Algebraic Invariants of the Powers of Edge Ideals of Caterpillars and some Forests

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I dedicate this thesis to my loving parents, venerable supervisor, respectable teachers and fellows for their limitless support and encouragement.

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Abstract

The purpose of this dissertation is to analyze various algebraic properties, including Stanley depth, depth, regularity, and the projective dimension of the quotient rings obtained from the powers of edge ideals corresponding to caterpillar trees or a forest comprising these trees. The study establishes lower and upper bounds for the Stanley depth, depth, and the projective dimension of these quotient rings. Furthermore, the precise values of the regularity for the above quotient rings are also given.

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Introduction

Abstract algebra is a branch of mathematics that focuses on algebraic structures, including rings and modules. In this context, algebraic and geometric invariants are utilized to characterize rings and modules. Stanley depth, depth, projective dimension and regularity are examples of such geometric and algebraic invariants used to describe modules and rings, respectively.

Stanley depth, introduced by Stanley in 1982, is an important invariant for finitely generated $\mathbb{Z}^n$-graded modules over commutative rings. Stanley also proposed a relationship between Stanley depth and module's depth, known as Stanley's conjecture given in [28]. However, Duval et al. in [11] demonstrated that Stanley's conjecture is invalid for modules of the form $\mathcal{X}/\mathcal{N}$, where $\mathcal{X}$ is the ring of polynomials over a field and $\mathcal{N}$ is a monomial ideal. Despite this, identifying classes of modules that still fulfill Stanley's inequality remains a challenging task. Few results related to Stanley depth are given in [7, 12, 15, 17, 18].

In recent decades, investigating the regularity of the powers of ideals that are homogeneous has emerged as a focal point. This research initiative originated from a significant theorem, demonstrated by Cutkosky-Herzog-Trung in [8] and Kodiyalam in [31]. The theorem establishes that for an ideal A which is homogenous in a standard graded algebra over a field, the regularity function $\operatorname{reg}(A^t)$ exhibits an asymptotic linearity. Several researchers have investigated the values and bounds for the projective dimension and regularity of edge ideals. See [2, 3, 5, 9] for more information on these two invariants.

In this thesis bounds for the Stanley depth, Depth and projective dimension of the quotient ring corresponding to the powers of the edge ideals of caterpillars and a forest of such caterpillars are computed. Also exact values of regularity for such structures are calculated.

Chapter 1 of this thesis provides an introduction to the fundamental concepts of Ring and Module Theory, covering the basic definitions, examples, and results related to these algebraic structures. Additionally, the basic definition and examples of graph theory are also included, which will be utilized in the study.

Chapter 2 presents an overview of Stanley depth, depth, regularity and projective dimension. It discusses different results associated with these algebraic invariants and provides a few examples. The chapter also states the Stanley's conjecture and outlines the method for determining the Stanley depth of squarefree monomial ideals and quotient rings associated to these ideals.

In chapter 3, bounds for the depth, projective dimension and Stanley depth of the quotient ring corresponding to the powers of the edge ideals of caterpillars and a forest of such caterpillars are computed. Moreover, we determine the exact value for the regularity of the considered quotient rings, along with the regularity values for the quotient rings corresponding to the powers of the edge ideals of the considered forests.

Chapter 1

Preliminaries

The basic concepts related to rings, modules and also a few basic definitions of graph theory are discussed in this chapter. A few examples are provided for a better understanding.

Ring Theory is the study of rings, initially studied in the 1800's. Emmy Noether introduced the general concept of commutative rings. Initially, the focus was on commutative rings, but later the study expanded to include non-commutative rings.

Module Theory, is the study of modules. A module is an algebraic object with a ring associated with it, where the ring acts on the module.

Graph Theory involves the study of graphs and encompasses operations on graphs. This chapter introduces fundamental terminologies of Graph Theory and provides examples of graphs that will be used in subsequent chapters of the study.

1.1 Ring Theory

This section explores various properties of rings, providing examples along the way. Several definitions outlining distinct categories of rings and their corresponding properties are examined and elaborated upon. The concept of ideals and a few related operations is also discussed.

Definition 1.1.1. A ring $\mathcal{X}$ with two binary operations namely addition " $+$ " and multiplication " $\times$ " is a set such that the following hold:

- with respect to addition $\mathcal{X}$ is an abelian group,
- multiplication is associative i.e $(d \times e) \times f = d \times (e \times f)$ and also distributive over addition i.e
  1. $d \times (e + f) = (d \times e) + (d \times f)$,
  2. $(e + f) \times d = (e \times d) + (f \times d), \ \forall \ d, e, f \in \mathcal{X}$.

Definition 1.1.2. Let $\mathcal{X}$ be a ring. Then,

- $\mathcal{X}$ is known as a ring having multiplicative identity if $\exists \ u$ such that $\forall \ d \in \mathcal{X}, \ d \times u = u \times d = d$,
- If $d \times e = e \times d, \ \forall \ d, e \in \mathcal{X}$, then $\mathcal{X}$ is said to be a commutative ring.

The rings considered throughout this thesis are commutative rings having identity.

Example 1.1.1. Following are a few examples of rings:

- $\mathbb{R}, \mathbb{Z}$ and $\mathbb{Q}$ are commutative rings with identity.
- For $n \geq 2$ the ring $n\mathbb{Z}$ is a commutative ring without identity.

1.1.1 Polynomial Ring

Definition 1.1.3. Let $\mathcal{X}$ be a commutative ring with unity. For $n \geq 0$ and $r_i \in \mathcal{X}$ $P(Y) = r_0 + r_1 Y + r_2 Y^2 + \cdots + r_{n-1} Y^{n-1} + r_n Y^n$ is termed as a polynomial in indeterminate Y having co-efficients r_i.

Definition 1.1.4. Let $\mathcal{X}$ be a commutative ring. The set of formal symbols $\mathcal{X}[Y] = r_0 + r_1 Y + r_2 Y^2 + \cdots + r_{n-1} Y^{n-1} + r_n Y^n : n \geq 0, r_i \in \mathcal{X}\}$ is called the ring of polynomials over $\mathcal{X}$ in the variable Y.

Example 1.1.2. $\mathbb{R}[Y]$, $\mathbb{Q}[Y]$ and $\mathbb{Z}[Y]$ are few examples of polynomial rings.

1.1.2 Noetherian Ring

Definition 1.1.5. *(Chain conditions)* Let $\mathcal{Q}$ be a poset w.r.t $\leq$. Then the following are equivalent:

- Every increasing sequence $x_1 \leq \ldots \leq x_p \leq \ldots$ in $\mathcal{Q}$ is stationary i.e $\exists\ p$ such that $x_m = x_p\ \forall\ m \geq p$,
- A maximal element is present in every non-empty subset of $\mathcal{Q}$.

If $\mathcal{Q}$ represents the set containing all the ideals of the ring $\mathcal{X}$ which is ordered by the relation $\subseteq$ then the first chain condition is called the ascending chain condition.

Definition 1.1.6. A ring $\mathcal{X}$ is termed to be Noetherian if it has an ascending chain condition on its ideal.

Example 1.1.3. $\mathbb{Z}$, which is the ring of integers is a Noetherian ring.

1.1.3 Graded Ring

Definition 1.1.7. Let $\mathcal{X}$ be a ring and G be an abelian group then $\mathcal{X}$ is graded (G-graded) if $\mathcal{X} = \bigoplus_{i \in G} \mathcal{X}_i$ is a direct sum decomposition of abelian groups and if $\mathcal{X}_i \mathcal{X}_j \subseteq \mathcal{X}_{i+j} \forall\ i, j \in G$.

Example 1.1.4. The ring $\mathcal{X}[x, y]$ is a $\mathbb{Z}$ - graded ring as $\mathcal{X}[x, y] = \mathcal{X} \oplus (\mathcal{X}x + \mathcal{X}y) \oplus (\mathcal{X}x^2 + \mathcal{X}y^2 + \mathcal{X}xy) \oplus (\mathcal{X}x^3 + \mathcal{X}y^3 + \mathcal{X}x^2y + \mathcal{X}xy^2) \oplus \cdots$.

Definition 1.1.8. Let $z \in \mathbb{Z}_n$; then $h \in \mathcal{X}$ is said to be homogeneous having degree z if h has the form ax^z where $a \in K$. The ring of polynomials $\mathcal{X} = \mathcal{X}[x_1, \ldots, x_d]$ in d variables is $\mathbb{Z}^d$ - graded having graded components

$$\mathcal{X}_{\mathbf{z}} = \begin{cases} Kx^{\mathbf{z}}, & \text{if } \mathbf{z} \in \mathbb{Z}_+^d, \\ 0\,, & \text{otherwise.} \end{cases}$$

1.1.4 Ring Homomorphisms and Quotient Rings

Definition 1.1.9. For rings $\mathcal{X}$ and $\mathcal{Y}$, a map $\zeta : \mathcal{X} \longrightarrow \mathcal{Y}$ is a homomorphism that satisfies the following:

- $\zeta(d+e) = \zeta(d) + \zeta(e)$,
- $\zeta(de) = \zeta(d)\zeta(e)$, where $d \in \mathcal{X}$ and $e \in \mathcal{Y}$.

Definition 1.1.10. Suppose $\mathcal{X}$ and $\mathcal{Y}$ are rings and if $\zeta : \mathcal{X} \longrightarrow \mathcal{Y}$ is a homomorphism then the kernal of ζ is given as $\ker \zeta = \{d \in \mathcal{X} : \zeta(d) = 0\}$ and the image of ζ is defined as $\zeta(\mathcal{X}) = \{e \in \mathcal{Y} : e = \zeta(d)\}$.

Example 1.1.5. The map $\zeta : \mathbb{Z} \longrightarrow \mathbb{Z}_n$ defined as $\zeta(z) = z(\text{mod } n)$ is a ring homomorphism with $\ker \zeta = n\mathbb{Z}$ and $\operatorname{Im} \zeta = \mathbb{Z}_n$.

Definition 1.1.11. Let $\mathcal{X}$ be a ring with ideal A, then the quotient ring $\mathcal{X}/A$ is the set of cosets of A in $\mathcal{X}$, with addition and multiplication defined as $(d_1 + A) + (d_2 + A) = (d_1 + d_2) + A$ and $(d_1 + A)(d_2 + A) = d_1 d_2 + A$.

1.1.5 Ideals and Their Properties

Definition 1.1.12. Let $A \subseteq \mathcal{X}$, then A is said to be an ideal of $\mathcal{X}$ if

- A is an additive abelian subgroup of $\mathcal{X}$,
- If $a \in A$ and $d \in \mathcal{X}$, then the product $da \in A$, i.e $dA \subseteq A$ and also $ad \in A$, i.e $Ad \subseteq A$.

Example 1.1.6. For the ring $\mathbb{Z}$ its ideals are of the type $n\mathbb{Z}$, where $n \in \mathbb{Z}$.

Every ideal is a subring but the converse doesnot necessarily hold. For example $\mathbb{Z}$ is a subring of $\mathbb{Q}$ but not an ideal of $\mathbb{Q}$.

Definition 1.1.13. Let A and B be the ideals of $\mathcal{X}$. Then

- $A+B$ is the sum of ideals A and B given by $A+B=\{a+b : a \in A, b \in B\}$,
- AB is the product of A and B given by $AB=\{a_1b_1+a_2b_2+\cdots+a_rb_r : a_1,\dots,a_r \in A, b_1,\dots,b_r \in B\}$,
- The intersection $A \cap B$ is defined as $A \cap B=\{a \in \mathcal{X} : a \in A \text{ and } a \in B\}$.

Example 1.1.7. Let $l,p \in \mathbb{Z}^+$. If $A=(l)=l\mathbb{Z}$ and $B=(p)=p\mathbb{Z}$ be ideals of $\mathcal{X}=\mathbb{Z}$. Then

- $A+B=gcd(l,p)\mathbb{Z}$,
- $A \cap B=lcm(l,p)\mathbb{Z}$,
- $AB=(lp)=lp\mathbb{Z}$.

Example 1.1.8. Let $A=2\mathbb{Z}$ and $B=8\mathbb{Z}$ then, $A+B=2\mathbb{Z}$, $A\cap B=8\mathbb{Z}$ and $AB=16\mathbb{Z}$.

Definition 1.1.14. Suppose A and B are two ideals of $\mathcal{X}$ then the ideal quotient of A and B also called the colon ideal is given by $(A:B)=\{a \in \mathcal{X} : aB \subseteq A\}$.

Definition 1.1.15. In a ring $\mathcal{X}$, a maximal ideal $\mathcal{N}$ is a proper ideal that has no other proper ideal lying between it and the entire ring $\mathcal{X}$.

1.1.6 Monomial Ideals

Definition 1.1.16. Let $\mathcal{X}=K[x_1,\dots,x_d]$ where K represents a field. The product $x_1^{a_1}x_2^{a_2}\dots x_n^{a_d}$, with $a_i \in \{0,1,2,\dots\}$ is called a monomial of $\mathcal{X}$. A monomial is square free when $a_i \in \{0,1\}$.

Definition 1.1.17. An ideal $A \subseteq \mathcal{X}=K[x_1,\dots,x_d]$ if generated by monomials is classified as a monomial ideal. Furthermore, we denote by $G(A)$ the set of monomials in A that are minimal w.r.t divisibility.

Example 1.1.9. Let $\mathcal{X}=K[x_1,\dots,x_7]$. Then $A=(x_6^2x_2,x_3^2x_1,x_4,x_3)$ is a monomial ideal and $B=(x_1x_2,x_3x_5,x_4)$ is a monomial ideal which is square free.

Definition 1.1.18. Let $\mathcal{X} = K[x_1, \ldots, x_n], A = (u_1, \ldots, u_n)$ and $B = (v_1, \ldots, v_m)$ are monomial ideals of $\mathcal{X}$. Then

- $A + B = (u_1, u_2, \ldots, u_n, v_1, v_2, \ldots, v_m)$,
- $AB = (u_1v_1, \ldots, u_1v_m, u_2v_1, \ldots, u_2v_m, u_nv_1, \ldots, u_nv_m)$,
- $A^2 = (u_1^2, u_1u_2, \ldots, u_2u_1, \ldots, u_2u_n, \ldots, u_nu_1, \ldots, u_n^2)$,
- $A \cap B = \{lcm(u, v) : u \in G(A), v \in G(B)\}$,
- $A : B = \cap_{v \in G(B)}(A : v) = \{\frac{u}{gcd(u,v)} : u \in G(A)\}$.

Example 1.1.10. Let A and B be monomial ideals of $\mathcal{X} = K[x_1, x_2, x_3]$ such that $A = (x_1x_2, x_1^2x_3, x_2x_3^2, x_1x_3^2)$ and $B = (x_1x_3, x_2^2x_3, x_1^2)$. Then

- $A + B = (x_1x_2, x_1^2x_3, x_2x_3^2, x_1x_3, x_2^2x_3, x_1^2) = (x_1x_2, x_2x_3^2, x_1x_3, x_2^2x_3, x_1^2)$,
- $AB = (x_1^2x_2x_3, x_1x_2^3x_3, x_1^3x_2, x_1^3x_3^2, x_1^2x_2^2x_3^2, x_1^4x_3, x_1x_2x_3^3, x_2^3x_3^3, x_1^2x_2x_3^2, x_1^2x_3^3, x_1x_2^2x_3^3,$
  $x_1^3x_3^2) = (x_1^2x_2x_3, x_1x_2^3x_3, x_1^3x_2, x_1^3x_3^2, x_1^4x_3, x_1x_2x_3^3, x_2^3x_3^3, x_1^2x_3^3)$,
- $A \cap B = (x_1x_2x_3, x_1x_2^2x_3, x_1^2x_2, x_1^2x_3, x_1^2x_2^2x_3, x_1^2x_3, x_1x_2x_3^2, x_2^2x_3^2, x_1^2x_2x_3^2, x_1x_3^2$
  $x_1x_2^2x_3^2, x_1^2x_3^2) = (x_1x_2x_3, x_1^2x_2, x_1^2x_3, x_2x_3^2, x_1x_3^2)$,
- $A : B = (A : x_1x_3) \cap (A : x_2^2x_3) \cap (A : x_1^2)$
  $= (\frac{x_1x_2}{gcd(x_1x_2,x_1x_3)}, \frac{x_1^2x_3}{gcd(x_1^2x_3,x_1x_3)}, \frac{x_2x_3^2}{gcd(x_2x_3^2,x_2^2x_3)}, \frac{x_1x_3^2}{gcd(x_1x_3^2,x_1x_3)}) \cap (\frac{x_1x_2}{gcd(x_1x_2,x_2^2x_3)},$
  $\frac{x_1^2x_3}{gcd(x_1^2x_3,x_2^2x_3)}, \frac{x_2x_3^2}{gcd(x_2x_3^2,x_2^2x_3)}, \frac{x_1x_3^2}{gcd(x_1x_3^2,x_2^2x_3)}) \cap \frac{x_1x_2}{gcd(x_1x_2,x_1^2)}, \frac{x_1^2x_3}{gcd(x_1^2x_3,x_1^2)},$
  $\frac{x_2x_3^2}{gcd(x_2x_3^2,x_1^2)}, \frac{x_1x_3^2}{gcd(x_1x_3^2,x_1^2)})$
  $= (x_2, x_1, x_3, x_3) \cap (x_1, x_1^2, x_3, x_1x_3) \cap (x_2, x_3, x_2x_3^2, x_3^2)$
  $= (x_1, x_2, x_3) \cap (x_1, x_3) \cap (x_2, x_3) = (x_1x_2, x_1x_3, x_2x_3, x_3)$.

1.2 Module Theory

This section delves into the properties of modules, presents examples, and discusses a few important results in Module Theory.

Definition 1.2.1. Let $\mathcal{X}$ be a commutative ring. An $\mathcal{X}$-module is a set $\mathcal{N}$ along with a binary operation on $\mathcal{N}$, with which $\mathcal{N}$ is an abelian group and an action $\mathcal{X}$ acts on $\mathcal{N}$ that is a map $\mathcal{X} \times \mathcal{N} \to \mathcal{N}$ defined as $*(r, n) = rn$ which satisfies the following :

- $(d + e)n = dn + en \; \forall \; d, e \in \mathcal{X}$ and $n \in \mathcal{N}$,
- $(de)(n) = d(en) \; \forall \; d, e \in \mathcal{X}$ and $n \in \mathcal{N}$,
- $d(n + m) = dn + dm \; \forall \; d \in \mathcal{X}$ and $n, m \in \mathcal{N}$.

If the ring contains unity then we have one more axiom that is,

- $1n = n \; \forall \; n \in \mathcal{N}$.

Example 1.2.1. If $\mathcal{X}$ represents a ring and $A \subseteq \mathcal{X}$ is the ideal of $\mathcal{X}$, then $\mathcal{X}/A$ is an $\mathcal{X}$-module under the scalar multiplication $d(b + A) = db + A \; \forall \; d \in \mathcal{X}, b + A \in \mathcal{X}/A$.

Definition 1.2.2. For a ring $\mathcal{X}$ and $\mathcal{X}$-module $\mathcal{N}$, a subset $\mathcal{O} \subseteq \mathcal{N}$ which is non empty is a submodule of $\mathcal{N}$ if $\mathcal{O}$ is a subgroup of $\mathcal{N}$ which is an additive group which also satisfies the module axioms using the scalar multiplication on $\mathcal{N}$.

1.2.1 Noetherian Modules

Definition 1.2.3. For a ring $\mathcal{X}$, an $\mathcal{X}$-module $\mathcal{N}$ is considered Noetherian if every increasing sequence of $\mathcal{X}$-submodules of $\mathcal{N}$ eventually becomes stationary. Additionally, a finitely generated $\mathcal{X}$-module $\mathcal{N}$ is also Noetherian.

Example 1.2.2. A finite additive abelain group G, which is a $\mathbb{Z}$-module is Noetherian.

1.2.2 Free Modules

Definition 1.2.4. Let $\mathcal{F}$ be an $\mathcal{X}$-module. If $\mathcal{F}$ has a non-empty basis A, then $\mathcal{F}$ is called a free $\mathcal{X}$-module.

Example 1.2.3. Let $\mathcal{X}$ be a ring. Then $\mathcal{X}^n = \{(a_1, a_2, \dots, a_n); a_i \in \mathcal{X}\}$ is a free module with basis $\{c_1, c_2, \dots, c_n\}$, where $c_i = (0, \dots, 0, 1, 0, \dots, 0)$ with 1 at the *ith* position.

1.2.3 Graded Modules

Definition 1.2.5. For a G-graded ring $\mathcal{X}$ and an $\mathcal{X}$-module $\mathcal{N}$ we say that $\mathcal{N}$ is graded (G-graded) if $\mathcal{N} = \bigoplus_{i \in G} \mathcal{N}_i$ is a direct sum decomposition of abelian groups and $\mathcal{N}$ is graded (G-graded) if $\mathcal{X}_i \mathcal{N}_j \subseteq \mathcal{N}_{i+j} \forall\ i, j \in G$.

Definition 1.2.6. An $\mathcal{X}$ - module $\mathcal{N}$ is said to be $\mathbb{Z}^n$-graded if $\mathcal{N} = \bigoplus_{i \in \mathbb{Z}^n} \mathcal{N}_i$ and $\mathcal{X}_i \mathcal{N}_j \subset \mathcal{N}_{i+j}$ for all $i, j \in \mathbb{Z}^n$.

1.2.4 Exact Sequences

Definition 1.2.7. A sequence of $\mathcal{X}$-modules and $\mathcal{X}$-homomorphisms given by

$$\dots \longrightarrow \mathcal{N}_{i-1} \xrightarrow{h_i} \mathcal{N}_i \xrightarrow{h_{i+1}} \mathcal{N}_{i+1} \xrightarrow{h_{i+2}} \dots$$

is termed as exact at $\mathcal{N}_i$ if the condition $Im(h_i) = ker(h_{i+1})$ is satisfied. The sequence is said to be exact if it is exact at every $\mathcal{N}_i$.

Definition 1.2.8. The sequence

$$0 \longrightarrow \mathcal{N}' \xrightarrow{h} \mathcal{N} \xrightarrow{k} \mathcal{N}'' \longrightarrow 0$$

is said to be an exact sequence if and only if h is one-one, k is onto and the condition $Im(h) = ker(k)$ is satisfied. Such a sequence is termed as a short exact sequence.

Example 1.2.4. Let $n \geq 2$. Then the sequence

$$0 \longrightarrow \mathbb{Z} \xrightarrow{h} \mathbb{Z} \xrightarrow{k} \mathbb{Z}/n\mathbb{Z} \longrightarrow 0$$

is said to be a short exact sequence where $h : \mathbb{Z} \longrightarrow \mathbb{Z}$ is defined as $h(x) = nx$ and $k : \mathbb{Z} \longrightarrow \mathbb{Z}/n\mathbb{Z}$ is defined as $k(x) = x + n\mathbb{Z}$.

1.2.5 Module Homomorphisms and Quotient Modules

Definition 1.2.9. For a ring $\mathcal{X}$ and $\mathcal{X}$-modules, $\mathcal{N}$ and $\mathcal{O}$, a map $\zeta : \mathcal{N} \longrightarrow \mathcal{O}$ is an $\mathcal{X}$-module homomorphism if it respects the $\mathcal{X}$-module structure of $\mathcal{N}$ and $\mathcal{O}$ that is

- $\zeta(n+p) = \zeta(n) + \zeta(p)\ \forall\ n, p \in \mathcal{N}$ and
- $\zeta(dn) = d\zeta(n)\ \forall\ n \in \mathcal{N}$ and $d \in \mathcal{X}$.

Definition 1.2.10. Let $\mathcal{X}$ be ring and if $\zeta : \mathcal{N} \longrightarrow \mathcal{O}$ is an $\mathcal{X}$-module homomorphism then we define the kernal of ζ as $\ker\zeta = \{n \in \mathcal{N} : \zeta(n) = 0\}$ and the image of ζ is defined as $\zeta(\mathcal{N}) = \{o \in \mathcal{O} : o = \zeta(n)\}$.

Example 1.2.5. Let $\mathcal{X}$ be ring and $\mathcal{N} = \mathcal{X}^n$, then for each $i \in \{1, 2, \ldots, n\}$, the projection map $\pi_i : \mathcal{X}^n \longrightarrow \mathcal{X}$ defined as $\pi_i((h_1, h_2, \ldots, h_n)) = h_i$ is a homomorphism and we have $\ker\pi_i = \{(h_1, \ldots, h_{i-1}, 0, h_{i+1}, \ldots, h_n) : h_1, \ldots, h_{i-1}, h_{i+1}, \ldots, h_n \in \mathcal{X}\}$ and $\operatorname{Im}\pi_i = \mathcal{X}^n$.

Definition 1.2.11. Consider a ring $\mathcal{X}$, an $\mathcal{X}$-module $\mathcal{N}$, and a submodule $\mathcal{O}$ of $\mathcal{N}$. The abelian quotient group $\mathcal{N}/\mathcal{O} = \{n + \mathcal{O} : n \in \mathcal{N}\}$, can be transformed into a $\mathcal{X}$-module by defining scalar multiplication as $d(n + \mathcal{O}) = dn + \mathcal{O}$.

1.3 Graph Theory

We discuss a few basics related to Graph Theory in this section.

Definition 1.3.1. A graph say G comprises of a set of vertices given by $V(G)$ and a set of edges given by $E(G)$. Additionally, each edge is associated with a relation that connects two vertices, known as its endpoints, which may or may not be distinct.

Definition 1.3.2. A loop represents an edge which has the same endpoints. Edges sharing the same pair of endpoints are called multiple edges. A graph without any multiple edges or loops is a simple graph.

Definition 1.3.3. For every pair of vertices in a graph G, if there exists a single path that joins these vertices, then that graph is termed as connected. The component of any connected graph G is a connected subgraph that is not present in any other subgraph which is larger.

Definition 1.3.4. The number of edges which are incident to a vertex represents the degree of a graph and a vertex having degree 1 is said to be a pendant vertex. The distance between two vertices is determined by the shortest length of a path that connects those vertices and the diameter of a connected graph G is the maximum distance between any two of the vertices.

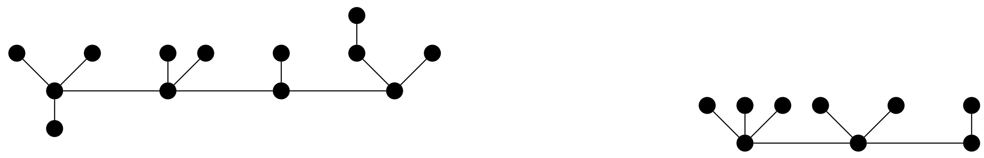
Definition 1.3.5. A graph having n vertices such that there is an edge which connects the adjacent vertices is said to be a path graph denoted as P_n. A graph having n vertices such that there is an edge which connects the adjacent vertices and also the first and the last vertex is known as a cycle graph and is denoted by C_n.

(a) Path

(b) Cycle

Figure 1.1

Definition 1.3.6. A graph in which any two vertices are joined by a unique path is termed as tree graph. A caterpillar tree is a tree which becomes a path after removing its pendant vertices. A graph in which every component is a tree is a forest.



(a) Tree (b) Caterpillar

Figure 1.2

Definition 1.3.7. The set of edges having no common vertices is a matching in the graph G. An induced matching is such a matching that forms a subgraph of G which is also induced and $indmat(G) = \max\{|M| : M \text{ is an induced matching in } G\}$ denotes the induced matching number.

Definition 1.3.8. A graph G is considered to be chordal if it lacks any induced cycle with a length that is greater than 3.

Chapter 2

Stanley Depth, Depth, Regularity and Projective Dimension

In this section the above mentioned invariants of an $\mathcal{X}$-module $\mathcal{N}$ and a few related results are discussed. The ring considered throughout this section is a ring of polynomial in d variables that is $\mathcal{X} = \mathcal{X}[x_1, \dots, x_d]$.

2.1 Depth

In this subsection the concept of depth in the context of graded modules is discussed. A few preliminary definitions followed by some well established results are given which set the groundwork for subsequent discussions.

Definition 2.1.1. For an $\mathcal{X}$-module $\mathcal{N}$, a non-zero element $d \in \mathcal{X}$ is a regular element on $\mathcal{N}$ if whenever the product $dn = 0$ where $n \in \mathcal{N}$ implies that $n = 0$.

Definition 2.1.2. Let $\mathcal{N}$ be an $\mathcal{X}$-module and $d_1, d_2, \dots, d_n$ be a sequence of elements of the ring $\mathcal{X}$. This sequence is defined as $\mathcal{N}$-regular if:

- d_j is regular on $\mathcal{N}/(d_1, d_2, \dots, d_n)$ for any j and
- $\mathcal{N} = \mathcal{N}/(d_1, d_2, ..., d_n)\mathcal{N}$.

Example 2.1.1. Consider $\mathcal{X} = K[x_1, x_2, x_3, x_4]$ as a module over itself. Then as x_1 is regular on $\mathcal{X}/(0)\mathcal{X}$, x_2 is regular on $\mathcal{X}/(x_1)\mathcal{X}$, x_3 is regular on $\mathcal{X}/(x_1, x_2)\mathcal{X}$ and x_4 is regular on $\mathcal{X}/(x_1, x_2, x_3)\mathcal{X}$, so x_1, x_2, x_3, x_4 is the $\mathcal{N}$-regular sequence in $\mathcal{X}$.

Definition 2.1.3. The depth of the $\mathcal{X}$-module $\mathcal{N}$ finitely generated over the Noetherian ring $\mathcal{N}$, w.r.t the unique maximal ideal $\mathbf{m}$, is defined to be the common length of all the maximal regular sequences in the ideal $\mathbf{m}$ when considered on $\mathcal{N}$.

2.1.1 Few Results Related to Depth

Lemma 2.1.4 ([4, Proposition 1.2.9]). *For a given short exact sequence* $0 \longrightarrow \mathcal{N}_1 \longrightarrow \mathcal{N}_2 \longrightarrow \mathcal{N}_3 \longrightarrow 0$ *of modules which is considered over a local ring or a graded Noetherian ring having local* $\mathcal{X}_0$*, we have*

- $\operatorname{depth}(\mathcal{N}_2) \geq \min\{\operatorname{depth}(\mathcal{N}_3), \operatorname{depth}(\mathcal{N}_1)\}$,
- $\operatorname{depth}(\mathcal{N}_1) \geq \min\{\operatorname{depth}(\mathcal{N}_2), \operatorname{depth}(\mathcal{N}_3) + 1\}$,
- $\operatorname{depth}(\mathcal{N}_3) \geq \min\{\operatorname{depth}(\mathcal{N}_1) - 1, \operatorname{depth}(\mathcal{N}_2)\}$.

Lemma 2.1.5 ([15, Lemma 3.6]). *Let* A *be the monomial ideal of a ring* $\mathcal{X}$*. If* $\mathcal{X}' = \mathcal{X}[y]$ *is a ring of polynomials over* $\mathcal{X}$ *in the variable* y*. Then* $\operatorname{depth}(\mathcal{X}'/A\mathcal{X}') = \operatorname{depth}(\mathcal{X}/A) + 1$.

Lemma 2.1.6 ([10, Lemma 2.12]). *Let* $A_1 \subset \mathcal{X}_1 = K[x_1, \ldots, x_r]$ *and* $A_2 \subset \mathcal{X}_2 = K[x_{r+1}, x_{r+2}, \ldots, x_d]$ *be monomial ideals where* $1 \leq r \leq d$*. If* $\mathcal{X} = \mathcal{X}_1 \otimes_K \mathcal{X}_2$*. Then* $\operatorname{depth}_{\mathcal{X}}(\mathcal{X}_1/A_1 \otimes_K \mathcal{X}_2/A_2) = \operatorname{depth}_{\mathcal{X}}(\mathcal{X}/(A_1\mathcal{X} + A_2\mathcal{X})) = \operatorname{depth}_{\mathcal{X}_1}(\mathcal{X}_1/A_1) + \operatorname{depth}_{\mathcal{X}_2}(\mathcal{X}_2/A_2)$.

Proposition 2.1.2 ([26, Corollary 1.3]). *Consider a monomial ideal* A *of the ring* $\mathcal{X}$*. Then for any monomial* $x \notin A \operatorname{depth}(\mathcal{X}/A) \leq \operatorname{depth}(\mathcal{X}/(A : x))$*.*

Corollary 2.1.3 ([6, Corollary 1.6]). *Let* $A \subseteq \mathcal{X}$ *be a monomial ideal. Then* $\operatorname{sdepth}(\mathcal{X}/A) = 0$ *iff* $\operatorname{depth}(\mathcal{X}/A) = 0$.

Theorem 2.1.4 ([13, Proposition 4.3]). *Consider a graph* G *having* c *connected components and let* $A = A(G)$ *and* $d = d(G)$ *represent the diameter of the graph* G*. Then for* $t \geq 1$*, we have* $\operatorname{depth} \geq c - t$*.*

Lemma 2.1.5 ([20, Lemma 2.8]). *Let* $A = A(P_n)$ *be an edge ideal of a path graph with* $n \geq 2$*. Then* $\operatorname{depth}(\mathcal{X}/A) = \lceil \frac{n}{3} \rceil$.

2.2 Stanley Depth

This subsection explores the concept of Stanley depth, a combinatorial property associated with modules. A concise overview of the Stanley's conjecture, an outline of a methodology for determining the Stanley depth and a few results are then discussed.

Definition 2.2.1. Consider a $\mathbb{Z}^d$-graded $\mathcal{X}$-module $\mathcal{N}$ which is finitely generated. Let $nK[Z]$ be the subspace of $\mathcal{N}$ which is generated by the elements of the type ny, where n is an element in $\mathcal{N}$ and is homogenous, y is a monomial in the ring of polynomial $K[Z]$, and Z is a subset of $\{x_1, \ldots, x_d\}$. If say $uK[Z]$ is a free $K[Z]$-module, then it is referred to as a Stanley space having dimension equal to $|Z|$. For the module $\mathcal{N}$ its Stanley decomposition is its representation as a finite direct sum of these Stanley spaces. The Stanley depth of

$$\mathcal{D} : \mathcal{N} = \bigoplus_{k=1}^{s} u_k K[Z_k]$$

is $\operatorname{sdepth}(\mathcal{D}) = \min\{|Z_k|\colon k = 1, \ldots, s\}$ and that of $\mathcal{N}$ is the number $\operatorname{sdepth}(\mathcal{N}) = \max\{\operatorname{sdepth}(\mathcal{D}) :\ \mathcal{D}$ is a Stanley decomposition of $\mathcal{N}\}$.

2.2.1 Stanley's Conjecture

Stanley in 1982 [28] gave a conjecture which stated that for a Z^d-graded $\mathcal{X}$-module $\mathcal{N}$ which is finitely generated, we have that $\operatorname{sdepth}(\mathcal{N}) \geq \operatorname{depth}(\mathcal{N})$. Duval et al in [11] disproved this conjecture later with the help of a counterexample for the module of the type $\mathcal{X}/\mathcal{N}$. An algorithm for the computation of Stanley depth is provided in [16] by Ichim et al..

2.2.2 Method for the Computation of Stanley Depth for Squarefree Monomial Ideals

In 2009, Herzog et al. in [15] introduced an innovative approach for computing the lower bound of the Stanley depth of monomial ideals. This method involves utilizing posets and is designed to achieve this computation in a fixed number of steps. Let A

be a squarefree monomial ideal having generating set as $G(A) = \{e_1, \ldots, e_m\}$. Now the characteristic poset of A w.r.t $g = (1, \ldots, 1)$, written as $\mathcal{Q}_I^{(1,\ldots,1)}$ is defined to be

$$\mathcal{Q}_I^{(1,\ldots,1)} = \{\gamma \subset [n] \mid \gamma \text{ contains } \operatorname{supp}(e_j) \text{ for, some } j\},$$

where $\operatorname{supp}(e_j) = \{i : x_i|e_j\} \subseteq [n] := \{1, \ldots, n\}$. For each $\rho, \sigma \in \mathcal{Q}_I^{(1,\ldots,1)}$ where $\rho \subseteq \sigma$, and

$$[\rho\,,\,\sigma] = \{\gamma \in \mathcal{Q}_I^{(1,\ldots,1)} : \rho \subseteq \gamma \subseteq \sigma\}.$$

Let $\mathcal{Q} : \mathcal{Q}_I^{(1,\ldots,1)} = \cup_{j=1}^{k}[\gamma_j\,,\,\eta_j]$ be a partition of $\mathcal{Q}_I^{(1,\ldots,1)}$, and for every j, suppose $s(j) \in \{0,1\}^n$ is the tuple with $\operatorname{supp}(x^{s(j)}) = \gamma_j$, then the Stanley decomposition $\mathcal{D}(\mathcal{Q})$ of A is given by

$$\mathcal{D}(\mathcal{Q}) : A = \bigoplus_{j=1}^{r} x^{s(j)} K[\{x_k \mid k \in \eta_j\}].$$

Clearly, $\operatorname{sdepth} \mathcal{D}(\mathcal{Q}) = \min\{|\eta_1|, \ldots, |\eta_r|\}$ and

$$\operatorname{sdepth}(A) = \max\{\operatorname{sdepth} \mathcal{D}(\mathcal{Q}) \mid \mathcal{Q} \text{ is a partition of } \mathcal{Q}_I^{(1,\ldots,1)}\}.$$

Example 2.2.1. Consider $A = (x_1x_4, x_2x_5, x_3x_4x_5) \subset K[x_1, x_2, x_3, x_4, x_5]$ and $B = 0$. Set $\sigma_1 = (1,0,0,1,0)$, $\sigma_2 = (0,1,0,0,1)$ and $\sigma_3 = (0,0,1,1,1)$. Thus A is generated by $x^{\sigma_1}, x^{\sigma_2}, x^{\sigma_3}$ and choose $g = (1,1,1,1,1)$. The poset $\mathcal{Q} = \mathcal{Q}^g_{A/B}$ is given by:

$$\begin{aligned}\mathcal{Q} = \{&(1,0,0,1,0), (0,1,0,0,1), (1,1,0,1,0), (1,1,0,0,1), (1,0,1,1,0), (1,0,0,1,1),\\ &(0,1,1,0,1), (0,1,0,1,1), (0,0,1,1,1), (1,1,1,1,0), (1,1,1,0,1), (1,1,0,1,1),\\ &(1,0,1,1,1), (0,1,1,1,1), (1,1,1,1,1)\}.\end{aligned}$$

Partitions of $\mathcal{Q}$ are given by:

$$\begin{aligned}
\mathcal{Q}_1: \ & [(1,0,0,1,0),(1,0,0,1,0)] \bigcup [(0,1,0,0,1),(0,1,0,0,1)] \bigcup \\
& [(1,1,0,1,0),(1,1,0,1,0)] \bigcup [(1,0,0,1,1),(1,0,0,1,1)] \bigcup \\
& [(1,1,0,0,1),(1,1,0,0,1)] \bigcup [(1,0,1,1,0),(1,0,1,1,0)] \bigcup \\
& [(0,1,1,0,1),(0,1,1,0,1)] \bigcup [(0,1,0,1,1),(0,1,0,1,1)] \bigcup \\
& [(0,0,1,1,1),(0,0,1,1,1)] \bigcup [(1,1,1,1,0),(1,1,1,1,0)] \bigcup \\
& [(1,1,1,0,1),(1,1,1,0,1)] \bigcup [(1,1,0,1,1),(1,1,0,1,1)] \bigcup \\
& [(1,0,1,1,1),(1,0,1,1,1)] \bigcup [(0,1,1,1,1),(0,1,1,1,1)] \bigcup \\
& [(1,1,1,1,1),(1,1,1,1,1)],
\end{aligned}$$

$$\begin{aligned}
\mathcal{Q}_2: \ & [(1,0,0,1,0),(1,1,0,1,0)] \bigcup [(0,1,0,0,1),(1,1,0,0,1)] \bigcup \\
& [(1,0,1,1,0),(1,1,1,1,0)] \bigcup [(1,0,0,1,1),(1,1,0,1,1)] \bigcup \\
& [(0,1,1,0,1),(1,1,1,0,1)] \bigcup [(0,1,0,1,1),(0,1,1,1,1)] \bigcup \\
& [(0,0,1,1,1),(1,0,1,1,1)] \bigcup [(1,1,1,1,1),(1,1,1,1,1)],
\end{aligned}$$

$$\begin{aligned}
\mathcal{Q}_3: \ & [(1,0,0,1,0),(1,1,1,1,0)] \bigcup [(0,1,0,0,1),(1,1,1,0,1)] \bigcup \\
& [(1,0,0,1,1),(1,1,0,1,1)] \bigcup [(0,1,0,1,1),(0,1,1,1,1)] \bigcup \\
& [(0,0,1,1,1),(1,0,1,1,1)] \bigcup [(1,1,1,1,1),(1,1,1,1,1)].
\end{aligned}$$

Now the corresponding Stanley decomposition is

$$\begin{aligned}
\mathcal{D}(\mathcal{Q}_1) \ := \ & x_1x_4K[x_1,x_4] \oplus x_2x_5K[x_2,x_5] \oplus x_1x_2x_4K[x_1,x_2,x_4] \oplus x_1x_2x_5K[x_1,x_2,x_5] \\
& \oplus x_1x_3x_4K[x_1,x_3,x_4] \oplus x_1x_4x_5K[x_1x_4x_5] \oplus x_2x_3x_5K[x_2,x_3,x_5] \oplus \\
& x_2\varrho_4x_5K[x_2,x_4,x_5] \oplus x_3x_4x_5K[x_3,x_4,x_5] \oplus x_1x_2x_3x_4K[x_1,x_2,x_3,x_4] \oplus \\
& x_1x_2x_3x_5K[x_1,x_2,x_3,x_5] \oplus x_1x_2x_4x_5K[x_1,x_2,x_4,x_5] \\
& \oplus x_1x_3x_4x_5K[x_1,x_3,x_4,x_5] \oplus x_2x_3x_4x_5K[x_2,x_3,x_4,x_5] \\
& \oplus x_1x_2x_3x_4x_5K[x_1,x_2,x_3,x_4,x_5],
\end{aligned}$$

$$\begin{aligned}
\mathcal{D}(\mathcal{Q}_2) \ := \ & x_1x_4K[x_1,x_2,x_4] \oplus x_2x_5K[x_1,x_2,x_5] \oplus x_1x_3x_4K[x_1,x_2,x_3,x_4] \oplus \\
& x_1x_4x_5K[x_1,x_2,x_4,x_5] \oplus x_2x_3x_5K[x_1,x_2,x_3,x_5] \oplus x_2x_4x_5K[x_2,x_3,x_4,x_5] \\
& \oplus x_3x_4x_5K[x_1,x_3,x_4,x_5] \oplus x_1x_2x_3x_4x_5K[x_1,x_2,x_3,x_4,x_5],
\end{aligned}$$

$$
\begin{aligned}
\mathcal{D}(\mathcal{Q}_3) \;:=\; & x_1x_4K[x_1,x_2,x_3,x_4] \oplus x_2x_5K[x_1,x_2,x_3,x_5] \oplus x_1x_4x_5K[x_1,x_2,x_4,x_5] \oplus \\
& x_2x_4x_5K[x_2,x_3,x_4,x_5] \oplus x_3x_4x_5K[x_1,x_3,x_4,x_5] \\
& \oplus x_1x_2x_3x_4x_5K[x_1,x_2,x_3,x_4,x_5].
\end{aligned}
$$

Then

$$
\begin{aligned}
\operatorname{sdepth}(A) &\geq \max\{\operatorname{sdepth}(\mathcal{D}(\mathcal{Q}_1)), \operatorname{sdepth}(\mathcal{D}(\mathcal{Q}_2)), \operatorname{sdepth}(\mathcal{D}(\mathcal{Q}_3))\} \\
&= \max\{2,3,4\} \\
&= 4.
\end{aligned}
$$

Example 2.2.2. Let $\mathcal{X} = K[x_1,x_2,x_3,x_4,x_5,x_6]$, consider $A = (x_1x_3, x_2x_5, x_4x_6, x_1x_4x_6)$. Then select $g = (1,1,1,1,1,1)$ and the poset $\mathcal{Q} = \mathcal{Q}^g_{\mathcal{X}/A}$ is given by:

$$
\begin{aligned}
\mathcal{Q} = \{ & (0,0,0,0,0,0), (1,0,0,0,0,0), (0,1,0,0,0,0), (0,0,1,0,0,0), (0,0,0,1,0,0), \\
& (0,0,0,0,1,0), (0,0,0,0,0,1), (1,1,0,0,0,0), (1,0,0,1,0,0), (1,0,0,0,1,0), \\
& (1,0,0,0,0,1), (0,1,1,0,0.0), (0,1,0,1,0.0), (0,0,1,1,0,0), (0,0,0,1,1,0), \\
& (0,0,1,0,0,1), (0,0,0,1,1,0), (0,0,0,1,0.1).(0,0,0,0,1,1), (1,1,0,1,0,0), \\
& (1,0,0,1,1,0), (1,0,0,0,1,1), (0,1,1,1,0,0), (0,0,1,1,1,0), (0,0,1,1,0,1), \\
& (0,0,1,0,1,1), (0,0,0,1,1,1), (0,0,1,1,1,1)\}.
\end{aligned}
$$

The partitions of $\mathcal{Q}$ can be written as :

$$\begin{aligned}
\mathcal{Q}_1 : \ & [(0,0,0,0,0,0),(1,0,0,0,0,0)] \bigcup [(0,1,0,0,0,0),(0,1,0,0,0,0)] \bigcup \\
& [(0,0,1,0,0,0),(0,0,1,0,0,0)] \bigcup [(0,0,0,1,0,0),(0,0,0,1,0,0)] \bigcup \\
& [(0,0,0,0,1,0),(0,0,0,0,1,0)] \bigcup [(0,0,0,0,0,1),(0,0,0,0,0,1)] \bigcup \\
& [(1,1,0,0,0,0),(1,1,0,0,0,0)] \bigcup [(1,0,0,1,0,0),(1,0,0,1,0,0)] \bigcup \\
& [(0,1,0,1,0,0),(0,1,0,1,0,0)] \bigcup [(0,0,1,1,0,0),(0,0,1,1,0,0)] \bigcup \\
& [(0,0,0,1,1,0),(0,0,0,1,1,0)] \bigcup [(0,0,0,1,0,1),(0,0,0,1,0,1)] \bigcup \\
& [(0,0,0,0,1,1),(0,0,0,0,1,1)] \bigcup [(1,1,0,1,0.0),(1,1,0,1,0,0)] \bigcup \\
& [(1,0,0,1,1,0),(1,0,0,1,1,0)] \bigcup [(1,0,0,0,1,1),(1,0,0,0,1,1)] \bigcup \\
& [(0,0,0,1,1,0),(0,0,0,1,1,0)] \bigcup [(0,0,0,0,1,1),(0,0,0,0,1,1)] \bigcup \\
& [(1,1,0,1,0,0),(1,1,0,1,0,0)] \bigcup [(1,0,0,1,1,0),(1,0,0,1,1,0)] \bigcup \\
& [(1,0,0,0,1,1),(1,0,0,0,1,1)] \bigcup [(0,1,1,1,0,0),(0,1,1,1,0,0)] \bigcup \\
& [(0,0,1,1,1,0),(0,0,1,1,1,0)] \bigcup [(0,0,1,1,0,1),(0,0,1,1,0,1,)] \bigcup \\
& [(0,0,0,1,1,1),(0,0,0,1,1,1)] \bigcup [(0,0,1,1,1,1),(0,0,1,1,1,1,)],
\end{aligned}$$

$$\begin{aligned}
\mathcal{Q}_2 : \ & [(0,0,0,0,0,0),(1,1,0,1,0,0)] \bigcup [(0,0,1,0,0,0),(0,1,1,1,0,0)] \bigcup \\
& [(0,0,0,0,1,0),(1,0,0,1,1,0)] \bigcup [(0,0,0,0,0,1),(1,0,0,0,1,1)] \bigcup \\
& [(0,0,0,1,1,0),(0,0,1,1,1,0)] \bigcup [(0,0,1,0,0,1),(0,0,0,1,0,1)] \bigcup \\
& [(0,0,0,1,0,1),(0,0,0,1,1,1)].
\end{aligned}$$

So the corresponding Stanley decomposition is of the partitions will be

$$\begin{aligned}
\mathcal{D}(\mathcal{Q}_1) \ := \ & K[x_1] \oplus x_2K[x_2] \oplus x_3K[x_3] \oplus x_4K[x_4] \oplus x_5K[x_5] \oplus x_6K[x_6] \oplus \\
& x_1x_2K[x_1,x_2] \oplus x_1x_4K[x_1,x_4] \oplus x_2x_4K[x_2,x_4] \oplus x_3x_4K[x_3,x_4] \\
& \oplus x_4x_5K[x_4,x_5] \oplus x_4x_6K[x_4,x_6] \oplus x_5x_6K[x_5,x_6] \oplus x_1x_2x_4K[x_1,x_2,x_4] \\
& \oplus x_1x_4x_5K[x_1,x_4,x_5] \oplus x_1x_5x_6K[x_1,x_5,x_6] \oplus x_4x_5K[x_4,x_5] \\
& \oplus x_4x_5K[x_4,x_5] \oplus x_5x_6K[x_5,x_6] \oplus x_1x_2x_4K[x_1,x_2,x_4] \oplus x_1x_4x_5K[x_1,x_4,x_5] \\
& \oplus x_1x_5x_6K[x_1,x_5,x_6] \oplus x_2x_3x_4K[x_2,x_3,x_4] \oplus x_3x_4x_5K[x_3,x_4,x_5] \\
& \oplus x_3x_4x_6K[x_3,x_4,x_6] \oplus x_4x_5x_6K[x_4,x_5,x_6] \oplus x_3x_4x_5x_6K[x_3,x_4,x_5,x_6],
\end{aligned}$$

$$\begin{aligned}\mathcal{D}(\mathcal{Q}_2) &:= K[x_1,x_2,x_4]\oplus x_3K[x_2,x_3,x_4]\oplus x_5K[x_1,x_4,x_5]\oplus x_6K[x_1,x_5,x_6]\oplus \\ & \quad x_4x_5K[x_3,x_4,x_5]\oplus x_3x_6K[x_4,x_6]\oplus x_4x_6K[x_4,x_5,x_6].\end{aligned}$$

Then

$$\begin{aligned}\operatorname{sdepth}(\mathcal{Q}/A) &\geq \max\{\operatorname{sdepth}(\mathcal{D}(\mathcal{Q}_1))\,,\operatorname{sdepth}(\mathcal{D}(\mathcal{Q}_2))\} \\ &= \max\{1,3\} \\ &= 3.\end{aligned}$$

2.2.3 Few Results Related to Sdepth

Lemma 2.2.2 ([26, Lemma 2.2]). *For a short exact sequence* $0 \longrightarrow \mathcal{N}_1 \longrightarrow \mathcal{N}_2 \longrightarrow \mathcal{N}_3 \longrightarrow 0$ *of* $\mathcal{X}$*- modules which are* $\mathbb{Z}^n$*-graded we have* $\operatorname{sdepth}(\mathcal{N}_2) \geq \min\{\operatorname{sdepth}(\mathcal{N}_1)$, $\operatorname{sdepth}(\mathcal{N}_3)\}$.

Lemma 2.2.3 ([15, Lemma 3.6]). *Let A be a monomial ideal of a ring $\mathcal{X}$. If $\mathcal{X}' = \mathcal{X}[y]$ is a ring of polynomials over $\mathcal{X}$ in the variable y. Then* $\operatorname{sdepth}(\mathcal{X}'/I\mathcal{X}') = \operatorname{sdepth}(\mathcal{X}/A)+1$.

Lemma 2.2.4 ([10, Lemma 2.13]). *Let $A_1 \subset \mathcal{X}_1 = K[x_1, ..., x_r]$ and $A_2 \subset \mathcal{X}_2 = K[x_{r+1}, x_{r+2}, \dots, x_d]$ be monomial ideals where $1 \leq r \leq d$. If $\mathcal{X} = \mathcal{X}_1 \otimes_K \mathcal{X}_2$. Then* $\operatorname{sdepth}_{\mathcal{X}}(\mathcal{X}_1/A_1 \otimes_K \mathcal{X}_2/A_2) \geq \operatorname{sdepth}_{\mathcal{X}_1}(\mathcal{X}_1/A_1) + \operatorname{sdepth}_{\mathcal{X}_2}(\mathcal{X}_2/A_2)$.

Proposition 2.2.3 ([7, Proposition 2.7]). *Consider a monomial ideal A of a ring $\mathcal{X}$. Then for any monomial $x \notin A$,* $\operatorname{sdepth}(\mathcal{X}/A) \leq \operatorname{sdepth}(\mathcal{X}/(A : x))$.

Proposition 2.2.4 ([15, Proposiiton 3.4]). *Let an ideal $A \subseteq \mathcal{X}$ be generated by u elements. Then* $\operatorname{sdepth}(A) \geq \max\{1, d-u+1\}$.

Proposition 2.2.5 ([15, Proposition 3.8]). *Let a complete intersection monomial ideal $A \subseteq \mathcal{X}$ be generated by a minimal set of u elements. Then* $\operatorname{sdepth}(A) = d - \lceil \frac{u}{2} \rceil$.

Theorem 2.2.6 ([23, Theorem 2.3]). *Let $A \subseteq \mathcal{X}$ be monomil ideal which is generated by a minimal set of u elements. Then, we have* $\operatorname{sdepth}(A) \geq \max\{1, d - \lfloor \frac{u}{2} \rfloor\}$.

Theorem 2.2.7 ([13, Theorem 4.18]). *Consider a graph G having c connected components and let $A = A(G)$ and $d = d(G)$ represent the diameter of the graph G. Then for the first three powers that is $1 \leq t \leq 3$, we have that* $\operatorname{sdepth}(\mathcal{X}/A^t) \geq \{\lceil \frac{d-4t+5}{3} \rceil + c - 1.\}$

2.3 Regularity

Definition 2.3.1. For a field K let $\mathcal{X} = K[x_1, \ldots, x_d]$ be a ring of polynomials in d variables over the given field. Then for a Z-graded $\mathcal{X}$-module $\mathcal{N}$ which is finitely generated having a minimal free resolution

$$0 \longrightarrow \bigoplus_{h\in Z} \mathcal{X}(-h)^{\beta_{p,h}(\mathcal{N})} \longrightarrow \bigoplus_{h\in Z} \mathcal{X}(-h)^{\beta_{p-1,h}(\mathcal{N})} \longrightarrow ..... \longrightarrow \bigoplus_{h\in Z} \mathcal{X}(-h)^{\beta_{0,h}(\mathcal{N})} \longrightarrow 0$$

the regularity of $\mathcal{N}$ is $\operatorname{reg}(\mathcal{N}) = \max\{h - k : \beta_{k,h}\ (\mathcal{N}) \neq 0\}$.

2.3.1 Few Results Related to Regularity

Lemma 2.3.1 ([5, Theorem 4.7])**.** *Consider a monomial ideal A in the ring $\mathcal{X}$, and let x_i be one of the variables in $\mathcal{X}$. Then*

- $\operatorname{reg}(\mathcal{X}/A) = reg(\mathcal{X}/(A : x_i)) + 1$, *if* $\operatorname{reg}(\mathcal{X}/(A : x_i)) \geq \operatorname{reg}(\mathcal{X}/(A, x_i))$,
- $\operatorname{reg}(\mathcal{X}/A) \in \operatorname{reg}(\mathcal{X}/(A, x_i)) + 1, \operatorname{reg}(\mathcal{X}/(A, x_i))$, *if* $\operatorname{reg}(\mathcal{X}/(A : x_i)) = \operatorname{reg}(\mathcal{X}/(A, x_i))$,
- $\operatorname{reg}(\mathcal{X}/A) = \operatorname{reg}(\mathcal{X}/(A, x_i))$ *if* $\operatorname{reg}(\mathcal{X}/(A : x_i)) \leq \operatorname{reg}(\mathcal{X}/(A, x_i))$.

Lemma 2.3.2 ([21, Lemma 3.6])**.** *Consider a monomial ideal A in the ring $\mathcal{X}$. If $\mathcal{X}' = \mathcal{X}[y]$ is a ring of polynomials over $\mathcal{X}$ in the variable y. Then* $\operatorname{reg}(\mathcal{X}'/A) = \operatorname{reg}(\mathcal{X}/A)$.

Lemma 2.3.2 ([32, lemma 8])**.** *Let $\mathcal{X}_1 = K[x_1, \ldots, x_r]$ and $\mathcal{X}_2 = K[x_{r+1}, \ldots, x_d]$ be rings of polynomials and A_1 and A_2 be edge ideals of $\mathcal{X}_1$ and $\mathcal{X}_2$, respectively. Then* $\operatorname{reg}(\mathcal{X}/(A_1\mathcal{X} + A_2\mathcal{X})) = \operatorname{reg}(\mathcal{X}_1/A_1) + \operatorname{reg}(\mathcal{X}_2/A_2)$.

Proposition 2.3.3 ([19, Lemma 2.2])**.** *For a simple graph G which is finite we have,* $\operatorname{reg}(\mathcal{X}/A(G)) \geq \operatorname{indmat}(G)$.

Lemma 2.3.4 ([3, Proposition 3.1.1])**.** *Let $A = A(P_n) \subset \mathcal{X}$, where P_n is a path of length n. Then* $\operatorname{reg}(\mathcal{X}/A(P_n)) = \lceil \frac{n}{3} \rceil$.

2.4 Projective Dimension

Definition 2.4.1. For a field K let $\mathcal{X} = K[x_1, \ldots, x_d]$ be the ring of polynomials in d variables over the given field. Then for a $\mathbb{Z}$-graded $\mathcal{X}$-module $\mathcal{N}$ which is finitely generated having a minimal free resolution

$$0 \longrightarrow \bigoplus_{h\in Z} \mathcal{X}(-h)^{\beta_{p,h}(\mathcal{N})} \longrightarrow \bigoplus_{h\in Z} \mathcal{X}(-h)^{\beta_{p-1,h}(\mathcal{N})} \longrightarrow ..... \longrightarrow \bigoplus_{h\in Z} \mathcal{X}(-h)^{\beta_{0,h}(\mathcal{N})} \longrightarrow 0$$

the projective dimension of $\mathcal{N}$ is given by $\operatorname{pdim}(\mathcal{N}) = \max\{k : \beta_{k,h}\ (\mathcal{N}) \neq 0\}$.

2.4.1 Few Results Related to Projective Dimension

Lemma 2.4.1 ([4, Theorem 1.3.3])**.** *(Auslander–Buchsbaum formula) Suppose $\mathcal{X}$ is a local Noetherian ring and is also commutative, and $\mathcal{N}$ is a non-zero $\mathcal{X}$-module which is finitely generated and has finite projective dimension. Then, the sum of the projective dimension and the depth of the module $\mathcal{N}$ is equal to the dimension of the ring $\mathcal{X}$ this is* $\operatorname{pdim}(\mathcal{N}) + \operatorname{depth}(\mathcal{N}) = \operatorname{depth}(\mathcal{X})$.

Lemma 2.4.2 ([9, Lemma 5.1])**.** *For A a square-free monomial ideal, and any subset of the variables T relabeled as $T = \{x_1, \ldots, x_d\}$, either $\exists$ a d with $1 \leq d \leq j$ such that* $\operatorname{pdim}(\mathcal{X}/A) = \operatorname{pdim}(\mathcal{X}/((A, x_1, \ldots, x_{d1}) : x_d))$ *or* $\operatorname{pdim}(\mathcal{X}/A) = \operatorname{pdim}(\mathcal{X}/(S, x_1, \ldots, x_j))$.

Lemma 2.4.3 ([3, Proposition 3.1.1])**.** *Let $A = A(P_n) \subset \mathcal{X}$, where P_n is a path of length n. Then* $\operatorname{pdim}(\mathcal{X}/A(P_n)) = \lceil \frac{2n}{3} \rceil$.

Theorem 2.4.4 ([9, Theorem 2.6])**.** *Let $A \subset \mathcal{X}$ be a monomial ideal which is square-free. Then* $\operatorname{reg}(A^{\nabla}) = \operatorname{pdim}(\mathcal{X}/A)$. *Also we have $A^{\nabla\nabla} = A$, so* $\operatorname{reg}(A) = \operatorname{pdim}(\mathcal{X}/A^{\nabla})$, *where ∇ represents the Alexander Dual.*

Proposition 2.4.5 ([3, Proposition 5.0.6])**.** *Let C_n be an n length cycle. Then*

$$\operatorname{pdim}(\mathcal{X}/A(C_n) = \begin{cases} \lceil \frac{2n}{3} \rceil & \text{if } 3 \mid (n-1) \\ \lceil \frac{2(n-1)}{3} \rceil & \text{if } 3 \nmid (n-1). \end{cases}$$

Chapter 3

Algebraic Invariants of the Powers of Edge Ideals of Caterpillars and some Forests

In this chapter a caterpillar graph is defined as in Definition 3.1.1. We compute bounds for the depth and sdepth of the quotient ring corresponding to the powers of the edge ideals of these caterpillars and also the bounds for the projective dimension. When considering a forest of p connected caterpillars a lower bound for the depth and sdepth of the quotient rings which are associated to the powers of the edge ideals of such forests is also calculated. The exact value for the regularity of the considered quotient rings is also computed along with the values of regularity for the quotient rings which are associated to the powers of the edge ideals of the considered forests.

3.1 Preliminaries

We discuss a few preliminaries first which aid in proving the Theorems given in this chapter. These basic results include a few Lemma's which help in simplifying the structures under study and a few other basic Theorems related to bipartite graphs and trees in general are given.

Lemma 3.1.1 ([1, Theorem 2.6 and 2.9])**.** *Let $A = A(\mathcal{S}_k)$. Then*

- $\operatorname{depth}(\mathcal{X}/A) = \operatorname{sdepth}(\mathcal{X}/A) = 1,$

- $\operatorname{depth}(\mathcal{X}/A^t), \operatorname{sdepth}(\mathcal{X}/A^t) \geq 1$.

Lemma 3.1.2 ([20, Lemma 2.6]). *Suppose G is bipartite having edge ideal A. Then for all $t \geq 1$ $\operatorname{depth}(\mathcal{X}/A^t) \geq 1$ and $\operatorname{depth}(\mathcal{X}/A^t) = 1$ for a large t if the graph is tree.*

Lemma 3.1.3 ([20, Lemma 2.10]). *Let G be a graph and $A = A(G)$. Let x_i be a leaf of G and x_j be the unique neighbor of x_i. Then $(A^t : x_i x_j) = A^{t-1}$ for any $t \geq 2$.*

Lemma 3.1.4 ([20, Lemma 2.5]). *Consider a monomial ideal A which is square free in a polynomial ring $\mathcal{X}$, and let M be a monomial in $\mathcal{X}$. Suppose y is a variable such that y doesnot divide M. Let B be the extension in $\mathcal{X}$ of the minor of A formed by keeping $y = 0$. For any $t \geq 1$, it holds that $((A^t : M), y) = ((B^t : M), y)$.*

Lemma 3.1.5. *If $A = A(CP_{m_1,..,m_p})$, then for $t \geq 1$, $(A^t, x_p) = (A^t(CP_{m_1,..,m_{p-1}}), x_p)$.*

Proof: It is clear that $(A^t(CP_{m_1,..,m_{p-1}}), x_n) \subseteq (A^t, x_p)$. Conversely, if $x \in A^t$ is a monomial which is not divisible by x_p, then, by the definition of $G(A)$, we get that $x \in A^t(CP_{m_1,..,m_{p-1}})$.

Theorem 3.1.6 ([20, Theorem 3.4]). *Consider a forest G having c connected components, denoted as $G_1, \ldots, G_c$, and let $A = A(G)$. For each connected component G_i, let d_i be its diameter, and define d as $d = \max\{d_i\}$. Then for $t \geq 1$, $\operatorname{depth}(\mathcal{X}/A^t) \geq \max\{\lceil \frac{d-t+2}{3} \rceil + c - 1, c\}$,*

Theorem 3.1.7 ([25, Theorem 2.7]). *Consider a forest G having c connected components, denoted as $G_1, \ldots, G_c$, and let $A = A(G)$. For each connected component G_i, let d_i be its diameter, and define d as $d = \max\{d_i\}$. Then for $t \geq 1$, $\operatorname{sdepth}(\mathcal{X}/A^t) \geq \max\{\lceil \frac{d-t+2}{3} \rceil + c - 1, c\}$.*

Corollary 3.1.8 ([20, Corollary 3.7]). *Consider a forest G having c connected components, denoted as $G_1, \ldots, G_c$, and let $A = A(G)$. For each connected component G_i, let d_i be its diameter, and define d as $d = \max\{d_i\}$ and let q denote the number of near leaves in the component with d as its diameter. Then for $t \geq 1.$, $\operatorname{depth}(\mathcal{X}/A^t) \geq \max\{\lceil \frac{d-t+q}{3} \rceil + c - 1, c\}$.*

Corollary 3.1.9 ([25, Corollary 3.2]). *Consider a forest G having c connected components, denoted as $G_1, \ldots, G_c$, and let $A = A(G)$. For each connected component G_i, let d_i be its diameter, and define d as $d = \max\{d_i\}$ and let q denote the number of near leaves in the component with d as its diameter. Then for $t \geq 1$.* $\operatorname{sdepth}(\mathcal{X}/A^t) \geq \max\{\lceil\frac{d-t+q}{3}\rceil + c - 1, c\}$.

Lemma 3.1.10 ([27, Lemma 2.26]). *Let $A = A(\mathcal{S}_k)$. Then* $\operatorname{reg}(\mathcal{X}/A(\mathcal{S}_k)) = 1$.

Theorem 3.1.11 ([2, Theorem 4.7]). *Consider a forest G with the edge ideal $A = A(G)$, and let indmat(G) be the induced matching number of G. For all $t \geq 1$, the regularity of the t-th power of the ideal A is given by* $\operatorname{reg}(A^t) = 2t + \textit{indmat(G)} - 1$.

Lemma 3.1.12 ([14, Corollary 6.9]). *For a chordal graph G, the regularity of the quotient ring $\mathcal{X}/A(G)$ is equal to the induced matching number of G that is* $\operatorname{reg}(\mathcal{X}/A(G)) = \operatorname{indmat}(G)$.

Remark 3.1.13. Consider a graph G having c connected components, and let k_1, k_2,...,k_c be the induced matching numbers of the connected components of G respectively. Then the induced matching number of G is equal to $k_1 + k_2 + ... + k_c$.

3.2 Bounds for the Invariants of the First Power of the Edge Ideals of Caterpillars

In this section a caterpillar is defined and bounds for the the three invariants the depth, sdepth and projective dimension of the squarefree case of the quotient ring $\mathcal{X}/A$ are given. For the fourth invariant the regularity, an exact value is calculated.

Definition 3.2.1. Let $p \geq 1$ and $m_i \geq 1$ be integers, where $M_i = \{x_{i1}, x_{i2}, ..., x_{im_i}\}$, $1 \leq i \leq p$ and P_p is a path on p vertices $\{x_1, ..., x_p\}$. We define a caterpillar by $CP_{(m_1,...,m_p)}$, where p equals the number of vertices on the path P_p and the set M_i denotes the pendant vertices on each x_i respectively.

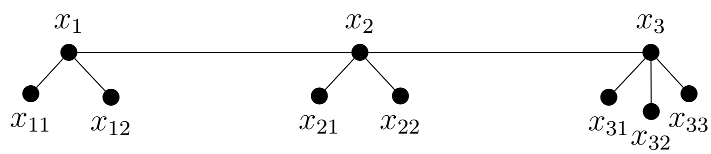


Figure 3.1: $CP_{(2,2,3)}$

Theorem 3.2.2. *Let* $p \geq 1, m_i \geq 1, m = \min\{m_1, \ldots, m_p\}, M = \max\{m_1, \ldots, m_p\}$ *and* $A = A(CP_{(m_1,\ldots,m_p)})$. *Then*

$$\lceil \frac{p}{2} \rceil + \lceil \frac{p-1}{2} \rceil (m) \leq \operatorname{depth}(\mathcal{X}/A), \operatorname{sdepth}(\mathcal{X}/A) \leq \lceil \frac{p}{2} \rceil + \lceil \frac{p-1}{2} \rceil (M).$$

Proof: For $p = 1$, the result is derived from Lemma 3.1.1. First we state the proof for $p = 2$.

Consider the short exact sequence

$$0 \longrightarrow \mathcal{X}/(A : x_2) \xrightarrow{.x_2} \mathcal{X}/A \longrightarrow \mathcal{X}/(A, x_2) \longrightarrow 0.$$

Note that

$$\mathcal{X}/(A : x_2) \cong K[x_2, x_{11}, x_{12}, ..., x_{1m_1}].$$

Thus

$$\operatorname{depth}(\mathcal{X}/A : x_2)) = 1 + m_1 \geq 1 + m.$$

Also we have

$$\mathcal{X}/(A, x_2) \cong K[CP_{(m_1)}]/A(CP_{(m_1)}) \otimes K[x_{21}, x_{22}, ..., x_{2m_2}].$$

Thus

$$\operatorname{depth}(\mathcal{X}/A, x_2)) = 1 + m_2 \geq 1 + m.$$

Also, using Lemma 2.1.2 we get

$$\operatorname{depth}(\mathcal{X}/A : x_2)) = 1 + m_1 \leq 1 + M.$$

So, by Lemma 2.1.4, we get

$$1 + m \leq \operatorname{depth}(\mathcal{X}/A) \leq 1 + M.$$

Without loss of generality if we let $m_1 = m$, then

$$\operatorname{depth}(\mathcal{X}/A) = 1 + m_1 = 1 + m.$$

For $n \geq 3$, consider the short exact sequence

$$0 \longrightarrow \mathcal{X}/(A : x_p) \xrightarrow{.x_p} \mathcal{X}/A \longrightarrow \mathcal{X}/(A, x_p) \longrightarrow 0.$$

Note that

$$\mathcal{X}/(A : x_p) \cong (K[CP_{(m_1,\ldots m_{p-2})}]/I(CP_{(m_1,\ldots m_{p-2})})) \otimes K[x_n, M_{p-1}]$$

and

$$\mathcal{X}/(A, x_p) \cong (K[CP_{(m_1,\ldots m_{p-1})}]/I(CP_{(m_1,\ldots m_{p-1})})) \otimes K[M_p].$$

So, by induction we have

$$\begin{aligned}
\operatorname{depth}(\mathcal{X}/A : x_p)) &= \operatorname{depth}(K[CP_{(m_1,\ldots m_{p-2})}]/I(CP_{(m_1,\ldots m_{p-2})})) + 1 + m_{p-1} \\
&\geq \lceil \tfrac{p-2}{2} \rceil + \lceil \tfrac{p-3}{2} \rceil (\min\{m_1, ..., m_{p-2}\}) + 1 + m_{p-1} \\
&\geq \lceil \tfrac{p}{2} \rceil + \lceil \tfrac{p-3}{2} \rceil (m) + m \\
&\geq \lceil \tfrac{p}{2} \rceil + \lceil \tfrac{p-3}{2} \rceil (m) + m \\
&= \lceil \tfrac{p}{2} \rceil + \lceil \tfrac{p-1}{2} \rceil (m)
\end{aligned}$$

and

$$\begin{aligned}
\operatorname{depth}(\mathcal{X}/A, x_p)) &= \operatorname{depth}(K[CP_{(m_1,\ldots m_{p-1})}]/I(CP_{(m_1,\ldots m_{p-1})})) + m_p \\
&\geq \lceil \tfrac{p-1}{2} \rceil + \lceil \tfrac{p-2}{2} \rceil (min\{m_1, ..., m_{p-1}\}) + m_p \\
&\geq \lceil \tfrac{p-1}{2} \rceil + \lceil \tfrac{p-2}{2} \rceil (m) + m \\
&= \lceil \tfrac{p-1}{2} \rceil + \lceil \tfrac{p}{2} \rceil (m).
\end{aligned}$$

Also

$$\begin{aligned}\operatorname{depth}(\mathcal{X}/A : x_p)) &= \operatorname{depth}(K[CP_{(m_1,\dots m_{p-2})}]/I(CP_{(m_1,\dots m_{p-2})})) + 1 + m_{p-1}\\ &\leq \lceil \tfrac{p-2}{2} \rceil + \lceil \tfrac{p-3}{2} \rceil (max\{m_1, ..., m_{p-2}\}) + 1 + m_{p-1}\\ &\leq \lceil \tfrac{p}{2} \rceil + \lceil \tfrac{p-3}{2} \rceil (M) + m_{p-1}\\ &\leq \lceil \tfrac{p}{2} \rceil + \lceil \tfrac{p-3}{2} \rceil (M) + M\\ &= \lceil \tfrac{p}{2} \rceil + \lceil \tfrac{p-1}{2} \rceil (M).\end{aligned}$$

So, by the Lemma 2.1.4 and Proposition 2.1.2

$$\lceil \frac{p}{2} \rceil + \lceil \frac{p-1}{2} \rceil (m) \leq \operatorname{depth}(\mathcal{X}/A) \leq \lceil \frac{p}{2} \rceil + \lceil \frac{p-1}{2} \rceil (M).$$

By using Lemma 2.2.2 instead of the Depth lemma we have a similar proof for sdepth.

Corollary 3.2.3 ([27, Theorem 2.5])**.** *Let* $p \geq 1, m_i \geq 1, m = \min\{m_1, \dots, m_p\}, M = \max\{m_1, m_2, \dots, m_p\}$ *and* $A = A(CP_{(m_1,\dots,m_p)})$. *If* $m = M$. *Then*

$$\operatorname{depth}(\mathcal{X}/A), \operatorname{sdepth}(\mathcal{X}/A) = \lceil \frac{p}{2} \rceil + \lceil \frac{p-1}{2} \rceil (M).$$

Corollary 3.2.4. *Let* $p \geq 1$, $m_i \geq 1$, $m = min\{m_1, m_2, \dots, m_p\}$, $M = max\{m_1, m_2, \dots, m_p\}$ *and* $A = A(CP_{(m_1,\dots,m_p)})$. *Then*

$$p + \sum_{i=1}^{p} m_i - \lceil \frac{p}{2} \rceil - \lceil \frac{p-1}{2} \rceil (m) \leq \operatorname{pdim}(\mathcal{X}/A) \leq p + \sum_{i=1}^{p} m_i - \lceil \frac{p}{2} \rceil - \lceil \frac{p-1}{2} \rceil (M).$$

Proof: As $\operatorname{depth}(\mathcal{X}) = p + \sum_{i=1}^{p} m_i$, so using Lemma 2.4.1 and Theorem 3.2.2 we get

$$p + \sum_{i=1}^{p} m_i - \lceil \frac{p}{2} \rceil - \lceil \frac{p-1}{2} \rceil (m) \leq \operatorname{pdim}(\mathcal{X}/A) \leq p + \sum_{i=1}^{p} m_i - \lceil \frac{p}{2} \rceil - \lceil \frac{p-1}{2} \rceil (M).$$

Theorem 3.2.5. *Let* $p \geq 1, m_i \geq 1$ *and* $A = I(CP_{(m_1,\dots,m_p)})$. *Then we have*

$$\operatorname{reg}(\mathcal{X}/A) = \lceil \frac{p}{2} \rceil.$$

Proof: For $p = 1$, we get the result from Lemma 3.1.10.

For $p = 2$, we have

$$\mathcal{X}/(A : x_1) \cong K[x_1, M_2],$$

so

$$\operatorname{reg}(\mathcal{X}/(A : x_1)) = 0$$

and

$$\mathcal{X}/(A, x_1) \cong K[CP_{(m_2)}]/I(CP_{(m_2)}) \otimes K[M_1].$$

So

$$\operatorname{reg}(\mathcal{X}/(A, x_1)) = 1.$$

So, by Lemma 2.3.1 we get

$$\operatorname{reg}(\mathcal{X}/A) = 1.$$

Let $p \geq 3$, then we have the given isomorphisms

$$\mathcal{X}/(A : x_{p-1}) \cong (K[CP_{(m_1,\ldots m_{p-3})}]/A(CP_{(m_1,\ldots m_{p-3})})) \otimes K[x_{p-1}, M_{p-2}, x_p, M_p]$$

and

$$\mathcal{X}/(A, x_{p-1}) \cong (K[CP_{(m_1,\ldots m_{p-2})}]/A(CP_{(m_1,\ldots m_{p-2})}))\otimes K[CP_{(m_p)}]/A(CP_{(m_p)})\otimes K[M_{p-1}].$$

So, by induction we have

$$\operatorname{reg}(\mathcal{X}/(A : x_{n-1})) = \operatorname{reg}(K[CP_{(m_1,\ldots m_{n-3})}]/A(CP_{(m_1,\ldots m_{n-3})})) = \lceil \frac{p-3}{2} \rceil$$

and

$$\operatorname{reg}(\mathcal{X}/(A, x_{p-1})) = \operatorname{reg}(K[CP_{(m_1,\ldots m_{p-2})}]/A(CP_{(m_1,\ldots m_{p-2})}))+\operatorname{reg}(K[CP_{(m_p)}]/A(CP_{(m_p)}))$$

$$= \lceil \tfrac{p-2}{2} \rceil + 1 = \lceil \tfrac{p}{2} \rceil.$$

Thus, using Lemma 2.3.1 we get

$$\operatorname{reg}(\mathcal{X}/A) = \lceil \frac{p}{2} \rceil.$$

3.3 Bounds for the Invariants of Higher Powers of the Edge Ideals of Caterpillars and some Forests

In this section bounds for the depth, sdepth and projective dimension of the $\mathcal{X}/A^t$ associated to the powers of the edge ideals are given. For regularity we have exact values.

Note : For convenience label the vertices $\{x_{p1}, x_{p2}, \ldots, x_{pm_p}\}$ by $l_1, l_2, \ldots, l_{m_p}$. Let $S_j = K[A]/(l_1, l_2, \ldots, l_j)$ and $I_j = I \cap S_j$, where $1 \leq j \leq m_p$.

Theorem 3.3.1. *Let $p \geq 2$, $t, m_i \geq 1$ and $A = I(CP_{(m_1,\ldots,m_p)})$. Then we have*

$$\operatorname{depth}(\mathcal{X}/A^t), \operatorname{sdepth}(\mathcal{X}/A^t) \geq \max\{\lceil \frac{2(p-t)+1}{2} \rceil, 1\}.$$

Proof : Since $CP_{(m_1,\ldots,m_p)}$ is a bipartite graph so for all $t \geq 1$ the result is derived from Lemma 3.1.2 and $\operatorname{depth}(\mathcal{X}/A^t) \geq 1$.

For $p \geq 2$ and $t = 1$, we get the result from Theorem 3.2.2. For $p = 2$ and $t \geq 1$ we get the desired result from Lemma 3.1.2. For $p = 3$ and $t \geq 3$ the result also follows from Lemma 3.1.2.

We first prove for $p = 3$ and $t = 2$.

Consider the short exact sequence

$$0 \longrightarrow \mathcal{X}/(A^2 : x_3) \xrightarrow{.x_3} \mathcal{X}/A^2 \longrightarrow \mathcal{X}/(A^2, x_3) \longrightarrow 0.$$

From Lemma 3.1.5 we have

$$\mathcal{X}/(A^2, x_3) \cong \mathcal{X}/(A^2(CP_{(m_1,m_2,m_3)}), x_3) \cong K[CP_{(m_1,m_2)}]/A^2(CP_{(m_1,m_2)}) \otimes K[M_3].$$

So

$$\operatorname{depth}(\mathcal{X}/(A^2, x_3)) \geq \max\{1, 1\} + m_3 \geq 1 + 1 = 2.$$

Now consider the family of short exact sequences as

$$0 \longrightarrow \mathcal{X}_0/(A_0^2 : x_3 l_1) \xrightarrow{.l_1} \mathcal{X}_0/(A_0^2 : x_3) \longrightarrow \mathcal{X}_0/((A_0^2 : x_3), l_1) \longrightarrow 0,$$

$$0 \longrightarrow \mathcal{X}_1/(A_1^2 : x_3 l_2) \xrightarrow{.l_2} \mathcal{X}_1/(A_1^2 : x_3) \longrightarrow \mathcal{X}_1/((A_1^2 : x_3), l_2) \longrightarrow 0,$$

$$\vdots \qquad\qquad \vdots \qquad\qquad \vdots$$

$$0 \longrightarrow \mathcal{X}_{j-1}/(A_{j-1}^2 : x_3 l_j) \xrightarrow{.l_j} \mathcal{X}_{j-1}/(A_{j-1}^2 : x_3) \longrightarrow \mathcal{X}_{j-1}/((A_{j-1}^2 : x_3), l_j) \longrightarrow 0.$$

By Lemma 3.1.3

$$\mathrm{depth}(\mathcal{X}_{j-1}/(A_{j-1}^2 : x_3 l_j)) = \mathrm{depth}(\mathcal{X}_{j-1}/(A_{j-1}))$$

and by Theorem 3.2.2

$$\mathrm{depth}(\mathcal{X}_{j-1}/(A_{j-1}^2 : x_3 l_j)) \geq \max\{\lceil \frac{2(3-1)+1}{2} \rceil, 1\} = \max\{\lceil \frac{5}{2} \rceil, 1\} = 3.$$

Since using Lemma 3.1.4 we have

$$\mathcal{X}_{j-1}/((A_{j-1}^2 : x_3), l_j) \cong \mathcal{X}_j/((A_j^2 : x_3).$$

So, consider another short exact sequence as

$$0 \longrightarrow \mathcal{X}_j/((A_j^2 : x_3 x_2) \xrightarrow{.x_2} \mathcal{X}_j/((A_j^2 : x_3) \longrightarrow \mathcal{X}_j/((A_j^2 : x_3), x_2) \longrightarrow 0.$$

By Lemma 3.1.3 and Theorem 3.2.2

$$\mathrm{depth}(\mathcal{X}_j/(A_j^2 : x_3 x_2)) = \mathrm{depth}(\mathcal{X}_j/A_j) \geq \max\{\lceil \frac{2(3-2)+1}{2} \rceil, 1\} = \max\{\lceil \frac{3}{2} \rceil, 1\} = 2$$

and

$$\mathcal{X}_j/((A_j^2 : x_3), x_2) \cong K[CP_{(m_1)}]/A^2(CP_{(m_1)}) \otimes K[x_3, M_2].$$

So

$$\mathrm{depth}(\mathcal{X}_j/((A_j^2 : x_3), x_2)) \geq \max\{1, 1\} + 1 + 1 \geq 3$$

and

$$\mathrm{depth}(\mathcal{X}_j/((A_j^2 : x_3)) \geq 3.$$

Hence

$$\operatorname{depth}(\mathcal{X}_{j-1}/(A_{j-1}^2 : x_3), l_j) \geq 3$$

and so

$$\operatorname{depth}(\mathcal{X}/A^2) \geq 2.$$

Now let $p \geq 4$ and $t \geq 2$, then consider the following short exact sequence

$$0 \longrightarrow \mathcal{X}/(A^t : x_p) \xrightarrow{.x_p} \mathcal{X}/A^t \longrightarrow \mathcal{X}/(A^t, x_p) \longrightarrow 0.$$

From Lemma 3.1.5 we have

$$\mathcal{X}/(A^t, x_p) \cong K[CP_{(m_1,\ldots,m_{p-1})}]/A^t(CP_{(m_1,\ldots,m_{p-1})}) \otimes K[M_p].$$

So

$$\operatorname{depth}\mathcal{X}/(A^t, x_p) \geq \max\{\lceil \frac{2(p-1-t)+1}{2} \rceil, 1\} + m_p \geq \max\{\lceil \frac{2p-2-2t+1}{2} \rceil, 1\} + 1$$

$$= \max\{\lceil \tfrac{2p-2t-1}{2} \rceil, 1\} + 1 \geq \max\{\lceil \tfrac{2(p-t)+1}{2} \rceil, 1\}.$$

Now consider the family of short exact sequences as

$$0 \longrightarrow \mathcal{X}_0/(A_0^t : x_p l_1) \xrightarrow{.l_1} \mathcal{X}_0/(A_0^t : x_p) \longrightarrow \mathcal{X}_0/((A_0^t : x_p), l_1) \longrightarrow 0,$$

$$0 \longrightarrow \mathcal{X}_1/(A_1^t : x_p l_2) \xrightarrow{.l_2} \mathcal{X}_1/(A_1^t : x_p) \longrightarrow \mathcal{X}_1/((A_1^t : x_p), l_2) \longrightarrow 0,$$

$$\vdots \qquad\qquad \vdots \qquad\qquad \vdots$$

$$0 \longrightarrow \mathcal{X}_{j-1}/(A_{j-1}^t : x_p l_j) \xrightarrow{.l_j} \mathcal{X}_{j-1}/(A_{j-1}^t : x_p) \longrightarrow \mathcal{X}_{j-1}/((A_{j-1}^t : x_p), l_j) \longrightarrow 0.$$

By Lemma 3.1.3

$$\operatorname{depth}(\mathcal{X}_{j-1}/(A_{j-1}^t : x_p l_j)) = \operatorname{depth}(\mathcal{X}_{-1}/(A_{j-1}^{t-1}))$$

and by Theorem 3.2.2

$$\operatorname{depth}(\mathcal{X}_{j-1}/(A_{j-1}^t : x_p x_j)) \geq \max\{\lceil \frac{2(p-(t-1))+1}{2} \rceil, 1\}$$

$$= \max\{\lceil\tfrac{2p-2t+2+1}{2}\rceil, 1\}$$
$$= \max\{\lceil\tfrac{2(p-t)+3}{2}\rceil, 1\}.$$
$$\geq \max\{\lceil\tfrac{2(p-t)+1}{2}\rceil, 1\}.$$

Since using Lemma 3.1.4 we get

$$\mathcal{X}_{j-1}/((A^t_{j-1} : x_p), l_j) \cong \mathcal{X}_j/((A^t_j : x_p).$$

So, consider another short exact sequence as

$$0 \longrightarrow \mathcal{X}_j/((A^t_j : x_p x_{p-1}) \xrightarrow{.x_{p-1}} \mathcal{X}_j/((A^t_j : x_{p-1}) \longrightarrow \mathcal{X}_j/((A^t_j : x_p), x_{p-1}) \longrightarrow 0.$$

By Lemma 3.1.3 and Theorem 3.2.2

$$\operatorname{depth}(\mathcal{X}_j/(A^t_j : x_p x_{p-1})) = \operatorname{depth}(\mathcal{X}_j/A^{t-1}_j)$$
$$\geq \max\{\lceil\tfrac{2(p-1-(t-1))+1}{2}\rceil, 1\}$$
$$= \max\{\lceil\tfrac{2(p-t)+1}{2}\rceil, 1\}$$

and

$$\mathcal{X}_j/((A^t_j : x_p), x_{p-1}) \cong K[CP_{(m_1,\ldots,m_{p-2})}]/A^t(CP_{(m_1,\ldots,m_{p-2})}) \otimes K[x_p, M_{p-1}].$$

So

$$\begin{aligned}\operatorname{depth}(\mathcal{X}_j/((A^t_j : x_p), x_{p-1})) &\geq \max\{\lceil\tfrac{2(p-2-t)+1}{2}\rceil, 1\} + 1 + m_{p-1}\\ &\geq \max\{\lceil\tfrac{2p-4-2t+1}{2}\rceil, 1\} + 1 + 1\\ &= \max\{\lceil\tfrac{2p-2t-3}{2}\rceil, 1\} + 2\\ &\geq \max\{\lceil\tfrac{2(p-t)+1}{2}\rceil, 1\}\end{aligned}$$

and

$$\operatorname{depth}(\mathcal{X}_j/((A^t_j : x_p)) \geq \max\{\lceil\frac{2(p-t)+1}{2}\rceil, 1\}.$$

Hence

$$\operatorname{depth}(\mathcal{X}_{j-1}/(A^t_{j-1} : x_p), l_j) \geq \max\{\lceil\frac{2(p-t)+1}{2}\rceil, 1\}$$

and so

$$\operatorname{depth}(\mathcal{X}/A^t) \geq \max\{\lceil \frac{2(p-t)+1}{2} \rceil, \}.$$

Proof for sdepth is same using Lemma 2.2.2 instead of Lemma 2.1.4.

Corollary 3.3.2. *Let $p \geq 1, m_i \geq 1$ and $A = I(CP_{(m_1,\ldots,m_p)})$, then*

$$\operatorname{indmat}(G) = \lceil \frac{p}{2} \rceil.$$

Proof: As, the considered caterpillar is a tree, so it is a chordal graph. Using Lemma 3.1.12 and Theorem 3.2.5 we get

$$\operatorname{indmat}(G) = \lceil \frac{p}{2} \rceil.$$

Corollary 3.3.3. *Let $p \geq 1, m_i \geq 1$ and $A = I(CP_{(m_1,\ldots,m_p)})$, then we have*

$$\operatorname{reg}(\mathcal{X}/A^t) = 2t + \lceil \frac{p}{2} \rceil - 2.$$

Proof: From Theorem 3.1.11 we have

$$\operatorname{reg}(A^t) = 2t + \operatorname{indmat}(G) - 1.$$

From Corollary 3.3.2

$$\operatorname{indmat}(G) = \lceil \frac{p}{2} \rceil.$$

So

$$\operatorname{reg}(A^t) = 2t + \lceil \frac{p}{2} \rceil - 1.$$

Hence

$$\operatorname{reg}(\mathcal{X}/A^t) = 2t + \lceil \frac{p}{2} \rceil - 2.$$

Theorem 3.3.4. *Let G be a forest with c connected components $CP_(m_{11}, m_{21}, \ldots, m_{p_1 1})$, $CP_(m_{12}, m_{22}, \ldots, m_{p_2 2}), \ldots, CP_(m_{1c}, m_{2c}, \ldots, m_{p_c c})$ and let $p_i \geq 1$, where $1 \leq i \leq p$ represent the number of vertices on the path of each $CP_(m_{1i}, m_{2i}, \ldots, m_{p_i i})$ respectively. Let $p = max\{p_i\}$, $m_i \geq 1$ and $A = A(G)$. Then we have*

$$\operatorname{depth}(\mathcal{X}/A^t), \operatorname{sdepth} \mathcal{X}/A^t) \geq \max\{\lceil \frac{2(p-t)+1}{2} \rceil + c - 1, c\}$$

for all $t \geq 1$.

Proof : For $t = 1$ and $c = 1$, the result is derived from Theorem 3.3.1. For $t = 1$ and $c \geq 2$, it follows from [26, Lemma 6.2.7]. Thus for $t = 1$ and any value of c the result holds. Let $t \geq 2$. If $p = 1$, then from Theorem 3.1.6 we have that $\operatorname{depth}(\mathcal{X}/A^t) \geq c$. Now let $p = 2$ and without loss of generality say $p = p_1$, then consider the following short exact sequence

$$0 \longrightarrow \mathcal{X}/(A^t : x_{21}) \xrightarrow{.x_{21}} \mathcal{X}/A^t \longrightarrow \mathcal{X}/(A^t, x_{21}) \longrightarrow 0.$$

We have

$$\mathcal{X}/(A^t, x_{21}) \cong V(G) - N(x_{21})/(J^t, x_{21}) \otimes K[M_{21}].$$

where J denotes the edge ideal formed by deleting the vertex x_{21}.

So $\operatorname{depth}(\mathcal{X}/(A^t, x_{21})) \geq \max\{\lceil \frac{2(1-t)+1}{2} \rceil + c - 1, c\} + m_{21} \geq \max\{\lceil \frac{2-2t+1}{2} \rceil + c - 1, c\} + 1.$

$$\geq \max\{\lceil \tfrac{3-2t}{2} \rceil + c - 1, c\} + 1$$
$$\geq \max\{\lceil \tfrac{2(p-t)+1}{2} \rceil + c - 1, c\}.$$

For convenience label the vertices of the set $M_{p_1 1}$ by $l_{11}, l_{21}, \ldots, l_{j1}$ and we also let $\mathcal{X}_{j1} = K[A]/(l_{11}, l_{21}, \ldots, l_{j1})$ and $I_{j1} = I \cap \mathcal{X}_{j1}$, where $1 \leq j1 \leq m_{p_1}$.

Now consider the family of short exact sequences as

$$0 \longrightarrow \mathcal{X}_0/(A_0^t : x_{21}l_{11}) \xrightarrow{.l_{11}} \mathcal{X}_0/(A_0^t : x_{21}) \longrightarrow \mathcal{X}_0/((A_0^t : x_{21}), l_{11}) \longrightarrow 0,$$

$$0 \longrightarrow \mathcal{X}_1/(A_1^t : x_{21}l_{21}) \xrightarrow{.l_{21}} \mathcal{X}_1/(A_1^t : x_{21}) \longrightarrow \mathcal{X}_1/((A_1^t : x_{21}), l_{21}) \longrightarrow 0,$$

$$\vdots \qquad\qquad \vdots \qquad\qquad \vdots$$

$$0 \longrightarrow \mathcal{X}_{j1-1}/(A_{j1-1}^t : x_{21}l_{j1}) \xrightarrow{.l_{j1}} \mathcal{X}_{j1-1}/(A_{j1-1}^t : x_{21}) \longrightarrow \mathcal{X}_{j1-1}/((A_{j1-1}^t : x_{21}), l_{j1}) \longrightarrow 0.$$

By Lemma 3.1.3 we get

$$\operatorname{depth}(\mathcal{X}_{j1-1}/(A_{j1-1}^t : x_{21}l_{j1})) = \operatorname{depth}(\mathcal{X}_{j1-1}/(A_{j1-1}^{t-1})).$$

Using induction we get

$$
\begin{aligned}
\operatorname{depth}(\mathcal{X}_{j1-1}/(A^t_{j1-1}:x_{21}l_{j1})) &\geq \max\{\lceil \tfrac{2(2-(t-1))+1}{2}\rceil + c-1, c\}\\
&= \max\{\lceil \tfrac{2(3-t)+1}{2}\rceil + c-1, c\}\\
&= \max\{\lceil \tfrac{6-2t+1}{2}\rceil + c-1, c\}\\
&= \max\{\lceil \tfrac{7-2t}{2}\rceil + c-1, c\}\\
&\geq \max\{\lceil \tfrac{2(p-t)+1}{2}\rceil + c-1, c\}.
\end{aligned}
$$

Since using Lemma 3.1.4 we have

$$\mathcal{X}_{j1-1}/((A^t_{j1-1}:x_{21}), l_{j1}) \cong \mathcal{X}_{j1}/((A^t_{j1})):x_{21}).$$

So we consider another short exact sequence as

$$0 \longrightarrow \mathcal{X}_{j1}/((A^t_{j1}:x_{21}x_{11}) \xrightarrow{.x_{11}} \mathcal{X}_{j1}/((A^t_{j1}:x_{21}) \longrightarrow \mathcal{X}_{j1}/((A^t_{j1}:x_{21}, x_{11}) \longrightarrow 0.$$

By Lemma 3.1.3 we have

$$\operatorname{depth}(\mathcal{X}_{j1}/(A^t_{j1}:x_{21}x_{11})) = \operatorname{depth}(\mathcal{X}_{j1}/A^{t-1}_{j1})$$

$$
\begin{aligned}
&\geq \max\{\lceil \tfrac{2(1-(t-1))+1}{2}\rceil + c-1, c\}\\
&= \max\{\lceil \tfrac{2(2-t)+1}{2}\rceil + c-1, c\}\\
&= \max\{\lceil \tfrac{5-2t}{2}\rceil + c-1, c\}.\\
&\geq \max\{\lceil \tfrac{2(p-t)+1}{2}\rceil + c-1, c\}
\end{aligned}
$$

and

$$\mathcal{X}_{j1}/((A^t_{j1}:x_{21}), x_{11}) \cong V(G) - N(x_{11}, x_{21})/(J^t:x_{21}) \otimes K[x_{21}, M_{11}].$$

So, $\operatorname{depth}(\mathcal{X}_{j1}/((A^t_{j1}:x_{21}), x_{11})) \geq \max\{\lceil \tfrac{2(0-t)+1}{2}\rceil + c-1-1, c-1\} + 1 + m_{11}$

$$
\begin{aligned}
&\geq \max\{\lceil \tfrac{1-2t}{2}\rceil + c-2, c-1\} + 2.\\
&\geq \max\{\lceil \tfrac{2(p-t)+1}{2}\rceil + c-1, c\}
\end{aligned}
$$

and

$$\operatorname{depth}(\mathcal{X}_{j1}/((A^t_{j1}:x_{21})) \geq \max\{\lceil \frac{2(p-t)+1}{2}\rceil + c-1, c\}.$$

Hence

$$\operatorname{depth}(\mathcal{X}_{j1}-1/(A_{j1}-1^t:x_{21}), l_{j1}) \geq \max\{\lceil \frac{2(p-t)+1}{2}\rceil + c-1, c\},$$

and so

$$\operatorname{depth}(\mathcal{X}/A^t) \geq \max\{\lceil \frac{2(p-t)+1}{2} \rceil + c - 1, c\}.$$

Let $p \geq 3$, then consider the following short exact sequence

$$0 \longrightarrow \mathcal{X}/(A^t : x_{p_11}) \xrightarrow{.x_{p_11}} \mathcal{X}/A^t \longrightarrow \mathcal{X}/(A^t, x_{p_11}) \longrightarrow 0.$$

We have

$$\mathcal{X}/(A^t, x_{p_11}) \cong V(G) - N(x_{p_11})/(J^t, x_{p_11}) \otimes K[M_{p_11}],$$

where J denotes the edge ideal formed by deleting the vertex x_{p_11}.

So

$$\begin{aligned} \operatorname{depth}(\mathcal{X}/(A^t, x_{p_11})) &\geq \max\{\lceil \frac{2(p-1-t)+1}{2} \rceil + c - 1, c\} + m_{p_11} \\ &\geq \max\{\lceil \tfrac{2p-2-2t+1}{2} \rceil + c - 1, c\} + 1 \\ &= \max\{\lceil \tfrac{2p-2t-1}{2} \rceil + c - 1, c\} + 1 \\ &\geq \max\{\lceil \tfrac{2(p-t)+1}{2} \rceil + c - 1, c\}. \end{aligned}$$

Now consider the family of short exact sequences as

$$0 \longrightarrow \mathcal{X}_0/(A_0^t : x_{p_11}l_{11}) \xrightarrow{.l_{11}} \mathcal{X}_0/(A_0^t : x_{p_11}) \longrightarrow \mathcal{X}_0/((A_0^t : x_{p_11}), l_{11}) \longrightarrow 0,$$

$$0 \longrightarrow \mathcal{X}_1/(A_1^t : x_{p_11}l_{21}) \xrightarrow{.l_{21}} \mathcal{X}_1/(A_1^t : x_{p_11}) \longrightarrow \mathcal{X}_1/((A_1^t : x_{p_11}), l_{21}) \longrightarrow 0,$$

$$\vdots \qquad\qquad \vdots \qquad\qquad \vdots$$

$$0 \longrightarrow \mathcal{X}_{j1-1}/(A_{j1-1}^t : x_{p_11}l_{j1}) \xrightarrow{.l_{j1}} \mathcal{X}_{j1-1}/(A_{j1-1}^t : x_{p_11}) \longrightarrow \mathcal{X}_{j1-1}/((A_{j1-1}^t : x_{p_11}), l_{j1} \longrightarrow 0.$$

By Lemma 3.1.3 we get

$$\operatorname{depth}(\mathcal{X}_{j1-1}/(A_{j1-1}^t : x_{p_11}l_{j1}) = \operatorname{depth}(\mathcal{X}_{j1-1}/(A_{j1-1}^{t-1}))$$

and using induction we get

$$\begin{aligned}
\operatorname{depth}(\mathcal{X}_{j1-1}/(A^t_{j1-1}:x_{p_11}l_{j1}) &\geq \max\{\lceil\frac{2(p-(t-1))+1}{2}\rceil+c-1,c\}\\
&= \max\{\lceil\tfrac{2p-2t+2+1}{2}\rceil+c-1,c\}\\
&= \max\{\lceil\tfrac{2(p-t)+3}{2}\rceil+c-1,c\}\\
&\geq \max\{\lceil\tfrac{2(p-t)+1}{2}\rceil+c-1,c\}.
\end{aligned}$$

Since using Lemma 3.1.4 we have

$$\mathcal{X}_{j1-1}/((A^t_{j1-1}:x_{p_11}),l_{j1}\cong\mathcal{X}_{j1}/((A^t_{j1}:x_{p_11}).$$

So, we consider another short exact sequence as

$$0\longrightarrow\mathcal{X}_{j1}/((A^t_{j1}:x_{p_11}x_{p_11-1})\xrightarrow{.x_{p_11-1}}\mathcal{X}_{j1}/((A^t_{j1}:x_{p_11})\longrightarrow\mathcal{X}_{j1}/((A^t_{j1}:x_{p_11}),x_{p_11-1})\longrightarrow 0.$$

By Lemma 3.1.3

$$\begin{aligned}
\operatorname{depth}(\mathcal{X}_{j1}/(A^t_{j1}:x_{p_11}x_{p_11-1})) &= \operatorname{depth}(\mathcal{X}_{j1}/A^{t-1}_{j1})\\
&\geq \max\{\lceil\tfrac{2(p-1-(t-1))+1}{2}\rceil+c-1,c\}\\
&= \lceil\tfrac{2(p-t)+1}{2}\rceil+c-1,c\}
\end{aligned}$$

and

$$\mathcal{X}_{j1}/((A^t_{j1}:x_{p_11}),x_{p_11-1})\cong V(G)-N(x_{p_11-1},x_{p_11})/(J^t:x_{p_11})\otimes K[x_{p_11},M_{p_11-1}].$$

So

$$\begin{aligned}
\operatorname{depth}(\mathcal{X}_{j1}/((A^t_{j1}:x_{p_11}),x_{p_1-1}) &\geq \max\{\lceil\frac{2(p-2-t)+1}{2}\rceil+c-1,c\}+1+m_{p_11-1}\\
&\geq \max\{\lceil\tfrac{2(p-t)-3}{2}\rceil+c-1,c\}+2\\
&\geq \max\{\lceil\tfrac{2(p-t)+1}{2}\rceil+c-1,c\}.
\end{aligned}$$

and

$$\operatorname{depth}(\mathcal{X}_j/((A^t_j:x_{p_11}))\geq\max\{\lceil\frac{2(p-t)+1}{2}\rceil+c-1,c\}.$$

Hence

$$\operatorname{depth}(\mathcal{X}_{j1-1}/(A^t_{j1-1} : x_{p_1 1}), l_{j1}) \geq \max\{\lceil \frac{2(p-t)+1}{2} \rceil + c - 1, c\}$$

and so

$$\operatorname{depth}(\mathcal{X}/A^t) \geq \max\{\lceil \frac{2(p-t)+1}{2} \rceil + c - 1, c\}.$$

Proof for sdepth is same using Lemma 2.2.2 instead of Lemma 2.1.4.

Corollary 3.3.5. *Let G be a forest with c connected components $CP_(m_{11}, m_{21}, \ldots, m_{p_1 1})$, $CP_(m_{12}, m_{22}, \ldots, m_{p_2 2}), \ldots, CP_(m_{1c}, m_{2c}, \ldots, m_{p_c c})$ and let $n_i \geq 1$, where $1 \leq i \leq p$ represent the number of vertices on the path of each $CP_(m_{1i}, m_{2i}, \ldots, m_{p_i i})$ respectively. Let $p = max\{p_i\}$, $m_i \geq 1$ and $A = A(G)$. Then we have*

$$\operatorname{pdim}(\mathcal{X}/A) \geq p + \sum_{i=1}^{p} m_i - \max\{\lceil \frac{2(p-t)+1}{2} \rceil + c - 1, c\}.$$

Proof: As $\operatorname{depth}(\mathcal{X}) = p + \sum_{i=1}^{p} m_i$, so using Lemma 2.4.1 and Theorem 3.3.4 we get,

$$\operatorname{pdim}(\mathcal{X}/A) \geq p + \sum_{i=1}^{p} m_i - \max\{\lceil \frac{2(p-t)+1}{2} \rceil + c - 1, c\}.$$

Corollary 3.3.6. Let G be a forest having c connected components $CP_(m_{11}, m_{21}, \ldots, m_{p_1 1})$, $CP_(m_{12}, m_{22}, \ldots, m_{p_2 2}), \ldots, CP_(m_{1c}, m_{2c}, \ldots, m_{cc})$ and let $n_i \geq 1$, where $1 \leq i \leq p$ represent the number of vertices on the path of each $CP_(m_{1i}, m_{2i}, \ldots, m_{p_i i})$ respectively. Let $p = p_1 + \cdots + p_i, m_i \geq 1$ and $A = A(G)$. Then for $t \geq 1$ we have

$$\operatorname{reg}(\mathcal{X}/A^t) = 2t + \lceil \frac{p}{2} \rceil - 2.$$

Proof: From Theorem 3.1.11 we have

$$\operatorname{reg}(A^t) = 2t + \lceil \frac{p}{2} \rceil - 1.$$

Hence using this and Remark 3.1.13

$$\operatorname{reg}(\mathcal{X}/A^t) = 2t + \lceil \frac{p}{2} \rceil - 2.$$

Chapter 4

Conclusion

In this dissertation different algebraic invariants namely the Stanley depth, depth, regularity and projective dimension, within the framework of quotient rings associated to the powers of edge ideals of caterpillar trees and their forest arrangements are studied. The bounds calculated for the depth here are stronger than the ones given in [20] for trees and the ones calculated for Stanley depth are stronger than the ones given for forests in [25]. The caterpillar considered here is the one for which $m_i \geq 1$ that is there is no vertex on the path P_n of the caterpillar on which there are no pendant vertices. In the future, focus could be directed towards caterpillars for which there exists at least one vertex on P_n that has no pendant vertices attached to it, thus completely covering the entire class of caterpillars.

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