Anisotropic Charge Matter with Non-Linear Equations of State

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Dedication

This thesis is dedicated to

my beloved parents

Sheikh Manzoor Ahmad and **Mah Anjum**

my supportive brother

Mateen Manzoor

and my supervisor

Prof. Azad A. Siddiqui

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Abstract

In this thesis, we find solutions to Einstein field equations using nonlinear equations of state. The spacetime is considered to be static and spherically symmetric. The solutions are found by extending Chaplygin equation of state together with a choice of gravitational potentials and electric field intensities in the presence of anisotropic pressure. Numerous physical features of the solutions have been explored. The metric potentials, densities, and pressures have no singularities and satisfy the required physical conditions. The stability requirements are satisfied and the balance of forces (F_a, F_h, F_g, F_e) in our models have been examined by analyzing the Tolman–Oppenheimer–Volkoff (TOV) equation. The anisotropic factor is zero at the center and increases afterward. We also study the behavior of mass, compactness factors, and redshifts which are found to be in acceptable ranges.

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Chapter 1

Introduction

1.1 Exact Solutions

The exact solutions of the Einstein-Maxwell field equations (EMFEs) for the static and spherically symmetric geometry are of great interest. Finding exact solutions to the EMFEs is challenging due to their non-linearity and complexity. In fact, only a limited number of exact solutions are known. In Relativistic Astrophysics solutions of the EMFEs which satisfy some physical criterion are of real interest because these can be used to model compact objects like stars, neutron stars, quark stars, etc. This helps us to understand the highly dense distribution of matter in the interior of relativistic compact objects. Although many solutions have been obtained, only a few exact solutions of the EMFEs fulfill the physical criterion to be considered as models of relativistic compact objects. Often an equation of state (EoS), which relates energy density to pressure, i.e. $p = p(\rho)$, is assumed to determine solutions. Many other exact solutions to the Einstein field equations have been obtained by various approaches using different kinds of equations of state. Tooba Feroze [49] has discovered two solutions for the EMFEs by using general linear EoS which is $p_r = \alpha\rho - \gamma$ in the energy-momentum conservation equation. Prasad and Kumar [38] present a class of relativistic and well-behaved solutions to EMFEs for anisotropic matter distribution using linear EoS. Feroze and Siddiqui [52, 53] obtained the solution with quadratic EoS, $p_r = \alpha\rho^2 + \beta\rho - \gamma$ where α, β and γ are arbitrary constants for charged anisotropic matter by analyzing the characteristics of a compact relativistic object. Malaver and Iyer [7] develop a model for the charged anisotropic distribution with linear $p_r = \gamma\rho$ and quadratic $p_r = \gamma\rho^2$ EoS. Sunzu [28] generates a new solution for a neutral anisotropic star by utilizing a quadratic EoS. Thirukkanesh et al. [35] investigate the stability and enhancement of the physical characteristics of compact, relativistic objects that follow a quadratic EoS. Notably, Sifiso et al. [36] provided a quadratic EoS to describe the matter distribution in spherical coordinates. Paul et al. [25] and Thirukkanesh et al. [26] discover new models of a compact star using color-flavor-locked EoS, $p_r = g\rho + h\rho^{1/2} - f$ for anisotropic matter in spherical symmetric spacetime. Malaver and Kasmaei [8] also presented the work on the polytropic equation of state which is $p_r = \beta\rho^\Gamma$ where $\Gamma = 1 + \dfrac{1}{\eta}$ and η is the polytropic index. Nasima and Azam [37] worked on anisotropic-charged physical models with a generalized polytropic EoS, investigating various scenarios involving linear, quadratic, and polytropic equations. Malaver worked on a modified Van Der Waals EoS, $p_r = g\rho^2 + \dfrac{f\rho}{1+h\rho}$ and found a new class of exact solution with charged [18] and uncharged [20] anisotropic matter. Errehymy et al. [34] investigate the non-linear Van Dar Waal EoS and introduce new models of compact stars within the framework of general relativity, characterized by a new class of exact solutions to the field equations. A number of solutions to EMFEs have been found using the Chaplygin EoS, $p_r = A\rho + \dfrac{B}{\rho}$ by Malaver [27,30], Sunzu [28], Bhar [39–41], Kumar [51] and Ortiz [50]. To construct such solutions, we need to understand some following basic principles that serve as the building blocks for their formation.

1.2 Tensors

Tensors are mathematical objects, described entirely in their respective coordinate systems by changing properties. Tensor formalization is studied in relativity to gain deeper geometric awareness because it explains how quantities transform under coordinate transformations because they are valid in all systems. Tensor is fundamentally a generalization of vectors and dual vectors. If T is a general tensor with the components having number of upper indices equal to 'r' and number of lower indices equal to 's', then in component notation T is written as

$$T = T^{i_1\cdots i_r}{}_{j_1\cdots j_s}\hat{e}_{i_1}\otimes\cdots\otimes\hat{e}_{i_r}\otimes\hat{v}^{j_1}\otimes\cdots\otimes\hat{v}^{j_s}, \tag{1.1}$$

where

$$\hat{e}_{i_1}\otimes\cdots\otimes\hat{e}_{i_r}\otimes\hat{v}^{j_1}\otimes\cdots\otimes\hat{v}^{j_s}, \tag{1.2}$$

is the basis of the (r,s) tensor, provided by the tensor product of vectors and dual vectors. Similarly, one can define components of contravariant and covariant tensors having all upper and lower indices respectively. The transformation law for mixed tensor components is defined as

$$T^{i'_1\cdots i'_r}{}_{j'_1\cdots j'_s} = T^{i_1\cdots i_r}{}_{j_1\cdots j_s}\frac{\partial x^{i'_1}}{\partial x^{i_1}}\cdots\frac{\partial x^{i'_r}}{\partial x^{i_r}}\frac{\partial x^{j_1}}{\partial x^{j'_1}}\cdots\frac{\partial x^{j_s}}{\partial x^{j'_s}}. \tag{1.3}$$

Basic operations such as addition and subtraction are valid for tensors but have the same rank and are of the same type, while the product of two tensors given by its components whose upper and lower indices consist of all upper and lower indices of the original tensor components is valid. It is expressed as follows

$$T^{\alpha\beta}{}_{\gamma} = U^{\alpha\beta}{}_{\gamma} \pm V^{\alpha\beta}{}_{\gamma}, \tag{1.4}$$

$$T^{\alpha\beta\gamma}{}_{\eta\xi} = U^{\alpha\beta}{}_{\eta}V^{\gamma}{}_{\xi}. \tag{1.5}$$

Contraction is the summation of one upper and one lower index (of the same type) otherwise tensor would not hold the transformation laws.
Rank n tensor $\xrightarrow{1\ Contrac}$ Rank $(n-2)$ tensor $\xrightarrow{1\ Contrac}$Rank $(n-4)$ tensor $\cdots$
As an example consider

$$S^{\alpha\beta}{}_{\gamma} = T^{\alpha\eta\beta}{}_{\gamma\eta}. \tag{1.6}$$

Given a tensor, its symmetric and skew-symmetric parts are defined as follows

$$T_{(i_1 \cdots i_n)} = \frac{1}{n!}\left(T_{i_1 \cdots i_n} + sum\ over\ permutation\ of\ indices\ i_1 \cdots i_n\right), \tag{1.7}$$

$$T_{[i_1 \cdots i_n]} = \frac{1}{n!}\left(T_{i_1 \cdots i_n} + alternating\ sum\ over\ permutation\ of\ indices\ i_1 \cdots i_n\right). \tag{1.8}$$

1.2.1 The Metric Tenor

In general relativity, the metric tensor is the most basic and important tensor to investigate space-time geometry. It is also broadly used as a tool for lowering and raising the indices of vectors and tensors, that is going from covariant to contravariant vectors and tensors and vice versa. The metric tensor, also known as the metric, is a symmetric tensor of rank 2 [6]. The components of the covariant metric tensor is presented in the form of matrix below where α, β = 0, 1, 2, 3.

$$g_{\alpha\beta} = \begin{pmatrix} g_{00} & g_{01} & g_{02} & g_{03} \\ g_{10} & g_{11} & g_{12} & g_{13} \\ g_{20} & g_{21} & g_{22} & g_{23} \\ g_{30} & g_{31} & g_{32} & g_{33} \end{pmatrix}. \tag{1.9}$$

The contravariant metric tensor, $g^{\alpha\beta}$, is the inverse of the above matrix. In tensor algebra, the product of a tensor and its inverse gives

$$g_{\alpha\gamma}g^{\gamma\beta} = \delta^{\beta}_{\alpha} = g^{\beta\gamma}g_{\gamma\alpha}, \tag{1.10}$$

where δ^{β}_{α} is known as the Kronecker Delta (named after Leopold Kronecker) or identity the matrix that is defined by

$$\delta^{\beta}_{\alpha} = \begin{cases} 1 & if \quad \alpha = \beta, \\ 0 & if \quad \alpha \neq \beta. \end{cases}$$

The determinant of the matrix given in eq. (1.10) is denoted by 'g' i.e.

$$g = det g_{\alpha\beta} \tag{1.11}$$

The line element in spherical coordinates, (t, r, θ, ϕ), is mathematically defined as

$$ds^2 = g_{\alpha\beta}dx^{\alpha}dx^{\beta}. \tag{1.12}$$

1.2.2 Curvature Tensor

The curvature tensor is a mathematical representation of the intrinsic geometric curvature of spacetime due to the presence of matter. It is also called the Riemann curvature tensor. The symbols used to write the Riemann curvature tensor are called the Christoffel symbols (named after Elwin Bruno Christoffel). These symbols allow us to measure distances on surfaces or space-times that are equipped with a metric. Mathematically, they are defined as

$$\Gamma^{\alpha}_{\beta\gamma} = \frac{1}{2} g^{\alpha\eta} \left(g_{\eta\beta,\gamma} + g_{\eta\gamma,\beta} - g_{\beta\gamma,\eta} \right). \tag{1.13}$$

The Riemann curvature tensor is then defined as

$$R^{\alpha}_{\beta\gamma\eta} = \Gamma^{\alpha}_{\beta\eta,\gamma} - \Gamma^{\alpha}_{\beta\gamma,\eta} + \Gamma^{\alpha}_{\gamma\xi}\Gamma^{\xi}_{\beta\eta} - \Gamma^{\alpha}_{\eta\xi}\Gamma^{\xi}_{\beta\gamma}, \tag{1.14}$$

where ‘,’ represents the partial derivative. $R^{\alpha}_{\beta\gamma\eta}$ can be transformed into covariant tensor form by using the transformation

$$R_{\alpha\beta\gamma\eta} = g_{\alpha\xi} R^{\xi}_{\beta\gamma\eta}, \tag{1.15}$$

where $R^{\alpha}_{\beta\gamma\eta} \neq R_{\alpha\beta\gamma\eta}$. The covariant curvature tensor in explicit form, by performing some algebra, can be written as

$$R_{\alpha\beta\gamma\eta} = \frac{1}{2} \left(g_{\eta\alpha,\beta\gamma} + g_{\beta\gamma,\eta\alpha} - g_{\beta\eta,\alpha\gamma} - g_{\gamma\alpha,\beta\eta} \right) + g_{\xi\zeta} \left(\Gamma^{\zeta}_{\eta\alpha}\Gamma^{\xi}_{\gamma\beta} - \Gamma^{\zeta}_{\gamma\alpha}\Gamma^{\xi}_{\eta\beta} \right). \tag{1.16}$$

Some properties of the Riemann curvature tensors are:

1. It is skew-symmetric either the order of the first two indices is swapped or of the last two.

$$R_{\alpha\beta\gamma\eta} = -R_{\beta\alpha\gamma\eta}, \qquad R_{\alpha\beta\gamma\eta} = -R_{\alpha\beta\eta\gamma}. \tag{1.17}$$

2. It satisfies Bianchi’s identity of the first and second kind given as

$$R_{\alpha[\beta\gamma\eta]} = R_{\alpha\beta\gamma\eta} + R_{\alpha\eta\beta\gamma} + R_{\alpha\gamma\eta\beta} = 0, \tag{1.18}$$

$$R^{\alpha}_{p[\beta\gamma;\eta]} = R^{\alpha}_{p\beta\gamma;\eta} + R^{\alpha}_{p\eta\beta;\gamma} + R^{\alpha}_{p\gamma\eta;\beta} = 0. \tag{1.19}$$

   Here ’;’ is the covariant derivative that is the generalization of the partial derivative for curved spacetime. For example, the components of the covariant derivative of a tensor of rank (1, 1) are defined as

$$A^{\alpha}_{\beta;\gamma} = A^{\alpha}_{\beta,\gamma} + \Gamma^{\alpha}_{\beta\eta} A^{\eta}_{\gamma} - \Gamma^{\eta}_{\gamma\beta} A^{\alpha}_{\eta}. \tag{1.20}$$

3. It is symmetric if the first two pair of indices are swapped with the last two

$$R_{\alpha\beta\gamma\eta} = R_{\gamma\eta\alpha\beta}. \tag{1.21}$$

By contracting the components of the Riemann tensor one can construct the Ricci tensor

$$R_{\alpha\beta} = R^{\gamma}_{\alpha\gamma\beta}. \tag{1.22}$$

and the trace of the Ricci tensor is the Ricci scalar (or curvature scalar) as given below

$$R = g^{\alpha\beta} R_{\alpha\beta}. \tag{1.23}$$

Ricci tensor is symmetric and an interesting feature of the Ricci scalar is that it determines the nature of singularity as it is invariant under coordinate transformations. There are two types of singularities coordinate and essential, one arises with a bad choice of coordinates and is removable, the other occurs due to a problem in geometry that can not be removed. By examining the invariant quantities given below, one can conclude that if curvature invariants are finite then there is coordinate singularity otherwise essential.

$$R_1 = R, \tag{1.24}$$

$$R_2 = R^{\alpha\beta}{}_{\gamma\eta} R^{\gamma\eta}{}_{\alpha\beta}, \tag{1.25}$$

$$R_3 = R^{\alpha\beta}{}_{\gamma\eta} R^{\gamma\eta}{}_{\xi\zeta} R^{\xi\zeta}{}_{\alpha\beta}, \tag{1.26}$$

$$R_4 = R^{\alpha\beta}{}_{\gamma\eta} R^{\gamma\eta}{}_{\xi\zeta} R^{\xi\zeta}{}_{\omega\rho} R^{\omega\rho}{}_{\alpha\beta}. \tag{1.27}$$

Now as the curvature tensor is discussed in detail then the principle of minimal gravitational coupling implicitly used by Einstein is quite easier to understand, which is stated as "No term explicitly containing the curvature tensor should be added in making the transition from special to general theory of relativity".

1.2.3 The Einstein Tensor

The Einstein tensor is defined over pseudo-Riemannian manifolds and is a tensor of order 2. The Einstein tensor is written as [48]

$$G_{\alpha\beta} = R_{\alpha\beta} - \frac{1}{2} R g_{\alpha\beta}, \tag{1.28}$$

The Einstein tensor is symmetric and divergence-free i.e

$$G_{\alpha\beta} = G_{\beta\alpha}, \tag{1.29}$$

and

$$G^{\alpha\beta}{}_{;\gamma} = 0. \tag{1.30}$$

1.2.4 Maxwell Tensor

The Maxwell tensor is also known as the electromagnetic field tensor. For its construction define the four-vector potential, A_α, and skew-symmetric tensor, $F_{\alpha\beta}$, as

$$A_\alpha = (-\phi, \mathbf{A}), \tag{1.31}$$

$$F_{\alpha\beta} = \partial_\alpha A_\beta - \partial_\beta A_\alpha. \tag{1.32}$$

Here, ϕ is the scalar potential and $\mathbf{A}$ is 3-vector potential. Electric and magnetic fields in terms of ϕ and $\mathbf{A}$ become

$$\mathbf{E} = -\boldsymbol{\nabla}\phi - \partial_t \mathbf{A}, \tag{1.33}$$

$$\mathbf{B} = \boldsymbol{\nabla} \times \mathbf{A}. \tag{1.34}$$

From equations (1.32)-(1.34), we get $F_{0i} = E_i$ and $F_{ij} = \epsilon_{ijk}B^k$ where $i, j, k = 1, 2, 3$ and ϵ_{ijk} is the Levi-Civita tensor defined as

$$\epsilon_{ijk} = \begin{cases} +1 & \text{if } ijk \text{ is an even permutaion of 123,} \\ \text{-1} & \text{if } ijk \text{ is an odd permutaion of 123,} \\ 0 & \text{otherwise.} \end{cases} \tag{1.35}$$

Further simplification yields the covariant form of electromagnetic field tensor as

$$F_{\alpha\beta} = \begin{pmatrix} 0 & E_1 & E_2 & E_3 \\ -E_1 & 0 & B^3 & -B^2 \\ -E_2 & -B^3 & 0 & B^1 \\ -E_3 & B^2 & -B^1 & 0 \end{pmatrix}, \tag{1.36}$$

and its contravarient form is represented as $F^{\alpha\beta} = g^{\alpha\gamma}g^{\beta\eta}F_{\gamma\eta}$.

1.2.5 Stress-Energy Tensor

The stress-energy tensor also called the energy-momentum tensor or the stress-energy-momentum tensor is a 2nd-rank symmetric tensor i.e. $T_{\alpha\beta} = T_{\beta\alpha}$. It is a physical quantity that demonstrates the distribution of matter at each region of spacetime. As the stress-energy tensor is of order two, so in 4-dimensions its components can be displayed in a 4×4 matrix form

$$T^{\alpha\beta} = \begin{pmatrix} T^{00} & T^{01} & T^{02} & T^{03} \\ T^{10} & T^{11} & T^{12} & T^{13} \\ T^{20} & T^{21} & T^{22} & T^{23} \\ T^{30} & T^{31} & T^{32} & T^{33} \end{pmatrix}. \tag{1.37}$$

The components of the above matrix demonstrate the following physical quantities:

- Energy density of the matter is denoted by T^{00},
- Energy flux $\times c^{-1}$ in the i^{th} direction is denoted by T^{0i},
- Momentum density $\times$c in the i^{th} direction is denoted by T^{i0},
- The flow of i^{th} component of momentum per unit area in the j^{th} direction is represented by T^{ij}.

Here i and j are the indices that represent the spatial coordinates. The stress-energy tensor in the covariant form is given by

$$T_{\alpha\beta} = T^{\gamma\eta} g_{\gamma\alpha} g_{\eta\beta}, \tag{1.38}$$

in the mixed form as

$$T^{\alpha}_{\beta} = T^{\alpha\gamma} g_{\gamma\beta}. \tag{1.39}$$

The stress-energy tensor is also divergence-free.

$$T^{\alpha\beta}_{;\gamma} = 0. \tag{1.40}$$

1.3 An Introduction to General Relativity

The special theory of relativity was Einstein's first step towards relativizing physics. Only non-accelerated systems are addressed by the special theory of relativity. After a ten-years effort to incorporate acceleration into the theory, Einstein presented his general theory of relativity in 1915. General relativity (GR) is concerned with one of the fundamental forces in the universe which is gravity. As gravity defines macroscopic behavior so GR also explains large-scale physical events. This theory describes and explains gravity as a manifestation of the curvature of spacetime, not a force as it was once thought in Newtonian physics. Geometry defines gravity. Gravity is the warping of spacetime induced by the existence of matter and energy, although the pathways followed by matter and energy in spacetime are determined by spacetime structure. In short, GR deals with these two important aspects:

- Effects of curvature on motion of matter.
- Effects of matter on the spacetime curvature.

According to John A. Wheeler, "Matter tells spacetime how to curve and spacetime tells matter how to move."

1.3.1 The Einstein Field Equations

The general theory of Relativity explains the interaction of gravity as a consequence of spacetime being curved by energy and matter. Einstein derived the Einstein field equations (EFEs) in 1915, which are in fact a set of tensor equations. EFEs are nonlinear partial differential equations (PDEs) that relate the energy and momentum present within a spacetime with the curvature of the spacetime. The Einstein field equations in the components form are [48]

$$G_{\alpha\beta} = kT_{\alpha\beta}, \tag{1.41}$$

Now by including the cosmological constant "Λ" and then Einstein field equations can be written as

$$G_{\alpha\beta} + \Lambda g_{\alpha\beta} = kT_{\alpha\beta}. \tag{1.42}$$

The EFEs are used to find out the geometry of the spacetime due to the energy, mass and linear momentum present in a given spacetime. These equations describe how spacetime is curved in the presence of matter and how the motion of the matter is influenced by the curvature of spacetime. The EFEs are tensor equations that give a relation between different 4×4 symmetric tensors. Each symmetric tensor has 10 independent components. Answers to many physical problems can be found by the solution of the field equations that explain GR such as planetary dynamics, the evolution of Universe, stars birth and death, and black holes.

1.4 Compact Ojects

Earlier in 1798, Laplace introduced the theoretical concept of the existence of a massive object from whose gravitational field no particle 'even light' can escape, but it did not receive much attention due to some strange properties. However, gravity has been in the spotlight due to its unique properties. Astronomers believe that stars form through the gravitational collapse of gaseous mass. In the 18th century, it was known as Laplace's nebular hypothesis which was overruled lately in the 1970s by solar nebular disk model, mainly due to its lack of information about the dissemination of angular momentum in between planets and the sun. Star formation is caused by nebulae, which means that there is a dense disk-like structure of interstellar dust and gas in the form of molecular clouds made of hydrogen (which is 75 %) and helium (which is 23%). This cloud starts spinning due to the conservation of momentum from the movement of particles, and this sharp rotation flattens the cloud into a protoplanetary disk. The protostar formation occurs as regions of increased gravity, allowing gas and dust to condense, and as these regions become larger, they collapse under the increased gravitational field, creating an increase in temperature and gradually becoming a protostar. Eventually, the star would remain that way for thousands of years until the fusion process ignites the compressed hydrogen atoms to become helium, then carbon until the iron is formed. The fusion process that creates iron does not produce any energy. Because in the process of nuclear fusion, if the gravitational force of inward pressure is greater than the pressure pushed outwards, the core of the star will collapse, which is a process dominated by gravity. From this point, stars begin to form, often referred to as “compact objects.” The kinds of compact objects are:

- White dwarfs,
- Neutron stars,
- Black holes.

White Dwarfs

The collapse of a star’s core into a white dwarf takes tens of thousands of years. Smaller stars, up to 1.4 solar masses, implode under the infliction of their own gravitational pull and become white dwarfs as a result of a lack of nuclear fuel. White dwarfs are very dense objects with masses equivalent to that of the sun. It has a volume similar to that of the Earth.

Neutron Stars

The cores of bigger stars collapse gravitationally to produce neutron stars. Instead of producing a white dwarf, a star with more than 1.4 solar masses must continue its destruction under the influence of gravity to form a neutron star. A neutron star has a mass of roughly 1.4 to 4 solar masses and a radius of about 10 kilometers. As a result, it is an incredibly dense object.

Black Holes

Black holes have been one of the key discoveries of GR. They were first predicted by Albert Einstein in 1916. In 1967 American theorist John Wheeler coined the term black hole. Black holes are among the weirdest and most interesting phenomena in the Universe. A black hole is a region of space with so tremendous gravity that even light cannot leave. Gravity is incredibly powerful because much of the matter has been crammed into a very very small region. This can occur when a star dies. “Nothing can travel faster than light” Einstein stated in his theory of relativity. As a result, if light cannot escape, neither can anything else; the gravitational field drags everything back. So, one has a series of occurrences, a region of space from which one cannot leave to reach a distant observer. This region is now known as a black hole [4, 5]. At the conclusion of their lifetimes, only the most massive stars those with a mass three times the mass of the Sun become black holes. A star as massive as 20 solar masses has a proclivity to collapse under its own gravity and become a black hole. A black hole is the death of a star and is also known as the last stage of a star. Black holes cannot be seen through the naked eye as they do not emit light. Space telescopes like the event horizon telescope in collaboration with several other telescopes located at different locations can form a combined image of a black hole by capturing the light of the accretion disk through different angles. Black holes can be as small as an atom or as large as a galaxy. Stellar black holes have a mass equal to 15-20 solar masses, and our galaxy Milky Way contains many such black holes. The enormous black holes are the ones with masses equal to roughly 106 solar masses and they are located at the center of the largest galaxies a recent discovery shows Sagittarius is a supermassive black hole located at the center of our galaxy Milky Way, which has a mass of roughly 4×10^6 times that of the Sun [9].

1.5 Solutions of Einstein Field Equations

In this section, we discuss some important and well-known black hole solutions of the Einstein field equations.

1.5.1 The Schwarzschild Solution

Let us consider an isolated point of mass m. We consider space-time to be static and spherically symmetry. The line element is specified as

$$ds^2 = -e^{2\nu(r)}dt^2 + e^{2\lambda(r)}dr^2 + r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right) \cdot \tag{1.43}$$

There is a vacuum all around the source so the stress-energy tensor is zero, i.e. $T_{\alpha\beta} = 0$, for all α and β. Then the EFEs become $R_{\alpha\beta} = 0$ called the vaccum EMFEs and Ricci scalar is zero. For a diagonal metric given in eq. (1.43), there are 4 independent components of the Ricci scalar. When the EFEs are solved, the metric in eq. (1.43) becomes

$$ds^2 = -\left(1 - \frac{2m}{r}\right)dt^2 + \left(1 - \frac{2m}{r}\right)^{-1}dr^2 + r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right) \cdot \tag{1.44}$$

This is known as the Schwarzschild solution [10]. This solution exhibits two singularities: the essential singularity at $r = 0$, which cannot be removed, and the coordinate singularity at $r = 2m$, which can be eliminated via suitable transformations of coordinates. For example, the Eddington-Finkelstein coordinate transformations can be used which are following

$$t = \tilde{t} \pm 2m\, ln\left(1 - \frac{\tilde{r}}{2m}\right) \quad \text{for} \quad \tilde{r} \leq 2m, \tag{1.45}$$

$$t = \tilde{t} \pm 2m\, ln\left(\frac{\tilde{r}}{2m} - 1\right) \quad \text{for} \quad \tilde{r} \geq 2m, \tag{1.46}$$

$$r = \tilde{r}. \tag{1.47}$$

The metric (1.43) takes the following form under the above transformations,

$$d\tilde{s}^2 = -\left(1 - \frac{2m}{\tilde{r}}\right) d\tilde{t}^2 \pm \frac{4m}{\tilde{r}} d\tilde{r} d\tilde{t} + \left(1 - \frac{2m}{\tilde{r}}\right) d\tilde{r}^2 + \tilde{r}^2 \left(d\theta^2 + \sin^2\theta d\phi^2\right) \cdot \tag{1.48}$$

Obviously, the coordinate singularity $\tilde{r} = 2m$ has been excluded, while the essential singularity $\tilde{r} = 0$ has been retained. In the outer region $\tilde{r} > 2m$, the observer cannot get any information about the events happening in the inner region $\tilde{r} < 2m$, because no object can escape the hypersurface $\tilde{r} = 2m$.

1.5.2 The Reissner-Nordström Solution

The Reissner-Nordström solution is a generalization of the Schwarzschild solution that is obtained by adding charge into the previous assumptions made for obtaining the Schwarzschild solution. Due to the addition of charge, we also need to incorporate the Maxwell field equations to find metrics in this case. We are well aware that this is two parameter solution, total mass m and total charge Q. The metric for this solution is as follows [3]:

$$ds^2 = -\left(1 - \frac{2m}{r} + \frac{Q^2}{r^2}\right) dt^2 + \left(1 - \frac{2m}{r} + \frac{Q^2}{r^2}\right)^{-1} dr^2 + r^2 \left(d\theta^2 + \sin^2\theta d\phi^2\right) \cdot \tag{1.49}$$

For $r = 0$, $g_{00} \to \infty$ and for $(1 - \frac{2m}{r} + \frac{Q^2}{r^2}) = 0$, $g_{11} \to \infty$, are the essential and coordinate singularities respectively. We can find the location of the horizon by setting

$$1 - \frac{2m}{r} + \frac{Q^2}{r^2} = 0, \tag{1.50}$$

or

$$r^2 - 2mr + Q^2 = 0. \tag{1.51}$$

The solution to the above equation is

$$r_{\pm} = m \pm (m^2 - Q^2)^{1/2}. \tag{1.52}$$

This solution corresponds to a black hole with r_+ representing the outer horizon and r_- is the location of the inner horizon. For $m > Q$ the black hole is non-extremal and for the case $m = Q$ we have extremal black hole. We will consider only the non-extremal case for our purpose. The solution is also asymptotically flat that is far away from the source the metric reduces to the flat Minkowski spacetime [?]. Mathematically this fact can be seen as:

$$r \to \infty \Rightarrow ds^2 = -dt^2 + dr^2 + r^2 \left(d\theta^2 + sin^2 d\theta\phi^2\right). \tag{1.53}$$

We conclude this section with a quick remark, that if we put a charge equal to zero, $(Q = 0)$, the Reissner-Nordström metric reduces to the Schwarzschild metric.

Chapter 2

Charged Anisotropic Model with $p_r = A\rho^2 + B\rho - \frac{G}{\rho}$

In this chapter, we present solution of the EMFEs for charged anisotropic matter distribution. The solution is obtained by extending the work of Malaver [27] by adding a quadratic term in the Chaplygin equation of state. The spacetime is considered to be spherically symmetric and static given as

$$ds^2 = -e^{2\nu(r)}dt^2 + e^{2\lambda(r)}dr^2 + r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right), \tag{2.1}$$

metric tensor of line element (2.1) can be written as

$$g_{\alpha\beta} = diag\left(-e^{2\nu}, e^{2\lambda}, r^2, r^2 sin^2\theta\right), \tag{2.2}$$

and inverse metric tensor for line element (2.1) can be written as

$$g^{\alpha\beta} = \left(-e^{-2\nu}, e^{-2\lambda}, \frac{1}{r^2}, \frac{1}{r^2 sin^2\theta}\right). \tag{2.3}$$

The functions $\nu(r)$ and $\lambda(r)$ are used to represent the gravitational potential in the static radial coordinate r. The energy-momentum tensor describes the distribution of matter within this star's interior in the following manner:

$$T_{\alpha\beta} = diag\left(-\rho - \frac{1}{2}E^2\ ,\ p_r - \frac{1}{2}E^2\ ,\ p_t - \frac{1}{2}E^2\ ,\ p_t - \frac{1}{2}E^2\right). \tag{2.4}$$

The Christofell symbols for the line element (2.1) are

$$\Gamma^1_{00} = \nu' e^{2(\nu-\lambda)}, \tag{2.5}$$

$$\Gamma^1_{22} = -re^{2\lambda}, \tag{2.6}$$

$$\Gamma^1_{11} = \lambda', \tag{2.7}$$

$$\Gamma^2_{21} = \Gamma^3_{31} = \frac{1}{r}, \tag{2.8}$$

$$\Gamma^1_{33} = -re^{-2\lambda} sin^2\theta, \tag{2.9}$$

$$\Gamma^2_{33} = -sin\theta, \tag{2.10}$$

$$\Gamma^2_{33} = cot\theta, \tag{2.11}$$

$$\Gamma^0_{01} = \nu'\cdot \tag{2.12}$$

The non-zero components of the Ricci tensor of the line element (2.1) are

$$R_{00} = \nu'' + \nu'(\nu' - \lambda') + \frac{2\nu'}{r}, \tag{2.13}$$

$$R_{11} = -\nu'' + \nu'(\nu' - \lambda') + \frac{2\lambda'}{r}, \tag{2.14}$$

$$R_{22} = 1 - e^{-2\lambda} + re^{-2\lambda}(\lambda' - \nu'), \tag{2.15}$$

$$R_{33} = sin^2\theta R_{22}\cdot \tag{2.16}$$

The equations describing the field generated by the anisotropic charged matter are

$$\frac{1}{r^2}[r(1 - e^{-2\lambda})]' = \rho + \frac{E^2}{2}, \tag{2.17}$$

$$\frac{1}{r^2}\left(\frac{1}{e^{2\lambda}} - 1\right) + \frac{2\nu'}{re^{2\lambda}} = p_r + \frac{E^2}{2}, \tag{2.18}$$

$$\frac{1}{e^{2\lambda}}\left(\nu'' + \nu'^2 - \nu'\lambda' - \frac{\lambda'}{r} + \frac{\nu'}{r}\right) = p_t + \frac{E^2}{2}, \tag{2.19}$$

$$\sigma = \frac{(r^2E)'}{r^2e^{\lambda}}\cdot \tag{2.20}$$

The intensity of the electric field is denoted by E, while the energy density is represented by ρ. The radial pressure is p_r and the tangential pressure is p_t. The primes in the equations signify differentiations with respect to r. The Einstein field equations can be expressed in a different form by applying the transformations

$$x = Cr^2, \quad e^{-2\lambda(r)} = Z, \quad A_1^2y^2 = e^{2\nu(r)}, \tag{2.21}$$

proposed by Durgapal and Bannerji [19], where A_1 and $C > 0$ are arbitrary constants.

$$\frac{1-Z}{x} - 2\dot{Z} = \frac{\rho}{C} + \frac{E^2}{2C}, \tag{2.22}$$

$$4Z\frac{\dot{y}}{y} - \frac{1-Z}{x} = \frac{p_r}{C} - \frac{E^2}{2C}, \tag{2.23}$$

$$4xZ\frac{\ddot{y}}{y} + (4Z + 2x\dot{Z})\frac{\dot{y}}{y} + \dot{Z} = \frac{p_t}{C} + \frac{E^2}{2C}, \tag{2.24}$$

$$\Delta = 4xCZ\frac{\ddot{y}}{y} + \dot{Z}C(1 + 2x\frac{\dot{y}}{y}) + \frac{C(1-Z)}{x} + \frac{E^2}{2}, \tag{2.25}$$

$$\sigma = \left(\frac{4CZ}{x}\right)^{1/2}(x\dot{E} + E), \tag{2.26}$$

where σ is the charge density, $\Delta = p_t - p_r$ is an anisotropic parameter, and the dot represents differentiation with respect to the variable 'x'. The total mass within the radius 'r' of the sphere is given by

$$M(x) = \frac{1}{4C^{3/2}}\int_0^x \sqrt{x}(\rho + E^2)dx\cdot \tag{2.27}$$

To develop a model with physical significance, we need a reasonable analysis of one of the variables $(\rho, p_r, p_t, y, \Delta, Z, E, \sigma)$ to simplify the integrability of the field equations. In this thesis, we choose the gravitational potential Z as

$$Z = \frac{bx+1}{ax+1}\cdot \tag{2.28}$$

In order to ensure the development of a well-behaved model, it is necessary to assign specific values to the non-zero arbitrary constants, denoted as a and b. The gravitational potential stated in eq (2.28) exhibits continuity and finiteness, with the condition $e^{-2\lambda(x)} = 1$ at the central point of the star $(x = 0)$. We choose the electric field intensity to be

$$\frac{E^2}{2C} = \frac{bx}{(ax+1)(bx+1)}\cdot \tag{2.29}$$

The equation of state of our model is chosen as

$$p_r = A\rho^2 + B\rho - \frac{G}{\rho}, \tag{2.30}$$

where A, B and G are arbitrary constants. By substituting eqs. (2.29) and (2.28) in eq. (2.22), we get

$$\rho = \frac{C\left((a-b)(ax+1)(bx+1) - 2(b-a)(bx+1) - bx(ax+1)\right)}{(ax+1)^2(bx+1)}\cdot \tag{2.31}$$

Putting eq. (2.31) in eq. (2.30), we get

$$p_r = \frac{AC^2\left((a-b)(ax+1)(bx+1) - 2(b-a)(bx+1) - bx(ax+1)\right)^2}{(ax+1)^4(bx+1)^2} \tag{2.32}$$

$$+ \frac{BC\left((a-b)(ax+1)(bx+1) - 2(b-a)(bx+1) - bx(ax+1)\right)}{(ax+1)^2(bx+1)} \tag{2.33}$$

$$- \frac{G(ax+1)^2(bx+1)}{C\left((a-b)(ax+1)(bx+1) - 2(b-a)(bx+1) - bx(ax+1)\right)}\cdot \tag{2.34}$$

Now putting the values from eqs. (2.29) and (2.28) in eq. (2.26). We get the expression for charge density as

$$\sigma^2 = \frac{2C^2 b(3+2(a+b)x+abx^2)^2}{(1+ax)^4(1+bx)^2}. \tag{2.35}$$

By eqs. (2.29) and (2.31) in eq. (2.27). The expression for the mass function becomes

$$\begin{aligned} M(x) = &-\frac{\left(a^2 b\,(b-a-1)\,(b-a)\sqrt{x} - a^3\sqrt{b}\arctan\left(\sqrt{b}\sqrt{x}\right) + a^{\frac{3}{2}} b^2 \arctan\left(\sqrt{a}\sqrt{x}\right)\right)x}{2\sqrt{C}\,a^2 b\,(b-a)\,(ax+1)} \\ &-\frac{ab\,(b-a)\sqrt{x} + a^2\sqrt{b}\arctan\left(\sqrt{b}\sqrt{x}\right) - \sqrt{a}\,b^2\arctan\left(\sqrt{a}\sqrt{x}\right)}{2\sqrt{C}\,a^2 b\,(b-a)\,(ax+1)}. \end{aligned} \tag{2.36}$$

Eq. (2.24) takes the following form after substituting eqs. (2.29), (2.30) and (2.34) in it

$$\begin{aligned} \frac{\dot{y}}{y} = &\frac{AC\left((a-b)\,(ax+1)\,(bx+1) - 2\,(b-a)\,(bx+1) - bx\,(ax+1)\right)^2}{4\,(ax+1)^3\,(bx+1)^3} \\ &- \frac{G\,(ax+1)^3}{4C^2\left((a-b)\,(ax+1)\,(bx+1) - 2\,(b-a)\,(bx+1) - bx\,(ax+1)\right)} - \frac{bx}{4\,(bx+1)} \\ &+ \frac{a-b}{4\,(bx+1)} + \frac{B\left((a-b)\,(ax+1)\,(bx+1) - 2\,(b-a)\,(bx+1) - bx\,(ax+1)\right)}{4\,(ax+1)\,(bx+1)^2}. \end{aligned} \tag{2.37}$$

Integration the above equation, we get

$$\begin{aligned} y(x) = &C_1(bx+1)^U(ax+1)^T(ab^2+(a-a^2)b)x^2 + \left(3b^2+(1-2a)b-a^2)x+3b-3a\right)^J \cdot \\ &\times exp(H) \end{aligned} \tag{2.38}$$

where C_1 is the constant of integration and the expressions for U, T, J, and H are shown in Appendix A1. Introducing Δ as p_t minus p_r, the anisotropic factor is defined. This factor measures the difference in pressure within the star. By combining eq. (2.25) and eq. (2.34), we can derive an expression for Δ which is shown in Appendix A1.

Using the gravitational potential y and $e^{2\lambda}$, as defined in eqs. (2.38) and (2.28), respectively, the line element takes the form

$$\begin{aligned} ds^2 = &-C_1^2 A_1^2 (bCr^2+1)^{2U}(aCr^2+1)^{2T}\left(ab^2+(a-a^2)b\right)(Cr^2)^2 \\ &+ \left(3b^2+(1-2a)b-a^2)Cr^2+3b-3a\right)^{2J} exp(2H)dt^2 + \left(\frac{1+aCr^2}{1+bCr^2}\right)dr^2 + r^2\left(d\theta^2+\sin^2\theta d\phi^2\right). \end{aligned}$$

2.1 Boundary Conditions

A necessary condition for a stable solution is that it must meet the exterior solution of Riessner-Nordström at the boundary. i.e.

$$e^{-2\lambda} = \left(1 - \frac{2M}{R} + \frac{Q^2}{R^2}\right) = \frac{bCR^2+1}{aCR^2+1}. \tag{2.39}$$

The total mass within the boundary in our case by using eq. (2.39) is

$$M = \frac{R}{2}\left(1 - \frac{1+bCR^2}{1+hCR^2} + \frac{2C^2bR^4}{(aCR^2+1)(bCR^2+1)}\right). \tag{2.40}$$

Thus by putting the eq. (2.40) and $Q = Er^2$ while comparing $e^{2\nu}$ with Riessner-Nordström, the constant C_1^2 takes the form

$$C_1^2 = \frac{(1+bCR^2)(1+aCR^2)^{-1}}{A_1^2 D(R)}, \tag{2.41}$$

where

$$D(R) = (bCR^2+1)^{2U}(aCR^2+1)^{2T}\left(ab^2+(a-a^2)b)(CR^2)^2+(3b^2+(1-2a)b\right.$$
$$\left.-a^2)CR^2+3b-3a\right)^{2J}exp^{2H(R)}. \tag{2.42}$$

By using the boundary condition $p_r(r=R)=0$, we get

$$G = \frac{AC^3\left(-2\left(b-a\right)\left(CR^2b+1\right)+\left(CR^2a+1\right)\left(a-b\right)\left(CR^2b+1\right)-CR^2\left(CR^2a+1\right)b\right)^3}{\left(CR^2a+1\right)^6\left(CR^2b+1\right)^3}$$
$$+\frac{BC^2\left(-2\left(b-a\right)\left(CR^2b+1\right)+\left(CR^2a+1\right)\left(a-b\right)\left(CR^2b+1\right)-CR^2\left(CR^2a+1\right)b\right)^2}{\left(CR^2a+1\right)^4\left(CR^2b+1\right)^2}. \tag{2.43}$$

2.2 Physcial Conditions

Metric Potentials

The most fundamental and most important condition is to check the regularity of the metric potentials. The metric potentials are free from singularities and both $e^{2\lambda}$ and $e^{2\nu}$ are monotonically increasing moreover $e^{2\lambda} = 1$ at the center. The behavior of metric potentials is plotted in Figures 2.1 and 2.2. The metric potentials are dimensionless in geometrical units.

Electric Field Intensity

The behavior of the electric strength is depicted in Figure 2.3, starting at zero and gradually increasing as it approaches the boundary of the star. It has dimensions L^{-2} in geometrical units.

Density and Pressures

At $r=0$, densities and pressures are finite. The values of energy density and pressure at $r=0$ are as follow

$$\rho(r=0) = 3C(a-b), \tag{2.44}$$

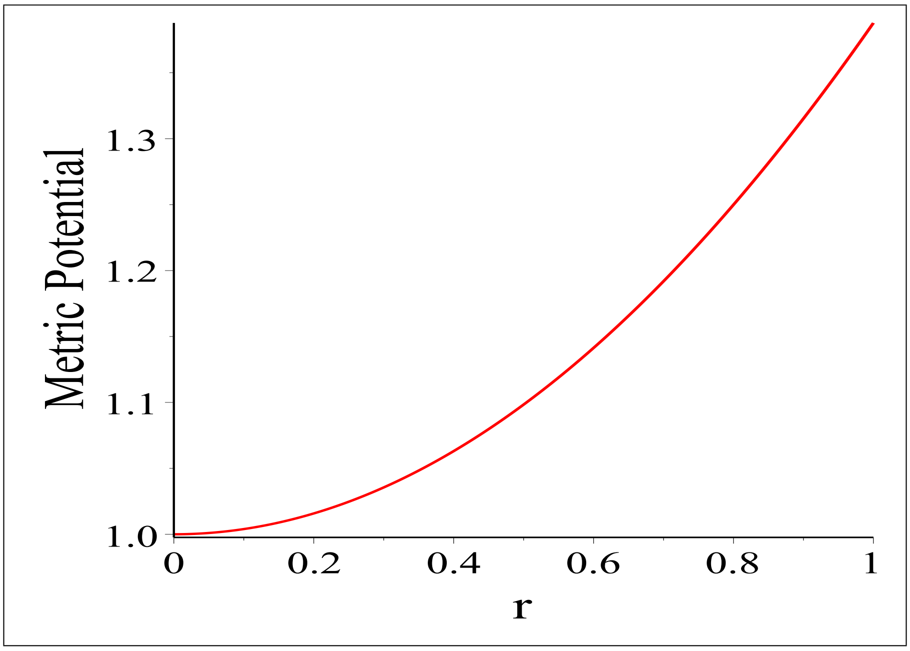

Figure 2.1: Graph of metric potential with respect to r. In this and all other figures of this chapter, the values of the constants are $a = 0.5$, $b = 0.0236$, $C = 0.83$, $A = 0.029$, $B = 0.0001$ and $A_1 = 1.1$.

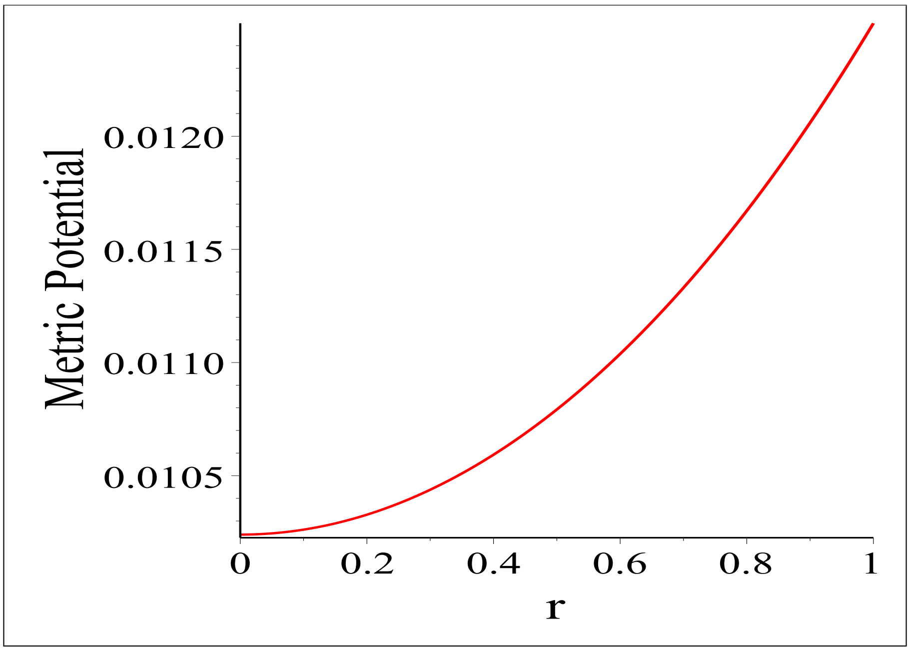

Figure 2.2: Graph of metric potential with respect to r

$$p_r(r=0) = p_t(r=0) = 9AC^2(a-b)^2 + 3BC(a-b) - \frac{G}{3C(a-b)}. \tag{2.45}$$

Note that radial and tangential pressures are equal and positive at $r = 0$. Figures 2.5 and 2.6 show the behavior of radial and tangential pressures, respectively, where p_r is monotonically

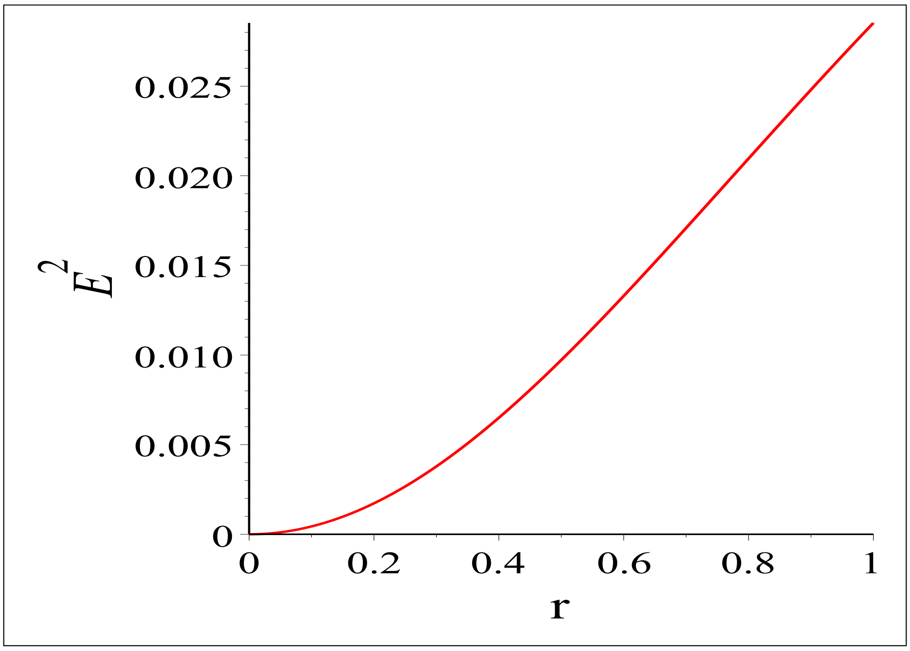

Figure 2.3: Graph of electric field intensity with respect to r

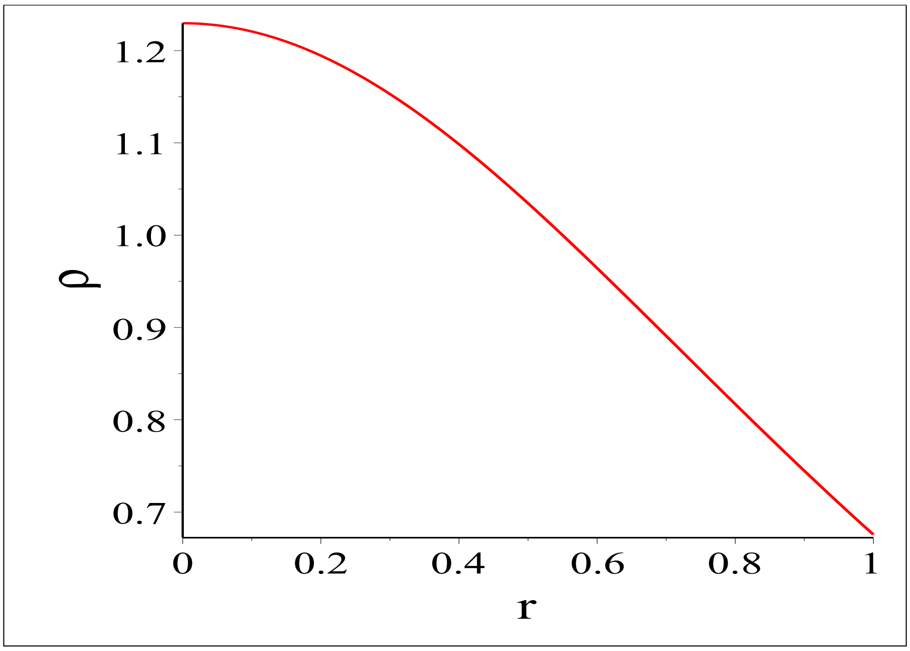

Figure 2.4: Graph of energy density with respect to r

decreasing and $p_r(r = R) = 0$. Density (ρ) and tangential pressure (p_t) also have decreasing nature as shown in the Figure 2.4. The densities and pressures have dimensions L^{-2} in geometrical units. According to the criterion of Zeldovich [13], the pressure-density ratio must be less than 1 to satisfy the requirement for compact objects. Figure 2.7 shows the pressure-density ratio.

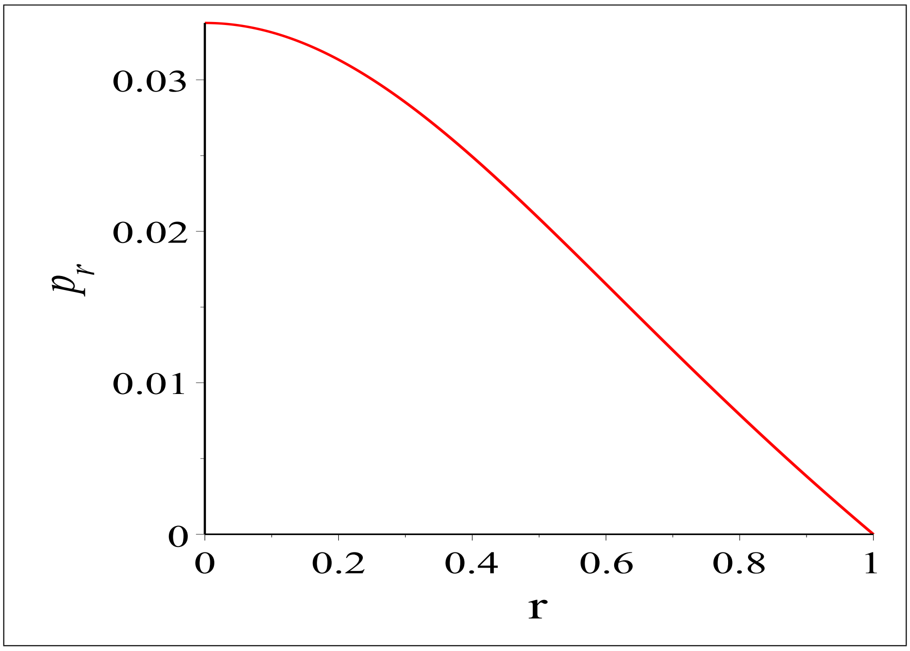

Figure 2.5: Graph of radial pressure with respect to r

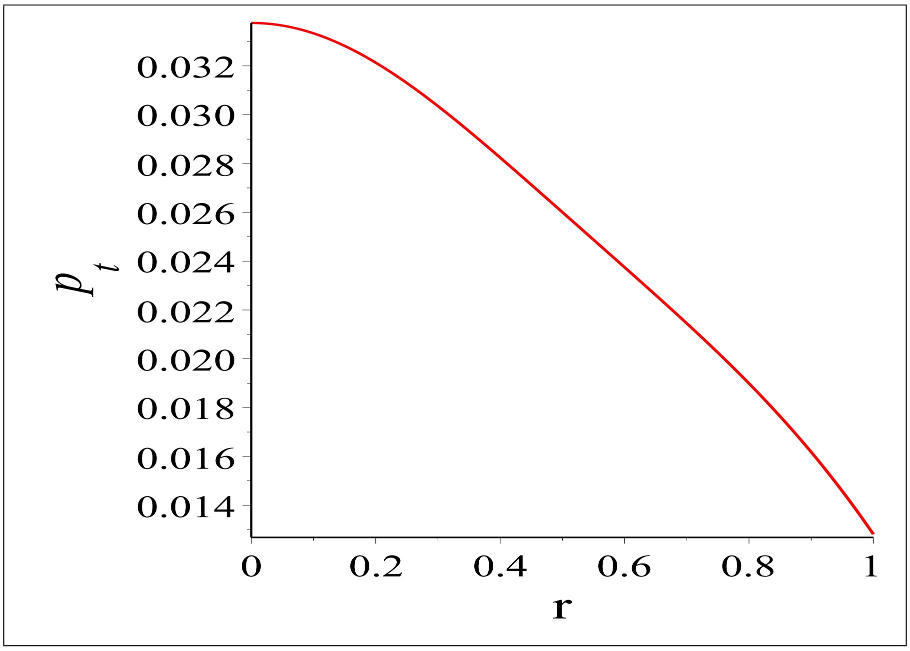

Figure 2.6: Graph of tangential pressure with respect to r

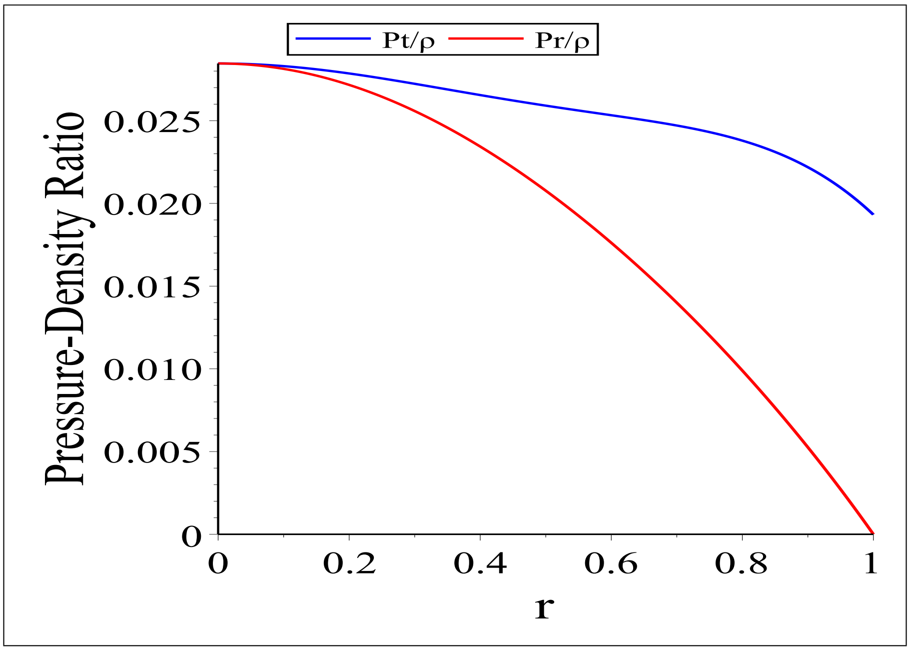

Figure 2.7: Graph of pressure density ratio with respect to r

Anisotropic Factor

Anisotropy is defined as the difference between tangential and radial pressure i.e. $\Delta = p_t - p_r$, the spherical symmetry $p_r(0) = p_t(0)$ implies $\Delta_{r=0} = 0$. For compact objects, it is necessary that $\Delta > 0$ in the interior structure meaning that the anisotropic force is repulsive in nature, it is possible in the case when $(p_t > p_r)$. Otherwise, if radial pressure dominates the tangential one i.e. $(p_r > p_t)$ then this implies the presence of a new force that is attractive in nature. The Δ curve for our case satisfies the required condition, as shown in Figure 2.8.

Gradients

The generalized expressions for density gradient, and pressure gradients in radial and transverse directions for our model are:

$$\frac{d\rho}{dr} = -\frac{2C^2br}{(Car^2+1)(Cbr^2+1)} + \frac{2C^3abr^3}{(Car^2+1)^2(Cbr^2+1)} + \frac{2C^3b^2r^3}{(Car^2+1)(Cbr^2+1)^2} - \frac{2C^2a(a-b)r}{(Car^2+1)^2} + \frac{8C^2a(b-a)r}{(Car^2+1)^3}, \tag{2.46}$$

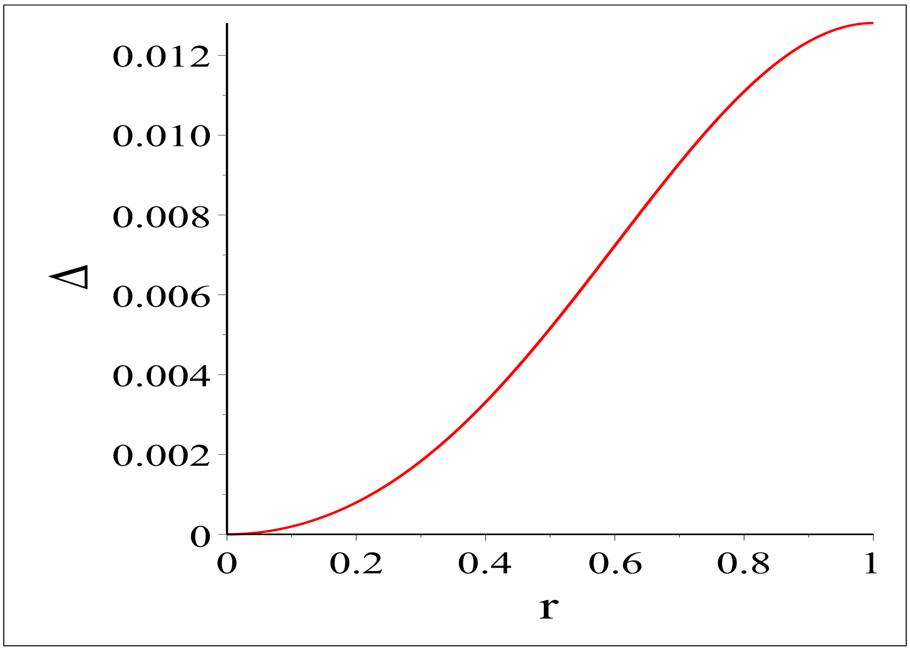

Figure 2.8: Graph of anisotropic factor with respect to r

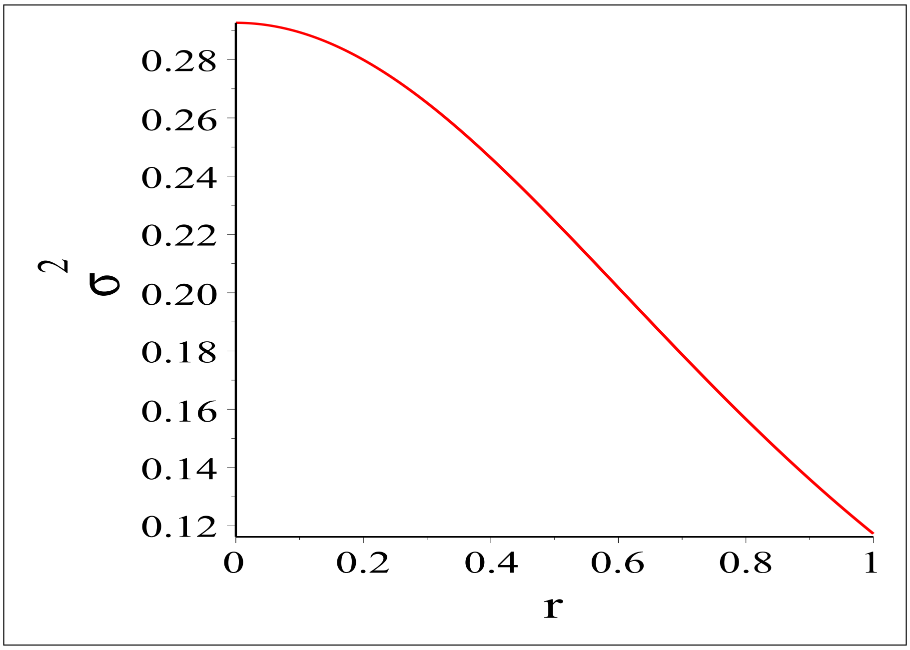

Figure 2.9: Graph of charge density with respect to r

$$
\begin{aligned}
\frac{dp_r}{dr} = &-\frac{8AC^3ar\left((a-b)\left(Car^2+1\right)\left(Cbr^2+1\right)-2\left(b-a\right)\left(Cbr^2+1\right)-Cbr^2\left(Car^2+1\right)\right)^2}{\left(Car^2+1\right)^5\left(Cbr^2+1\right)^2} \\
&-\frac{4AC^3br\left((a-b)\left(Car^2+1\right)\left(Cbr^2+1\right)-2\left(b-a\right)\left(Cbr^2+1\right)-Cbr^2\left(Car^2+1\right)\right)^2}{\left(Car^2+1\right)^4\left(Cbr^2+1\right)^3} \\
&-\frac{2Gbr\left(Car^2+1\right)^2}{(a-b)\left(Car^2+1\right)\left(Cbr^2+1\right)-2\left(b-a\right)\left(Cbr^2+1\right)-Cbr^2\left(Car^2+1\right)} \\
&-\frac{4Gar\left(Car^2+1\right)\left(Cbr^2+1\right)}{(a-b)\left(Car^2+1\right)\left(Cbr^2+1\right)-2\left(b-a\right)\left(Cbr^2+1\right)-Cbr^2\left(Car^2+1\right)} \\
&+\frac{G\left(Car^2+1\right)^2\left(Cbr^2+1\right)\left(-2C^2abr^3+2Ca\left(a-b\right)r\left(Cbr^2+1\right)\right)}{C\left((a-b)\left(Car^2+1\right)\left(Cbr^2+1\right)-2\left(b-a\right)\left(Cbr^2+1\right)-Cbr^2\left(Car^2+1\right)\right)^2} \\
&+\frac{G\left(Car^2+1\right)^2\left(Cbr^2+1\right)\left(2C\left(a-b\right)br\left(Car^2+1\right)-2Cbr\left(Cax^2+1\right)-4Cb\left(b-a\right)r\right)}{C\left((a-b)\left(Car^2+1\right)\left(Cbr^2+1\right)-2\left(b-a\right)\left(Cbr^2+1\right)-Cbr^2\left(Car^2+1\right)\right)^2} \\
&+\frac{BC\left(-2C^2abr^3+2Ca\left(a-b\right)r\left(Cbr^2+1\right)\right)}{\left(Car^2+1\right)^2\left(Cbr^2+1\right)} \\
&+\frac{BC\left(2C\left(a-b\right)br\left(Car^2+1\right)-2Cbr\left(Car^2+1\right)-4Cb\left(b-a\right)r\right)}{\left(Car^2+1\right)^2\left(Cbr^2+1\right)} \\
&+\frac{2AC^2\left((a-b)\left(Car^2+1\right)\left(Cbr^2+1\right)-2\left(b-a\right)\left(Cbr^2+1\right)-Cbr^2\left(Car^2+1\right)\right)}{\left(Car^2+1\right)^4} \\
&\times\left(\frac{-2C^2abr^3+2Ca\left(a-b\right)r\left(Cbr^2+1\right)+2Cbr\left(Car^2+1\right)\left(a-b-1\right)-4Cb\left(b-a\right)r}{\left(Cbr^2+1\right)^2}\right) \\
&-\frac{4BC^2ar\left((a-b)\left(Car^2+1\right)\left(Cbr^2+1\right)-2\left(b-a\right)\left(Cbr^2+1\right)-Cbr^2\left(Car^2+1\right)\right)}{\left(Car^2+1\right)^3\left(Cbr^2+1\right)} \\
&-\frac{2BC^2br\left((a-b)\left(Cax^2+1\right)\left(Cbr^2+1\right)-2\left(b-a\right)\left(Cbr^2+1\right)-Cbr^2\left(Car^2+1\right)\right)}{\left(Car^2+1\right)^2\left(Cbr^2+1\right)^2}.
\end{aligned}
\tag{2.47}
$$

The decreasing nature of gradients is observed, which is displayed for our model in Figures 2.10, 2.11 and 2.12.

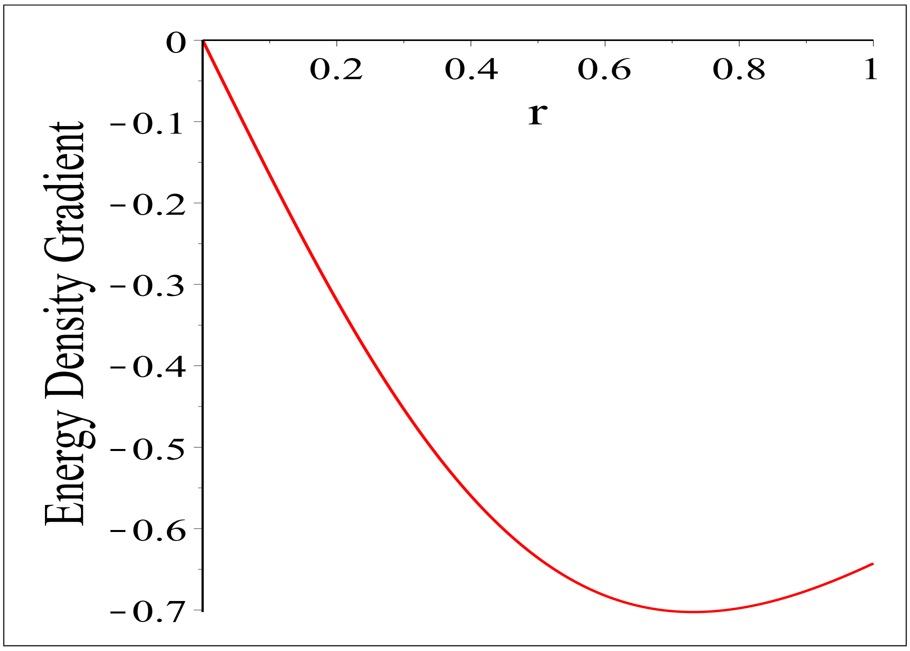

Figure 2.10: Graph of energy density gradient with respect to r

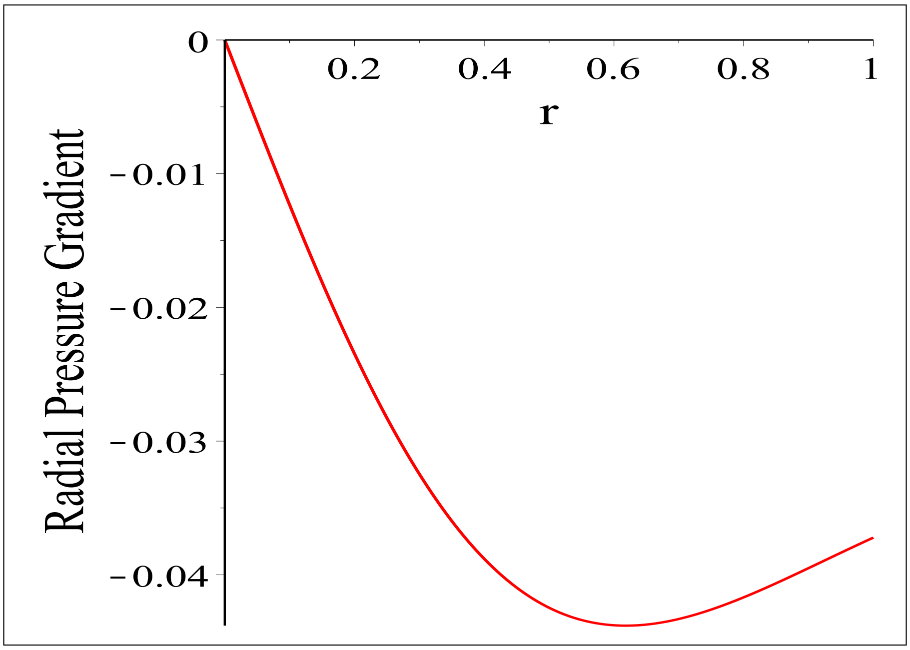

Figure 2.11: Graph of radial pressure gradient with respect to r

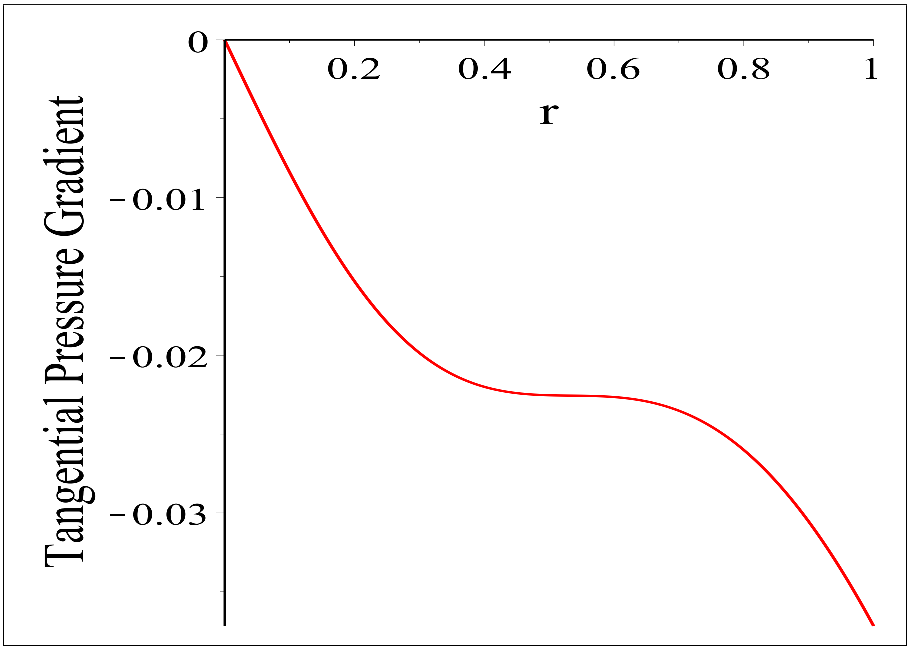

Figure 2.12: Graph of tangential pressure gradient with respect to r

Mass-Radius Relation

We have previously obtained an expression for the mass function provided in eq. (2.40). Buchdahl [14] introduced the concept that the inner mass-radius ratio of a compact star must be smaller than 4/9, that is to say, $M/R < 4/9$. This is true for any solution with an energy density greater than zero, for the present model, the mass-radius ratio is $M = 0.138 < 4/9$. The profile of the mass function is shown in Figure 2.13 which is positive and regular inside the stellar interior and has an increasing nature with m. The compactness factor for stellar configuration is given as

$$\mu(r) = \frac{M(r)}{r}. \tag{2.48}$$

The profile for our case is shown in Figure 2.14 which is increasing monotonically and is less than 4/9.

Surface and Gravitational Redshifts

A surface redshift of less than 1 is considered physically acceptable and is plotted in Figure 2.15, which exhibits features that increase towards the boundary, which is what nature depicts. Furthermore, the gravitational redshift (z) shown in Figure 2.16 has a decreasing nature for our model.

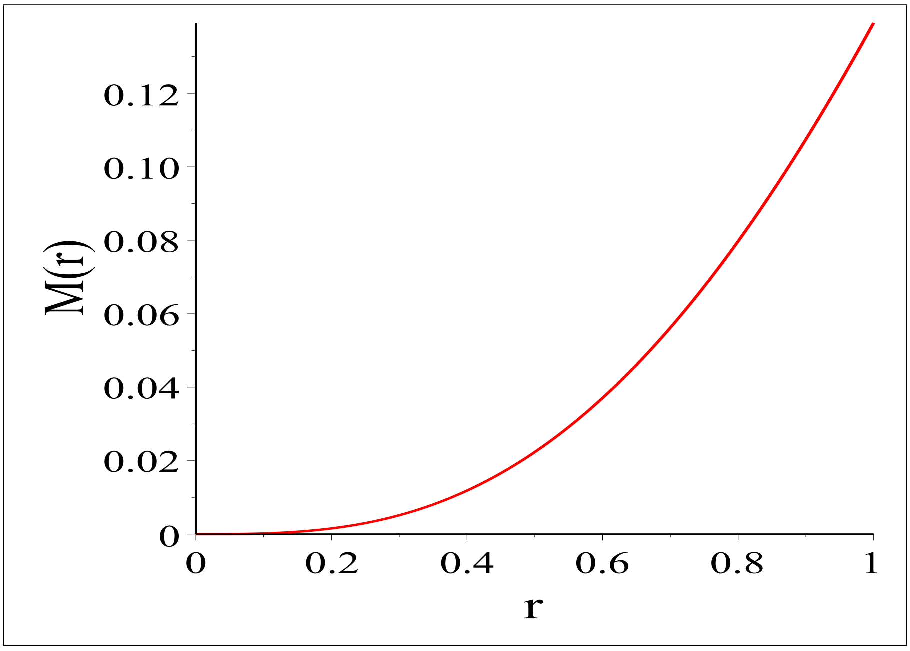

Figure 2.13: Graph of mass $M(r)$ with respect to r

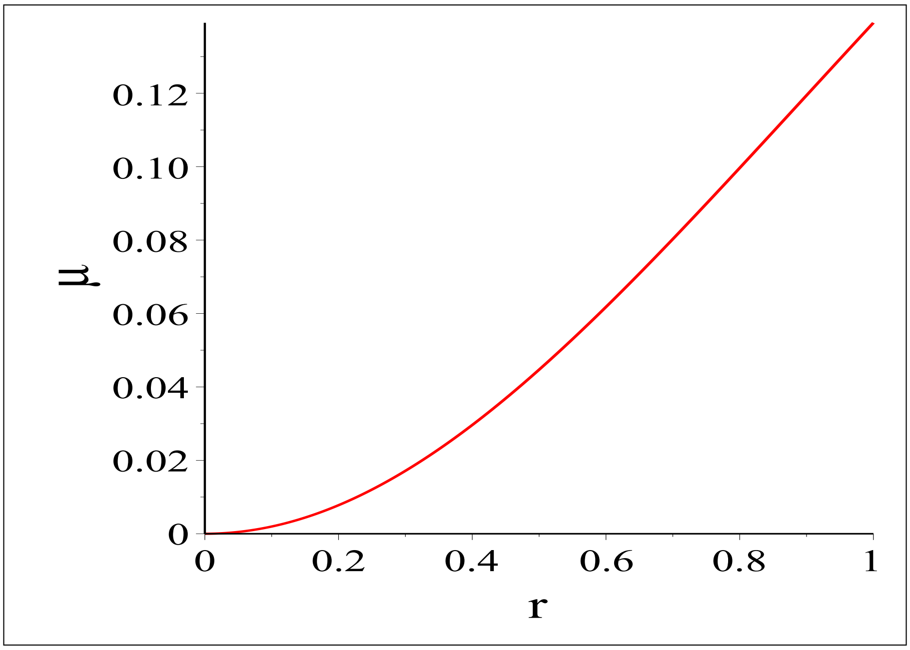

Figure 2.14: Graph of compactness factor with respect to r

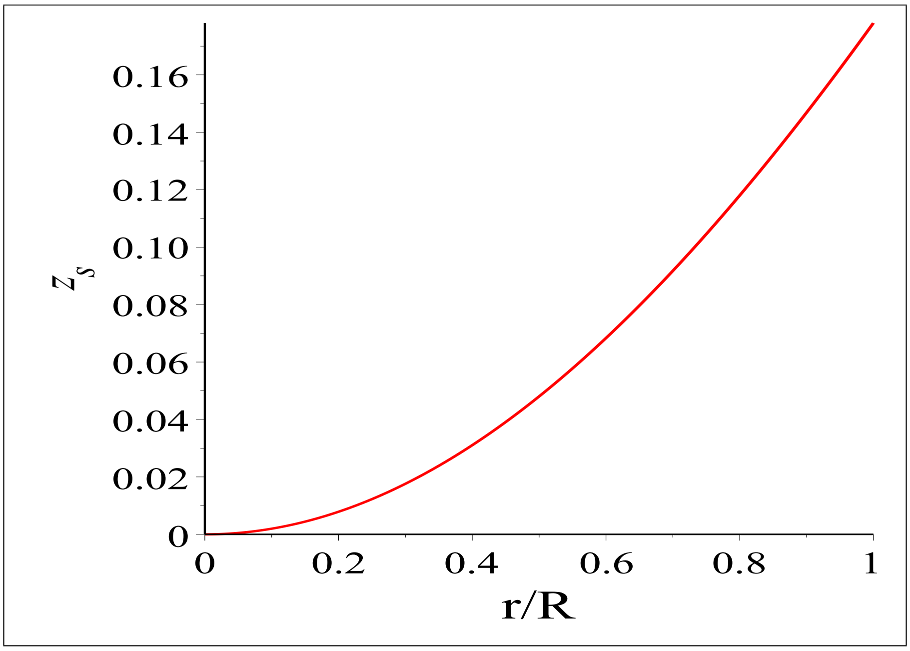

Figure 2.15: Graph of surface redshift with respect to r

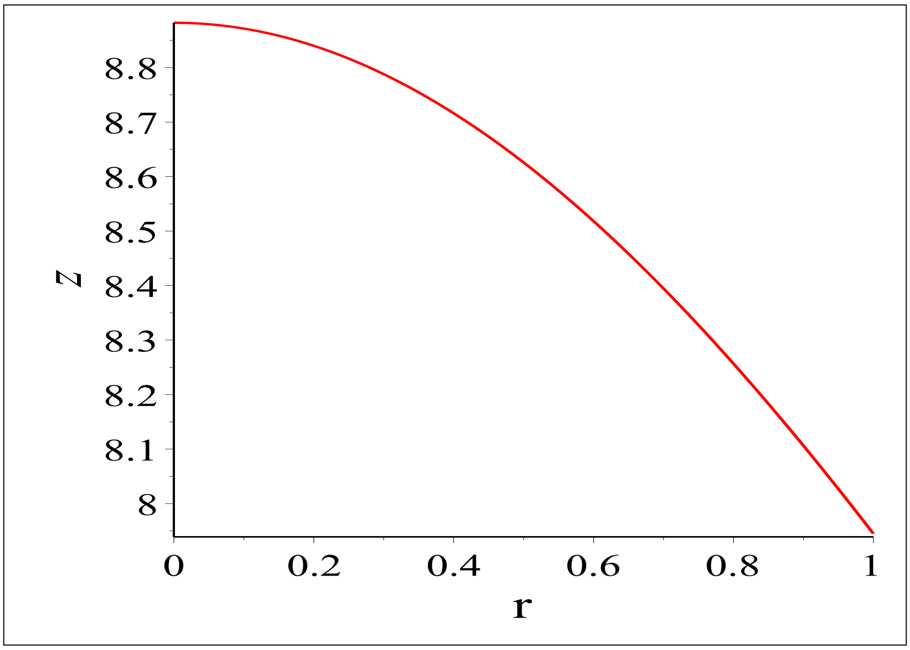

Figure 2.16: Graph of gravitational redshift with respect to r

Trace of Energy Tensor

The energy tensor trajectory of our proposed compact star model is based on r and is shown in Figure 2.17, which is positive in our case and meets Bundy's [47] condition for an anisotropic fluid sphere, namely $\rho - p_r - 2p_t > 0$.

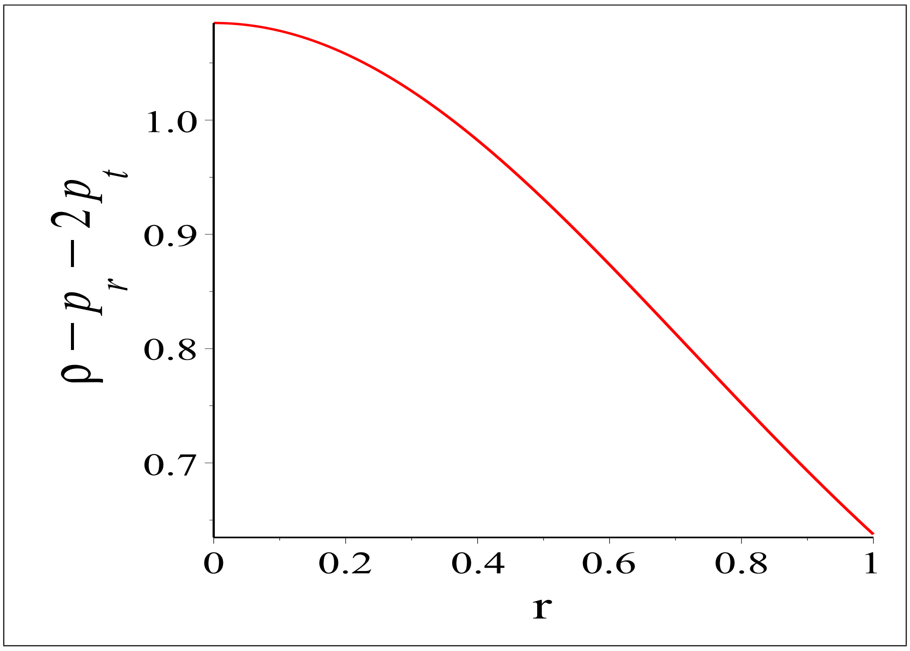

Figure 2.17: Graph of a trace of energy tensor with respect to r

2.3 The Stability Analysis

In this section, we will check the stability conditions of our model.

Casuality Condition

The radial and tangential velocity is defined as $v_r^2 = \frac{dp_r}{d\rho}$ and $v_t^2 = \frac{dp_t}{d\rho}$ are plotted in Figure 2.18. The expression for radial velocity for our model is

$$v_{sr}^2 = \frac{G\left(Car^2+1\right)^4\left(Cbr^2+1\right)^2}{C^2\left((a-b)\left(Car^2+1\right)\left(Cbr^2+1\right)-2\left(b-a\right)\left(Cbr^2+1\right)-Cbr^2\left(Car^2+1\right)\right)^2} + \frac{2AC\left((a-b)\left(Car^2+1\right)\left(Cbr^2+1\right)-2\left(b-a\right)\left(Cbr^2+1\right)-Cbr^2\left(Car^2+1\right)\right)}{\left(Car^2+1\right)^2\left(Cbr^2+1\right)} + B\cdot \tag{2.49}$$

and the expression for tangential velocity is shown in Appendix A1. The study of the speed of sound in stellar matter is important to understand stellar behavior. The stability of a stellar sphere needs to meet the conditions of radial velocity v_r^2 and tangential velocity v_t^2, expressed as $0 < v_r^2 < 1$ and $0 < v_t^2 < 1$, where $v_r^2 > v_t^2$. For stable stellar configurations, the inequality $0 \leq |v_t^2 - v_r^2| \leq 1$ is also satisfied as shown in Figure 2.19.

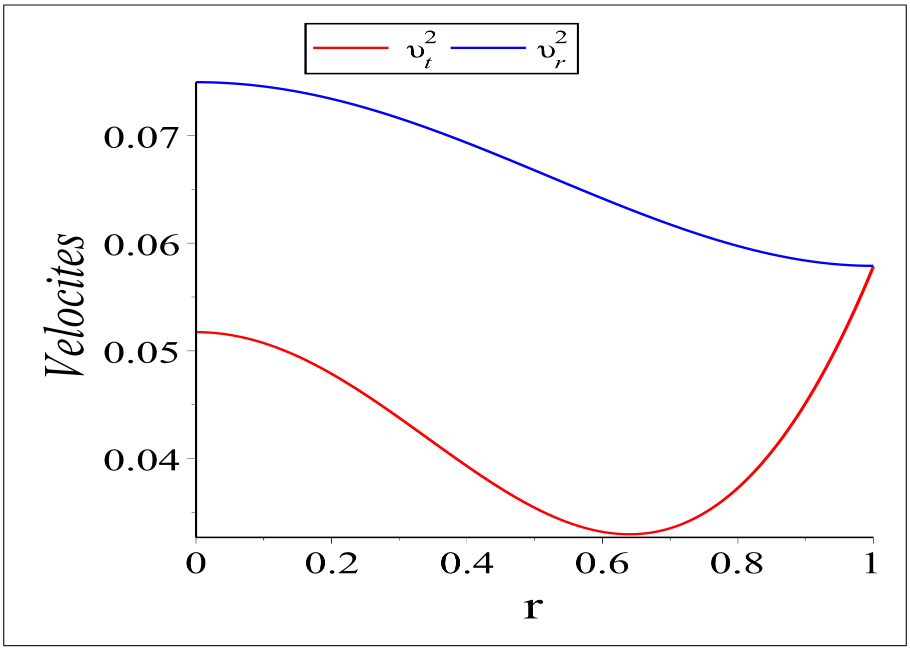

Figure 2.18: Graph of radial and tangential velocities with respect to r

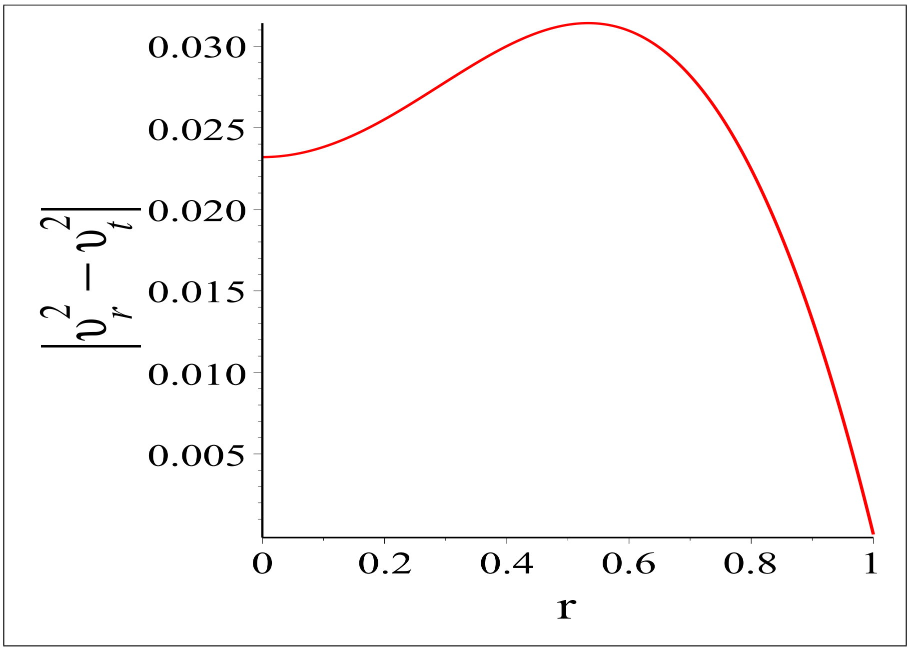

Figure 2.19: Graph of stability factor with respect to r

Energy Conditions

For a physically stable solution, it should surely satisfy the following energy conditions throughout the stellar interior.

1. Null Energy Condition: $\rho(r) \geq 0$,
2. Weak Energy Condition: $\rho + p_r \geq 0$, $\rho + p_t \geq 0$,
3. Strong Energy Condition: $\rho + p_r + 2p_t \geq 0$.

For stable configuration Figures 2.20, 2.21, and 2.22 clearly show the well-behaved nature of energy conditions for our case.

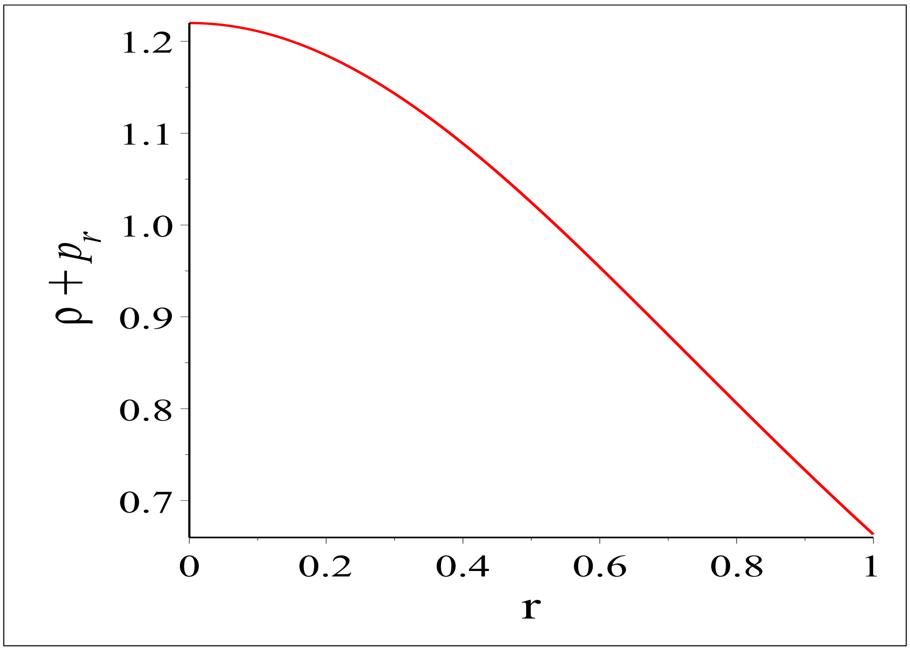

Figure 2.20: Graph of weak energy condition $(\rho + p_r)$ with respect to r

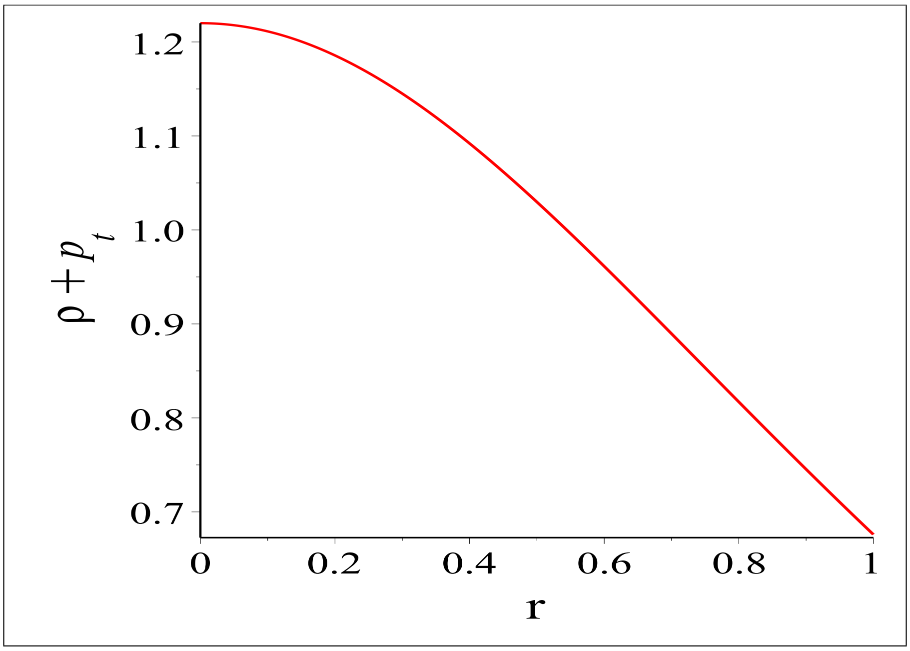

Figure 2.21: Graph of weak energy condition $(\rho + p_t)$ with respect to r

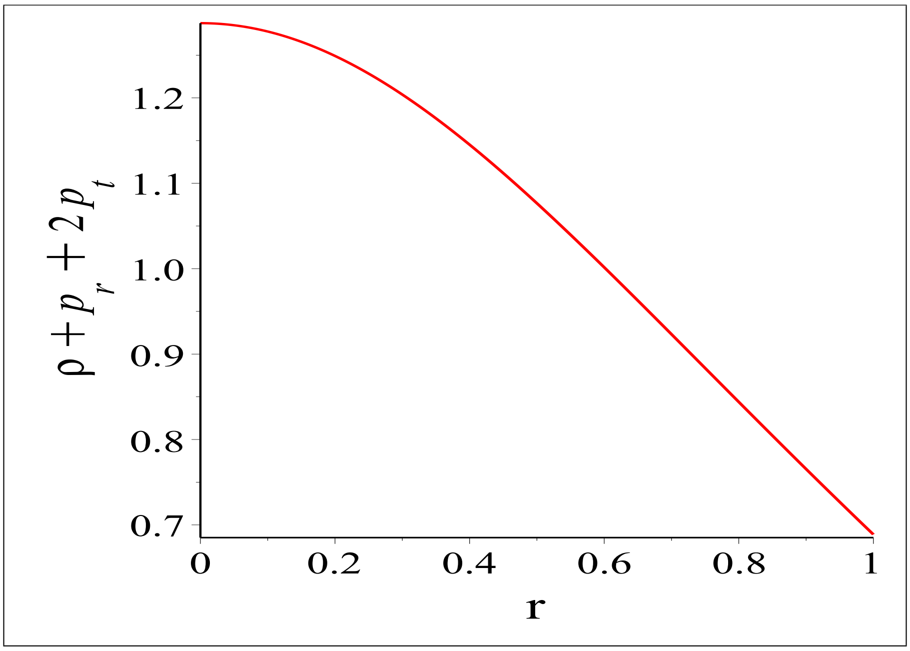

Figure 2.22: Graph of strong energy condition $(p_r + \rho + 2p_t)$ with respect to r

Adiabatic Index

Heinzmann and Hillebrandt [45] and Chen et al. [46] explained the two specific heat ratios for stable systems which are

$$\Gamma_r = \frac{\rho + p_r}{p_r}\frac{dp_r}{dr}, \tag{2.50}$$

$$\Gamma_t = \frac{\rho + p_t}{p_t}\frac{dp_t}{dr}\cdot \tag{2.51}$$

The value of Γ_i should be greater than 4/3 in Figures 2.23 and 2.24, the conditions for our model are shown, and we can clearly observe that this condition satisfies the stability in the relativistic fluid sphere $\Gamma_i > 4/3$.

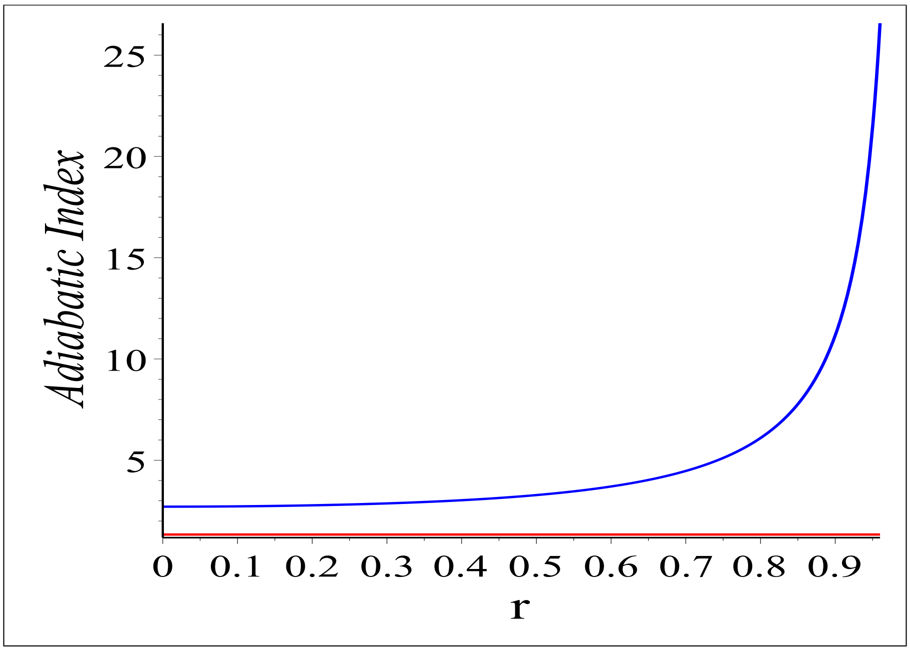

Figure 2.23: Graph of adiabatic index (Γ_r) with respect to r

Equilibrium State Under Various Forces

The hydrostatic equilibrium state of the model is studied by analyzing the Tolman-Oppenheimer-Volkov (TOV) equation [15, 16]. This equation dictates how the forces should be balanced, achieving hydrostatic equilibrium in a stellar sphere. The TOV equation for a charged anisotropic compact object is given by:

$$\frac{2}{r}(p_t - p_r) - \frac{dp_r}{dr} - (\rho + p_r)\nu' + \sigma E e^{\lambda} = 0\cdot \tag{2.52}$$

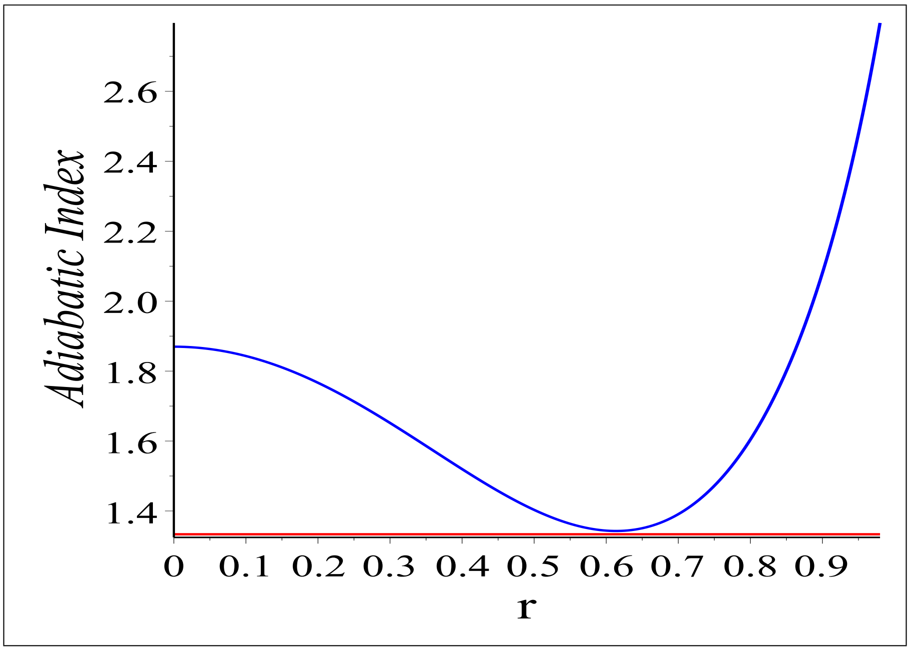

Figure 2.24: Graph of adiabatic index (Γ_t) with respect to r

the eq. (3.23) can also be written as

$$F_h + F_g + F_e + F_a = 0\cdot \tag{2.53}$$

where F_h, F_g, F_e, F_a are the hydrostatic, gravitational, electric and anisotropic force respectively, given as

$$F_h = -\frac{dp_r}{dr}, \tag{2.54}$$

$$F_g = -(\rho + p_r)\nu', \tag{2.55}$$

$$F_e = \sigma E e^{\lambda}, \tag{2.56}$$

$$F_a = \frac{2}{r}(p_t - p_r)\cdot \tag{2.57}$$

In Figure 2.25, the nature of these forces are displayed. The graph demonstrates how F_g, which is dominant in nature, is balanced by the combination of three forces i.e. F_h, F_e and F_a.

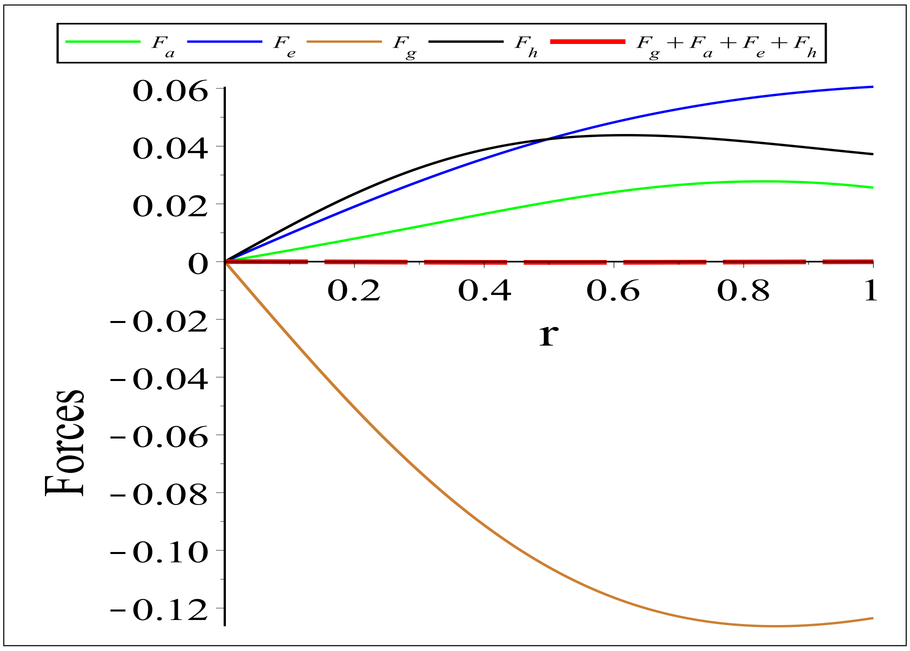

Figure 2.25: Graph of forces with respect to r

Chapter 3

Charged Anisotropic Model with $p_r = A\rho^2 - \dfrac{B}{\rho^2}$

In this chapter, we present another static and spherically symmetric solution of the EMFEs for anisotropic charged matter distribution. To obtain this solution, we choose a new version of modified Chaplygin equation of state which is $p_r = Ap^{\alpha} - \dfrac{B}{\rho^{\beta}}$, where $\alpha = 2$ and $\beta = 2$. The field equations are same as given by equations (2.22)-(2.26).

3.1 Electrified Anisotropic Framework

Inspired by the work of Manuel Malaver [**?**,17,22,23], Sunzu at el. [32] and Ratanpal [24] we opt for a particular expression of the gravitational potential Z that exhibits smooth behavior and regularity at the center while ensuring that the electric field strength, E, must be zero at the center and has increasing nature, to have:

$$Z(x) = \frac{1}{(1+bx)^2}, \tag{3.1}$$

$$\frac{E^2}{2C} = \frac{ax}{(1+bx)^2}, \tag{3.2}$$

where a and b are the arbitray constants. The equation of state we choose for this model is the following

$$p_r = A\rho^2 - \frac{B}{\rho^2}, \tag{3.3}$$

where A and B are arbitray constants. By substituting eqs. (3.1) and (3.2) in eq. (2.22), we get

$$\rho = \frac{C((b^2x + 2b - ax)(1+bx) + 4b)}{(1+bx)^3}. \tag{3.4}$$

Putting eq. (3.4) in eq. (3.3), we get

$$P_r = A\left(\frac{C((b^2x+2b-ax)(1+bx)+4b)}{(1+bx)^3}\right)^2 - B\left(\frac{(1+bx)^3}{C\left((b^2x+2b-ax)(1+bx)+4b\right)}\right)^2\cdot \quad (3.5)$$

Now putting the values from eqs. (3.1) and (2.26) in eq. (2.26), we get the expression for charge density as

$$\sigma^2 = \frac{C^2a(3+bx)^2}{(1+bx)^6}\cdot \quad (3.6)$$

The total mass with in the radius 'r' of the sphere is given by

$$M(x) = \frac{1}{4C^{3/2}}\int_0^x \sqrt{x}(\rho+E^2)dx\cdot \quad (3.7)$$

By using eqs. (3.4) and (3.2) within eq. (3.7), the expression for the mass function is

$$M(x) = \frac{\sqrt{x}\left(2b^4x^2+2ab^2x^2+4b^3x+5abx+3a\right)}{4\sqrt{C}\,b^2\cdot(bx+1)^2} - \frac{3a\arctan\left(\sqrt{b}\sqrt{x}\right)}{4\sqrt{C}\,b^{\frac{5}{2}}}\cdot \quad (3.8)$$

Substituting eqs. (3.1), (3.2) and (3.5) in eq. (2.25), we get

$$\begin{aligned}\frac{\dot{y}}{y} &= \frac{A((b^2x+2b-ax)(1+bx)+4b)^2}{4(1+bx)^4} - \frac{B(1+bx)^8}{4C^2(b^2x+2b-ax)(1+bx)+4b)^2}\\ &+ \frac{b^2x-ax+2b}{4}\cdot\end{aligned} \quad (3.9)$$

Integrating eq. (3.9), we get

$$y(x) = C_1(b(b^2-a)x^2+(3b^2-a)x+6b)^H(1+bx)^Ke^{V(x)}\cdot \quad (3.10)$$

where C_1 is the constant of integration and expression for H, K, and $V(x)$ are shown in Appendix A2. and the factor of anisotropy is presented as follows:

$$\begin{aligned}\Delta(x) = &\left(\frac{4Cx}{(1+bx)^2}\right)\Bigg[\Bigg(\frac{AC\left((bx+1)\left(b^2x-ax+2b\right)+4b\right)^2}{4\left(bx+1\right)^4} - \frac{b^2x-ax+2b}{4}\\ &+\frac{B\left(bx+1\right)^8}{4C^3\left((bx+1)\left(b^2x-ax+2b\right)+4b\right)^2}\Bigg)^2 - \frac{ACb\left((bx+1)\left(b^2x-ax+2b\right)+4b\right)^2}{(bx+1)^5}\\ &-\frac{2Bb\left(bx+1\right)^7}{C^3\left((bx+1)\left(b^2x-ax+2b\right)+4b\right)^2} + \frac{B\left(bx+1\right)^8\left(b\left(b^2x-ax+2b\right)+\left(b^2-a\right)(bx+1)\right)}{2C^3\left((bx+1)\left(b^2x-ax+2b\right)+4b\right)^3}\\ &+\frac{AC\left(b\left(b^2x-ax+2b\right)+\left(b^2-a\right)(bx+1)\right)\left((bx+1)\left(b^2x-ax+2b\right)+4b\right)}{2\left(bx+1\right)^4} + \frac{b^2-a}{4}\Bigg]\\ &+\left(\frac{4C}{(bx+1)^2} - \frac{4bCx}{(bx+1)^3}\right)\times\Bigg(\frac{AC\left((bx+1)\left(b^2x-Cax^2+2b\right)+4b\right)^2}{4\left(bx+1\right)^4}\\ &-\frac{B\left(bx+1\right)^8}{4C^3\left((bx+1)\left(b^2x-ax+2b\right)+4b\right)^2} + \frac{b^2x-ax+2b}{4}\Bigg) - \frac{aCx}{(bx+1)^2} - \frac{2Cb}{(bx+1)^3}\\ &-A\left(\frac{C((b^2x+2b-ax)(1+bx)+4b)}{(1+bx)^3}\right)^2 + B\left(\frac{(1+bx)^3}{C\left((b^2x+2b-ax)(1+bx)+4b\right)}\right)^2.\end{aligned} \quad (3.11)$$

The spacetime metric (2.1) becomes

$$\begin{aligned}ds^2 = &-C_1^2A_1^2(b(b^2-a)(Cr^2)^2+(3b^2-a)Cr^2+6b)^{2H}(1+bCr^2)^{2K}exp(2V)dt^2+(1+bCr^2)^2dr^2\\ &+r^2\left(d\theta^2+\sin^2\theta d\phi^2\right).\end{aligned}$$

3.2 Boundary Conditions

A necessary condition for a stable solution is that it must meet the exterior solution of Riessner-Nordström at the boundary. i.e.

$$e^{-2\lambda} = \left(1 - \frac{2M}{R} + \frac{Q^2}{R^2}\right) = \frac{1}{(1+bCR^2)^2}. \tag{3.12}$$

The total mass within the boundary in our case by using eq. (3.12) which is

$$M = \frac{R}{2}\left(\frac{(1+bCR^2)^2 - 1 + 2C^2aR^4}{(1+bCR^2)^2}\right). \tag{3.13}$$

Thus by putting the eq. (3.13) and $Q = Er^2$ while comparing $e^{2\nu}$ with Riessner-Nordström, solution the constant C_1^2 is obtained as

$$C_1^2 = \frac{(1+bCR^2)^{-2}}{A_1^2 \times [(b(b^2-a)(CR^2)^2 + (3b^2-a)CR^2 + 6b)^{2H}(1+bCR^2)^{2K}e^{2V(R)}]}. \tag{3.14}$$

By using the boundary condition $p_r(r = R) = 0$, we get

$$B = \frac{AC^4\left(\left(CR^2b+1\right)\left(CR^2b^2+2b-CR^2a\right)+4b\right)^4}{\left(CR^2b+1\right)^{12}}. \tag{3.15}$$

3.3 Physical Conditions

Metric potentials

The most fundamental and most important condition is to check the regularity of the metric potentials. The metric potentials are free from singularities and both $e^{2\lambda}$ and $e^{2\nu}$ are monotonically increasing moreover $e^{2\lambda} = 1$ at the center. The behavior of metric potentials is plotted in Figures 3.1 and 3.2.

Electric Field Intensity

The behavior of the electric field intensity is depicted in Figure 3.3 starting at zero and gradually increasing as it approaches the boundary of the star.

Energy and Charge Density

The expression of density at the center is obtained as

$$\rho(0) = 6Cb > 0\cdot \tag{3.16}$$

Thus this implies that energy density is free from central singularity. The profile of energy and charged density is shown in Figures 3.4 and 3.5, which depicts its decreasing nature.

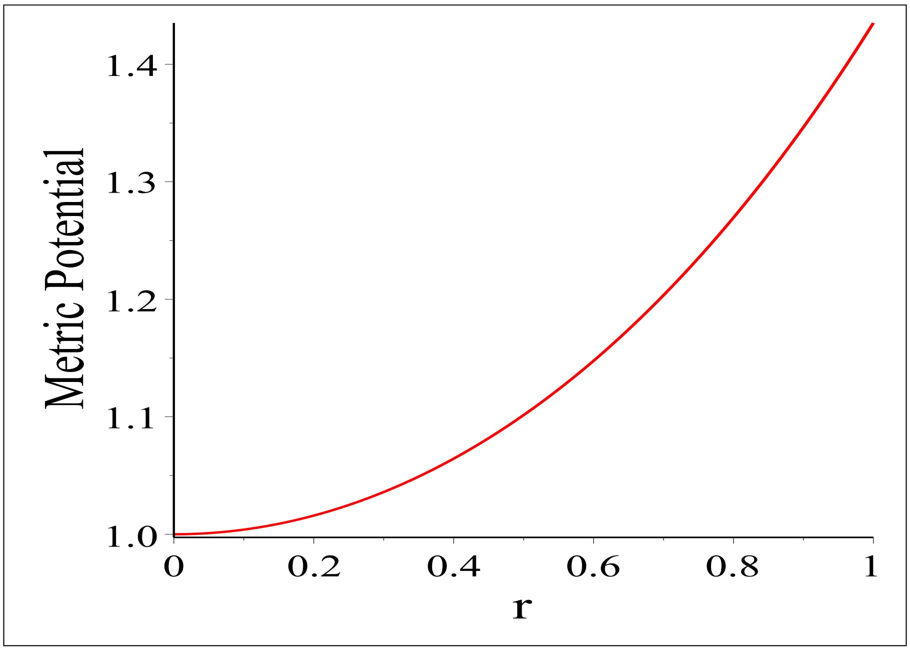

Figure 3.1: Graph of metric potential ($e^{2\lambda}$) with respect to r. In this and all other figures of this chapter the values of the constants are $a = 0.75$, $b = 1.65$, $C = 0.12$, $A = 0.038$ and $A_1 = 0.2$.

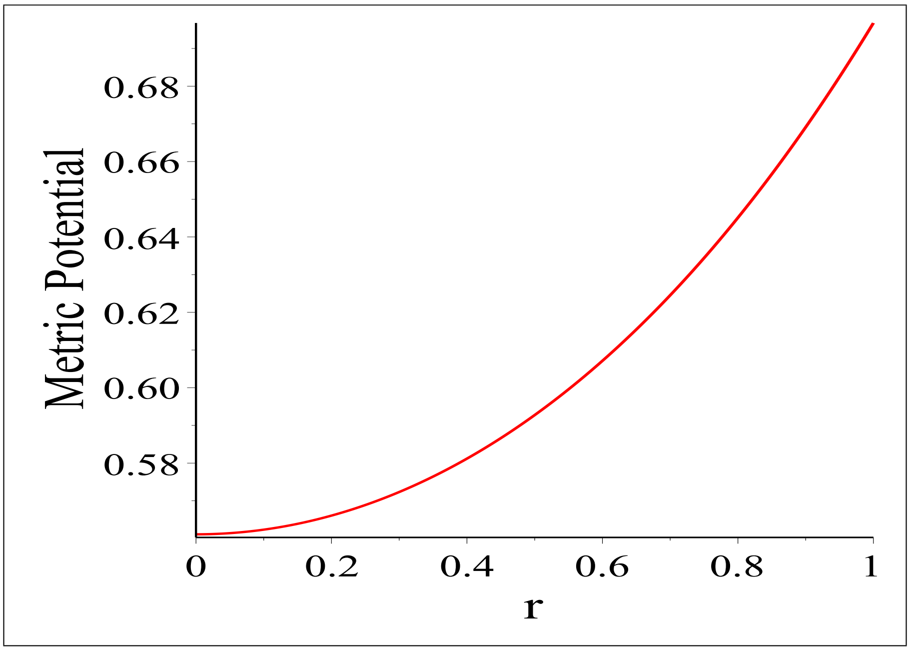

Figure 3.2: Graph of metric potential ($e^{2\nu}$) with respect to r

Radial and Transtential pressure

At $r = 0$, there is no singularity for pressures. The calculation formula for pressures value at $r = 0$ are as follow:

$$p_r(r = 0) = p_t(r = 0) = A(6Cb)^2 - \frac{B}{(6Cb)^2}. \quad (3.17)$$

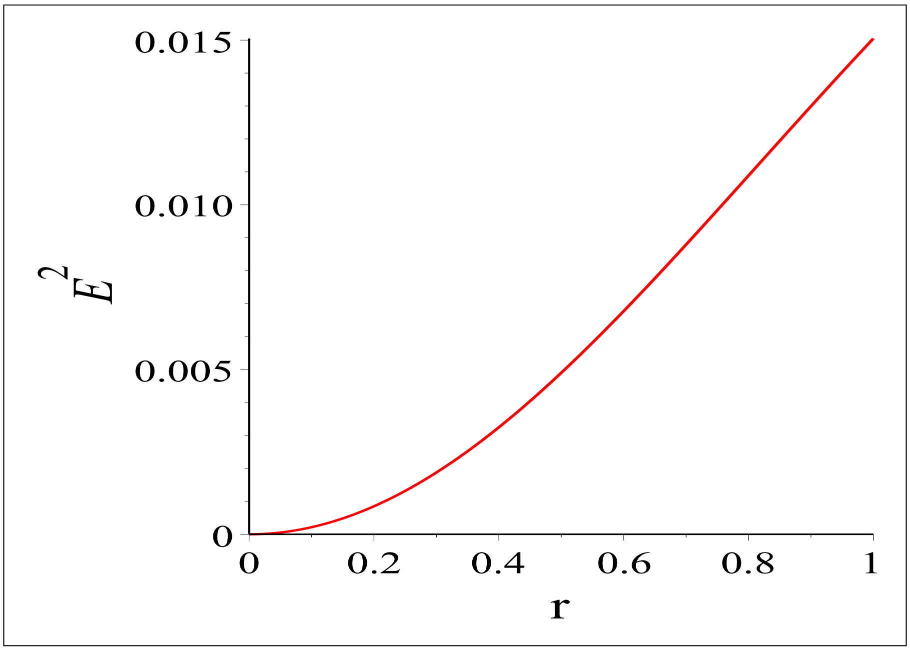

Figure 3.3: Graph of electric field intensity with respect to r

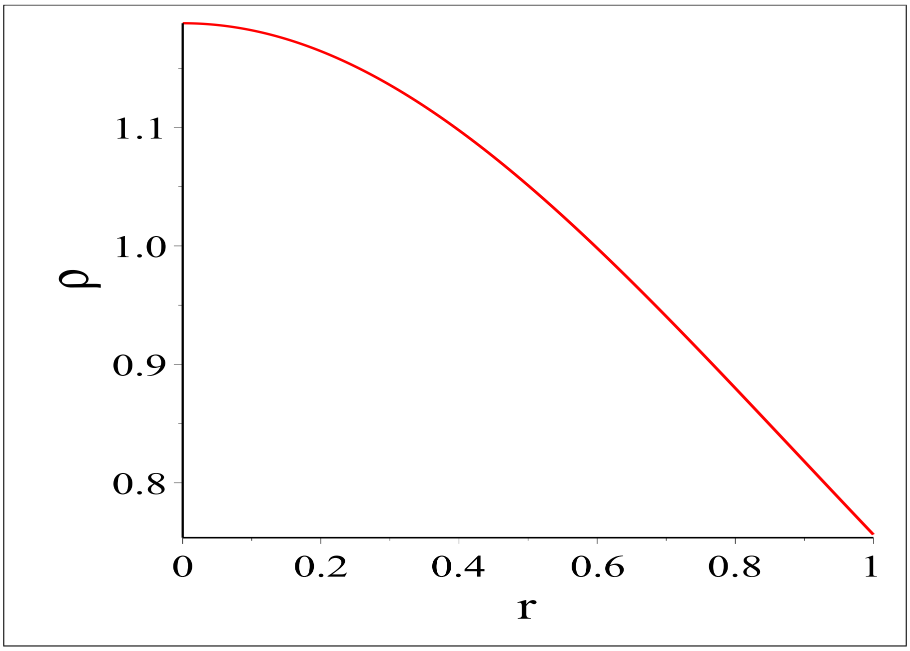

Figure 3.4: Graph of energy density with respect to r

Note that radial and tangential pressures are equal and positive at $r = 0$. Figures 3.6 and 3.7 show the behavior of radial and transverse pressure, respectively, where p_r is monotonically decreasing and $p_r(r = R) = 0$ and tangential pressure (p_t) also have decreasing nature.

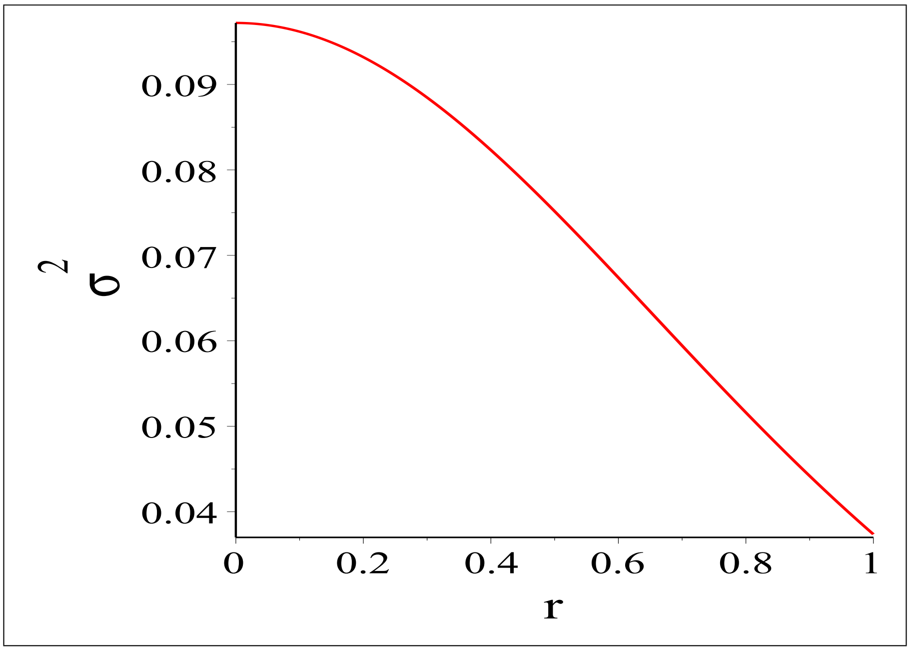

Figure 3.5: Graph of charge density with respect to r

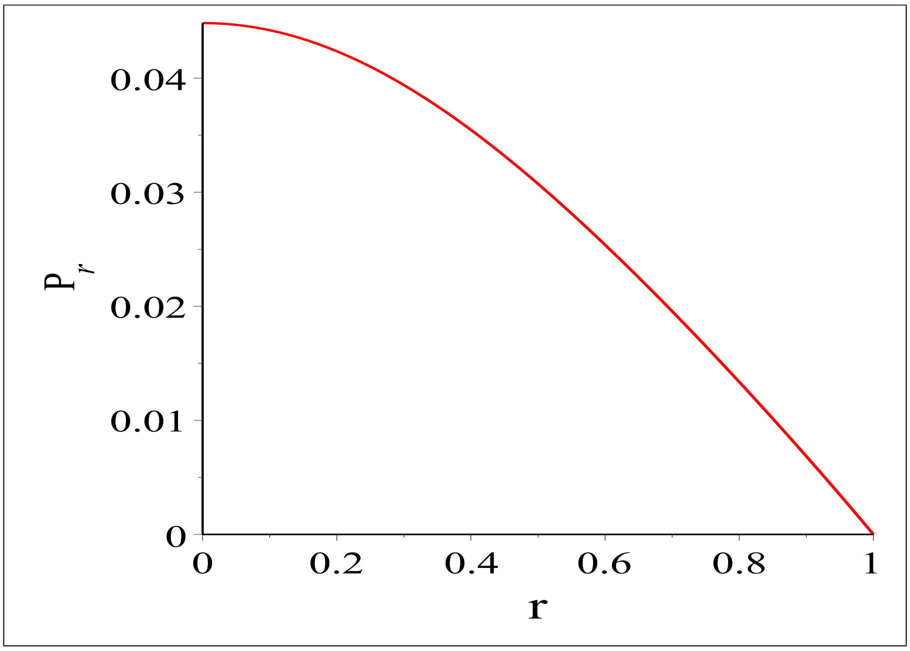

Figure 3.6: Graph of radial pressure with respect to r

Anisotropy

Anisotropy is defined as the difference between tangential and radial pressure. For the compact object, it is necessary that the anisotropic factor must be zero at the center and increases afterward. The anisotropic factor satisfies the required condition and the graph is shown in Figure 3.9.

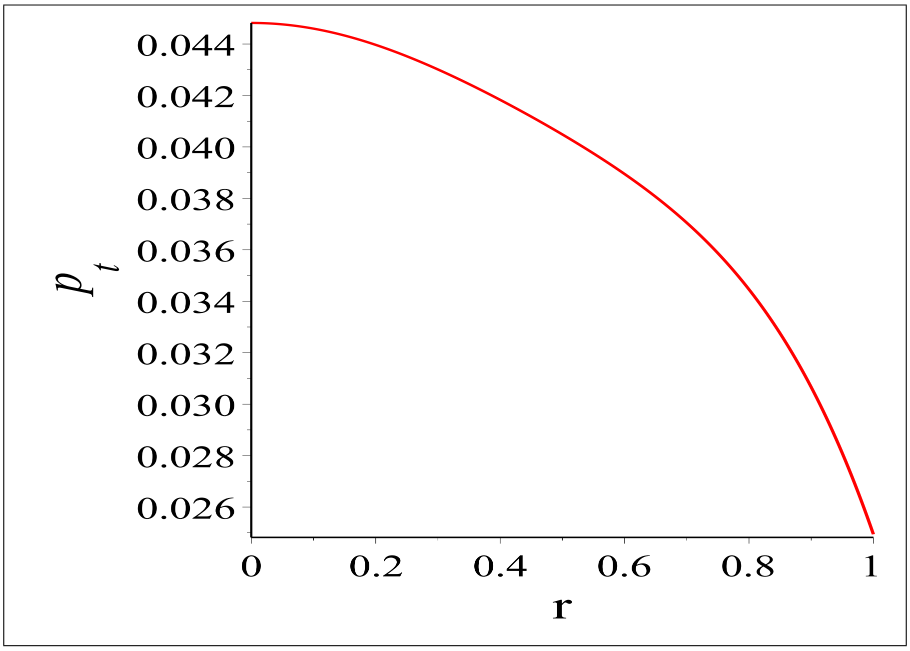

Figure 3.7: Graph of tangential pressure with respect to r

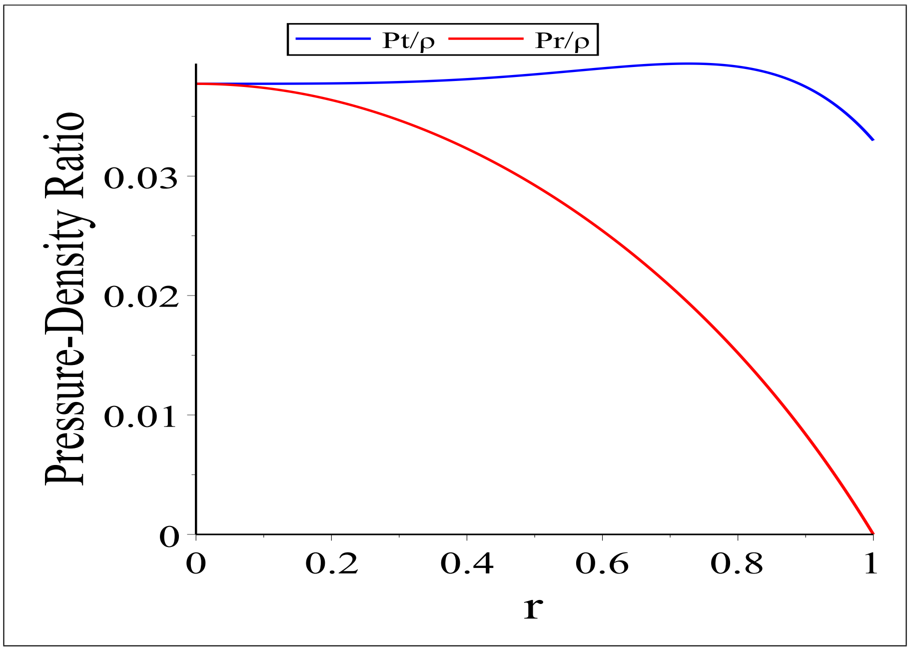

Figure 3.8: Graph of pressure density ratio with respect to r

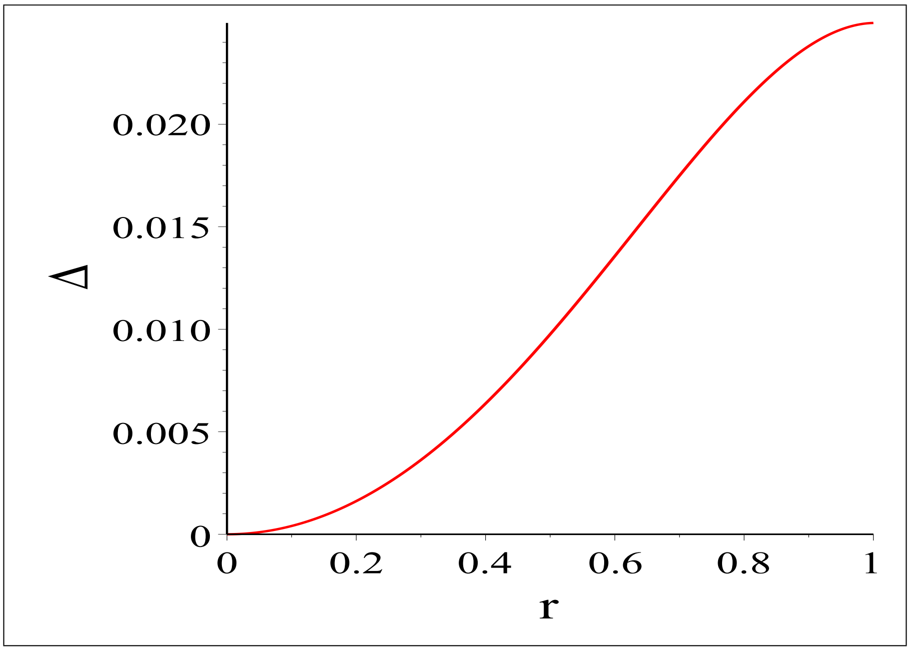

Figure 3.9: Graph of anisotropic Δ with respect to r

Trace of Energy Tensor

Bondi's [47] condition for an anisotropic fluid sphere states that the trace of energy-momentum tensor must be positive for compact objects to be acceptable. For our model the condition, $\rho - p_r - 2p_t > 0$, is satisfied, and a graph is shown in Figure 3.10.

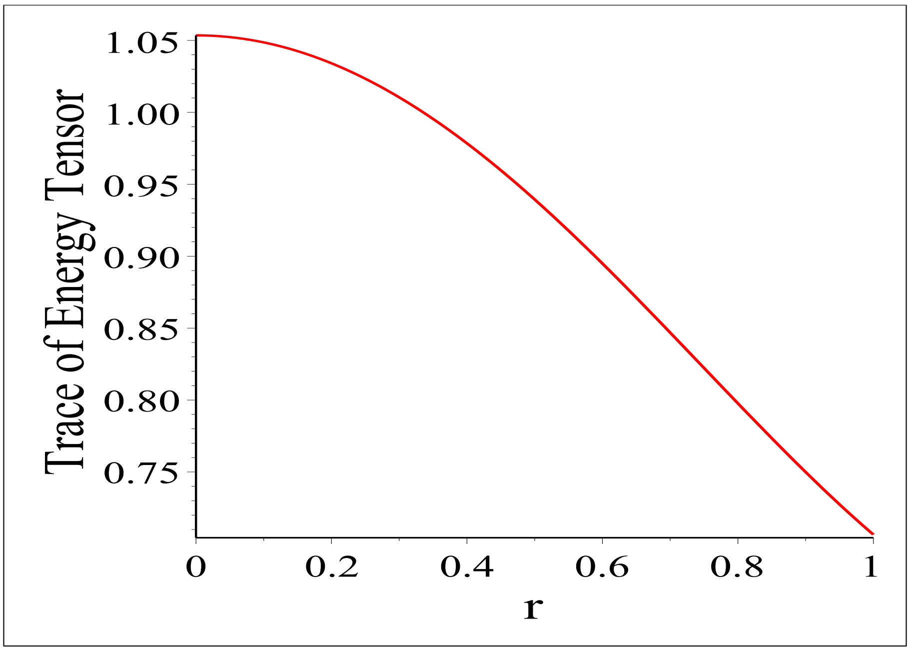

Figure 3.10: Graph of a trace of energy tensor with respect to r

Pressure-Density Ratio

$\frac{p}{\rho}$ is defined as the ratio of pressure to density. This ratio must be less than 1 throughout the stellar configuration. Figure 3.11 illustrates the graph of the pressure-density ratio which clearly satisfies the following property.

$$0 < \frac{p_r}{\rho} < 1 \quad , \quad 0 < \frac{p_t}{\rho} < 1 \cdot \tag{3.18}$$

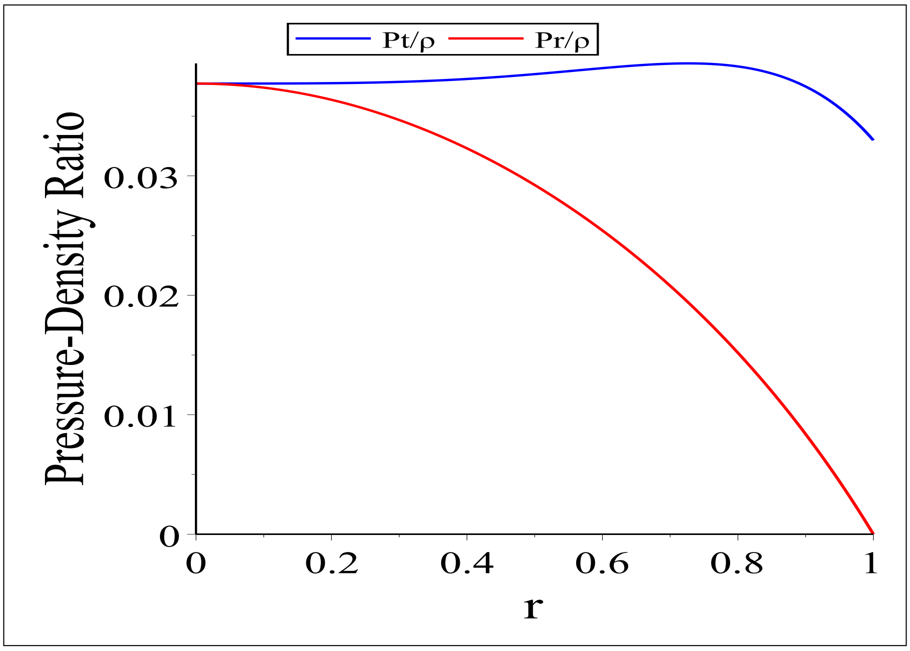

Figure 3.11: Graph of pressure density ratio with respect to r

Mass and Compactness factor

We previously derived the mass function using eq. (3.13). Buchdal [44] proposed that the ratio of mass to radius inside a compact star should be smaller than 4/9, meaning ($M/R < 4/9$). In our current models, the mass-radius ratio is M/R = 0.15, which is less than 4/9 as shown in Figure 3.12 illustrates the shape of the mass function, which is positive and smooth within the star's interior. The compactness factor for the stellar configuration is expressed as follows:

$$\mu = \frac{M(r)}{r} \tag{3.19}$$

The profile for our case is shown in Figure 3.13 which is increasing monotonically and is less than 4/9.

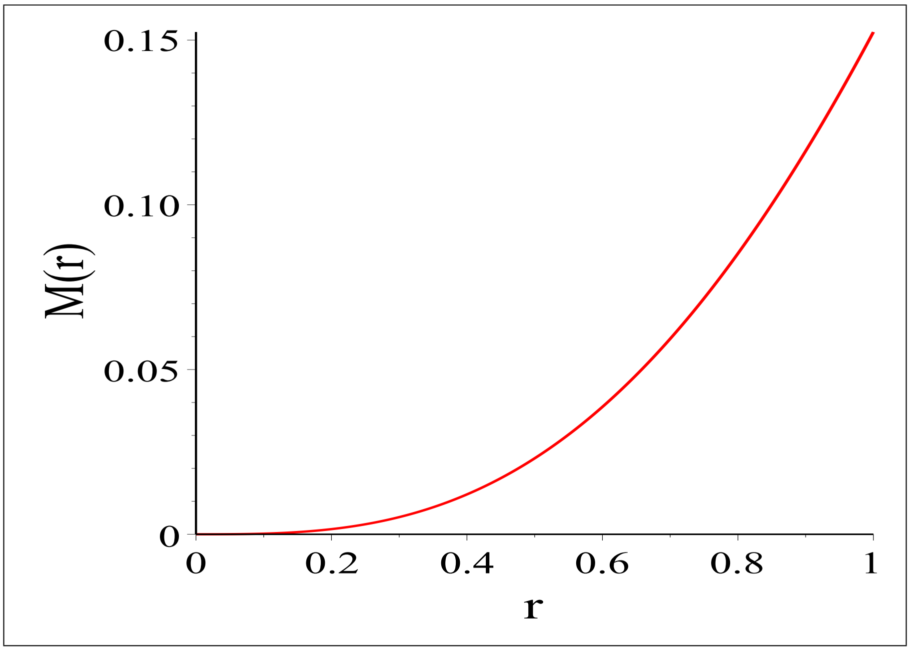

Figure 3.12: Graph of mass $M(r)$ with respect to r

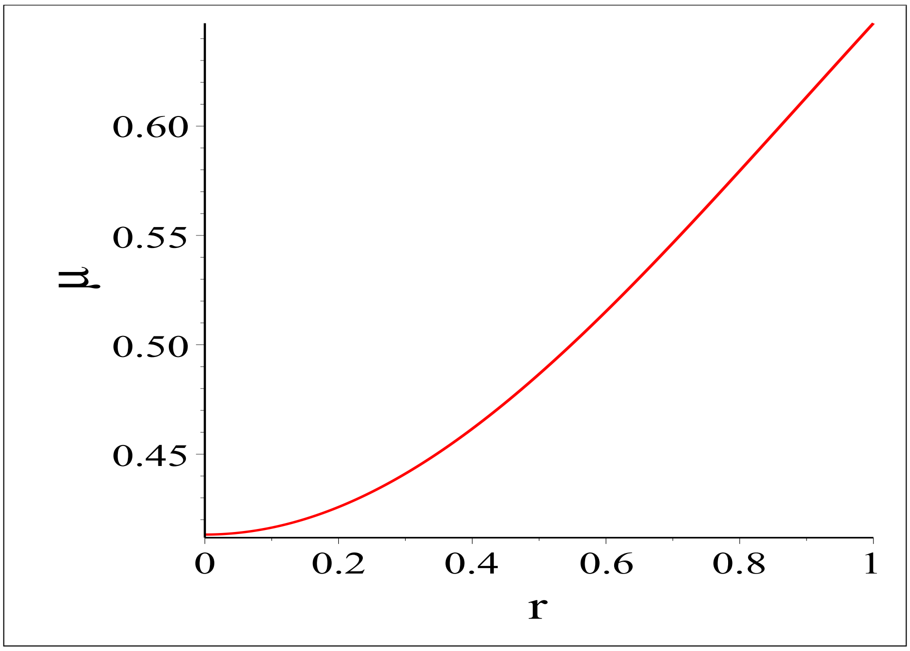

Figure 3.13: Graph of compactness factor with respect to r

Surface and Gravitational Redshift

A surface redshift of less than 1 is considered physically acceptable and is plotted in Figure 3.14, which exhibits features that increase towards the boundary, which is what nature depicts. Furthermore, the gravitational redshift (z) shown in Figure 3.15 has a decreasing nature for our model.

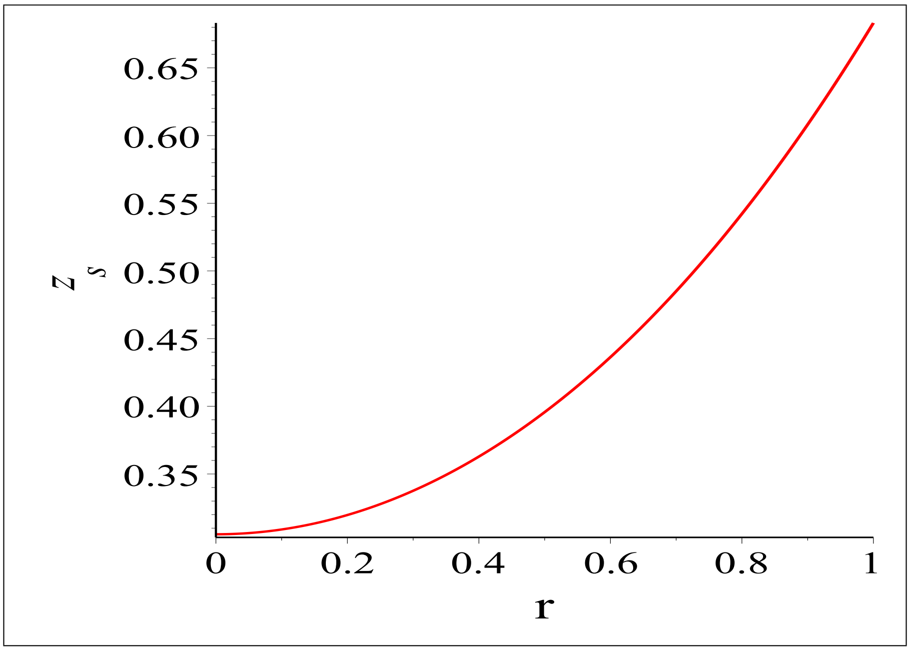

Figure 3.14: Graph of surface redshift with respect to r

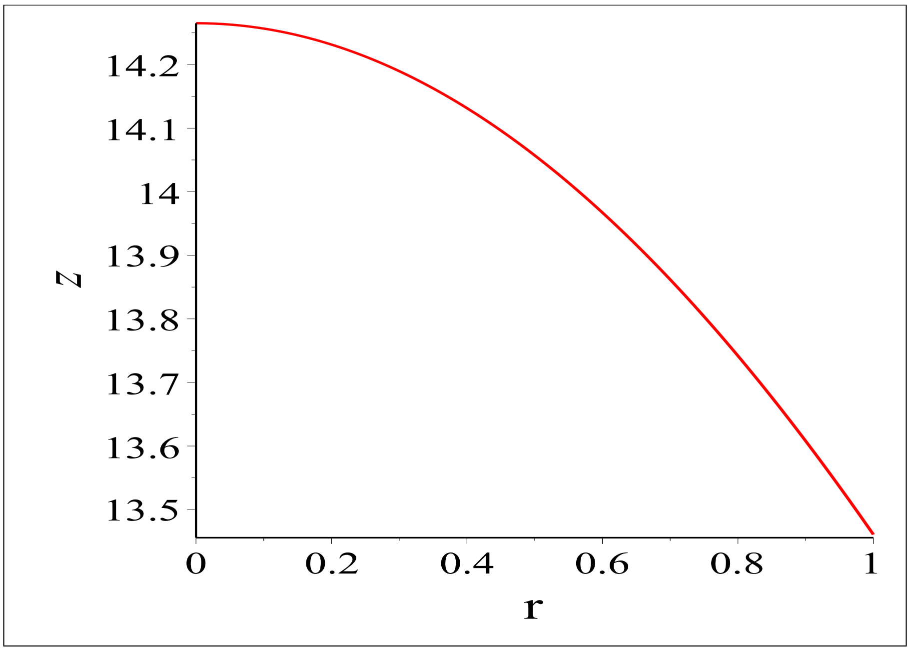

Figure 3.15: Graph of gravitational redshift with respect to r

Gradients

The calculation for the gradient of density and pressure in the radial direction for our model is presented below

$$\frac{d\rho}{dr} = -\frac{2C^2 r \left(\left(C^2 b^4 - C^2 a b^2 \right) r^4 + 4Cb^3 r^2 + 15b^2 + a \right)}{\left(Cbr^2 + 1 \right)^4}, \tag{3.20}$$

$$\begin{aligned}\frac{dp_r}{dr} = &-\frac{12AC^3br\left(\left(Cbr^2+1\right)\left(Cb^2r^2-Car^2+2b\right)+4b\right)^2}{\left(Cbr^2+1\right)^7}\\&-\frac{12Bbr\left(Cbr^2+1\right)^5}{C\left(\left(Cbr^2+1\right)\left(Cb^2r^2-Car^2+2b\right)+4b\right)^2}\\&+\frac{2B\left(Cbr^2+1\right)^6\left(2Cbr\left(Cb^2r^2-Car^2+2b\right)+\left(2Cb^2r-2Car\right)\left(Cbr^2+1\right)\right)}{C^2\left(\left(Cbr^2+1\right)\left(Cb^2r^2-Car^2+2b\right)+4b\right)^3}\\&+\frac{2AC^2\left(2Cbr\left(Cb^2r^2-Car^2+2b\right)+\left(2Cb^2r-2Car\right)\left(Cbr^2+1\right)\right)}{\left(Cbr^2+1\right)^3}\\&\times\frac{\left(\left(Cbr^2+1\right)\left(Cb^2r^2-Car^2+2b\right)+4b\right)}{\left(Cbr^2+1\right)^3}.\end{aligned}$$

The pressures and density gradients are graphed, and all of them have negative values as shown in Figures 3.16, 3.17 and 3.18.

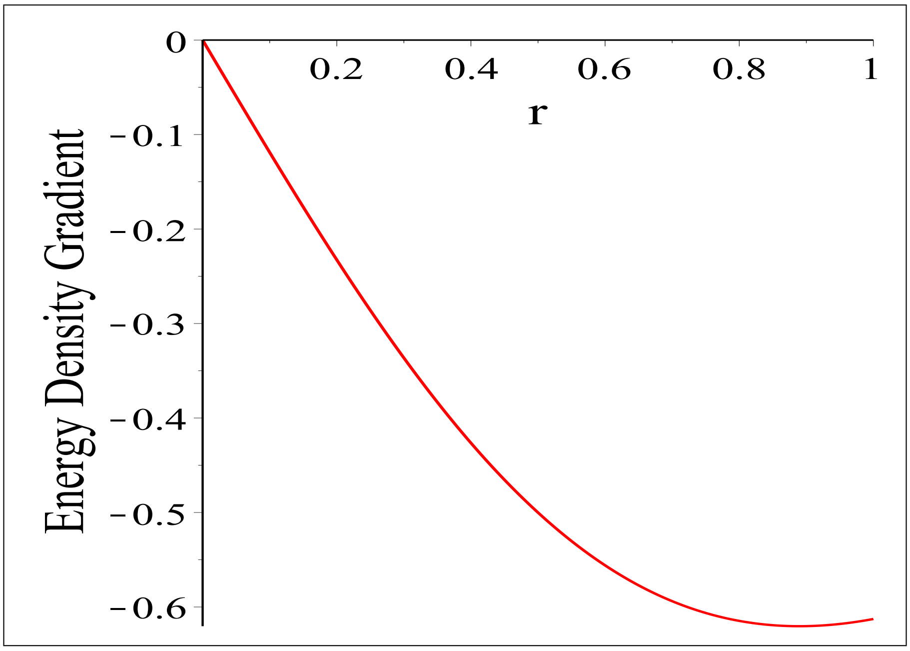

Figure 3.16: Graph of energy density gradient with respect to r

3.4 The Stability Analysis

In this section, we will check the stability conditions of our model.

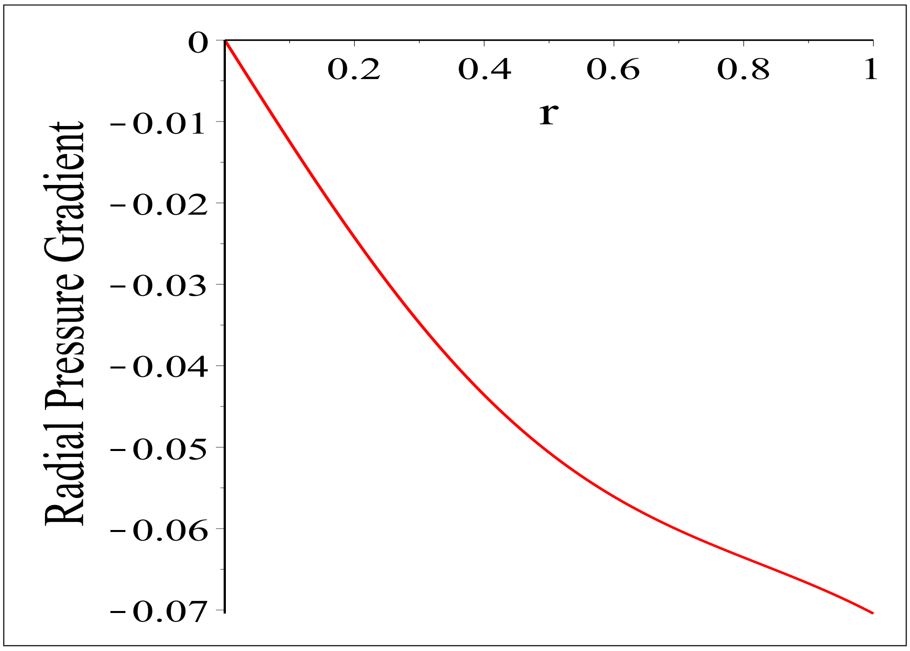

Figure 3.17: Graph of radial pressure gradient with respect to r

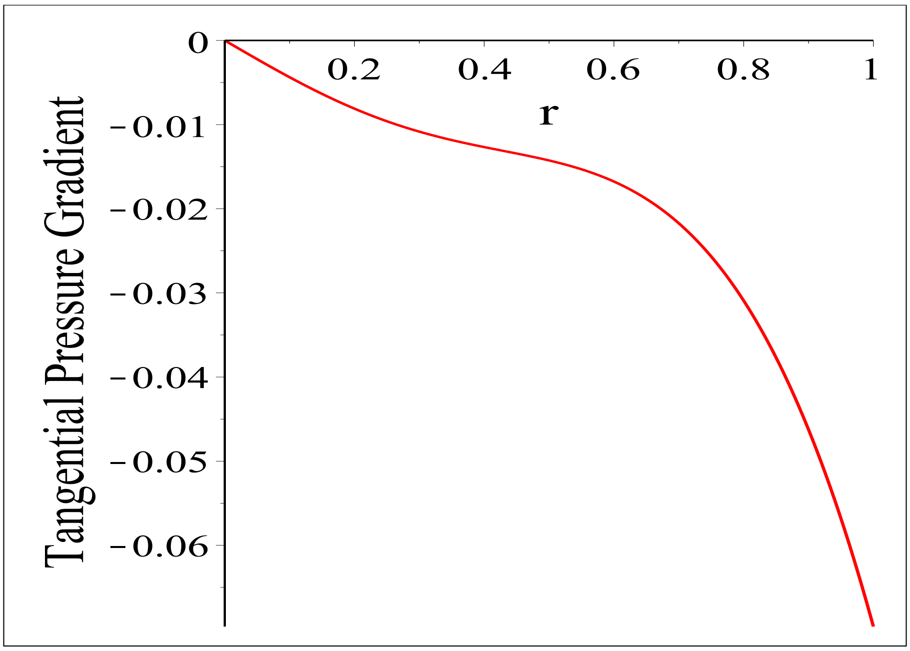

Figure 3.18: Graph of tangential pressure gradient with respect to r

Casuality Condition

The radial and tangential velocity is defined as $v_r^2 = \dfrac{dp_r}{d\rho}$ and $v_t^2 = \dfrac{dp_t}{d\rho}$ are plotted in Figure 3.19. The expression for radial velocity for our model is

$$v_r^2 = \frac{2B\left(Cbr^2+1\right)^9}{C^3\left(\left(Cbr^2+1\right)\left(Cb^2r^2-Car^2+2b\right)+4b\right)^3} + \frac{2AC\left(\left(Cbr^2+1\right)\left(Cb^2r^2-Car^2+2b\right)+4b\right)}{\left(Cbr^2+1\right)^3}. \tag{3.21}$$

and the expression for tangential velocity is shown in Appendix A2. The study of the speed of sound in stellar matter is important to understand stellar behavior. The stability of a stellar sphere needs to meet the conditions of radial velocity v_r^2 and tangential velocity v_t^2, expressed as $0 < v_r^2 < 1$ and $0 < v_t^2 < 1$, where $v_r^2 > v_t^2$. For stable stellar configurations, the inequality $0 \leq |v_t^2 - v_r^2| \leq 1$ is also satisfied as shown in Figure 3.20.

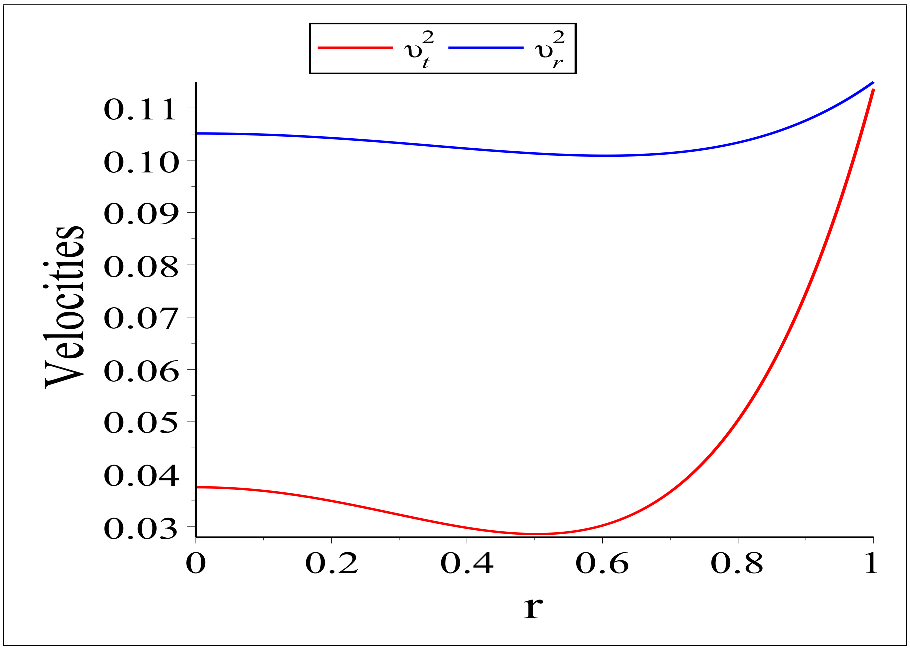

Figure 3.19: Graph of radial and tangential velocities with respect to r

Energy Conditions

For a physically permissible system, it should surely satisfy the following energy conditions throughout the stellar interior:

1. Null Energy Condition: $\rho(r) \geq 0$,

2. Weak Energy Condition: $\rho + p_r \geq 0$, $\rho + p_t \geq 0$,

3. Strong Energy Condition: $\rho + p_r + 2p_t \geq 0$.

For stable configuration Figures 3.21, 3.22 and 3.23 clearly show the well-behaved nature of energy requirements for our case.

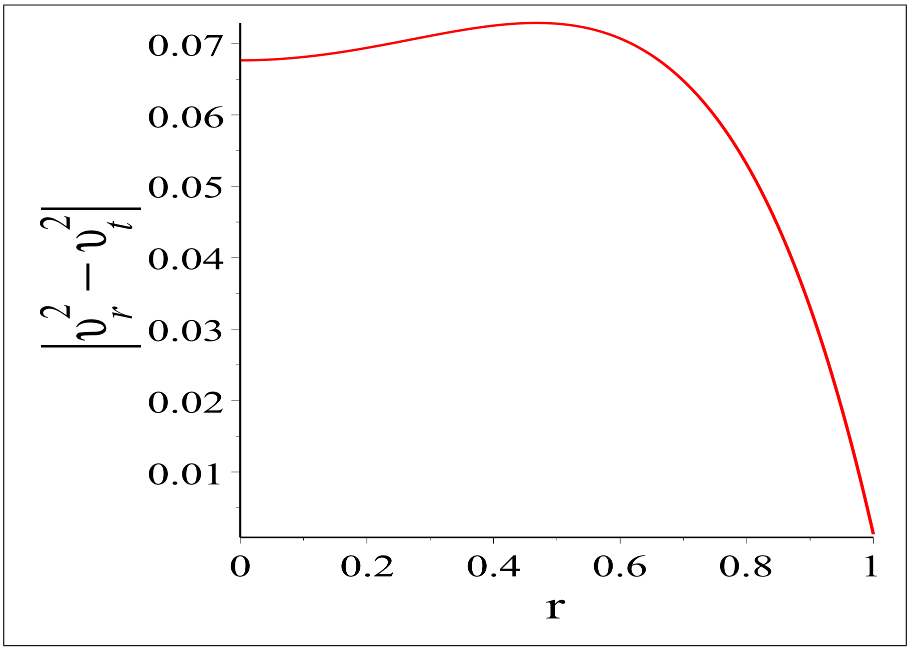

Figure 3.20: Graph of stability factor with respect to r

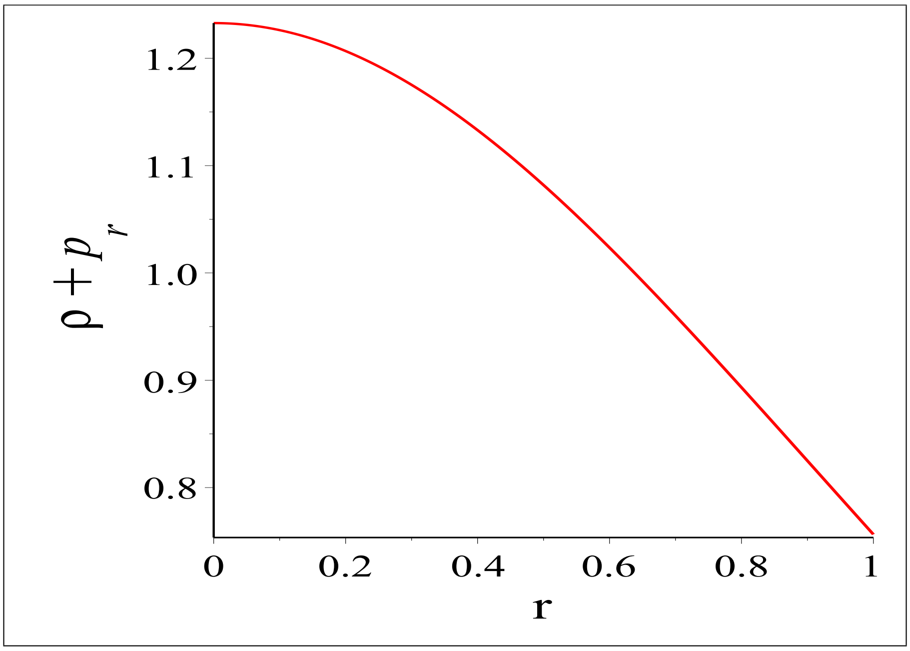

Figure 3.21: Graph of weak energy condition $(\rho + p_r)$ with respect to r

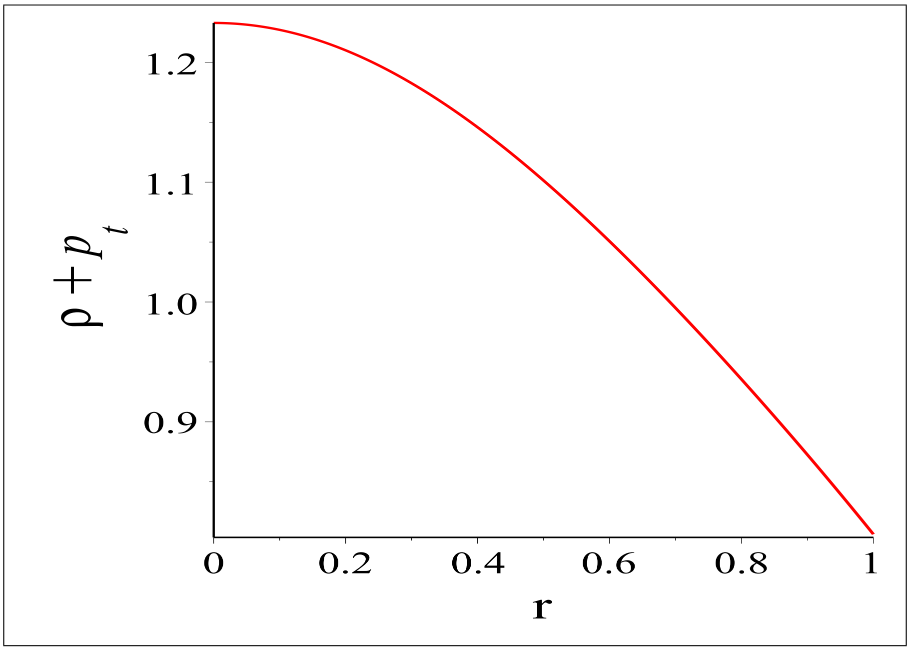

Figure 3.22: Graph of weak energy condition $(\rho + p_t)$ with respect to r

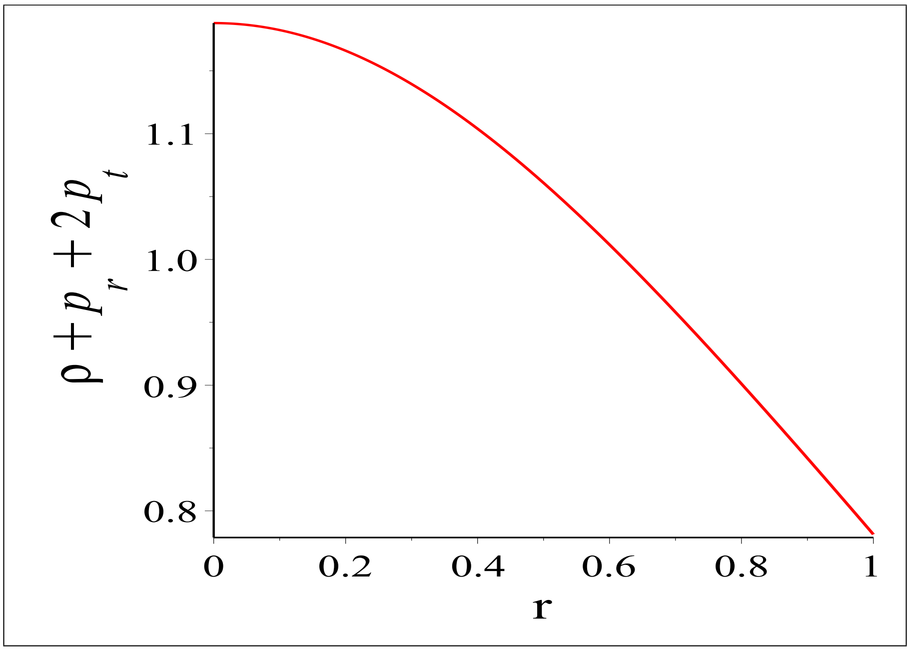

Figure 3.23: Graph of strong energy condition $(p_r + \rho + 2p_t)$ with respect to r

Adiabatic Index

Heinzmann and Hillebrandt [45] and Chen et al. [46] explained the two specific heat ratios for stable systems which are

$$\Gamma_r = \frac{\rho + p_r}{p_r} \frac{dp_r}{dr}. \tag{3.22}$$

The value of Γ_r should be greater than 4/3 in Figure 3.24, the condition for our model is shown, and we can clearly observe that this condition satisfies the stability in the relativistic fluid sphere $\Gamma_r > 4/3$.

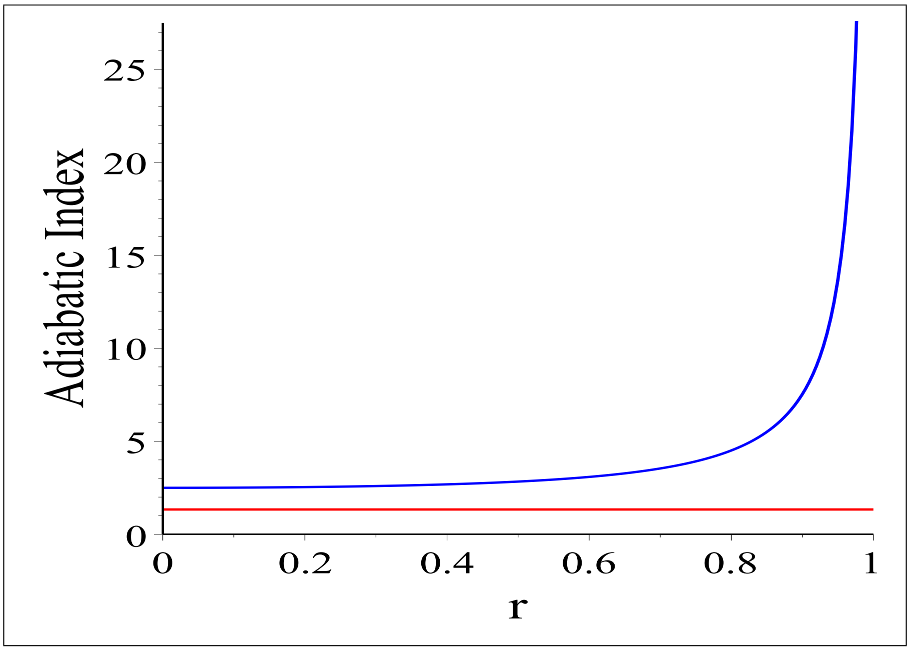

Figure 3.24: Graph of adiabatic index (Γ_r) with respect to r

Equilibrium State Under Various Forces

The hydrostatic equilibrium state of the model is studied by analyzing the Tolman-Oppenheimer-Volkov (TOV) equation [15, 16]. This equation dictates how the forces should be balanced, achieving hydrostatic equilibrium in a stellar sphere. The TOV equation for a charged anisotropic compact object is given by:

$$\frac{2}{r}(p_t - p_r) - \frac{dp_r}{dr} - (\rho + p_r)\nu' + \sigma E e^{\lambda} = 0\cdot \tag{3.23}$$

the eq. (3.23) can also be written as

$$F_h + F_g + F_e + F_a = 0\cdot \tag{3.24}$$

where F_h, F_g, F_e, F_a are the hydrostatic, gravitational, electric and anisotropic force respectively, given as

$$F_h = -\frac{dp_r}{dr}, \tag{3.25}$$

$$F_g = -(\rho + p_r)\nu', \tag{3.26}$$

$$F_e = \sigma E e^{\lambda}, \tag{3.27}$$

$$F_a = \frac{2}{r}(p_t - p_r)\cdot \tag{3.28}$$

In Figure 2.25, the nature of these forces are displayed. The graph demonstrates how F_g, which is dominant in nature, is balanced by the combination of three forces i.e. F_h, F_e and F_a.

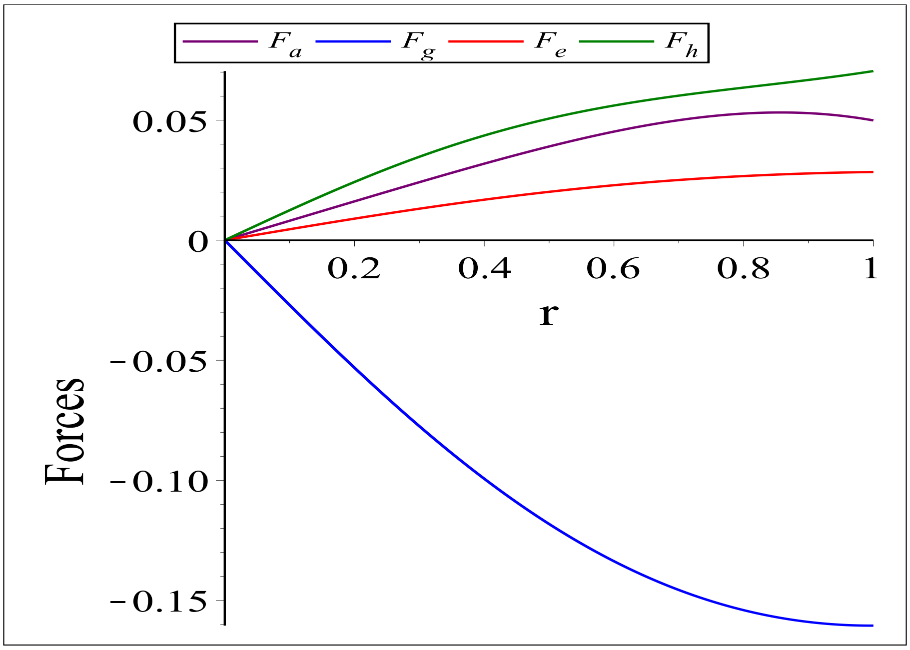

Figure 3.25: Graph of forces with respect to r

Chapter 4

Conclusion

In this thesis, we first discuss how numerous researchers have successfully obtained solutions to the Einstein field equations by employing various equations of state. In Chapter 1, we briefly discussed various tensors that are essential for the formulation of Einstein's field equations and also discussed in depth about these equations. This chapter also includes some important solutions of the Einstein field equations for point masses such as the Schwarzschild and the Reissner-Nordstrom solutions. In Chapter 2, we present a static and spherically symmetric model for a charged anisotropic compact star by choosing an equation of state, $p_r = A\rho^2 + B\rho - \frac{G}{\rho}$. This form of the equation of state reduces to the quadratic equation by putting $G = 0$, and to the Chaplygin equation of state by putting $A = 0$. We also choose gravitational metric potential of the form $Z = \frac{bx+1}{ax+1}$ and electric field intensity $\frac{E^2}{2C} = \frac{bx}{(ax+1)(bx+1)}$ for this model. In Chapter 3, we find another solution of the EMFEs. We generate a model that describes the distribution of electrically charged matter with spherically symmetric spacetime. For this, we choose a modified Chaplygin equation of state, $P_r = A\rho^2 - B/\rho^2$, where A and B are constants. The metric potential is $Z = \frac{1}{(1+bx)^2}$ and the electric field intensity is $\frac{E^2}{2C} = \frac{ax}{(bx+1)^2}$. We have plotted the graphs for both models for gravitational potentials, radial pressure, tangential pressure, anisotropy, pressure-density ratio, adiabatic index, mass, gradients, compactness, hydrostatic forces, and redshifts, all of which satisfy the physical requirements. In addition, energy conditions for both models are also satisfied. The causality condition is also obeyed for both models. Hence, all the conditions for a stable compact star are satisfied by both models.

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Appendix A

1

$$
\begin{aligned}
U = {} & \frac{a-b-1}{4b} + \frac{AC(9b^4 + (6-24a)b^3 + (22a^2 - 4a + 1)b^2 + (-8a^3 - 2a^2)b + a^4)}{4(b^3 - 3ab^2 + 3a^2b - a^3)} \\
& - \frac{B(3b - a + 1)}{4b} && \text{(A.1)} \\
T = {} & \frac{B}{2} - \frac{AC(9b^4 + (6-24a)b^3 + (22a^2 - 4a + 1)b^2 + (-8a^3 - 2a^2)b + a^4)}{4(b^3 - 3ab^2 + 3a^2b - a^3)} && \text{(A.2)} \\
J = {} & \frac{-G(3b^4 - 6ab^3 + (4a^2 - 4a + 1)b^2 + (4a^2 - 2a^3)b + a^4)}{4C^2(2b^6 + (6-6a)b^5 + (6a^2 - 12a + 6)b^4 + (-2a^3 + 6a^2 - 6a + 2)b^3)} && \text{(A.3)}
\end{aligned}
$$

$$
\begin{aligned}
H(x) = & - G\Big(7b^6 + (-18a - 3)\, b^5 + \left(9a^2 + 6a - 3\right) b^4 + \left(12a^3 - 12a^2 + 12a - 1\right) b^3 + \big(-15a^4 \\
& + 18a^3 - 9a^2\big)\, b^2 + \left(6a^5 - 9a^4\right) b - a^6\Big) \\
& \ln\left(\frac{\left|\left(2ab^2 + \left(2a - 2a^2\right)b\right)x - \sqrt{9b^4 + (6-24a)b^3 + (22a^2 - 16a + 1)b^2 + (10a^2 - 8a^3)b + a^4} + 3b^2 + (1-2a)b - a^2\right|}{\left|(2ab^2 + (2a - 2a^2)b)x + \sqrt{9b^4 + (6-24a)b^3 + (22a^2 - 16a + 1)b^2 + (10a^2 - 8a^3)b + a^4} + 3b^2 + (1-2a)b - a^2\right|} \right) \\
& + \frac{G\left(a^2b^2 + \left(a^2 - a^3\right)b\right)x^2 + \left(\left(4a - 2a^2\right)b + 2a^3\right)x}{4C^2\left(2b^4 + (4-4a)\, b^3 + (2a^2 - 4a + 2)\, b^2\right)} - \frac{B}{4(b^2x + b)} - \frac{x}{4} + AC\,\Big[\big(16ab^5 \\
& + \left(8a - 40a^2\right) b^4 + \left(32a^3 + 4a^2\right) b^3 + \left(-8a^4 - 16a^3 + 4a^2\right) b^2 + \left(4a^4 - 2a^3\right) b\big)\, x^3 + \big(20b^5 \\
& + (8 - 20a)\, b^4 + \left(32a - 36a^2\right) b^3 + \left(52a^3 - 36a^2 + 8a\right) b^2 + \big(-16a^4 - 8a^3 - a^2\big)\, b \\
& + 4a^4 - a^3\big)\, x^2 + \big(40b^4 + (28 - 88a)\, b^3 + \left(48a^2 + 4\right) b^2 + \left(8a^3 - 36a^2 + 4a\right) b - 8a^4 + 8a^3 \\
& - 2a^2\big)\, x + 20b^3 + (20 - 52a)\, b^2 + \left(44a^2 - 24a + 3\right) b - 12a^3 + 4a^2 - a\Big]\, \big(4(\big(2a^2b^4 - 4a^3b^3 \\
& + 2a^4b^2\big)\, x^4 + \left(4ab^4 - 4a^2b^3 - 4a^3b^2 + 4a^4b\right) x^3 + \left(2b^4 + 4ab^3 - 12a^2b^2 + 4a^3b + 2a^4\right) x^2 \\
& + \left(4b^3 - 4ab^2 - 4a^2b + 4a^3\right) x + 2b^2 - 4ab + 2a^2\big)^{-1}
\end{aligned}
\tag{A.4}
$$

$$
\begin{aligned}
\Delta(x) =&C \left(\frac{4x(1+bx)}{(1+ax)} \left[\left(\frac{AC\left((a-b)(ax+1)(bx+1)-2(b-a)(bx+1)-bx(ax+1)\right)^2}{4(ax+1)^3(bx+1)^3} \right.\right.\right.\\
&- \frac{G(ax+1)^3}{4C^2\left((a-b)(ax+1)(bx+1)-2(b-a)(bx+1)-bx(ax+1)\right)} + \frac{a-b}{4(bx+1)} - \frac{bx}{4(bx+1)^2}\\
&\left. + \frac{B\left((a-b)(ax+1)(bx+1)-2(b-a)(bx+1)-bx(ax+1)\right)}{4(ax+1)(bx+1)^2} \right)^2\\
&- \frac{3ACa\left(-2(b-a)(bx+1)+(a-b)(ax+1)(bx+1)-bx(ax+1)\right)^2}{4(ax+1)^4(bx+1)^3}\\
&- \frac{3ACb\left((a-b)(ax+1)(bx+1)-2(b-a)(bx+1)-bx(ax+1)\right)^2}{4(ax+1)^3(bx+1)^4}\\
&- \frac{3G(ax+1)^2}{4C^2\left((a-b)(ax+1)(bx+1)-2(b-a)(bx+1)-bx(ax+1)\right)}\\
&+ \frac{G(ax+1)^3\left(a(a-b)(bx+1)+(a-b)b(ax+1)-b(ax+1)-abx-2b(b-a)\right)}{4C^2\left((a-b)(ax+1)(bx+1)-2(b-a)(bx+1)-bx(ax+1)\right)^2}\\
&- \frac{(a-b)b}{4(bx+1)^2} - \frac{b}{4(bx+1)^2} + \frac{b^2x}{2(bx+1)^3}\\
&- \frac{Ba\left((a-b)(ax+1)(bx+1)-2(b-a)(bx+1)-bx(ax+1)\right)}{4(ax+1)^2(bx+1)^2}\\
&- \frac{Bb\left((a-b)(ax+1)(bx+1)-2(b-a)(bx+1)-bx(ax+1)\right)}{2(ax+1)(bx+1)^3}\\
&+ \frac{AC\left(a(a-b)(bx+1)+(a-b)b(ax+1)-b(ax+1)-abx-2b(b-a)\right)}{2(ax+1)^3(bx+1)^3}\\
&\times \frac{\left((a-b)(ax+1)(bx+1)-2(b-a)(bx+1)-bx(ax+1)\right)}{2(ax+1)^3(bx+1)^3}\\
&\left. + \frac{B\left(a(a-b)(bx+1)+(a-b)b(ax+1)-b(ax+1)-abx-2b(b-a)\right)}{4(ax+1)(bx+1)^2} \right]\\
&+ \left(\frac{2(b-a)x}{(ax+1)^2} + \frac{4(bx+1)}{ax+1} \right) \left(\frac{AC\left((a-b)(ax+1)(bx+1)-2(b-a)(bx+1)-bx\cdot(ax+1)\right)^2}{4(ax+1)^3(bx+1)^3} \right.\\
&- \frac{G(ax+1)^3}{4C^2\left((a-b)(ax+1)(bx+1)-2(b-a)(bx+1)-bx(ax+1)\right)} + \frac{a-b}{4(bx+1)}\\
&\left. - \frac{bx}{4(bx+1)^2} + \frac{B\left((a-b)(ax+1)(bx+1)-2(b-a)(bx+1)-bx(ax+1)\right)}{4(ax+1)(bx+1)^2} \right)\\
&\left. - \frac{bx}{(ax+1)(bx+1)} + \frac{b-a}{(ax+1)^2} \right)\\
&- \frac{AC^2\left((a-b)(ax+1)(bx+1)-2(b-a)(bx+1)-bx(ax+1)\right)^2}{(ax+1)^4(bx+1)^2}\\
&- \frac{BC\left((a-b)(ax+1)(bx+1)-2(b-a)(bx+1)-bx(ax+1)\right)}{(ax+1)^2(bx+1)}\\
&+ \frac{G(ax+1)^2(bx+1)}{C\left((a-b)(ax+1)(bx+1)-2(b-a)(bx+1)-bx(ax+1)\right)}
\end{aligned} \tag{A.5}
$$

$$
\begin{aligned}
v_t^2 = C \Bigg(& -\frac{2C^2br}{(Car^2+1)(Cbr^2+1)} + \frac{2C^3abr^3}{(Car^2+1)^2(Cbr^2+1)} + \frac{2C^3b^2r^3}{(Car^2+1)(Cbr^2+1)^2} \\
& - \frac{2C^2a(a-b)r}{(Car^2+1)^2} + \frac{8C^2a(b-a)r}{(Car^2+1)^3} \Bigg)^{-1} \Bigg(\frac{4Cr^2\left(Cbr^2+1\right)}{Car^2+1} \\
& \Bigg[\frac{6AC^2a^2r\left((a-b)\left(Car^2+1\right)\left(Cbr^2+1\right)-2(b-a)\left(Cbr^2+1\right)-Cbr^2\left(Car^2+1\right)\right)^2}{(Car^2+1)^5(Cbr^2+1)^3} \\
& + \frac{9AC^2abr\left((a-b)\left(Car^2+1\right)\left(Cbr^2+1\right)-2(b-a)\left(Cbr^2+1\right)-Cbr^2\left(Car^2+1\right)\right)^2}{(Car^2+1)^4(Cbr^2+1)^4} \\
& + \frac{6AC^2b^2r\left((a-b)\left(Car^2+1\right)\left(Cbr^2+1\right)-2(b-a)\left(Cbr^2+1\right)-Cbr^2\left(Car^2+1\right)\right)^2}{(Car^2+1)^3(Cbr^2+1)^5} \\
& - \frac{3Ga^2r\left(Cax^2+1\right)}{C\left((a-b)(Car^2+1)(Cbr^2+1)-2(b-a)(Cbr^2+1)-Cbr^2(Car^2+1)\right)} \\
& + \frac{G(4Ca(a-b)br-4Cabr)\left(Car^2+1\right)^3}{4C^2\left((a-b)(Car^2+1)(Cbr^2+1)-2(b-a)(Cbr^2+1)-Cbr^2(Car^2+1)\right)^2} \\
& + \frac{3Gar\left(Car^2+1\right)^2\left(a(a-b)\left(Cbr^2+1\right)+(a-b)b\left(Car^2+1\right)-b\left(Car^2+1\right)-Cabr^2-2b(b-a)\right)}{2C\left((a-b)(Car^2+1)(Cbr^2+1)-2(b-a)(Cbr^2+1)-Cbr^2(Car^2+1)\right)^2} \\
& + \frac{B(4Ca(a-b)br-4Cabr)}{4(Car^2+1)(Cbr^2+1)^2} + \frac{C(a-b)b^2r}{(Cbr^2+1)^3} + \frac{2Cb^2r}{(Cbr^2+1)^3} - \frac{3C^2b^3r^3}{(Cbr^2+1)^4} \\
& + \frac{3Ga\left(Car^2+1\right)^2\left(-2C^2abr^3+2Ca(a-b)r\left(Cbr^2+1\right)+2C(a-b)br\left(Car^2+1\right)\right)}{4C^2\left((a-b)(Car^2+1)(Cbr^2+1)-2(b-a)(Cbr^2+1)-Cbr^2(Car^2+1)\right)^2} \\
& - \frac{3Ga\left(Car^2+1\right)^2\left(2Cbr\left(Car^2+1\right)+4Cb(b-a)r\right)}{4C^2\left((a-b)(Car^2+1)(Cbr^2+1)-2(b-a)(Cbr^2+1)-Cbr^2(Car^2+1)\right)^2} \\
& - \frac{G\left(Car^2+1\right)^3\left(a(a-b)\left(Cbr^2+1\right)+(a-b)b\left(Car^2+1\right)-b\left(Car^2+1\right)-Cabr^2-2b(b-a)\right)}{\left((a-b)(Cax^2+1)(Cbr^2+1)-2(b-a)(Cbr^2+1)-Cbr^2(Car^2+1)\right)^3} \\
& \times \frac{\left(-2C^2abr^3+2Ca(a-b)r\left(Cbr^2+1\right)+2C(a-b)br\left(Car^2+1\right)-2Cbr\left(Car^2+1\right)-4Cb(b-a)r\right)}{2C^2} \\
& - \frac{Ba\left(-2C^2abr^3+2Ca(a-b)r\left(Cbr^2+1\right)+2C(a-b)br\left(Car^2+1\right)\right)}{4(Car^2+1)^2(Cbr^2+1)^2} \\
& - \frac{Ba\left(2Cbr\left(Car^2+1\right)+4Cb(b-a)r\right)}{4(Car^2+1)^2(Cbr^2+1)^2} - \frac{Bb\left(2Cbr\left(Car^2+1\right)+4Cb(b-a)r\right)}{2(Car^2+1)(Cbr^2+1)^3} \\
& - \frac{Bb\left(-2C^2abr^3+2Ca(a-b)r\left(Cbr^2+1\right)+2C(a-b)br\left(Car^2+1\right)\right)}{2(Car^2+1)(Cbr^2+1)^3} \\
& - \frac{3ACa\left((a-b)\left(Car^2+1\right)\left(Cbr^2+1\right)-2(b-a)\left(Cbr^2+1\right)-Cbr^2\left(Car^2+1\right)\right)}{(Cbr^2+1)^3} \\
& \times \frac{\left(-2C^2abr^3+2Ca(a-b)r\left(Cbr^2+1\right)+2C(a-b)br\left(Car^2+1\right)-2Cbr\left(Car^2+1\right)-4Cb(b-a)r\right)}{2(Car^2+1)^4} \\
& - \frac{3ACb\left((a-b)\left(Car^2+1\right)\left(Cbr^2+1\right)-2(b-a)\left(Cbr^2+1\right)-Cbx^2\left(Car^2+1\right)\right)}{(Cbr^2+1)^4} \\
& \times \frac{\left(-2C^2abr^3+2Ca(a-b)r\left(Cbr^2+1\right)+2C(a-b)br\left(Car^2+1\right)-2Cbr\left(Car^2+1\right)-4Cb(b-a)r\right)}{2(Car^2+1)^3} \\
& + \frac{AC\left(a(a-b)\left(Cbr^2+1\right)+(a-b)b\left(Car^2+1\right)-b\left(Car^2+1\right)-Cabr^2-2b(b-a)\right)}{(Cbr^2+1)^3}
\end{aligned}
$$

$$\times \frac{\left(-2C^2abr^3 + 2Ca\left(a-b\right)r\left(Cbr^2+1\right) + 2C\left(a-b\right)br\left(Cax^2+1\right) - 2Cbr\left(Car^2+1\right) - 4Cb\left(b-a\right)r\right)}{2\left(Car^2+1\right)^3}$$

$$+\frac{BCa^2r\left(\left(a-b\right)\left(Car^2+1\right)\left(Cbr^2+1\right) - 2\left(b-a\right)\left(Cbr^2+1\right) - Cbr^2\left(Car^2+1\right)\right)}{\left(Car^2+1\right)^3\left(Cbr^2+1\right)^2}$$

$$+\frac{2BCabr\left(\left(a-b\right)\left(Car^2+1\right)\left(Cbr^2+1\right) - 2\left(b-a\right)\left(Cbr^2+1\right) - Cbr^2\left(Car^2+1\right)\right)}{\left(Car^2+1\right)^2\left(Cbr^2+1\right)^3}$$

$$+\frac{AC\left(4Ca\left(a-b\right)br - 4Cabr\right)\left(\left(a-b\right)\left(Car^2+1\right)\left(Cbr^2+1\right) - 2\left(b-a\right)\left(Cbr^2+1\right) - Cbr^2\left(Car^2+1\right)\right)}{2\left(Car^2+1\right)^3\left(Cbr^2+1\right)^3}$$

$$-\frac{AC\left(4Ca\left(a-b\right)br - 4Cabr\right)\left(Cbr^2\left(Car^2+1\right)\right)}{2\left(Car^2+1\right)^3\left(Cbr^2+1\right)^3}$$

$$+\frac{3BCb^2r\left(\left(a-b\right)\left(Car^2+1\right)\left(Cbr^2+1\right) - 2\left(b-a\right)\left(Cbr^2+1\right) - Cbr^2\left(Car^2+1\right)\right)}{\left(Cax^2+1\right)\left(Cbx^2+1\right)^4}$$

$$-\frac{3AC^2ar\left(a\left(a-b\right)\left(Cbr^2+1\right) + \left(a-b\right)b\left(Car^2+1\right) - b\left(Car^2+1\right) - Cabr^2 - 2b\left(b-a\right)\right)}{\left(Cbr^2+1\right)^3}$$

$$\times \frac{\left(\left(a-b\right)\left(Car^2+1\right)\left(Cbr^2+1\right) - 2\left(b-a\right)\left(Cbr^2+1\right) - Cbr^2\left(Car^2+1\right)\right)}{\left(Car^2+1\right)^4}$$

$$-\frac{3AC^2br\left(a\left(a-b\right)\left(Cbr^2+1\right) + \left(a-b\right)b\left(Car^2+1\right) - b\left(Car^2+1\right) - Cabr^2 - 2b\left(b-a\right)\right)}{\left(Cbr^2+1\right)^4}$$

$$\times \frac{\left(\left(a-b\right)\left(Car^2+1\right)\left(Cbr^2+1\right) - 2\left(b-a\right)\left(Cbr^2+1\right) - Cbr^2\left(Car^2+1\right)\right)}{\left(Car^2+1\right)^3}$$

$$-\frac{BCar\left(a\left(a-b\right)\left(Cbr^2+1\right) + \left(a-b\right)b\left(Car^2+1\right) - b\left(Car^2+1\right) - Cabr^2 - 2b\left(b-a\right)\right)}{2\left(Car^2+1\right)^2\left(Cbr^2+1\right)^2}$$

$$-\frac{BCbr\left(a\left(a-b\right)\left(Cbr^2+1\right) + \left(a-b\right)b\left(Car^2+1\right) - b\left(Car^2+1\right) - Cabr^2 - 2b\left(b-a\right)\right)}{\left(Car^2+1\right)\left(Cbr^2+1\right)^3}$$

$$+2\left(-\frac{3AC^2ar\left(\left(a-b\right)\left(Car^2+1\right)\left(Cbr^2+1\right) - 2\left(b-a\right)\left(Cbr^2+1\right) - Cbr^2\left(Car^2+1\right)\right)^2}{2\left(Car^2+1\right)^4\left(Cbr^2+1\right)^3}\right.$$

$$-\frac{3AC^2br\left(\left(a-b\right)\left(Car^2+1\right)\left(Cbr^2+1\right) - 2\left(b-a\right)\left(Cbr^2+1\right) - Cbr^2\left(Car^2+1\right)\right)^2}{2\left(Car^2+1\right)^3\left(Cbr^2+1\right)^4}$$

$$-\frac{3Gar\left(Car^2+1\right)^2}{2C\left(\left(a-b\right)\left(Car^2+1\right)\left(Cbr^2+1\right) - 2\left(b-a\right)\left(Cbr^2+1\right) - Cbr^2\left(Car^2+1\right)\right)}$$

$$-\frac{C\left(a-b\right)br}{2\left(Cbr^2+1\right)^2} - \frac{Cbr}{2\left(Cbr^2+1\right)^2} + \frac{C^2b^2r^3}{\left(Cbr^2+1\right)^3}$$

$$+\frac{G\left(Car^2+1\right)^3\left(-2C^2abr^3 + 2Ca\left(a-b\right)r\left(Cbr^2+1\right) + 2C\left(a-b\right)br\left(Car^2+1\right)\right)}{4C^2\left(\left(a-b\right)\left(Car^2+1\right)\left(Cbr^2+1\right) - 2\left(b-a\right)\left(Cbr^2+1\right) - Cbr^2\left(Car^2+1\right)\right)^2}$$

$$-\frac{G\left(Car^2+1\right)^3\left(2Cbr\left(Car^2+1\right) + 4Cb\left(b-a\right)r\right)}{4C^2\left(\left(a-b\right)\left(Car^2+1\right)\left(Cbr^2+1\right) - 2\left(b-a\right)\left(Cbr^2+1\right) - Cbr^2\left(Car^2+1\right)\right)^2}$$

$$+\frac{B\left(-2C^2abr^3 + 2Ca\left(a-b\right)r\left(Cbr^2+1\right) + 2C\left(a-b\right)br\left(Car^2+1\right)\right)}{4\left(Car^2+1\right)\left(Cbr^2+1\right)^2}$$

$$-\frac{B\left(2Cbr\left(Car^2+1\right) + 4Cb\left(b-a\right)r\right)}{4\left(Car^2+1\right)\left(Cbr^2+1\right)^2}$$

$$+\frac{AC\left(\left(a-b\right)\left(Car^2+1\right)\left(Cbr^2+1\right) - 2\left(b-a\right)\left(Cbr^2+1\right) - Cbr^2\left(Car^2+1\right)\right)}{\left(Cbr^2+1\right)^3}$$

$$\times \frac{\left(-2C^2abr^3 + 2Ca\left(a-b\right)r\left(Cbr^2+1\right) + 2C\left(a-b\right)br\left(Car^2+1\right) - 2Cbr\left(Car^2+1\right) - 4Cb\left(b-a\right)r\right)}{2\left(Car^2+1\right)^3}$$

$$-\frac{BCar\left((a-b)\left(Car^2+1\right)\left(Cbr^2+1\right)-2\left(b-a\right)\left(Cbr^2+1\right)-Cbr^2\left(Car^2+1\right)\right)}{2\left(Car^2+1\right)^2\left(Cbr^2+1\right)^2}$$

$$-\frac{BCbr\left((a-b)\left(Car^2+1\right)\left(Cbr^2+1\right)-2\left(b-a\right)\left(Cbr^2+1\right)-Cbr^2\left(Car^2+1\right)\right)}{\left(Car^2+1\right)\left(Cbr^2+1\right)^3}\Bigg)$$

$$\Bigg(\frac{AC\left((a-b)\left(Car^2+1\right)\left(Cbr^2+1\right)-2\left(b-a\right)\left(Cbr^2+1\right)-Cbr^2\left(Car^2+1\right)\right)^2}{4\left(Car^2+1\right)^3\left(Cbr^2+1\right)^3}$$

$$-\frac{G\left(Car^2+1\right)^3}{4C^2\left((a-b)\left(Car^2+1\right)\left(Cbr^2+1\right)-2\left(b-a\right)\left(Cbr^2+1\right)-Cbr^2\left(Car^2+1\right)\right)}$$

$$+\frac{a-b}{4\left(Cbr^2+1\right)}-\frac{Cbr^2}{4\left(Cbr^2+1\right)^2}$$

$$+\frac{B\left((a-b)\left(Car^2+1\right)\left(Cbx^2+1\right)-2\left(b-a\right)\left(Cbr^2+1\right)-Cbr^2\left(Car^2+1\right)\right)}{4\left(Car^2+1\right)\left(Cbr^2+1\right)^2}\Bigg)$$

$$+\left(\frac{8Cbr}{Car^2+1}+\frac{4C\left(b-a\right)r}{\left(Car^2+1\right)^2}-\frac{8C^2a\left(b-a\right)r^3}{\left(Car^2+1\right)^3}-\frac{8Car\left(Cbr^2+1\right)}{\left(Car^2+1\right)^2}\right)$$

$$\Bigg(\frac{AC\left((a-b)\left(Car^2+1\right)\left(Cbr^2+1\right)-2\left(b-a\right)\left(Cbr^2+1\right)-Cbr^2\left(Car^2+1\right)\right)^2}{4\left(Car^2+1\right)^3\left(Cbr^2+1\right)^3}$$

$$-\frac{G\left(Car^2+1\right)^3}{4C^2\left((a-b)\left(Car^2+1\right)\left(Cbr^2+1\right)-2\left(b-a\right)\left(Cbr^2+1\right)-Cbr^2\left(Car^2+1\right)\right)}$$

$$+\frac{a-b}{4\left(Cbr^2+1\right)}-\frac{Cbr^2}{4\left(Cbr^2+1\right)^2}$$

$$+\frac{B\left((a-b)\left(Car^2+1\right)\left(Cbr^2+1\right)-2\left(b-a\right)\left(Cbr^2+1\right)-Cbr^2\left(Car^2+1\right)\right)}{4\left(Car^2+1\right)\left(Cbr^2+1\right)^2}\Bigg)$$

$$+\left(\frac{2C\left(b-a\right)r^2}{\left(Car^2+1\right)^2}+\frac{4\left(Cbr^2+1\right)}{Car^2+1}\right)$$

$$\Bigg(-\frac{3AC^2ar\left((a-b)\left(Car^2+1\right)\left(Cbr^2+1\right)-2\left(b-a\right)\left(Cbx^2+1\right)-Cbr^2\left(Car^2+1\right)\right)^2}{2\left(Car^2+1\right)^4\left(Cbr^2+1\right)^3}$$

$$-\frac{3AC^2br\left((a-b)\left(Car^2+1\right)\left(Cbr^2+1\right)-2\left(b-a\right)\left(Cbr^2+1\right)-Cbr^2\left(Car^2+1\right)\right)^2}{2\left(Car^2+1\right)^3\left(Cbr^2+1\right)^4}$$

$$-\frac{3Gar\left(Car^2+1\right)^2}{2C\left((a-b)\left(Car^2+1\right)\left(Cbr^2+1\right)-2\left(b-a\right)\left(Cbr^2+1\right)-Cbr^2\left(Car^2+1\right)\right)}$$

$$-\frac{C\left(a-b\right)br}{2\left(Cbr^2+1\right)^2}-\frac{Cbr}{2\left(Cbr^2+1\right)^2}+\frac{C^2b^2r^3}{\left(Cbr^2+1\right)^3}$$

$$+\frac{G\left(Car^2+1\right)^3\left(-2C^2abr^3+2Ca\left(a-b\right)r\left(Cbr^2+1\right)+2C\left(a-b\right)br\left(Car^2+1\right)\right)}{4C^2\left((a-b)\left(Car^2+1\right)\left(Cbr^2+1\right)-2\left(b-a\right)\left(Cbr^2+1\right)-Cbr^2\left(Car^2+1\right)\right)^2}$$

$$-\frac{G\left(Car^2+1\right)^3\left(2Cbr\left(Car^2+1\right)+4Cb\left(b-a\right)r\right)}{4C^2\left((a-b)\left(Car^2+1\right)\left(Cbr^2+1\right)-2\left(b-a\right)\left(Cbr^2+1\right)-Cbr^2\left(Car^2+1\right)\right)^2}$$

$$+\frac{B\left(-2C^2abr^3+2Ca\left(a-b\right)r\left(Cbr^2+1\right)+2C\left(a-b\right)br\left(Car^2+1\right)-2Cbr\left(Car^2+1\right)-4Cb\left(b-a\right)r\right)}{4\left(Car^2+1\right)\left(Cbr^2+1\right)^2}$$

$$-\frac{\left(2Cbr\left(Car^2+1\right)+4Cb\left(b-a\right)r\right)}{4\left(Car^2+1\right)\left(Cbr^2+1\right)^2}$$

$$+\frac{AC\left((a-b)\left(Car^2+1\right)\left(Cbr^2+1\right)-2\left(b-a\right)\left(Cbr^2+1\right)-Cbr^2\left(Car^2+1\right)\right)}{\left(Cbr^2+1\right)^3}$$

$$
\times \frac{\left(-2C^2abr^3 + 2Ca\,(a-b)\,r\left(Cbr^2+1\right) + 2C\,(a-b)\,br\left(Car^2+1\right) - 2Cbr\left(Car^2+1\right) - 4Cb\,(b-a)\,r\right)}{2\left(Car^2+1\right)^3}
$$

$$
- \frac{BCar\left((a-b)\left(Car^2+1\right)\left(Cbr^2+1\right) - 2\,(b-a)\left(Cbr^2+1\right) - Cbr^2\left(Car^2+1\right)\right)}{2\left(Car^2+1\right)^2\left(Cbr^2+1\right)^2}
$$

$$
\left. - \frac{BCbr\left((a-b)\left(Car^2+1\right)\left(Cbr^2+1\right) - 2\,(b-a)\left(Cbr^2+1\right) - Cbr^2\left(Car^2+1\right)\right)}{\left(Car^2+1\right)\left(Cbr^2+1\right)^3}\right)
$$

$$
\left. - \frac{2Cbr}{\left(Car^2+1\right)\left(Cbr^2+1\right)} + \frac{2C^2abx^3}{\left(Car^2+1\right)^2\left(Cbr^2+1\right)} + \frac{2C^2b^2x^3}{\left(Car^2+1\right)\left(Cbr^2+1\right)^2} - \frac{4Ca\,(b-a)\,r}{\left(Car^2+1\right)^3}\right]
$$

Appendix A

2

$$V(x) = \Bigg[C^3\sqrt{6b - \frac{(3b^2-a)^2}{4b(b^2-a)}}\left(2b^3-2ab\right)\left(b^{12}-6ab^{10}+15a^2b^8-20a^3b^6+15a^4b^4-6a^5b^2+a^6\right)$$
$$\left(15b^4-18ab^2-a^2\right)\Big]^{-1}\Big[\Big(-\left(3b^2-a\right)\left(885b^{14}-3477ab^{12}+1849a^2b^{10}+4439a^3b^8-2817a^4b^6\right.$$
$$\left.-1151a^5b^4-109a^6b^2-3a^7\right)\left(2\left(b^2-a\right)\right)^{-1}+1049b^{14}-8013ab^{12}+11493a^2b^{10}+1799a^3b^8-3a^7$$
$$-5829a^4b^6-1527a^5b^4-121a^6b^2\Big)\left(b^{\frac{3}{2}}B\sqrt{b^2-a}\right)\arctan\left(\frac{2b\left(b^2-a\right)x+3b^2-a}{2\sqrt{b}\sqrt{b^2-a}\sqrt{6b-\frac{(3b^2-a)^2}{4b(b^2-a)}}}\right)\Bigg]$$
$$+B\Big[\left(223b^{17}+536ab^{15}-4540a^2b^{13}+5480a^3b^{11}+602a^4b^9-2008a^5b^7-508a^6b^5-40a^7b^3\right.$$
$$\left.-a^8b\right)x+1227b^{16}-3524ab^{14}-2272a^2b^{12}+10716a^3b^{10}-2786a^4b^8-3436a^5b^6-648a^6b^4$$
$$-44a^7b^2-a^8\Big]\Big[4C^3\left(15b^6-33ab^4+17a^2b^2+a^3\right)\left(b^{12}-6ab^{10}+15a^2b^8-20a^3b^6+15a^4b^4\right.$$
$$\left.-6a^5b^2+a^6\right)\left((bx+1)\left(\left(b^2-a\right)x+2b\right)+4b\right)\Big]^{-1}-\frac{AC\left(\left(b^2+a\right)^2+8b^2\left(b^2-a\right)\right)}{4b^3(bx+1)}$$
$$-\frac{AC\left(b^2+a\right)}{b(bx+1)^2}-\frac{4ACb}{3(bx+1)^3}-\frac{B\left(b^{14}-4ab^{12}+6a^2b^{10}-4a^3b^8+a^4b^6\right)x^5}{20C^3\left(b^{12}-6ab^{10}+15a^2b^8-20a^3b^6+15a^4b^4-6a^5b^2+a^6\right)}$$
$$-\frac{B\left(2b^{13}-12ab^{11}+24a^2b^9-20a^3b^7+6a^4b^5\right)x^4}{16C^3\left(b^{12}-6ab^{10}+15a^2b^8-20a^3b^6+15a^4b^4-6a^5b^2+a^6\right)}+\frac{\left(b^2-a\right)x^2+4bx}{8}$$
$$-\frac{B\left(-5b^{12}+12ab^{10}+6a^2b^8-28a^3b^6+15a^4b^4\right)x^3}{12C^3\left(b^{12}-6ab^{10}+15a^2b^8-20a^3b^6+15a^4b^4-6a^5b^2+a^6\right)}+\frac{AC\left(b^2-a\right)^2x}{4b^2}$$
$$-\frac{B\left(8b^{11}+12ab^9-60a^2b^7+20a^3b^5+20a^4b^3\right)x^2}{8C^3\left(b^{12}-6ab^{10}+15a^2b^8-20a^3b^6+15a^4b^4-6a^5b^2+a^6\right)}$$
$$-\frac{B\left(19b^{10}\right)x}{4C^3(b^{12}-6ab^{10}+15a^2b^8-20a^3b^6+15a^4b^4-6a^5b^2+a^6)}$$
$$-\frac{Bx132ab^8}{4C^3(b^{12}-6ab^{10}+15a^2b^8-20a^3b^6+15a^4b^4-6a^5b^2+a^6)}$$
$$+\frac{Bx78a^2b^6}{4C^3(b^{12}-6ab^{10}+15a^2b^8-20a^3b^6+15a^4b^4-6a^5b^2+a^6)}$$
$$+\frac{Bx100a^3b^4}{4C^3(b^{12}-6ab^{10}+15a^2b^8-20a^3b^6+15a^4b^4-6a^5b^2+a^6)}$$

$$+\frac{Bx15a^4b^2}{4C^3(b^{12}-6ab^{10}+15a^2b^8-20a^3b^6+15a^4b^4-6a^5b^2+a^6)}.$$

$$H = Bb\left(885b^{14}-3477ab^{12}+1849a^2b^{10}+4439a^3b^8-2817a^4b^6-1151a^5b^4-109a^6b^2-3a^2\right)$$
$$\times[4C^3(b^2-a)(15b^4-18ab^2-a^2)(b^{12}-6ab^{10}+15a^2b^8-20a^3b^6+15a^4b^8-20a^3b^6$$
$$+15a^4b^4-6a^5b62+a^6)]^{-1} \tag{A.1}$$

$$K = \frac{AC(b^2-a)(b^2+a)}{2b^3}. \tag{A.2}$$

$$v_t^2 = -\frac{C\left(Cbr^2+1\right)^4}{2C^2r\left(\left(C^2b^4-C^2ab^2\right)r^4+4Cb^3r^2+15b^2+a\right)}$$
$$\left[\frac{8Cr}{\left(Cbr^2+1\right)^2}\left(\left(\frac{AC\left(\left(Cbr^2+1\right)\left(Cb^2r^2-Car^2+2b\right)+4b\right)^2}{4\left(Cbr^2+1\right)^4}\right.\right.\right.$$
$$\left.-\frac{B\left(Cbr^2+1\right)^8}{4C^3\left(\left(Cbr^2+1\right)\left(Cb^2r^2-Car^2+2b\right)+4b\right)^2}+\frac{Cb^2r^2-Car^2+2b}{4}\right)^2$$
$$-\frac{ACb\left(\left(Cbr^2+1\right)\left(Cb^2r^2-Car^2+2b\right)+4b\right)^2}{\left(Cbr^2+1\right)^5}-\frac{2Bb\left(Cbr^2+1\right)^7}{C^3\left(\left(Cbr^2+1\right)\left(Cb^2r^2-Car^2+2b\right)+4b\right)^2}$$
$$+\frac{B\left(Cbr^2+1\right)^8\left(b\left(Cb^2r^2-Car^2+2b\right)+\left(b^2-a\right)\left(Cbr^2+1\right)\right)}{2C^3\left(\left(Cbr^2+1\right)\left(Cb^2r^2-Car^2+2b\right)+4b\right)^3}$$
$$+\frac{AC\left(b\left(Cb^2r^2-Car^2+2b\right)+\left(b^2-a\right)\left(Cbr^2+1\right)\right)}{2}$$
$$\left.\times\frac{\left(\left(Cbr^2+1\right)\left(Cb^2r^2-Car^2+2b\right)+4b\right)}{\left(Cbr^2+1\right)^4}+\frac{b^2-a}{4}\right)$$
$$-\frac{16C^2br^3}{\left(Cbr^2+1\right)^3}\left(\left(\frac{AC\left(\left(Cbr^2+1\right)\left(Cb^2r^2-Car^2+2b\right)+4b\right)^2}{4\left(Cbr^2+1\right)^4}\right.\right.$$
$$\left.-\frac{B\left(Cbr^2+1\right)^8}{4C^3\left(\left(Cbr^2+1\right)\left(Cb^2r^2-Car^2+2b\right)+4b\right)^2}+\frac{Cb^2r^2-Car^2+2b}{4}\right)^2$$
$$-\frac{ACb\left(\left(Cbr^2+1\right)\left(Cb^2r^2-Car^2+2b\right)+4b\right)^2}{\left(Cbr^2+1\right)^5}-\frac{2Bb\left(Cbx^2+1\right)^7}{C^3\cdot\left(\left(Cbr^2+1\right)\left(Cb^2r^2-Car^2+2b\right)+4b\right)^2}$$
$$+\frac{B\left(Cbr^2+1\right)^8\left(b\left(Cb^2r^2-Car^2+2b\right)+\left(b^2-a\right)\left(Cbx^2+1\right)\right)}{2C^3\left(\left(Cbr^2+1\right)\left(Cb^2r^2-Car^2+2b\right)+4b\right)^3}$$
$$+\frac{AC\left(b\left(Cb^2r^2-Car^2+2b\right)+\left(b^2-a\right)\left(Cbr^2+1\right)\right)\left(\left(Cbr^2+1\right)\left(Cb^2r^2-Car^2+2b\right)+4b\right)}{2\left(Cbr^2+1\right)^4}$$
$$\left.+\frac{b^2-a}{4}\right)+\frac{4Cr^2}{\left(Cbr^2+1\right)^2}\left(2\left(-\frac{2AC^2br\left(\left(Cbr^2+1\right)\left(Cb^2r^2-Car^2+2b\right)+4b\right)^2}{\left(Cbr^2+1\right)^5}\right.\right.$$
$$-\frac{4Bbr\left(Cbr^2+1\right)^7}{C^2\left(\left(Cbr^2+1\right)\left(Cb^2r^2-Car^2+2b\right)+4b\right)^2}$$
$$+\frac{B\left(Cbr^2+1\right)^8\left(2Cbr\left(Cb^2r^2-Car^2+2b\right)+\left(2Cb^2r-2Car\right)\left(Cbr^2+1\right)\right)}{2C^3\left(\left(Cbr^2+1\right)\left(Cb^2r^2-Car^2+2b\right)+4b\right)^3}$$
$$+\frac{AC\left(2Cbr\left(Cb^2r^2-Cax^2+2b\right)+\left(2Cb^2r-2Car\right)\left(Cbr^2+1\right)\right)}{\left(Cbr^2+1\right)^4}$$

$$\times \frac{\left(\left(Cbr^2+1\right)\left(Cb^2r^2-Car^2+2b\right)+4b\right)}{2}+\frac{2Cb^2r-2Car}{4}\Bigg)$$

$$\Bigg(\frac{AC\left(\left(Cbr^2+1\right)\left(Cb^2r^2-Car^2+2b\right)+4b\right)^2}{4\left(Cbr^2+1\right)^4}+\frac{Cb^2r^2-Car^2+2b}{4}$$

$$-\frac{B\left(Cbr^2+1\right)^8}{4C^3\left(\left(Cbr^2+1\right)\left(Cb^2r^2-Car^2+2b\right)+4b\right)^2}\Bigg)$$

$$+\frac{10AC^2b^2r\left(\left(Cbx^2+1\right)\left(Cb^2r^2-Car^2+2b\right)+4b\right)^2}{\left(Cbr^2+1\right)^6}$$

$$-\frac{28Bb^2r\left(Cbr^2+1\right)^6}{C^2\left(\left(Cbr^2+1\right)\left(Cb^2x^2-Car^2+2b\right)+4b\right)^2}$$

$$+\frac{B\left(b\left(2Cb^2r-2Car\right)+2Cb\left(b^2-a\right)r\right)\left(Cbr^2+1\right)^8}{2C^3\left(\left(Cbr^2+1\right)\left(Cb^2r^2-Car^2+2b\right)+4b\right)^3}$$

$$+\frac{4Bb\left(Cbx^2+1\right)^7\left(2Cbr\left(Cb^2r^2-Car^2+2b\right)+\left(2Cb^2r-2Car\right)\left(Cbr^2+1\right)\right)}{C^3\left(\left(Cbr^2+1\right)\left(Cb^2r^2-Car^2+2b\right)+4b\right)^3}$$

$$+\frac{8Bbr\left(Cbr^2+1\right)^7\left(b\left(Cb^2r^2-Car^2+2b\right)+\left(b^2-a\right)\left(Cbr^2+1\right)\right)}{C^2\left(\left(Cbr^2+1\right)\left(Cb^2r^2-Car^2+2b\right)+4b\right)^3}$$

$$-\frac{3B\left(Cbr^2+1\right)^8\left(b\left(Cb^2r^2-Car^2+2b\right)+\left(b^2-a\right)\left(Cbr^2+1\right)\right)}{2C^3\left(\left(Cbr^2+1\right)\left(Cb^2r^2-Car^2+2b\right)+4b\right)^4}$$

$$\times\frac{\left(2Cbr\left(Cb^2x^2-Car^2+2b\right)+\left(2Cb^2r-2Car\right)\left(Cbr^2+1\right)\right)}{2C^3\left(\left(Cbr^2+1\right)\left(Cb^2r^2-Car^2+2b\right)+4b\right)^4}$$

$$+\frac{AC\left(b\left(2Cb^2r-2Car\right)+2Cb\left(b^2-a\right)x\right)\left(\left(Cbr^2+1\right)\left(Cb^2r^2-Car^2+2b\right)+4b\right)}{2\left(Cbr^2+1\right)^4}$$

$$-\frac{2ACb\left(2Cbr\left(Cb^2r^2-Car^2+2b\right)+\left(2Cb^2r-2Car\right)\left(Cbr^2+1\right)\right)}{\left(Cbr^2+1\right)^3}$$

$$\cdot\frac{\left(\left(Cbr^2+1\right)\left(Cb^2r^2-Car^2+2b\right)+4b\right)}{\left(Cbr^2+1\right)^2}$$

$$-\frac{4AC^2br\left(b\left(Cb^2r^2-Car^2+2b\right)+\left(b^2-a\right)\left(Cbr^2+1\right)\right)}{\left(Cbr^2+1\right)^2}$$

$$\times\frac{\left(\left(Cbr^2+1\right)\left(Cb^2r^2-Car^2+2b\right)+4b\right)}{\left(Cbr^2+1\right)^3}+\frac{AC\left(b\left(Cb^2r^2-Cax^2+2b\right)+\left(b^2-a\right)\left(Cbr^2+1\right)\right)}{\left(Cbr^2+1\right)^4}$$

$$\times\frac{\left(2Cbr\left(Cb^2r^2-Car^2+2b\right)+\left(2Cb^2r-2Car\right)\left(Cbr^2+1\right)\right)}{2}\Bigg)$$

$$+\left(\frac{24C^2b^2r^3}{\left(Cbr^2+1\right)^4}-\frac{24Cbr}{\left(Cbr^2+1\right)^3}\right)\Bigg(\frac{AC\cdot\left(\left(Cbr^2+1\right)\left(Cb^2r^2-Car^2+2b\right)+4b\right)^2}{4\left(Cbr^2+1\right)^4}$$

$$-\frac{B\left(Cbr^2+1\right)^8}{4C^3\left(\left(Cbr^2+1\right)\left(Cb^2r^2-Car^2+2b\right)+4b\right)^2}+\frac{Cb^2r^2-Car^2+2b}{4}\Bigg)$$

$$
\begin{aligned}
&+\left(\frac{4}{\left(Cbr^2+1\right)^2}-\frac{4Cbr^2}{\left(Cbr^2+1\right)^3}\right)\left(-\frac{2AC^2br\left(\left(Cbr^2+1\right)\left(Cb^2r^2-Car^2+2b\right)+4b\right)^2}{\left(Cbr^2+1\right)^5}\right.\\
&-\frac{4Bbr\left(Cbr^2+1\right)^7}{C^2\left(\left(Cbr^2+1\right)\left(Cb^2r^2-Car^2+2b\right)+4b\right)^2}+\frac{2Cb^2r-2Car}{4}\\
&+\frac{B\left(Cbr^2+1\right)^8\left(2Cbr\left(Cb^2r^2-Car^2+2b\right)+\left(2Cb^2r-2Car\right)\left(Cbr^2+1\right)\right)}{2C^3\left(\left(Cbr^2+1\right)\left(Cb^2r^2-Car^2+2b\right)+4b\right)^3}\\
&+\frac{AC\left(2Cbr\left(Cb^2r^2-Car^2+2b\right)+\left(2Cb^2r-2Car\right)\left(Cbr^2+1\right)\right)}{2}\\
&\left.\times\frac{\left(\left(Cbr^2+1\right)\left(Cb^2r^2-Car^2+2b\right)+4b\right)}{\left(Cbr^2+1\right)^4}\right)-\frac{2Car}{\left(Cbr^2+1\right)^2}+\frac{4C^2abr^3}{\left(Cbr^2+1\right)^3}+\frac{12Cb^2r}{\left(Cbr^2+1\right)^4}\Bigg].
\end{aligned}
\tag{A.3}
$$