Separation Axioms and Connectedness in Bounded Uniform Filter Spaces

By

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To my beloved

parents,

husband,

&

daughter.

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Sana Khadim

Contents

ABSTRACT

The well-known concepts such as uniform convergence, Cauchy convergence, Cartesian closedness, quotient reflectiveness, quotients are not productive, boundedness, etc., are missing from **Top**, the category of topological spaces and continuous mappings. Mathematicians have taken several routes toward defining these ideas in Topology. In 2018, Dieter Leseberg introduced bounded uniform filter spaces that generalize topological spaces, bounded spaces, filter convergence spaces, semiuniform convergence spaces, and bornological spaces based on the concept of filters of a set. This generalization overcomes almost all of the known deficiencies that appeared in **Top**. The most basic reason for introducing bounded uniform filter spaces was to bring boundedness, convergence theories, uniformity, and topological concepts under a single umbrella. Bounded uniform filter spaces can be further subdivided into several isomorphic subcategories corresponding to various concepts in topological spaces.

The dissertation is divided into six parts.

In the first chapter, we review a few elementary categorical concepts. We recall some useful categorical spaces including preuniform convergence spaces, semiuniform convergence spaces, boundedness, bornological spaces, and bounded uniform filter (b-UFIL) spaces along with their corresponding morphisms respectively. The category of bounded uniform filter (b-UFIL) spaces and bounded uniformly continuous (buc) mappings, **b-UFIL**, is also proven to be a topological category.

The second chapter provides a characterization of each of local T_0 and local T_1 objects in the category of b-UFIL spaces and examines their mutual relations. It is shown that every local T_1 b-UFIL space is equivalent to a local T_0 b-UFIL space, but the converse implication is not always true. We illustrate this with a few examples.

In chapter three, we characterize each of T_0 and T_1 objects in the categories of several types of b-UFIL spaces and examine their mutual relations. Moreover, we investigate the productivity and

hereditary properties of T_0 (resp. T_1) bounded uniform filter spaces, and compare our findings to the usual T_0 and T_1 spaces. We show that every T_0 (resp. T_1) bounded uniform filter space satisfies the usual T_0 (resp. the usual T_1), although the converse is not always true. It is further proved that if a bounded uniform filter space is T_0 at p and T_1 at p for any $p \in Z$, then the space is T_0 (resp. T_1) overall. Finally, it is shown that the categories $\overline{T}_0$**b-UFIL**, T_0**b-UFIL** and T_1**b-UFIL** satisfy several properties such as epireflective and quotient-reflective subcategories of **b-UFIL**. Further, it is proved that $T_0^{'}$**b-UFIL** is a normalized, cartesian closed, and hereditary topological construct. This is summarized in a diagram.

In chapter four, we characterize both closed and strongly closed subobjects in the category of b-UFIL spaces. We define two notions of closure operators and prove that they are (weakly) hereditary, idempotent, and productive closure operators of **b-UFIL**. Also, we introduce four different closure operators in bounded uniform filter spaces namely $\textbf{b-UFIL}_{0cl}$, $\textbf{b-UFIL}_{0scl}$, $\textbf{b-UFIL}_{1cl}$ and $\textbf{b-UFIL}_{1scl}$. Using these closure operators we further characterize each of T_j $(j = 0, 1)$ b-UFIL spaces and examine that each of them forms quotient-reflective subcategories of **b-UFIL**. Moreover, these implications are summarized in a diagram.

In the fifth chapter, we illustrate the concepts of connected and strongly connected bounded uniform filter spaces and prove that every strongly connected bounded uniform filter space is a connected bounded uniform filter space, but the inverse implication is not always true. Finally, we introduce ultraconnected objects in a topological category and examine the relationship among connected, irreducible, and ultraconnected b-UFIL spaces. This is further summarized in a diagram.

The findings from chapters two, three, four, and five are summed up in chapter six, along with some unresolved proposals and topics for future study.

Keywords: *topological category; bounded uniform filter spaces; bounded uniformly continuous mappings; local T_0 and local T_1 bounded uniform filter spaces; T_0 and T_1 bounded uniform filter spaces; quotient reflective subcategories of **b-UFIL**; closed and strongly closed objects in **b-UFIL**; closure operators in **b-UFIL**; connectedness and strongly connectedness in **b-UFIL**; irreducibility and strongly irreducibility in **b-UFIL**; ultraconnectedness and strongly ultraconnectedness in **b-UFIL***

CHAPTER 0

INTRODUCTION

In the year 1906, Fréchet introduced "metric spaces" which is a constructive concept from Analysis point of view. But this was not big enough to describe pointwise convergence in function spaces. In order to fix it, a structure named "topological spaces" was introduced by Felix Hausdorff in 1914 (that is called a Hausdorff space nowadays) and by Kuratowski in 1922.

But it is stated in [81] that the category **Top** has other deficiencies mentioned below:

a) Suppose that $C(K_1,K_2)$ is the set containing all surjective continuous mappings $f:(K_1,\tau_1)\longrightarrow(K_2,\tau_2)$ between two topological spaces. Then there is no topology τ that defines continuous convergence on $C(K_1,K_2)$.

b) **Top** is not a cartesian closed category.

c) In **Top**, the product of quotients is not supposed to be quotient always.

d) Quotients are not hereditary in **Top**.

e) In **Top**, the concept of uniformity is not available such as uniform convergence, uniform continuity, Cauchy sequence or Cauchy filters, and completeness.

Having all these deficiencies in mind, one may come up with a question: *What are the remedies to overcome these deficiencies?*

Several attempts have been made to overcome these deficiencies including Kent convergence spaces [5], quasiuniform spaces [34], generalized topological spaces [4], seminearness spaces [6] and nearness spaces [17]. But none of them have led to fulfilling all the above properties. In 1995, Preuss [60] developed the concept of semiuniform convergence spaces (a basic structure in the domain of convenient topology) that solves almost all the flaws that appeared in **Top**. Moreover, by exempting symmetric condition from the semi-uniform convergence spaces, Preuss obtained preuniform convergence spaces and as a special case semiuniform convergence spaces Pre. Later, in 2018 and 2019, Leseberg [91, 92] extended the idea of Preuss and Whyler [46], and defined his concept of bounded structures in terms of bounded uniform filter spaces. Interestingly, not only **PUConv** (the category of preuniform convergence spaces and uniformly continuous mappings) are embedded in **b-UFIL** but also **BORN** (category of bornological spaces and continuous mappings that are embedded in bounded spaces) can easily be embedded in b-UFIL as its subcategories. Also, the category **b-UFIL** forms a strong topological universe [91].

Not only in Algebraic Topology where another substitute characterization of locally semi-simple coverings in the form of light morphisms, for example, [78], but in Computer Science as well, where this concept corresponds to approach the values via observations [72], the classical T_0 separation axiom of topology plays a crucial role. The T_0 axiom has also been used in the construction of cellular complexes in computer graphs and image processing, digital lines in digital topology, and to fill in gaps where Hausdorff topologies fall short in areas like denotational semantics of programming languages and lambda calculus [70, 33].

Many mathematicians, including Brümmer [13], Marny [16], Hoffmann [20], Harvey [27], and Baran [50] in 1971, 1973, 1974, 1977 and 1991 respectively, extended the concept of separation axiom T_0 to the concept of topological categories. Furthermore, Weck-Schwarz investigated the connection between these several generalizations of T_0 objects in 1991 [51] and Baran did the same in 1995 [57]. Among the primary motivations for broadening T_0 separation axiom was the need to define T_2 objects in topological categories chosen at random [62].

It was not until 1992 that Baran [50] generalized the separation axioms to any topological category at all. Two notions of T_0, one of T_1, and four of each of T_2, T_3, and T_4 were his final tally. In the end, he concluded that, in the context of topological spaces, all of these notions can be reduced to their classical counterparts. It was with this goal in mind that we set out to generalize the concept of closed (resp. strongly closed) sets to any topological category over **Set** [52]. The notions of compactness in [71], Hausdorffness in [50], regular and normal objects in [66], perfectness in [71], and sobriety in [101] have all been generalized by employing the closed (resp. strongly closed) sets in the well known topological category over **Set**. And in numerous well-known topological classes [88, 100, 96], the notion of closedness is amenable to the creation of closure operators [39].

Closure operators are one of the foremost ingredients in not only Categorical Algebra but also Categorical Topology. Coreflections have been characterized in the category **Top** in terms of eventually non-idempotent closure operators [12]. Hence non-topological closure operators are required in this endeavor. Epireflective subcategories have been defined by Hong [21] and Salbany [26] in **Top** using closure operators. Galois equivalence between conclusive factorization systems in idempotent and weakly hereditary closure operators in **Top** is given by Nakagawa [25]. Moreover, closure operators have played a vital role in defining diagonal theorems (referred to in [36, 35, 42, 37, 43, 40, 31, 23, 38]) in various topological categories, i.e. generalization of the classical topological fact that a space S is T_2 iff the diagonal Δ_S is closed in $S \times S$. In a topological category, closure operators are defined by Dikranjan and Guili [39] where the epimorphisms of the full-subcategories of topological categories are characterized and suitable closure operators were formed in arbitrary topological categories (see in [67, 71, 80, 96, 100]). Both in Topology and Algebra various examples can be found where closure operators and their relations with other subcategories are studied inclusively [39, 84].

In higher analysis, geometry, and topology, the notion of connectedness plays a crucial role. It is also very important in almost all the areas where the notion of topological spaces is itself relevant. Then various characterizations of connectedness have been studied in topological categories such as the concept of D-connectedness, introduced by Preuss, and connected (respectively strongly connected) objects in numerous topological categories were investigated by Baran. These generalizations of the notion of connectedness in a topological category can be found in [75, 55, 64, 32, 44].

This thesis comprises five chapters.

The first chapter covers some fundamental concepts starting from Category Theory. We restate the definitions of a category, subcategory, epireflective, quotient reflective, and bireflective subcategories along with some examples. We also revisit functors and their various kinds such as full functor, faithful functor, amnestic functor, concrete functor, inclusion functor, identity functor, etc. along with their examples. Further, in the second section, we talk about categorical topology and discuss topological functors, initial and final lifts, discrete and indiscrete functors, normalized functors, etc. Moreover, the third section consists of a brief literature review on bounded uniform filter spaces. We start with the concept of filters and boundedness, then we define the categories **BOUND**, **b-UFIL**, **CROSSb-UFIL**, **sb-UFIL**, **LIMsb-UFIL**, **BONb-UFIL**,

S-Conv (resp. **ROS-Conv**), **PU-Conv** (resp. **SU-Conv**) and **BORN**. In the fourth section, we show that the category **b-UFIL** forms a topological construct that is normalized and define the initial and the final b-UFIL structures. The fifth section consists of the main objectives of this complete study that are fulfilled in this research. In the last section, we give importance to the research work.

In the second chapter, we define two notions of T_0 and one notion of T_1 bounded uniform filter spaces at some point t. In the first section of this chapter, we recall the definition of the wedge product of a set Z with itself at some point, say $t \in Z$, principal t-axis map, skewed t-axis map, folding mapping at t, and local $\overline{T}_0$ and T_0' objects in a topological category using the wedge product of Z at t. Furthermore, this section contains the characterization of local T_0' and $\overline{T}_0$ objects in **b-UFIL** and examines their mutual relationship. The second section contains the definition of local T_1 objects in a topological category using the wedge product of Z at t, then we give a characterization of local T_1 objects in **b-UFIL** using skewed t-axis and folding mapping, and examine their mutual relationship. The content of this research work has been published in **AIMS Mathematics (2022), https://www.aimspress.com/article/doi/10.3934/math.2022911** [103].

In the third chapter, we define generically three notions of T_0 and one notion of T_1 in bounded uniform filter spaces which leads to the quotient reflective subcategories of **b-UFIL**. In the first section of this chapter, we define the wedge product of Z^2 with itself diagonally, principal axis map, skewed axis map, folding mapping, and generic $\overline{T}_0$, T_0' and T_0 objects in a topological category using the self wedge product of Z^2 diagonally. This section also contains the characterization of T_0', $\overline{T}_0$ and T_0 objects in **b-UFIL** and we examine their mutual relationship. The second section contains the definition of generic T_1 objects in a topological category using the wedge product of Z^2 diagonally, then we give a characterization of T_1 objects in **b-UFIL** using skewed axis and folding mapping, and examine their mutual relationship. Furthermore, in the third section of this chapter, we discuss the quotient reflective subcategories of **b-UFIL** and it is shown that every $\overline{T}_0$**b-UFIL** (resp. T_0**b-UFIL**, T_1**b-UFIL**) is a quotient-reflective subcategory of **b-UFIL**. Also, it is shown that T_0'**b-UFIL** is a normalized, cartesian closed, and hereditary topological construct. The key findings of this chapter have been published in **AIMS Mathematics (2022), https://www.aimspress.com/article/doi/10.3934/math.2022911** [103].

The fourth chapter contains the notion of closedness and closure operators in bounded uniform filter spaces. We start by giving some motivation and the definition of closed sets in general topology. In the first section, we define the infinite wedge product of Z at t, infinite principal t-axis map, infinite folding mapping at t, quotient map, closedness of a set using infinite principal t-axis map and infinite folding mapping at t, strongly closedness of a set, etc. Then we characterize closed and strongly closed objects in bounded uniform filter spaces. We have also seen the closedness (resp. strongly closedness) of the inverse and subset of a closed (resp. strongly closed) set. In the second section, we define closure operators in bounded uniform filter spaces and prove that $scl^{\textbf{b-UFIL}}(F)$ and $cl^{\textbf{b-UFIL}}(F)$ are (weakly) hereditary, idempotent and productive closure operators of **b-UFIL**. Further we give characterizations of $\textbf{b-UFIL}_{0cl}$, $\textbf{b-UFIL}_{0scl}$, $\textbf{b-UFIL}_{1cl}$ and $\textbf{b-UFIL}_{1scl}$. The last section consists of relationships between closure operators in **b-UFIL** and separation axioms in **b-UFIL**. The content of this chapter is published in **Filomat (2022), https://www.pmf.ni.ac.rs/filomat-content/2022/36-20/36-20-17-17791.pdf** [102].

In chapter five, we discuss the concept of connectedness and irreducibility in bounded uniform filter spaces. In the first section, we define open (resp. closed) and strongly open (resp. strongly closed) subobjects in a topological category. Then we characterize open (resp. strongly open) subobjects in bounded uniform filter spaces. Also, we define connected (resp. strongly connected) objects in a topological category and characterize them in **b-UFIL**. In the second section, we define irreducible (resp. strongly irreducible) objects in a topological category and give characterizations of irreducible and strongly irreducible objects in **b-UFIL**. Further, we restate the concept of ultraconnectedness (resp. strongly ultraconnectedness) in a topological category and characterize them in the category of b-UFIL spaces. We also define the notion of ultraconnectedness in a set-based topological category and examine the relationship among ultraconnectedness, strong ultraconnectedness, connectedness, and strong connectedness in a b-UFIL space. The outcomes of the conducted research work have been published in **Filomat (2022), https://www.pmf.ni.ac.rs/filomat-content/2022/36-20/36-20-17-17791.pdf** [102].

The last chapter contains a discussion and conclusion of the entire study with some future research directions.

CHAPTER 1

PRELIMINARIES

We state a few fundamental categorical concepts and several equivalent forms of bounded uniform filter spaces in this chapter. Also, we prove that the category **b-UFIL** with objects as bounded uniform filter spaces and morphisms as bounded uniformly continuous mappings, forms a topological category that is normalized. Moreover, the main objectives of this research work along with its importance of are presented.

1.1 Category Theory

In 1947, Mac Lane and Eilenberg introduced a theory known as Category Theory. This theory puts processes on equal footing with things (here by "things" we mean "objects" in the category and by "processes" we mean "morphisms" between the objects).

Category theory serves several purposes. It is an interesting subject to study on its own but the most exciting part of it is that it shows how interconnected different areas of mathematics actually are and gives a new perspective of the subject as a unified whole, rather than a scattered collection of seemingly different topics.

Definition 1.1 (Category)**.** *(cf. [47]) We define a category $\boldsymbol{E}$ by a quadruple $\boldsymbol{E} = (Obj, hom, \circ, id)$ that contains:*

1. *family of objects denoted as $Obj(\boldsymbol{E})$.*

2. *set of morphisms $Hom_{\boldsymbol{E}}(A_1, A_2)$ for each pair of objects $A_1, A_2 \in Obj(\boldsymbol{E})$.*

3. *function called composition;*

$$Hom_{\boldsymbol{E}}(A_2,A_3)\times Hom_{\boldsymbol{E}}(A_1,A_2)\longrightarrow Hom_{\boldsymbol{E}}(A_1,A_3)$$
$$(f_{23},f_{12})\longmapsto f_{23}\circ f_{12}.$$

for each object $A_1,A_2,A_3 \in Obj(\boldsymbol{E})$ *such that*

(i) For each $A_1 \in Obj(\boldsymbol{E})$*, there is an identity morphism* $\mathbb{1}_{A_1} \in Hom_{\boldsymbol{E}}(A_1,A_1)$ *such that for all* $f_{12} \in Hom_{\boldsymbol{E}}(A_1,A_2)$ *and all* $f_{21} \in Hom_{\boldsymbol{E}}(A_2,A_1)$*, we have*

$$f_{12}\circ \mathbb{1}_{A_1} = f_{12} \ \text{ and } \ \mathbb{1}_{A_1}\circ f_{21} = f_{21}.$$

(ii) For objects $A_1,A_2,A_3,A_4 \in Obj(\boldsymbol{E})$ *and morphisms* $f_{12} \in Hom_{\boldsymbol{E}}(A_1,A_2)$, $f_{23} \in Hom_{\boldsymbol{E}}(A_2,A_3)$ *and* $f_{34} \in Hom_{\boldsymbol{E}}(A_3,A_4)$*, we have*

$$f_{34}\circ(f_{23}\circ f_{12}) = (f_{34}\circ f_{23})\circ f_{12}.$$

Examples 1.1.1. *1. The category* ***Set*** *with objects as sets, morphisms as mappings between sets and composition as the usual one between mappings.*

2. ***Grp*** *denotes the category of groups and group homomorphism.*

3. ***Semi-Grp*** *is the category of semi-groups and group homomorphism.*

4. ***Mon*** *denotes the category of monoids and group homomorphisms.*

5. *The category* ***Abgrp*** *with the objects as abelian groups and the morphism set as the set of group homomorphisms which are closed under composition.*

6. *The category of rings and ring homomorphisms is denoted as* ***Ring****.*

7. *The category* ***Top*** *with the objects as topological spaces and the set of morphisms as continuous mappings.*

8. ***Haus*** *denotes the category of Hausdorff spaces and continuous mappings.*

9. *The category of metric spaces and contraction mappings is denoted by* ***Met****.*

Note that we represent a category in bold letters.

Definition 1.2 (Subcategory)**.** *(cf. [85]) Let* ***E*** *be a category. A category* ***H*** *is called a subcategory of* ***E*** *(*$\boldsymbol{H} \subseteq \boldsymbol{E}$*) if the following conditions hold.*

(i) $Obj(\boldsymbol{H}) \subseteq Obj(\boldsymbol{E})$,

(ii) $Hom_{\boldsymbol{H}}(H_1, H_2) \subseteq Hom_{\boldsymbol{E}}(H_1, H_2)$, *for all* $H_1, H_2 \in Obj(\boldsymbol{H})$,

(iii) *For all* $H_1 \in Obj(\boldsymbol{H})$, *the identity morphism* $\mathbb{1}_{H_1}$ *is same as identity* $\mathbb{1}_{H_1}$ *in* $\boldsymbol{E}$,

(iv) *Composition law is preserved.*

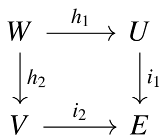
Definition 1.3 (Full Subcategory)**.** *(cf. [85]) We define a subcategory* $\boldsymbol{H}$ *of* $\boldsymbol{E}$ *as a full subcategory if* $Hom_{\boldsymbol{H}}(H_1, H_2) = Hom_{\boldsymbol{E}}(H_1, H_2)$ *for every pair of* $H_1, H_2 \in Obj(\boldsymbol{H})$.

Examples 1.1.2. *1.* $\boldsymbol{H} =$ ***Semi-Grp*** *is a subcategory of* $\boldsymbol{E} =$ ***Mon***. *Also, it is not a full subcategory.*

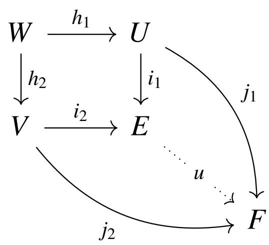
2. $\boldsymbol{H} =$ ***Haus*** *is a full subcategory of* $\boldsymbol{E} =$ ***Top***.

3. $\boldsymbol{H} =$ ***Mon*** *is a full subcategory of* $\boldsymbol{E} =$ ***Grp***.

Definition 1.4 (Push-out)**.** *(cf. [47]) The pushout of the morphisms* h_1 *and* h_2 *consists of an object* E *and two morphisms* $i_1 : U \longrightarrow E$ *and* $i_2 : V \longrightarrow E$ *such that the diagram*



commutes and such that (E, i_1, i_2) *is universal with respect to this diagram. That is, for any other such set* (F, j_1, j_2) *for which the following diagram commutes, there must exist a unique* $u : E \longrightarrow F$ *also making the diagram commutes:*



As with all universal constructions, the pushout, if it exists, is unique up to a unique isomorphism.

Definition 1.5 (Functors)**.** *(cf. [85]) Consider two categories* $\boldsymbol{E}$ *and* $\boldsymbol{F}$. *Then* $\mathfrak{U} : \boldsymbol{E} \longrightarrow \boldsymbol{F}$ *is known as a functor if*

(i) $\forall E_1 \in Obj(\boldsymbol{E}) \Rightarrow \mathfrak{U}(E_1) \in Obj(\boldsymbol{F})$.

(ii) $e : E_1 \longrightarrow E_2 \in Hom(\mathbf{E}) \Rightarrow \mathfrak{U}(e) : \mathfrak{U}(E_1) \longrightarrow \mathfrak{U}(E_2) \in Hom(\boldsymbol{F})$.

(iii) $\mathfrak{U}(\mathbb{1}_E) = \mathbb{1}_{\mathfrak{U}(E)},\ \ for\ all\ E \in Obj(\boldsymbol{E})$.

(iv) *If* $E_1 \xrightarrow{e_{12}} E_2 \xrightarrow{e_{23}} E_3 \in Hom(\mathbf{E})$ *then,*

$$\mathfrak{U}(e_{23} \circ e_{12}) = \mathfrak{U}(e_{23}) \circ \mathfrak{U}(e_{12}).$$

Definition 1.6 (Identity Functor)**.** *(cf. [85]) For a category* $\boldsymbol{H}$*, a functor* $id_{\boldsymbol{H}} : \boldsymbol{H} \longrightarrow \boldsymbol{H}$ *is known as the identity functor defined in such a way that for all* $H_1, H_2 \in Obj(\boldsymbol{H})$ *and* $\Theta : H_1 \longrightarrow H_2$ *implies that* $id_{\boldsymbol{H}}(\Theta) = \Theta$*.*

Definition 1.7 (Constant Functor)**.** *(cf. [85]) For two given categories* $\boldsymbol{L}$ *and* $\boldsymbol{M}$ *and* $F_1 \in Obj(\boldsymbol{M})$*. A functor* $\mathscr{K}_{F_1} : \boldsymbol{L} \longrightarrow \boldsymbol{M}$ *is called a constant functor with value* F_1 *such that*

$$\mathscr{K}_{F_1}(\Theta) = id_{F_1},$$

where $\Theta : E_1 \longrightarrow E_2$ *is morphism in* $\boldsymbol{L}$ *for all* $E_1, E_2 \in Obj(\boldsymbol{L})$ *and* $id_{F_1} : F_1 \longrightarrow F_1$ *is the identity functor.*

Definition 1.8 (Forgetful Functor)**.** *(cf. [85]) For given a topological construct* $\boldsymbol{L}$*. A functor* $\mathscr{K} : \boldsymbol{L} \longrightarrow \boldsymbol{Set}$ *is called a forgetful functor such that for every* $L \in Obj(\boldsymbol{L})$*,* $\mathscr{K}(L)$ *is the underlying set of* L *and* $\mathscr{K}(l) = l$ *is the underlying mapping of the morphism* l*.*

Example 1.1.3. *Consider* $\mathfrak{U} : \boldsymbol{Top} \longrightarrow \boldsymbol{Set}$ *given by* $\mathfrak{U}(E, \tau) = E$ *for some set* E *and* $\mathfrak{U}(e) = e$ *for a continuous mapping* $e : (E, \tau) \longrightarrow (F, \sigma)$*. Then* $\mathfrak{U}$ *is a functor. A functor* $\mathfrak{U}$ *defined in this way is called a forgetful functor.*

Definition 1.9 (Full Functor)**.** *(cf. [85]) Consider a functor* $\mathfrak{U} : \boldsymbol{E} \longrightarrow \boldsymbol{F}$*. If for all* $E_1, E_2 \in Obj(\boldsymbol{E})$ *and* $g \in Hom(\mathfrak{U}(E_1), \mathfrak{U}(E_2))$ *then there is* $h : E_1 \longrightarrow E_2$ *such that* $\mathfrak{U}(h) = g$ *then* $\mathfrak{U}$ *is called a full functor.*

Examples 1.1.4.

1. *A functor* $\mathfrak{U} : \boldsymbol{Set} \longrightarrow \boldsymbol{Top}$ *given by* $\mathfrak{U}(E) = (E, \{\emptyset, E\})$ *for some set* E*, is a full functor.*
2. *The forgetful functor* $\mathfrak{U} : \boldsymbol{Top} \longrightarrow \boldsymbol{Set}$ *is not a full functor.*

Definition 1.10 (Faithful Functor)**.** *(cf. [85]) For a functor* $\mathfrak{U} : \boldsymbol{E} \longrightarrow \boldsymbol{F}$*, if for all* $E_1, E_2 \in Obj(\boldsymbol{E})$ *and* $e, f \in Hom(E_1, E_2)$*, gives* $\mathfrak{U}(e) = \mathfrak{U}(f)$ *implies* $e = f$*. Then* $\mathfrak{U}$ *is called a faithful functor.*

Example 1.1.5. *The forgetful functor $\mathfrak{U} : \boldsymbol{Top} \longrightarrow \boldsymbol{Set}$ is a faithful functor.*

Definition 1.11 (Amnestic Functor)**.** *(cf. [85]) Consider a functor $\mathfrak{U} : \boldsymbol{E} \longrightarrow \boldsymbol{F}$. For given $E_1 \in Obj(\boldsymbol{E})$ and $g \in Hom(E_1, E_1)$, if $\mathfrak{U}(g) = 1_{E_1} = 1_{\mathfrak{U}(E_1)}$ and g is isomorphism implies g is identity, then $\mathfrak{U}$ is called amnestic functor.*

Examples 1.1.6. 1. *The forgetful functor $\mathfrak{U} : \boldsymbol{Top} \longrightarrow \boldsymbol{Set}$ is amnestic functor.*

2. *A functor $\mathfrak{U} : \boldsymbol{Met} \longrightarrow \boldsymbol{Top}$ given by $\mathfrak{U}(E,d) = (E, \tau_d)$, where $\tau_d = \{U \subset E : U = \cup_{x\in E, r>0} B(x,r)\}$ and $\mathfrak{U}(e) = e$ is continuous for a contraction map $e : (E, d_E) \longrightarrow (F, d_F)$, is not amnestic functor.*

Definition 1.12 (Concrete Functor)**.** *(cf. [85]) Let $\mathfrak{U} : \boldsymbol{E} \longrightarrow \boldsymbol{F}$ be a functor, if $\mathfrak{U}$ is amnestic and faithful then it is called concrete functor.*

Examples 1.1.7. 1. *The forgetful functor $\mathfrak{U} : \boldsymbol{Grp} \longrightarrow \boldsymbol{Set}$ is concrete functor.*

2. *The forgetful functor $\mathfrak{U} : \boldsymbol{Top} \longrightarrow \boldsymbol{Set}$ is concrete functor.*

Definition 1.13 (Inclusion Functor)**.** *(cf. [85]) Let $\boldsymbol{H}$ and $\boldsymbol{E}$ be two categories with $\boldsymbol{H}$ as a subcategory of $\boldsymbol{E}$. The naturally associated functor $\mathscr{K} : \boldsymbol{H} \hookrightarrow \boldsymbol{E}$ is called an inclusion functor.*

Definition 1.14 (Composite Functor)**.** *(cf. [85]) Let $\mathscr{K} : \boldsymbol{E} \longrightarrow \boldsymbol{F}$ and $\mathscr{H} : \boldsymbol{F} \longrightarrow \boldsymbol{G}$ be two fuctors between categories $\boldsymbol{E}$, $\boldsymbol{F}$ and $\boldsymbol{G}$. The composite $\mathscr{H} \circ \mathscr{K} : \boldsymbol{E} \longrightarrow \boldsymbol{G}$ is a functor defined by*

$$(\mathscr{H} \circ \mathscr{K})(\Theta) = \mathscr{H}(\mathscr{K}\Theta),$$

where $\Theta : E_1 \longrightarrow E_2$ is morphism in $\boldsymbol{E}$ for all $E_1, E_2 \in Obj(\boldsymbol{E})$ and $\mathscr{H}(\mathscr{K}\Theta) : \mathscr{H}(\mathscr{K}F_1) \longrightarrow \mathscr{H}(\mathscr{K}F_1)$.

Definition 1.15 (Isomorphic Functors or Isomorphism)**.** *(cf. [85]) For two given categories $\boldsymbol{L}$ and $\boldsymbol{M}$, a functor $\mathscr{K}_{LM} : \boldsymbol{L} \longrightarrow \boldsymbol{M}$ is known as an isomorphism if there is a functor $\mathscr{K}_{ML} : \boldsymbol{M} \longrightarrow \boldsymbol{L}$ so that*

$$\mathscr{K}_{ML} \circ \mathscr{K}_{LM} = id_{\boldsymbol{L}} \text{ and } \mathscr{K}_{LM} \circ \mathscr{K}_{ML} = id_{\boldsymbol{M}}.$$

Such a functor $\mathscr{K}_{ML}$ is called an inverse of $\mathscr{K}_{LM}$.

Corollary 1.1.8. *(cf. [85]) If $\mathscr{H}_1$ and $\mathscr{H}_2$ are inverses of a functor $\mathscr{K}$, then $\mathscr{H}_1 = \mathscr{H}_2$.*

Remark 1.1.9. *Due to the above corollary, we may speak of the inverse of an isomorphism $\mathscr{K}$. It will be denoted by $\mathscr{K}^{-1}$.*

Definition 1.16 (Isomorphic Category)**.** *(cf. [85]) For given two categories* $\boldsymbol{L}$ *and* $\boldsymbol{M}$*, if a functor* $\mathscr{K} : \boldsymbol{L} \longrightarrow \boldsymbol{M}$ *is an isomorphism then* $\boldsymbol{L}$ *and* $\boldsymbol{M}$ *are known as isomorphic categories.*

Definition 1.17 (Retraction)**.** *(cf. [47]) Let* $\Theta : E_1 \longrightarrow E_2$ *be a morphism in category* $\boldsymbol{E}$ *for all* $E_1, E_2 \in Obj(\boldsymbol{E})$*. Then* Θ *is a retraction if there exists a morphism* $\Psi : E_2 \longrightarrow E_1$ *satisfying* $\Theta \circ \Psi = \mathbb{1}_{E_2}$*.* E_2 *is a retract of* E_1 *if there exists such a retraction.*

Definition 1.18 (Cartesian Closedness)**.** *(cf. [92]) For a category* $\boldsymbol{E}$ *to be cartesian closed the following must hold:*

(a) There exists Cartesian product $A \times B$ *in* $\boldsymbol{E}$ *against every pair* (A, B) *of* $\boldsymbol{E}$*-objects;*

(b) Every $A \in Obj(\boldsymbol{E})$ *satisfies: For each* $B \in Obj(\boldsymbol{E})$ *there is a power object* $B^A \in Obj(\boldsymbol{E})$ *and an evaluation morphism* $ev_{A,B} : A \times B^A \longrightarrow B$ *in* $Hom(\boldsymbol{E})$ *such that for all* $C \in Obj(\boldsymbol{E})$ *and for every* $\boldsymbol{E}$*-morphism* $h : A \times C \longrightarrow B$ *there is an unique morphism* $\hat{h} : C \longrightarrow B^A$ *in* $Hom(\boldsymbol{E})$ *so that the diagram*

$$\begin{array}{ccc} A \times B^A & \xrightarrow{\quad ev_{A,B} \quad} & B \\ & {\scriptstyle \mathbb{1}_A \times \hat{h}} \nwarrow \quad \nearrow {\scriptstyle h} & \\ & A \times C & \end{array}$$

commute.

Example 1.1.10. *1. The category* $\boldsymbol{E} = \boldsymbol{Set}$ *is cartesian closed.*

2. The categories ***Top*** *and* ***Grp*** *are not cartesian closed.*

Definition 1.19 (Natural Transformation (cf. [47]))**.** *Consider* $\boldsymbol{E}$ *and* $\boldsymbol{F}$ *as categories, and* $\mathscr{H}$ $\mathscr{K}$ *as functors from* $\boldsymbol{E}$ *to* $\boldsymbol{F}$*. A family of morphisms* $\mathfrak{N} : \mathscr{H} \to \mathscr{K}$ *is called natural transformation if it satisfies*

(i) $\forall\, a \in Obj(\boldsymbol{E})$*,* $\mathfrak{N}$ *picks a morphism* $\mathfrak{N}_a : \mathscr{H}(a) \to \mathscr{K}(a)$ *in* $\boldsymbol{F}$*, where* $\mathfrak{N}_a$ *is component of* $\mathfrak{N}$ *at a.*

(ii) for all morphism $g : a \to b$ *of* $\boldsymbol{E}$ *the following diagram commutes*

$$\begin{array}{ccc} \mathscr{H}(a) & \xrightarrow{\mathscr{H}(g)} & \mathscr{H}(b) \\ \downarrow{\scriptstyle \mathfrak{N}_a} & & \downarrow{\scriptstyle \mathfrak{N}_b} \\ \mathscr{K}(a) & \xrightarrow{\mathscr{K}(g)} & \mathscr{K}(b) \end{array}$$

i.e $\mathscr{K}(g) \circ \mathfrak{N}_a = \mathfrak{N}_b \circ \mathscr{H}(g)$.

Remarks 1.1.11. *Let $R,S,T : \mathbf{D} \longrightarrow \mathbf{D}'$ be any three functors from a category $\mathbf{D}$ to a category $\mathbf{D}'$ such that $\varphi : T \longrightarrow S$ and $\vartheta : R \longrightarrow T$ are natural transformations, that is, for each $A_1 \in Obj\mathbf{D}$, $\varphi_{A_1} \in Hom_{\mathbf{D}'}(TA_1, SA_1)$ and $\vartheta_{A_1} \in Hom_{\mathbf{D}'}(RA_1, TA_1)$.*

1. *Closure Property: Natural transformations are closed under composition i.e.*

$$(\varphi \circ \vartheta)_{A_1} = \varphi_{A_1} \circ \vartheta_{A_1}.$$

By the axiom 3 in 1.1 of category $\mathbf{D}'$, we have $\varphi_{A_1} \circ \vartheta_{A_1} : R(A_1) \longrightarrow S(A_1)$ and for $f_{12} \in Hom_{\mathbf{D}}(A_1, A_2)$ and using commutative square diagrams 1.19 of φ_{A_1} and ϑ_{A_1}, we get

$$\begin{aligned}(\varphi_{A_2} \circ \vartheta_{A_2}) \circ R(f_{12}) &= \varphi_{A_2} \circ (\vartheta_{A_2} \circ R(f_{12})) \\ &= \varphi_{A_2} \circ (T(f_{12}) \circ \vartheta_{A_1}) \\ &= (\varphi_{A_2} \circ T(f_{12})) \circ \vartheta_{A_1} \\ &= (S(f_{12}) \circ \varphi_{A_1}) \circ \vartheta_{A_1} \\ &= S(f_{12}) \circ (\varphi_{A_1} \circ \vartheta_{A_1}).\end{aligned}$$

$$\begin{array}{ccccc} R(A_1) & \xrightarrow{\vartheta_{A_1}} & T(A_1) & \xrightarrow{\varphi_{A_1}} & S(A_1) \\ {\scriptstyle R(f_{12})}\downarrow & & {\scriptstyle T(f_{12})}\downarrow & & \downarrow{\scriptstyle S(f_{12})} \\ R(A_2) & \xrightarrow{\vartheta_{A_2}} & T(A_2) & \xrightarrow{\varphi_{A_2}} & S(A_2) \end{array}$$

Thus $\varphi_{A_1} \circ \vartheta_{A_1}$ is natural transformation which is same as the natural transformation $(\varphi \circ \vartheta)_{A_1}$.

2. *Isomorphism: Two functors T and S are said to be naturally equivalent (or isomorphic) if there exist natural transformations $\varphi : T \longrightarrow S$ and $\psi : S \longrightarrow T$ such that $\forall\ \varphi_{A_1} \in Hom(TA_1, SA_1)$ and $\psi_{A_1} \in Hom(SA_1, TA_1)$*

$$(\varphi \circ \psi)_{A_1} := id_{S(A_1)},$$

and

$$(\psi \circ \varphi)_{A_1} := id_{T(A_1)}.$$

In other words, two functors are (naturally) isomorphic if the natural transformation between them is an isomorphism.

Definition 1.20 (Left and Right Adjoint) (cf. [47])**.** *For two given categories $\boldsymbol{L}$ and $\boldsymbol{M}$, and two functors $\mathscr{K}_{LM} : \boldsymbol{L} \to \boldsymbol{M}$ and $\mathscr{K}_{ML} : \boldsymbol{M} \to \boldsymbol{L}$. We called $\mathscr{K}_{ML}$ as right adjoint of $\mathscr{K}_{LM}$ and*

$\mathscr{K}_{LM}$ as left adjoint of $\mathscr{K}_{ML}$ together with natural transformation $\Psi : id_{\mathbf{L}} \to \mathscr{K}_{ML} \circ \mathscr{K}_{LM}$ and $\Theta : \mathscr{K}_{LM} \circ \mathscr{K}_{ML} \to id_{\mathbf{M}}$ such that below diagrams commutes

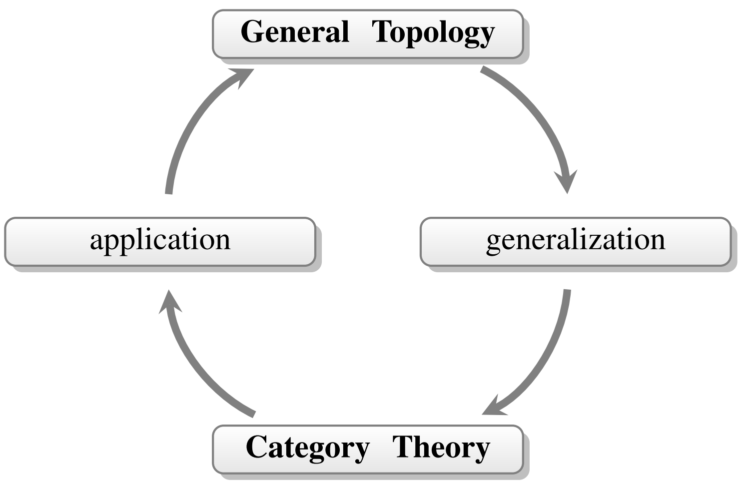
$$\begin{array}{ccc} \mathscr{K}_{LM} & \xrightarrow{\mathscr{K}_{LM}\cdot\Psi} & \mathscr{K}_{LM} \circ \mathscr{K}_{ML} \circ \mathscr{K}_{LM} \\ & \searrow^{id} & \downarrow{\scriptstyle \Theta\cdot\mathscr{K}_{LM}} \\ & & \mathscr{K}_{LM} \end{array}$$

and

$$\begin{array}{ccc} \mathscr{K}_{ML} & \xrightarrow{\Psi\cdot\mathscr{K}_{ML}} & \mathscr{K}_{ML} \circ \mathscr{K}_{LM} \circ \mathscr{K}_{ML} \\ & \searrow^{id} & \downarrow{\scriptstyle \mathscr{K}_{ML}\cdot\Theta} \\ & & \mathscr{K}_{ML} \end{array}$$

1.2 Categorical Topology

In the year 1971, Horst Herrlich [18] presented a novel sub-branch of mathematics named "Categorical Topology". It is the field of Mathematics where general topology and category theory overlap. The purpose of introducing this remarkable field was to implement categorical concepts and findings to topological settings and also to elaborate not only the original topological phenomenon but also similar phenomenon throughout topology as well as in other fields. In other words, Categorical Topology is the study of Topology using categorical methods.

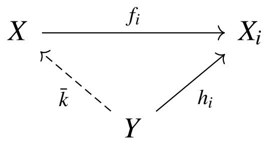


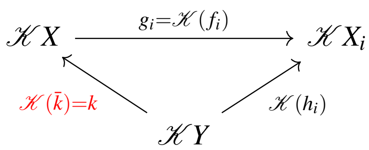
In our work, we assume that $\mathbf{F} = \mathbf{M} := \mathbf{Set}$.

Definition 1.21 (Initial and Final Lifts)**.** *(cf. [47])*

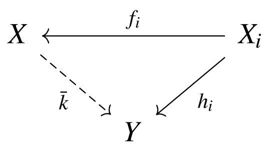
1. *Let $\mathscr{K} : \mathbf{E} \longrightarrow \mathbf{F}$ be a functor between two categories. For a $\mathscr{K}$-source, i.e. family of mappings $\mathscr{K}X \xrightarrow{g_i} \mathscr{K}X_i$ in $\mathbf{F}$ there is a family $X \xrightarrow{f_i} X_i$ in $\mathbf{E}$ such that $\mathscr{K}(f_i) = g_i$ and*

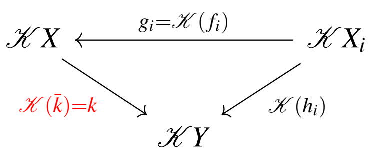
if $\mathscr{K}(h_i) = g_i k$, for every source $Y \xrightarrow{h_i} X_i$ in $\boldsymbol{E}$ along the same domain as in f_i. Then there exists a lift $Y \xrightarrow{\bar{k}} X$ of $\mathscr{K}Y \xrightarrow{k} \mathscr{K}X$ that is, $\mathscr{K}(\bar{k}) = k$. In other words, if there exists a morphism k in the codomain then we say that for every $\boldsymbol{F}$-morphism k, there is a $\boldsymbol{E}$-morphism $\bar{k}$ (in the domain) so that the following diagrams commute.





2. *Let $\mathscr{K} : \boldsymbol{E} \longrightarrow \boldsymbol{F}$ be a functor between two categories. For a $\mathscr{K}$-sink, i.e. family of mappings $\mathscr{K}X \xleftarrow{g_i} \mathscr{K}X_i$ in $\boldsymbol{F}$ there is a family $X \xleftarrow{f_i} X_i$ in $\boldsymbol{E}$ such that $\mathscr{K}(f_i) = g_i$ and if $\mathscr{K}(h_i) = kg_i$, for every sink $Y \xleftarrow{h_i} X_i$ in $\boldsymbol{E}$ along the same domain as in f_i. Then there exists a lift $Y \xleftarrow{\bar{k}} X$ of $\mathscr{K}Y \xleftarrow{k} \mathscr{K}X$ that is, $\mathscr{K}(\bar{k}) = k$. In other words, if there exists a morphism k in the codomain then we say that for every $\boldsymbol{F}$-morphism k, there is a $\boldsymbol{E}$-morphism $\bar{k}$ (in the domain) so that the following diagrams commute.*





Definition 1.22 (Topological Functor). *(cf. [47]) For any two categories $\boldsymbol{L}$ and $\boldsymbol{M}$, we call $\mathscr{K} : \boldsymbol{L} \longrightarrow \boldsymbol{M}$ a topological functor or $\boldsymbol{L}$ a topological category over $\boldsymbol{M}$ if the following conditions are satisfied:*

(i) *$\mathscr{K}$ is concrete.*

(ii) *$\mathscr{K}$ consists of small fibers, i.e. $\forall M_1 \in Obj(\boldsymbol{M}), \mathscr{K}^{-1}(M_1) = \{L_1 \in Obj(\boldsymbol{L}) | \mathscr{K}(L_1) = M_1\}$ is a set.*

(iii) *Every $\mathscr{K}$-source (family of mappings in $\boldsymbol{M}$) has an initial lift.*

Example 1.2.1. *(i) Let **E=Top** and **F=Set**, then the forgetful functor $\mathscr{K}\ :\ \boldsymbol{E} \longrightarrow \boldsymbol{F}$ is a topological functor. Its initial lift is its initial topology, i.e., the topology τ_* on a set X for the family of topological spaces $(X_i,\tau_i)_{i\in I}$ and the family of mappings $f_i\ :\ X \longrightarrow (X_i,\tau_i)$ is defined as*

$$\tau_* := \bigcup_{i\in I}\bigcap_{k=1}^{n}\{f_{ik}^{-1}(U_{ik})\ :\ U_{ik}\in\tau_i\}.$$

*(ii) The forgetful functor $\mathscr{K}$: **Grp** $\rightarrow$ **Set** is not a topological functor since initial lift does not exist in group. In other words, the subset of a group may not always be its subgroup.*

*(iii) The forgetful functor $\mathscr{K}$: **Met** $\rightarrow$ **Set** is not a topological functor since infinite product of metric spaces may not be a metric space.*

In fact, all algebraic structures do not preserve the initial lift, so they (**Ring**, **Module**, etc.) are not topological categories.

Definition 1.23 (Epireflective)**.** *(cf. [47]) Consider a topological functor $\mathfrak{F}:\mathscr{C}\rightarrow$ **Set** with $\mathscr{G}'\subset\mathscr{C}$ as a full isomorphism-closed subcategory of $\mathscr{C}$. Then $\mathscr{G}'$ is known as epireflective in $\mathscr{C}$ iff $\mathscr{G}'$ is closed under the formation of products and extremal subobjects (i.e., subspaces).*

Definition 1.24 (Quotient-reflective)**.** *(cf. [47]) Consider a topological functor $\mathfrak{F}:\mathscr{C}\rightarrow$ **Set** with $\mathscr{G}'\subset\mathscr{C}$ as a full isomorphism-closed subcategory $\mathscr{C}$. Then $\mathscr{G}'$ is said to be quotient-reflective in $\mathscr{C}$ iff $\mathscr{G}'$ is epireflective and is closed under finer structures (i.e., if $M\in\mathscr{G}'$, $N\in\mathscr{C}$, $\mathfrak{F}(M)=\mathfrak{F}(N)$, and $id:M\rightarrow N$ is a $\mathscr{C}$-morphism, then $N\in\mathscr{G}'$).*

Definition 1.25 (Bireflective Subcategories)**.** *(cf. [47]) Consider a topological functor $\mathfrak{F}:\mathscr{C}\rightarrow$ **Set** with $\mathscr{G}'\subset\mathscr{C}$ as a full isomorphism-closed subcategory $\mathscr{C}$. Then $\mathscr{G}'$ is bireflective in $\mathscr{C}$ iff $\mathscr{G}'$ is epireflective and contains the subcategory of all indiscrete objects.*

Example 1.2.2. *1. T_0**Top** is epireflective in **Top**, where T_0**Top** is the category of T_0 topological spaces and continuous mappings.*

*2. Suppose **Prord** denotes the category of preordered spaces and order-preserving mappings. Then **Prord$_{0C}$** is quotient-reflective in **Prord**, where **Prord$_{0C}$** is a closure operator of **Prord**.*

*3. **Dim(Prord)** is bireflective in **Prord**, where **Dim(Prord)** is the category of zero-dimensional preordered spaces and order-preserving mappings.*

Definition 1.26 (Discrete Functor)**.** *(cf. [47]) Let **E** and **F** be two categories. A left adjoint mapping $D\ :\ \boldsymbol{F}\longrightarrow\boldsymbol{E}$ of the topological functor $\mathscr{K}\ :\ \boldsymbol{E}\longrightarrow\boldsymbol{F}$ is called discrete functor.*

An object of the form $c = D\mathscr{K}c$ is called a *discrete object* in **E**.

Definition 1.27 (Indiscrete Functor)**.** *(cf. [47]) Let* **L** *and* **M** *be two categories. A right adjoint mapping* $D : \boldsymbol{M} \longrightarrow \boldsymbol{L}$ *of the topological functor* $\mathscr{K} : \boldsymbol{L} \longrightarrow \boldsymbol{M}$ *is called indiscrete functor.*

An object of the form $c = c\mathscr{K}D$ is called an *indiscrete object* in **L**.

Definition 1.28 (Normalized Topological Functor) (cf. [47, 81])**.** *A normalized topological functor is a functor* $\mathfrak{U} : \boldsymbol{C} \longrightarrow \boldsymbol{D}$ *whose constant objects have a unique structure.*

Examples 1.2.3. *The forgetful functor* $\mathfrak{U} : \boldsymbol{Top} \longrightarrow \boldsymbol{Set}$ *is a normalized functor due to the fact that single topology exists on a set with one element.*

Definition 1.29 (One-Point Extension or Extensionality)**.** *(cf. [92]) For a topological category* $\boldsymbol{E}$ *to be extensional the following must hold: every* $A \in Obj(\boldsymbol{E})$ *has one-point extension, i.e., every* $E \in Obj(\boldsymbol{E})$ *can be embedded into* $E^* \in Obj(\boldsymbol{E})$ *by adding a single point* ∞ *s.t. for every morphism* $e : A \longrightarrow E \in Hom(\boldsymbol{E})$ *with domain as a subobject of* $F \in Obj(\boldsymbol{E})$*, the mapping given by*

$$e^*(f) := \begin{cases} e(f), & if\ f \in A, \\ \infty, & if\ f \in F \backslash A, \end{cases}$$

is in $Hom(\boldsymbol{E})$*. That is the following diagram*

$$\begin{array}{ccc} A & \xrightarrow{\quad j \quad} & F \\ \downarrow{\scriptstyle e} & & \downarrow{\scriptstyle e^*} \\ E & \xrightarrow{\ i\ } & E^* = E \cup \{\infty\} \end{array}$$

commute.

1.3 Bounded Uniform Filter Spaces

Here we discuss the structure of bounded uniform filter spaces introduced by Leseberg [92] that overcomes almost all the deficiencies that appeared in the category **Top**. Moreover, **b-UFIL**, the category of bounded uniform filter spaces and bounded uniformly continuous mappings, is rich in structure as it contains **PUConv** and **SUConv** as its subcategories.

In this section, we would like to define the concept of boundedness and bounded uniform filter spaces introduced by Leseberg. Before moving ahead, let us recall some definitions as follows.

Definition 1.30 (Filter)**.** *(cf. [92]) For a non-empty set* E*, a non-empty collection* $\mathscr{F}$ *of subsets of* E *is a filter on* E *if the following axioms hold:*

(i) $\mathscr{F}$ *does not contain empty set* ($\phi \notin \mathscr{F}$);

(ii) $\forall\, F_1,\, F_2 \in \mathscr{F} \quad \Rightarrow \quad F_1 \cap F_2 \in \mathscr{F}$;

(iii) $\forall\, F \in \mathscr{F}$ *with* $F \subset F' \subset E \quad \Rightarrow \quad F' \in \mathscr{F}$.

If the condition (i) does not hold, i.e., $\phi \in \mathscr{F}$, *then* $\mathscr{F}$ *is an improper filter.*

A filter on E is denoted by $[E]$ and the set of all filters on E is represented by $\mathscr{F}(E)$. A filter on a singleton set $\{x\}$ is denoted by $[x]$ or $\dot{x}$.

Examples 1.3.1. *Let* X *be any set and* $x, y \in X$ *be its elements. Then*

1. $[x] = \{U \subset X; x \in U\}$ *is a filter.*
2. $[V] = \{U \subset X; V \subset U\}$ *is a filter.*
3. $[x] \times [x] = \{(U,V); U,V \in [x]\}$ *is a filter.*
4. $[x] \times [y] = \{(U,V); U \in [x] \wedge V \in [y]\}$ *is a filter.*

Definition 1.31 (Union and Intersection of Filters)**.** *(cf. [92]) Let* α *and* β *be proper filters on* E_1, γ *be a proper filter on* E_2, *and* $e : E_1 \longrightarrow E_2$ *be a function.*

1. $\alpha \cup \beta = \{U \mid U \subset E_1 \text{ and } V \cap W \subset U \text{ for some } V \in \alpha \text{ and } W \in \beta\}$.
2. $\alpha \cap \beta = \{U \mid U \subset E_1 \text{ and } U \in \alpha \text{ and } U \in \beta\}$.
3. $e(\alpha) = \{V \mid V \subset E_2 \text{ and } e(U) \subset V \text{ for some } U \in \alpha\}$.
4. $e^{-1}(\gamma) = \{U \mid U \subset E_1 \text{ and } e^{-1}(V) \subset U \text{ for some } V \in \gamma\}$.

Definition 1.32 (Pre-Uniform Convergence Space)**.** *(cf. [92]) A pre-uniform convergence space is a pair* (X, ψ_X), *where* X *is a set and* ψ_X *is a set of filters of* $X \times X$ *such that the following hold:*

I) *the filter generated by* $\{(x,x)\}$ *that is,* $[x] \times [x] \in \psi_X$, *for all* $x \in X$;

II) *whenever* $F \in \psi_X$ *and* $F \subset G$ *implies that* $G \in \psi_X$.

Definition 1.33. *(cf. [92]) A mapping* $f : (X, \psi_X) \longrightarrow (Y, \psi_Y)$ *between pre-uniform convergence spaces is called uniformly continuous mapping if*

$$(f \times f)(F) \in \psi_Y, \quad for\ each\ F \in \psi_X.$$

Or in other words, $(f \times f)(\psi_X) \subset \psi_Y$.

Definition 1.34 (Semiuniform Convergence Space)**.** *(cf. [92]) A semiuniform convergence space is a pre-uniform convergence space s.t. whenever $F \in \psi_X$ implies that $F^{-1} \in \psi_X$.*

Example 1.3.2. *1. The discrete semiuniform convergent structure ψ_d on X is given by*

$$\psi_d = \{[\emptyset],\ [x] \times [x]\ :\ x \in X\}.$$

2. The indiscrete semiuniform convergent structure on X is given by $\psi = \mathscr{F}(X \times X)$.

Definition 1.35 (Bornological Space)**.** *(cf. [92]) A bornology Θ on a set X is a family of subsets of X that is*

(i) closed under the finite union of sets;

(ii) closed under subsets;

(iii) forms a cover of X.

The pair (X, Θ^X) is called a bornological space.

Definition 1.36 (Bounded Mapping)**.** *(cf. [92]) Let (X, Θ^X) and (Y, Θ^Y) be two bornological spaces. A mapping $f : X \longrightarrow Y$ is called bounded mapping if the image under f of every Θ^X-bounded set is a Θ^Y-bounded set; that is, if for every $A \in \Theta^X$, $f(A) \in \Theta^Y$.*

Example 1.3.3. *(a) Collection of all finite subsets of X forms a bornology on X.*

(b) The power set $P(X)$ of X is a bornology on X.

We denote **PUConv** (respectively **SUConv**) as category of preuniform (respectively semiuniform) convergence spaces and uniformly continuous mappings. Similarly, **BORN** as category of bornological spaces and bounded mappings.

Definition 1.37 (b-set or Boundedness)**.** *(cf. [92]) Let X be a set and $P(X)$ be the power set of X. A subset $\Theta \subset P(X)$ is called a b-set if the following are satisfied:*

(i) the empty set is bounded i.e. $\phi \in \Theta$;

(ii) $x \in \Theta$ for all $x \in X$;

(iii) if $B \in \Theta$ and there is some $B' \subset B$, then $B' \in \Theta$.

Note that the first axiom of 1.37 is only needed in case X is empty.

Definition 1.38 (Bounded mapping). *(cf. [92]) Given a pair of b-sets Θ^X and Θ^Y and a mapping $f : X \longrightarrow Y$. Then f is called **bounded** if and only if $B \in \Theta^X$ implies $f(B) \in \Theta^Y$.*

Definition 1.39 (Bounded Space). *(cf. [92]) Let X be a set and Θ^X be a b-set, then the pair (X, Θ^X) is called a bounded space.*

Remark 1.3.4. *Every bornological space is a bounded space but the converse is not true.*

Remark 1.3.5. *The category of b-sets and bounded mappings **BOUND** defines a topological universe [81].*

Example 1.3.6. 1. *The set Θ^X is called **discrete** if $\Theta^X := \{\phi\} \cup \{\{x\} : x \in X\} := D^X$.*

2. *The set Θ^X is called **indiscrete** if $\Theta^X := P(X)$.*

Definition 1.40 (b-UFIL space) (cf. [91]). *Let X be a non-empty set, Θ^X be a non-empty subset of $P(X)$ and $\psi \subset \mathscr{F}(X \times X)$ be a non-empty set of uniform filters on the cartesian product of X with itself. A pair (Θ^X, ψ) is said to be a bounded uniform filter structure (or b-UFIL structure) on X and the corresponding triplet (X, Θ^X, ψ) is known as bounded uniform filter space (or b-UFIL space) on X if the following axioms hold:*
(b-UFIL1) $E' \subset E \in \Theta^X$ implies $E' \in \Theta^X$;
(b-UFIL2) $x \in X$ implies $\{x\} \in \Theta^X$;
(b-UFIL3) $E \in \Theta^X \backslash \emptyset$ implies $[E] \times [E] \in \psi$;
(b-UFIL4) $\sigma \in \psi$ and $\sigma \subset \sigma' \in \mathscr{F}(X \times X)$ implies $\sigma' \in \psi$.

A b-UFIL space (Z, Θ^Z, ψ) is a *symmetric b-UFIL space* provided that the following axiom holds:
(b-UFIL5) $\sigma \in \psi$ implies $\sigma^{-1} \in \psi$.

A symmetric b-UFIL space (Z, Θ^Z, ψ) is a *symmetric bounded uniform limit space* provided that the following axiom holds:
(b-UFIL6) $\sigma \in \psi$ and $\sigma' \in \psi$ implies $\sigma \cap \sigma' \in \psi$.

A b-UFIL space (Z, Θ^Z, ψ) is a *crossbounded uniform filter space* provided it satisfies the following condition:
(crb) $\sigma \in \psi$ implies $E \times E \in \sigma$ for some $E \in \Theta^Z$.

Definition 1.41 (buc mappings) (cf. [91]). *Let (X, Θ^X, ψ_X) and (Y, Θ^Y, ψ_Y) be two b-UFIL spaces and $h : X \to Y$ be a mapping. Then h is called bounded uniformly continuous (or buc) mapping if* *(buc1)* $E \in \Theta^X$ *implies* $h(E) \in \Theta^Y$*; and*

(buc2) $\sigma \in \psi_X$ *implies* $(h \times h)(\sigma) \in \psi_Y$*; where* $(h \times h)(\sigma) := \{V \subset Y \times Y : \exists U \in \sigma \mid (h \times h)[U] \subset V\}$ *with* $(h \times h)[U] := \{(h \times h)(x,y) : (x,y) \in U\} = \{(h(x),h(y)) : (x,y) \in U\}$.

We denote by **b-UFIL** the category of b-UFIL spaces and buc mappings. Similarly, **sb-UFIL** (respectively **LIMsb-UFIL**) as the category of symmetric b-UFIL spaces (respectively category of symmetric b-UFIL limit spaces) and buc mappings. Furthermore, **BONb-UFIL** is the category of bornological b-UFIL spaces and buc mappings.

If we denote by **CROSSb-UFIL** the corresponding defined full subcategory of **b-UFIL**, then it is clear that **BOUND** and **CROSSb-UFIL** are isomorphic. Hence we can introduce the following:

A cross bounded uniform filter space (Z, Θ^Z, ψ) is a *bornological b-UFIL space* provided that the following axiom holds:

(b-UFIL7) $E, E' \in \Theta^Z$ implies $E \cup E' \in \Theta^Z$.

Examples 1.3.7. 1. A b-UFIL structure on A is discrete if $(\Theta^A, \psi) := (D^A, \psi_{dis})$, where $\psi_{dis} := \{[a] \times [a] : a \in A\} \cup \{P(A^2)\}$.

2. A b-UFIL structure on A is indiscrete if $(\Theta^A, \psi) := (P(A), \mathscr{F}(A^2))$.

3. Let $A = \{1,2,3\}$, $\Theta^A = \{\emptyset, \{1\}, \{2\}, \{3\}\}$ and $\psi = \{[\emptyset], [1] \times [1], [2] \times [2], [3] \times [3], [1] \times [2], [2] \times [3]\}$. Then, (Θ^A, ψ) is a b-UFIL structure on X and (A, Θ^A, ψ) is a bounded uniform filter space.

Remark 1.3.8. *The category **BOUND** and the category **PUConv** can be embedded into the category **b-UFIL**.*

Remark 1.3.9. *(i) A bornological b-UFIL structure on X is discrete if* $(\Theta^X, \psi) := (D^X_{born}, \psi_{dis})$*, where* $D^X_{born} := \{E \subset X : E \text{ is finite}\}$ *[92].*

*(ii) The category **PUConv** is isomorphic to **DISb-UFIL** (category of discrete b-UFIL spaces and buc mappings) [91].*

*(iii) The category **SUConv** is isomorphic to **DISsb-UFIL** (category of discrete symmetric b-UFIL spaces and buc mappings) [91].*

Definition 1.42. *(Set convergence) (cf. [93]) For an arbitrary set A, let* $\Theta^A \subset P(A)$ *be a nonempty boundedness of A and* $q \subset \Theta^A \times \mathscr{F}(A)$.

(i) *A pair* (Θ^A, q) *is called a set-convergence on* A *and the triplet* (A, Θ^A, q) *is known as a set-convergence space (or S-Conv space) if the following axioms hold:*
(S-Conv1) $E \in \Theta^A$ *implies* $(E, [E]) \in q$;
(S-Conv2) $(\emptyset, \xi) \in q$ *implies* $\xi = P(A)$;
(S-Conv3) $(E, \xi) \in q$ *and* $\xi \subset \xi' \in \mathscr{F}(A)$ *implies* $(E, \xi') \in q$.

(ii) *A set-convergence space* (A, Θ^A, q) *is called a reordered set-convergence space (or ROS-Conv space) provided that the following axiom holds:*
(S-Conv4) If $E \in \Theta^A$ *and* $(E, \xi) \in q$ *then for* $E' \subset E \in \Theta^A$ *with* $E' \neq \emptyset$ *implies* $(E', \xi) \in q$.

Let (A, Θ^A, q) and (B, Θ^B, p) be a pair of S-Conv spaces and $h : A \to B$ be a mapping. Then h is called a bounded continuous (or b-continuous) mapping if h is bounded and h transfers convergent filters.

We denote **S-Conv** (respectively **ROS-Conv**) as the category of S-Conv spaces (respectively reordered S-Conv spaces) and b-continuous mappings. Also, we write $\xi \; q \; E$ for $(E, \xi) \in q$.

Note that if we restrict Θ^A to be the discrete bounded structure on A, then many point-convergence spaces in the classical sense, such as, limit spaces, Kent-convergence spaces, topological spaces etc., can be embedded into **ROS-Conv** spaces. Also note that the category **ROS-Conv** can be regarded as a full subcategory of **b-UFIL** as mentioned in [93].

Proposition 1.3.10. *(cf. [92]) For a b-UFIL space* (Z, Θ^Z, μ)*, the corresponding tripet* (Z, Θ^Z, q_μ) *can be achieved provided that the following axioms hold:*

(i) $\xi \; q_\mu \; \emptyset$ *iff* $\xi = P(X)$;

(ii) $\xi \; q_\mu \; E$ *iff* $[E] \times \xi \in \mu$, $\forall E \in \Theta^Z \setminus \{\emptyset\}$.

Then the b-uniform filter space (Z, Θ^Z, q_μ) *is called b-uniform convergence space.*

Proposition 1.3.11. *(cf. [92]) For a ROS-Conv space* (A, Θ^A, q)*, the corresponding triplet* (A, Θ^A, μ_q) *can be achieved provided that the* μ_q *is defined by the following setting:*

$$\mu_q := \{\sigma \in \mathscr{F}(A \times A) : \exists \xi \in \mathscr{F}(A) \; \exists E \in \Theta^A \setminus \{\emptyset\} \; (\xi \; q \; E \text{ and } [E] \times \xi \subset \sigma)\} \cup \{P(A \times A)\}.$$

Then (Θ^A, μ_q) *forms a so-called bounded uniform convergence, and the triple* (A, Θ^A, μ_q) *a bounded uniform convergence space.*

Remark 1.3.12. *By Propositions 1.3.10 and 1.3.11 it is clear that the categories **ROS-Conv** and **b-UConv** are isomorphic.*

1.4 b-UFIL as a Topological Category

Theorem 1.1 (cf. [92]). *Consider* $\mathfrak{F} : $ ***b-UFIL*** $\longrightarrow$ ***Set*** *which is given by for every* (Z,Θ^Z,ψ) *b-UFIL space,* $\mathfrak{F}(Z,\Theta^Z,\psi) = Z$ *a set and for every* $h : (Z,\Theta^Z,\psi_Z) \longrightarrow (Y,\Theta^Y,\psi_Y)$ *buc mapping,* $\mathfrak{F}(h) = h : Z \longrightarrow Y$ *is a topological functor.*

Proof. 1. We first prove that the functor $\mathfrak{F}$ is concrete. Let $h,\ g : (Z,\Theta^Z,\psi_Z) \longrightarrow (Y,\Theta^Y,\psi_Y)$ be buc mappings and $\mathfrak{F}(h) = \mathfrak{F}(g)$. As $\mathfrak{F}(h) = h$ and $\mathfrak{F}(g) = g \implies h = g$. Therefore, functor $\mathfrak{F}$ is faithful. Consider a buc mapping $h : (Z,\Theta^Z,\psi_Z) \longrightarrow (Z,\Theta^Y,\psi_Y)$ with $\mathfrak{F}(h) = \mathbb{1}_Z$. Let h be an isomorphism. We claim that $(\Theta^Z,\psi_Z) = (\Theta^Y,\psi_Y)$. By definition of bounded uniformly continuous mappings, we have

(a) for all $E \in \Theta^Z$ implies $h(E) \in \Theta^Y$;

(b) for all $\sigma \in \psi_Z$ implies $(h \times h)(\sigma) \in \psi_Y$.

Since h is an isomorphism, so there is a unique $g : (Z,\Theta^Y,\psi_Y) \longrightarrow (Z,\Theta^Z,\psi_Z)$ s.t.

$$h \circ g = \mathbb{1}_{(Z,\Theta^Y,\psi_Y)},\quad g \circ h = \mathbb{1}_{(Z,\Theta^Z,\psi_Z)};$$

$$\textit{and}\quad \mathfrak{F}(h) = \mathfrak{F}(g) = \mathbb{1}_Z.$$

It can be followed easily that $\mathfrak{F}$ is amnestic and thus it is concrete.

2. Next we prove that the functor $\mathfrak{F}$ contains small fibers, i.e. for every $Z \in$ **Set**, $\mathfrak{F}^{(-1)}(Z)$ is a set.

   Consider $\mathfrak{F}^{(-1)}(Z) = \{(Z,\Theta^Z,\psi_Z) \in Obj(\textbf{b-UFIL}) : \mathfrak{F}(Z,\Theta^Z,\psi_Z) = Z\}$. On the other hand, let $\Phi = \{(\Theta^Z,\psi_Z) : (\Theta^Z,\psi_Z)\ is\ a\ b-UFIL\ structure\ defined\ on\ X\}$ and $\Gamma : \mathfrak{F}^{(-1)}(Z) \longrightarrow \Phi$ be a function defined by $\Gamma(Z,\Theta^Z,\psi_Z) = Z$. Clearly it can be observed that Γ is one-to-one (injective) and onto (surjective). Therefore $\mathfrak{F}^{(-1)}(Z) \simeq \Gamma \subset P(Z \times Z)$; i.e. $\mathfrak{F}^{(-1)}(Z)$ is a set. Hence $\mathfrak{F}$ contains small fibers.

3. Lastly we prove that (Θ^Z,ψ_Z) contains an initial structure on Z. Consider b-UFIL spaces $(Z_i,\Theta^{Z_i},\psi_i)_{i\in I}$ and a non-empty set Z with a source $h_i : Z \longrightarrow Z_i$ in **Set** $\forall\ i \in I$. Then (Θ^Z_I,ψ^I_Z) is the initial b-UFIL structure on Z, where

$$\Theta^Z_I := \{E \subset Z\ :\ \forall i \in I,\ h_i[E] \in \Theta^{Z_i}\}$$

   and

$$\psi^I_Z := \{\sigma \in \mathfrak{F}(Z \times Z)\ :\ \forall i \in I,\ (h_i \times h_i)(\sigma) \in \psi_i\}.$$

Evidently, (Θ_I^Z, ψ_Z^I) satisfies the axioms (b-UFIL1), (b-UFIL2) and (b-UFIL4) respectively.

To check (b-UFIL3): For $E \in \Theta^X \backslash \{\emptyset\}$ we are getting

$$(h_i \times h_i)([E] \times [E]) = h_i([E]) \times h_i([E]) = h_i[E] \times h_i[E] \in \psi_i$$

$\forall i \in I$, hence $[E] \times [E] \in \Theta_I^Z$ follows. Here, in general for $\sigma_1, \sigma_2 \in \mathfrak{F}(Z \times Z)$ their cross-product is defined by setting:

$$\sigma_1 \times \sigma_2 := \{R \subset Z \times Z \,:\, \exists \alpha_1 \in \sigma_1, \exists \alpha_2 \in \sigma_2 \; s.t. \; R \supset \alpha_1 \times \alpha_2\}.$$

By definition of (Θ_I^Z, ψ_Z^I) each f_i is bounded uniformly continuous. Now let (Y, Θ^Y, ψ_Y) be b-UFIL space and $g : Y \longrightarrow Z$ be a mapping such that for each $i \in I$, $h_i \circ g : (Y, \Theta^Y, \psi_Y) \longrightarrow (Z_i, \Theta^{Z_i}, \psi_i)$ is bounded uniformly continuous, we have to show that $g : (Y, \Theta^Y, \psi_Y) \longrightarrow (Z, \Theta_I^Z, \psi_Z^I)$ is bounded uniformly continuous.

To check (b-UFIL1): $E \in \Theta^Y$ implies $(h_i \circ g)[E] = h_i[g[E]] \in \Theta^{Z_i}$ for each $i \in I$ by the assumption. Hence $g[E] \in \Theta_I^Z$ follows.

To check (b-UFIL2): $\sigma \in \psi_Y$ implies

$$(h_i \times h_i)((g \times g)(\sigma)) = ((h_i \circ g) \times (h_i \circ g))(\sigma) \in \psi_i$$

for each $i \in I$ by the hypothesis. Consequently, $(g \times g)(\sigma) \in \psi_Z$ results, showing that $g : (Y, \Theta^Y, \psi_Y) \longrightarrow (Z, \Theta_I^Z, \psi_Z^I)$ is bounded uniformly continuous. Since being valid $(\Theta^Z, \psi) \in P(P(Z)) \times P(\mathfrak{F}(Z \times Z))$ the class of all bounded uniform filter structures on Z is a set.

□

Theorem 1.4.1. *Consider a topological functor $\mathscr{F} : \mathscr{C} \to$ **Set** with $\mathscr{C} =$ **b-UFIL**. Then $\mathscr{F}$ is normalized.*

Proof. Since a unique b-UFIL structure $\Theta^Z = \{\emptyset\}$, and $\psi_Z = \{[\emptyset]\}$ exists whenever $Z = \emptyset$ and a unique b-UFIL structure $\Theta^Z = \{\emptyset, \{a\}\}$, and $\psi_Z = \{[\emptyset], [a] \times [a]\}$ exists whenever $Z = \{a\}$. □

Now we give the proper definitions of the initial and the final structures on bounded uniform filter spaces.

Definition 1.43 (Initial and Final b-UFIL structures) (cf. [91, 92]). *(i) For given a family of b-UFIL spaces* $(Z_j, \Theta^{Z_j}, \psi_j)_{j\in I}$ *and mappings* $(h_j : Z \to Z_j)_{j\in I}$*. The initial b-UFIL structure on* Z *is represented by* (Θ^Z, ψ)*, where* $\Theta^Z := \{E \subset Z : \forall j \in I, h_j[E] \in \Theta^{Z_j}\}$ *and* $\psi := \{\sigma \in \mathscr{F}(Z^2) : \forall j \in I, (h_j \times h_j)(\sigma) \in \psi_j\}$ *with* $Z^2 := Z \times Z$.

(ii) For given a family of b-UFIL spaces $(Z_j, \Theta^{Z_j}, \psi_j)_{j\in I}$ *and mappings* $(h_j : Z_j \to Z)_{j\in I}$*. The final b-UFIL structure on* Z *is represented by* (Θ^Z, ψ)*, where* $\Theta^Z := \{E \subset Z : \exists j \in I, \exists E_j \in \Theta^{Z_j} : \ E \subset h_j[E_j]\} \cup D^Z := \{\emptyset\} \cup \{\{a\} : a \in Z\}$ *and* $\psi := \{\sigma \in \mathscr{F}(Z^2) : \exists j \in I, \exists \sigma_j \in \psi_j : \ (h_j \times h_j)(\sigma_j) \subset \sigma\} \cup \{[z] \times [z] : z \in Z\} \cup \{P(Z^2)\}$.

Remark 1.4.2. *(cf. [92])*

1. *The category **b-UFIL** is catresian closed.*
2. *Every object in the category **b-UFIL** has one-point extension.*
3. *In the category **b-UFIL**, product of quotient mappings is quotient.*

1.5 Prime Goals

The prime goals of this work are stated below:

(i) to characterize local $\overline{T_0}$, local T_0' and local T_1 objects in the category **b-UFIL**, and evaluate their relationship among each other;

(ii) to give the characterization of $\overline{T_0}$, T_0' andT_1 objects in the category **b-UFIL**, and examine their mutual relationship;

(iii) to examine that under conditions of T_0 and T_1, preuniform (respectively semiuniform) spaces are isomorphic to bornological (respectively symmetric) bounded uniform filter spaces;

(iv) to examine the quotient-reflective properties of several bounded uniform filter spaces;

(v) to characterize both closed and strongly closed subobjects in the category **b-UFIL** and to prove that they form favourable closure operators in the sense of [39] which satisfy the fundamental properties such as (weakly) hereditary, idempotent and productivity;

(vi) to characterize $\overline{T_0}$ and T_1 b-UFIL spaces with respect to these closure operators and examine that each of these subcategories of $\overline{T_0}$ and T_1 b-UFIL spaces are quotient-reflective and discuss the relationship among them;

(vii) to give characterization of both connected and strongly connected bounded uniform filter spaces in the sense of Baran;

(viii) to introduce ultraconnected objects in topological category, and to characterize irreducible (resp. ultraconnected) bounded uniform filter spaces and examine their relationship with connected objects.

1.6 Significance of this Research

The importance of T_0 and T_1 axioms is not only confined to mathematics but their applications have been seen in almost every field of science such as computer graphics and image processing [56, 49, 45], lambda calculus and denotational semantics of programming language [70, 33], axiomatic quantum mechanics [69, 68] and in continuation theory of computer science [72]. Also in mathematics, many important theories including compactification [53] and completion theories [74] are only applicable in the presence of T_0 spaces. Because of their tremendous importance, T_0 and T_1 spaces have been characterized to the topological category by several mathematicians named as Brümmer [13] (1971), Marny [16] (1973), Hoffmann [20] (1974), Harvey [27] (1977) and Baran [50] (1991). They have also examined the applications of these concepts there. In this thesis, we give a brief characterization of several notions for local T_0, local T_1, T_0, and T_1 objects in the category of b-UFIL spaces, which is an extension of semiuniform convergence spaces and boundedness. Some interesting conclusions are obtained with respect to their relationship with each other and with usual separation axioms. We have also investigated quotient reflective subcategories of the category **b-UFIL** and compared the results.

For the sake of defining the notion of closedness in categorical language, Baran [50] introduced local T_0 and local T_1 of topology in a topological category using initial, final lifts, and discrete objects. In 1993, Baran [52] investigated the concept of closedness and closure operators in a topological category. These notions of closedness (strongly closedness) are used to extend several famous theorems of general topology such as Urysohn lemma and Tietze extension theorem. In our study, we characterize both closed and strongly closed objects in the category of b-UFIL spaces. Also, we examine the relationship between them.

Closure operators are one of the main concepts in general topology that can be defined in terms of closed sets. In a topological category, closure operators are defined by Dikranjan and Guili [39] where the epimorphisms of the full-subcategories of topological categories are characterized by using them and suitable closure operators were formed in arbitrary topological categories (see in [67, 71, 80, 96, 100]). Both in Topology and Algebra various examples can be found where closure operators and their relations with other subcategories are studied inclusively [39, 84]. In this thesis, we define two notions of closure operators in the category of b-UFIL spaces and prove that they are (weakly) hereditary, idempotent, and productive. Using these closure operators we characterize each of T_j $(j = 0,1)$ b-UFIL spaces and examine that each of them forms quotient-reflective subcategories of the category **b-UFIL**.

The notion of connectedness plays a very crucial role in almost all areas of mathematics where the notion of topological spaces is itself relevant. With the help of the connected spaces concept, we can define the intermediate value theorem in topology. Then various characterization of connectedness has been studied in topological categories such as the concept of D-connectedness was introduced by Preuss and connected (respectively strongly connected) objects in numerous topological categories were investigated by Baran. These generalizations of the notion of connectedness in a topological category can be found in [75, 55, 64, 32, 44]. In our work, we give a brief characterization of connected and strongly connected objects in the bounded uniform filter spaces category. We introduce the concept of ultraconnected and strongly ultraconnected objects in a set-based topological category. Also, we investigate the relation among connected (resp. strongly connected), irreducible (resp. strongly irreducible), and ultraconnected (resp. connected) bounded uniform filter spaces.

CHAPTER 2

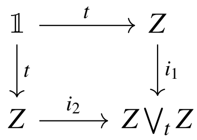
LOCAL T_0 AND LOCAL T_1 BOUNDED UNIFORM FILTER SPACES

In general topology, all the basic concepts including compactness, connectedness, terfectness, soberness, Hausdorffness, and closure operators can be defined in terms of closedness. In order to define these notions of closedness in categorical language, Baran [50] introduced local T_0 and local T_1 of topology in a topological category using initial, final lifts, and discrete objects. Moreover, these notions of closedness (strongly closedness) are used to extend several famous theorems of general topology such as the Urysohn lemma and Tietze extension theorem.

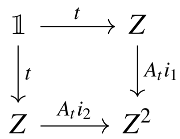
2.1 Local T_0 Bounded Uniform Filter Spaces

In this section, we recall the definition of local T_0 b-UFIL spaces (at some fixed point t). Consider a set Z with $t \in Z$. The wedge product of Z at t is denoted as $Z \bigvee_t Z$ and is referred to as the two disjoint copies of Z at t. Any element $z \in Z \bigvee_t Z$ is written as z_1 (or z_2 resp.) if z belongs to the first (or second resp.) component of the wedge product. Also, the cartesian product of Z with itself is denoted Z^2.

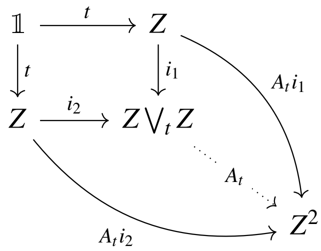
Categorically the wedge product of Z at t is a push out, i.e., on a one-point set $\mathbb{1}$, a retraction map $t : \mathbb{1} \longrightarrow Z$, and inclusion maps $i_1,\ i_2 : Z \longrightarrow Z \bigvee_t Z$ of Z in the first and second factor respectively, the following representation is a push out denoted $Z \bigvee_t Z$.



i.e., $i_1 t = i_2 p$ [50]. To put it another way, for



the following is a push-out diagram



i.e. $A_t i_1 \circ t = A_t i_2 \circ t = (t,t)$ where $A_t\ :\ Z\bigvee_t Z \longrightarrow Z^2$.

Definition 2.1. *(cf. [50])*

(i) A principal t-axis map $A_t\ :\ Z\bigvee_t Z \longrightarrow Z^2$ *is defined as follow*

$$A_t(z_j) := \begin{cases} (z,t), & j = 1, \\ (t,z), & j = 2. \end{cases}$$

(ii) A skewed t-axis map $S_t\ :\ Z\bigvee_t Z \longrightarrow Z^2$ *is defined as follow*

$$S_t(z_j) := \begin{cases} (z,z), & j = 1, \\ (t,z), & j = 2. \end{cases}$$

(iii) A fold map at t, $\nabla_t\ :\ Z\bigvee_t Z \longrightarrow Z$ *is defined as follow*

$$\nabla_t(z_j) := z, \qquad j = 1,2.$$

Consider a topological functor $\mathfrak{F}\ :\ \mathscr{G} \longrightarrow$ **Set** with $Z \in Obj(\mathscr{G})$ and $\mathfrak{F}Z = E$ and $t \in E$.

Definition 2.2. *(cf. [50])*

(i) *Z is $\overline{T_0}$ at t if the $\mathfrak{F}$-source $\{E\bigvee_t E \xrightarrow{A_t} \mathfrak{F}(Z^2) = E^2$ and $E\bigvee_t E \xrightarrow{\nabla_t} \mathfrak{F}DE = E\}$ has the discrete initial lift.*

(ii) *Z is $T_0^{'}$ at t if the $\mathfrak{F}$-source $\{E\bigvee_t E \xrightarrow{id} \mathfrak{F}(Z\bigvee_t Z) = E\bigvee_t E$ and $E\bigvee_t E \xrightarrow{\nabla_t} \mathfrak{F}DE = E\}$ has the discrete initial lift, where $Z\bigvee_t Z$ represents the wedge product in $\mathscr{G}$, i.e., final lift of $\mathfrak{F}$-sink $\{\mathfrak{F}Z = E \xrightarrow{i_1,\, i_2} E\bigvee_t E\}$, where $i_1,\ i_2$ denote the canonical injections.*

Remark 2.1.1. *(i) In **Top**, $\overline{T_0}$ and T_0' at t are equivalent to the usual T_0 at t, i.e., for all $z \in Z$ with $z \neq t$, there is a neighborhood N_z of z with $t \notin N_z$ or there is a neighborhood N_t of t with $z \notin N_t$ [61].*

*(ii) Consider a topological functor $\mathscr{U} : \mathscr{G} \to$ **Set** with $Z \in Obj(\mathscr{G})$. Let $t \in \mathscr{U}(Z)$ be a retract of Z. If Z is $\overline{T_0}$ at t then Z is $T_0^{'}$ at t but the converse implication is not valid generally [54].*

Theorem 2.1 (cf. [103])**.** *Let (Z,Θ^Z,ψ) be a b-UFIL space and $t \in Z$. Then (Z,Θ^Z,ψ) is $\overline{T_0}$ at t iff for all $z_a \in Z$ with $z_a \neq t$, the following hold.*

(i) $\{z_a,t\} \notin \Theta^Z$;

(ii) $[z_a]\times[t] \notin \psi$ *or* $[t]\times[z_a] \notin \psi$;

(iii) $([z_a]\times[z_a])\cap([t]\times[t]) \notin \psi$.

Proof. Let (Z,Θ^Z,ψ) be $\overline{T_0}$ at t. We shall prove that the above conditions $(i)-(iii)$ hold. Let $\{z_a,t\} \in \Theta^Z$ for $z_a \neq t$ and $W = \{z_{a1},z_{a2}\} \in \Theta^{Z\vee_t Z}$. Since $\nabla_t W = \{z_a\} \in \mathscr{D}^Z$, and $\pi_k A_t W = \{z_a,t\} \in \Theta^Z$ for $k = 1,2$, where $\pi_k : Z^2 \to Z$ for $k = 1,2$ are the projection maps. Since (Z,Θ^Z,ψ) is $\overline{T_0}$ at t, by the Definitions 1.43 and 2.2, we get a contradiction. Hence, $\{z_a,t\} \notin \Theta^Z$.

Next, suppose that $[z_a]\times[t] \in \psi$ for some $z_a \neq t$. Let $\sigma = [z_{a1}]\times[z_{a2}]$. Clearly, $(\nabla_t\times\nabla_t)\sigma = [z_a]\times[z_a] \in \psi_{dis}$, $(\pi_1 A_t\times\pi_1 A_t)\sigma = [z_a]\times[t] \in \psi$ and $(\pi_2 A_t\times\pi_2 A_t)\sigma = [t]\times[z_a] \in \psi$, a contradiction. It follows that $[z_a]\times[t] \notin \psi$ or $[t]\times[z_a] \notin \psi$.

Further, if $([z_a]\times[z_a])\cap([t]\times[t]) \in \psi$ for some $z_a \neq t$. Let $\sigma = ([z_{a1}]\times[z_{a1}])\cap([z_{a2}]\times[z_{a2}])$. Since $(\nabla_t\times\nabla_t)\sigma = [z_a]\times[z_a] \in \psi_{dis}$, $(\pi_1 A_t\times\pi_1 A_t)\sigma = ([z_a]\times[z_a])\cap([t]\times[t]) \in \psi$ and $(\pi_2 A_t\times\pi_2 A_t)\sigma = ([t]\times[t])\cap([z_a]\times[z_a]) \in \psi$, a contradiction since (Z,Θ^Z,ψ) is $\overline{T_0}$ at t. Thus, $([z_a]\times[z_a])\cap([t]\times[t]) \notin \psi$.

Conversely, let us assume that the conditions $(i)-(iii)$ hold. Let $(\Theta^{Z\vee_t Z},\overline{\psi})$ be the initial structure induced by $A_t : Z\vee_t Z \to (Z^2,\Theta^{Z^2},\psi^2)$ and $\nabla_t : Z\vee_t Z \to (X,\mathscr{D}^Z,\psi_{dis})$, where (Θ^{Z^2},ψ^2)

represents the product b-UFIL structure on Z^2 and $(\mathscr{D}^Z, \psi_{dis})$ the discrete b-UFIL structure on Z, respectively. We show that $(\Theta^{Z\vee_t Z}, \overline{\psi})$ is the discrete b-UFIL structure on $Z \vee_t Z$. Let $W \in \Theta^{Z\vee_t Z}$ and $\nabla_t W \in \mathscr{D}^Z$.

If $\nabla_t W = \emptyset$, then $W = \emptyset$. Suppose $\nabla_t W \neq \emptyset$, it follows that $\nabla_t W = \{z_a\}$ for some $z_a \in Z$. If $z_a = t$, then $W = \{t\}$. Suppose $z_a \neq t$. It follows that $W = \{z_{a1}\}, \{z_{a2}\}$ or $\{z_{a1}, z_{a2}\}$. The case, $W = \{z_{a1}, z_{a2}\}$ cannot happen since $\pi_k A_t W = \{z_a, t\} \notin \Theta^Z$ $(k = 1, 2)$ by the assumption. Hence, $W = \{z_{a1}\}, \{z_{a2}\}$ and consequently, $\Theta^{Z\vee_t Z} = \mathscr{D}^{Z\vee_t Z}$, the discrete b-UFIL structure on $Z \vee_t Z$.

Next, let $\sigma \in \overline{\psi}$. By Definition 1.43 (i), $(\nabla_t \times \nabla_t)\sigma \in \mathscr{D}^Z$ and $(\pi_k A_t \times \pi_k A_t)\sigma \in \psi$ for $k = 1, 2$. We need to show that $\sigma = [z_{ai}] \times [z_{ai}]$ $(i = 1, 2)$, $\sigma = [t] \times [t]$ or $\sigma = [\emptyset] = P((Z \vee_t Z) \times (Z \vee_t Z))$.

If $(\nabla_t \times \nabla_t)\sigma = [\emptyset]$, then $\sigma = [\emptyset] = P((Z \vee_t Z) \times (Z \vee_t Z))$. Suppose $(\nabla_t \times \nabla_t)\sigma = [z_a] \times [z_a]$ for some $z_a \in Z$. If $z_a = t$, then $\sigma = [t] \times [t]$.

If $z_a \neq t$, then $(\nabla_t \times \nabla_t)\sigma = [z_a] \times [z_a]$, then $\{z_{a1}, z_{a2}\} \times \{z_{a1}, z_{a2}\} \in \sigma$. Thus there is a finite subset N_0 of σ such that $\sigma = [N_0]$. Clearly, $N_0 \subseteq \{z_{a1}, z_{a2}\} \times \{z_{a1}, z_{a2}\}$ and if $i \neq j$, then $\{\{z_{ai}\} \times \{z_{aj}\}\} \neq N_0$ and $\{\{z_{a1}\} \times \{z_{a1}\}, \{z_{a2}\} \times \{z_{a2}\}\} \neq N_0$ since in particular for $k = 1$, $i = 1$, and $j = 2$, $(\pi_1 A_t \times \pi_1 A_t)([z_{a1}] \times [z_{a2}]) = [z_a] \times [t] \notin \psi$ and $(\pi_1 A_t \times \pi_1 A_t)(([z_{a1}] \times [z_{a1}]) \cap ([z_{a2}] \times [z_{a2}])) = ([z_a] \times [z_a]) \cap ([t] \times [t]) \notin \psi$ by using the second and the third conditions respectively.

Therefore, we must have $\sigma = [z_{ai}] \times [z_{ai}]$ $(i = 1, 2)$ and consequently, by Definitions 2.2, 1.40 and 1.43, (Z, Θ^Z, ψ) is $\overline{T_0}$ at t. □

Theorem 2.2 (cf. [103]). *All b-UFIL spaces are T_0' at t.*

Proof. Suppose (Z, Θ^Z, ψ) is a b-UFIL space and $t \in Z$. By Definition 2.2 (ii), we show that for any $W \in \Theta^{X\vee_t X}$, $W \subset i_j(V)$ $(j = 1$ *or* $2)$ for some $V \in \Theta^Z$, and $\nabla_t W \in \mathscr{D}^Z$, and for any $\sigma \in \mathscr{F}((Z \vee_t Z) \times (Z \vee_t Z))$, $\sigma \supset (i_j \times i_j)\alpha$ $(j = 1$ *or* $2)$ for some $\alpha \in \psi$ and $(\nabla_t \times \nabla_t)\sigma \in \psi_{dis}$. Then $W = \emptyset$, $\{t\}$ or $\{z_{ak}\}$ for $k = 1, 2$.

If $\nabla_t W = \emptyset$, then $W = \emptyset$. Let $\nabla_t W \neq \emptyset$. It follows that $\nabla_t W = \{z_a\}$ for some $z_a \in Z$.

If $z_a = t$, then $\nabla_t W = \{t\}$, it follows that $W = \{t\}$.

Suppose $z_a \neq t$, it follows that $W = \{z_{a1}\}, \{z_{a2}\}$ or $\{z_{a1}, z_{a2}\}$. If $W = \{z_{a1}, z_{a2}\}$, then $\{z_{a1}, z_{a2}\} \subset i_1(V)$ for some $V \in \Theta^Z$ which shows that z_{a2} should be in first component of the wedge product $Z \vee_t Z$, a contradiction. In similar manner, $\{z_{a1}, z_{a2}\} \not\subset i_2(V)$ for some $V \in \Theta^Z$. Hence, $W \neq \{z_{a1}, z_{a2}\}$. Thus, we must have $W = \{z_{ak}\}$ for $k = 1, 2$ only and consequently, $\Theta^{Z\vee_t Z} = \mathscr{D}^{Z\vee_t Z}$, the discrete b-UFIL structure on $Z \vee_t Z$.

Next, for any $\sigma \in \mathscr{F}((Z \vee_t Z) \times (Z \vee_t Z))$, if $(\nabla_t \times \nabla_t)\sigma = [\emptyset]$, then $\sigma = [\emptyset] = P((Z \vee_t Z) \times (Z \vee_t Z))$.

Now, assume that $(\nabla_t \times \nabla_t)\sigma = [z_a] \times [z_a]$ for some $z_a \in Z$. If $z_a = t$, then $(\nabla_t \times \nabla_t)\sigma = [t] \times [t]$, and consequently $\sigma = [t] \times [t]$.

Suppose that $z_a \neq t$, then $(\nabla_t \times \nabla_t)\sigma = [z_a] \times [z_a]$, hence $\{z_{a1}, z_{a2}\} \times \{z_{a1}, z_{a2}\} \in \sigma$. Thus there exists a finite subset M of σ such that $\sigma = [M]$. Clearly, $M \subseteq \{z_{a1}, z_{a2}\} \times \{z_{a1}, z_{a2}\}$ and if $k \neq l$, then $\{\{z_{ak}\} \times \{z_{al}\}\} \neq M$ and $\{\{z_{a1}\} \times \{z_{a1}\}, \{z_{a2}\} \times \{z_{a2}\}\} \neq M$. Suppose that $M = \{\{z_{ak}\} \times \{z_{al}\}\}$, then for $k = 1$, $l = 2$, and $j = 1$ (respectively $j = 2$), $[z_{a1}] \times [z_{a2}] \supset (i_1 \times i_1)\alpha$ for some $\alpha \in \psi$. It follows that $(z_{a1}, z_{a2}) \in (i_1 \times i_1)(U)$ for all $U \in \alpha$, which implies that z_{a2} (respectively z_{a1}) is in the first (respectively second) component of the wedge product $Z \vee_t Z$, a contradiction. In similar manner, if $M = \{\{z_{a1}\} \times \{z_{a1}\}, \{z_{a2}\} \times \{z_{a2}\}\}$, then for $j = 1$ (respectively $j = 2$), $(([z_{a1}] \times [z_{a1}]) \cap ([z_{a2}] \times [z_{a2}])) \supset (i_1 \times i_1)\alpha$ for some $\alpha \in \psi$. It follows that $\{(z_{a1}, z_{a1}), (z_{a2}, z_{a2})\} \in (i_1 \times i_1)(U)$ for all $U \in \alpha$, which implies that z_{a2} (respectively z_{a1}) is in the first (respectively second) component of the product $Z \vee_t Z$, a contradiction.

Thus, we must have $\sigma = [z_{ak}] \times [z_{ak}]$ $(k = 1, 2)$ and consequently, by Definitions 2.2, 1.43, and Theorems 2.1, 2.3, (Z, Θ^Z, ψ) is T_0' at t. □

Example 2.1.2. Let $Z = \{1, 2, 3\}$ and (Θ^Z, ψ) be a b-UFIL structure on Z with $\Theta^Z = \{\emptyset, \{1\}, \{2\}, \{3\}\}$ and $\psi = \{[\emptyset], [1] \times [1], [2] \times [2], [3] \times [3], [1] \times [2], [3] \times [2]\}$. Then, (Z, Θ^Z, ψ) is $\overline{T_0}$ at 2.

Example 2.1.3 (local T_0' but not local $\overline{T_0}$)**.** Let $Z = \{1, 2, 3\}$ and (Θ^Z, ψ) be a b-UFIL structure on Z with $\Theta^Z = \{\emptyset, \{1\}, \{2\}, \{3\}\}$ and $\psi = \{[\emptyset], [1] \times [1], [2] \times [2], [3] \times [3], [1] \times [2], [2] \times [1]\}$. Then, (Z, Θ^Z, ψ) is T_0' at 2 but not $\overline{T_0}$ at 2 as condition (ii) of Theorem 2.1 fails to hold.

2.2 Local T_1 Bounded Uniform Filter Spaces

In this section, we recall the definition of local T_1 b-UFIL spaces (at some fixed point t) using S_t and ∇_t maps (Definition 2.1).

Assume that $\mathfrak{F} : \mathscr{G} \longrightarrow$ **Set** is a topological functor, $Z \in Obj(\mathscr{G})$ with $\mathfrak{F}Z = E$ and $t \in E$.

Definition 2.3 (cf. [50])**.** *Z is T_1 at t if the $\mathfrak{F}$-source $\{E \bigvee_t E \xrightarrow{S_t} \mathfrak{F}(Z^2) = E^2$ and $E \bigvee_t E \xrightarrow{\nabla_t} \mathfrak{F}DE = E\}$ has the discrete initial lift.*

Remark 2.2.1. *(i) In **Top**, T_1 at t is equivalent to the classical T_1 at t, i.e., for every $z \in Z$ distinct from t, there is a neighborhood N_z of z with $t \notin N_z$ and there is a neighborhood N_t*

of t not containing $z \notin N_t$ [61].

(ii) *Consider a topological functor $\mathscr{F} : \mathscr{G} \to$ **Set** with $Z \in Obj(\mathscr{G})$. For a retract $t \in \mathscr{F}(Z)$ of Z. If Z is T_1 at t, then Z is $T_0^{'}$ at t but not conversely in general [54].*

Theorem 2.3 (cf. [103]). *Consider a b-UFIL space (Z, Θ^Z, ψ) with $t \in Z$. Then (Z, Θ^Z, ψ) is T_1 at t iff for all $z_a \in Z$ with $z_a \neq t$, the followings hold.*

(i) $\{z_a, t\} \notin \Theta^Z$;

(ii) $[z_a] \times [t] \notin \psi$ *and* $[t] \times [z_a] \notin \psi$;

(iii) $([z_a] \times [z_a]) \cap ([t] \times [t]) \notin \psi$.

Proof. Let (Z, Θ^Z, ψ) be T_1 at t. We shall prove that the above conditions $(i)-(iii)$ hold. Let $\{z_a, t\} \in \Theta^Z$ for $z_a \neq t$ and $W = \{z_{a1}, z_{a2}\} \in \Theta^{Z \vee_t Z}$. Since $\nabla_t W = \{z_a\} \in \mathscr{D}^Z$, $\pi_1 S_t W = \{z_a, t\} \in \Theta^Z$ and $\pi_2 S_t W = \{z_a\} \in \Theta^Z$, where $\pi_k : Z^2 \to Z$ for $k = 1,2$ are the projection maps. By Definitions 2.3, 1.40, and 1.43, a contradiction. Hence, $\{z_a, t\} \notin \Theta^Z$.

Next, suppose that $[z_a] \times [t] \in \psi$ and $[t] \times [z_a] \in \psi$, for some $z_a \neq t$. Let $\sigma = [z_{a1}] \times [z_{a2}]$. Clearly, $(\nabla_t \times \nabla_t)\sigma = [z_a] \times [z_a] \in \psi_{dis}$, $(\pi_1 S_t \times \pi_1 S_t)\sigma = [z_a] \times [t] \in \psi$ and $(\pi_2 S_t \times \pi_2 S_t)\sigma = [z_a] \times [z_a] \in \psi$, a contradiction. Similarly, for $\sigma = [z_{a2}] \times [z_{a1}]$, $(\nabla_t \times \nabla_t)\sigma = [z_a] \times [z_a] \in \psi_{dis}$, $(\pi_1 S_t \times \pi_1 S_t)\sigma = [t] \times [z_a] \in \psi$ and $(\pi_2 S_t \times \pi_2 S_t)\sigma = [z_a] \times [z_a] \in \psi$, again a contradiction. It follows that $[z_a] \times [t] \notin \psi$ and $[t] \times [z_a] \notin \psi$.

Further, if $([z_a] \times [z_a]) \cap ([t] \times [t]) \in \psi$ for some $z_a \neq t$. Let $\sigma = ([z_{a1}] \times [z_{a1}]) \cap ([z_{a2}] \times [z_{a2}])$. Since $(\nabla_t \times \nabla_t)\sigma = [z_a] \times [z_a] \in \psi_{dis}$, $(\pi_1 S_t \times \pi_1 S_t)\sigma = ([z_a] \times [z_a]) \cap ([t] \times [t]) \in \psi$ and $(\pi_2 S_t \times \pi_2 S_t)\sigma = ([z_a] \times [z_a]) \in \psi$, a contradiction since (Z, Θ^Z, ψ) is T_1 at t. Thus, $([z_a] \times [z_a]) \cap ([t] \times [t]) \notin \psi$.

Conversely, let us assume that the conditions $(i)-(iii)$ hold. Let $(\Theta^{Z \vee_t Z}, \overline{\psi})$ be the initial structure induced by $S_t : Z \vee_t Z \to (Z^2, \Theta^{Z^2}, \psi^2)$ and $\nabla_t : Z \vee_t Z \to (X, \mathscr{D}^Z, \psi_{dis})$, where (Θ^{Z^2}, ψ^2) and $(\mathscr{D}^Z, \psi_{dis})$ are product b-UFIL structure on Z^2 and discrete b-UFIL structure on Z, respectively. We show that $(\Theta^{Z \vee_t Z}, \overline{\psi})$ is a discrete b-UFIL structure on $Z \vee_t Z$. Let $W \in \Theta^{Z \vee_t Z}$ and $\nabla_t W \in \mathscr{D}^Z$.

If $\nabla_t W = \emptyset$, then $W = \emptyset$.

Suppose $\nabla_t W \neq \emptyset$, it follows that $\nabla_t W = \{z_a\}$ for some $z_a \in Z$. If $z_a = t$, then $W = \{t\}$. Suppose $z_a \neq t$. It follows that $W = \{z_{a1}\}, \{z_{a2}\}$ or $\{z_{a1}, z_{a2}\}$. The case, $W = \{z_{a1}, z_{a2}\}$ cannot happen since $\pi_1 S_t W = \{z_a, t\} \notin \Theta^Z$ (for $k = 1$) by assumption. Hence, $W = \{z_{a1}\}, \{z_{a2}\}$ and consequently, $\Theta^{Z \vee_t Z} = \mathscr{D}^{Z \vee_t Z}$, the discrete b-UFIL structure on $Z \vee_t Z$.

Next, let $\sigma \in \overline{\psi}$. By Definition 1.43 (i), $(\nabla_t \times \nabla_t)\sigma \in \mathscr{D}^Z$ and $(\pi_k S_t \times \pi_k S_t)\sigma \in \psi$ for $k = 1,2$. We need to show that $\sigma = [z_{ai}] \times [z_{ai}]$ $(i = 1,2)$, $\sigma = [t] \times [t]$ or $\sigma = [\emptyset] = P((Z \vee_t Z) \times (Z \vee_t Z))$. If $(\nabla_t \times \nabla_t)\sigma = [\emptyset]$, then $\sigma = [\emptyset] = P((Z \vee_t Z) \times (Z \vee_t Z))$.
Suppose $(\nabla_t \times \nabla_t)\sigma = [z_a] \times [z_a]$ for some $z_a \in Z$. If $z_a = t$, then $\sigma = [t] \times [t]$.
If $z_a \neq t$, then $(\nabla_t \times \nabla_t)\sigma = [z_a] \times [z_a]$, then $\{z_{a1}, z_{a2}\} \times \{z_{a1}, z_{a2}\} \in \sigma$. Thus there is a finite subset N_0 of σ such that $\sigma = [N_0]$. Clearly, $N_0 \subseteq \{z_{a1}, z_{a2}\} \times \{z_{a1}, z_{a2}\}$ and if $i \neq j$, then $\{\{z_{ai}\} \times \{z_{aj}\}\} \neq N_0$ and $\{\{z_{a1}\} \times \{z_{a1}\}, \{z_{a2}\} \times \{z_{a2}\}\} \neq N_0$ since in particular for $k = 1$, $i = 1$, and $j = 2$, $(\pi_1 S_t \times \pi_1 S_t)([z_{a1}] \times [z_{a2}]) = [z_a] \times [t] \notin \psi$ and $(\pi_1 S_t \times \pi_1 S_t)(([z_{a1}] \times [z_{a1}]) \cap ([z_{a2}] \times [z_{a2}])) = ([z_a] \times [z_a]) \cap ([t] \times [t]) \notin \psi$, using the second and the third conditions respectively.

Therefore, we must have $\sigma = [z_{ai}] \times [z_{ai}]$ $(i = 1,2)$ and consequently, by Definitions 2.3, 1.40, and 1.43, (Z, Θ^Z, ψ) is T_1 at t. $\square$

Example 2.2.2. Let $A = \{1,2,3\}$ and (Θ^A, ψ) be a b-UFIL structure on A with $\Theta^A = \{\emptyset, \{1\}, \{2\}, \{3\}\}$ and $\psi = \{[\emptyset], [1] \times [1], [2] \times [2], [3] \times [3], [1] \times [2], [2] \times [1]\}$. Then, (A, Θ^A, ψ) is T_1 at 3 but not $\overline{T_1}$ at $t = 1$ or $t = 2$ as condition (ii) of Theorem 2.3 fails to hold.

2.3 Relationship between Local T_0 and Local T_1 Bounded Uniform Filter Spaces

Remark 2.3.1. *Let (Z, Θ^Z, ψ) be a symmetric b-UFIL space and $t \in Z$, then every T_1 at t is a $\overline{T_0}$ and t but the converse is not valid generally.*

Example 2.3.2. Let $A = \{1,2,3\}$ and (Θ^A, ψ) be a b-UFIL structure on A with $\Theta^A = \{\emptyset, \{1\}, \{2\}, \{3\}\}$ and $\psi = \{[\emptyset], [1] \times [1], [2] \times [2], [3] \times [3], [1] \times [2], [1] \times [3]\}$. Then clearly, (A, Θ^A, ψ) is $\overline{T_0}$ at 1 but not T_1 at 1.

Corollary 2.3.3 (cf. [103]). *Let (Z, Θ^Z, ψ) be a symmetric b-UFIL space and $t \in Z$, then the following statements are equivalent:*

(i) Z is $\overline{T_0}$ at t.

(ii) Z is T_1 at t.

(iii) $\{z, t\} \notin \Theta^Z$; $[z] \times [t] \notin \psi$; and $([z] \times [z]) \cap ([t] \times [t]) \notin \psi$.

Proof. The proof of this can be easily deduced using Theorem 2.1, Theorem 2.3, and Definition 1.40. □

Corollary 2.3.4 (cf. [103]). *Let (Z, Θ^Z, ψ) be a symmetric b-UFIL limit space and $t \in Z$ be any point, then the following statements are equivalent:*

(i) *Z is $\overline{T_0}$ at t.*

(ii) *Z is T_1 at t.*

(iii) $\{z, t\} \notin \Theta^Z$*; and* $[z] \times [t] \notin \psi$.

Proof. The proof of this can be easily deduced using Theorem 2.1, Theorem, 2.3, and Definition 1.40. □

CHAPTER 3

QUOTIENT REFLECTIVE SUBCATEGORIES OF BOUNDED UNIFORM FILTER SPACES

To generalize the classical T_0 objects, various approaches have been discussed by Topologists since 1971 such as Brümmer, Marny, Hoffman, Harvey and Baran in [13, 16, 20, 27, 50, 57, 51]. In addition, the relationships between several forms of generalized T_0 objects in the topological category have been examined in [57, 51]. In arbitrary topological categories, the concept of Hausdorff objects is defined by generalizing T_0 objects [50]. But also T_0 is used to define non-T_2 spaces such as sober spaces [101] that are used in theoretical computer science. Further, a generalization of the classical T_1 objects of topology in topological categories has been investigated by Baran [50] in 1991. One of its important uses is to define each of T_3, T_4, normal, completely regular, and regular objects in an abstract topological category [66]. Baran's approach was to characterize these separation axioms in terms of their initial and final lifts, and discreteness.

Since points do not make sense in topos theory, Baran [50] used the generic element method defined by Johnstone [28] to characterize separation axioms that are applicable in topos theory as well, where we replace the wedge product $Z\bigvee_t Z$ at t by $Z^2\bigvee_\Delta Z^2$ at diagonal Δ. Note that $Z^2\bigvee_\Delta Z^2$ is the *wedge product of* Z^2 *diagonally intersected with* Z^2. Any element $(z_a,z_b)\in Z^2\bigvee_\Delta Z^2$ is written as $(z_a,z_b)_1$ (or $(z_a,z_b)_2$ resp.) if it lies in the first (or second resp.) component of $Z^2\bigvee_\Delta Z^2$. Clearly, $(z_a,z_b)_1=(z_a,z_b)_2$ if and only if $z_a=z_b$.

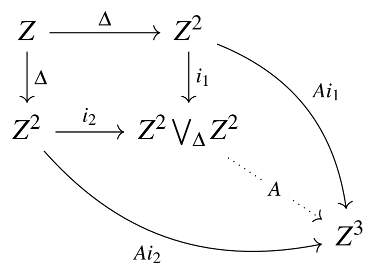
Categorically the wedge product $Z^2\bigvee_\Delta Z^2$ is the push out of the coproduct of Z^2 with itself, i.e., for a diagonal map $\Delta:\mathbb{1}\longrightarrow Z^2$, and inclusion maps $i_1,\ i_2:Z^2\longrightarrow Z^2\bigvee_\Delta Z^2$ of Z^2 in the first and second factors respectively, the following representation is a push out denoted by $Z^2 \bigvee_{\Delta} Z^2$.

$$\begin{array}{ccc} Z & \xrightarrow{\Delta} & Z^2 \\ \downarrow{\scriptstyle \Delta} & & \downarrow{\scriptstyle i_1} \\ Z^2 & \xrightarrow{i_2} & Z^2 \bigvee_{\Delta} Z^2 \end{array}$$

i.e., $i_1\Delta = i_2\Delta$ [50]. To put it another way, for

$$\begin{array}{ccc} Z & \xrightarrow{\Delta} & Z^2 \\ \downarrow{\scriptstyle \Delta} & & \downarrow{\scriptstyle Ai_1} \\ Z^2 & \xrightarrow{Ai_2} & Z^3 \end{array}$$

the following is a push-out diagram



i.e. $Ai_1 \circ \Delta = Ai_2 \circ \Delta = (\Delta, \Delta)$ where $A : Z^2 \bigvee_{\Delta} Z^2 \longrightarrow Z^3$.

3.1 T_0 Bounded Uniform Filter Spaces

We begin this section by recalling the definition of T_0 spaces in the categorical topology, and then we characterize them to T_0 b-UFIL spaces.

Definition 3.1. *(cf. [50])*

(i) A principal axis map $A : Z^2 \bigvee_{\Delta} Z^2 \longrightarrow Z^3$ is defined as follows

$$A((z_a, z_b)_l) := \begin{cases} (z_a, z_b, z_a), & l = 1, \\ (z_a, z_a, z_b), & l = 2. \end{cases}$$

(ii) A skewed axis map $S : Z^2 \bigvee_{\Delta} Z^2 \longrightarrow Z^3$ is defined as folows

$$S((z_a, z_b)_l) := \begin{cases} (z_a, z_b, z_b), & l = 1, \\ (z_a, z_a, z_b), & l = 2. \end{cases}$$

(iii) A fold map $\nabla : Z^2 \bigvee_\Delta Z^2 \longrightarrow Z^3$ *is defined as follows*

$$\nabla((z_a, z_b)_l) := (z_a, z_b), \qquad l = 1, 2,$$

Definition 3.2. *(cf. [61]) Consider a topological space* (Z, τ) *with* $z_a, z_b \in Z$*. For all* $z \neq x$*, there exists an open set of* z_b *missing* z_a *(or an open set of* z_a *missing* z_b*), then* (Z, τ) *is said to be a* T_0 *topological space.*

Theorem 3.1. *Consider a topological space* (Z, τ)*. Then following statements are equivalent.*

(a) (Z, τ) *is* T_0*.*

(b) Each initial morphism is a monomorphism whose domain is (Z, τ) *[13].*

(c) Each morphism is constant upto two elements indiscrete space (Z, τ) *[16].*

(d) (Z, τ) *contains no indiscrete subspace with (at least) two elements [16].*

(e) Each initial source is a monosource whose domain is (Z, τ) *[20].*

(f) Each initial epimorphism is a homeomorphism whose domain is (Z, τ) *[27].*

(g) The topology τ_* *induced by* $A : Z^2 \bigvee_\Delta Z^2 \longrightarrow (Z^3, \tau_*)$ *and* $\Delta : Z^2 \bigvee_\Delta Z^2 \longrightarrow (Z^2, P(Z^2))$ *is discrete [61].*

(h) Suppose the final topology induced by $i_1, i_2 : Z^2 \longrightarrow Z^2 \bigvee_\Delta Z^2$ *is* τ^**. Then the topology induced by* $id : Z^2 \bigvee_\Delta Z^2 \longrightarrow (Z^2 \bigvee_\Delta Z^2, \tau^*)$ *and* $\Delta : Z^2 \bigvee_\Delta Z^2 \longrightarrow (Z^2, P(Z^2))$ *is discrete [61].*

Using (b), the idea of T_0 objects in category theory was defined by Brümmer [13] in 1971. Using (c) and (d), (e) and (f) respectively, Marny [16], Hoffmann [20] and Harvey [27] introduced T_0 objects in topological categories in 1973, 1974 and 1977 respectively. Weck-Schwarz [51] investigated the relation among (b), (c), (d), (e), and (f) in 1991 and named the output as a separated object if it fulfills (b), (e) and (f), or a T_0 object if it satisfies (b) and (c). Also, he proved that all T_0 objects are separated objects but the converse is not generally true. Baran [50] further generalized (g) and (h) T_0 objects to a topological category in 1991 and named them as $\overline{T_0}$ objects and T_0' objects. Moreover, he [57] analyzed the relationship among various T_0 objects from (a) to (h) in 1995 and proved that every $\overline{T_0}$ is T_0' but the converse is not valid generally, and there is no relation among $\overline{T_0}$, T_0 and separatedness, and no relationship among T_0', T_0 and separatedness.

Using (g) and (h) T_0 objects, Baran [62] introduced different kinds of T_2 objects in a topological category in 1996.

Definition 3.3. *Consider a topological functor $\mathfrak{F} : \mathscr{G} \longrightarrow$ **Set** with $Z \in Obj(\mathscr{G})$ and $\mathfrak{F}Z = E$.*

(i) Z is $\overline{T_0}$ if the $\mathfrak{F}$-source $\{E^2 \bigvee_\Delta E^2 \xrightarrow{A} \mathfrak{F}(Z^3) = E^3$ and $E^2 \bigvee_\Delta E^2 \xrightarrow{\nabla} \mathfrak{F}D(E^2) = E^2\}$ has the discrete initial lift [50].

(ii) Z is T_0' if the $\mathfrak{F}$-source $\{E^2 \bigvee_\Delta E^2 \xrightarrow{id} \mathfrak{F}(E^2 \bigvee_\Delta E^2)' = E^2 \bigvee_\Delta E^2$ and $E^2 \bigvee_\Delta E^2 \xrightarrow{\nabla} \mathfrak{F}D(E^2) = E^2\}$ has the discrete initial lift, where $(E^2 \bigvee_\Delta E^2)'$ represents the final lift of the $\mathfrak{F}$-sink $\{\mathfrak{F}(Z^2) = E^2 \xrightarrow{i_1,\, i_2} E^2 \bigvee_\Delta E^2\}$ [50, 57].

(iii) Z is T_0 property if there is no indiscrete subspace of Z with at least two elements [16, 51].

Remark 3.1.1. *Let $\mathfrak{F} : \mathscr{G} \longrightarrow \mathscr{E}$ be a topological functor, where $\mathscr{E} =$ (elementary) **topos** defined in [28], then the above definitions 3.3 also make sense in the topos theory as well [50].*

Remark 3.1.2. *(i) In **Top**, all T_0, $\overline{T_0}$ and T_0' are equivalent to the usual T_0, i.e., for each distinct $z_a, z_b \in Z$, there is a neighborhood N_{z_a} of z_a with $z_b \notin N_{z_a}$ or there is a neighborhood N_{z_b} of z_b with $z_a \notin N_{z_b}$ [61].*

(ii) A topological space Z is T_0 iff for all $t \in Z$, Z is T_0 at t [61].

*(iii) In any topological category, $\overline{T_0} \implies T_0'$ but the converse implication is not valid generally. Moreover, all of $\overline{T_0}$ and T_0' objects do not have any relation with T_0 object[57]. For Example, $\overline{T_0}$ could be all objects such as in **Born** [57], and $\overline{T_0} \implies T_0 \implies T_0'$ such as in **SUConv** [83] and $\overline{T_0} = T_0 \implies T_0'$ such as in **Lim** [57].*

*(iv) Consider a normalized topological functor $\mathscr{F} : \mathscr{G} \to$ **Set**, then $T_0 \implies$ local T_0 [54].*

(v) In general, there is no relation between T_0, $\overline{T_0}$ and T_0' [57].

Theorem 3.2 (cf. [103])**.** *Let (Z, Θ^Z, ψ) be a b-UFIL space. (Z, Θ^Z, ψ) is $\overline{T_0}$ iff for every distinct $z_a, z_b \in Z$, the following hold:*

(i) $\{z_a, z_b\} \notin \Theta^Z$;

(ii) $[z_a] \times [z_b] \notin \psi$ or $[z_b] \times [z_a] \notin \psi$;

(iii) $([z_a] \times [z_a]) \cap ([z_b] \times [z_b]) \notin \psi$.

Proof. Let (Z,Θ^Z,ψ) be $\overline{T_0}$. We shall prove that the above conditions $(i)-(iii)$ hold. Let $\{z_a,z_b\}\in\Theta^Z$ for $z_a\neq z_b$ and $W=\{(z_a,z_b)_1,(z_a,z_b)_2\}\in\Theta^{Z^2\vee_\Delta Z^2}$. Since $\nabla W=\{(z_a,z_b)\}\in\mathscr{D}^{Z^2}$, $\pi_1 AW=\{z_a\}\in\Theta^Z$, and $\pi_k AW=\{z_a,z_b\}\in\Theta^Z$ for $k=2,3$, where $\pi_k:Z^3\to Z^2$ for $k=1,2,3$ are the projection maps. Since (Z,Θ^Z,ψ) is $\overline{T_0}$, by the Definitions 1.43 and 3.3, we get a contradiction. Hence, $\{z_a,z_b\}\notin\Theta^Z$.

Next, suppose that $[z_a]\times[z_b]\in\psi$ for some $z_a\neq z_b$. Let $\sigma=[(z_a,z_b)_1]\times[(z_a,z_b)_2]$. Clearly, $(\nabla\times\nabla)\sigma=[(z_a,z_b)]\times[(z_a,z_b)]\in\psi^2_{dis}$, $(\pi_1A\times\pi_1A)\sigma=[z_a]\times[z_a]\in\psi$, $(\pi_2A\times\pi_2A)\sigma=[z_b]\times[z_a]\in\psi$, and $(\pi_3A\times\pi_3A)\sigma=[z_a]\times[z_b]\in\psi$, a contradiction. It follows that $[z_a]\times[z_b]\notin\psi$ or $[z_b]\times[z_a]\notin\psi$.

Further, if $([z_a]\times[z_a])\cap([z_b]\times[z_b])\in\psi$ for some $z_a\neq z_b$. Let $\sigma=([(z_a,z_b)_1]\times[(z_a,z_b)_1])\cap([(z_a,z_b)_2]\times[(z_a,z_b)_2])$. Since $(\nabla\times\nabla)\sigma=[(z_a,z_b)]\times[(z_a,z_b)]\in\psi^2_{dis}$, $(\pi_1A\times\pi_1A)\sigma=[z_a]\times[z_a]\in\psi$, $(\pi_2A\times\pi_2A)\sigma=([z_b]\times[z_b])\cap([z_a]\times[z_a])\in\psi$, and $(\pi_3A\times\pi_3A)\sigma=([z_a]\times[z_a])\cap([z_b]\times[z_b])\in\psi$, a contradiction since (Z,Θ^Z,ψ) is $\overline{T_0}$. Thus, $([z_a]\times[z_a])\cap([z_b]\times[z_b])\notin\psi$.

Conversely, suppose that the conditions $(i)-(iii)$ hold. Let $(\Theta^{Z^2\vee_\Delta Z^2},\overline{\psi})$ be the initial structure induced by $A:Z^2\vee_\Delta Z^2\to(Z^3,\Theta^{Z^3},\psi^3)$ and $\nabla:Z^2\vee_\Delta Z^2\to(Z^2,\mathscr{D}^{Z^2},\psi^2_{dis})$, where (Θ^{Z^3},ψ^3) represents the product b-UFIL structure on Z^3 and $(\mathscr{D}^{Z^2},\psi^2_{dis})$ the discrete b-UFIL structure on Z^2, respectively. We show that $(\Theta^{Z^2\vee_\Delta Z^2},\overline{\psi})$ is a discrete b-UFIL structure on $Z^2\vee_\Delta Z^2$. Let $W\in\Theta^{Z^2\vee_\Delta Z^2}$ and $\nabla W\in\mathscr{D}^{Z^2}$.

If $\nabla W=\emptyset$, then $W=\emptyset$. Suppose $\nabla W\neq\emptyset$, it follows that $\nabla W=\{(z_a,z_b)\}$ for some $(z_a,z_b)\in Z^2$. Suppose $z_a\neq z_b$. It follows that $W=\{(z_a,z_b)_1\}$ or $\{(z_a,z_b)_2\}$ or $\{(z_a,z_b)_1,(z_a,z_b)_2\}$. The case, $W=\{(z_a,z_b)_1,(z_a,z_b)_2\}$ cannot happen since $\pi_1AW=\{z_a\}\in\Theta^Z$ but $\pi_kAW=\{z_a,z_b\}\notin\Theta^Z$ $(k=2,3)$ by the assumption. Hence, $W=\{(z_a,z_b)_1\}$ or $\{(z_a,z_b)_2\}$ and consequently, $\Theta^{Z^2\vee_\Delta Z^2}=\mathscr{D}^{Z^2\vee_\Delta Z^2}$, the discrete b-UFIL structure on $Z^2\vee_\Delta Z^2$.

Next, let $\sigma\in\overline{\psi}$. By Definition 1.43(i), $(\nabla\times\nabla)\sigma\in\mathscr{D}^{Z^2}$ and $(\pi_kA\times\pi_kA)\sigma\in\psi$ for $k=1,2,3$. We need to prove that $\sigma=[(z_a,z_b)_i]\times[(z_a,z_b)_i]$ $(i=1,2,3)$, or $\sigma=[\emptyset]=P((Z^2\vee_\Delta Z^2)\times(Z^2\vee_\Delta Z^2))$.

If $(\nabla\times\nabla)\sigma=[\emptyset]$, then $\sigma=[\emptyset]=P((Z^2\vee_\Delta Z^2)\times(Z^2\vee_\Delta Z^2))$. Suppose $(\nabla\times\nabla)\sigma=[(z_a,z_b)]\times[(z_a,z_b)]$ for some $(z_a,z_b)\in Z^2$. If $z_a\neq z_b$, then $(\nabla\times\nabla)\sigma=[(z_a,z_b)]\times[(z_a,z_b)]$, hence $\{(z_a,z_b)_1,(z_a,z_b)_2\}\times\{(z_a,z_b)_1,(z_a,z_b)_2\}\in\sigma$. Thus there is a finite subset N_0 of σ such that $\sigma=[N_0]$. Clearly, $N_0\subseteq\{(z_a,z_b)_1,(z_a,z_b)_2\}\times\{(z_a,z_b)_1,(z_a,z_b)_2\}$ and if $i\neq j$, then it can be easily seen that $N_0\neq\{\{(z_a,z_b)_i\}\times\{(z_a,z_b)_j\}\}$ by the second condition and that by the third condition $N_0\neq\{\{(z_a,z_b)_1\}\times\{(z_a,z_b)_1\},\{(z_a,z_b)_2\}\times\{(z_a,z_b)_2\}\}$.

Therefore, we must have $\sigma = [(z_a,z_b)_i] \times [(z_a,z_b)_i]$ $(i = 1,2)$ and consequently, by Definitions 3.3, 1.40, and 1.43, (Z,Θ^Z,ψ) is $\overline{T_0}$. □

Theorem 3.3 (cf. [103]). *Let (Z,Θ^Z,ψ) be a b-UFIL space. (Z,Θ^Z,ψ) is T_0 iff for every distinct $z_a,z_b \in Z$, the following hold:*

(i) $\{z_a,z_b\} \notin \Theta^Z$;

(ii) $([z_a]\times[z_a])\cap([z_b]\times[z_b]) \notin \psi$.

Proof. Let (Z,Θ^Z,ψ) be T_0, and $\{z_a,z_b\} \in \Theta^Z$, and $[\{z_a,z_b\}]\times[\{z_a,z_b\}] \in \psi$ for some $z_a,z_b \in Z$ with $z_a \neq z_b$. Suppose that $W = \{z_a,z_b\}$. Note that (W,Θ^W,ψ_W) is the subspace of (Z,Θ^Z,ψ), where (Θ^W,ψ_W) is the initial b-UFIL structure on W induced by the inclusion map $i : W \to Z$. By Definition 1.43(i), for any $B \subset W$, $B \in \Theta^W$ precisely when $i(B) = B \in \Theta^Z$, and for $\alpha \in \mathscr{F}(W\times W)$, $\alpha \in \psi_W$ precisely when $(i\times i)\alpha = \alpha \in \psi$. Specifically, for $B = W = \{z_a,z_b\}$, $i(W) = W \in \Theta^Z$ and for $\alpha = [W]\times[W] = [\{z_a,z_b\}]\times[\{z_a,z_b\}] = ([z_a]\times[z_a])\cap([z_b]\times[z_b])$, $(i\times i)\alpha = [W]\times[W] \in \psi$, by the assumption. It follows that $(\Theta^W,\psi_W) = (P(W),\mathscr{F}(W\times W))$, the indiscrete b-UFIL structure on W, a contradiction. Therefore, $\{z_a,z_b\} \notin \Theta^Z$ and $([z_a]\times[z_a])\cap([z_b]\times[z_b]) \notin \psi$.

Conversely, suppose that $\{z_a,z_b\} \notin \Theta^Z$ and $([z_a]\times[z_a])\cap([z_b]\times[z_b]) \notin \psi$ for all $z_a,z_b \in Z$ with $z_a \neq z_b$. We show that the initial structure (Θ^W,ψ_W) is not an indiscrete b-UFIL structure on W. Assume that $W = \{z_a,z_b\} \subset Z$. By the assumption and using the Definition 1.43(i), $\{z_a,z_b\} \notin \Theta^W$ and $([z_a]\times[z_a])\cap([z_b]\times[z_b]) \notin \psi_W$. Thus, (W,Θ^W,ψ_W) is not an indiscrete bounded uniform subspace of (Z,Θ^Z,ψ) and therefore by the Definition 3.3(iii), (Z,Θ^Z,ψ) is T_0. □

Example 3.1.3. Let $Z = \{1,2,3\}$ and (Θ^Z,ψ) be a b-UFIL structure on Z with $\Theta^Z = \{\emptyset,\{1\},\{2\},\{3\}\}$ and $\psi = \{[\emptyset],[1]\times[1],[2]\times[2],[3]\times[3],[1]\times[2],[2]\times[3]\}$. Then, (Z,Θ^Z,ψ) is a T_0 bounded uniform filter space.

Theorem 3.4 (cf. [103]). *All b-UFIL spaces are T_0'.*

Proof. Suppose (Z,Θ^Z,ψ) is a b-UFIL space. By the Definition 3.3(ii), we show that for any $W \in \Theta^{Z^2\vee_\Delta Z^2}$, $W \subset i_j(V)$ $(j = 1$ *or* $2)$ for some $V \in \Theta^{Z^2}$, and $\nabla W \in \mathscr{D}^{Z^2}$, and for any $\sigma \in \mathscr{F}((Z^2\vee_\Delta Z^2)\times(Z^2\vee_\Delta Z^2))$, $\sigma \supset (i_j\times i_j)\alpha$ $(j = 1$ *or* $2)$ for some $\alpha \in \psi^2$ and $(\nabla\times\nabla)\sigma \in \psi^2_{dis}$. Then $W = \emptyset$, $\{t\}$ or $\{z_{ak}\}$ for $k = 1,2$.

If $\nabla_t W = \emptyset$, then $W = \emptyset$. Let $\nabla_t W \neq \emptyset$. It follows that $\nabla_t W = \{(z_a, z_b)\}$ for some $x \in Z$.

Suppose $x \neq y$, it follows that $W = \{(z_a, z_b)_1\}$ or $\{(z_a, z_b)_2\}$ or $\{(z_a, z_b)_1, (z_a, z_b)_2\}$. If $W = \{(z_a, z_b)_1, (z_a, z_b)_2\}$, then $\{(z_a, z_b)_1, (z_a, z_b)_2\} \subset i_1(V)$ for some $V \in \Theta^{Z^2}$ which shows that $(z_a, z_b)_2$ must be in the first component of $Z^2 \vee_\Delta Z^2$, a contradiction. Similarly, $\{(z_a, z_b)_1, (z_a, z_b)_2\} \not\subset i_2(V)$ for some $V \in \Theta^{Z^2}$. Hence, $W \neq \{(z_a, z_b)_1, (z_a, z_b)_2\}$. Thus, we must have $W = \{(z_a, z_b)_k\}$ for $k = 1, 2$ only and consequently, $\Theta^{Z^2 \vee_\Delta Z^2} = \mathscr{D}^{Z^2 \vee_\Delta Z^2}$, the discrete b-UFIL structure on $Z^2 \vee_\Delta Z^2$.

Next, for $\sigma \in \mathscr{F}((Z^2 \vee_\Delta Z^2) \times (Z^2 \vee_\Delta Z^2))$, if $(\nabla \times \nabla)\sigma = [\emptyset]$, then $\sigma = [\emptyset] = P((Z^2 \vee_\Delta Z^2) \times (Z^2 \vee_\Delta Z^2))$.

Now, assume that $(\nabla \times \nabla)\sigma = [(z_a, z_b)] \times [(z_a, z_b)]$ for some $(z_a, z_b) \in Z^2$. Suppose that $z_a \neq z_b$, then $\{(z_a, z_b)_1, (z_a, z_b)_2\} \times \{(z_a, z_b)_1, (z_a, z_b)_2\} \in \sigma$. Thus there exists a finite subset M of σ such that $\sigma = [M]$. Clearly, $M \subseteq \{(z_a, z_b)_1, (z_a, z_b)_2\} \times \{(z_a, z_b)_1, (z_a, z_b)_2\}$ and if $k \neq l$, then $\{\{(z_a, z_b)_k\} \times \{(z_a, z_b)_l\}\} \neq M$ and $\{\{(z_a, z_b)_1\} \times \{(z_a, z_b)_1\}, \{(z_a, z_b)_2\} \times \{(z_a, z_b)_2\}\} \neq M$. Suppose that $M = \{\{(z_a, z_b)_k\} \times \{(z_a, z_b)_l\}\}$, then for $k = 1$, $l = 2$, and $j = 1$ (resp. $j = 2$), $[(z_a, z_b)_1] \times [(z_a, z_b)_2] \supset (i_1 \times i_1)\alpha$ for some $\alpha \in \psi$. It follows that $((z_a, z_b)_1, (z_a, z_b)_2) \in (i_1 \times i_1)(U)$ for all $U \in \alpha$, which implies that $(z_a, z_b)_2$ (resp. $(z_a, z_b)_1$) is in the first (resp. second) component of $Z^2 \vee_\Delta Z^2$, a contradiction. Similarly, if $M = \{\{(z_a, z_b)_1\} \times \{(z_a, z_b)_1\}, \{(z_a, z_b)_2\} \times \{(z_a, z_b)_2\}\}$, then for $j = 1$ (resp. $j = 2$), $(([(z_a, z_b)_1] \times [(z_a, z_b)_1]) \cap ([(z_a, z_b)_2] \times [(z_a, z_b)_2])) \supset (i_1 \times i_1)\alpha$ for some $\alpha \in \psi$. It follows that $\{((z_a, z_b)_1, (z_a, z_b)_1), ((z_a, z_b)_2, (z_a, z_b)_2)\} \in (i_1 \times i_1)(U)$ for all $U \in \alpha$, which implies that $(z_a, z_b)_2$ (resp. $(z_a, z_b)_1$) is in the first (resp. second) component of $Z^2 \vee_\Delta Z^2$, a contradiction.

Thus, we must have $\sigma = [(z_a, z_b)_k] \times [(z_a, z_b)_k]$ $(k = 1, 2)$ and consequently, by the Definitions 3.3, 1.43, and Theorems 3.2, 3.5, (Z, Θ^Z, ψ) is T_0'. □

Remark 3.1.4 (cf. [103]). *Let Z be a b-UFIL space.*

(i) By Theorems 2.1 and 3.2, Z is $\overline{T_0}$ iff Z is $\overline{T_0}$ at t, $\forall t \in Z$.

(ii) By Theorems 2.2 and 3.4, Z is T_0' iff Z is T_0' at t, $\forall t \in Z$.

Example 3.1.5. Let $Z = \{1, 2, 3\}$ and (Θ^Z, ψ) be a b-UFIL structure on Z with $\Theta^Z = \{\emptyset, \{1\}, \{2\}, \{3\}\}$ and $\psi = \{[\emptyset], [1] \times [1], [2] \times [2], [3] \times [3], [1] \times [2], [2] \times [3], [3] \times [1]\}$. Then, (Z, Θ^Z, ψ) is T_0 at $t = 1$, $t = 2$ and $t = 3$ so it is a T_0 bounded uniform filter space.

Example 3.1.6. Let $Z = \{1, 2, 3\}$ and (Θ^Z, ψ) be a b-UFIL structure on Z with $\Theta^Z = \{\emptyset, \{1\}, \{2\}, \{3\}\}$

and $\psi = \{[\emptyset], [1]\times[1], [2]\times[2], [3]\times[3], [1]\times[2]\}$. Then, (Z, Θ^Z, ψ) is T_0' at $t = 1$, $t = 2$ and $t = 3$ so it is a T_0' bounded uniform filter space.

Corollary 3.1.7 (cf. [103]). *Let (Z, Θ^Z, ψ) be a bornological b-UFIL space. Then, (Z, Θ^Z, ψ) is $\overline{T_0}$ iff for every distinct $z_a, z_b \in Z$, the following hold:*

(i) $[z_a]\times[z_b] \notin \psi$ *or* $[z_b]\times[z_a] \notin \psi$*;*

(ii) $([z_a]\times[z_a]) \cap ([z_b]\times[z_b]) \notin \psi$*.*

Proof. By using the similar argument in Theorem 3.2, and by applying Remark 1.3.9(i), we obtain the claim. □

3.2 T_1 Bounded Uniform Filter Spaces

In this section, we first recall the definition of T_1 spaces in the categorical topology, then we characterize them to T_1 b-UFIL spaces.

Definition 3.4. *Consider a topological space (Z, ρ) with $z_a, z_b \in Z$. For every z_a distinct from z_b, there is an open set N_{z_b} of z_b with $z_a \notin N_{z_b}$ and there is an open set N_{z_a} of z_a with $z_b \notin N_{z_a}$, then (Z, ρ) is a T_1 space [61].*

Definition 3.5. *Consider a topological functor $\mathfrak{F} : \mathscr{G} \longrightarrow$ **Set** with $Z \in Obj(\mathscr{G})$ and $\mathfrak{F}Z = E$. Z is T_1 if the $\mathfrak{F}$-source $\{E^2 \bigvee_\Delta E^2 \xrightarrow{S} \mathfrak{F}(Z^3) = E^3$ and $E^2 \bigvee_\Delta E^2 \xrightarrow{\nabla} \mathfrak{F}D(E^2) = E^2\}$ has the discrete initial lift [50].*

Remark 3.2.1. *(i) In* **Top***, T_1 is equivalent to the usual T_1, i.e., for every distinct $z_a, z_b \in Z$, there is a neighborhood N_{z_a} of z_a with $z_b \notin N_{z_a}$ and there is a neighborhood N_{z_b} of z_b with $z_a \notin N_{z_b}$ [61].*

(ii) A topological space Z is T_1 if and only if Z is T_1 at t for all $t \in Z$ [61].

*(iii) Consider a normalized topological functor $\mathscr{F} : \mathscr{G} \to$ **Set**, then $T_1 \implies$ local T_1 [54].*

*(iv) Let $\mathscr{F} : \mathscr{G} \to$ **Set** be a topological functor and $Z \in Obj(\mathscr{G})$. If $X \in$* **PreT₂***$(\mathscr{G})$, then Z is $\overline{T_0}$ iff Z is T_1, where* **PreT₂***$(\mathscr{G})$ is the category of pre-Hausdorff spaces and continuous maps defined by Baran in [76].*

Theorem 3.5 (cf. [103]). *Let (Z, Θ^Z, ψ) be a b-UFIL space. (Z, Θ^Z, ψ) is T_1 iff for every distinct $z_a, z_b \in Z$, the following hold:*

(i) $\{z_a, z_b\} \notin \Theta^Z$;

(ii) $[z_a] \times [z_b] \notin \psi$ *and* $[z_b] \times [z_a] \notin \psi$;

(iii) $([z_a] \times [z_a]) \cap ([z_b] \times [z_b]) \notin \psi$.

Proof. Let (Z, Θ^Z, ψ) be T_1. We shall prove that the above conditions $(i) - (iii)$ hold. Let $\{z_a, z_b\} \in \Theta^Z$ for $z_a \neq z_b$ and $W = \{(z_a, z_b)_1, (z_a, z_b)_2\} \in \Theta^{Z^2 \vee_\Delta Z^2}$. Since $\nabla W = \{(z_a, z_b)\} \in \mathscr{D}^{Z^2}$, $\pi_1 SW = \{z_a\} \in \Theta^Z$, $\pi_2 SW = \{z_a, z_b\} \in \Theta^Z$ and $\pi_3 SW = \{z_b\} \in \Theta^Z$, where $\pi_k : Z^3 \to Z^2$ for $k = 1, 2, 3$ are the projection maps. Then by Definitions 3.5, 1.40, and 1.43, (Z, Θ^Z, ψ) is not T_1, a contradiction. Hence, $\{z_a, z_b\} \notin \Theta^Z$.

Next, suppose that $[z_a] \times [z_b] \in \psi$ and $[z_b] \times [z_a] \in \psi$, for some $z_a \neq z_b$. Let $\sigma = [(z_a, z_b)_1] \times [(z_a, z_b)_2]$. Clearly, $(\nabla \times \nabla)\sigma = [(z_a, z_b)] \times [(z_a, z_b)] \in \psi^2_{dis}$, $(\pi_1 S \times \pi_1 S)\sigma = [z_a] \times [z_a] \in \psi$, $(\pi_2 S \times \pi_2 S)\sigma = [z_b] \times [z_a] \in \psi$, and $(\pi_3 S \times \pi_3 S)\sigma = [z_b] \times [z_b] \in \psi$, a contradiction. Similarly, for $\sigma = [(z_a, z_b)_2] \times [(z_a, z_b)_1]$, $(\nabla \times \nabla)\sigma = [(z_a, z_b)] \times [(z_a, z_b)] \in \psi^2_{dis}$, $(\pi_1 S \times \pi_1 S)\sigma = [z_a] \times [z_a] \in \psi$, $(\pi_2 S \times \pi_2 S)\sigma = [z_a] \times [z_b] \in \psi$, and $(\pi_3 S \times \pi_3 S)\sigma = [z_b] \times [z_b] \in \psi$, again a contradiction. It follows that $[z_a] \times [z_b] \notin \psi$ and $[z_b] \times [z_a] \notin \psi$.

Further, if $([z_a] \times [z_a]) \cap ([z_b] \times [z_b]) \in \psi$ for some $z_a \neq z_b$. Let $\sigma = ([(z_a, z_b)_1] \times [(z_a, z_b)_1]) \cap ([(z_a, z_b)_2] \times [(z_a, z_b)_2])$. Since $(\nabla \times \nabla)\sigma = [(z_a, z_b)] \times [(z_a, z_b)] \in \psi^2_{dis}$, $(\pi_1 S \times \pi_1 S)\sigma = [z_a] \times [z_a] \in \psi$, $(\pi_2 S \times \pi_2 S)\sigma = ([z_b] \times [z_b]) \cap ([z_a] \times [z_a]) \in \psi$, and $(\pi_3 S \times \pi_3 S)\sigma = ([z_b] \times [z_b]) \in \psi$, a contradiction since (Z, Θ^Z, ψ) is T_1. Thus, $([z_a] \times [z_a]) \cap ([z_b] \times [z_b]) \notin \psi$.

Conversely, suppose that the conditions $(i) - (iii)$ hold. Let $(\Theta^{Z^2 \vee_\Delta Z^2}, \overline{\psi})$ be the initial structure induced by $S : Z^2 \vee_\Delta Z^2 \to (Z^3, \Theta^{Z^3}, \psi^3)$ and $\nabla : Z^2 \vee_\Delta Z^2 \to (Z^2, \mathscr{D}^{Z^2}, \psi^2_{dis})$, where (Θ^{Z^3}, ψ^3) and $(\mathscr{D}^{Z^2}, \psi^2_{dis})$ are product b-UFIL structure on Z^3 and discrete b-UFIL structure on Z^2, respectively. We show that $(\Theta^{Z^2 \vee_\Delta Z^2}, \overline{\psi})$ is a discrete b-UFIL structure on $Z^2 \vee_\Delta Z^2$. Let $W \in \Theta^{Z^2 \vee_\Delta Z^2}$ and $\nabla W \in \mathscr{D}^{Z^2}$.

If $\nabla W = \emptyset$, then $W = \emptyset$.

Suppose $\nabla W \neq \emptyset$, it follows that $\nabla W = \{(z_a, z_b)\}$ for some $(z_a, z_b) \in Z^2$. Suppose $z_a \neq z_b$. It follows that $W = \{(z_a, z_b)_1\}$ or $\{(z_a, z_b)_2\}$ or $\{(z_a, z_b)_1, (z_a, z_b)_2\}$. The case, $W = \{(z_a, z_b)_1, (z_a, z_b)_2\}$ cannot happen since $\pi_1 SW = \{z_a\}, \pi_3 SW = \{z_b\} \in \Theta^Z$ but $\pi_2 SW = \{z_a, z_b\} \notin \Theta^Z$ by assumption. Hence, $W = \{(z_a, z_b)_1\}$ or $\{(z_a, z_b)_2\}$ and consequently, $\Theta^{Z^2 \vee_\Delta Z^2} = \mathscr{D}^{Z^2 \vee_\Delta Z^2}$, the discrete b-UFIL structure on $Z^2 \vee_\Delta Z^2$.

Next, let $\sigma \in \overline{\psi}$. By Definition 1.43(i), $(\nabla \times \nabla)\sigma \in \mathscr{D}^{Z^2}$ and $(\pi_k S \times \pi_k S)\sigma \in \psi$ for $k = 1, 2, 3$. We

need to prove that $\sigma = [(z_a, z_b)_i] \times [(z_a, z_b)_i]$ $(i = 1,2,3)$, or $\sigma = [\emptyset] = P((Z^2 \vee_\Delta Z^2) \times (Z^2 \vee_\Delta Z^2))$. If $(\nabla \times \nabla)\sigma = [\emptyset]$, then $\sigma = [\emptyset] = P((Z^2 \vee_\Delta Z^2) \times (Z^2 \vee_\Delta Z^2))$.
Suppose $(\nabla \times \nabla)\sigma = [(z_a, z_b)] \times [(z_a, z_b)]$ for some $(z_a, z_b) \in Z^2$. If $z_a \neq z_b$, then $(\nabla \times \nabla)\sigma = [(z_a, z_b)] \times [(z_a, z_b)]$, then $\{(z_a, z_b)_1, (z_a, z_b)_2\} \times \{(z_a, z_b)_1, (z_a, z_b)_2\} \in \sigma$. Thus there is a finite subset N_0 of σ such that $\sigma = [N_0]$. Clearly, $N_0 \subseteq \{(z_a, z_b)_1, (z_a, z_b)_2\} \times \{(z_a, z_b)_1, (z_a, z_b)_2\}$ and if $i \neq j$, then it can be easily seen that $N_0 \neq \{\{(z_a, z_b)_i\} \times \{(z_a, z_b)_j\}\}$ by the second condition and that by the third condition $N_0 \neq \{\{(z_a, z_b)_1\} \times \{(z_a, z_b)_1\}, \{(z_a, z_b)_2\} \times \{(z_a, z_b)_2\}\}$.

Therefore, we must have $\sigma = [(z_a, z_b)_i] \times [(z_a, z_b)_i]$ $(i = 1,2)$ and consequently, by Definitions 3.5, 1.40, and 1.43, (Z, Θ^Z, ψ) is T_1. □

3.3 Relationship between T_0 and T_1 Bounded Uniform Filter Spaces

Remark 3.3.1. *Let (Z, Θ^Z, ψ) be a symmetric b-UFIL space, then every T_1 implies T_0 but the converse implication is not valid generally.*

Example 3.3.2 (cf. [103])**.** Consider $Z = \{1,2,3\}$. Let (Θ^Z, ψ) be a b-UFIL structure on Z with $\Theta^Z = \{\emptyset, \{1\}, \{2\}, \{3\}\}$ and $\psi = \{[\emptyset], [1] \times [1], [2] \times [2], [3] \times [3], [1] \times [2], [1] \times [3], [2] \times [3]\}$. Then, (Z, Θ^Z, ψ) is $\overline{T_0}$ but not T_1 b-UFIL space.

Remark 3.3.3 (cf. [103])**.** *Let Z be a b-UFIL space.*

(i) By the Theorems 2.3 and 3.5, Z is T_1 iff Z is T_1 at t, $\forall t \in Z$.

(ii) By the Theorems 3.5, 3.2, 3.3 and 3.4, $T_1 \implies \overline{T_0} \implies T_0 \implies T_0'$, but generally the converse does not hold.

Example 3.3.4. Consider $Z = \{1,2,3\}$. Let (Θ^Z, ψ) be a b-UFIL structure on Z with $\Theta^Z = \{\emptyset, \{1\}, \{2\}, \{3\}\}$ and $\psi = \{[\emptyset], [1] \times [1], [2] \times [2], [3] \times [3]\}$. Then, (Z, Θ^Z, ψ) is T_1 at $t = 3$, $t = 1$ and $t = 2$ so it is a T_1 bounded uniform filter space.

Example 3.3.5. Let $Z = \{1,2,3\}$ and (Θ^Z, ψ) be a b-UFIL structure on Z with $\Theta^Z = \{\emptyset, \{1\}, \{2\}, \{3\}\}$ and $\psi = \{[\emptyset], [1] \times [1], [2] \times [2], [3] \times [3], [1] \times [2], [2] \times [3], ([2] \times [2]) \cap ([3] \times [3])\}$. Then, (Z, Θ^Z, ψ) is a T_0' bounded uniform filter space but not T_0.

Example 3.3.6. Let $Z = \{1,2,3\}$ and (Θ^Z, ψ) be a b-UFIL structure on Z with $\Theta^Z = \{\emptyset, \{1\}, \{2\}, \{3\}\}$ and $\psi = \{[\emptyset], [1] \times [1], [2] \times [2], [3] \times [3], [1] \times [2], [2] \times [1]\}$. Then, (Z, Θ^Z, ψ) is a T_0 bounded uniform filter space but not $\overline{T_0}$ as property (ii) of Theorem 3.2 fails to hold.

Example 3.3.7. Let $Z = \{1,2,3\}$ and (Θ^Z, ψ) be a b-UFIL structure on Z with $\Theta^Z = \{\emptyset, \{1\}, \{2\}, \{3\}\}$ and $\psi = \{[\emptyset], [1] \times [1], [2] \times [2], [3] \times [3], [1] \times [2], [2] \times [3]\}$. Then, (Z, Θ^Z, ψ) is a $\overline{T_0}$ bounded uniform filter space but not T_1 as property (ii) of Theorem 3.5 fails to hold.

Corollary 3.3.8 (cf. [103]). *Let (Z, Θ^Z, ψ) be a bornological b-UFIL space. Then, (Z, Θ^Z, ψ) is T_1 iff for every distinct $z_a, z_b \in Z$, the following hold:*

(i) $[z_a] \times [z_b] \notin \psi$ *and* $[z_b] \times [z_a] \notin \psi$;

(ii) $([z_a] \times [z_a]) \cap ([z_b] \times [z_b]) \notin \psi$.

Proof. By using the similar argument in Theorem 3.2, applying Remark 1.3.9(i), and by replacing the map S by the map A, the results are obtained. □

Corollary 3.3.9 (cf. [103]). *Let (Z, Θ^Z, ψ) be a symmetric b-UFIL space, then the following statements are equivalent:*

(i) Z *is* $\overline{T_0}$.

(ii) Z *is* T_1.

(iii) $\forall z_a, z_b \in Z$ *with* $z_a \neq z_b$, $\{z_a, z_b\} \notin \Theta^Z$; $[z_a] \times [z_b] \notin \psi$; *and* $([z_a] \times [z_a]) \cap ([z_b] \times [z_b]) \notin \psi$.

Proof. The proof of this can be easily deduced using Theorems 3.2, 3.5, and Definition 1.40. □

Corollary 3.3.10 (cf. [103]). *Let (Z, Θ^Z, ψ) be a symmetric b-UFIL limit space, then the following statements are equivalent:*

(i) Z *is* $\overline{T_0}$.

(ii) Z *is* T_1.

(iii) $\forall z_a, z_b \in Z$ *with* $z_a \neq z_b$, $\{z_a, z_b\} \notin \Theta^Z$; *and* $[z_a] \times [z_b] \notin \psi$.

Proof. The proof of the corollary can be easily deduced from Theorems 3.2, 3.5, and Definition 1.40. □

Definition 3.6. *(cf. [93]) Consider a ROS-Conv space (Z, Θ^Z, q). A reordered set-convergence pair (Θ^Z, q) is said to be:*

(1) T_0 *set- convergence iff the following condition holds, i.e.*

(T_0) $\forall z_a, z_b \in Z$, $[z_a]\ q\ \{z_b\}$ *and* $[z_b]\ q\ \{z_a\}$ *implies that* $z_a = z_b$;

(2) T_1 *set- convergence iff the following condition holds, i.e.*

(T_1) $\forall z_a, z_b \in Z$ *and* $[z_a]\ q\ \{z_b\}$ *implies that* $z_a = z_b$.

Remark 3.3.11. *(cf. [93]) Let* (Z, Θ^Z, μ) *be a b-UFIL space. The pair* (Θ^Z, μ) *of b-UFIL structure on Z is said to be* T_0 *(respectively* T_1*) iff the corresponding pair* (Θ^Z, q_μ) *is* T_0 *(respectively* T_1*) set- convergence. Note that we refer it as usual.*

Corollary 3.3.12 (cf. [103])**.** *Let* (Z, Θ^Z, ψ) *be a discrete symmetric b-UFIL limit space, then the following statements are equivalent.*

(i) Z *is* $\overline{T_0}$;

(ii) Z *is* T_1;

(iii) Z *is* T_0 *(in the usual sense);*

(iv) Z *is* T_1 *(in the usual sense);*

(v) $\forall z_a, z_b \in Z$ *with* $z_a \neq z_b$, $[z_a] \times [z_b] \notin \psi$.

Proof. It follows from Corollary 3.3.10, Remark 3.3.11, and Definition 1.43(iii). □

Corollary 3.3.13 (cf. [103])**.** *The following categories are isomorphic.*

(i) $\overline{T_0}$***DISb-UFIL****;*

(ii) $\overline{T_0}$***PUConv****;*

(iii) $\overline{T_0}$***BONb-UFIL****.*

Proof. It follows from Theorem 3.2, Corollary 3.1.7, Definition 1.40 and Theorem 3.1.10 of [86]. □

Corollary 3.3.14 (cf. [103])**.** *The following categories are isomorphic.*

(i) $\overline{T_0}$***DISsb-UFIL****;*

(ii) $\overline{T_0}$***SUConv****;*

(iii) T_1***SUConv****;*

(iv) $\overline{T_0}$***BONsb-UFIL****;*

(v) T_1***BONsb-UFIL****.*

Proof. It follows from Corollaries 3.1.7–3.3.9, Theorems 4.4 and 4.6 of [83]. □

3.4 Quotient Reflective Subcategories of Bounded Uniform Filter Spaces

Theorem 3.6 (cf. [103])**.** *Every* $\overline{T_0}$***b-UFIL*** *(resp.* T_0***b-UFIL****,* T_1***b-UFIL****) is a quotient-reflective subcategory of* ***b-UFIL****.*

Proof. Let $\mathscr{G} = \overline{T_0}$**b-UFIL** and $(A, \Theta^A, \mu_A) \in \mathscr{G}$. It is straightforward to check that A is a full subcategory, isomorphism-closed, and closed under finer structures. We are left to show that it is also closed under extremal sub-objects and closed under the formation of products.

Let $Z \subset A$ and (Θ^Z, μ_X) be the sub-b-UFIL structure on Z induced by $i : Z \to A$. We show that (Z, Θ^Z, μ_X) is a $\overline{T_0}$b-UFIL space. Suppose that $\{z_a, z_b\} \in \Theta^Z$ for any $z_a, z_b \in Z$ with $z_a \neq z_b$. Then $i(\{z_a, z_b\}) = \{i(z_a), i(z_b)\} = \{z_a, z_b\} \in \Theta^A$, a contradiction by Theorem 3.2. Thus, $\{z_a, z_b\} \notin \Theta^Z$. Similarly, let $[z_a] \times [z_b] \in \mu_X$ and $([z_a] \times [z_a]) \cap ([z_b] \times [z_b]) \in \mu_X$, then $(i \times i)([z_a] \times [z_b]) = [z_a] \times [z_b] \in \mu_A$ and $(i \times i)(([z_a] \times [z_a]) \cap ([z_b] \times [z_b])) = ([z_a] \times [z_a]) \cap ([z_b] \times [z_b]) \in \mu_A$, again a contradiction. Thus $[z_a] \times [z_b] \notin \mu_X$ and $([z_a] \times [z_a]) \cap ([z_b] \times [z_b]) \notin \mu_X$. Hence, A is closed under extremal subobjects.

Next, suppose that $A = \prod_{j \in I} A_j$, where $(\Theta^{A_j}, \mu_{A_j})$ are the $\overline{T_0}$b-UFIL structures on A_j induced by $\pi_j : A \to A_j$ for all $j \in I$, i.e., $(A_j, \Theta^{A_j}, \mu_{A_j}) \in \mathscr{G}$. We show that (A, Θ^A, μ_A) is a $\overline{T_0}$b-UFIL space. Let $\{z_a, z_b\} \in \Theta^A$ for any $z_a, z_b \in A$ with $z_a \neq z_b$. Then $\pi_j(\{z_a, z_b\}) = \{\pi_j(z_a), \pi_j(z_b)\} = \{z_{aj}, z_{bj}\} \in \Theta^{A_j}$, a contradiction to Theorem 3.2. Thus $\{z_a, z_b\} \notin \Theta^A$. Similarly, suppose that $[z_a] \times [z_b] \in \mu_A$ and $([z_a] \times [z_a]) \cap ([z_b] \times [z_b]) \in \mu_A$, then there exists $j \in I$ for which $z_{aj} \neq z_{bj} \in A_j$, such that $(\pi_j \times \pi_j)([z_a] \times [z_b]) = [z_{aj}] \times [z_{bj}] \in \mu_{A_j}$ and $(\pi_j \times \pi_j)(([z_a] \times [z_a]) \cap ([z_b] \times [z_b])) = ([z_{aj}] \times [z_{aj}]) \cap ([z_{bj}] \times [z_{bj}]) \in \mu_{A_j}$, a contradiction. Thus $[z_a] \times [z_b] \notin \mu_A$ and $([z_a] \times [z_a]) \cap ([z_b] \times [z_b]) \notin \mu_A$. Hence, A is closed under the formation of products.

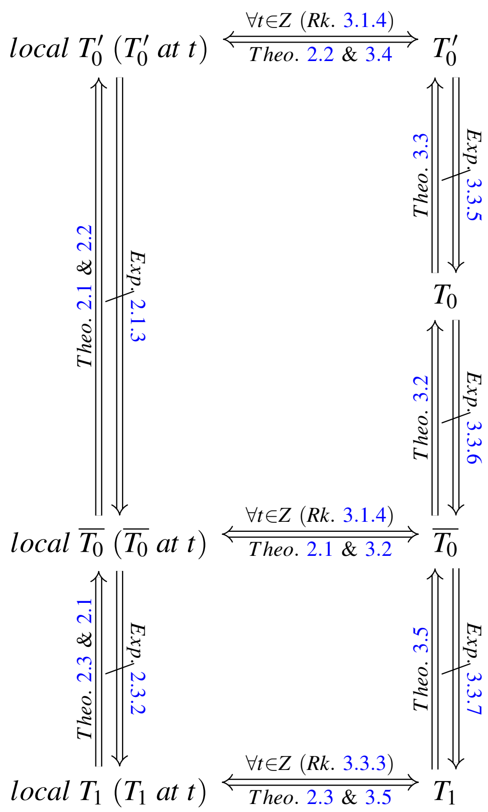
Thus, $\overline{T_0}$**b-UFIL** is a quotient-reflective subcategory of **b-UFIL**.

Similar to the discussion above, the proof can be clearly deduced by putting $\mathscr{G} = T_0$**b-UFIL** or T_1**b-UFIL**, using Theorem 3.3 or Theorem 3.5, respectively. □

Theorem 3.7 (cf. [103])**.** *Every T_0'**b-UFIL** is a cartesian closed and hereditary topological construct.*

Proof. By Theorem 3.4, both **b-UFIL** and T_0'**b-UFIL** are isomorphic categories, and consequently, by Theorems 2.9.4 and 2.9.5 of [92], T_0'**b-UFIL** is a cartesian closed and hereditary topological construct. □

3.4.1 A Pictorial Representation of T_0 and T_1 Separation Axioms in Bounded Uniform Filter Spaces



Chapter 4

NOTION OF CLOSEDNESS AND CLOSURE OPERATORS IN BOUNDED UNIFORM FILTER SPACES

In general topology, all the basic concepts including compactness, connectedness, perfectness, soberness, Hausdorffness and closure operators can be defined in terms of closedness. In order to define these notions of closedness in categorical language, Baran [50] introduced local T_0 and local T_1 of topology in a topological category using initial and final lifts, and discrete objects. Also, he [52] investigated the concept of closedness and closure operators in a topological category in 1993. Moreover, these notions of closedness (strongly closedness) are used to extend several famous theorems of general topology such as Urysohn lemma and Tietze extension theorem.

We first recall the definition of closedness in classical topology.

Definition 4.1 (Closed Set)**.** *(cf. [52]) Consider a topological space (Z,τ) with $M \subset Z$ as a subset of Z. Then M is closed in Z iff $Z \backslash M$ is open in Z.*

Example 4.0.1. Consider a topological space (Z,τ) with $Z = \{1,2,3\}$ and $\tau = \{\emptyset,\{2\},\{1,2\},\{3,2\},Z\}$. Then the following are closed sets in (Z,τ):

$$\emptyset,\ Z,\ \{1\},\ \{3\},\ \textit{and}\ \{1,3\}.$$

Definition 4.2 (Strongly Closed Set)**.** *(cf. [52]) Consider a topological space (Z,τ). A set*

$M \subset Z$ is strongly closed iff M is closed and for all $z \notin M$, there is a $U_M \in \tau$ such that $M \subset U_M$ and $z \notin U_M$.

Example 4.0.2. Consider $Z = \{1,2,3,4\}$ and $\tau = \{\emptyset, \{2\}, \{1,2\}, \{3,4\}, \{2,3,4\}, Z\}$. Then the following are closed sets in (Z, τ):

$$\{1\},\ \{1,2\},\ \{3,4\},\ \textit{and}\ \{1,3,4\}.$$

Clearly, $\{1,2\}$ and $\{3,4\}$ are strongly closed sets.

Remark 4.0.3. *(cf. [52])*

(i) In the case of Analysis (i.e. real numbers equipped with the standard topology), closed and strongly closed sets are equivalent.

(ii) Strongly closedness implies closedness but the converse is not valid generally.

Theorem 4.1. *(cf. [52]) If the topological space Z is T_1 then strongly closedness coincides with closedness.*

4.1 Closed and Strongly Closed Subobjects in the category of Bounded Uniform Filter Spaces

In this section, we define the notion of closedness in b-UFIL spaces by characterizing closed and strongly closed subobjects in the category **b-UFIL**.

Consider a set Z with $t \in Z$. The *wedge product of Z at t* is represented as $Z \bigvee_t Z$ and is defined as the two disjoint copies of Z at t. Any element $z \in Z \bigvee_t Z$ is written as z_1 (or z_2 resp.) if z lies in the first (or second resp.) component of $Z \bigvee_t Z$. Furthermore, the cartesian product of Z with itself is written as Z^2.

Similarly, we define the *infinite wedge product of Z at t* as the infinitely countable disjoint copies of Z identifying at t and denote it as $\bigvee_t^\infty Z$.

Any element $z \in \bigvee_t^\infty Z$ is written as z_j if it lies in the j^{th} component of $\bigvee_t^\infty Z$. In a categorical sense, it represents multiple or countable push out at t.

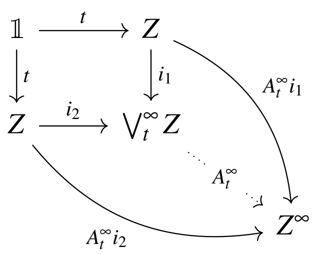
Categorically the infinite wedge product of Z at t is a multiple (or countable) push out, i.e., for a map $t : \mathbb{1} \longrightarrow Z$, and inclusion maps $i_1,\ i_2,\ i_3, \cdots : Z \longrightarrow Z \bigvee_t Z$ of Z in the first, second, third factor, and so on, the following representation is a push out denoted by $\bigvee_t^\infty Z$

$$\begin{array}{ccc} \mathbb{1} & \xrightarrow{\;t\;} & Z \\ \big\downarrow t & & \big\downarrow i_1 \\ Z & \xrightarrow{\;i_2\;} & \bigvee_t^{\infty} Z \end{array}$$

i.e., $i_1 t = i_2 t = i_3 t = \cdots$ [50]. To put it another way, for

$$\begin{array}{ccc} \mathbb{1} & \xrightarrow{\;t\;} & Z \\ \big\downarrow t & & \big\downarrow A_t^{\infty} i_1 \\ Z & \xrightarrow{\;A_t^{\infty} i_2\;} & Z^{\infty} \end{array}$$

the following is push-out diagram



i.e. $A_t^{\infty} i_1 \circ t = A_t^{\infty} i_2 \circ t = A_t^{\infty} i_3 \circ t = \cdots = (t, t, \cdots)$ where $A_t^{\infty} \; : \; \bigvee_t^{\infty} Z \longrightarrow Z^{\infty}$.

Definition 4.3. (cf. [50, 52])

(i) An infinite principal t axis map $A_t^{\infty} \; : \; \bigvee_t^{\infty} Z \longrightarrow Z^{\infty}$ is defined as follow

$$A_t^{\infty}(z_j) := (t, t, \cdots, t, \underbrace{z}_{j^{th}\ place}, t, \cdots), \quad \forall j \in I.$$

(ii) An infinite fold map at t $\nabla_t^{\infty} \; : \; \bigvee_t^{\infty} Z \longrightarrow Z$ is defined as follow

$$\nabla_t^{\infty}(z_j) := z, \quad \forall j \in I.$$

Definition 4.4. (cf. [52]) A map $Q : \mathfrak{L}Z = M \rightarrow M/L$, where $L \subset M$ and $M/L = (M \backslash L) \cup \{\star\}$, is said to be the quotient map or the epi map provided that it identifies L to $\star$ and is identity at $M \backslash L$.

Definition 4.5. (cf. [50, 52]) Let $\mathfrak{F} \; : \; \mathscr{G} \longrightarrow$ **Set** be a topological functor and $z_a \in Obj(\mathscr{G})$ with $\mathfrak{F}(Z) = M$ and $t \in M$.

(i) $\{t\}$ is closed provided that initial lift of $\mathfrak{F}$-source $\{\bigvee_t^{\infty} M \xrightarrow{A_t^{\infty}} \mathfrak{F}Z^{\infty} = M^{\infty} \; and \; \bigvee_t^{\infty} M \xrightarrow{\nabla_t^{\infty}} \mathfrak{F}DM = M\}$ is discrete.

(ii) $L \subset Z$ is closed provided that $\{\star\}$ (image of L) is closed in Z/L or $L = \emptyset$.

Remark 4.1.1. In **Top**, all closed sets reduce to the classical closed sets i.e., a set L is closed iff L is closed in the usual sense.

Definition 4.6. (cf. [50, 52]) Let $\mathfrak{F} : \mathscr{G} \longrightarrow$ **Set** be a topological functor and $z_a \in Obj(\mathscr{G})$ with $\mathfrak{F}(Z) = M$ and $t \in M$.

(i) $L \subset Z$ is strongly closed provided that Z/L is T_1 at $\star$ or $L = \emptyset$.

(ii) $L = M = \emptyset$ provided that L is both closed and strongly closed.

Remark 4.1.2.

1. Specifically for $\mathscr{G} =$ **Top**, a set A is strongly closed provided that A is closed and for $a \notin A$ there exists an open set $\mathscr{N}_A$ containing A such that $a \notin \mathscr{N}_A$ [67].
2. If $\mathscr{G} = T_1$**Top**, then closed sets and strongly closed sets coincide with each other [67].
3. In general, there is no relation between closed and strongly closed sets of an arbitrary topological category [52].
4. The closed and strongly closed sets are still independent even if $E \in Obj(\mathscr{G})$ is T_1 [65].
5. Let $\mathfrak{F} : \mathscr{G} \longrightarrow \mathscr{E}$ be a topological functor, where $\mathscr{E}$ is (elementary) **topos** with infinite products defined in [28], then the above definitions make sense in the topos theory as well [50].

Lemma 4.1.3. ([88]) *Let A be any set, η and γ be filters on $A \times A$, and $h : A \to Z$ be a function. Then:*

(i) $(h \times h)(\eta \cap \gamma) = (h \times h)\eta \cap (h \times h)\gamma$ *and* $(h \times h)\eta \cup (h \times h)\gamma \subset (h \times h)(\eta \cup \gamma)$.

(ii) *If $\eta \subset \gamma$, then $(h \times h)\eta \subset (h \times h)\gamma$, and if γ is a proper filter on $X \times X$, then $\gamma \subset (hh^{-1} \times hh^{-1})\gamma$.*

Theorem 4.2. *Let (Z, Θ^Z, ψ) be a b-UFIL space and $t \in Z$. Then $\{t\}$ is closed in Z if and only if for all $z_a \in Z$ with $z_a \neq t$, the conditions below hold.*

(i) $\{z_a, t\} \notin \Theta^Z$,

(ii) $[z_a] \times [t] \notin \psi$ *or* $[t] \times [z_a] \notin \psi$,

(iii) $([z_a] \times [z_a]) \cap ([t] \times [t]) \notin \psi$.

Proof. Let $\{t\}$ be closed in Z. We show that the above conditions $(i)-(iv)$ hold. Let $\{z_a,t\}\in\Theta^Z$ for $z_a\neq t$ and $W=\{z_{a_1},z_{a_2}\}\in\Theta^{\vee_t^\infty Z}$. Since $\nabla_t^\infty W=\{z_a\}\in\mathscr{D}^Z$, and $\pi_1A_t^\infty W=\pi_2A_t^\infty W=\{z_a,t\}\in\Theta^Z$, $\pi_kA_t^\infty W=\{t\}\in\Theta^Z$ for $k\geq 3$, where $\pi_k:Z^\infty\to Z$ for $k\in I$ are the projection maps. By Definitions 1.40, 1.43, and 4.5(i), a contradiction. Hence, $\{z_a,t\}\notin\Theta^Z$.

Next, suppose that $[z_a]\times[t]\in\psi$ for some $z_a\neq t$. Let $\varsigma=[z_{a_1}]\times[z_{a_2}]$. Clearly, $(\nabla_t^\infty\times\nabla_t^\infty)\varsigma=[z_a]\times[z_a]\in\psi_{dis}$, $(\pi_1A_t^\infty\times\pi_1A_t^\infty)\varsigma=[z_a]\times[t]\in\psi$, $(\pi_2A_t^\infty\times\pi_2A_t^\infty)\varsigma=[t]\times[z_a]\in\psi$, and $(\pi_kA_t^\infty\times\pi_kA_t^\infty)\varsigma=[t]\times[t]\in\psi$ for $k\geq 3$, a contradiction. It follows that either $[z_a]\times[t]\notin\psi$ or $[t]\times[z_a]\notin\psi$.

Further, suppose that $([z_a]\times[z_a])\cap([t]\times[t])\in\psi$ for some $z_a\neq t$. Assume that $\varsigma=([z_{a_1}]\times[z_{a_1}])\cap([z_{a_2}]\times[z_{a_2}])$. Since $(\nabla_t^\infty\times\nabla_t^\infty)\varsigma=[z_a]\times[z_a]\in\psi_{dis}$, $(\pi_1A_t^\infty\times\pi_1A_t^\infty)\varsigma=(\pi_2A_t^\infty\times\pi_2A_t^\infty)\varsigma=([z_a]\times[z_a])\cap([t]\times[t])\in\psi$, and $(\pi_kA_t^\infty\times\pi_kA_t^\infty)\varsigma=[t]\times[t]\in\psi$ for $k\geq 3$, a contradiction to the closedness of $\{t\}$. Thus, $([z_a]\times[z_a])\cap([t]\times[t])\notin\psi$.

Conversely, let us assume that the conditions $(i)-(iv)$ hold. Let $(\Theta^{X\vee_t X},\overline{\psi})$ be the initial structure induced by $\nabla_t^\infty:\vee_t^\infty Z\to(z_a,\mathscr{D}^Z,\psi_{dis})$ and $A_t^\infty:\vee_t^\infty Z\to(Z^\infty,\Theta^{Z^\infty},\psi^\infty)$, where $(\mathscr{D}^Z,\psi_{dis})$ and $(\Theta^{Z^\infty},\psi^\infty)$ are discrete b-UFIL structure on Z and product b-UFIL structure on Z^∞, respectively. We show that $(\Theta^{X\vee_t X},\overline{\psi})$ is a discrete b-UFIL structure on $\vee_t^\infty Z$. Let $W\in\Theta^{\vee_t^\infty Z}$ and $\nabla_t^\infty W\in\mathscr{D}^Z$.

If $\nabla_t^\infty W=\emptyset$, then $W=\emptyset$.

Suppose $\nabla_t^\infty W\neq\emptyset$, it indicates that $\nabla_t^\infty W=\{z_a\}$ for some $z_a\in Z$. If $z_a=t$, then $W=\{t\}$. Suppose $z_a\neq t$. Then we show that $W=\{z_aj\}$ for all $j\in I$ and the case $W\subset\{z_{a_1},z_{a_2},z_a3,\dots\}$ can not happen. Let $W=\{z_{a_1},z_{a_2}\}$ then, $\pi_kA_t^\infty W=\{z_a,t\}\notin\Theta^Z$ (for $k=1,2$) by the assumption and by Definition 1.40(b-UFIL 1), any set containing W can not be in $\Theta^{\vee_t^\infty Z}$. Hence, $W=\{z_aj\}$ ($j\in I$) and consequently, $\Theta^{\vee_t^\infty Z}=\mathscr{D}^{\vee_t^\infty Z}$, the discrete b-UFIL structure on $\vee_t^\infty Z$.

Next, let $\varsigma\in\overline{\psi}$. By Definition 1.43(i), $(\nabla_t^\infty\times\nabla_t^\infty)\varsigma\in\mathscr{D}^Z$ and $(\pi_kA_t^\infty\times\pi_kA_t^\infty)\varsigma\in\psi$ for $k\in I$. We need to show that $\varsigma=[z_aj]\times[z_aj]$ ($j\in I$), $\varsigma=[t]\times[t]$ or $\varsigma=[\emptyset]=P(\vee_t^\infty Z)^2$.

If $(\nabla_t^\infty\times\nabla_t^\infty)\varsigma=[\emptyset]$, then $\varsigma=[\emptyset]=P(\vee_t^\infty X)^2$.

Suppose $(\nabla_t^\infty\times\nabla_t^\infty)\varsigma=[z_a]\times[z_a]$ for some $z_a\in Z$. If $z_a=t$, since $(\nabla_t^\infty)^{-1}\{t\}=\{p_j=(t,t,t,\dots)\}$, so $\varsigma=[(t,t,t,\dots)]\times[(t,t,t,\dots)]$.

If $z_a\neq t$, then $(\nabla_t^\infty\times\nabla_t^\infty)\varsigma=[z_a]\times[z_a]$, then either $\{z_{aj_1},z_{aj_2},\dots,z_{aj_m}\}\times\{z_{aj_1},z_{aj_2},\dots,z_{aj_m}\}\in\varsigma$ or $\{z_a1,z_a2,\dots\}\times\{z_a1,z_a2,\dots\}\in\varsigma$.

If $B=\{z_{aj_1},z_{aj_2},\dots,z_{aj_m}\}\times\{z_{aj_1},z_{aj_2},\dots,z_{aj_m}\}\in\varsigma$, there exists a finite subset N_0 of ς so that

$\varsigma = [N_0]$. Clearly, $N_0 \subseteq B = \{z_{aj_1}, z_{aj_2}, \ldots, z_{aj_m}\} \times \{z_{aj_1}, z_{aj_2}, \ldots, z_{aj_m}\}$ and if $j_r \neq j_s$ ($r, s = 1, 2, \ldots, m$), then $\{\{z_{aj_r}\} \times \{z_{aj_s}\}\} \neq N_0$ and $\{\{z_{a_1}\} \times \{z_{a_1}\}, \{z_{a_2}\} \times \{z_{a_2}\}, \ldots, \{z_{aj_m}\} \times \{z_{aj_m}\}\} \neq N_0$ since in particular for $k = 1$, $j_r = 1$, and $j_s = 2$, $(\pi_1 A_t^\infty \times \pi_1 A_t^\infty)([z_{a_1}] \times [z_{a_2}]) = [z_a] \times [t] \notin \psi$, and $(\pi_1 A_t^\infty \times \pi_1 A_t^\infty)(([z_{a_1}] \times [z_{a_1}]) \cap ([z_{a_2}] \times [z_{a_2}]) \cap \cdots \cap ([z_{aj_m}] \times [z_{aj_m}])) = ([z_a] \times [z_a]) \cap ([t] \times [t]) \notin \psi$, using the second and the third conditions respectively.

If $B = \{z_{a1}, z_{a2}, \ldots\} \times \{z_{a1}, z_{a2}, \ldots\} \in \varsigma$, there exists a finite subset N_0 of ς so that $\varsigma = [N_0]$. Clearly, $N_0 \subseteq B = \{z_{a1}, z_{a2}, \ldots\} \times \{z_{a1}, z_{a2}, \ldots\}$. The following cases for N_0 can not happen.

(a) $\{\{z_{ai}\} \times \{z_{aj}\},\ i, j \in I\} \neq N_0$ since $(\pi_j A_t^\infty \times \pi_j A_t^\infty)(\varsigma) = [z_a] \times [t] \notin \psi$ or $[t] \times [z_a] \notin \psi$ (for all $j \in I$), using the condition (ii).

(b) $\{z_{a1}, z_{a2}, \ldots\} \times \{z_{a1}, z_{a2}, \ldots\} \neq N_0$ since $(\pi_k A_t^\infty \times \pi_k A_t^\infty)([z_{a_1}] \times [z_{a_2}]) = ([z_a] \times [z_a]) \cap ([t] \times [t]) \cap ([z_a] \times [t]) \cap ([t] \times [z_a]) \notin \psi$ (for $k \in I$). By Definition 1.40(b-UFIL 3) and the condition (i) of our supposition, if $\{z_a, t\} \notin \Theta^Z$ then $[z_a, t] \times [z_a, p] = ([z_a] \times [z_a]) \cap ([t] \times [t]) \cap ([z_a] \times [t]) \cap ([t] \times [z_a]) \notin \psi$. Otherwise, $[\{z_a, t\}] \times [\{z_a, t\}] \subset [z_a] \times [t]$ and by Definition 1.40(b-UFIL 4), it concludes that $[z_a] \times [t] \in \psi$, a contradiction to the condition (ii).

(c) For $r, s > 1$ and $s \leq r$, $\{z_{ar}, z_{ar+1}, \ldots\} \times \{z_{as}, z_{as+1}, \ldots\} \neq N_0$ as $(\pi_j A_t^\infty \times \pi_j A_t^\infty)(\varsigma) = ([z_a] \times [z_a]) \cap ([t] \times [t]) \cap ([z_a] \times [t]) \cap ([t] \times [z_a]) \notin \psi$ (for all $j \in I$), by the similar argument as in above part (b).

(d) $\{\{z_{aj}\} \times \{z_{aj}\},\ j \in I\} \neq N_0$ since $(\pi_j A_t^\infty \times \pi_j A_t^\infty)(\varsigma) = ([z_a] \times [z_a]) \cap ([t] \times [t]) \notin \psi$ (for all $j \in I$), using the condition (iii).

(e) For some fixed r, $\{\{z_{aj}\} \times \{z_{aj}\},\ j \in I\} \cup \{\{z_{ar}\} \times \{z_{ar+1}\}\} \neq N_0$ or $\{\{z_{aj}\} \times \{z_{aj}\},\ j \in I\} \cup \{\{z_{ar}\} \times \{z_{ar+1}\}, \{z_{ar+5}\} \times \{z_{ar}\}\} \neq N_0$ or $\{\{z_{aj}\} \times \{z_{aj}\},\ j \in I\} \cup \{\{z_{ar}\} \times \{z_{ar+1}\}, \{z_{ar+1}\} \times \{z_{ar}\}, \{z_{ar+5}\} \times \{z_{ar+20}\}\} \neq N_0$, since $(\pi_j A_t^\infty \times \pi_j A_t^\infty)(\varsigma) = ([z_a] \times [z_a]) \cap ([t] \times [t]) \notin \psi$ (for all $j \in I$), using the condition (iii).

Therefore, we must have $\varsigma = [z_{aj}] \times [z_{aj}]$ ($j \in I$) or $\varsigma = [\emptyset]$ or $\varsigma = [(t, t, t, \ldots)] \times [(t, t, t, \ldots)]$, and consequently, by definitions 1.40, 1.43 and 4.5(i), the singleton $\{t\}$ is closed in Z. □

Theorem 4.3. *Let (Z, Θ^Z, ψ) be a b-UFIL space, $\emptyset \neq F \subset Z$, $W \in \Theta^Z$ and $\varsigma \in \psi$. For every $z_a, z_b \in Z$ with $z_a \notin F$ and $z_b \in F$,*

(i) *$Q(W) \supseteq \{z_a, \star\}$ if and only if $W \supseteq \{z_a, z_b\}$.*

(ii) $(Q\times Q)\varsigma\subset[z_a]\times[\star]$ *if and only if* $\varsigma\subset[z_a]\times[z_b]$ *or* $\varsigma\cup([z_a]\times[F])$ *is proper.*

(iii) $(Q\times Q)\varsigma\subset[\star]\times[z_a]$ *if and onl if* $\varsigma\subset[z_b]\times[z_a]$ *or* $\varsigma\cup([F]\times[z_a])$ *is proper.*

(iv) $(Q\times Q)\varsigma\subset([z_a]\times[z_a])\cap([\star]\times[\star])$ *if and only if* $\varsigma\cap([F]\times[F])\subset([z_a]\times[z_a])\cap([F]\times[F])$ *and* $\varsigma\cup([F]\times[F])$ *is proper,*

where $Q:Z\to Z/F$ *is a quotient map defined in Definition 4.4.*

Proof. (i) Let $\{z_a,\star\}\subseteq Q(W)$. Then it follows that $Q^{-1}(\{z_a,\star\})\subseteq Q^{-1}(Q(W))\subseteq W$ and therefore, $\{z_a,z_b\}\subseteq W$, for all $z_a\notin F$ and $z_b\in F$.

Conversely, suppose that $W\supseteq\{z_a,z_b\}$, for all $z_a\notin F$ and $z_b\in F$. By Definition 4.4, it follows that $Q(W)\supseteq Q(\{z_a,z_b\})=\{z_a,\star\}$.

(ii) Let $(Q\times Q)\varsigma\subset[z_a]\times[\star]$ for $\varsigma\in\psi$ and $z_a\notin F$. If $\varsigma\not\subset[z_a]\times[z_b]$ and $\varsigma\cup([z_a]\times[F])$ is improper for some $z_b\in F$, then $U\cap(\{z_a\}\times F)=\emptyset$ for some $U\in\varsigma$. It follows that $(z_a,z_b)\notin U$ for all $z_b\in F$, and $(Q\times Q)(\{z_a\}\times\{z_b\})\notin(Q\times Q)(U)\in(Q\times Q)\varsigma$, which implies that $(\{z_a\}\times\{\star\})\notin(Q\times Q)\varsigma$. Therefore, $(Q\times Q)\varsigma\not\subset[z_a]\times[\star]$, a contradiction to the assumption. Thus, $\varsigma\subset[z_a]\times[z_b]$ or $\varsigma\cup([z_a]\times[F])$ is proper for all $z_a\notin F$ and $z_b\in F$.

Conversely, assume that $\varsigma\subset[z_a]\times[z_b]$ or $\varsigma\cup([z_a]\times[F])$ is proper. We claim that $(Q\times Q)\varsigma\subset[z_a]\times[\star]$. If $\varsigma\subset[z_a]\times[z_b]$, then we get $(Q\times Q)\varsigma\subset(Q\times Q)([z_a]\times[z_b])=(Q\times Q)([z_a]\times[\star])$.

If $\varsigma\cup([z_a]\times[F])$ is proper, then $V\cap(x\times F)\neq\emptyset$ for all $V\in\varsigma$. Let $M\in(Q\times Q)\varsigma$. Then, there exists some $U\in\varsigma$ so that $(Q\times Q)(U)\subset M$. Hence, $U\cap(x\times F)\neq\emptyset$, as $\varsigma\cup([z_a]\times[F])$ is proper. It follows that for some $z_b\in F$, $(\{z_a\}\times\{z_b\})\in U$, and $(Q\times Q)(\{z_a\}\times\{z_b\})\in(Q\times Q)(U)\subset M$, which implies that $(\{z_a\}\times\{\star\})\subset M$. Consequently, $M\in([z_a]\times[\star])$ and hence, $(Q\times Q)\varsigma\subset[z_a]\times[\star]$.

(iii) The proof is similar as we have done above in part (ii).

(iv) Let $(Q\times Q)\varsigma\subset([z_a]\times[z_a])\cap([\star]\times[\star])$. We first show that $\varsigma\cup([F]\times[F])$ is proper. As opposed, assume that $\varsigma\cup([F]\times[F])$ is improper, then $U\cap(F\times F)=\emptyset$ for some $U\in\varsigma$. We note that $(Q\times Q)(U)\in(Q\times Q)\varsigma\subset([z_a]\times[z_a])\cap([\star]\times[\star])\subset([\star]\times[\star])$, by the assumption. It follows that $(\{\star\}\times\{\star\})\in(Q\times Q)(U)$. Thus, for some $(\{a\}\times\{b\})\in U$, we have $(Q\times Q)(\{a\}\times\{b\})=\{\star\}\times\{\star\}$ implying that $(\{a\}\times\{b\})\in U\cap(F\times F)$, a contradiction, and it shows that $\varsigma\cup([F]\times[F])$ must be proper.

Next, we show that $\varsigma\cap([F]\times[F])\subset([z_a]\times[z_a])\cap([F]\times[F])$. Let $U\in\varsigma\cap([F]\times[F])$. We prove that $U'\in([z_a]\times[z_a])\cap([F]\times[F])$, because $U\in\varsigma\cap([F]\times[F])$ implies $U\in\varsigma$ and $F\times F\subset U'$.

By the assumption, we get $(Q\times Q)(U') \in (Q\times Q)\varsigma \subset ([z_a]\times[z_a])\cap([\star]\times[\star]) = (Q\times Q)([z_a]\times[z_a])\cap(Q\times Q)([F]\times[F]) = (Q\times Q)(([z_a]\times[z_a])\cap([F]\times[F]))$, hence $(Q\times Q)(U') \in (Q\times Q)(([z_a]\times[z_a])\cap([F]\times[F]))$. It follows that there exists some $V \in ([z_a]\times[z_a])\cap([F]\times[F])$ such that $(Q\times Q)(V) \subset (Q\times Q)(U')$. Further, $V \in ([z_a]\times[z_a])\cap([F]\times[F])$ implies that $V \in ([z_a]\times[z_a])$ and $V \in ([F]\times[F])$, i.e., $V\cap(F\times F) \neq \emptyset$, and $V \subset V\cap(F\times F)$. Also, we have $V \subset V\cap(F\times F) = (Q\times Q)^{-1}((Q\times Q)(V)) \subset (Q\times Q)^{-1}((Q\times Q)(U')) \subset U'$. Therefore, $V\subset U'$ and $U' \in ([z_a]\times[z_a])\cap([F]\times[F])$ and thus by the arbitrariness of U', $\varsigma\cap([F]\times[F]) \subset ([z_a]\times[z_a])\cap([F]\times[F])$.

Conversely, let $\varsigma\cap([F]\times[F]) \subset ([z_a]\times[z_a])\cap([F]\times[F])$ and $\varsigma\cup([F]\times[F])$ is proper. We claim that $(Q\times Q)\varsigma \subset ([z_a]\times[z_a])\cap([\star]\times[\star])$. First, we show that $(Q\times Q)\varsigma \subset ([\star]\times[\star])$. As opposed assume that $(Q\times Q)\varsigma \not\subset ([\star]\times[\star])$. Then there exists some $M \subset (Q\times Q)\varsigma$ such that $(\{\star\}\times\{\star\}) \not\subset M$. Since $M \subset (Q\times Q)\varsigma$, it follows that there exists some $U \in \varsigma$ such that $(Q\times Q)(U) \subset M$. Hence, $U\cap(F\times F) \neq \emptyset$, since $\varsigma\cup([F]\times[F])$ is proper, and we have $(Q\times Q)(U\cap(F\times F)) \subset (Q\times Q)(U) \subset M$, which implies that $(\{\star\}\times\{\star\}) \subset M$, a contradiction. Thus, we must have $(Q\times Q)\varsigma \subset ([\star]\times[\star])$. Now, $(Q\times Q)(\varsigma\cap([F]\times[F])) = (Q\times Q)\varsigma\cap(Q\times Q)([F]\times[F]) = (Q\times Q)\varsigma\cap([\star]\times[\star]) = (Q\times Q)\varsigma$. Also, $\varsigma\cap([F]\times[F]) \subset ([z_a]\times[z_a])\cap([F]\times[F])$ by the assumption, therefore $(Q\times Q)\varsigma = (Q\times Q)(\varsigma\cap([F]\times[F])) \subset (Q\times Q)(([z_a]\times[z_a])\cap([F]\times[F])) = ([z_a]\times[z_a])\cap([\star]\times[\star])$.

□

Theorem 4.4. *Let (Z,Θ^Z,ψ) be a b-UFIL space, $\emptyset \neq F \subset Z$ is closed if and only if for each $z_a, z_b \in Z$ with $z_a \notin F$, $z_b \in F$ and $\varsigma \in \psi$, the conditions below hold:*

(i) $\{z_a, z_b\} \notin \Theta^Z$,

(ii) $\varsigma \not\subseteq [z_a]\times[z_b]$ *and* $\varsigma\cup([z_a]\times[F])$ *is improper (or* $\varsigma \not\subseteq [z_b]\times[z_a]$ *and* $\varsigma\cup([F]\times[z_a])$ *is improper),*

(iii) $\varsigma\cap([F]\times[F]) \not\subseteq ([z_a]\times[z_a])\cap([F]\times[F])$ *or* $\varsigma\cup([F]\times[F])$ *is improper.*

Proof. Let L be non-empty closed set. Then, by Definition 4.5, $\{\star\}$ is closed in Z/F since L is nonempty. By Theorem 4.2, for all $z_a \in Z/F$ with $z_a \neq \star$, $\{z_a,\star\} \notin \Theta^{Z/F}$, $[z_a]\times[\star] \notin \psi$ (or $[\star]\times[z_a] \notin \psi_{Z/F}$), and $([z_a]\times[z_a])\cap([\star]\times[\star]) \notin \psi_{Z/F}$, where $(\Theta^{Z/F},\psi_{Z/F})$ is the quotient b-UFIL structure on Z/F induced by $Q: Z\to Z/F$. By Definition 1.43(ii), for all $\varsigma\in\psi$, $z_a \notin F$, and $W \in \Theta^Z$, we get $Q(W) \not\supseteq \{z_a,\star\}$, $(Q\times Q)\varsigma \not\subset [z_a]\times[\star]$ (or $(Q\times Q)\varsigma \not\subset [\star]\times[z_a]$) and

$(Q\times Q)\varsigma \not\subset ([z_a]\times[z_a])\cap([\star]\times[\star])$ if and only if by Theorem 4.3, $\{z_a,z_b\}\notin W$, and it follows by Definition 1.40(b-UFIL 1) that $\{z_a,z_b\}\notin \Theta^Z$, $\varsigma \nsubseteq [z_a]\times[z_b]$ and $\varsigma\cup([z_a]\times[F])$ is improper (or $\varsigma \nsubseteq [z_b]\times[z_a]$ and $\varsigma\cup([F]\times[z_a])$ is improper), and $\varsigma\cap([F]\times[F]) \nsubseteq ([z_a]\times[z_a])\cap([F]\times[F])$ or $\varsigma\cup([F]\times[F])$ is improper. □

Theorem 4.5. *Let (Z,Θ^Z,ψ) be a b-UFIL space, $\emptyset\neq F\subset Z$ is strongly closed if and only if for each $z_a,z_b\in Z$ with $z_a\notin F$, $z_b\in F$ and $\varsigma\in\psi$, the conditions below hold:*

(i) $\{z_a,z_b\}\notin\Theta^Z$,

(ii) $\varsigma\nsubseteq[z_a]\times[z_b]$ *and* $\varsigma\cup([z_a]\times[F])$ *is improper,*

(iii) $\varsigma\nsubseteq[z_b]\times[z_a]$ *and* $\varsigma\cup([F]\times[z_a])$ *is improper,*

(iv) $\varsigma\cap([F]\times[F])\nsubseteq([z_a]\times[z_a])\cap([F]\times[F])$ *or* $\varsigma\cup([F]\times[F])$ *is improper.*

Proof. Let L be strongly closed. Then, by Definition 4.5, Z/F is T_1 at $\star$ since L is non-empty. By Theorem 2.3, for all $z_a\in Z/F$ with $z_a\neq\star$, $\{z_a,\star\}\notin\Theta^{Z/F}$, $[z_a]\times[\star]\notin\psi$, $[\star]\times[z_a]\notin\psi_{Z/F}$ and $([z_a]\times[z_a])\cap([\star]\times[\star])\notin\psi_{Z/F}$, where $(\Theta^{Z/F},\psi_{Z/F})$ is the quotient b-UFIL structure on Z/F induced by $Q:Z\to Z/F$. By Definition 1.43(ii), for all $\varsigma\in\psi$, $z_a\notin F$ and $W\in\Theta^Z$, hence we get $Q(W)\nsupseteq\{z_a,\star\}$, $(Q\times Q)\varsigma\not\subset[z_a]\times[\star]$, $(Q\times Q)\varsigma\not\subset[\star]\times[z_a]$, and $(Q\times Q)\varsigma\not\subset([z_a]\times[z_a])\cap([\star]\times[\star])$ if and only if by Theorem 4.3, $\{z_a,z_b\}\notin W$ which concludes by Definition 1.40(b-UFIL 1) that $\{z_a,z_b\}\notin\Theta^Z$, $\varsigma\nsubseteq[z_a]\times[z_b]$ and $\varsigma\cup([z_a]\times[F])$ is improper, $\varsigma\nsubseteq[z_b]\times[z_a]$ and $\varsigma\cup([F]\times[z_a])$ is improper, and $\varsigma\cap([F]\times[F])\nsubseteq([z_a]\times[z_a])\cap([F]\times[F])$ or $\varsigma\cup([F]\times[F])$ is improper. □

Remark 4.1.4. *Every strongly closed bounded uniform filter space is closed but the converse is not valid in general.*

Theorem 4.6. *1. Let $h:(Z,\Theta^Z,\psi_X)\to(Y,\Theta^Y,\psi_Y)$ be a buc map between two b-UFIL spaces. If $G\subset Y$ is closed, then $h^{-1}(G)$ is closed in Z.*

2. Let (Z,Θ^Z,ψ) be a b-UFIL space. If $F\subset Z$ is closed and $E\subset F$ is closed, then $E\subset Z$ is closed.

Proof. (1) Let $G\subset Y$ be closed and for all $z_a,z_b\in Z$ with $x\notin h^{-1}(G)$, $z_b\in h^{-1}(G)$ and $\varsigma\in\psi_X$, we show that

(i) $\{z_a,z_b\}\notin\Theta^Z$.

(ii) $\varsigma \nsubseteq [z_a] \times [z_b]$ and $\varsigma \cup ([z_a] \times [h^{-1}(G)])$ is improper (or $\varsigma \nsubseteq [z_b] \times [z_a]$ and $\varsigma \cup ([h^{-1}(G)] \times [z_a])$ is improper).

(iii) $\varsigma \cap ([h^{-1}(G)] \times [h^{-1}(G)]) \nsubseteq ([z_a] \times [z_a]) \cap ([h^{-1}(G)] \times [h^{-1}(G)])$ or $\varsigma \cup ([h^{-1}(G)] \times [h^{-1}(G)])$ is improper.

Note that $h(z_a), h(z_b) \in Y$, $h(z_a) \notin G$, $h(z_b) \in G$ and $(h \times h)\varsigma \in \psi_Y$. Since G is closed, by Theorem 4.4, we have

(i) $\{h(z_a), h(z_b)\} \notin \Theta^Y$.

(ii) $(h \times h)\varsigma \nsubseteq [h(z_a)] \times [h(z_b)]$ and $(h \times h)\varsigma \cup ([h(z_a)] \times [G])$ is improper (or $(h \times h)\varsigma \nsubseteq [h(z_b)] \times [h(z_a)]$ and $(h \times h)\varsigma \cup ([G] \times [h(z_a)])$ is improper).

(iii) $(h \times h)\varsigma \cap ([G] \times [G]) \nsubseteq ([h(z_a)] \times [h(z_a)]) \cap ([G] \times [G])$ or $(h \times h)\varsigma \cup ([G] \times [G])$ is improper.

Suppose $\{h(z_a), h(z_b)\} \notin \Theta^Y$. Clearly, $\{z_a, z_b\} \notin \Theta^Z$, otherwise, if $W = \{z_a, z_b\} \in \Theta^Z$, then $h(W) = h(\{z_a, z_b\}) = \{h(z_a), h(z_b)\} \in \Theta^Y$, a contradiction.

Suppose $(h \times h)\varsigma \nsubseteq [h(z_a)] \times [h(z_b)]$, then by Lemma 4.1.3, clearly it appears that $\varsigma \nsubseteq [z_a] \times [z_b]$. Next, we conclude that $\varsigma \cup ([z_a] \times [h^{-1}(G)])$ is improper. On the contrary, suppose that it is proper. By Lemma 4.1.3, $(h \times h)\varsigma \cup ([h(z_a)] \times [G]) \subset (h \times h)\varsigma \cup ([h(z_a)] \times [h(h^{-1}(G))]) \subset (h \times h)\varsigma \cup (h \times h)([z_a] \times [h^{-1}(G)])) \subset (h \times h)(\varsigma \cup ([z_a] \times [h^{-1}(G)]))$, and consequently $(h \times h)\varsigma \cup ([h(z_a)] \times [G])$ is proper, a contradiction. Thus, $\varsigma \cup ([z_a] \times [h^{-1}(G)])$ is improper. In a similar manner, $\varsigma \nsubseteq [z_b] \times [z_a]$ and $\varsigma \cup ([h^{-1}(G)] \times [z_a])$ is improper.

Suppose $(h \times h)\varsigma \cap ([G] \times [G]) \nsubseteq ([h(z_a)] \times [h(z_a)]) \cap ([G] \times [G])$, then clearly $\varsigma \cap ([h^{-1}(G)] \times [h^{-1}(G)]) \nsubseteq ([z_a] \times [z_a]) \cap ([h^{-1}(G)] \times [h^{-1}(G)])$ by Lemma 4.1.3. Now we show that $\varsigma \cup ([h^{-1}(G)] \times [h^{-1}(G)])$ is improper. As opposed assume that it is proper. Then, $(h \times h)\varsigma \cup ([G] \times [G]) \subset (h \times h)\varsigma \cup ([h(h^{-1}(G))] \times [h(h^{-1}(G))]) \subset (h \times h)\varsigma \cup (h \times h)([h^{-1}(G)] \times [h^{-1}(G)]) \subset (h \times h)(\varsigma \cup ([h^{-1}(G)] \times [h^{-1}(G)]))$, and consequently $(h \times h)\varsigma \cup ([G] \times [G])$ is proper, a contradiction. Thus, $\varsigma \cup ([h^{-1}(G)] \times [h^{-1}(G)])$ is improper.

(2) Let $F \subset Z$ and $E \subset F$ be closed, and for all $z_a, z_b \in Z$ with $z_a \notin E$, $z_b \in E$, and $\varsigma \in \psi_X$, we show that

(i) $\{z_a, z_b\} \notin \Theta^Z$,

(ii) $\varsigma \nsubseteq [z_a] \times [z_b]$ and $\varsigma \cup ([z_a] \times [E])$ is improper (or $\varsigma \nsubseteq [z_b] \times [z_a]$ and $\varsigma \cup ([E] \times [z_a])$ is improper).

(iii) $\varsigma \cap ([E] \times [E]) \nsubseteq ([z_a] \times [z_a]) \cap ([E] \times [E])$ or $\varsigma \cup ([E] \times [E])$ is improper.

If $z_a \notin F$. Since $F \subset Z$ is closed, then by Theorem 4.4, we have $\{z_a, z_b\} \notin \Theta^Z$, $\varsigma \nsubseteq [z_a] \times [z_b]$ and $\varsigma \cup ([z_a] \times [F])$ is improper (or $\varsigma \nsubseteq [z_b] \times [z_a]$ and $\varsigma \cup ([F] \times [z_a])$ is improper), and $\varsigma \cap ([F] \times [F]) \nsubseteq ([z_a] \times [z_a]) \cap ([F] \times [F])$ or $\varsigma \cup ([F] \times [F])$ is improper. Consequently, since $E \subset F$ is closed, we get $\{z_a, z_b\} \notin \Theta^Z$, $\varsigma \nsubseteq [z_a] \times [z_b]$ and $\varsigma \cup ([z_a] \times [E])$ is improper (or $\varsigma \nsubseteq [z_b] \times [z_a]$ and $\varsigma \cup ([E] \times [z_a])$ is improper), and $\varsigma \cap ([E] \times [E]) \nsubseteq ([z_a] \times [z_a]) \cap ([E] \times [E])$ or $\varsigma \cup ([E] \times [E])$ is improper.

If $z_a \in F$. Since the inclusion map $i : (F, \Theta^F, \psi_F) \to (Z, \Theta^Z, \psi_X)$ is an initial lift and $\varsigma \in \psi_X$. By Definition 1.43(i), it follows that $(i \times i)^{-1}\varsigma \in \psi_F$. Note that $(i \times i)^{-1}\varsigma = \varsigma \cup ([F] \times [F])$ and $\varsigma \subset (i \times i)((i \times i)^{-1}\varsigma)$. Since $E \subset F$ is closed and $z_a, z_b \in F$ with $z_a \notin E$, $z_b \in E$, by Theorem 4.4

(i) $\{z_a, z_b\} \notin \Theta^Z$,

(ii) $(i \times i)^{-1}\varsigma \nsubseteq [z_a] \times [z_b]$ and consequently $\varsigma \nsubseteq [z_a] \times [z_b]$, and $(i \times i)^{-1}\varsigma \cup ([z_a] \times [E]) = \varsigma \cup ([z_a] \times [E])$ is improper (or $\varsigma \nsubseteq [z_b] \times [z_a]$ and $\varsigma \cup ([E] \times [z_a])$ is improper),

(iii) $(i \times i)^{-1}\varsigma \cap ([E] \times [E]) = \varsigma \cap ([E] \times [E]) \nsubseteq ([z_a] \times [z_a]) \cap ([E] \times [E])$ or $(i \times i)^{-1}\varsigma \cup ([E] \times [E]) = \varsigma \cup ([E] \times [E])$ is improper.

Thus, $E \subset Z$ is closed (since $E \subset F$). □

Theorem 4.7. *1. Let $h : (Z, \Theta^Z, \psi_X) \to (Y, \Theta^Y, \psi_Y)$ be a buc map between two b-UFIL spaces. If $G \subset Y$ is strongly closed, then $h^{-1}(G)$ is strongly closed in Z.*

2. Let (Z, Θ^Z, ψ) be a b-UFIL space. If $F \subset Z$ is strongly closed and $E \subset F$ is strongly closed, then $E \subset Z$ is strongly closed.

Proof. The proof is analogous to the proof of Theorem 4.6 by using Theorem 4.5 instead of Theorem 4.4. □

Theorem 4.8. *Let (Z, Θ^Z, ψ) be a b-UFIL space and $\emptyset \neq L \subset Z$. Then if Z is $\overline{T_0}$ then Z/L is $\overline{T_0}$ iff L is closed in Z.*

Proof. Suppose that (Z,Θ^Z,ψ) is $\overline{T_0}$, i.e., by Theorem 3.2 for every distinct $z_a,z_b \in Z$, the following hold:

(i) $\{z_a,z_b\} \notin \Theta^Z$;

(ii) $[z_a]\times[z_b] \notin \psi$ (or $[z_b]\times[z_a] \notin \psi$);

(iii) $([z_a]\times[z_a])\cap([z_b]\times[z_b]) \notin \psi$.

We show that Z/L is $\overline{T_0}$ iff L is closed. Let $z_a,\star \in Z/L$ with $z_a \neq \star$ and $(Z/L,\Theta^{Z/L},\psi_{Z/L})$ be the quotient bounded uniform filter space induced by $Q:(Z,\Theta^Z,\psi)\to(Z/L,\Theta^{Z/L},\psi_{Z/L})$. Now we suppose that Z/L is T_1. It follows that

(i) $\{z_a,\star\} \notin \Theta^{Z/L}$;

(ii) $[z_a]\times[\star] \notin \psi_{Z/L}$ (or $[\star]\times[z_a] \notin \psi_{Z/L}$);

(iii) $([z_a]\times[z_a])\cap([\star]\times[\star]) \notin \psi_{Z/L}$.

Note that $Q(z_a)=\star,\ \forall z_a \in L$; and $Q(z_a)=z_a,\ \forall z_a \notin L$. By Definition 1.43(ii), for all $\varsigma \in \psi$, $z_a \notin L$ and $W \in \Theta^Z$, we get $Q(W) \not\supseteq \{z_a,\star\}$, $(Q\times Q)\varsigma \not\subset [z_a]\times[\star]$ (or $(Q\times Q)\varsigma \not\subset [\star]\times[z_a]$), and $(Q\times Q)\varsigma \not\subset ([z_a]\times[z_a])\cap([\star]\times[\star])$ if and only if by Theorem 4.3, $\{z_a,z_b\} \notin W$ which concludes by Definition 1.40(b-UFIL 1) that $\{z_a,z_b\} \notin \Theta^Z$, $\varsigma \nsubseteq [z_a]\times[z_b]$ and $\varsigma \cup ([z_a]\times[L])$ is improper (or $\varsigma \nsubseteq [z_b]\times[z_a]$ and $\varsigma \cup ([L]\times[z_a])$ is improper), and $\varsigma \cap ([L]\times[L]) \nsubseteq ([z_a]\times[z_a])\cap([L]\times[L])$ or $\varsigma \cup ([L]\times[L])$ is improper, iff L is closed by Theorem 4.4. □

Theorem 4.9. *Let (Z,Θ^Z,ψ) be a b-UFIL space and $\emptyset \neq L \subset Z$. Then if Z is T_1 then Z/L is T_1 iff L is strongly closed in Z.*

Proof. Suppose that (Z,Θ^Z,ψ) is T_1, i.e., by Theorem 3.5 for every distinct $z_a,z_b \in Z$, the following hold:

(i) $\{z_a,z_b\} \notin \Theta^Z$;

(ii) $[z_a]\times[z_b] \notin \psi$;

(iii) $[z_b]\times[z_a] \notin \psi$;

(iv) $([z_a]\times[z_a])\cap([z_b]\times[z_b]) \notin \psi$.

We show that Z/L is T_1 iff L is strongly closed. Let $z_a, \star \in Z/L$ with $z_a \neq \star$ and $(Z/L, \Theta^{Z/L}, \psi_{Z/L})$ be the quotient bounded uniform filter space induced by $Q : (Z, \Theta^Z, \psi) \to (Z/L, \Theta^{Z/L}, \psi_{Z/L})$. Now we suppose that Z/L is T_1. It follows that

(i) $\{z_a, \star\} \notin \Theta^{Z/L}$;

(ii) $[z_a] \times [\star] \notin \psi_{Z/L}$;

(iii) $[\star] \times [z_a] \notin \psi_{Z/L}$;

(iv) $([z_a] \times [z_a]) \cap ([\star] \times [\star]) \notin \psi_{Z/L}$.

Note that $Q(z_a) = \star$, $\forall z_a \in L$; and $Q(z_a) = z_a$, $\forall z_a \notin L$. By Definition 1.43(ii), for all $\varsigma \in \psi$, $z_a \notin L$ and $W \in \Theta^Z$, we get $Q(W) \not\supseteq \{z_a, \star\}$, $(Q \times Q)\varsigma \not\subset [z_a] \times [\star]$, $(Q \times Q)\varsigma \not\subset [\star] \times [z_a]$, and $(Q \times Q)\varsigma \not\subset ([z_a] \times [z_a]) \cap ([\star] \times [\star])$ if and only if by Theorem 4.3, $\{z_a, z_b\} \notin W$ which concludes by Definition 1.40(b-UFIL 1) that $\{z_a, z_b\} \notin \Theta^Z$, $\varsigma \not\subseteq [z_a] \times [z_b]$ and $\varsigma \cup ([z_a] \times [L])$ is improper, $\varsigma \not\subseteq [z_b] \times [z_a]$ and $\varsigma \cup ([L] \times [z_a])$ is improper, and $\varsigma \cap ([L] \times [L]) \not\subseteq ([z_a] \times [z_a]) \cap ([L] \times [L])$ or $\varsigma \cup ([L] \times [L])$ is improper, iff L is strongly closed by Theorem 4.5. □

Theorem 4.10. *Let (Z, Θ^Z, ψ) be a b-UFIL space and $\emptyset \neq L \subset Z$. Then if Z is T_1 then L is always closed in Z.*

Proof. Suppose that Z is T_1. It follows that Z is $\overline{T_0}$. By Theorem 4.8, Z/L is $\overline{T_0}$ iff L is closed in Z. Thus L is always closed in Z. □

Theorem 4.11. *Let (Z, Θ^Z, ψ) be a b-UFIL space and $\emptyset \neq L \subset Z$. Then if Z is T_1 then L is strongly closed in Z.*

Proof. Suppose that Z is T_1. By Theorem 4.9, Z/L is T_1 iff L strongly closed. Thus L is strongly closed in Z. □

4.2 Closure Operators in Bounded Uniform Filter Spaces

Let $\mathscr{G}$ be a set based topological category, $z_a \in Obj(\mathscr{G})$ and C be the closure operator of $\mathscr{G}$ in the sense of [39, 59] defined below.

Definition 4.7. *(cf. [39, 59]) Let $\mathscr{G}$ be a set-based topological category. Let $Z \in Ob(\mathscr{G})$ be the underlying set and $E \subset Z$. A closure operator $C : Z \longrightarrow Z$ of $\mathscr{G}$ is an assignment to each E a subset CE of Z such that the conditions below hold:*

(a) $E \subset CE$;

(b) *whenever* $L \subset M$ *impies* $CF \subset CE$;

(c) *(Continuity condition) For each* $h : Z \longrightarrow Y$ *in* $\mathscr{G}$ *and* $E \subset Y$, $C(h^{-1}(E)) \subset h^{-1}(CE)$, *or equivalently,* $h(CE) \subset C(h(E))$.

$E \subset Z$ is said to be C-closed (resp. C-densed) in Z if $CE = E$ (resp. $CE = Z$). A $\mathscr{G}$-morphism $h : Z \longrightarrow Y$ is said to be C-closed if for all C-closed E in Z implies $h(E)$ is C-closed in Y [59].

Definition 4.8. *(cf. [39, 59]) Let* C *be a closure operator of* $\mathscr{G}$ *as defined in 4.7. Then* C *is said to be idempotent if* $C(CE) = CE$, *and if every subobject of any object in* $\mathscr{G}$ *is* C*-dense in its* C*-closure then* C *is called weakly hereditary.*

Examples 4.2.1.

1. The discrete closure operator δ is defined by setting $\delta(E) = E$ for each $Z \in \mathscr{G}$ and $E \subset Z$.
2. The trivial or indiscrete closure operator ∂ is defined by setting $\partial(E) = Z$ for each $Z \in \mathscr{G}$ and $E \subset Z$ [59, 41].

Definition 4.9. Let (Z, Θ^Z, ψ) be a b-UFIL space and $F \subset Z$.

(i) $cl^{\textbf{b-UFIL}}(F) = \cap\{M \subset Z : F \subset M \textit{ and } M \textit{ is closed}\}$ is known as the closure of L.

(ii) $scl^{\textbf{b-UFIL}}(F) = \cap\{M \subset Z : F \subset M \textit{ and } M \textit{ is strongly closed}\}$ is known as the strong closure of L.

Theorem 4.12. $scl^{\textbf{b-UFIL}}(F)$ *and* $cl^{\textbf{b-UFIL}}(F)$ *are (weakly) hereditary, idempotent and productive closure operators of* ***b-UFIL****.*

Proof. The proof is straightforward by combining Theorems 4.6, 4.7, Definition 4.9, and Theorems 2.3, 2.4, Proposition 2.5 and Exercise 2.D of [59]. □

Definition 4.10. *For a topological category* $\mathscr{G}$ *and a closure operator* C *of* $\mathscr{G}$.

(i) $\mathscr{G}_{0C} = \{Z \in \mathscr{G} : z_a \in C(\{z\})$ *and* $z \in C(\{z_a\})$ *implies* $z = z_a$ *with* $z, z_a \in Z\}$.

(ii) $\mathscr{G}_{1C} = \{Z \in \mathscr{G} : C(\{z\}) = \{z\}$ *for each* $z \in Z\}$.

Remark 4.2.2. For $\mathscr{G} = \textbf{Top}$, $C = K$ (the ordinary closure operator), $\textbf{Top}_{jC}$ reduces to T_j space for $j = 0, 1$ respectively.

Theorem 4.13. *An object (Z,Θ^Z,ψ) is in **b-UFIL**$_{0scl}$ iff for every $z,z_a \in Z$ with $z \neq z_a$, one of the following is true.*

(i) there exists $F_1 \subset Z$ strongly closed subset of Z such that $z \notin F_1$ and $z_a \in F_1$,

(ii) there exists $F_2 \subset Z$ strongly closed subset of Z such that $z_a \notin F_2$ and $z \in F_2$.

Proof. Suppose that $(Z,\Theta^Z,\psi) \in$ **b-UFIL**$_{0scl}$ and $z,z_a \in Z$ with $z \neq z_a$. We get $z_a \notin scl(\{z\})$ and $z \notin scl(\{z_a\})$. If $z \notin scl(\{z_a\})$, then it follows by Definition 4.9(ii) that there exists $F_1 \subset Z$ strongly closed subset of Z such that $z \notin F_1$ and $z_a \in F_1$. Similarly, if $z_a \notin scl(\{z\})$, then again by Definition 4.9(ii) it follows that there exists $F_2 \subset Z$ strongly closed subset of Z such that $z_a \notin F_2$ and $z \in F_2$.

Conversely, suppose the first condition holds, i.e., for all $z,z_a \in Z$ with $z \neq z_a$, there exists $F_1 \subset Z$ strongly closed subset of Z such that $z \notin F_1$ and $z_a \in F_1$. By Definition 4.9(ii), we get $z \notin scl(\{z_a\})$. If the latter holds, i.e., for all $z,z_a \in Z$ with $z \neq z_a$, there exists $F_2 \subset Z$ strongly closed subset of Z such that $z_a \notin F_2$ and $z \in F_2$. Then again by Definition 4.9(ii), it results that $z_a \notin scl(\{z\})$ and consequently $(Z,\Theta^Z,\psi) \in$ **b-UFIL**$_{0scl}$. □

Theorem 4.14. *An object (Z,Θ^Z,ψ) is in **b-UFIL**$_{0cl}$ iff for every $z,z_a \in Z$ with $z \neq z_a$, $\{z,z_a\} \notin \Theta^Z$, one of the conditions below hold.*

(i) there exists $F_1 \subset Z$ closed subset of Z such that $z \notin F_1$ and $z_a \in F_1$,

(ii) there exists $F_2 \subset Z$ closed subset of Z such that $z_a \notin F_2$ and $z \in F_2$.

Proof. By using a similar argument in Theorem 4.13 and by using part (i) of Definition 4.9 instead of the part (ii), the results are obtained. □

Remark 4.2.3. *Every object (Z,Θ^Z,ψ) that is in **b-UFIL**$_{0scl}$ is in **b-UFIL**$_{0cl}$ but the converse is not valid generally.*

Example 4.2.4. *Let $Z = \{1,2,3\}$ and (Θ^Z,ψ) be a b-UFIL structure on Z with $\Theta^Z = \{\emptyset,\{1\},\{2\},\{3\}\}$ and $\psi = \{[\emptyset],[1]\times[1],[2]\times[2],[3]\times[3],[1]\times[2]\}$. For $F_1 = \{1,2\}$ and $F_2 = \{2,3\}$, (Z,Θ^Z,ψ) is in **b-UFIL**$_{0cl}$ but not **b-UFIL**$_{0scl}$.*

Theorem 4.15. *An object (Z,Θ^Z,ψ) is in **b-UFIL**$_{1scl}$ iff for every distinct $z,z_a \in Z$,*

(i) $\{z,z_a\} \notin \Theta^Z$,

(ii) $[z] \times [z_a] \notin \psi$,

(iii) $[z_a] \times [z] \notin \psi$,

(iv) $([z] \times [z]) \cap ([z_a] \times [z_a]) \notin \psi$.

Proof. Suppose that $(Z, \Theta^Z, \psi) \in$ **b-UFIL**$_{1scl}$ and $z, z_a \in Z$ with $z \neq z_a$. We get $scl(\{z\}) = \{z\}$ for all $z \in Z$. It follows that $\{z\}$ is strongly closed and consequently by Theorem 4.5, for any $z_a \in Z$ with $z \neq z_a$, $\{z, z_a\} \notin \Theta^Z$, $[z] \times [z_a] \notin \psi$, $[z_a] \times [z] \notin \psi$ and $([z] \times [z]) \cap ([z_a] \times [z_a]) \notin \psi$ (for all $z \neq z_a$).

Conversely, suppose the conditions hold, i.e., $\{z, z_a\} \notin \Theta^Z$, $[z] \times [z_a] \notin \psi$, $[z_a] \times [z] \notin \psi$ and $([z] \times [z]) \cap ([z_a] \times [z_a]) \notin \psi$ (for all $z \neq z_a$). It follows that $\{z\}$ is strongly closed by Theorem 4.2. Consequently, $scl(\{z\}) = \{z\}$ for all $z \in Z$ and hence $(Z, \Theta^Z, \psi) \in$ **b-UFIL**$_{1scl}$. □

Theorem 4.16. *An object (Z, Θ^Z, ψ) is in **b-UFIL**$_{1cl}$ iff for every distinct $z, z_a \in Z$,*

(i) $\{z, z_a\} \notin \Theta^Z$,

(ii) $[z] \times [z_a] \notin \psi$ *(or* $[z_a] \times [z] \notin \psi$*)*,

(iii) $([z] \times [z]) \cap ([z_a] \times [z_a]) \notin \psi$.

Proof. By using a similar argument in Theorem 4.15 and by using Theorem 4.4 instead of Theorem 4.5, the results are obtained. □

Remark 4.2.5. *Every object (Z, Θ^Z, ψ) that is in **b-UFIL**$_{1scl}$ is in **b-UFIL**$_{1cl}$ but the converse is not valid generally.*

Example 4.2.6. *Let $Z = \{1, 2, 3\}$ and (Θ^Z, ψ) be a b-UFIL structure on Z with $\Theta^Z = \{\emptyset, \{1\}, \{2\}, \{3\}\}$ and $\psi = \{[\emptyset], [1] \times [1], [2] \times [2], [3] \times [3], [1] \times [2], [2] \times [3]\}$. Then, (Z, Θ^Z, ψ) is in **b-UFIL**$_{1cl}$ but not in **b-UFIL**$_{1scl}$ as properties (ii) & (ii) of Theorem 4.15 fails to hold.*

4.3 Closure Operators and their Relation with Separation Axioms

In this section, we investigate the relationship of closure operators in bounded uniform filter spaces among themselves and with T_0 and T_1 bounded uniform filter spaces. Also we will give a comparison between respective outcomes.

Theorem 4.17. *Let (Z, Θ^Z, ψ) be a b-UFIL space. Then the following are equivalent:*

(i) (Z,Θ^Z,ψ) *is* $\overline{T_0}$.

(ii) $(Z,\Theta^Z,\psi) \in$ ***b-UFIL***$_{1cl}$.

(iii) *(a)* $\{z,z_a\} \notin \Theta^Z$,

(b) $[z]\times[z_a] \notin \psi$ *(or* $[z_a]\times[z] \notin \psi$*)*,

(c) $([z]\times[z])\cap([z_a]\times[z_a]) \notin \psi$.

Proof. The proof can be easily deduced from Definition 4.5, Theorem 4.4 and Theorem 4.4 of [103]. □

Theorem 4.18. *Let* (Z,Θ^Z,ψ) *be a b-UFIL space. Then, for all* $z,z_a \in Z$ *with* $z \neq z_a$*., the following are equivalent:*

(i) (Z,Θ^Z,ψ) *is* T_1.

(ii) $(Z,\Theta^Z,\psi) \in$ ***b-UFIL***$_{1scl}$.

(iii) *(a)* $\{z,z_a\} \notin \Theta^Z$,

(b) $[z]\times[z_a] \notin \psi$,

(c) $[z_a]\times[z] \notin \psi$,

(d) $([z]\times[z])\cap([z_a]\times[z_a]) \notin \psi$.

Proof. The proof can be easily deduced from Definition 4.5, Theorem 4.5 and Theorem 4.6 of [103]. □

Theorem 4.19. *Each of the subcategories* ***b-UFIL***$_{kcl}$*,* $k=0,1$ *and* ***b-UFIL***$_{kscl}$*,* $k=0,1$ *are quotient-reflective in* ***b-UFIL****, i.e., they are isomorphism-closed, full, closed under the formation of finer structures, products, and subspaces.*

Proof. By using a similar argument in Theorem 3.6, the results are obtained. As deduced from above Theorems 4.17 and 4.18 that **b-UFIL**$_{1cl} \cong \overline{T_0}$**b-UFIL** and **b-UFIL**$_{1scl} \cong T_1$**b-UFIL** respectively. □

Corollary 4.3.1. *Let* (Z,Θ^Z,ψ) *be a bornological b-UFIL space. Then the following are equivalent:*

(i) (Z,Θ^Z,ψ) *is* $\overline{T_0}$.

(ii) $(Z,\Theta^Z,\psi)\in$ ***BONb-UFIL***$_{1cl}$.

(iii) $[z]\times[z_a]\notin\psi$ *(or* $[z_a]\times[z]\notin\psi$*) and* $([z]\times[z])\cap([z_a]\times[z_a])\notin\psi$*, for all* $z,z_a\in Z$ *with* $z\neq z_a$.

Proof. The proof can be easily deduced from Definition 4.5 and Corollary 4.9 of [103]. □

Corollary 4.3.2. *Let* (Z,Θ^Z,ψ) *be a bornological b-UFIL space. Then the following are equivalent:*

(i) (Z,Θ^Z,ψ) *is* T_1.

(ii) $(Z,\Theta^Z,\psi)\in$ ***BONb-UFIL***$_{1scl}$.

(iii) $[z]\times[z]\notin\psi$*,* $[z_a]\times[z]\notin\psi$ *and* $([z]\times[z])\cap([z_a]\times[z_a])\notin\psi$*, for all* $z,z_a\in Z$ *with* $z\neq z_a$.

Proof. The proof can be easily deduced from Definition 4.5 and Corollary 4.10 of [103]. □

Corollary 4.3.3. *Let* (Z,Θ^Z,ψ) *be a discrete symmetric b-UFIL space. Then the following are equivalent:*

(i) (Z,Θ^Z,ψ) *is* $\overline{T_0}$.

(ii) (Z,Θ^Z,ψ) *is* T_1.

(iii) $(Z,\Theta^Z,\psi)\in$ ***BONsb-UFIL***$_{1cl}$.

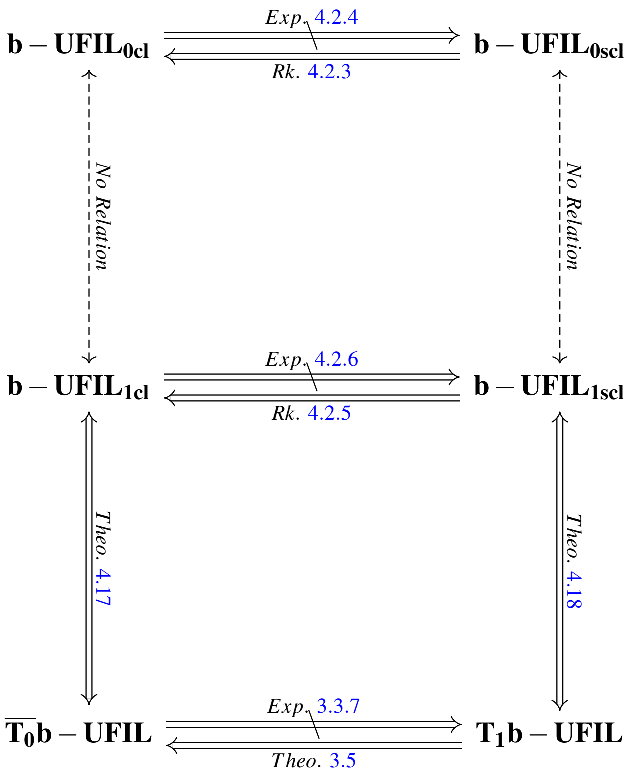
(iv) $(Z,\Theta^Z,\psi)\in$ ***BONsb-UFIL***$_{1scl}$.

(v) $(Z,\Theta^Z,\psi)\in$ ***SUConv***$_{1scl}$.

(vi) $[z]\times[z_a]\notin\psi$ *and* $([z]\times[z])\cap([z_a]\times[z_a])\notin\psi$*, for all* $z,z_a\in Z$ *with* $z\neq z_a$.

Proof. The proof can be easily deduced from Definition 4.5, Corollary 4.18 of [103] and Theorem 4.5 of [88]. □

4.3.1 A Pictorial Representation of Closure Operators in Bounded Uniform Filter Spaces



CHAPTER 5

CONNECTEDNESS AND IRREDUCIBILITY IN BOUNDED UNIFORM FILTER SPACES

In classical topology, we define the intermediate value theorem using the concept of connected spaces which is the generalization of the intermediate value theorem of calculus where we replace the closed set by a connected space.

Definition 5.1 (Connected Spaces)**.** *Consider a topological space* (Z,τ)*. By separation of Z we mean a pair L, M of disjoint open subsets of Z with* $L\cup M = Z$*. Also, Z is known as connected if one cannot find any such separation of Z [22].*

Theorem 5.1 (Classical Connectedness)**.** *A topological space Z is connected iff one of the following equivalent conditions holds:*

(i) The only clopen (both closed and open) subsets of Z are $\emptyset$ *and Z [22].*

(ii) A separation of Z cannot be found [95].

(iii) Every continuous function $h: Z \to \{0,1\}$ *is constant, where* $\{0,1\}$ *is equipped with the discrete topology [95].*

5.1 Connected and Strongly Connected Bounded Uniform Filter Spaces

Definition 5.2. (cf. [75]) Consider a topological functor $\mathfrak{F} : \mathscr{G} \to \mathbf{Set}$ with $Z \in Obj(\mathscr{G})$ and $L \subset Z$.

(i) L is open iff its complement L^c is closed in Z.

(ii) L is strongly open iff its complement L^c is strongly closed in Z.

Theorem 5.2. *Consider a b-UFIL space (Z, Θ^Z, ψ). Then $\emptyset \neq L \subset Z$ is open iff for each $z, z_a \in Z$ with $z \in L$, $z_a \in L^c = M$, and $\varsigma \in \psi$, the conditions below hold:*

(i) $\{z, z_a\} \notin \Theta^Z$.

(ii) $\varsigma \nsubseteq [z] \times [z_a]$ *and* $\varsigma \cup ([z] \times [M])$ *is improper (or* $\varsigma \nsubseteq [z_a] \times [z]$ *and* $\varsigma \cup ([M] \times [z])$ *is improper).*

(iii) $\varsigma \cap ([M] \times [M]) \nsubseteq ([z] \times [z]) \cap ([M] \times [M])$ *or* $\varsigma \cup ([M] \times [M])$ *is improper.*

Proof. The proof can be easily deduced from Definition 5.2 and Theorem 4.4. □

Example 5.1.1. *Consider a b-UFIL structure (Θ^Z, ψ) on $Z = \{1, 2, 3\}$ with $\Theta^Z = \{\emptyset, \{1\}, \{2\}, \{3\}\}$ and $\psi = \{[\emptyset], [1] \times [1], [2] \times [2], [3] \times [3], [1] \times [2], [1] \times [3], [2] \times [1]\}$. Clearly, if we take $\emptyset \neq L = \{1\} \subset Z$, $L^c = M = \{2, 3\}$, $z = 1 \in L$ and $z_a = 2, 3 \in M$ then*

(i) $\{1, 2\}, \{1, 3\} \notin \Theta^Z$.

(ii) $\varsigma \nsubseteq [z_a] \times [1]$ *and* $\varsigma \cup ([M] \times [1])$ *is improper.*

(iii) $\varsigma \cap ([M] \times [M]) \nsubseteq ([1] \times [1]) \cap ([M] \times [M])$ *or* $\varsigma \cup ([M] \times [M])$ *is improper.*

Thus, $L \subset Z$ is open.

Theorem 5.3. *Consider a b-UFIL space (Z, Θ^Z, ψ). Then $\emptyset \neq L \subset Z$ is strongly open iff for each $z, z_a \in Z$ with $z \in L$, $z_a \in L^c = M$, and $\varsigma \in \psi$, the conditions below hold:*

(i) $\{z, z_a\} \notin \Theta^Z$.

(ii) $\varsigma \nsubseteq [z] \times [z_a]$ *and* $\varsigma \cup ([z] \times [M])$ *is improper.*

(iii) $\varsigma \nsubseteq [z_a] \times [z]$ *and* $\varsigma \cup ([M] \times [z])$ *is improper.*

(iv) $\varsigma \cap ([M] \times [M]) \nsubseteq ([z] \times [z]) \cap ([M] \times [M])$ *or* $\varsigma \cup ([M] \times [M])$ *is improper.*

Proof. The proof can be easily deduced from Definition 5.2 and Theorem 4.5. □

Example 5.1.2. *Consider a b-UFIL structure* (Θ^Z, ψ) *on* $Z = \{1,2,3\}$ *with* $\Theta^Z = \{\emptyset, \{1\}, \{2\}, \{3\}\}$ *and* $\psi = \{[\emptyset], [1] \times [1], [2] \times [2], [3] \times [3], [1] \times [2], [1] \times [3], [2] \times [1], [3] \times [1]\}$. *Clearly, if we take* $\emptyset \neq L = \{1\} \subset Z$, $M = L^c = \{2,3\}$, $z = 1 \in L$ *and* $z_a = 2,3 \in M$ *then*

(i) $\{1,2\}, \{1,3\} \notin \Theta^Z$.

(ii) $\varsigma \nsubseteq [z_a] \times [1]$ *and* $\varsigma \cup ([M] \times [1])$ *is improper.*

(iii) $\varsigma \nsubseteq [1] \times [z_a]$ *and* $\varsigma \cup ([1] \times [M])$ *is improper.*

(iv) $\varsigma \cap ([M] \times [M]) \nsubseteq ([1] \times [1]) \cap ([M] \times [M])$ *or* $\varsigma \cup ([M] \times [M])$ *is improper.*

Thus, $L \subset Z$ *is strongly open.*

Remark 5.1.3. *Let* (Z, Θ^Z, ψ) *be a b-UFIL space, then every strongly open subobject in* Z *is open but the converse is not valid generally.*

Example 5.1.4. *Consider a b-UFIL structure* (Θ^Z, ψ) *on* $Z = \{1,2,3\}$ *with* $\Theta^Z = \{\emptyset, \{1\}, \{2\}, \{3\}\}$ *and* $\psi = \{[\emptyset], [1] \times [1], [2] \times [2], [3] \times [3], [1] \times [2], [1] \times [3], [2] \times [1]\}$. *Clearly, if we take* $\emptyset \neq L = \{1\} \subset Z$, $M = L^c = \{2,3\}$, $z = 1 \in L$ *and* $z_a = 2,3 \in M$ *then*

(i) $\{1,2\}, \{1,3\} \notin \Theta^Z$.

(ii) $\varsigma \nsubseteq [z_a] \times [1]$ *and* $\varsigma \cup ([M] \times [1])$ *is improper.*

(iii) $\varsigma \cap ([M] \times [M]) \nsubseteq ([1] \times [1]) \cap ([M] \times [M])$ *or* $\varsigma \cup ([M] \times [M])$ *is improper.*

Thus, $L \subset Z$ *is open but not strongly open since the property (iii) of Theorem 5.3 does not hold.*

Definition 5.3. (cf. [75]) Consider a topological functor $\mathfrak{F} : \mathscr{G} \to \mathbf{Set}$ with $Z \in Obj(\mathscr{G})$ and $\mathfrak{F}(Z) = E$.

(i) Z is said to be connected if and only if $\emptyset$ and Z are the only subsets of Z that are both strongly open and strongly closed.

(ii) Z is said to be strongly connected if and only if $\emptyset$ and Z are the only subsets of Z that are both open and closed.

Remark 5.1.5. 1. In **Top**, the notion of strongly connectedness reduce to the usual connectedness [75].

2. In T_1**Top**, the notions of connectedness and strongly connectedness reduce to the usual connectedness and coincide [75].

3. In general, there is no relation between connectedness and strongly connectedness [75].

Theorem 5.4. *Consider a b-UFIL space (Z, Θ^Z, ψ) with a proper subset $\emptyset \neq L \subset Z$. Then (Z, Θ^Z, ψ) is connected iff one of the conditions below holds:*

(i) *for some $z, z_a \in Z$ with $z \notin L$, $z_a \in L$, and $\varsigma \in \psi$, either $\{z, z_a\} \in \Theta^Z$, or $\varsigma \subseteq [z] \times [z_a]$ or $\varsigma \cup ([z] \times [L])$ is proper (or $\varsigma \subseteq [z_a] \times [z]$ or $\varsigma \cup ([L] \times [z])$ is proper), or $\varsigma \cap ([L] \times [L]) \subseteq ([z] \times [z]) \cap ([L] \times [L])$ and $\varsigma \cup ([L] \times [L])$ is proper.*

(ii) *for some $z, z_a \in Z$ with $z \in L$, $z_a \in L^c = M$, and $\varsigma \in \psi$, either $\{z, z_a\} \in \Theta^Z$, or $\varsigma \subseteq [z] \times [z_a]$ or $\varsigma \cup ([z] \times [M])$ is proper (or $\varsigma \subseteq [z_a] \times [z]$ or $\varsigma \cup ([M] \times [z])$ is proper), or $\varsigma \cap ([M] \times [M]) \subseteq ([z] \times [z]) \cap ([M] \times [M])$ and $\varsigma \cup ([M] \times [M])$ is proper.*

Proof. The proof can be easily deduced from Definition 5.3(ii) and Theorem 4.5. □

Example 5.1.6. *Consider a b-UFIL structure (Θ^Z, ψ) on $Z = \{1,2,3\}$ with $\Theta^Z = \{\emptyset, \{1\}, \{2\}, \{3\}\}$ and $\psi = \{[\emptyset], [1] \times [1], [2] \times [2], [3] \times [3], [1] \times [2], [1] \times [3]\}$. Clearly, if we take $\emptyset \neq L = \{1\} \subset Z$, then*

$$\varsigma = [1] \times [2] \subseteq [1] \times [2] \;\; and \;\; \varsigma = [1] \times [3] \subseteq [1] \times [3], \quad for\ z = 2,3 \notin L;$$

$$\varsigma = [2] \times [1] \subseteq [2] \times [1] \;\; and \;\; \varsigma = [3] \times [1] \subseteq [3] \times [1], \quad for\ z = 2,3 \notin L.$$

Thus, (Z, Θ^Z, ψ) is connected.

Theorem 5.5. *Consider a b-UFIL space (Z, Θ^Z, ψ) with a proper subset $\emptyset \neq L \subset Z$. Then (Z, Θ^Z, ψ) is strongly connected iff one of the conditions below holds:*

(i) *for some $z, z_a \in Z$ with $z \notin L$, $z_a \in L$, and $\varsigma \in \psi$, either $\{z, z_a\} \in \Theta^Z$, or $\varsigma \subseteq [z] \times [z_a]$ or $\varsigma \cup ([z] \times [L])$ is proper and $\varsigma \subseteq [z_a] \times [z]$ or $\varsigma \cup ([L] \times [z])$ is proper, or $\varsigma \cap ([L] \times [L]) \subseteq ([z] \times [z]) \cap ([L] \times [L])$ and $\varsigma \cup ([L] \times [L])$ is proper.*

(ii) *for some* $z, z_a \in Z$ *with* $z \in L$, $z_a \in L^c = M$, *and* $\varsigma \in \psi$, *either* $\{z, z_a\} \in \Theta^Z$, *or* $\varsigma \subseteq [z] \times [z_a]$ *or* $\varsigma \cup ([z] \times [M])$ *is proper and* $\varsigma \subseteq [z_a] \times [z]$ *or* $\varsigma \cup ([M] \times [z])$ *is proper, or* $\varsigma \cap ([M] \times [M]) \subseteq ([z] \times [z]) \cap ([M] \times [M])$ *and* $\varsigma \cup ([M] \times [M])$ *is proper.*

Proof. The proof can be easily deduced from Definition 5.3(iii) and Theorem 4.4. □

Example 5.1.7. *Consider a b-UFIL structure* (Θ^Z, ψ) *on* $Z = \{1,2,3\}$ *with* $\Theta^Z = \{\emptyset, \{1\}, \{2\}, \{3\}\}$ *and* $\psi = \{[\emptyset], [1] \times [1], [2] \times [2], [3] \times [3], [1] \times [2], [1] \times [3], [2] \times [1], [3] \times [1]\}$. *Clearly, if we take* $\emptyset \neq L = \{1\} \subset Z$*, then*

$$\varsigma = [1] \times [2] \subseteq [1] \times [2] \;\; and \;\; \varsigma = [2] \times [1] \subseteq [2] \times [1], \quad for\ z = 2 \notin L.$$

But

$$\varsigma = [1] \times [3] \subseteq [1] \times [3] \;\; and \;\; \varsigma = [3] \times [1] \subseteq [3] \times [1], \quad for\ z = 3 \notin L.$$

Thus, (Z, Θ^Z, ψ) *is strongly connected.*

Theorem 5.6. *Let* (Z, Θ^Z, ψ) *be a b-UFIL space. If* (Z, Θ^Z, ψ) *is strongly connected, then* (Z, Θ^Z, ψ) *is connected.*

Proof. The proof can be easily deduced from Theorems 5.4 and 5.5. □

Example 5.1.8. *Consider a b-UFIL structure* (Θ^Z, ψ) *on* $Z = \{1,2,3\}$ *with* $\Theta^Z = \{\emptyset, \{1\}, \{2\}, \{3\}\}$ *and* $\psi = \{[\emptyset], [1] \times [1], [2] \times [2], [3] \times [3], [1] \times [2], [1] \times [3]\}$. *Clearly, if we take* $\emptyset \neq L = \{1\} \subset Z$*, then*

$$\varsigma = [1] \times [2] \subseteq [1] \times [2] \;\; and \;\; \varsigma = [1] \times [3] \subseteq [1] \times [3], \quad for\ z = 2,3 \notin L;$$

$$\varsigma = [2] \times [1] \subseteq [2] \times [1] \;\; and \;\; \varsigma = [3] \times [1] \subseteq [3] \times [1], \quad for\ z = 2,3 \notin L.$$

Thus, (Z, Θ^Z, ψ) *is connected but not strongly connected.*

5.2 Irreducible and Ultraconnected Bounded Uniform Filter Spaces

Irreducibility or hyperconnectedness is one of the important concept of Topology and Algebraic geometry. The Zariski topology on a prime ideal and the cofinite topology on any infinite set both are irreducible spaces. However, standard topology is not irreducible.

In 2020, T.M. Baran [94] extended the classical irreducibility of topology to set based topological category.

Definition 5.4. (cf. [94]) Consider a topological functor $\mathfrak{F}:\mathscr{G}\to\textbf{Set}$ with $Z\in Obj(\mathscr{G})$.

(i) Z is said to be irreducible if for all closed subobjects E and L of Z with $E\cup L=Z$, then either $Z=E$ or $Z=L$.

(ii) Z is said to be strongly irreducible if for each strongly closed subobject E and L of Z with $E\cup L=Z$, then either $Z=E$ or $Z=L$.

Remark 5.2.1. (i) In **Top**, the notion of irreducibility becomes equivalent to the classical irreducibility [64].

(ii) In **Top**, every irreducible space is connected but the converse implication is not valid generally [94].

(iii) In T_1**Top**, the notion of irreducibility and strongly irreducibility coincide [94].

Theorem 5.7. *Let (Z,Θ^Z,ψ) be a b-UFIL space.*

(i) If (Z,Θ^Z,ψ) is irreducible, then (Z,Θ^Z,ψ) is strongly irreducible.

(ii) If (Z,Θ^Z,ψ) is irreducible, then (Z,Θ^Z,ψ) is strongly connected.

(iii) If (Z,Θ^Z,ψ) is strongly irreducible, then (Z,Θ^Z,ψ) is connected.

Proof. (i) Let (Z,Θ^Z,ψ) be irreducible. Suppose E and L are two strongly closed subsets of Z with $E\cup L=Z$. By Theorems 4.4 and 4.5, E and F are closed subsets of Z. Since (Z,Θ^Z,ψ) is irreducible and by Definition 5.4, $E=Z$ or $L=Z$, and consequently, (Z,Θ^Z,ψ) is strongly irreducible.

(ii) Let (Z,Θ^Z,ψ) be irreducible but not strongly connected. By the Theorem 5.5, there exists a non-empty proper subset L of Z satisfying for every $z_a,z_b\in Z$ with $x\notin L$, $z_b\in L$, and $\varsigma\in\psi$, $\{z_a,z_b\}\notin\Theta^Z$, $\varsigma\not\subseteq[z_a]\times[z_b]$ and $\varsigma\cup([z_a]\times[F])$ is improper, $\varsigma\not\subseteq[z_b]\times[z_a]$ and $\varsigma\cup([L]\times[z_a])$ is improper, and $\varsigma\cap([L]\times[L])\not\subseteq([z_a]\times[z_a])\cap([L]\times[F])$ or $\varsigma\cup([L]\times[L])$ is improper, and for all $z_a,z_b\in Z$ with $z_a\in L$, $z_b\in L^c=M$, and $\varsigma\in\psi$, $\{z_a,z_b\}\notin\Theta^Z$, $\varsigma\not\subseteq[z_a]\times[z_b]$ and $\varsigma\cup([z_a]\times[L^c])$ is improper, $\varsigma\not\subseteq[z_b]\times[z_a]$ and $\varsigma\cup([M]\times[z_a])$ is improper, and $\varsigma\cap([M]\times[M])\not\subseteq([z_a]\times[z_a])\cap([M]\times[M])$ or $\varsigma\cup([M]\times[M])$ is improper. By Theorem 4.4, L and M are closed, and $L\cup M=Z$, which leads to a contradiction.

(iii) By using a similar argument as in (ii), the result can be obtained.

□

Example 5.2.2 (Strongly Irreducible but not Irreducible). *Let* $Z = \{1,2,3,4\}$ *and* (Θ^Z, ψ) *be a b-UFIL structure on* Z *with* $\Theta^Z = \{\emptyset, \{1\}, \{2\}, \{3\}, \{4\}\}$ *and* $\psi = \{[\emptyset], [1]\times[1], [2]\times[2], [3]\times[3], [4]\times[4], [1]\times[2], [1]\times[3], [1]\times[4], [2]\times[1], [3]\times[1], [4]\times[1], [1]\times[\{2,4\}], [\{2,4\}]\times[1], [2]\times[\{2,4\}], [\{1,2\}]\times[2], [\{1,2\}]\times[4], [\{1,2\}]\times[\{2,4\}]\}$. *Then* (Z, Θ^Z, ψ) *is strongly irreducible but not irreducible.*

Example 5.2.3 (Strongly Connected but not Irreducible). *Let* $Z = \{1,2,3,4\}$ *and* (Θ^Z, ψ) *be a b-UFIL structure on* Z *with* $\Theta^Z = \{\emptyset, \{1\}, \{2\}, \{3\}, \{4\}\}$ *and* $\psi = \{[\emptyset], [1]\times[1], [2]\times[2], [3]\times[3], [4]\times[4], [1]\times[2], [1]\times[3], [1]\times[4], [2]\times[1], [3]\times[1], [4]\times[1], [1]\times[\{2,4\}], [\{2,4\}]\times[1], [2]\times[\{2,4\}], [\{1,2\}]\times[2], [\{1,2\}]\times[4], [\{1,2\}]\times[\{2,4\}]\}$. *Then* (Z, Θ^Z, ψ) *is strongly connected but not irreducible.*

Example 5.2.4. Let $Z = \{1,2,3,4\}$ and (Θ^Z, ψ) be a b-UFIL structure on Z with $\Theta^Z = \{\emptyset, \{1\}, \{2\}, \{3\}, \{4\}\}$ and $\psi = \{[\emptyset], [1]\times[1], [2]\times[2], [3]\times[3], [4]\times[4], [1]\times[2], [1]\times[3], [1]\times[4], [2]\times[4], [1]\times[\{2,4\}], [2]\times[\{2,4\}], [\{1,2\}]\times[2], [\{1,2\}]\times[4], [\{1,2\}]\times[\{2,4\}]\}$. Then (Z, Θ^Z, ψ) is connected but not strongly irreducible.

The concept of ultraconnectedness is also one of the primary concepts of Topology since it is stronger than path-connectedness, and it has been studied by several authors under the name of strongly connected [7, 15, 29].

We first introduce the notion of ultraconnectedness in a set-based topological category and examine the relationship among ultraconnectedness, strongly ultraconnectedness, connectedness and strongly connectedness in a b-UFIL space.

Definition 5.5. Consider a topological functor $\mathfrak{F} : \mathscr{G} \to \mathbf{Set}$ with $Z \in Obj(\mathscr{G})$.

(i) Z is called ultraconnected if E and H are open subobjects of Z with $Z = E \cup H$, then either $E = Z$ or $H = Z$.

(ii) Z is called strongly ultraconnected if E and H are strongly open subobjects of Z with $Z = E \cup H$, then either $E = Z$ or $H = Z$.

Remark 5.2.5. In **Top**, the notion of ultraconnectedness coincides with classical ultraconnectedness [7].

Theorem 5.8. *Consider a topological space (Z,τ).*

(i) If (Z,τ) is ultraconnected, then (Z,τ) is connected but in general the converse implication is not true.

(ii) In general, there is no relationship between irreducible and ultraconnected topological spaces.

Proof. (i) It follows from Theorem 1 of [7] but the converse is not true in general. For example, $Z=\{1,2,3\}$ and $\tau=\{\emptyset,\{1\},\{1,2\},\{1,3\},Z\}$ is connected but not ultraconnected.

(ii) Let $Z=\{1,2,3\}$ and $\tau_1=\{\emptyset,\{1\},\{1,2\},\{1,3\},Z\}$, and $\tau_2=\{\emptyset,\{2\},\{3\},\{2,3\},Z\}$ be two topological spaces on Z. Then (Z,τ_1) is irreducible but not ultraconnected. Similarly, (Z,τ_2) is ultraconnected but not irreducible.

□

Theorem 5.9. *Let (Z,Θ^Z,ψ) be a b-UFIL space.*

(i) If (Z,Θ^Z,ψ) is ultraconnected, then (Z,Θ^Z,ψ) is strongly ultraconnected.

(ii) If $((Z,\Theta^Z,\psi)$ is ultraconnected, then (Z,Θ^Z,ψ) is strongly connected.

(iii) If (Z,Θ^Z,ψ) is strongly ultraconnected, then (Z,Θ^Z,ψ) is connected.

Proof. (i) Let (Z,Θ^Z,ψ) be ultraconnected. Suppose E and L are two strongly open subsets of Z with $E\cup L=Z$. By Theorems 5.2 and 5.3, E and L are open subsets of Z. Since (Z,Θ^Z,ψ) is ultraconnected and by Definition 5.5, $E=Z$ or $L=Z$, and consequently, (Z,Θ^Z,ψ) is strongly ultraconnected.

(ii) Let (Z,Θ^Z,ψ) be ultraconnected but not strongly connected. By the Theorem 5.5, for a non-empty proper subset F of Z satisfying $\forall z_a,z_b\in Z$ with $x\notin L$, $z_b\in L$, and $\varsigma\in\psi$, $\{z_a,z_b\}\notin\Theta^Z$, $\varsigma\nsubseteq[z_a]\times[z_b]$ and $\varsigma\cup([z_a]\times[L])$ is improper, $\varsigma\nsubseteq[z_b]\times[z_a]$ and $\varsigma\cup([L]\times[z_a])$ is improper, and $\varsigma\cap([L]\times[L])\nsubseteq([z_a]\times[z_a])\cap([L]\times[L])$ or $\varsigma\cup([L]\times[L])$ is improper, and $\forall z_a,z_b\in Z$ with $z_a\in L$, $z_b\in L^c=M$, and $\varsigma\in\psi$, $\{z_a,z_b\}\notin\Theta^Z$, $\varsigma\nsubseteq[z_a]\times[z_b]$ and $\varsigma\cup([z_a]\times[M])$ is improper, $\varsigma\nsubseteq[z_b]\times[z_a]$ and $\varsigma\cup([M]\times[z_a])$ is improper, and $\varsigma\cap([M]\times[M])\nsubseteq([z_a]\times[z_a])\cap([M]\times[M])$ or $\varsigma\cup([M]\times[M])$ is improper. By Theorem 5.2, L and M are open and $L\cup M=Z$, which leads to a contradiction.

(iii) This can be proved easily parallel to the proof of (ii).

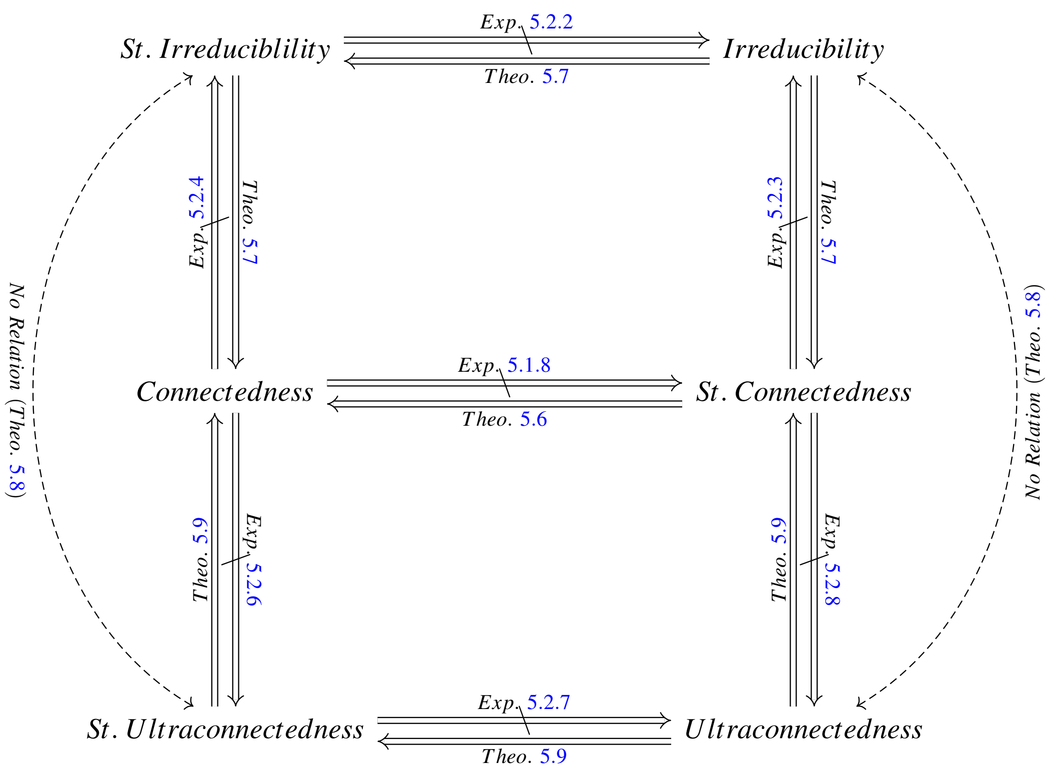
□

Example 5.2.6. Let $Z = \{1,2,3,4\}$ and (Θ^Z, ψ) be a b-UFIL structure on Z with $\Theta^Z = \{\emptyset, \{1\}, \{2\}, \{3\}, \{4\}\}$ and $\psi = \{[\emptyset], [1]\times[1], [2]\times[2], [3]\times[3], [4]\times[4], [1]\times[2], [1]\times[3], [1]\times[4], [2]\times[4], [1]\times[\{2,4\}], [2]\times[\{2,4\}], [\{1,2\}]\times[2], [\{1,2\}]\times[4], [\{1,2\}]\times[\{2,4\}]\}$. Then (Z, Θ^Z, ψ) is connected but neither strongly irreducible nor strongly ultraconnected.

Example 5.2.7 (Strongly Ultraconnected but not Ultraconnected)**.** *Let* $Z = \{1,2,3,4\}$ *and* (Θ^Z, ψ) *be a b-UFIL structure on* Z *with* $\Theta^Z = \{\emptyset, \{1\}, \{2\}, \{3\}, \{4\}\}$ *and* $\psi = \{[\emptyset], [1]\times[1], [2]\times[2], [3]\times[3], [4]\times[4], [1]\times[2], [1]\times[3], [1]\times[4], [2]\times[4], [1]\times[\{2,4\}], [2]\times[\{2,4\}], [\{1,2\}]\times[2], [\{1,2\}]\times[4], [\{1,2\}]\times[\{2,4\}]\}$*. Then* (Z, Θ^Z, ψ) *is strongly ultraconnected but not ultraconnected.*

Example 5.2.8 (Strongly Connected but not Ultraconnected)**.** *Let* $Z = \{1,2,3,4\}$ *and* (Θ^Z, ψ) *be a b-UFIL structure on* Z *with* $\Theta^Z = \{\emptyset, \{1\}, \{2\}, \{3\}, \{4\}\}$ *and* $\psi = \{[\emptyset], [1]\times[1], [2]\times[2], [3]\times[3], [4]\times[4], [1]\times[2], [1]\times[3], [1]\times[4], [2]\times[1], [3]\times[1], [4]\times[1], [1]\times[\{2,4\}], [\{2,4\}]\times[1], [2]\times[\{2,4\}], [\{1,2\}]\times[2], [\{1,2\}]\times[4], [\{1,2\}]\times[\{2,4\}]\}$*. Then* (Z, Θ^Z, ψ) *is strongly connected but not ultraconnected.*

5.2.1 A Pictorial Representation of Connectedness, Irreducibility, and Ultraconnectedness in Bounded Uniform Filter Spaces



CHAPTER 6

SUMMARY OF RESULTS

In this chapter, it is concluded by summarizing all the essential findings and results obtained in this study. Different outcomes in b-UFIL spaces will be discussed. Also, their relationships among themselves and with the classical ones will be stated. In the end, some future research directions in the category of b-UFIL spaces as an extension of this research work will be given.

6.1 Summary

First of all, local $\overline{T_0}$, local T_0' and local T_1 b-UFIL spaces are characterized. It is shown that every local T_1 b-UFIL is local $\overline{T_0}$ b-UFIL but the converse is not true in general. Moreover, generic $\overline{T_0}$, T_0', T_0 and T_1 in the category **b-UFIL** are characterized. It is shown that $T_1 \implies \overline{T_0} \implies T_0 \implies T_0'$, but the converse is not valid generally and provided some related results. Further it is shown that under $\overline{T_0}$ condition, $\overline{T_0}$**DISb-UFIL** $\cong$ $\overline{T_0}$**PUConv** $\cong$ $\overline{T_0}$**BONb-UFIL** which is not isomorphic in general. Also, the isomorphic relation among $\overline{T_0}$**DISsb-UFIL**, $\overline{T_0}$**SUConv**, T_1**SUConv**, $\overline{T_0}$**BONsb-UFIL** and T_1**BONsb-UFIL** is stated, and their relationships with the usual ones are examined. Then, it is proved that $\overline{T_0}$**b-UFIL**, T_0**b-UFIL** and T_1**b-UFIL** are quotient-reflective subcategories of **b-UFIL**, and T_0'**b-UFIL** is a hereditary and cartesian closed topological category.

Also, the characterizations of closed and strongly closed b-UFIL spaces using quotient maps have been given. It is shown that the subobject and the inverse of a closed (resp. strongly closed) b-UFIL space is a closed (resp. strongly closed) b-UFIL space. In addition, it is proved that $scl^{\textbf{b-UFIL}}(F)$ and $cl^{\textbf{b-UFIL}}(F)$ are (weakly) hereditary, idempotent, and productive closure operators of **b-UFIL** for $F \subset Z$, where $(Z, \Theta^Z, \psi) \in$ **b-UFIL**. Then **b-UFIL**$_{0cl}$, **b-UFIL**$_{0scl}$, **b-**

$\mathbf{UFIL}_{1cl}$ and $\mathbf{b\text{-}UFIL}_{1scl}$ closure operators are characterized and their relationship with T_0 and T_1 b-UFIL spaces is stated as follow:

(a) $\mathbf{b\text{-}UFIL}_{1cl} \cong \overline{T_0}\mathbf{b\text{-}UFIL}$,

(b) $\mathbf{b\text{-}UFIL}_{1scl} \cong T_1\mathbf{b\text{-}UFIL}$,

(c) $\mathbf{BONb\text{-}UFIL}_{1cl} \cong \overline{T_0}\mathbf{BONb\text{-}UFIL}$,

(d) $\mathbf{BONb\text{-}UFIL}_{1scl} \cong T_1\mathbf{BONb\text{-}UFIL}$,

(e) $\overline{T_0}\mathbf{DISsb\text{-}UFIL} \cong T_1\mathbf{DISsb\text{-}UFIL} \cong \mathbf{BONsb\text{-}UFIL}_{1cl} \cong \mathbf{BONsb\text{-}UFIL}_{1scl} \cong \mathbf{SUConv}_{1scl}$.

Moreover, a characterization of open (resp. strongly open) objects in **b-UFIL** has been given. Also, connected (resp. strongly connected) bounded uniform filter spaces are characterized. It is proved that a strongly connected object in the category **b-UFIL** is connected but the converse is not valid generally.

Furthermore, the concept of hyperconnectedness or irreducibility (resp. strongly irreducibility) in the category of b-UFIL spaces is stated. Also, the relationship among irreducibility, strongly irreducibility, connectedness, and strongly connectedness in a b-UFIL space is analyzed. It is shown that

(a) Every irreducible b-UFIL space is strongly irreducible but not conversely.

(b) Every irreducible b-UFIL space is strongly connected b-UFIL spaces but not conversely.

(c) Every strongly irreducible b-UFIL space is connected b-UFIL spaces but not conversely.

In the end, the notion of ultraconnectedness in a set-based topological category is defined by us and the relationship among ultraconnectedness, strongly ultraconnectedness, connectedness, and strongly connectedness in a b-UFIL space has been examined. Also, it is proved that

(a) Every ultraconnected b-UFIL space is strongly ultraconnected but not conversely.

(b) Every ultraconnected b-UFIL space is strongly connected b-UFIL spaces but not conversely.

(c) Every strongly ultraconnected b-UFIL space is connected b-UFIL spaces but not conversely.

6.2 Future Research Directions

In light of the above study, the following can be studied in the category **b-UFIL** as future research work:

(i) How can one characterize the notion of sobriety, compactness, and D-connectedness in the category **b-UFIL**?

(ii) How one can define pre-Hausdorff, Hausdorff, regular and normal objects in **b-UFIL**, and what would be their relation to the classical ones?

(iii) Can Urysohn Lemma and Tietze Extension Theorem be extended in the category of b-UFIL spaces?

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