}, y_{n-1}, y_n\big)S_{n,2}.$$

As $y_{n-1} \notin I(\mathcal{P}_{n,2}) \cap S'$, it follows that

$$I(\mathcal{P}_{n,2}) \cap S' = (I(\mathcal{P}_{n,2}) \cap S') \cap S'' \oplus y_{n-1}\big(I(\mathcal{P}_{n,2}) \cap S' : y_{n-1}\big)S',$$

where $S'' = K[x_1, \ldots, x_{n-2}, x_n, y_1, \ldots, y_{n-2}, y_n]$. Thus

$$I(\mathcal{P}_{n,2}) = (I(\mathcal{P}_{n,2}) \cap S') \cap S'' \oplus y_{n-1}\big(I(\mathcal{P}_{n,2}) \cap S' : y_{n-1}\big)S' \oplus x_{n-1}\big(I(\mathcal{P}_{n,2}) : x_{n-1}\big)S_{n,2},$$

where

$$(I(\mathcal{P}_{n,2}) \cap S') \cap S'' = (\mathcal{G}(I(\mathcal{P}_{n-2,2})), x_n y_n)S''$$

and

$$(I(\mathcal{P}_{n,2}) \cap S' : y_{n-1}\big)S' = \big(\mathcal{G}(I(\mathcal{P}_{n-3,2})), x_{n-2}, y_{n-2}, x_n, y_n\big)S'.$$

The induction hypothesis and Lemma 7.2.1 yield

$$\operatorname{sdepth}((I(\mathcal{P}_{n,2}) \cap S') \cap S'') \geq \operatorname{sdepth}(S_{n-2,2}/I(\mathcal{P}_{n-2,2})) + \operatorname{sdepth}_{K[x_n,y_n]}(x_n y_n).$$

Again by induction on n, Lemmas 7.2.1 and 3.7.5 it follows that

$$\operatorname{sdepth}((I(\mathcal{P}_{n,2}) \cap S' : y_{n-1}\big)S') \geq \operatorname{sdepth}(S_{n-3,2}/I(\mathcal{P}_{n-3,2})) + \operatorname{sdepth}_T(x_{n-2}, x_n, y_n, y_{n-2}) + 1$$

and

$$\operatorname{sdepth}\big(\big(I(\mathcal{P}_{n,2}) : x_{n-1}\big)S_{n,2}\big) \geq \operatorname{sdepth}(S_{n-3,2}/I(\mathcal{P}_{n-3,2})) + \operatorname{sdepth}_R(x_{n-2}, y_{n-2}, y_{n-1}, x_n, y_n\big) + 1,$$

where $T = [x_{n-2}, x_n, y_n, y_{n-2}]$ and $R = K[x_{n-2}, x_n, y_n, y_{n-2}, y_{n-1}]$. Thus $\operatorname{sdepth}((I(\mathcal{P}_{n,2}) \cap S') \cap S'') > \lceil \frac{n}{3} \rceil$ as $\operatorname{sdepth}_{K[x_n,y_n]}(x_n y_n) = 2$. Applying [7, Theorem 2.2], it results that $\operatorname{sdepth}((I(\mathcal{P}_{n,2}) \cap S' : y_{n-1}\big)S') > \lceil \frac{n}{3} \rceil$ and $\operatorname{sdepth}(\big(I(\mathcal{P}_{n,2}) : x_{n-1}\big)S_{n,2}) > \lceil \frac{n}{3} \rceil$. This completes the proof. □

Now, for the case $m = 3$, some notations are introduced. For $3 \leq l \leq n-2$, let $J_l := (x_{n-l}, x_{n-l+1}, x_{n-l-1}, z_{n-l}, y_{n-l-1}, z_{n-l+1}, z_{n-l-1})$, $I(P'_{l-1}) := (x_{n-l+2}x_{n-l+3}, \ldots, x_{n-1}x_n)$ and $I(P''_{l-1}) := (z_{n-l+2}z_{n-l+3}, \ldots, z_{n-1}z_n)$ be the monomial ideals of $S_{n,3}$. Now consider the subsets $D_l := \{x_{n-l+2}, x_{n-l+3}, \ldots, x_n\}$, $D'_l := \{z_{n-l+2}, z_{n-l+3}, \ldots, z_{n-1},$

$z_n\}$ and $D''_l := \{x_{n-l}, x_{n-l+1}, z_{n-l}, y_{n-l-1}, z_{n-l+1}, z_{n-l-1}, x_{n-l-1}\}$. Let L_l be a monomial ideal of $S_{n,3}$ such that $L_l = I(P'_{l-1}) + I(P''_{l-1}) + J_l$. With these notation, the next result is presented as follows.

Lemma 7.2.2. *Let* $3 \leq l \leq n-2$. *Then* $\operatorname{sdepth}_{K[D_l \cup D'_l \cup D''_l]}(L_l) \geq \lceil \frac{l+2}{3} \rceil + 1$.

Proof. Since $L_l = I(P'_{l-1}) + I(P''_{l-1}) + J_l$, by [14, Theorem 1.3], it follows that

$$\operatorname{sdepth}_{K[D_l \cup D'_l \cup D''_l]}(L_l) \geq \min\big\{ \operatorname{sdepth}_{K[D_l \cup D'_l \cup D''_l]}(J_l), \min\{\operatorname{sdepth}_{K[D_l \cup D'_l]}(I(P'_{l-1})), \operatorname{sdepth}_{K[D_l]}(K[D_l]/I(P'_{l-1})) + \operatorname{sdepth}_{K[D'_l]}(I(P''_{l-1}))\}\big\}. \quad (7.7)$$

By using [60, Theorem 2.3] and [67, Proposition 2.1], Eq. 7.7 implies that

$$\operatorname{sdepth}_{K[D_l \cup D'_l \cup D''_l]}(L_l) \geq \min\{2l-4+4, \min\{2l-2-\lfloor \tfrac{l-2}{2} \rfloor, \lceil \tfrac{l-1}{3} \rceil + l - 1 - \lfloor \tfrac{l-2}{2} \rfloor\}\} \geq \lceil \frac{l+2}{3} \rceil + 1.$$

□

Theorem 7.2.2. *Let* $n \geq 1$. *Then* $\operatorname{sdepth}(I(\mathcal{P}_{n,3})) > \operatorname{sdepth}(S_{n,3}/I(\mathcal{P}_{n,3}))$.

Proof. Let $1 \leq t \leq n$, then by Lemma 7.1.2, $\operatorname{sdepth}(S_{t,3}/I(\mathcal{P}_{t,3})) = \lceil \frac{t}{3} \rceil$. The Lemma 7.1.2 is used in the proof several times without referring it. Using the same lemma it is enough to show that $\operatorname{sdepth}(I(\mathcal{P}_{n,3})) > \lceil \frac{n}{3} \rceil$. To do it, the induction on n is used. If $n = 1$, then by Remark 7.2.1 the required result follows. If $n = 2$, the result holds true by using Theorem 7.2.1. If $n = 3$ then by [53, Lemma 2.1] $\operatorname{sdepth}(I(\mathcal{P}_{3,3})) > \lceil \frac{3}{3} \rceil$. If $n \geq 4$, then $I(\mathcal{P}_{n,3})$ can be decomposed as a vector space in the following way:

$$I(\mathcal{P}_{n,3}) = I(\mathcal{P}_{n,3}) \cap R_1 \oplus y_n(I(\mathcal{P}_{n,3}) : y_n)S_{n,3}.$$

Similarly, decompose $I(\mathcal{P}_{n,3}) \cap R_1$ as follows:

$$I(\mathcal{P}_{n,3}) \cap R_1 = I(\mathcal{P}_{n,3}) \cap R_2 \oplus y_{n-1}(I(\mathcal{P}_{n,3}) \cap R_1 : y_{n-1})R_1.$$

Continuing in the same way for $1 \leq l \leq n-1$, it follows that

$$I(\mathcal{P}_{n,3}) \cap R_l = I(\mathcal{P}_{n,3}) \cap R_{l+1} \oplus y_{n-l}(I(\mathcal{P}_{n,3}) \cap R_l : y_{n-l})R_l,$$

where $R_l := K[x_1, \dots x_n, y_1, \dots, y_{n-l}, z_1, \dots, z_n]$. Finally, the following decomposition of $I(\mathcal{P}_{n,3})$ can be obtained:

$$I(\mathcal{P}_{n,3}) = I(\mathcal{P}_{n,3}) \cap R_n \oplus \oplus_{l=1}^{n-1} y_{n-l}(I(\mathcal{P}_{n,3}) \cap R_l : y_{n-l})R_l \oplus y_n(I(\mathcal{P}_{n,3}) : y_n)S_{n,3}.$$

Therefore

$$\begin{aligned} \operatorname{sdepth}(I(\mathcal{P}_{n,3})) \geq \min \Big\{ & \operatorname{sdepth}(I(\mathcal{P}_{n,3}) \cap R_n), \operatorname{sdepth}((I(\mathcal{P}_{n,3}) : y_n)S_{n,3}), \\ & \min_{l=1}^{n-1}\{\operatorname{sdepth}((I(\mathcal{P}_{n,3}) \cap R_l : y_{n-l})R_l)\}\Big\}. \end{aligned} \tag{7.8}$$

Since

$$\begin{aligned} I(\mathcal{P}_{n,3}) \cap R_n = \big(& (z_1z_2, z_2z_3, \dots, z_{n-1}z_n) \\ & + (x_1x_2, x_2x_3, \dots, x_{n-1}x_n)\big)K[x_1, \dots, x_n, z_1, \dots, z_n], \end{aligned}$$

thus by [14, Theorem 1.3] and [67, Proposition 2.1], it follows that $\operatorname{sdepth}(I(\mathcal{P}_{n,3}) \cap R_n) > \lceil \frac{n}{3} \rceil$. As

$$(I(\mathcal{P}_{n,3}) : y_n)S_{n,3} = (\mathcal{G}(I(\mathcal{P}_{n-2,3})) + (x_n, z_n, x_{n-1}, z_{n-1}, y_{n-1}))[y_n].$$

Let $B := K[x_n, z_n, z_{n-1}, x_{n-1}, y_{n-1}]$, thus by induction on n, Lemmas 7.2.1 and 3.7.5, it follows that

$$\begin{aligned} \operatorname{sdepth}((I(\mathcal{P}_{n,3}) : y_n)S_{n,3}) > & \operatorname{sdepth}(S_{n-2,3}/I(\mathcal{P}_{n-2,3})) \\ & + \operatorname{sdepth}_B(x_n, z_n, z_{n-1}, x_{n-1}, y_{n-1}) + 1. \end{aligned}$$

By applying [7, Theorem 2.2], it results that $\operatorname{sdepth}((I(\mathcal{P}_{n,3}) : y_n)S_{n,3}) > \lceil \frac{n}{3} \rceil$.

(1) If $l = 1$, then $(I(\mathcal{P}_{n,3}) \cap R_1 : y_{n-1})R_1 = \Big(\mathcal{G}(I(\mathcal{P}_{n-3,3})) + J_1\Big)[y_{n-1}]$, where $J_1 := (x_{n-1}, x_{n-2}, x_n, z_{n-1}, y_{n-2}, z_n, z_{n-2})$, then by induction on n, Lemmas 7.2.1 and 3.7.5, it follows that

$$\begin{aligned}\operatorname{sdepth}((I(\mathcal{P}_{n,3}) \cap R_1 : y_{n-1})R_1) > \operatorname{sdepth}(S_{n-3,3}/I(\mathcal{P}_{n-3,3}))\\ + \operatorname{sdepth}_{K[\operatorname{supp}(J_1)]}(J_1) + 1,\end{aligned}$$

by [7, Theorem 2.2], it follows that $\operatorname{sdepth}((I(\mathcal{P}_{n,3}) \cap R_1 : y_{n-1})R_1) > \lceil \frac{n}{3} \rceil$.

(2) If $l = 2$ and $n \neq 4$, then

$$(I(\mathcal{P}_{n,3}) \cap R_2 : y_{n-2})R_2 = \Big(\mathcal{G}(I(\mathcal{P}_{n-4,3})) + J_2\Big)[y_{n-2}, x_n, z_n],$$

where $J_2 := (x_{n-2}, x_{n-1}, x_{n-3}, z_{n-2}, z_{n-1}, y_{n-3}, z_{n-3})$, using the similar arguments as in case(1), $\operatorname{sdepth}((I(\mathcal{P}_{n,3}) \cap R_2 : y_{n-2})R_2) > \lceil \frac{n}{3} \rceil$.

(3) If $3 \leq l \leq n-3$, then $(I(\mathcal{P}_{n,3}) \cap R_l : y_{n-l})R_l = \Big(\mathcal{G}(I(\mathcal{P}_{n-(l+2),3})) + \mathcal{G}(L_l)\Big)[y_{n-l}]$, by induction on n, Lemmas 7.2.1 and 3.7.5, it follows that

$$\begin{aligned}\operatorname{sdepth}((I(\mathcal{P}_{n,3}) \cap R_l : y_{n-l})R_l) > \operatorname{sdepth}(S_{n-(l+2),3}/(I(\mathcal{P}_{n-(l+2),3})))\\ + \operatorname{sdepth}_{K[D_l \cup D'_l \cup D''_l]}(L_l) + 1, \quad (7.9)\end{aligned}$$

By Eq. 7.9 and Lemma 7.2.2, it follows that

$$\operatorname{sdepth}((I(\mathcal{P}_{n,3}) \cap R_l : y_{n-l})R_l) > \lceil \frac{n-(l+2)}{3} \rceil + \lceil \frac{l+2}{3} \rceil + 1 + 1 > \lceil \frac{n}{3} \rceil.$$

(4) If $l = n-2$, then $(I(\mathcal{P}_{n,3}) \cap R_{n-2} : y_2)R_{n-2} = (\mathcal{G}(L_{n-2}))[y_2]$, by Lemmas 7.2.2 and 3.7.5 it follows that $\operatorname{sdepth}((I(\mathcal{P}_{n,3}) \cap R_{n-2} : y_2)R_{n-2}) > \lceil \frac{n}{3} \rceil$.

(5) If $l = n-1$, then

$$(I(\mathcal{P}_{n,3}) \cap R_{n-1} : y_1)R_{n-1} = \big(I(P'_{n-2}) + I(P''_{n-2}) + J_{n-1}\big)K[D_{n-1} \cup D'_{n-1} \cup D''_{n-1} \cup \{y_1\}],$$

where $\mathcal{G}(J_{n-1}) = \{x_1, z_1, x_2, z_2\}$, $D_{n-1} = \{x_3, x_4, \ldots, x_n\}$, $D'_{n-1} = \{z_3, \ldots, z_n\}$ and $D''_{n-1} = \{x_1, z_1, x_2, z_2\}$. Using the proof of Lemma 7.2.2 and by Lemma 3.7.5 it follows that

$$\operatorname{sdepth}_{K[D_{n-1}\cup D'_{n-1}\cup D''_{n-1}\cup\{y_1\}]}\big(I(P'_{n-2}) + I(P''_{n-2}) + J_{n-1}\big) > \lceil \frac{n}{3} \rceil,$$

that is $\operatorname{sdepth}((I(\mathcal{P}_{n,3}) \cap R_{n-1} : y_1)R_{n-1}) > \lceil \frac{n}{3} \rceil$.

Thus by Eq. 7.8, the required inequality $\operatorname{sdepth}(I(\mathcal{P}_{n,3})) > \lceil \frac{n}{3} \rceil$ holds. □

Proposition 7.2.1. *Let* $n \geq 3$. *Then* $\operatorname{sdepth}(I(\mathcal{C}_{n,2})/I(\mathcal{P}_{n,2})) \geq \lceil \frac{n+2}{3} \rceil$.

Proof. For $3 \leq n \leq 5$, the method which is given in [36] can be used to show that there exist Stanley decompositions of desired Stanley depth. If $n = 3$ or $n = 4$, then

$$I(\mathcal{C}_{n,2})/I(\mathcal{P}_{n,2}) = x_1x_nK[x_1, x_n] \oplus x_1y_nK[x_1, y_n] \oplus y_1x_nK[y_1, x_n] \oplus y_1y_nK[y_1, y_n].$$

If $n = 5$, then

$$\begin{aligned} I(\mathcal{C}_{5,2})/I(\mathcal{P}_{5,2}) = x_1x_5K[x_1, x_5, x_3] \oplus x_1y_5K[x_1, y_5, x_3] \oplus y_1x_5K[y_1, x_3, x_5] \oplus \\ y_1y_5K[y_1, x_3, y_5] \oplus x_1y_3x_5K[x_1, y_3, x_5] \oplus x_1y_3y_5K[x_1, y_3, y_5] \oplus \\ y_1y_3y_5K[y_1, y_3, y_5] \oplus y_1y_3x_5K[y_1, y_3, x_5]. \end{aligned}$$

Let $n \geq 6$ and $T := \big(\bigcup_{i=3}^{n-3}\{x_iy_i, x_iy_{i+1}, x_{i+1}y_i, x_ix_{i+1}, y_iy_{i+1}\}, x_{n-2}y_{n-2}\big) \subset \tilde{S}$, where $\tilde{S} := K[x_3, x_4, \ldots, x_{n-2}, y_3, y_4 \ldots, y_{n-2}]$. Then consider the K-vector space isomorphism as follows:

$$I(\mathcal{C}_{n,2})/I(\mathcal{P}_{n,2}) \cong x_1x_n\frac{\tilde{S}}{T}[x_1, x_n] \oplus y_1y_n\frac{\tilde{S}}{T}[y_1, y_n] \oplus x_1y_n\frac{\tilde{S}}{T}[x_1, y_n] \oplus y_1x_n\frac{\tilde{S}}{T}[y_1, x_n].$$

Thus by Lemmas 7.1.1 and 3.7.5, it follows that $\operatorname{sdepth}(I(\mathcal{C}_{n,2})/I(\mathcal{P}_{n,2})) \geq \lceil \frac{n+2}{3} \rceil$. □

For $n \geq 6$, let $Q = \{x_{n-1}, y_{n-1}, x_n, y_n, x_2, y_2, x_1, y_1\}$, and consider a subgraph $\mathcal{C}^{\diamond}_{n,3}$ of $\mathcal{C}_{n,3}$ with $V(\mathcal{C}^{\diamond}_{n,3}) = V(\mathcal{C}_{n,3}) \setminus Q$ and

$$E(\mathcal{C}^{\diamond}_{n,3}) = E(\mathcal{C}_{n,3}) \setminus \{e \in E(\mathcal{C}_{n,3}) : \text{ where } e \text{ has at least one end vertex in } Q\}.$$

For example of $\mathcal{C}^{\diamond}_{n,3}$ see Fig. 7.2.

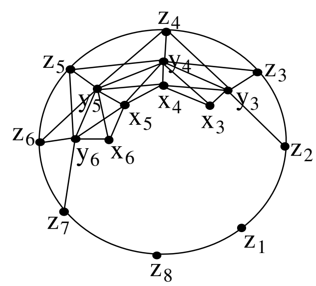

Figure 7.2: $\mathcal{C}^{\diamond}_{8,3}$.

Lemma 7.2.3. *Let $n \geq 6$, if $n \equiv 0 \pmod 3$, then* $\operatorname{sdepth}(S^{\diamond}_{n,3}/I(C^{\diamond}_{n,3})) = \lceil \frac{n-2}{3} \rceil$. *Otherwise,* $\lceil \frac{n-2}{3} \rceil \leq \operatorname{sdepth}(S^{\diamond}_{n,3}/I(C^{\diamond}_{n,3})) \leq \lceil \frac{n}{3} \rceil$.

Proof. Assume the short exact sequence as follows:

$$0 \longrightarrow S^{\diamond}_{n,3}/(I(C^{\diamond}_{n,3}) : z_1) \xrightarrow{\cdot z_1} S^{\diamond}_{n,3}/I(C^{\diamond}_{n,3}) \longrightarrow S^{\diamond}_{n,3}/(I(C^{\diamond}_{n,3}), z_1) \longrightarrow 0, \quad (7.10)$$

by Lemma 3.7.4

$$\operatorname{sdepth}(S^{\diamond}_{n,3}/I(C^{\diamond}_{n,3})) \geq \min\{\operatorname{sdepth}(S^{\diamond}_{n,3}/(I(C^{\diamond}_{n,3}) : z_1)), \operatorname{sdepth}(S^{\diamond}_{n,3}/(I(C^{\diamond}_{n,3}), z_1))\}.$$

As $(I(C^{\diamond}_{n,3}) : z_1) = \big((\cup_{i=3}^{n-3}\{x_iy_i, x_iy_{i+1}, x_ix_{i+1}, x_{i+1}y_i, y_iy_{i+1}, y_iz_i, y_iz_{i+1}, y_{i+1}z_i, z_iz_{i+1}\}, x_{n-2}y_{n-2}, y_{n-2}z_{n-2}), y_{n-2}z_{n-1}, z_{n-2}z_{n-1}, z_2, z_n\big) = (I(\mathcal{P}^*_{n-4,3}), z_2, z_n)$,

so it follows that $S^{\diamond}_{n,3}/(I(C^{\diamond}_{n,3}) : z_1) \cong S^*_{n-4,3}/I(\mathcal{P}^*_{n-4,3})[z_1]$. Therefore, by Lemmas 3.7.5 and 7.1.3, it results that

$$\operatorname{sdepth}(S^{\diamond}_{n,3}/(I(C^{\diamond}_{n,3}) : z_1)) = \lceil \frac{n-4+1}{3} \rceil + 1 = \lceil \frac{n}{3} \rceil.$$

Now suppose that

$$B := (I(C^{\diamond}_{n,3}), z_1) = \big((\cup_{i=3}^{n-3}\{x_iy_i, x_iy_{i+1}, x_ix_{i+1}, x_{i+1}y_i, y_iy_{i+1}, y_iz_i, y_iz_{i+1}, y_{i+1}z_i,$$
$$z_iz_{i+1}\}, x_{n-2}y_{n-2}, y_{n-2}z_{n-2}), y_{n-2}z_{n-1}, z_{n-2}z_{n-1}, z_{n-1}z_n, y_3z_2, z_2z_3, z_1\big),$$

Applying Lemma 3.7.4 on the following short exact sequence

$$0 \longrightarrow S^{\diamond}_{n,3}/(B:z_n) \xrightarrow{\cdot z_n} S^{\diamond}_{n,3}/B \longrightarrow S^{\diamond}_{n,3}/(B,z_n) \longrightarrow 0,$$

gives $\operatorname{sdepth}(S^{\diamond}_{n,3}/B) \geq \min\{\operatorname{sdepth}(S^{\diamond}_{n,3}/(B:z_n)), \operatorname{sdepth}(S^{\diamond}_{n,3}/(B,z_n))\}$. Here

$$(B:z_n) = \big((\cup_{i=3}^{n-3}\{x_iy_i, x_iy_{i+1}, x_ix_{i+1}, x_{i+1}y_i, y_iy_{i+1}, y_iz_i, y_iz_{i+1}, y_{i+1}z_i, z_iz_{i+1}\},$$
$$x_{n-2}y_{n-2}, y_{n-2}z_{n-2}), y_3z_2, z_2z_3, z_1, z_{n-1}\big) = (I(\mathcal{P}^{*}_{n-4,3}), z_1, z_{n-1}),$$

that gives $S^{\diamond}_{n,3}/(B:z_n) \cong S^{*}_{n-4,3}/I(\mathcal{P}^{*}_{n-4,3})[z_n]$. Therefore by Lemmas 3.7.5 and 7.1.3, it follows that $\operatorname{sdepth}(S^{\diamond}_{n,3}/(B:z_n)) = \lceil\frac{n-4+1}{3}\rceil + 1 = \lceil\frac{n}{3}\rceil$. Now

$$(B, z_n) = \big((\bigcup_{i=3}^{n-3}\{x_iy_i, x_iy_{i+1}, x_ix_{i+1}, x_{i+1}y_i, y_iy_{i+1}, y_iz_i, y_iz_{i+1}, y_{i+1}z_i, z_iz_{i+1}\},$$
$$x_{n-2}y_{n-2}, y_{n-2}z_{n-2}), y_{n-2}z_{n-1}, z_{n-2}z_{n-1}, y_3z_2, z_2z_3, z_1, z_n\big) = (I(\mathcal{P}^{**}_{n-4,3}), z_1, z_n),$$

thus $S^{\diamond}_{n,3}/(B,z_n) \cong S^{**}_{n-4,3}/I(\mathcal{P}^{**}_{n-4,3})$. Therefore by Lemma 7.1.4, it follows that

$$\operatorname{sdepth}(S^{\diamond}_{n,3}/(B,z_n)) = \lceil\frac{n-4+2}{3}\rceil = \lceil\frac{n-2}{3}\rceil.$$

For upper bound, as $z_1 \notin I(C^{\diamond}_{n,3})$ so by Proposition 3.7.3

$$\operatorname{sdepth}(S^{\diamond}_{n,3}/I(C^{\diamond}_{n,3})) \leq \operatorname{sdepth}(S^{\diamond}_{n,3}/(I(C^{\diamond}_{n,3}):z_1)).$$

Since $(S^{\diamond}_{n,3}/(I(C^{\diamond}_{n,3}):z_1)) \cong (S^{*}_{n-4,3}/I(\mathcal{P}^{*}_{n-4,3}))[z_1]$. Thus by Lemmas 3.7.5 and 7.1.3, it follows that

$$\operatorname{sdepth}(S^{\diamond}_{n,3}/I(C^{\diamond}_{n,3})) \leq \lceil\frac{n}{3}\rceil,$$

if $n \equiv 0 (\operatorname{mod} 3)$ then $\lceil\frac{n-2}{3}\rceil = \lceil\frac{n}{3}\rceil$, otherwise

$$\lceil\frac{n-2}{3}\rceil \leq \operatorname{sdepth}(S^{\diamond}_{n,3}/I(C^{\diamond}_{n,3})) \leq \lceil\frac{n}{3}\rceil.$$

□

Proposition 7.2.2. *Let $n \geq 3$. Then* $\operatorname{sdepth}(I(\mathcal{C}_{n,3})/I(\mathcal{P}_{n,3})) \geq \lceil \frac{n+2}{3} \rceil$.

Proof. For $3 \leq n \leq 4$, as the minimal generators of $I(\mathcal{C}_{n,3})/I(\mathcal{P}_{n,3})$ have degree 2, so by [53, Lemma 2.1], $\operatorname{sdepth}(I(\mathcal{C}_{n,3})/I(\mathcal{P}_{n,3})) \geq 2 = \lceil \frac{n+2}{3} \rceil$. If $n = 5$, then the method given in [36] can be used to show that there exist Stanley decompositions of desired Stanley depth. Let

$$H := x_1x_5K[x_1, x_3, x_5] \oplus x_1y_5K[x_1, x_3, y_5] \oplus y_1x_5K[x_3, x_5, y_1] \oplus y_1y_5K[x_3, y_1, y_5] \oplus z_1y_5K[x_3, y_5, z_1] \oplus z_1z_5K[z_1, z_3, z_5] \oplus y_1z_5K[y_1, y_3, z_5]$$

Clearly, $H \subset I(\mathcal{C}_{5,3})/I(\mathcal{P}_{5,3})$. Let $v \in I(\mathcal{C}_{5,3})/I(\mathcal{P}_{5,3})$ be a sqaurefree monomial such that $v \notin H$ then $\deg(v) \geq 3$. Since

$$I(\mathcal{C}_{5,3})/I(\mathcal{P}_{5,3}) = H \oplus_v vK[\operatorname{supp}(v)].$$

That gives $\operatorname{sdepth}(I(\mathcal{C}_{5,3})/I(\mathcal{P}_{5,3})) \geq 3 = \lceil \frac{5+2}{3} \rceil$. Now for $n \geq 6$, let

$$U := \big(\cup_{i=3}^{n-3}\{x_iy_i, x_iy_{i+1}, x_ix_{i+1}, x_{i+1}y_i, y_iy_{i+1}, y_iz_i, y_iz_{i+1}, y_{i+1}z_i, z_iz_{i+1}\}, x_{n-2}y_{n-2}, y_{n-2}z_{n-2}\big)$$

be a squarefree monomial ideal of $R := K[x_3, \dots, x_{n-2}, y_3, \dots, y_{n-2}, z_3, \dots, z_{n-2}]$. Then consider the K-vector space isomorphism as follows:

$$\begin{aligned}
I(\mathcal{C}_{n,3})/I(\mathcal{P}_{n,3}) \cong{}& y_1y_n\frac{R}{U}[y_1, y_n] \oplus x_1y_n\frac{R[z_2]}{\big(\mathcal{G}(U), y_3z_2, z_2z_3\big)}[x_1, y_n] \\
&\oplus z_1y_n\frac{R[x_2]}{\big(\mathcal{G}(U), y_3x_2, x_2x_3\big)}[z_1, y_n] \oplus y_1x_n\frac{R[z_{n-1}]}{\big(\mathcal{G}(U), y_{n-2}z_{n-1}, z_{n-2}z_{n-1}\big)}[y_1, x_n] \\
&\oplus y_1z_n\frac{R[x_{n-1}]}{\big(\mathcal{G}(U), y_{n-2}x_{n-1}, x_{n-2}x_{n-1}\big)}[y_1, z_n] \\
&\oplus x_1x_n\frac{R[z_1, z_2, z_{n-1}, z_n]}{\big(\mathcal{G}(U), y_{n-2}z_{n-1}, z_{n-2}z_{n-1}, z_{n-1}z_n, z_nz_1, z_1z_2, y_3z_2, z_2z_3\big)}[x_1, x_n] \\
&\oplus z_1z_n\frac{R[x_1, x_2, x_{n-1}, x_n]}{\big(\mathcal{G}(U), y_{n-2}x_{n-1}, x_{n-2}x_{n-1}, x_{n-1}x_n, x_nx_1, x_1x_2, y_3x_2, x_2x_3\big)}[z_1, z_n].
\end{aligned}$$

Clearly, it can be noted that $R/U \cong S_{n-4,3}/I(\mathcal{P}_{n-4,3})$,

$$\frac{R[z_2]}{\big(\mathcal{G}(U), y_3z_2, z_2z_3\big)} \cong \frac{R[x_2]}{\big(\mathcal{G}(U), y_3x_2, x_2x_3\big)} \cong \frac{R[z_{n-1}]}{\big(\mathcal{G}(U), y_{n-2}z_{n-1}, z_{n-2}z_{n-1}\big)}$$
$$\cong \frac{R[x_{n-1}]}{\big(\mathcal{G}(U), y_{n-2}x_{n-1}, x_{n-2}x_{n-1}\big)} \cong S^*_{n-4,3}/I(\mathcal{P}^*_{n-4,3}),$$

and

$$\frac{R[z_1, z_2, z_{n-1}, z_n]}{\big(\mathcal{G}(U), y_{n-2}z_{n-1}, z_{n-2}z_{n-1}, z_{n-1}z_n, z_nz_1, z_1z_2, y_3z_2, z_2z_3\big)}$$
$$\cong \frac{R[x_1, x_2, x_{n-1}, x_n]}{\big(\mathcal{G}(U), y_{n-2}x_{n-1}, x_{n-2}x_{n-1}, x_{n-1}x_n, x_nx_1, x_1x_2, y_3x_2, x_2x_3\big)} \cong S^{\diamond}_{n,3}/I(\mathcal{C}^{\diamond}_{n,3}).$$

Thus by Lemmas 7.1.2, 7.1.3 and 7.2.3, it follows that

$$\operatorname{sdepth}(I(\mathcal{C}_{n,3})/I(\mathcal{P}_{n,3})) \geq \min\Big\{\lceil\frac{n-4}{3}\rceil+2, \lceil\frac{n-4+1}{3}\rceil+2, \lceil\frac{n-2}{3}\rceil+2\Big\} = \lceil\frac{n+2}{3}\rceil.$$

□

Theorem 7.2.3. *Let* $1 \leq m \leq 3$ *and* $n \geq 3$*. Then*

$$\operatorname{sdepth}(I(\mathcal{C}_{n,m})) \geq \operatorname{sdepth}(S_{n,m}/I(\mathcal{C}_{n,m})).$$

Proof. For $m = 1$, $I(\mathcal{C}_{n,1}) = C_n$. Then the result follows by [16, Theorem 1.9] and [60, Theorem 2.3]. If $m = 2$ or 3, then assume the short exact sequence as follows:

$$0 \longrightarrow I(\mathcal{P}_{n,m}) \longrightarrow I(\mathcal{C}_{n,m}) \longrightarrow I(\mathcal{C}_{n,m})/I(\mathcal{P}_{n,m}) \longrightarrow 0,$$

then by Lemma 3.7.4, it follows that

$$\operatorname{sdepth}(I(\mathcal{C}_{n,m})) \geq \min\{\operatorname{sdepth}(I(\mathcal{P}_{n,m})), \operatorname{sdepth}(I(\mathcal{C}_{n,m})/I(\mathcal{P}_{n,m}))\}.$$

By Theorems 7.2.1 and 7.2.2, $\operatorname{sdepth}(I(\mathcal{P}_{n,m})) \geq \lceil\frac{n}{3}\rceil + 1$, and by Propositions 7.2.1 and 7.2.2, it results that $\operatorname{sdepth}(I(\mathcal{C}_{n,m})/I(\mathcal{P}_{n,m})) \geq \lceil\frac{n+2}{3}\rceil = \lceil\frac{n-1}{3}\rceil + 1$, this completes the proof. □

7.3 Upper bounds for depth and Stanley depth of cyclic modules associated to $\mathcal{P}_{n,m}$ and $\mathcal{C}_{n,m}$

Let $m \leq n$, in general, one don't know the values of depth and Stanley depth of $S_{n,m}/I(\mathcal{P}_{n,m})$. However, in the light of above observations, the following open question can be proposed.

Question 7.3.1. *Is* $\operatorname{depth}(S_{n,m}/I(\mathcal{P}_{n,m})) = \operatorname{sdepth}(S_{n,m}/I(\mathcal{P}_{n,m})) = \lceil\frac{n}{3}\rceil\lceil\frac{m}{3}\rceil$?

For $n \geq 2$, this question is confirmed for the cases when $1 \leq m \leq 3$ see Remark 7.1.1, Lemma 7.1.1 and Lemma 7.1.2. If $m = 4$, some calculations are made for depth and Stanley depth by using CoCoA, (for Stanley depth, SdepthLib:coc [71] is used). Calculations show that $\operatorname{depth}(S_{4,4}/I(\mathcal{P}_{4,4})) = \operatorname{sdepth}(S_{4,4}/I(\mathcal{P}_{4,4})) = 4 = \lceil\frac{4}{3}\rceil\lceil\frac{4}{3}\rceil$, $\operatorname{sdepth}(S_{5,4}/I(\mathcal{P}_{5,4})) = 4 = \lceil\frac{5}{4}\rceil\lceil\frac{4}{3}\rceil$, and $\operatorname{sdepth}(S_{6,4}/I(\mathcal{P}_{6,4})) = 4 = \lceil\frac{6}{3}\rceil\lceil\frac{4}{3}\rceil$. The upcoming result provide a partial answer to the Question 7.3.1.

Theorem 7.3.1. *Let $n \geq 2$. Then*

$$\operatorname{depth}(S_{n,m}/I(\mathcal{P}_{n,m})), \operatorname{sdepth}(S_{n,m}/I(\mathcal{P}_{n,m})) \leq \lceil\frac{n}{3}\rceil\lceil\frac{m}{3}\rceil.$$

Proof. Without loss of generality consider that $m \leq n$. First, the result is proved for depth. If $m = 1$, then $I(\mathcal{P}_{n,1}) = I(P_n)$, that gives the required result by Remark 7.1.1. For $m = 2, 3$ the result follows from Lemmas 7.1.1 and 7.1.2, respectively. For $m \geq 4$, this result is proved by using induction on m. Let u be a monomial such that

$$u := \begin{cases} x_{2(m-1)}x_{5(m-1)}\dots x_{(n-3)(m-1)}x_{n(m-1)}, & \text{if } n \equiv 2 (\operatorname{mod} 3); \\ x_{1(m-1)}x_{4(m-1)}\dots x_{(n-3)(m-1)}x_{n(m-1)}, & \text{if } n \equiv 1 (\operatorname{mod} 3); \\ x_{2(m-1)}x_{5(m-1)}\dots x_{(n-4)(m-1)}x_{(n-1)(m-1)}, & \text{if } n \equiv 0 (\operatorname{mod} 3). \end{cases}$$

Clearly $u \notin I(\mathcal{P}_{n,m})$ so by Corollary 3.7.2, it follows that

$$\operatorname{depth}(S_{n,m}/I(\mathcal{P}_{n,m})) \leq \operatorname{depth}(S_{n,m}/(I(\mathcal{P}_{n,m}) : u)).$$

In all cases $|\operatorname{supp}(u)| = \lceil \frac{n}{3} \rceil$ and $S_{n,m}/(I(\mathcal{P}_{n,m}) : u) \cong (S_{n,m-3}/I(\mathcal{P}_{n,m-3}))[\operatorname{supp}(u)]$, using induction hypothesis and Lemma 3.7.5, it follows that

$$\operatorname{depth}(S_{n,m}/I(\mathcal{P}_{n,m})) \leq \operatorname{depth}(S_{n,m}/(I(\mathcal{P}_{n,m}) : u)) \leq \lceil \frac{n}{3} \rceil \lceil \frac{m-3}{3} \rceil + \lceil \frac{n}{3} \rceil = \lceil \frac{m}{3} \rceil \lceil \frac{n}{3} \rceil.$$

Similarly, one can prove the result for Stanley depth by using Proposition 3.7.3.

□

Remark 7.3.1. For a positive answer to Question 7.3.1, one needs to prove that $\lceil \frac{n}{3} \rceil \lceil \frac{m}{3} \rceil$ is a lower bound for depth and Stanley depth of $S_{n,m}/I(\mathcal{P}_{n,m})$. The lower bound $\lceil \frac{\operatorname{diam}(P_{n,m})+1}{3} \rceil$ (Theorem 3.7.14) which was helpful for the cases when $1 \leq m \leq 3$ is no more useful if $m \geq 4$. For instance, $\operatorname{depth}(S_{4,4}/I(\mathcal{P}_{4,4})) = \operatorname{sdepth}(S_{4,4}/I(\mathcal{P}_{4,4})) = 4$, but this lower bound shows that $\operatorname{depth}(S_{4,4}/I(\mathcal{P}_{4,4})) \geq 2 = \lceil \frac{\operatorname{diam}(P_{4,4})+1}{3} \rceil$ and $\operatorname{sdepth}(S_{4,4}/I(\mathcal{P}_{4,4})) \geq 2 = \lceil \frac{\operatorname{diam}(P_{4,4})+1}{3} \rceil$.

Theorem 7.3.2. *Let $n \geq 3$ and $m \geq 1$. Then*

$$\operatorname{depth}(S_{n,m}/I(\mathcal{C}_{n,m})) \leq \begin{cases} \lceil \frac{n}{3} \rceil \lceil \frac{m}{3} \rceil, & \text{if } m \equiv 0 (\operatorname{mod} 3); \\ \lceil \frac{n-1}{3} \rceil + (\lceil \frac{m}{3} \rceil - 1) \lceil \frac{n}{3} \rceil, & \text{if } m \equiv 1, 2 (\operatorname{mod} 3). \end{cases}$$

Proof. This result is proved by using induction on m. If $m = 1$, then $I(\mathcal{C}_{n,1}) = I(C_n)$, by [16, Proposition 1.3] the required result holds. For $m = 2$ or $m = 3$, the result follows by Theorems 7.1.1 and 7.1.2 respectively. For $m \geq 4$,

$$v := \begin{cases} x_{2(m-1)} x_{5(m-1)} \dots x_{(n-3)(m-1)} x_{n(m-1)}, & \text{if } n \equiv 2 (\operatorname{mod} 3); \\ x_{1(m-1)} x_{4(m-1)} \dots x_{(n-6)(m-1)} x_{(n-3)(m-1)} x_{(n-1)(m-1)}, & \text{if } n \equiv 1 (\operatorname{mod} 3); \\ x_{3(m-1)} x_{6(m-1)} \dots x_{(n-3)(m-1)} x_{n(m-1)}, & \text{if } n \equiv 0 (\operatorname{mod} 3). \end{cases}$$

Clearly $v \notin I(\mathcal{C}_{n,m})$ and $S_{n,m}/(I(\mathcal{C}_{n,m}) : v) \cong (S_{n,m-3}/I(\mathcal{C}_{n,m-3}))[\operatorname{supp}(v)]$, since in all the cases $|\operatorname{supp}(v)| = \lceil \frac{n}{3} \rceil$, if $m \equiv 1, 2 (\operatorname{mod} 3)$ using induction hypothesis and Lemma 3.7.5, it follows that

$$\begin{aligned} \operatorname{depth}(S_{n,m}/(I(\mathcal{C}_{n,m}) : v)) &\leq \lceil \frac{n-1}{3} \rceil + (\lceil \frac{m-3}{3} \rceil - 1) \lceil \frac{n}{3} \rceil + \lceil \frac{n}{3} \rceil \\ &= \lceil \frac{n-1}{3} \rceil + (\lceil \frac{m}{3} \rceil - 1) \lceil \frac{n}{3} \rceil. \end{aligned}$$

Otherwise, by Lemma 3.7.5 and induction, it follows that

$$\operatorname{depth}(S_{n,m}/(I(\mathcal{C}_{n,m}):v)) \leq \lceil\frac{n}{3}\rceil\lceil\frac{m-3}{3}\rceil + \lceil\frac{n}{3}\rceil = \lceil\frac{n}{3}\rceil\lceil\frac{m}{3}\rceil.$$

□

Theorem 7.3.3. *Let* $n \geq 3$ *and* $m \geq 1$. *Then* $\operatorname{sdepth}(S_{n,m}/I(\mathcal{C}_{n,m})) \leq \lceil\frac{n}{3}\rceil\lceil\frac{m}{3}\rceil$.

Proof. This result can be proved by using the similar arguments as in Theorem 7.3.2 by using Corollary 7.1.1 and Theorem 7.1.2 instead of Theorems 7.1.1 and 7.1.2. □

Remark 7.3.2. The upper bounds of $\operatorname{sdepth}(S_{n,m}/I(\mathcal{P}_{n,m}))$ and $\operatorname{sdepth}(S_{n,m}/I(\mathcal{C}_{n,m}))$ as proved in Theorems 7.3.1 and 7.3.3 are too sharp. On the bases of above observations, the following question can be formulated. A positive answer to this question proves the Conjecture 1.2.1.

Question 7.3.2. *Is* $\operatorname{sdepth}(I(\mathcal{P}_{n,m})), \operatorname{sdepth}(I(\mathcal{C}_{n,m})) \geq \lceil\frac{n}{3}\rceil\lceil\frac{m}{3}\rceil$?

7.4 Conclusion

In this chapter, the strong product of two graphs is considered, when either both graphs are arbitrary paths or one is an arbitrary path and the other is an arbitrary cycle. It is verified that for $n \geq 2$, $\lceil\frac{n}{3}\rceil$ is the precise value of both Stanley depth and depth for the cyclic modules associated to $\mathcal{P}_{n,m}$ when $1 \leq m \leq 3$. Also, for $n \geq 3$, values and tight bounds are given for the cyclic modules associated to $\mathcal{C}_{n,m}$ when $1 \leq m \leq 3$. Also, it is proved that the Conjecture 1.2.1 holds for the edge ideals associated with some subclasses of $\mathcal{H}$ and $\mathcal{K}$. Furthermore, some sharp upper bounds for depth and Stanley depth in the general cases are established.

Chapter 8

Conclusion and future work

8.1 Conclusion

In this dissertation, Stanley depth and depth of the quotient ring of the edge ideal associated with the t^{th} power of a path and the t^{th} power of a cycle are determined. Lower bounds for the Stanley depth of the edge ideals associated with the t^{th} power of a path, and t^{th} power of a cycle are given and with the help of these bounds, a conjecture of Herzog is proved for these ideals. Some upper and lower bounds for Stanley depth and depth of edge ideals associated with line graphs of the ladder, and circular ladder graphs are established. Also, some bounds for the dimension of the quotient rings of the edge ideals associated to these graphs are also determined. Furthermore, the study is conducted for Stanley depth and depth of the edge ideals and quotient rings of the edge ideals, associated with classes of graphs obtained by taking the strong product of two arbitrary paths or one is an arbitrary path, and the other is an arbitrary cycle. The exact formulae for values of Stanley depth and depth for some subclasses of the strong product of these graphs are given. Also, some sharp upper bounds are provided for Stanley depth and depth in the general cases.

8.2 Future work

(1) In some cases, we are unable at the moment to fix values of depth and Stanley depth of some classes of graphs we considered so one can try to fix these values by using some other techniques.

(2) We proved the conjecture 1.2.1 in some special cases, since it is relatively new conjecture, so one may try to prove it in general or at least for some special classes of ideals.

(3) For some classes of edge ideals, we confirmed that Stanley's inequality holds, so one may try to prove Stanley's inequality for other classes of ideals.

(4) We determined Stanley depth and depth of edge ideals of powers of paths and cycles, so one may try to find these invariants for the edge ideals of powers of some other graphs.

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