Production of B-Meson in HQET Factorization at $\mathcal{O}(\alpha_s)$

by

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Abstract

We investigate the factorizability of the decay amplitude $W^+ \to B\gamma$ at tree level and with one loop QCD corrections. It was found that working in the heavy quark limit allows these amplitudes to be factorized as this limit sets the tone for the kinematical hierarchy of this process which is
$m_W \sim m_b >> \Lambda_{QCD}$. In light of this one can employ the Factorization Theorem of Heavy Quark Effective Theory to express each amplitude as a convolution of a hard perturbative part, namely the Hard Kernel and a non-perturbative part, the LCDA.
The decay amplitude can be decomposed in terms of two scalar form factors F_V and F_A which can be shown to be equal up-to one loop level. They can be explicitly calculated using the Hard Kernel and a model dependant LCDA.

Dedication

This thesis is dedicated to **Pookey**, **Mitten** and **Muffin**. Also to coffee...a lot of coffee

Acknowledgments

I extend my gratitude to two individuals who were not part of the department or the supervisory team but provided vital assistance in this task.
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Chapter 1

Introduction

One of the most thoroughly tested theories of contemporary physics is the Standard Model (SM) of Particle Physics. It describes the Electromagnetic, Strong and Weak forces of nature. However, in recent years, experimental evidence has shown its shortcomings and inconsistencies hence it is deemed to be incomplete.

Unlike the gauge sector of the SM, the flavour sector of the SM has not been verified to a high degree of precision. One of the most interesting problems in flavour physics is the determination of the CKM matrix elements which dictate the degree of mixing between the quark weak and mass eigen states in the weak interaction. The numerical values of these parameters are extracted from experimental data, prior to theoretical calculations.
Another interesting topic in flavour physics is the violation of CP symmetry which was first observed in 1964 in the study of neutral K mesons. Later it was also observed in a large number of decays involving the B meson. These decays are one of the most direct ways of determining the CKM elements. They are also highly suitable for the study of non-perturbative QCD which is responsible for the confinement of quarks and gluons within hadrons. This is due to the asymptotic freedom of QCD which causes the quark-gluon coupling to be strong at low energy scales and weak at high energy scales.

The B meson is interesting on account of the b quark which is heaviest of the quarks able to form a hadron. Due to its mass being far larger than typical hadronic scales, the strong interaction involving the b quark are perturbative because at the heavy quark mass scale the effective QCD coupling is sufficiently small for perturbative calculations of short distance effects. Hence studies involving the B meson help improve our understanding of QCD at different dynamical scales.

To deal with the disparate kinematical scales in the study of hadrons, various theoretical approaches have been developed. Among these, QCD Factorization is of crucial importance. It provides the theoretical basis to control QCD effects in exclusive processes. Factorization requires the high and low energy dynamics of a system to be independent of each other [1,2]. This is not really the case but can be a useful approximation under certain conditions and up to a certain degree of accuracy. This is something that has to be checked and proven for a particular process at a given order and may involve certain prerequisite conditions or approximations, outside of which Factorisation might not hold.
The earliest applications of the factorization framework in collider physics were in hard probing experiments like Deep Inelastic Scattering [3] and in hard hadron scattering like the Drell-Yan process [4]. The description of such processes relies on the parton model wherein the partons composing a hadron are considered mutually free [1,2]. The total cross sections of such scatterings are equal to a convolution integral of a hard perturbative parton-level cross-section with a process-independent, non-perturbative parton distribution function (PDF) [1,3].

Amplitudes of weak decays/productions of hadrons can also be understood in the factorization framework as there are two different types of interactions governing such processes. There is the electroweak sector which is responsible for the flavour changing and there is the low energy strong interaction which is responsible for binding the quarks together as a hadronic state. The issue is that the former has to be treated perturbatively and the latter non-perturbatively, even though they occur in the same process. For this, the technique of QCD Factorisation is used. Its factorization formula expresses the Feynman amplitude as a convolution of a perturbative Hard Kernel and a universal non-perturbative distribution amplitude [5,6]. The long distance hadronic information for exclusive hadron reactions is described by Light-Cone Distribution Amplitudes (LCDA's) which are analogous to parton distribution functions (PDFs) in the inclusive processes [2].
This approach is applied to factorize a decay process of the B meson in [7] and a similar approach for the factorization of a W boson to produce a B meson in [8] where the order by order computation of the B meson LCDA

uses bi-local operators comprising Wilson lines [2,9]. This is in contrast to the approaches of [10] using QCD sum rules and [11] using operator product expansion (OPE). LCDA evolution with respect to the factorization scale can be computed for both light mesons [12] and heavy mesons [13].
The B mesons distribution amplitudes play a central role in factorization approaches and were introduced as a direct analogue to LCDA's of light mesons [12]. To study the properties of B meson LCDA's, the radiative leptonic B decays such as $B \to \gamma l \bar{\nu}_l$ [14] are among the simplest processes. The form factor for such radiative processes can be evaluated in terms of B meson LCDA's up to one loop level accuracy.
The validity of factorisation approaches can also be tested through production processes of the B meson which involves the decay of Z or W bosons. The first detailed studies of radiative decays of W boson into D_c [15] and B_c in the framework of non-relativistic QCD (NRQCD) and light cone factorisation (LCF).

In light of all this we investigated the production of heavy-light mesons within the framework of Heavy Quark Effective Theory (HQET) factorization. For this we mainly focus on the process $W^+ \to B\gamma$. This process involves three different kinematical scales. Namely, the Weak boson mass M_W, b-quark mass m_b, and the QCD scale Λ_{QCD}. At leading twist $m_b \sim M_B$ because they differ only by $\mathcal{O}(\Lambda_{QCD})$. By an explicit computation up to one-loop level, we found that hard-scattering kernel is free of infrared (IR) divergences. This implies that this amplitude, up to one-loop level, can be factorized into a convolution of perturbatively calculable hard kernel and non-perturbative LCDA.

The structure of this thesis is as follows: After the Introduction follows a brief overview of some of the tools used to tackle this problems. Chapter 2 introduces the rudiments of Factorisation and Wilson lines. Chapter 3 gives a brief overview of HQET. The remaining chapters constitute the core problem of the thesis. In Chapter 4 we introduce the kinematics of the process and calculate the Feynman amplitudes, LCDA's and Hard Kernels. In Chapter 5 the Hard Kernels and LCDA's are used to calculate the vector and axial vector Form Factors for this process at tree level and one loop level. In Chapter 6 we conclude by proving that the factorization theorem holds for this process at tree level and one loop level. All of the Feynman and LCDA diagrams in this thesis have been drawn using the software JaxoDraw [16].

Chapter 2

Factorization

In this chapter we discuss the idea and motivation behind the Factorisation framework. We start off with a discussion on Naive Factorisation and point out its drawbacks. Then we introduce the approach of QCD Factorisation, which is what we will use to study this problem and we explain the perturbative and non-perturbative quantities associated with it like the Hard Kernel, LCDA's and Form Factors.

When dealing with weak decays of heavy-light mesons, there are three main kinematical scales [5]. Namely the mass scales of the weak boson m_W, heavy quark m_b (in our case, the b quark) and the scale of non-perturbative QCD interactions Λ_{QCD}. When considering this process in the heavy quark limit, these scales follow a hierarchy given as

$$m_W \sim m_b >> \Lambda_{QCD} \tag{2.1}$$

Hadronic processes involving the weak interactions are of the form

$$\mathcal{A} = \frac{G_F}{\sqrt{2}} \sum_i \lambda_i C_i(\mu) \langle F| \mathcal{O}_i(\mu) |I\rangle \tag{2.2}$$

Where G_F is the four fermion effective coupling, λ_i are the CKM elements and $C_i(\mu)$ are the short distance Wilson coefficients of the long distance effective operators $\mathcal{O}_i(\mu)$. These are all defined at the scale μ_F which is the energy scale separating the large and short distance physics of the process. Solving the matrix element is complicated on this account and requires partitioning or factorizing the amplitude, written out in terms of the constituent particles, into a part dependant upon the high energy and small distance dynamics and another dependant upon the low energy scale which contains the non-perturbative dynamics. This is due to the three point gluon self interaction term in the QCD Lagrangian. The QCD effective coupling diverges at the scale $\mathcal{O}(\Lambda_{QCD})$ [17], which is the momentum scale of interactions at the hadronic length scale hence perturbative QCD does not apply here.

2.1 Naive Factorization

A useful decay process to motivate the discussion would be

$$B \to \pi^+\pi^-$$

The idea behind Naive Factorisation is to divide a matrix element of a four-fermion operator of a heavy quark decay (mediated by a weak boson) into matrix elements corresponding to two separate current operators as in [5]:

$$\langle \pi^+\pi^- | \, (\bar{u}b)_{V-A}(\bar{d}u)_{V-A} \, |\bar{B}\rangle \to \langle \pi^- | \, (\bar{d}u)_{V-A} \, |0\rangle \, \langle \pi^+ | \, (\bar{u}b)_{V-A} \, |\bar{B}\rangle \tag{2.3}$$

This simplifies the matrix element into a product of a decay constant and a $\bar{B} \to \pi^+$ Form Factor. As the π^- and $(\bar{B}\pi^+)$ system can interact via gluon exchange so it introduces non-factorisable gluon contributions which can only be ignored if the gluons are soft. Such an assumption is insufficient as it does not allow for hard gluon interactions and re-scattering in the final state.

2.2 QCD Factorization

In this study, a subcategory of QCD Factorization is used, called HQET Factorization as we study a process involving heavy-light mesons. It is the presence of widely separated kinematical scales which allow the factorization decay/production amplitudes of heavy-light mesons into a convolution of Hard Kernel (T), containing the hard scales and Light Cone Distribution Amplitudes or LCDA's (Φ) and form factors (F).

In the weak decay or production of B mesons the typical mass and energy scales are the masses of the b quark and W bosons as m_b=4.8 GeV, m_W=80 GeV and the momentum scale of non-perturbative QCD interaction Λ_{QCD}=0.2 GeV. Studying this process using Light Cone Factorisation (LCF)

requires treating m_W as being far larger than $m_b \sim \Lambda_{QCD}$ hence the former gets treated as a hard scale whilst the latter are two different soft scales.
In [7] this decay is studied in QCD Factorisation which requires taking the b quark in the heavy quark limit, which makes it and all other scales above it as hard scales. Hence $m_W \sim m_b$ are classified as a hard scale and Λ_{QCD} is the only soft scale, making it a process with only two kinematical scales. The factorization formula, as given in [5,7] is

$$\begin{aligned}\left\langle M_1 M_2 \left| Q_i \right| \overline{B} \right\rangle = & \sum_j F_j^{B\to M_1}\left(m_2^2\right) \int_0^1 du\ T_{ij}^I(u)\ \Phi_{M_2}(u) \\ & + \sum_k F_k^{B\to M_2}\left(m_1^2\right) \int_0^1 dv\ T_{ij}^I(v)\ \Phi_{M_1}(v) \\ & + \int_0^1 d\xi\ du\ dv\ T_i^{II}(\xi,u,v)\ \Phi_B(\xi)\Phi_{M_1}(v)\Phi_{M_2}(u)\end{aligned} \tag{2.4}$$

$$\left\langle H_1 M_2 \left| Q_i \right| \overline{B} \right\rangle = \sum_j F_j^{B\to H_1}\left(m_2^2\right) \int_0^1 du\ T_{ij}^I(u)\ \Phi_{M_2}(u) \tag{2.5}$$

$F_j^{B\to M}$ denotes a $B \to M$ form factor and Φ_M is LCDA for the quark-antiquark Fock state of meson M. Both of these are non-perturbative quantities and are much simpler relative to the original matrix elements. $T_{ij}^I(u)$, $T_{ik}^I(v)$ and $T_i^{II}(\xi,u,v)$ are the hard-scattering functions of the light-cone momentum fractions u, v and ξ of the quarks inside the final state mesons and the B meson respectively. These hard-scattering functions can be computed perturbatively.

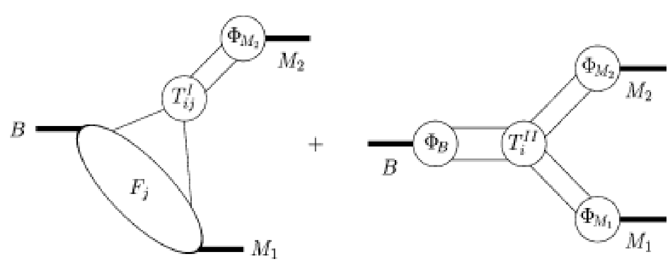

Figure 2.1: Graphical representation of factorization formula [7]

The decay of B meson into two light mesons is given by (2.3). The first two terms in it correspond to the spectator quark of the B meson going to either of the two final state (light) mesons, while the other quark engages in hard interactions, hence the Hard Kernel T^I. The third term in (2.3) is for the case when both quarks engage in hard interactions and there are no spectators hence the Hard Kernel T^{II}. In the hard exclusive processes where the decay is dominated by exchange of hard gluons, the amplitude can be expressed as the convolution of Hard Kernel with the LCDA's of the participating mesons.
B meson decaying into a heavy and a light meson is represented by (2.4). Factorization does not hold for the case of a spectator quark going to a light meson but the other meson being heavy, because the heavy meson is neither fast nor does it have a small mass.

2.2.1 Non-perturbative quantities

Form Factors

Form Factors are scalar functions consisting of independent terms, which are obtained by the decomposition of current matrix elements, using Lorentz and gauge symmetry. In QCD factorization the matrix elements of the vector currents are parameterized by two scalar form factors: $F_+^{B\to P}(q^2)$ and $F_0^{B\to P}(q^2)$. The B to pseudo-scalar (P) vector matrix element is parameterized as

$$\langle P(k)|\,\bar{q}\gamma^\mu b\,|\overline{B}(p)\rangle = F_+^{B\to P}\,(p^\mu + k^\mu) + \left[F_0^{B\to P} - F_+^{B\to P}\right]\frac{m_B^2 - m_P^2}{q^2}q^\mu \quad (2.6)$$

where q is the momentum difference between the two mesons. The two form factors tend to get equal as the momentum difference between the final and initial mesons approaches zero.
The above form factors are known as physical form factors and to use them is advantageous for they are directly related to measurable quantities or to other form factors that can be calculated from lattice QCD or QCD sum rules.

LCDA's of Light Mesons

The momenta distribution of light mesons is given by their respective LCDA's in exclusive processes. This is analogous to PDF's for inclusive processes.

In general they are vacuum to meson (or vice versa) matrix elements of a two-quark bi-local operator. Up to the heavy quark limit, the leading twist LCDA's for Pseudo-scalar (P), longitudinally polarized Vector ($V_{\parallel}$) and transversely polarized Vector ($V_{\perp}$) mesons are given as follows

$$\langle P(q)|\,\overline{q}(y)_{\alpha}q'(x)_{\beta}\,|0\rangle\,|_{(x-y)^2=0} = \frac{if_P}{4}\left(\not{q}\gamma_5\right)_{\beta\alpha}\int_0^1 du e^{i(\overline{u}qx+uqy)}\Phi_P(u,\mu) \quad (2.7)$$

$$\langle V_{\parallel}(q)|\,\overline{q}(y)_{\alpha}q'(x)_{\beta}\,|0\rangle\,|_{(x-y)^2=0} = -\frac{if_V}{4}\not{q}_{\beta\alpha}\int_0^1 du e^{i(\overline{u}qx+uqy)}\Phi_{\parallel}(u,\mu) \quad (2.8)$$

$$\langle V_{\perp}(q)|\,\overline{q}(y)_{\alpha}q'(x)_{\beta}\,|0\rangle\,|_{(x-y)^2=0} = -\frac{if_T(\mu)}{8}\left[\not{\epsilon}^*_{\perp},\not{q}\right]_{\beta\alpha}\int_0^1 du e^{i(\overline{u}qx+uqy)}\Phi_{\perp}(u,\mu) \quad (2.9)$$

With $f_{P,V_{\parallel},V_{\perp}}(\mu)$ being the form factors of the their respective mesons.

LCDA's of B Mesons

As LCDA's are defined in light-cone coordinates so the relevant definitions of these coordinates are given in detail in Chapter 4.
The reason for the introduction of the B meson LCDA in the QCD factorization formula is the hard interaction of spectator quark. This hard spectator interaction term depends upon $p'.l$ where p' and l are the momenta of the light meson and spectator quark, respectively. As only the p'_{-} component of the light meson momentum is non-zero so $p'l = p'_{-}l_{+}$. The decay amplitude for two-particle Fock state of the B meson is

$$\Psi_B(z,p) = \left\langle 0\,|\overline{q}_{\alpha}(z)[z,0]b_{\beta}(0)|\,\overline{B}_d(p)\right\rangle = \int \frac{d^4l}{(2\pi)^4}e^{-ilz}\hat{\Psi}_B(l,p) \quad (2.10)$$

where $\hat{\Psi}_B$ represents the full Bethe-Salpeter wave function and $[z,0]$ refers to the Wilson line to ensure bi-local gauge invariance of the matrix element. Then approximating the result as

$$\int \frac{d^4l}{(2\pi)^4}A(l,\ldots)\hat{\Psi}_B(l,p) = \int dl^{+}A\left(l^{+},\ldots\right)\int \frac{d^2l_{\perp}dl^{-}}{(2\pi)^4}\hat{\Psi}_B(l,p) \quad (2.11)$$

As the integration of (2.10) is only over $l_\perp$ and l_- so $z_\perp$ and z_+ are zero. Under these conditions the B meson LCDA, at leading order in $1/m_b$, can be parameterized in terms of two scalar wave functions as:

$$\begin{aligned} \langle 0|\, \bar{q}_\alpha(z)[z,0]b_\beta(0)\, |\overline{B}_d(p)\rangle = -\frac{if_B}{4}\delta_{ij} \left[\left(\not{p} + m_b \right) \gamma_5 \right]_{\beta\gamma} \\ \times \int_0^1 d\xi e^{-i\xi p^+ z^-} \left[\Phi_{B1}(\xi) + \not{n}^- \Phi_{B2}(\xi) \right]_{\gamma\alpha} \end{aligned} \tag{2.12}$$

with the renormalization conditions

$$\int_0^1 d\xi \Phi_{B1}(\xi) = 1, \quad \int_0^1 d\xi \Phi_{B2}(\xi) = 0 \tag{2.13}$$

As seen from the indices in (2.12), the B meson distribution amplitude is anti-symmetric at scales near or smaller than that of the heavy quark mass. At scales much higher than $\mathcal{O}(m_b)$, it tends to a more symmetric form.
The heavy quark distribution amplitude can be defined in either HQET or QCD, which entails defining the heavy quark field (in this case for the bottom quark) b in HQET. The light quark field q will remain the same in QCD or HQET.

2.3 Wilson Lines

The idea of Wilson lines can be motivated from a geometric investigation of local gauge invariance of the Dirac Lagrangian [9]. A local gauge transforms the Dirac fields as $\psi \to e^{-ig\alpha(x)}\psi$ and $\bar{\psi} \to e^{ig\alpha(x)}\bar{\psi}$
Considering the Lagrangian

$$\mathcal{L} = \bar{\psi}(i\not{\partial} - m)\psi$$

It is clear that the mass term preserves invariance under a local gauge transform but not the derivative term. For there to be local gauge invariance the derivative has to transform similarly to a Dirac field and that is a problem when one considers the derivative by its fundamental definition.

$$n^\mu \partial_\mu \psi(x) = lim_{\epsilon\to 0} \frac{\psi(x+\epsilon) - \psi(x)}{\epsilon} \tag{2.14}$$

There cannot be a common phase factor pulled from the derivative which will cancel the phase due to the accompanying $\bar{\psi}$. The conventional work around is to modify the derivative to include a gauge field. In the Wilson line approach we end up with a gauge field as well but it is introduced by inserting a two point object $U(x, y)$ in the derivative

$$n^{\mu} D_{\mu}\psi(x) = lim_{\epsilon\to 0}\frac{\psi(x+\epsilon) - U(x+\epsilon, x)\psi(x)}{\epsilon} \tag{2.15}$$

which is defined to have this bi-local gauge transformation

$$U(x, y) \to e^{ig\alpha(x)}U(x, y)e^{-ig\alpha(y)} \tag{2.16}$$

Further properties of $U(x, y)$ are inferred from the approach of [9].

- Transitive Property : If $U(x, y)$ connects the gauge transformation of two fields at two separate points then $U(x, b)U(b, y)$ will connect those gauge transformations with that of an intermediate point as well.

- Unitarity : The object $\bar{\psi}(x)U(x, y)\psi(y)$ is locally gauge invariant and by the transitive property it is $\bar{\psi}(x)U(x, b)U(b, y)\psi(y)$
  $\psi(x) = U(x, b)\psi(b)$ and $\bar{\psi}(b) = \bar{\psi}(x)U^{\dagger}(b, x)$ by their gauge transformations.
  Now considering a local operator from the above definitions
  $\bar{\psi}(b)\psi(b) = \bar{\psi}(x)U^{\dagger}(b, x)U(b, x)\psi(x)$
  For it to be gauge invariant, U must be unitary : $U^{\dagger}(b, x)U(b, x) = I$
  The transitive property also allows $\bar{\psi}(x)\psi(x) = \bar{\psi}(x)U(x, b)U(b, x)\psi(x)$ and this shows that the action of Hermitian conjugation on the Wilson line is $U^{\dagger}(b, x) = U(x, b)$

A unitary operator is a pure phase hence $U(x, y) = e^{ig\alpha(x,y)}$ where $\alpha(x, y)$ also obeys the transitive property along with being anti-symmetric. The Wilson line approach is useful when applied to QCD, where $U(x, y)$ contains non-ableian generators. This enforces path ordering. For example

$$U(x, y) = U(x, b_1)U(b_1, b_2)....U(b_{N-1}, b_N)U(b_N, y)$$

The path ordered exponentials for this product are written such that all the points along the Wilson path are going in a descending order from left to right (Fig 2.2). In parametrizing this path from x to y , $\alpha(x, y)$ gets treated like a vector function with integration over the path parameters. The path for the different cases of Wilson lines can be parameterized as

$$\mathbf{z}^{\mu} = \mathbf{b}^{\mu} + \lambda\mathbf{n}^{\mu}$$

$\mathbf{b}^\mu$ is the end point with $\lambda = [-\infty, \infty]$ for fully infinite and $\lambda = [-\infty, 0]$ for semi-infinite lines.
For a finite line one of the points can be set as zero and and the entire path parameterized with $\alpha = [0, 1]$

$$\mathbf{z}^\mu = \alpha z \mathbf{n}^\mu$$

The general expression of a Wilson line, with path ordering, is therefore given as

$$U(x, y) = \mathcal{P} e^{ig \int dz.A} \tag{2.17}$$

Just like the addition of $g\not{A}$ in the ordinary derivative in a Dirac Lagrangian introduces an interaction term so here the Wilson line introduces a locally gauge invariant interaction term of two spinor fields at non-infinitesimally separate space time points.

$$n.D\psi = \frac{\psi(y) - \mathcal{P} e^{ig \int dz.A} \psi(x)}{y - x} \tag{2.18}$$

up to NLO in the Dirac Lagrangian

$$\mathcal{L} = \bar{\psi} n.\partial\psi - \frac{ig\bar{\psi}(y) \int dz.A\psi(x)}{y - x} - m\bar{\psi}\psi \tag{2.19}$$

2.3.1 Wilson Line Feynman Rules

The Wilson line integral can be evaluated explicitly and the result succinctly be expressed in a set of Feynman rules.

$$U = \mathcal{P} e^{ig \int dz.A} = \sum_{0}^{\infty} \frac{(ig)^n}{n!} \mathcal{P} \int_c dz_n.A_n(z) \tag{2.20}$$

$$U = 1 + ig \int dz.A + ...$$

$$A_n^\mu(z) = \int d^4 k_n \mathcal{A}(k_n) e^{-i \sum_n k_n.z_n}$$

with $z^\mu = b^\mu + \lambda n^\mu$ thus breaking up the exponential into two factors gives two integrals ; one with the gauge field and other of the parameter integration hence

$$\mathcal{U} = \sum_{n=0}^{\infty} \frac{(-ig)^n}{n!} \prod_{j=1}^{n} \int d^4 k_j \mathcal{A}(k_j) e^{-i \sum_{l=0}^{j} b.k_l} I_n(\lambda) \tag{2.21}$$

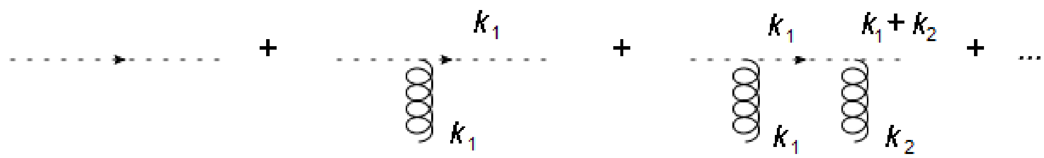

Figure 2.2: *The dotted line denotes the Wilson line. This figure shows the expansion of the path ordered exponential. Each nth term has n-1 gluons attached to it because the leading term is just an identity. All of these semi-infinite lines start from $-\infty$ and end at some point b^μ. The line from negative infinity carries zero momentum.*

which is

$$\mathcal{U} = \sum_{n=0}^{\infty} \mathcal{U}_n$$

where some of the terms in the sum are

$$\mathcal{U}_0 = 1$$

$$\mathcal{U}_1 = -ig \int d^4k_1 n.A(k_1) e^{-ia.k_1} I_1$$

$$\mathcal{U}_2 = \frac{(-ig)^2}{2!} \int d^4k_1 d^4k_2 n.A(k_1) n.A(k_2) I_2$$

The general expression of I_n is given below and explicitly solving individual I_i's as in the following section, gives the general result of I_n.

$$I_n(\lambda) = \prod_{j=1}^{n} \int_{-\infty}^{\lambda_{j+1}} d\lambda_j e^{-in.\sum_{j=1}^{n} \lambda_j k_j} \tag{2.22}$$

The integration limits of λ determine what case of Wilson line it is. Generally the semi-infinite case is obtained first and then using the Wilson line properties like transitivity and unitarity, other cases of Wilson lines can be derived.

2.3.2 Semi-Infinite Wilson Line

Evaluating the parameter integrals :

$$I_1 = \int_{-\infty}^{0} d\lambda_1 e^{-in.k_1\lambda_1} = \frac{i}{n.k_1 + i\epsilon}$$

In the explicit form of the above integrand the result should be undefined due to the infinite limit which gives $e^{-in.k_1(-\infty)}$. This issue is avoided by considering k to be complex ie $k_1 \to k_1 + i\epsilon$ which makes the exponential with infinity to vanish. This is also why there is an $i\epsilon$ in the denominator.

$$I_2 = \int_{-\infty}^{0} d\lambda_2 e^{-in.k_2\lambda_2} \int_{-\infty}^{\lambda_2} d\lambda_1 e^{-in.k_1\lambda_1} = \frac{i^2}{(n.k_1)(n.k_2 + n.k_1) + i\epsilon}$$

hence

$$I_n = \frac{i^n}{\prod_{i=1}^{n}\left(n.\sum_{l=1}^{i} k_l\right) + i\epsilon} \tag{2.23}$$

Putting together the two integrals gives the solved semi-infinite Wilson line

$$\mathcal{U}(b,-\infty) = \sum_{n=0}^{\infty} \frac{(-ig)^n}{n!} \prod_{j=1}^{n} \int d^4k_j \mathcal{A}(k_j) e^{-i\sum_{l=0}^{j} b.k_l} \frac{i^n}{\prod_{i=1}^{n}\left(n.\sum_{l=1}^{i} k_l\right) + i\epsilon} \tag{2.24}$$

which can be constructed from these rules :

- propagator : $\frac{i}{n.k+i\epsilon}$
- Wilson line-gluon vertex factor : $-ign^{\mu}T^a_{ij}$
  Note that T^a_{ij} is implicit in $\mathcal{A}^{\mu}$
- end point : $e^{-ib.k}$

2.3.3 Reversed Semi-Infinite Wilson Line

This path goes from a finite point b^{μ} to an infinite point with the parametrization $\lambda = [0, \infty]$ but the path ordering remains unaffected. The only change is in the parameter integral which has its limits flipped :

$$I_n(\lambda) = \prod_{j=1}^{n} \int_{\lambda_{j-1}}^{\infty} d\lambda_j e^{-in.\prod_{j=1}^{n}\lambda_j k_j} = \frac{(-i)^n}{\prod_{i=1}^{n}\left(n.\sum_{l=1}^{i} k_l\right) - i\epsilon} \tag{2.25}$$

For this to work the promotion of k_i to complex has to be done with the opposite sign as the infinity limit has opposite sign too.

$$\mathcal{U}(\infty, a) = \sum_{n=0}^{\infty} \frac{(-ig)^n}{n!} \prod_{j=1}^{n} \int d^4k_j \mathcal{A}(k_j) e^{-i\sum_{l=0}^{j} b.k_l} \frac{(-i)^n}{\prod_{i=1}^{n} \left(n. \sum_{l=1}^{i} k_l\right) - i\epsilon} \tag{2.26}$$

In its Feynman rules the only thing different is the propagator : $\frac{-i}{n.k-i\epsilon}$

2.3.4 Hermitian Conjugated Semi-Infinite Wilson Line

$$\mathcal{U}^\dagger(b, -\infty) = \sum_{n=0}^{\infty} \frac{(ig)^n}{n!} \prod_{j=1}^{n} \int d^4k_j \mathcal{A}^\dagger(k_j) e^{i\sum_{l=0}^{j} b.k_l} \frac{(-i)^n}{\prod_{i=1}^{n} \left(n. \sum_{l=1}^{i} k_l\right) - i\epsilon} \tag{2.27}$$

The † has done two things; it has flipped the imaginary signs and it has reversed the ordering of the gauge fields, which have also been hermitian conjugated.

The reversal of the order of summation of j is anti-path ordering which is evident from the earlier definition of Wilson lines : $\mathcal{U}^\dagger = \bar{\mathcal{P}} e^{ig\int dz.A^\dagger}$

Furthermore $\mathcal{A}^\dagger(k) = \mathcal{A}(-k)$ and substituting k with $-k$ gives

$$\mathcal{U}^\dagger(b, -\infty) = \sum_{n=0}^{\infty} \frac{(ig)^n}{n!} \prod_{j=n}^{1} \int d^4k_j \mathcal{A}(k_j) e^{-i\sum_{l=0}^{j} b.k_l} \frac{i^n}{\prod_{i=1}^{n} \left(n. \sum_{l=1}^{i} k_l\right) + i\epsilon} \tag{2.28}$$

Hence the Feynman rules to Hermitian conjugate a semi-infinite line are

- propagator: $\frac{i}{n.k+i\epsilon}$
- Wilson line-gluon vertex factor : $ign^\mu T^a_{ij}$
- end point : $e^{-ib.k}$

Hence we see that the action of Hermitian conjugation reverses the path ordering and flips the sign of the vertex factor. The reversal of path ordering is equivalent to switching the sign and limits of infinity in the parameter integral hence

$$\mathcal{U}^\dagger(b, -\infty) = \mathcal{U}(\infty, b)$$

2.3.5 Finite Wilson Lines

For a finite path the paramterization is similar to the previous cases but with the limits restricted to $\lambda = [0, |b-a|]$ hence

$$\mathcal{U}(b,a) = \sum_{n=0}^{\infty} \frac{(ig)^n}{n!} \prod_{i=1}^{n} \int d^4k_i n.A(k_i) e^{-ia.\sum_{l=1}^{n} k_l} I_n \tag{2.29}$$

where

$$I_n = \prod_{i=1}^{n} \int_0^{|b-a|} \int_0^{\lambda_{i+1}} d\lambda_i e^{-in.\sum_{l=1}^{n} k_l \lambda_l} \tag{2.30}$$

A finite line can also be made by joining two semi-infinite lines by the transitive property.

$$\mathcal{U}(b,a) = \mathcal{U}^{\dagger}(\infty, b)\mathcal{U}(\infty, a) \tag{2.31}$$

From the previous two subsections, these semi-infinite Wilson lines are

$$\begin{aligned} \mathcal{U}^{\dagger}(\infty, b) = \sum_{m=0}^{\infty} \frac{(ig)^m}{(2\pi)^4 m!} \prod_{i=1}^{m} \int d^4k_i n.A^{\dagger}(k_i) e^{i\sum_{l=1}^{m} b.k_l} \\ \times \frac{(-i)^m e^{-in.\sum_{j=1}^{m} k_j}}{\prod_{j=1}^{m} n.\sum_{l=1}^{m} k_l - i\epsilon} I_m^{\dagger} \end{aligned} \tag{2.32}$$

$$\begin{aligned} \mathcal{U}(\infty, a) = \sum_{n=0}^{\infty} \frac{(-ig)^n}{(2\pi)^4 n!} \prod_{r=1}^{n} \int d^4k_r n.A(k_r) e^{-i\sum_{l=1}^{n} a.k_l} \\ \times \frac{(i)^n e^{in.\sum k_j}}{\prod_{j=1}^{n} n.\sum_{l=1}^{n} k_l + i\epsilon} I_n \end{aligned} \tag{2.33}$$

and

$$I_n = \frac{(i)^n e^{-in.\sum_{j=1}^{n} k_j}}{\prod_{j=1}^{n} n.\sum_{l=1}^{n} k_l + i\epsilon} \tag{2.34}$$

$$I_m^{\dagger} = \frac{(-i)^m e^{in.\sum_{j=1}^{m} k_j}}{\prod_{j=1}^{m} n.\sum_{l=1}^{m} k_l - i\epsilon} \tag{2.35}$$

2.3.6 Eikonal Approximation

If an on-shell quark is interacting with a gluon soft enough to result in a negligible change in momentum, then the Dirac propagator and the QCD

quark-gluon vertex can be replaced by those of a Wilson line [9]. This approximation is known as Eikonal approximation. Such a thing becomes useful later when we discuss QCD loop corrections to heavy quarks hadronic systems.
Here we discuss the case of an on-shell quark interacting with a single soft virtual gluon. The part of the amplitude of interest is

$$\mathcal{M} \sim ... \frac{\not{p} + \not{q} + M}{(p+q)^2 - M^2}(gT^a\gamma^\mu)u(p) \tag{2.36}$$

treating the gluon momentum q as soft gives

$$\mathcal{M} \sim ... \frac{\not{p} + M}{2p.q}(gT^a\gamma^\mu)u(p) \tag{2.37}$$

rewriting the mass term using the Dirac equation and expanding out $\not{p}$ gives

$$\mathcal{M} \sim ... \frac{p_\nu\gamma^\nu\gamma^\mu u(p) + p_\nu\gamma^\mu\gamma^\nu u(p)}{2p.q}(gT^a) \tag{2.38}$$

The anti-commutator in the numerator being a Minkowsky metric tensor, allows the the quark‘s four momentum to be cast in the light cone basis giving

$$\mathcal{M} \sim ... \frac{n^\mu u(p)}{2n.q}(gT^a) \tag{2.39}$$

This is basically a spinor with the propagator and vertex factor of a Wilson line and the entire amplitude can be rewritten as a NLO Wilson line being multiplied with the spinor. For the quark emitting/absorbing more gluons, one can keep multiplying in Wilson line propagators and vertex factors to give a spinor being multiplied with n^{th} order Wilson line. Such a situation expressed as an order by order expansion is given as

$$\mathcal{M} \sim ... U_{[0,-\infty]}u(p) \tag{2.40}$$

Chapter 3

Heavy Quark Effective Theory

In this chapter we discuss the motivation behind Heavy Quark Effective Theory and give its Lagrangian and Feynman rules, which we then use to solve the heavy quark self energy loop. We use the HQET formalism to define the B meson LCDA, projection operator and phenomenological parameters.

Quarks and gluons exist in bound states called Hadrons, due to the non-perturbative nature of the QCD effective coupling. Hence the physics of hadrons is influenced by the mass and momentum scales of the fundamental particles involved. The typical length scale of hadrons is $R_h \sim \mathcal{O}(1/\Lambda_{QCD})$, about the order of femtometer at which the order of magnitude of the exchange momenta is $\mathcal{O}(\Lambda_{QCD})$ [19] where $\Lambda_{QCD} = 0.2$ GeV. At this scale the strong force becomes non-perturbative as the effective QCD coupling diverges and the effect is seen as quarks and gluons binding together as a hadronic system.

While the interactions between gluons and all flavours of quarks can be described by QCD but for heavy-light mesons HQET is a suitable approximation. This works because of the mass scale of one of the composing quarks being significantly greater than that of the other quark and the exchange momentum holding the meson together. This is aptly applicable to quarks like b, c, and t which are generally classified as being heavy due to their mass scale being significantly greater than Λ_{QCD} [19,20]. This effective theory is derived from the large quark mass limit of QCD which gives it some interesting symmetries. The large mass limit is the extreme end or leading order of this approximation ($M \to \infty$) and at this order all terms with the heavy quark mass in the denominator drop off. Hence the interacting quarks are insensitive to their flavour and spin effects. Only when moving away from this approximation (order by order in powers of $1/M$) do these effects become apparent. In this text only the heavy quark limit ($M \to \infty$) is considered when constructing the kinematics and Lagrangian.

3.1 HQET Lagrangian

The intuitive picture of HQET is that of a very heavy quark of total momentum $p^\mu = Mv^\mu + k^\mu$ acting as a static source of charge with lighter particles interacting with it. The soft momentum k^μ is insufficient to change the velocity of the heavy quark, which remains a conserved quantity [19,20]. In keeping with this picture it is useful to adopt the heavy quark rest frame with conserved velocity as $v^\mu = (1, \vec{0})$
Starting off with the quark-gluon sector of the QCD Lagrangian

$$\mathcal{L}_{QCD} = \bar{\Psi}(i\not{D} - m)\Psi$$

where $\Psi = e^{-iMv.x}(\psi_v + \chi_v)$ with the ψ_v and χ_v being the spinor and anti-spinor components. The exponential serves as a momentum translation operator to subtract off the original Mv^μ momentum of the heavy quark. From the Dirac equation for a heavy quark

$$(M\not{v} + \not{k})\Psi = M\Psi$$

it follows that

$$\left(1 - \not{v} - \frac{\not{k}}{M}\right)\Psi = 0 \tag{3.1}$$

To satisfy this equation of Ψ the components have to satisfy $\left(\frac{1+\not{v}+\frac{\not{k}}{M}}{2}\right)\Psi = \psi_v$ and $\left(\frac{1-\not{v}-\frac{\not{k}}{M}}{2}\right)\Psi = \chi_v$ where $\not{k}/M$ is small enough to be ignored. Furthermore, in the heavy quark rest frame $\not{v} = \gamma^0$ which implies that $1 \pm \not{v}$ acts as a projection operator on the heavy quark field to project out spinor and anti-spinor components. Due to the small $\not{k}/M$ approximation the Dirac equation is $(1 - \not{v})\Psi \approx 0$ and hence
$\psi_v \approx \left(\frac{1+\not{v}}{2}\right)\Psi$ and $\chi_v \approx 0$
This implies that at leading order the anti-spinors of heavy quarks are suppressed which in turn means that heavy quark vacuum polarization loops are also suppressed. Hence there is no contribution to the running coupling in

the heavy quark limit [19,20].
Returning to the QCD Lagrangian, $\Psi = e^{-iMv.x}\psi_v$ gets substituted in to give

$$\mathcal{L}_{HQET} = e^{iMv.x}\bar{\psi}_v(i\not{D} - M)e^{-iMv.x}\psi_v \tag{3.2}$$

where

$$\not{D} = \not{v}v.D + \not{D}_\perp \tag{3.3}$$

$\not{D}$ is a sum of components of the derivative parallel and perpendicular to the direction of v^μ and the parallel term makes use of $v^2 = 1$ to get the following

$$\not{v}v.D = \gamma_\mu v^\mu v_\alpha D^\alpha = \gamma_\mu v^\alpha v_\alpha D^\mu = \not{D} \tag{3.4}$$

$$\mathcal{L}_{HQET} = e^{iMv.x}\bar{\psi}_v \left(i\not{v}v.(\partial + igA) + i\not{D}_\perp - M\right) e^{-iMv.x}\psi_v \tag{3.5}$$

$$\mathcal{L}_{HQET} = \bar{\psi}_v(-(1-\not{v})M + iv.(\partial + igA))\psi_v$$

as $(\not{v} - 1)\psi_v = 0$ so

$$\mathcal{L}_{HQET} = \bar{\psi}_v(iv.D)\psi_v + \mathcal{O}(1/M) \tag{3.6}$$

At leading order this Lagrangian does not have any gamma matrices so it has an SU(2) spin symmetry. The absence of a mass term in it implies heavy flavour symmetry as there is no way to differentiate between the quarks of different flavours. The derivation of this term hinges on the assumption that $M >> \Lambda_{QCD}$ [19,20].

3.1.1 Reparameterization Invariance

Considering the following decomposition of the heavy quark momentum

$$p^\mu = Mv^\mu + k^\mu$$

The derivation of HQET hinges on the assumption that the residual momentum k^μ must be $\mathcal{O}(\Lambda_{QCD})$ and be much smaller than the heavy quark mass. These conditions allow for the expansion in orders of $\mathcal{O}(k/M)$. This expansion in orders of $\mathcal{O}(k/M)$ does not necessarily have to be in the heavy quark momentum but can also be taken in its velocity

$$v \to v + \varepsilon/M \tag{3.7}$$

where $\varepsilon \sim \mathcal{O}(\Lambda_{QCD})$ and to preserve the same parameterization of p^μ, the residual momentum should transform similarly

$$k \to k - \varepsilon \tag{3.8}$$

Upon dropping higher order terms and using $v^2 = 1$ and $v.\varepsilon = 0$ and transforming the heavy quark field as $\psi_h^v \to \psi_h^v + \delta\psi_h^v$ to conserve $\not{v}\psi_h^v = \psi_h^v$ On ends up with the result

$$(1 - \not{v})\delta\psi_h^v = \frac{\not{\varepsilon}}{M}\psi_h^v \tag{3.9}$$

at leading order in (ε/M). Upon operating with $\not{v}$, one ends up with $\not{v}\delta\psi_h^v = -\delta\psi_h^v$. Hence the appropriate choice for the change in h_v is

$$\delta h_v = \frac{\not{\varepsilon}}{2m_Q}h_v \tag{3.10}$$

For the HQET Lagrangian to be reparameterization invariant, it should be conserved under the following combined transformations

$$\begin{aligned} v &\to v + \varepsilon/M \\ \psi_h^v &\to e^{i\varepsilon\cdot x}\left(1 + \frac{\not{\varepsilon}}{2M}\right)\psi_h^v \end{aligned} \tag{3.11}$$

where $e^{i\varepsilon\cdot x}$ is the shift in residual momentum. This results in the following altered Lagrangians under the velocity shift and heavy quark field shift, respectively

$$\mathcal{L}_0 \to \mathcal{L}_0 + \frac{1}{M}\overline{\psi}_h^v(i\varepsilon\cdot D)\psi_h^v \tag{3.12}$$

$$\mathcal{L}_1 \to \mathcal{L}_1 - \frac{1}{M}\overline{\psi}_h^v(i\varepsilon\cdot D)\psi_h^v \tag{3.13}$$

3.2 HQET Feynman Rules

$$\mathcal{L}_{HQET} = i\bar{\psi}_v v.\partial\psi_v + ig\bar{\psi}_v v.A\psi_v \tag{3.14}$$

The Feynman rules can be read off from this Lagrangian
Heavy quark-gluon vertex factor : $-igT^a v^\mu$
Heavy quark propagator : $\frac{1}{v.p}\delta_{ij}$

They can also be derived in an alternate fashion by taking the heavy quark limit of QCD Feynman rules.

$$\frac{\not{p} + M}{p^2 - M^2}\delta_{ij}$$

Where $p^\mu = Mv^\mu + k^\mu$. Substituting this in and dividing through by M

$$\frac{\not{v} + 1 + k/M}{2v.k + k^2/M}\delta_{ij}$$

$$\frac{\not{v} + 1}{2v.k}\delta_{ij} + \mathcal{O}(k/M)$$

In the denominator, the momentum k being dotted with v is what is called the residual momentum and is soft and responsible for taking the heavy quark off-shell [21]. For heavy anti-quarks, the numerator will have $\not{v} - 1$ due to the anti-spinor completeness relation giving $\not{p} - M$.
The Lagrangian and Feynman rules of the light quark and gluon sectors remain unchanged.

3.3 Heavy Quark Self Energy

Just like the fermion self energy diagram, the one involving heavy quarks also has a UV divergence. Applying the Feynman rules on this section of the diagram in Fig 3.1(a) and using dimensional regularisation

$$\Sigma_Q = -C_F g^2 \mu^{2\varepsilon} \int d^D l \frac{v^\mu g_{\mu\nu} v^\nu}{[l^2][v.(l+p)]} \tag{3.15}$$

The loop correction is on the heavy quark propagator and hence p is also residual momentum [20]. As the denominator has a quadratic and a linear term of loop momentum so it is more convenient to use this parametrisation from [20]

$$\frac{1}{AB} = \int_0^\infty dx \frac{1}{(A + xB)^2} \tag{3.16}$$

where $A = l^2$ and $B = v.(l + p)$

$$\Sigma_Q = -C_F g^2 \mu^{2\varepsilon} \int_0^\infty dx \int d^D l \frac{1}{(l^2 + xv.(l+p))^2} \tag{3.17}$$

Using

$$\int d^D l \frac{1}{[l^2 + 2l.Q - R^2]^n} = \frac{(-1)^n i\pi^{D/2}\Gamma(n - D/2)}{\Gamma(n)[Q^2 + R^2]^{n-D/2}} \tag{3.18}$$

from [22] for integrating loop momenta where $Q^\mu = xv^\mu/4$ (different from the subscript Q of Σ) and $R^2 = -xv.p$

$$\Sigma_Q = -C_F g^2 \mu^{2\varepsilon} \int_0^\infty dx \frac{i\pi^{2-\varepsilon}\Gamma(\varepsilon)}{\left(\frac{x^2}{4} - xv.p\right)^\varepsilon} \tag{3.19}$$

$$\Sigma_Q = -iC_F g^2 \pi^{1.5-\varepsilon} \mu^{2\epsilon} (-v.p)^{1-2\varepsilon} \Gamma(1-\varepsilon)\Gamma(\varepsilon - 1/2)\Gamma(\varepsilon)$$

expanding out in powers of ε

$$\Sigma_Q = -2iC_F g^2 \pi^{2-\varepsilon}(p.v) \left[2 \ln\left(\frac{\mu}{p.v}\right) + \Gamma[0, -1/2]\right] + \mathcal{O}(\varepsilon) \tag{3.20}$$

Here the UV divergence is in the $1/\varepsilon$ term and has to be subtracted off by a counter-term diagram.
As a further note, the self energy diagram can also be made with loop momentum directed opposite which will mean that the propagator denominator will be $[l^2][v.(p - l)]$ but this negative sign of l (and consequently of Q^μ) makes no difference when the general loop integration formula is applied, it squares Q^μ hence both conventions of the diagram are equal.

3.4 Renormalization of HQET Propagator

The counter-term is present in the heavy quark bare fields.

$$\psi_0 = \left(1 + \frac{\delta_\psi}{2}\right)\psi = \sqrt{Z_\psi}\psi$$

hence the bare propagator is [1]

$$G_0 = Z_\psi G \tag{3.21}$$

Considering a sum of propagators at tree level and with an increasing number of 1PI loops, as in Fig 3.1(b)

$$iG_0 = \frac{i(1+\not{v})}{v.k} + \frac{i(1+\not{v})}{v.k} i\Sigma_Q \frac{i(1+\not{v})}{v.k} + ... \tag{3.22}$$

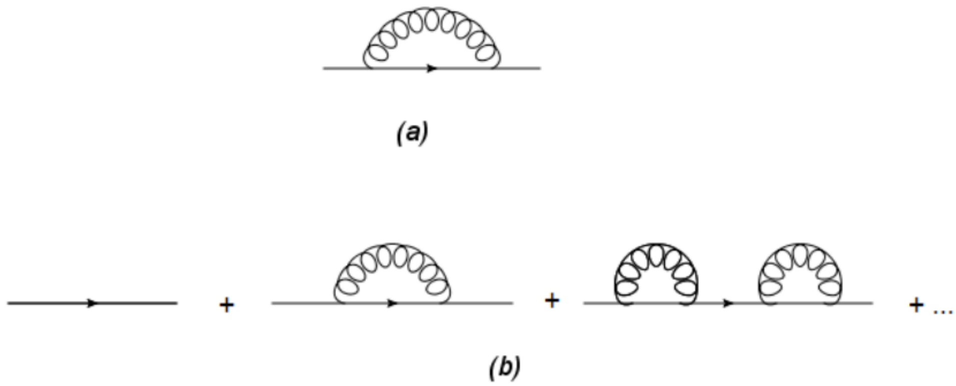

Figure 3.1: (*a*) *a single heavy quark propagator with one loop self energy correction.* (*b*) *summed heavy quark propagators including tree level and up-to N 1PI self energy loop corrections.*

Taking the first term common and expressing the rest as a geometric series

$$iG_0 = \frac{i(1+\not{v})}{v.k}\left[\frac{1}{1+\frac{(1+\not{v})\Sigma_Q}{v.k}}\right] \tag{3.23}$$

The renormalized propagator will be

$$iG^R = \frac{1}{1+\delta_\psi}\frac{i(1+\not{v})}{v.k+(1+\not{v})\Sigma_Q} \tag{3.24}$$

Expanding out the denominator gives

$$iG^R = \frac{i(1+\not{v})}{v.k+(1+\not{v})\Sigma_Q+\delta_\psi v.k+\mathcal{O}(1/\varepsilon^2)} \tag{3.25}$$

only the terms with $1/\varepsilon$ type of divergence have been retained in forming the counter-term because Σ_Q has divergence to the first order. From here a divergence-less self energy loop is $\Sigma_Q^R = \Sigma_Q + \frac{v.k}{1+\not{v}}\delta_Q$

3.5 HQET Factorization for Radiative B Decay

To focus on the essential features of the discussion we shall restrict ourselves to a simpler process as given in [7]

$$B \to \gamma l \bar{\nu}_l$$

The part which is to be factorised consists of only the hard photon and weak boson as the final states. In [7] its factorisation is done at tree level and up to first order gluon loop corrections in QCD. Due to the absence of any mesons in the final state, the HQET factorisation formula for this only consists of the second convolution integral.

$$\mathcal{M} = \int_0^\infty dk T^{II}(k, \mu_F)\Phi(k, \mu_F) + \mathcal{O}(1/m_b) \tag{3.26}$$

Where k is the light quark momentum and μ_F is the factorization scale separating the long and short distance physics. Thence it can be expressed as an order by order expression.

$$\mathcal{M}^{(0)} + \mathcal{M}^{(1)} + ... = \Phi^{(0)} \otimes T^{(0)} + \Phi^{(1)} \otimes T^{(0)} + \Phi^{(0)} \otimes T^{(1)} + ... \tag{3.27}$$

From here it can be proven that factorisation holds up to one loop order and that it does not require dependance on any transverse components.

3.5.1 B Meson Light-Cone Distribution Amplitude in HQET

For the two quarks in the B meson separated by light-like distance, in the rest frame of the B meson, the most general parametrization of the B meson LCDA which satisfies Lorentz and gauge symmetries, can be written in term of two independent variables: $\widetilde{\phi}_B^{\pm}$ as

$$\langle B(v)| \overline{u}_\beta(z)[z,0]h_{v,\alpha}(0) |0\rangle = \frac{i\hat{f}_B m_B}{4} \left\{ \left[2\widetilde{\phi}_B^+(t) - \frac{\not{z}}{t}\Big(\widetilde{\phi}_B^-(t) - \widetilde{\phi}_B^+(t)\Big) \right] P_L \right\}_{\alpha\beta} \tag{3.28}$$

where $z^2 = 0$ and $t = v \cdot z$ and $\widetilde{\phi}_B^{\pm}$ both are non-perturbative functions of t whereas α and β represent the spinor indices. m_B denotes the mass of B meson, [z,0] is the Wilson line for bi-local gauge invariance, u is the light quark field, defined in QCD. h_v is the field of a b quark moving with velocity v, defined in HQET. $\hat{f}_B$ is the B meson decay constant defined in HQET, upto one loop order.

$$\hat{f}_B(\mu_F) = \left[1 + \frac{\alpha_s C_F}{4\pi}\left(3\ln\frac{\mu_F}{m_b} + 2\right)\right] f_B \tag{3.29}$$

The parameterizing functions of the LCDA are defined in [8] as

$$\hat{\Phi}_B^{\pm} = i m_B \hat{f}_B \hat{\phi}_{\alpha\beta}^B = \frac{1}{v^{\pm}} \int dt e^{i\omega v\cdot z} \langle B(v)|\overline{u}(z)[z,0]\not{n}_{\mp}\gamma^5 h_v(0)|0\rangle\Big|_{z^+,z^\perp=0} \tag{3.30}$$

3.5.2 B Meson Momentum Space Projector

To derive the B meson momentum space operator, we start with (3.28). The pre-factor is chosen such that for $z = 0$ and $\tilde{\phi}_+(t) = \tilde{\phi}_-(t) = 0$

$$\langle B(p)|\, \bar{q}_\beta[\gamma^\mu\gamma^5]_{\beta\alpha} b_\alpha\,|0\rangle = -i f_B m_B \tag{3.31}$$

Then using the identity

$$\begin{aligned}\int d^4z M(z)T(z) &= \int \frac{d^4k}{(2\pi)^4} T(k) \int d^4z e^{-ikz} M(z) \\ &\equiv \int_0^\infty dk^+ M^B T(k)\Big|_{k=k\cdot n_-}\end{aligned} \tag{3.32}$$

where $M(z)$ is the position space projector and $T(z)$ the hard scattering amplitude in position space. To extract from here the B meson position space operator, we first decompose k momentum into light cone components. The factors $\not{z}$ and $1/(v.z)$ are removed by taking the derivative and then the partial integral of the hard scattering amplitude. This gives

$$\begin{aligned}\int d^4z M(z)T(z) = \frac{i\hat{f}_B m_B}{4}\Bigg\{\frac{1-\not{v}}{2}\int_0^\infty d\omega\Bigg[2\phi_B^+(\omega) \\ + \int_0^\omega d\eta \left(\phi_B^-(\eta) - \phi_B^+(\eta)\right)\gamma_\mu \frac{\partial}{\partial k^\mu}\Bigg]\gamma_5\Bigg\}_{\alpha\beta} T_{\beta\alpha}(k)\Bigg|_{k=\omega v}\end{aligned} \tag{3.33}$$

k_- component does not contribute to the momentum space hard scattering amplitude at leading order in the heavy quark limit and hence $T(k)$ can be written as

$$T(k) = T^{(0)}\left(k^+\right) + k_\perp^\mu T_\mu^{(1)}\left(k^+\right) \tag{3.34}$$

Decomposing $\partial/\partial k^\mu$ in terms of light-cone components with the derivative with respect to k^- not contributing, we have

$$\mathcal{M}^B_{\beta\alpha} = \frac{i\hat{f}_B m_B}{4}\left\{\frac{1-\not{v}}{2}\left[\phi_B^+(\omega)\not{n}^+ + \phi_B^-(\omega)\not{n}^- - \int_0^\omega d\eta\left(\phi_B^-(\eta)-\phi_B^+(\eta)\right)\gamma_\mu\frac{\partial}{\partial k_\perp^\mu}\right]\gamma_5\right\}_{\alpha\beta} \tag{3.35}$$

3.5.3 B Meson Phenomenological Parameters

Vector and Axial vector form factors associated with processes involving the B meson, have to be expressed in terms of a class of integrals called inverse moments. These integrals are unique to the meson in question. For the purposes of this text, we shall be dealing with the first inverse moment

$$\frac{1}{\lambda_B^{(0)}} = \int_0^\infty d\tilde{k}_+ \frac{\Phi_+(\tilde{k}_+)}{\sqrt{2}\tilde{k}_+} \tag{3.36}$$

and the logarithmic inverse moments ($n = 1, 2$ for the process in this text)

$$\frac{1}{\lambda_B^{(n)}} = \int_0^\infty d\tilde{k}_+ \frac{\Phi_+(\tilde{k}_+)}{\sqrt{2}\tilde{k}_+}\ln^{(n)}\left(\frac{\sqrt{2}m_b\tilde{k}_+}{\mu_F^2}\right) \tag{3.37}$$

Here Φ_+ is the μ_F scale dependant leading contributor to the full LCDA and its evolution equation is defined by the Lange-Neubert equation [7,8]. The solution has a number of different forms based upon the type of model being considered. In [7] one of the models is of this form :

$$\Phi_+(\tilde{k}_+) = \frac{2\tilde{k}_+}{\left(\lambda_B^{(0)}\right)^2} e^{-\frac{\sqrt{2}\tilde{k}_+}{\lambda_B^{(0)}}} \tag{3.38}$$

The moments are dependant upon the scale μ_F which occurs in both the logarithm and in Φ_+. The solution proceeds by first defining Φ_+ and $\lambda^{(0)}$ at the scale $\mu_F = 1$ GeV where $\lambda_B^{(0)} = 0.35 \pm 0.15$ GeV [7] and then getting the above solution for any other μ_F. The values of $\lambda^{(0)}$ at $\mu_F = 1$ GeV can be different for other types of mesons and LCDA models.

Chapter 4

Factorization of the Amplitude $W^+ \to B\gamma$

In this chapter, we compute the Feynman amplitudes of the $W^+ \to \gamma B$ process and the LCDAs of the B meson, both at tree level and one loop level. Then using the factorisation theorem at leading and next to leading orders we can extract the Hard Kernels to be used in calculating the vector and axial vector form factors.

In this thesis the radiative production of a B meson is studied in the process

$$W^+ \to B\gamma$$

as described in [8]. While the initial and final states are different as compared to the process of [7], the momenta scaling of the various particles is the same which gives the calculations quite similar features. The amplitudes are computed in the full theory of QCD while the LCDA's can be computed in either HQET or QCD, resulting in a different Hard Kernel in each case. However, this will result in the convolutions being the same in both cases [7].

There are two types of tree level amplitudes contributing to the leading order of this process. From each of these tree level amplitudes arise six different QCD one loop amplitudes and all of these have their corresponding LCDA's and Hard Kernels.
To prove factorisation the procedure that we will follow is thus; essentially a Feynman amplitude is split between the part comprised by its partons (which are joined by a Wilson line) and the rest of the diagram and the two parts

are then convoluted together. At tree level it is

$$\mathcal{M}^{(0)} = \Phi^{(0)} \otimes T^0 \tag{4.1}$$

and at one loop level

$$\mathcal{M}^{(1)} = \Phi^{(1)} \otimes T^{0)} + \Phi^{(0)} \otimes T^{(1)} \tag{4.2}$$

This is because when the amplitude has a loop correction, that loop can equally well be a part of the LCDA or the Hard Kernel. Hence one term has the LCDA with the loop and Hard Kernel at tree level and vice versa for the second term.

4.1 Kinematics

The situation is of a very massive W boson decaying to a B meson and a photon. The meson and photon move off in opposite directions in keeping with the conservation of momentum.

4.1.1 In the rest frame of the W boson

One of the possible ways of viewing this process is in the rest frame of the W boson, in Light-Cone Factorization (LCF). With respect to the stationary W, the B meson and photon move off in opposite directions.
The four momenta of the particles are

$$P_W^\mu = (M_W, \vec{0}) \tag{4.3}$$

$$P_B^\mu = (E_B, \vec{P}_B) \tag{4.4}$$

$$q^\mu = (E_\gamma, -\vec{q}) \tag{4.5}$$

To express all these energies and three-momenta in terms of known masses we make use of the on-shell condition and the principle of energy conservation. $M_W = E_B + E_\gamma$. Using

$$q^2 = 0 = (P_W^\mu - P_B^\mu)^2 \tag{4.6}$$

we get

$$E_B = \frac{M_W^2 + M_B^2}{2M_W} \tag{4.7}$$

and

$$E_\gamma = \frac{M_W^2 - M_B^2}{2M_W} \tag{4.8}$$

Then using the energy-mass-momentum relation

$$\vec{P}_B = \frac{M_W^2 - M_B^2}{2M_W}\hat{z} \tag{4.9}$$

and from the fact that $\vec{P}_W = 0 = \vec{P}_B + \vec{q}$

$$\vec{q} = \frac{M_B^2 - M_W^2}{2M_W}\hat{z} \tag{4.10}$$

So now we have in Minkowsky coordinates

$$P_W^\mu = (M_W, 0, 0, 0) \tag{4.11}$$

$$P_B^\mu = \left(\frac{M_W^2 + M_B^2}{2M_W}, 0, 0, \frac{M_W^2 - M_B^2}{2M_W}\right) \tag{4.12}$$

and

$$q^\mu = \left(\frac{M_W^2 - M_B^2}{2M_W}, 0, 0, \frac{M_B^2 - M_W^2}{2M_W}\right) \tag{4.13}$$

As the meson is a bound state of two quarks so we use the light cone coordinate system. A vector in the light cone representation is

$$V^\mu = (V_+, V_-, V_\perp)$$

where $V_\pm = \frac{V^0 \pm V^3}{\sqrt{2}}$ and $V_\perp = V^1, V^2$ and a scalar product of two vectors in this representation is [9]

$$V^\mu U_\mu = V_+.U_- + V_-.U_+ + V_\perp.U_\perp \tag{4.14}$$

A vector can also be represented in light cone basis vectors: $n^\mu = \frac{1}{\sqrt{2}}(1, 0, 0, 1)$ and $\bar{n}^\mu = \frac{1}{\sqrt{2}}(1, 0, 0, -1)$ which implies that the contraction of same basis is zero

$$V^\mu = (V.n)\bar{n}^\mu + (V.\bar{n})n^\mu + V_\perp^\mu = V_+^\mu + V_-^\mu + V_\perp^\mu \tag{4.15}$$

Working with this notation, these momenta can be written in Light Cone coordinates as

$$P_W^\mu = \frac{1}{\sqrt{2}}(M_W, M_W, 0) \tag{4.16}$$

$$P_B^\mu = \frac{1}{\sqrt{2}}\left(M_W, \frac{M_B^2}{M_W}, 0\right) \tag{4.17}$$

4.1.2 In the rest frame of B meson

As this process will be studied in HQET which holds in the rest frame of the B meson so here we set up the kinematics of the process in the frame of a stationary B meson with the W boson and photon moving off in opposite directions along $\hat{z}$. The photon's four-momentum is boosted along $\hat{z}$ with respect to the stationary B meson reference frame. Hence its boosted energy is

$$E'_\gamma = \gamma(E_\gamma - \vec{\beta}.\vec{q}_z) \tag{4.18}$$

Where the γ outside the bracket is the Lorentz factor and with the B being at rest $\gamma = 1$ and $\vec{\beta} = 0$ hence the photon's four momentum is frame invariant. The momenta of the three particles in Light Cone coordinates are

$$P^\mu_B = \frac{1}{\sqrt{2}}(M_B, M_B, 0) \tag{4.19}$$

$$q^\mu = \left(0, \frac{M_W^2 - M_B^2}{M_B^2}, 0\right) \tag{4.20}$$

All the amplitude calculations to follow will be done in terms of quark and photon momenta in light cone coordinates, defined in the B meson rest frame. As the B meson is composed of a $\bar{b}$ and u quark so its total momentum will be split between them. The momentum of the $\bar{b}$ quark will be

$$p^\mu_b = P^\mu_B - k^\mu$$

where k^μ is the u quark momentum. As $m_{u,d} \sim 0.001$ GeV the $\mathcal{O}(\Lambda_{QCD})$ exchange momenta make all the components of light quarks of order Λ_{QCD}. Hence they are grouped together with Λ_{QCD} in a single soft scale.
Based upon the kinematical hierarchy

$$m_W \sim m_b >> \Lambda_{QCD}$$

the exchange momenta of $\mathcal{O}(\Lambda_{QCD})$ are not sufficient to accelerate a 5 GeV mass appreciably as differences of $\mathcal{O}(\Lambda_{QCD})$ are ignored.
Quark and photon momenta in light cone coordinates, whilst using the kinematical hierarchy, are :

- $p^\mu \sim (m_b, m_b, 0)$
- $k^\mu \sim (\Lambda_{QCD}, \Lambda_{QCD}, \Lambda_{QCD})$
- $q^\mu \sim (0, m_b, 0)$

The Feynman amplitudes ($\mathcal{M}$) are calculated using the usual Feynman rules of the known Standard Model. The software FeynCalc [23] was utilized to compute the loop Feynman amplitudes. The output takes the form of PV scalar integrals [24] and coefficients. Terms suppressed in the heavy quark limit are dropped from the output, identified by power counting the coefficients which are functions of the various masses and momenta scalar products.
The meson LCDA's require solving the following matrix element [7,8,18] with HQET.

$$\Phi(\tilde{k}_+) = \int dz_- e^{i\tilde{k}_+ z_-} \langle \bar{b}, u| \bar{\psi}_u(z) e^{-ig\int dz.A} \psi_{\bar{b}}(0) |0\rangle |_{z_{+,\perp}=0} \tag{4.21}$$

Both the Wilson line and the S matrix (implicit in the matrix element) can be expanded out to give LCDA's of any arbitrary order of correction. The Fourier integral is only over a single variable because being on the light cone $z_\perp, z_+ = 0$. Then the hard kernel $T(\tilde{k}_+)$ can be extracted by solving the convolution integrals. It is convenient to calculate the hard kernel at the end because the amplitudes and LCDA's can be obtained from Feynman rules and the LCDA's contain Dirac delta functions which simplify the integration of the convolution.

4.2 Tree Level

Of the two tree level diagrams shown for this process in Fig 4.1(a) and Fig 4.1(b), the one with the heavy quark propagator is $\mathcal{O}(1/m_b)$ so it is highly suppressed. Hence at all orders, only the diagrams with the weak current at the heavy quark have the dominant contribution.
Two tree level diagrams can be constructed for this process. Only the one with the light quark propagator (4.22) contributes in the heavy quark limit while the second one with the b quark propagator (4.23) is suppressed and hence will not contribute at tree or loop level.

$$\mathcal{M}_u^{(0)} = -e^2 V_{ub} \bar{u} \not{\epsilon}_\gamma^* \frac{\not{q} + \not{k} + m}{(q+k)^2 - m^2} \not{\epsilon}_W P_L \nu \sim \mathcal{O}(1/\Lambda_{QCD}) \tag{4.22}$$

$$\mathcal{M}_{\bar{b}}^{(0)} = -e^2 V_{ub} \bar{u} \not{\epsilon}_W P_L \frac{\not{p} + \not{q} - \not{k} - m_b}{(p+q-k)^2 - m_b^2} \not{\epsilon}_\gamma^* \nu \sim \mathcal{O}(1/m_b) \tag{4.23}$$

The tree level LCDA in Fig 4.1(c) is given by the matrix element

$$\Phi(\tilde{k}_+) = \int dz_- e^{i\tilde{k}_+ z_-} \langle \bar{b}, u|0\rangle \tag{4.24}$$

and can be constructed from the following Feynman-Wilson rules :

- Spinor or anti-spinor for external fermion.
- e^{-ipx} for each point where any external or internal line joins the Wilson line. Momentum p is being carried to or from the Wilson line. x being the relative distance from that point to some origin on the Wilson line
- Fourier transform it to momentum space by integrating over position space

Hence

$$\Phi(\tilde{k}_+) = \int dz_- e^{i\tilde{k}+z_-} e^{-i(-k_+)z_-} e^{-i(0)} \bar{u}\nu = 2\pi\delta(\tilde{k}_+ + k_+)\bar{u}\nu \tag{4.25}$$

Extracting the tree level hard kernel

$$-e^2 V_{ub} \bar{u} \not{\epsilon}^*_\gamma \frac{\not{q} + \not{k} + m}{(q+k)^2 - m^2} \not{\epsilon}_W P_L \nu = \int d\tilde{k}_+ 2\pi\delta(\tilde{k}_+ + k_+) \bar{u} T^{(0)}(\tilde{k}_+)\nu \tag{4.26}$$

$$T^{(0)}(\tilde{k}_+) = e^2 V_{ub} \frac{\not{\epsilon}^*_\gamma \not{q} \not{\epsilon}_W P_L}{2q_-.\tilde{k}_+} \tag{4.27}$$

The above expression of the tree level hard kernel can be further simplified by invoking the following projector at leading order in α_s

$$\bar{u}(k)\nu(p-k) = \frac{\delta_{ij}}{N} \frac{1-\not{v}}{4} \gamma 5 \tag{4.28}$$

which ensures that the B meson is a spin and colour singlet. So the Hard Kernel becomes [8]

$$T^{(0)}(\omega) = \frac{-iR}{\omega} \left(\frac{\varepsilon_{\mu\nu\alpha\beta} p^\mu q^\nu \epsilon^\alpha_W \epsilon^{*\beta}_\gamma}{p.q} + i\epsilon_W.\epsilon^*_\gamma \right) \tag{4.29}$$

For the validity of the Factorisation Theorem, a necessary condition is that the Hard Kernel should be independent of the external states [7]. An expansion of the Fock state of the meson is given in [25] as follows :

$$|B\rangle = |b, \bar{u}\rangle + |b, g, \bar{u}\rangle + ... \tag{4.30}$$

The Hard Kernel associated with each of these states turns out to be the same [7].

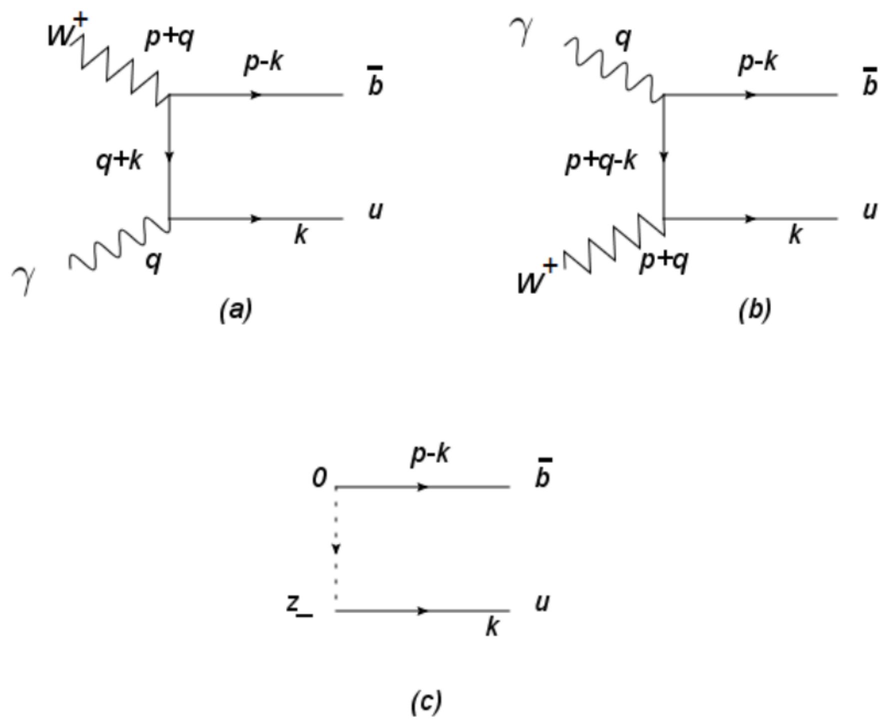

Figure 4.1: (*a*) *Tree level diagram with light quark propagator* (*b*) *Tree level diagram with b quark propagator.* (*c*) *Tree level LCDA for the B meson, common to both.*

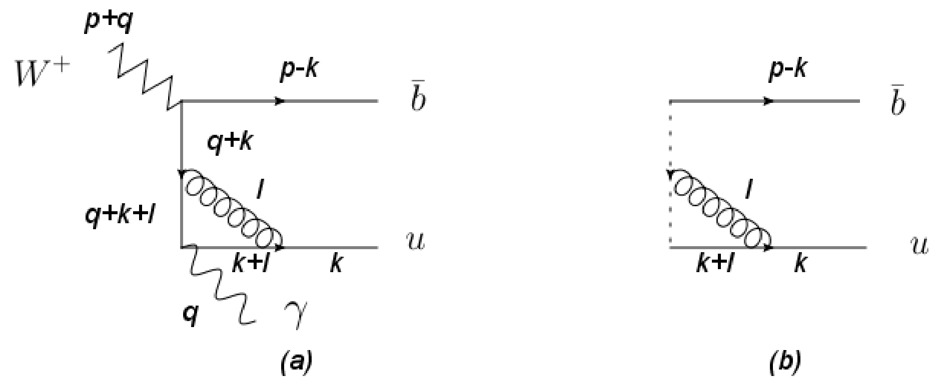

Figure 4.2: *(a) Feynman amplitude and (b) LCDA for the electromagnetic vertex correction. The vertex loop occurs over the electromagnetic current vertex.*

4.3 Electromagnetic Vertex Correction

4.3.1 Feynman Amplitude

$$\mathcal{M}_{em} = \beta_R \int d^D l \times \frac{\bar{u}\gamma^{\phi}(\not{k} + \not{l} + m)\not{\epsilon}^*_{\gamma}(\not{q} + \not{k} + \not{l} + m)\gamma_{\phi}(\not{q} + \not{k} + m)\not{\epsilon}_W P_L \nu}{[(k+l)^2 - m^2][l^2][(q+k+l)^2 - m^2][(q+k)^2 - m^2]} \tag{4.31}$$

where $\beta_{R,F} = -iC_F V_{ub} g^2 e^2 \mu^{2\varepsilon}_{R,F}$. Some of the scalar products required for simplifying this calculation are :
$p^2 = m_b^2$, $k^2 = m^2$, $q^2 = 0$, $q.\epsilon^* = 0$ and $2q.k = 2q_- k_+$.
As this contains a vertex loop so there will be a UV divergence.
Running this in FeynCalc [23] gives a result in terms of the one, two and three point scalar functions but retaining only the leading terms of $\mathcal{O}(1/\Lambda_{QCD})$ and dropping terms which have coefficients of $\mathcal{O}(\Lambda_{QCD}/m_b)$ leaves an output containing only bubble integrals which are shown in Appendix A.
A further series expansion [26] of the remaining result in powers of the regulator ε and the light quark mass m, both around zero, leaves the following

form at the leading order :

$$\mathcal{M}_{em} = \frac{\alpha C_F}{4\pi} T^{(0)}(\tilde{k}_+) \left[\frac{1}{\varepsilon_{UV}} + \ln\left(\frac{-2q.k\mu_R^2}{m^4} \right) \right] \tag{4.32}$$

The UV divergence is subtracted off by the counter-term diagram.
As a final step we also put in the $\frac{1}{(2\pi)^D}$ pre-factors in conjunction with the μ^2 in modified minimal subtraction scheme, in order to pull out a $\frac{1}{16\pi^2}$ to get a $\frac{\alpha}{4\pi} = \frac{g^2}{16\pi^2}$ term upfront. From now onwards this is done for all amplitude, LCDA and convolution results.

4.3.2 LCDA

The EM vertex LCDA in Fig 4.2(b) has similar Feynman-Wilson rules to compute it but with a few additions :

- integrate over loop momentum
- integrate over the length of the meson Wilson line, the point at which a loop propagator joins a Wilson line
- loop momentum is taken to be soft if it is connected to a Wilson line. As all the LCDA's being considered here are in HQET so the soft scale of loop momentum is Λ_{QCD}
- gluon-Wilson line vertex factor : $-igT^a n_\phi$

$$\begin{aligned} \Phi_{em}(\tilde{k}_+) =& C_F g^2 \mu_F^{2\varepsilon} \int dz_- \int d^D l \\ &\times \int_0^1 d\alpha z_- e^{i(\tilde{k}+k+l-\alpha l)z} \int \frac{\bar{u}\gamma^\phi (\not{k} + \not{l} + m) n_\phi \nu}{[l^2][(k+l)^2 - m^2]} \end{aligned} \tag{4.33}$$

$$\begin{aligned} \Phi_{em} =& C_F g^2 \mu_F^{2\varepsilon} \int dz_- e^{i(\tilde{k}+k)z} \\ &\times \left(z_- \frac{e^{-il_+ z_-} - 1}{-il_+ z_-} \right) \int d^D l \frac{\bar{u}\gamma^\phi (\not{k} + \not{l} + m) n_\phi \nu}{[l^2][(k+l)^2 - m^2]} \end{aligned} \tag{4.34}$$

$$\Phi_{em} = 2\pi C_F g^2 \mu_F^{2\varepsilon} \int d^D l [\delta(\tilde{k}+k+l) - \delta(\tilde{k}+k)] \frac{\bar{u}\gamma^\phi (\not{k} + \not{l} + m) n_\phi \nu}{[l_+][l^2][(k+l)^2 - m^2]} \tag{4.35}$$

Although in all LCDA diagrams the separation scale of the partons implies that the virtual gluons carry momentum of $\mathcal{O}(\Lambda_{QCD})$ which breaks the validity of perturbative QCD but the loop gluons are being exchanged over smaller length scales hence carry more momentum which means that the coupling parameters g_s are small enough to allow the perturbative expansion of the path ordered exponential. From the LCDA one of the convolutions is calculated thus

$$\Phi_{em} \otimes T^{(0)} = -\beta_F \int d^D l \times \int d\tilde{k}_+ [\delta(\tilde{k} + k + l) - \delta(\tilde{k} + k)] \frac{\bar{u}\gamma^\phi(\not{k} + \not{l} + m)\not{\epsilon}^*_\gamma n_\phi \not{q} \not{\epsilon}_W P_L \nu}{[l_+][l^2][(k+l)^2 - m^2][2q\tilde{k}_+]} \tag{4.36}$$

$$\Phi_{em} \otimes T^{(0)} = -\beta_F \int d^D l \times \frac{\bar{u}\gamma^\phi n_\phi(\not{k} + \not{l} + m)\not{\epsilon}^*_\gamma \not{q} \not{\epsilon}_W P_L \nu}{-2l_+ l^2 [(k+l)^2 - m^2]} \left(\frac{-q l_+}{(qk)(q(k+l))} \right) \tag{4.37}$$

The above loop integral has a zero degree of divergence which implies that it is the upper limit of the integral which will cause the divergence. Even though l is taken to be soft, the upper limit of this integral is much greater than the lower limit which characterises this divergence as UV.

$$\Phi_{em} \otimes T^{(0)} = -\beta_F \int d^D l \frac{\bar{u}\gamma^\phi(\not{k} + \not{l} + m)\not{\epsilon}^*_\gamma \not{q} \gamma_\phi \not{q} \not{\epsilon}_W P_L \nu}{[l^2][(k+l)^2 - m^2][q.(k+l)]b} \tag{4.38}$$

The m in the numerator of the light quark propagator gets dropped due to the on-shell photon. The numerator can be broken up thus

$$\bar{u}\gamma^\phi(\not{k} + \not{l})\not{\epsilon}^*_\gamma \not{q} \gamma_\phi \not{q} \not{\epsilon}_W P_L \nu + \bar{u}\gamma^\phi(m)\not{\epsilon}^*_\gamma \not{q} \gamma_\phi \not{q} \not{\epsilon}_W P_L \nu \tag{4.39}$$

The first term makes use of the identity $\gamma^\phi \not{a}\not{b}\not{c}\gamma_\phi = -2\not{c}\not{b}\not{a}$ and hence

$$\gamma^\phi(\not{k} + \not{l})\not{\epsilon}^*_\gamma \not{q} \gamma_\phi = -2\not{q}\not{\epsilon}^*_\gamma(\not{k} + \not{l}).$$

Using $\{\not{q}, \not{\epsilon}^*_\gamma\} = 0$ and the identity

$$\not{a}\not{b} = 2a.b - \not{b}\not{a}$$

the first term becomes

$$2\bar{u}\not{\epsilon}^*_\gamma[q.(k+l) - (\not{k}+\not{l})\not{q}]\not{q}\not{\epsilon}_W P_L \nu$$
$$= 2\bar{u}\not{\epsilon}^*_\gamma[q.(k+l)]\not{q}\not{\epsilon}_W P_L \nu$$

because $\not{q}\not{q} = q^2 = 0$
The second term can be rearranged using

$$\gamma^\phi \not{a}\not{b}\gamma_\phi = 4a.b = 2(\not{a}\not{b} + \not{b}\not{a}).$$

As $\not{\epsilon}^*_\gamma \not{q} = -\not{q}\not{\epsilon}^*_\gamma$ so the second term becomes zero as

$$\gamma^\phi \not{\epsilon}^*_\gamma \not{q}\gamma_\phi = 2(\not{\epsilon}^*_\gamma \not{q} - \not{\epsilon}^*_\gamma \not{q}) = 0.$$

This also proves $\epsilon^*_\gamma.q = 0$. This allows the cancellation of $q.(k+l)$ between the numerator and denominator, leaving a two point PV scalar function which can be evaluated by dimensional regularisation and its divergence subtracted off by a counter-term for the EM LCDA.

$$\Phi_{em} \otimes T^{(0)} = -\beta_F \frac{\bar{u}\not{\epsilon}^*_\gamma \not{q}\not{\epsilon}_W P_L \nu}{q.k} \int d^D l \frac{1}{[l^2][(k+l)^2 - m^2]} \tag{4.40}$$

$$\Phi_{em} \otimes T^{(0)} = \frac{\alpha C_F}{4\pi} T^{(0)}(\tilde{k}_+) \left[4 + \frac{2}{\varepsilon_{UV}} + \ln\left(\frac{\mu_F^4}{m^4}\right)\right] \tag{4.41}$$

The $1/\varepsilon_{UV}$ divergence is subtracted off by the convolution due to the LCDA vertex correction counter-term diagram.
Another way to obtain the convolution is by Eikonal approximation as shown in [7]. Those LCDA's which have a loop momentum attached to a Wilson line, can be constructed by taking the Eikonal approximation of the Feynman amplitude as that argument works both ways. As the tensor structure of the vertex correction is similar to a modified gamma matrix so a vertex correction is essentially a modified interaction term. Hence the counter-term required to renormalize it will come from the interaction Lagrangian density of the quark-gluon interaction.

$$\mathcal{L}_{int} = g_{(0)}\bar{\psi}_{(0)}\not{A}_{(0)}\psi_{(0)} = (1+\delta_g)(1+\delta_A/2)(1+\delta_\psi)g\bar{\psi}\not{A}\psi \tag{4.42}$$

retaining only the terms with $1/\varepsilon_{UV}$ because electromagnetic vertex correction only has a first-order divergence.

$$\mathcal{L}_{int} = (1+\delta_g+\delta_\psi+\delta_A/2)g\bar{\psi}\not{A}\psi = (1+\delta_1)g\bar{\psi}\not{A}\psi \tag{4.43}$$

So the counter-term vertex factor will be $-ig\delta_1 T^a$ and the counter-term diagram to renormalize the electromagnetic vertex correction amplitude will be

$$\mathcal{M}^c_{em} = -ig\delta_1 \bar{u}\not{\epsilon}^*_\gamma \frac{\not{q}+\not{k}+m}{(q+k)^2-m^2}\not{\epsilon}_W P_L \nu \tag{4.44}$$

The renormalized amplitude, which will be used in the factorization theorem and contains no divergences, is given by

$$\mathcal{M}^R_{em} = \mathcal{M}_{em} + \mathcal{M}^c_{em} \tag{4.45}$$

The renormalization of the vertex LCDA is different as the LCDA modified vertex is fundamentally different from that of Feynman amplitudes. As it additionally consists of a finite Wilson line propagator and a gluon-Wilson line vertex. The EM vertex LCDA is essentially

$$\Phi_{em} \sim \langle \bar{b}, u| \bar{\psi}_{\bar{b}}(0) \left(ig_s \int dz.A \right) \psi_u(z) \left(ig \int d^4x \bar{\psi} \not{A} \psi \right) |0\rangle \tag{4.46}$$

The stuff in between the bra-ket is what differentiates it from a tree level LCDA so these parameters will contribute the counter-term. As this LCDA does not use HQET so there are four quark counter-terms in the expression, two coupling and two gauge field counter-terms. All together they give

$$2\delta_\psi + 2\delta_g + \delta_A \tag{4.47}$$

which is the same as $2\delta_1$

4.3.3 Hard Kernel

The NLO hard kernel can be obtained from the earlier expression of the factorisation theorem

$$\mathcal{M}_{em} - \Phi_{em} \otimes T^{(0)} = \Phi^{(0)} \otimes T_{em}$$

$$\Phi^{(0)} \otimes T_{em} = -e^2 g^2 C_F V_{ub} \frac{\bar{u}\not{\epsilon}^*_\gamma \not{q} \not{\epsilon}_W P_L \nu}{2q.k} \left[\ln\left(\frac{-2q.k\mu_R^2}{\mu_F^4} \right) + 4 \right] \tag{4.48}$$

If $\mathcal{M}_{em}$ and $\Phi_{em} \otimes T^{(0)}$ are unrenormalized, then the UV divergence in T_{em} will be apparent, which is expected because it has a hard scale loop momentum and its vertex loop is identical to $\mathcal{M}_{em}$.

$$T_{em}(\tilde{k_+}) = \frac{\alpha C_F}{4\pi} T^{(0)}(\tilde{k}_+) \left[\ln \left(\frac{-2q.\tilde{k}\mu_R^2}{\mu_F^4} \right) - 4 \right] \tag{4.49}$$

where $2q.\tilde{k} = 2q_-\tilde{k}_+$. Mass singularities can arise when the light quark mass terms tend to zero, being much smaller as compared to the hard scale contained in the hard kernel. This NLO hard kernel has no mass singularities as m's from both logarithms mutually cancel. This is a validation of the factorisation theorem because the hard kernel by definition has to contain hard dynamics and hence should be free of infrared effects [7].

4.4 Light Quark Propagator Correction

$$\begin{aligned} \mathcal{M}_{u\bar{u}} =& \beta_R \int d^D l \\ &\times \frac{\bar{u}\not{\epsilon}_\gamma^*(\not{q} + \not{k} + m)\gamma^\phi(\not{q} + \not{k} - \not{l} + m)\gamma_\phi(\not{q} + \not{k} + m)\not{\epsilon}_W P_L \nu}{[(q+k)^2 - m^2][(q+k-l)^2 - m^2][l^2][(q+k)^2 - m^2]} \end{aligned} \tag{4.50}$$

Series expanding around zero in both ε and m yields

$$\mathcal{M}_{u\bar{u}} = \frac{\alpha C_F}{4\pi} T^{(0)}(\tilde{k}_+) \left[\frac{1}{\varepsilon} - 1 + \ln \left(\frac{\mu_R^2}{-2q.k} \right) \right] \tag{4.51}$$

The UV divergence has to be removed by renormalizing the propagator as follows
$\psi_0 = \left(1 + \frac{\delta_\psi}{2}\right)\psi = \sqrt{Z_\psi}\psi$ hence the bare propagator is $G_0 = Z_\psi G$.
Considering a sum of propagators at tree level and with increasing number of 1PI loops.

$$G_0 = \frac{i}{\not{p} - m_0} + \frac{i}{\not{p} - m_0} \Sigma \frac{i}{\not{p} - m_0} + ... \tag{4.52}$$

This can be expressed as a geometric series

$$G_0 = \frac{i}{\not{p} - m_0 + \Sigma} \tag{4.53}$$

The renormalized propagator will be

$$G = \frac{1}{1 + \delta_\psi} \frac{i}{(\not{p} - m(1 + \delta_m) + \Sigma)} \tag{4.54}$$

Expanding out the denominator gives

$$G = \frac{i}{\not{p} + \delta_2 \not{p} - m - \delta_m m + \Sigma + \delta_2 m} \tag{4.55}$$

only the terms with $1/\varepsilon$ type of divergence have been retained in forming the counter-term because $\mathcal{M}_{u\bar{u}}$ has divergence to the first order. From [1], the counter-term to cancel the divergence in the loop is

$$\not{p}\delta_2 - m(\delta_2 + \delta_m) \tag{4.56}$$

After renormalization, a one-loop dressed propagator will be

$$\frac{i}{\not{p} - m}(\Sigma + \not{p}\delta_2 - m(\delta_2 + \delta_m))\frac{i}{\not{p} - m} \tag{4.57}$$

As in Fig 4.3(b) this LCDA has a term

$$n^{\phi}\frac{g_{\phi\alpha}}{l^2}n^{\alpha}$$

from the gluon propagator attaching to the Wilson line at two points. And this gives $n^{\phi}n_{\phi} = 0$ hence

$$\Phi_{u\bar{u}} = 0 \tag{4.58}$$

Due to the NLO LCDA for this process being zero, the factorisation theorem reduces to

$$\mathcal{M}_{u\bar{u}} = \Phi^{(0)} \otimes T_{u\bar{u}} \tag{4.59}$$

$$T_{u\bar{u}}(\tilde{k}_+) = \frac{\alpha C_F}{4\pi}T^{(0)}(\tilde{k}_+)\left[-1 + \ln\left(\frac{-2q.\tilde{k}}{\mu_R^2}\right)\right] \tag{4.60}$$

This hard kernel has no mass singularity and has no factorisation scale dependance either as one of the NLO convolutions (which gives the μ_F) is zero.

4.5 Light Quark External Leg Correction

4.5.1 Feynman Amplitude and LCDA

From Fig 4.4(a)

$$\mathcal{M}_u = \beta_R \bar{u}\int d^D l \frac{\gamma^{\phi}(\not{k} - \not{l} + m)\gamma_{\phi}}{[(k-l)^2 - m^2][l^2]}\not{\epsilon}^*_{\gamma}\frac{(\not{q} + \not{k} + m)}{(q+k)^2 - m^2}\not{\epsilon}_W P_L \nu \tag{4.61}$$

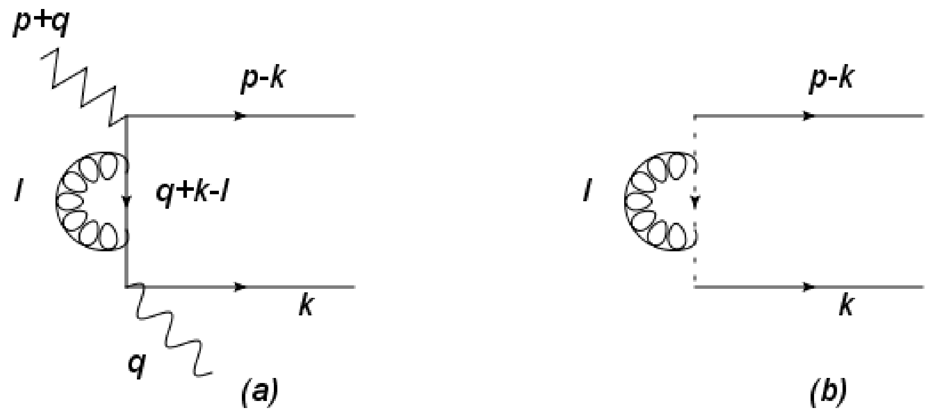

Figure 4.3: *(a) Feynman Amplitude and (b) LCDA with self energy correction on internal light quark line. The LCDA (b) is zero from the Wilson line Feynman rules.*

With the self energy loop on the external light quark leg means that this amplitude can also be written thus in the on-shell subtraction scheme as in [7].

$$\mathcal{M}_u = \mathcal{M}^{(0)} \frac{\delta_2^u}{2} \tag{4.62}$$

and can be renormalized by the corresponding counter-term vertex as in [1].

$$\mathcal{M}_u^c = \beta_R \bar{u} \left(\not{k} \delta_2^u - m(\delta_2^u + \delta_m^u) \right) \not{\epsilon}_\gamma^* \frac{\not{q} + \not{k} + m}{2q.k} \not{\epsilon}_W P_L \nu \tag{4.63}$$

where

$$\delta_2^u = i \frac{d\Sigma_2^u}{d\not{k}} |_{\not{k}=m}$$

and

$$\delta_m^u = \frac{\Sigma_2^u}{m}$$

$$\Sigma_2^u = -i \frac{\alpha C_F}{4\pi} \mu_{R,F}^{2\varepsilon} \int d^D l \frac{\gamma^\phi (\not{k} - \not{l} + m) \gamma_\phi}{[l^2][(k-l)2 - m^2]} \tag{4.64}$$

The LCDA is similar to a tree level LCDA but with a self energy correction on the external light quark and after dropping linear l terms from the numerator

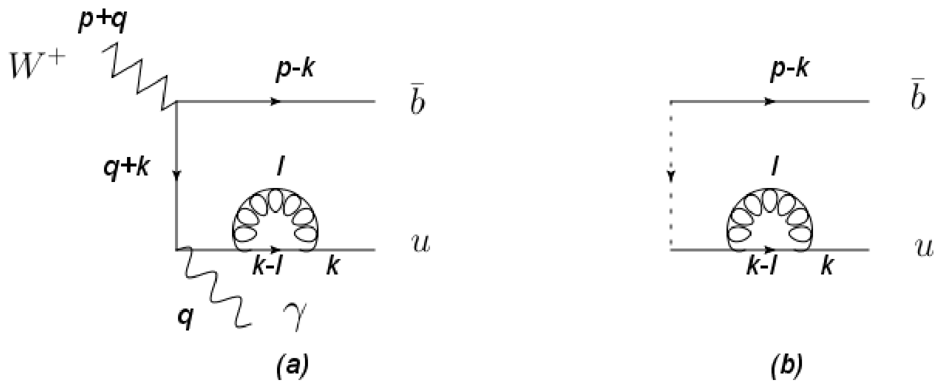

Figure 4.4: *Light quark external leg self energy correction (a) Feynman Amplitude and (b) LCDA. The self energy loop is the same in both cases as it is on the light quark.*

one of the NLO convolutions is

$$\Phi_u \otimes T^{(0)} = \beta_F \int d\tilde{k}\delta(\tilde{k}+k) \int d^D l \frac{\bar{u}\gamma^{\phi}(\not{k}+m)\gamma_{\phi}\not{\epsilon}^*_{\gamma}\not{q}\not{\epsilon}_W P_L \nu}{[2q.\tilde{k}][l^2][(k-l)^2-m^2]} \tag{4.65}$$

Which can similarly be written in the on-shell subtraction scheme as

$$\Phi_u \otimes T^{(0)} = \Phi^{(0)} \otimes T^{(0)} \frac{\delta_2^u}{2} \tag{4.66}$$

but for the purpose of evaluating the NLO hard kernel, we see that all divergences cancel among the terms in the Factorisation Theorem.

4.5.2 Evaluating the Light Quark Counter-term

Just like the full expression of this amplitude in the above equation, Σ_2^u can be evaluated in FeynCalc [23] in terms of PV scalar integrals and differentiated with respect to $\not{k}$ and series expanded in m and ε around zero to give

$$\delta_2^u = \frac{\alpha C_F}{4\pi}\left[-\frac{2}{\varepsilon} - 4 - 4\ln\left(1-\frac{k^2}{m^2}\right) + 2\ln\left(\frac{m}{\mu_{R,F}}\right)\right] \tag{4.67}$$

The light quark here is taken to be off-shell [7] hence $k^2 \neq m^2$. μ_R is for the counter-term in the amplitude and μ_F for the convolution.

Hard Kernel

Employing the above result in the expression of the factorisation theorem :

$$\mathcal{M}_u - \Phi_u \otimes T^{(0)} = \Phi^{(0)} \otimes T_u \tag{4.68}$$

$$T_u(\tilde{k}_+) = \frac{\alpha C_F}{4\pi} T^{(0)}(\tilde{k}_+) \frac{1}{2} \ln\left(\frac{\mu_F^2}{\mu_R^2}\right) \tag{4.69}$$

This hard kernel also lacks mass singularities but due to the NLO convolution, it has dependance on both the factorisation and the renormalisation scales.

4.6 b Quark External Leg Correction

The Feynman amplitude involving heavy quark propagators is evaluated in QCD but the LCDA's in HQET [7,8].

4.6.1 Feynman Amplitude and Convolution

From Fig 4.5(a)

$$\mathcal{M}_{\bar{b}} = \beta_R \bar{u} \not{\epsilon}_\gamma^* \frac{\not{q} + \not{k} + m}{2q.k} \not{\epsilon}_W P_L \int d^D l \frac{\gamma^\phi(\not{p} - \not{k} - \not{l} - m_b)\gamma_\phi}{[l^2][(p-k-l)^2 - m_b^2]} \nu \tag{4.70}$$

and from Fig 4.5(b) the convolution is

$$\Phi_{\bar{b}} \otimes T^{(0)} = \frac{\beta_F}{2q.k} \int d^D l \frac{\bar{u} \not{\epsilon}_\gamma^* \not{q} \not{\epsilon}_W P_L (1 - \not{v}) \nu}{[l^2][2v.(k+l)]} \tag{4.71}$$

In the on-shell scheme these are, as before

$$\mathcal{M}_{\bar{b}} = \mathcal{M}^{(0)} \frac{\delta_2^{\bar{b}}}{2} \tag{4.72}$$

and

$$\Phi_{\bar{b}} \otimes T^{(0)} = \Phi^{(0)} \otimes T^{(0)} \frac{\delta_2^Q}{2} \tag{4.73}$$

where the field on-shell scheme counter-term for the Feynman amplitude is similar to the light quark case, in that it is evaluated in QCD but the counter-term for convolution is evaluated in HQET.

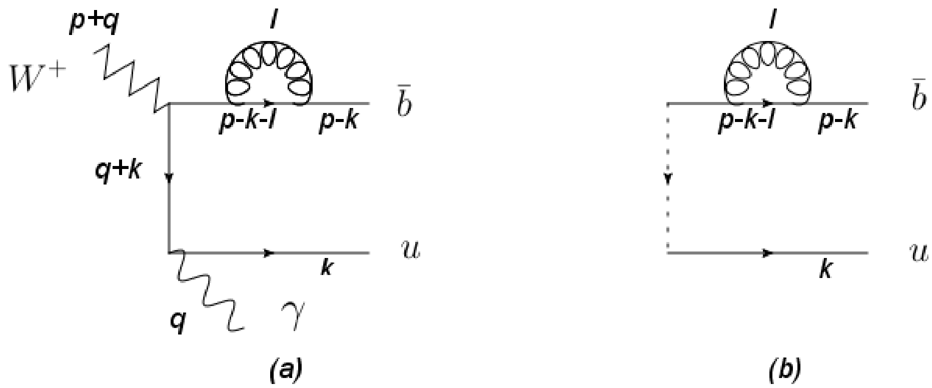

Figure 4.5: *The Feynman Amplitude (a) of the b quark external leg correction is calculated in QCD while the corresponding LCDA (b) in HQET.*

4.6.2 Evaluating the Counter-terms and Hard Kernel

The self energy loop on the b quark is in QCD

$$\Sigma_2^{\bar{b}} = -i\frac{\alpha C_F}{4\pi}\mu_R^{2\varepsilon}\int d^D l \frac{\gamma^\phi(\not{p} - \not{k} - \not{l} - m_b)\gamma_\phi}{(p-k-l)^2 - m_b^2} \tag{4.74}$$

and $\delta_2^{\bar{b}}$ can be evaluated in direct analogy with the earlier counter-term δ_2^u by considering the b quark with momentum $p-k$ to be off-shell. This basically amounts to the replacement $k \to p-k$ in the expression of δ_2^u with

$$(p-k)^2 = m_b^2 - 2m_b v.k$$

to give

$$\delta_2^{\bar{b}} = i\frac{d\Sigma_2^{\bar{b}}}{d\not{p}}|_{\not{p}=m_b} = \frac{2\alpha C_F}{4\pi}\left[-2 + \ln\left(\frac{m_b^3}{4(v.k)^2\mu_R}\right)\right]|_{v.k=0} \tag{4.75}$$

The heavy quark self energy loop in HQET is

$$\Sigma_2^Q = -i\frac{\alpha C_F}{4\pi}\mu_F^{2\varepsilon}\int d^D l \frac{1}{[l^2][2v.(k+l)]} \tag{4.76}$$

Here the momentum p is on-shell and k and l are residual momenta. Using [7] and after expanding to leading order in ε around zero

$$\delta_2^Q = i\frac{d\Sigma_2^Q}{d(v.k)}|_{v.k=0} = \frac{2\alpha C_F}{4\pi}\ln\left(\frac{\mu_F^2}{4(v.k)^2}\right)|_{v.k=0} \tag{4.77}$$

A divergence due to $v.k = 0$ in the logarithm is avoided in the Hard Kernel as the $v.k$ terms ends up cancelling upon subtraction between the amplitude and convolution.

$$T_b(\tilde{k}_+) = \frac{\alpha C_F}{4\pi}T^{(0)}(\tilde{k_+})\left[-2+\ln\left(\frac{m_b^3}{\mu_F^2\mu_R}\right)\right] \tag{4.78}$$

This hard kernel only has the heavy quark mass term in the logarithm, which does not cause a mass singularity.

4.7 Bottom-Up-Gluon Box

The hard kernel of this process is best illustrated via the technique of regions, as in [22], being applied to its Feynman amplitude and Convolution. From Fig 4.6(a)

$$\begin{aligned}\mathcal{M}_{Box} &= \beta_R\bar{u}\int d^Dl \\ &\times \frac{\gamma^\phi(\not{k}+\not{l}+m)\not{\epsilon}_\gamma^*(\not{q}+\not{k}+\not{l}+m)\not{\epsilon}_W P_L(\not{p}-\not{k}-\not{l}-m_b)\gamma_\phi}{[l^2][(k+l)^2-m^2][(q+k+l)^2-m^2][(p-k-l)^2-m_b^2]}\nu\end{aligned} \tag{4.79}$$

By the technique of regions the loop momentum spans four regions and is characterised by the momentum scaling of SCET I: hard (m_b), soft(Λ_{QCD}) and co-linear $\left(\sqrt{\Lambda_{QCD}m_b}\right)$ scales. As the two energetic particles are moving off in opposite directions so there are two co-linear regions that have a longitudinal momentum component directed opposite to each other.

- $l_h^\mu \sim (m_b, m_b, m_b)$
- $l_s^\mu \sim (\Lambda_{QCD}, \Lambda_{QCD}, \Lambda_{QCD})$
- $l_c^\mu \sim \left(\Lambda_{QCD}, m_b, \sqrt{\Lambda_{QCD}m_b}\right)$
- $l_{\bar{c}}^\mu \sim \left(m_b, \Lambda_{QCD}, \sqrt{\Lambda_{QCD}m_b}\right)$

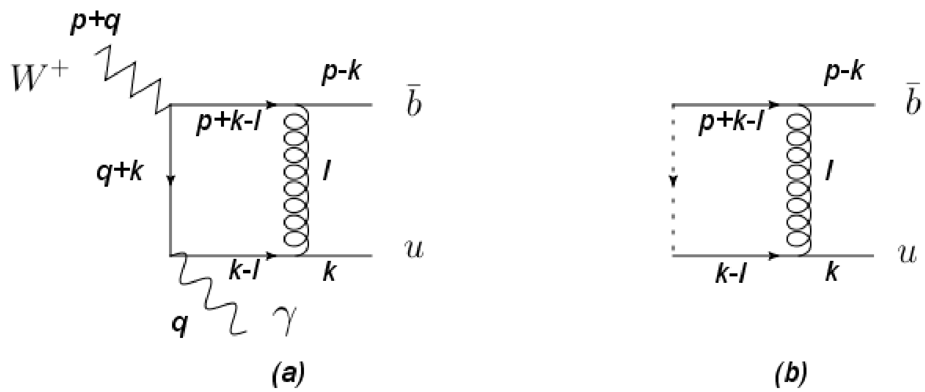

Figure 4.6: *(a) Box Feynman amplitude and (b) Box LCDA.*

This allows the approximation of a loop integral as

$$A(l) = A_h(l) + A_s(l) + A_c(l) + A_{\bar{c}}(l) \tag{4.80}$$

To apply this technique we power count all the momenta in the integral and determine the order of magnitude of each. Then the full integral is approximated as the leading order term. In the following, we use this technique to determine the dominant contributions of $\mathcal{M}_{Box}$ and $\Phi_{Box} \otimes T^{(0)}$
Recalling the on-shell kinematics and considering that $d^4l = dl_+dl_-dl_xdl_y$ the four regions of the box amplitude are :

- $\mathcal{M}^h \sim 1/m_b$
- $\mathcal{M}^s \sim 1/\Lambda_{QCD}$
- $\mathcal{M}^c \sim /\Lambda_{QCD}$
- $\mathcal{M}^{\bar{c}} \sim \ 1/m_b$

Convolutions (but not LCDAs and hard kernels individually) are same in HQET and QCD [7]. From Fig 4.6(b) the loop gluons are going from the hard b quark line to the soft light quark line so the loop momentum is soft in the LCDA.

$$\Phi_{Box} \otimes T^{(0)} = \beta_F \int d^D l$$
$$\times \frac{\bar{u}\gamma\phi(\not{k} - \not{l} + m)\not{\epsilon}^* \not{q}\not{\epsilon} P_L(\not{p} - m_b)\gamma_\phi \nu}{[l^2][(k-l)^2 - m^2][(p-k+l)^2 - m_b^2][2q.k]} \sim 1/\Lambda_{QCD} \tag{4.81}$$

The convolution is the same order of magnitude as the leading term of the box amplitude. Extracting the hard kernel correction

$$\Phi^{(0)} \otimes T_{Box} = \mathcal{M}_h + \mathcal{M}_s + \mathcal{M}_c + \mathcal{M}_{\bar{c}} - \Phi_{Box} \otimes T^{(0)} \tag{4.82}$$

$$\Phi^{(0)} \otimes T_{Box} = \mathcal{M}_h + \mathcal{M}_{\bar{c}} \sim 1/m_b$$

T_{Box} is of a very small order of magnitude so gets dropped from the final result.

4.8 Weak Vertex Correction

4.8.1 Feynman Amplitude

From Fig 4.7(a)

$$\mathcal{M}_{wk} = \beta_R \int d^D l$$
$$\times \frac{\bar{u}\not{\epsilon}_\gamma^*(\not{q} + \not{k} + m)\gamma^\phi(\not{q} + \not{k} + \not{l} + m)\not{\epsilon}_W P_L(\not{p} - \not{k} - \not{l} - m_b)\gamma_\phi \nu}{[(q+k)^2 - m^2][l^2][(q+k+l)^2 - m^2][(p-k-l)^2 - m_b^2]} \tag{4.83}$$

Dropping $\mathcal{O}(1/m_b)$ and smaller scaling terms from the output leaves behind $\mathcal{M}_{wk}$ which contains both B_0 and C_0 PV scalar functions. The evaluation of the C_0 integrals requires the aforementioned technique of regions [22].

$$\mathrm{C}_0 = \mathrm{C}_0^h + \mathrm{C}_0^s + \mathrm{C}_0^c + \mathrm{C}_0^{\bar{c}} \tag{4.84}$$

The above C_0 integrals were evaluated using the parameterizations from [22]

$$\frac{1}{ABC} = \int_0^1 d \int_0^x dy \frac{\Gamma(3)}{[Ay + B(x-y) + C(1-x)]^3} \tag{4.85}$$

and

$$\frac{1}{ABC} = \int_0^\infty dx \int_0^\infty dy \frac{\Gamma(3)}{[A + Bx + Cy]^3} \tag{4.86}$$

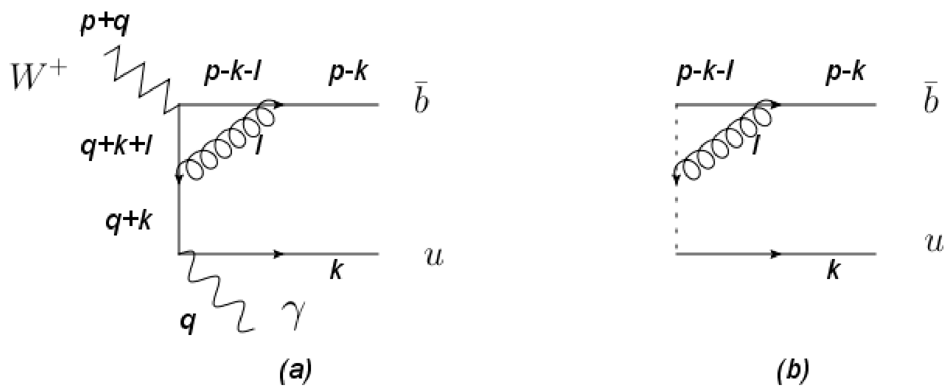

Figure 4.7: *Weak Vertex correction's (a) Feynman amplitude is written in QCD and requires the technique of regions to completely solve. Its LCDA (b) can be written in either QCD or HQET.*

and C_0^h had to be fully expanded using the package HypExp [27]. Finally this gives the following weak vertex amplitude where $a = 2p.q = 2p_+q_-$

$$\begin{aligned}\mathcal{M}_{wk} = \frac{\alpha C_F}{4\pi}\mathcal{M}^{(0)}\Bigg[&\frac{2m_b^2+3a}{a+m_b^2}\ln\left(\frac{-a}{m_b^2}\right) - 2\ln\left(1+\frac{m_b^2}{a}\right)\ln\left(\frac{-m_b^2}{a}\right) \\ &- 2\mathrm{Li}_2\left(\frac{-m_b^2}{a}\right) - \frac{2}{3}\pi^2 + \ln\left(\frac{a}{2q.k}\right)\ln\left(\frac{a\,2q.k}{m_b^4}\right) - 2i\pi\ln\left(\frac{a}{2q.k}\right) \\ &+ 2\ln\left(\frac{-\mu_R\, m_b}{2q.k}\right) - 2\ln^2\left(\frac{a}{2q.k}\right)\Bigg]\end{aligned} \tag{4.87}$$

4.8.2 LCDA and Hard Kernel

The weak vertex LCDA can be evaluated using similar Feynman-Wilson rules as those of the EM vertex where the LCDA is evaluated with HQET Feynman rules. From Fig 4.7(b)

$$\Phi_{wk}(\tilde{k}_+) = -ig^2\mu_F^{2\varepsilon}C_F\int dz_- d^D l\int_0^1 d\alpha z_- e^{i(\tilde{k}+k+l)z}\frac{\bar{u}n^\phi(1-\not{v})v_\phi\nu}{[l^2][2l.v]} \tag{4.88}$$

or the weak vertex amplitude can be written in the Eikonal approximation [7], thus making it equal to the convolution of with the NLO LCDA in QCD.

$$\Phi_{wk} \otimes T^{(0)} = \beta_F \int d^D l \frac{\bar{u}\not{\epsilon}^*_\gamma \not{q} \gamma^\phi \not{q} \not{\epsilon}_W P_L(\not{p} - m_b)\gamma_\phi \nu}{[2q.k][l^2][2q.(k+l)][-2p.l]} \tag{4.89}$$

From this point on it proceeds in the same fashion as the EM vertex. Using the identities

$$\gamma^\phi \not{q} \not{\epsilon} \gamma_\phi = 4q.\epsilon$$

along with $q^2 = 0$ to get rid of the m_b term in the numerator. The remaining part gets simplified using

$$\gamma^\phi \not{q} \not{\epsilon} P_L \not{p} \gamma_\phi = -2\not{p}\not{\epsilon} P_L \not{q}$$

Hence the numerator becomes

$$\bar{u}\not{\epsilon}^*_\gamma \not{q} \gamma^\phi \not{q} \not{\epsilon}_W P_L(\not{p} - m_b)\gamma_\phi \nu = (2a)\bar{u}\not{\epsilon}^*_\gamma \not{q} \not{\epsilon}_W P_L \nu$$

$$\Phi_{wk} \otimes T^{(0)} = \beta_F \frac{\bar{u}\not{\epsilon}^*_\gamma \not{q} \not{\epsilon}_W P_L \nu}{b} \int d^D l \frac{2a}{[l^2][2q.(k+l)][-2p.l]} \tag{4.90}$$

Solving this integral using (B.3) and (B.4) from [22] where
$A = l^2$, $B = 2q.(k+l)$ and $C = -2p.l$

$$\int d^D l \frac{1}{ABC} = 2a \int_0^\infty dx \int_0^\infty dy \frac{\Gamma(3) i\pi^{D/2}(-1)^3 \Gamma(1+\varepsilon)}{\Gamma(3)[m_b^2 y^2 - axy - bx]^{1+\varepsilon}} \tag{4.91}$$

$$\Phi_{wk} \otimes T^{(0)} = g^2 C_F \mathcal{M}^{(0)} \frac{(2a)\left(\frac{a\mu_F}{m_b b e^{\gamma_E/2}}\right)^{2\varepsilon} \pi \mathrm{Cosec}(2\pi\varepsilon)\Gamma(1+\varepsilon)}{(2\pi)^D a\varepsilon} \tag{4.92}$$

series expanding around zero in ε and dropping the divergent terms

$$\begin{aligned} \Phi_{wk} \otimes T^{(0)} &= \frac{\alpha}{4\pi} C_F \mathcal{M}^{(0)} \\ &\times \left[\frac{3\pi^2}{4} + 2\ln^2\left(\frac{a}{2q.k}\right) + \frac{1}{2}\ln^2\left(\frac{\mu_F^2}{m_b^2}\right) + 2\ln\left(\frac{a}{2q.k}\right)\ln\left(\frac{\mu_F^2}{m_b^2}\right)\right] \end{aligned} \tag{4.93}$$

and substituting in explicit expressions of the scalar products

$$\Phi_{wk} \otimes T^{(0)} = \frac{\alpha}{4\pi} C_F \mathcal{M}^{(0)} \left[\frac{3\pi^2}{4} + 2\ln^2\left(\frac{\mu_F}{\sqrt{2}k_+}\right)\right] \tag{4.94}$$

Using the Factorisation Theorem to get the Hard Kernel

$$\begin{aligned}
T_{wk}(\tilde{k}_+) = & \frac{\alpha C_F}{4\pi} T^{(0)}(\tilde{k}_+) \\
& \times \Bigg[\frac{2m_b^2 + 3a}{a + m_b^2} \ln\left(\frac{-a}{m_b^2}\right) - 2\ln\left(1 + \frac{m_b^2}{a}\right)\ln\left(\frac{-m_b^2}{a}\right) \\
& - 2\text{Li}_2\left(\frac{-m_b^2}{a}\right) - \frac{2}{3}\pi^2 + \ln\left(\frac{a}{2q.\tilde{k}}\right)\ln\left(\frac{a\ 2q.\tilde{k}}{m_b^4}\right) - 2i\pi\ln\left(\frac{a}{2q.\tilde{k}}\right) \\
& + 2\ln\left(\frac{-\mu_R\, m_b}{2q.\tilde{k}}\right) - 2\ln^2\left(\frac{a}{2q.\tilde{k}}\right) - 2\ln^2\left(\frac{\mu_F}{\sqrt{2}\tilde{k}_+}\right) - \frac{3\pi^2}{4} \Bigg]
\end{aligned} \tag{4.95}$$

This hard kernel also lacks light quark mass terms as all such terms were removed when performing power counting in the various propagator denominator factors in both the amplitude and the convolution. Like three of the previous cases of the NLO Hard Kernel, this one also has a dependance on both the factorisation and renormalisation scale. μ_F dependance comes from the loop level convolution integrals while μ_R comes from $\mathcal{M}_{wk}$, from the B_0 functions to be precise, as all μ_R dependance cancels among the three contributing regions of C_0. The fact that some of our Hard Kernels still contain μ_R, does not spoil the factorisabiltiy of the amplitude as independance from μ_R is something that has to hold at an entire order rather than individual terms of that order. It shall be shown how the different contributors to $T^{(1)}$ end up canceling all μ_R terms when they are added together.

Chapter 5

Vector and Axial Vector Form Factors

Due to the fact that hadronic systems are bound states with their constituent quarks and gluons held together by non-perturbative interactions, describing their amplitudes with Feynman diagrams is not sufficient as the formalism of Feynman diagrams really only applies to asymptotically free particles. To compensate for this, the idea is to equate the amplitude at the level of the constituents (as done earlier in this text), with the decomposed form of the amplitude as in [7,8,25], based on Lorentz symmetries.
Using [8] for this particular process

$$\begin{aligned} &\langle \bar{B}(p), \gamma(\epsilon^*, q)| \, (\bar{\Psi}_u \not{A} P_L \Psi_b)(\bar{\Psi}_u \not{A} \Psi_u) \, |W(\epsilon, p+q)\rangle \sim \mathcal{M} \\ &= R \left(\varepsilon_{\mu\nu\phi\alpha} \frac{p^\mu q^\nu \epsilon_W^\phi \epsilon_\gamma^{*\alpha}}{p.q} F_V + i\epsilon_W . \epsilon_\gamma^* F_A \right) \end{aligned} \tag{5.1}$$

Here $R = \frac{e_u e^2 V_{ub}}{4\sqrt{2}\sin\theta_w}$ and F_V and F_A are the vector and axial vector form factors, respectively and they are Lorentz scalars so can be calculated similarly in any reference frame [8]. The general Lorentz structure only depends upon the external states and so the above expression will be equally valid for tree or any loop order. The only difference between the two will be in the nature of the form factors which can be expressed as an order by order expansion. In both cases the information about the magnitude of the amplitude resides in the form factors as their coefficients are of unit magnitude.
Before moving onto actually calculating these form factors, it is prudent to prove their equivalence at both tree and loop level.

5.1 Equivalence of the Vector and Axial Vector Form Factors

5.1.1 $F_V^{(0)} = F_A^{(0)}$

For ease of notation, the decomposed amplitude (5.1) will be expressed as

$$\mathcal{M}^{(0)} = R\left(XF_V^{(0)} + YF_A^{(0)}\right) \tag{5.2}$$

where

$$X = \frac{\varepsilon_{\mu\nu\alpha\beta}p^\mu q^\nu \epsilon_W^\alpha \epsilon_\gamma^{*\beta}}{p.q}$$

and

$$Y = i\epsilon_W.\epsilon_\gamma^*$$

.

Following [7,8] the LCDA Φ will have only Φ_+ at leading order (Φ_- being suppressed by a factor of $1/m_b$) and those LCDA's calculated earlier will now be

$$\Phi_+ = i\hat{f}_B m_B \phi_+ \tag{5.3}$$

where $\hat{f}_B$ is the B meson decay constant in HQET [8,18]. Then decomposing the tree level hard kernel obtained earlier in (4.29) and applying it in the HQET factorisation theorem

$$\begin{aligned}\mathcal{M}^{(0)} &= \int_0^\infty d\omega \Phi_+^{(0)}(\omega)T^{(0)}(\omega) \\ &= i\hat{f}_B m_B \int_0^\infty d\omega \frac{-iR}{\omega}(X+Y)\phi_B^+(\omega) = \frac{\hat{f}_B m_B R(X+Y)}{\lambda_B^{(0)}}\end{aligned} \tag{5.4}$$

where $1/\lambda_B$ is the first inverse moment of ϕ_+. Equating (5.2) and (5.5)

$$\mathcal{M}^{(0)} = R\left(XF_V^{(0)} + YF_A^{(0)}\right) = \frac{\hat{f}_B m_B R(X+Y)}{\lambda_B^{(0)}} \tag{5.5}$$

and comparing coefficients of X and Y gives

$$F_V^{(0)} = F_A^{(0)} = \frac{\hat{f}_B m_B}{\lambda_B^{(0)}} \tag{5.6}$$

5.1.2 $F_V^{(1)} = F_A^{(1)}$

Lorentz decomposition of the NLO amplitude looks similar to the one at LO but with correspondingly different form factors

$$\mathcal{M}^{(1)} = R\left(XF_V^{(1)} + YF_A^{(1)}\right) = \Phi^{(1)} \otimes T^{(0)} + \Phi^{(0)} \otimes T^{(1)} \tag{5.7}$$

All the NLO convolutions and hard kernels calculated earlier can be expressed as a factor being multiplied with its LO counterpart.

$$\Phi^{(1)} \otimes T^{(0)} = \Phi^{(0)} \otimes T^{(0)} G = \mathcal{M}^{(0)} G \tag{5.8}$$

$$T^{(1)}(\omega) = T^{(0)}(\omega) H(\omega) \tag{5.9}$$

$$\Phi^{(0)} \otimes T^{(1)} = i\hat{f}_B m_B \int_0^\infty d\omega \phi_B^+(\omega) \frac{-iR}{\omega}(X+Y)H(\omega) \tag{5.10}$$

Using the earlier results of leading order form factors and $\mathcal{M}^{(0)}$

$$\begin{aligned} \mathcal{M}^{(1)} &= R\left(XF_V^{(1)} + YF_A^{(1)}\right) = RGF_{V,A}^{(0)}(X+Y) \\ &+ R\hat{f}_B m_B (X+Y) \int_0^\infty \frac{d\omega}{\omega} \phi_B^+(\omega) H(\omega) \end{aligned} \tag{5.11}$$

comparing the coefficients of X and Y gives

$$F_V^{(1)} = F_A^{(1)} = F_{V,A}^{(0)} G + \hat{f}_B m_B \int_0^\infty \frac{d\omega}{\omega} \phi_B^+(\omega) H(\omega) \tag{5.12}$$

5.2 Evaluating the Form Factors

From [7,8] the form factors take this general form

$$F_{V,A} = \int_0^\infty d\tilde{k}_+ \Phi_+(\tilde{k}_+) T(\tilde{k}_+) \tag{5.13}$$

Such integrals involving Φ_+ can be expressed in terms of the first inverse moment and nth logarithmic inverse moments of the meson in question.
The Hard Kernel has an order by order expansion in α_s and so the expressions

for the LO and NLO form factors can be obtained by substituting in the relevant Hard Kernels.

$$\begin{aligned}F^{(0)}_{V,A} &= \int_0^\infty d\tilde{k}_+ \Phi_+(\tilde{k}_+) T^{(0)}(\tilde{k}_+) \\ &= e^2 V_{ub} \frac{\not{\epsilon}^*_\gamma \not{q} \not{\epsilon}_W P_L}{\sqrt{2} q_-} \int_0^\infty dk_+ \frac{\Phi_+(k_+)}{\sqrt{2}\tilde{k}_+} = e^2 V_{ub} \frac{\not{\epsilon}^*_\gamma \not{q} \not{\epsilon}_W P_L}{\sqrt{2} q_-} \frac{1}{\lambda_B^{(0)}}\end{aligned} \tag{5.14}$$

As the NLO Hard Kernel consists of a sum of constant terms as well as double and single logarithms of $\tilde{k}_+$ so the NLO Form Factors can be expected to be composed of both the first inverse moment and the first and second logarithmic inverse moments.

First writing out the total NLO Hard Kernel as

$$T^{(1)} = \frac{\alpha C_F}{4\pi} T^{(0)}(\tilde{k}_+) \left[c_0 + c_1 \ln\left(\frac{\sqrt{2} m_b \tilde{k}_+}{\mu_F^2}\right) + c_2 \ln^2\left(\frac{\sqrt{2} m_b \tilde{k}_+}{\mu_F^2}\right)\right] \tag{5.15}$$

$$F^{(1)}_{V,A} = \frac{\alpha C_F}{4\pi} F^{(0)}_{V,A} \lambda_B^{(0)} \left[c_0 \frac{1}{\lambda_B^{(0)}} + c_1 \frac{1}{\lambda_B^{(1)}} + c_2 \frac{1}{\lambda_B^{(2)}}\right] \tag{5.16}$$

where the coefficients of the terms containing single and double logarithms of $\tilde{k}_+$ are in c_1 and c_2 and those terms entirely without $\tilde{k}_+$ are in c_0. For the case evaluated in this text, these coefficients are

$$\begin{aligned}c_0 = \ln&\left(\frac{-m_b^5}{\mu_F^5}\right) - \frac{2}{3}\pi^2 - 2\ln^2\left(\frac{\mu_F}{m_b}\right) - 2\ln^2\left(\frac{\sqrt{2}\mu_F^2 q_-}{a\, m_b}\right) - \frac{3}{4}\pi^2 \\ &- 7 - 2\mathrm{Li}_2\left(\frac{-m_b^2}{a}\right) + \frac{3a + 2m_b^2}{a + m_b^2} \ln\left(\frac{-a}{m_b^2}\right) + 2i\pi \ln\left(\frac{\sqrt{2} q_- \mu_F^2}{a\, m_b}\right) \\ &- 2\ln\left(1 + \frac{m_b^2}{a}\right)\ln\left(\frac{-m_b^2}{a}\right) - \ln\left(\frac{\sqrt{2} q_- \mu_F^2}{a\, m_b}\right)\ln\left(\frac{a\sqrt{2} q_- \mu_F^2}{m_b^5}\right) \\ &+ 2i\pi\end{aligned} \tag{5.17}$$

$$\begin{aligned}c_1 = &-\ln\left(\frac{\sqrt{2}\mu_F^2 q_-}{a\, m_b}\right) - \ln\left(\frac{a\sqrt{2}\mu_F^2 q_-}{m_b^5}\right) - 4\ln\left(\frac{\sqrt{2}\mu_F^2 q_-}{a\, m_b}\right) \\ &- 4\ln\left(\frac{\mu_F}{m_b}\right) + 2i\pi\end{aligned} \tag{5.18}$$

and

$$c_2 = -5 \tag{5.19}$$

Chapter 6

Conclusion

In this thesis we investigated the factorisability of the following amplitude

$$W^+ \to \gamma B$$

It was found that using the heavy quark limit affects not only the kinematical hierarchy but also the QCD Lagrangian which gets modified to the HQET Lagrangian. This allows the Factorisation Theorem to hold for this process at tree level and at one gluon loop level. All the amplitudes are computed in the following hierarchy

$$m_W \sim m_b >> \Lambda_{QCD}$$

and loop corrections on the b quark use the HQET Lagrangian.
From the factorisation theorem we extracted two functions associated with this process, namely the Hard Kernel and LCDA. They describe the hard and soft dynamics in the system respectively. In this section we see how all our previous calculations come together to prove the Factorisation Theorem. From the form of the Hard Kernel which contains no infrared effects, it can also be seen that it lacks any mass singularities and soft loop divergences. The Hard Kernel at NLO is also free of dependance on the renormalisation scale μ_R which is also a condition of factorisability as μ_R is associated with the regularized one loop Feynman amplitudes while the Hard Kernels occur in the convolution integrals which receive μ_F scale as a result of regularization. A further validation of the factorisation theorem is that the Hard Kernel is proven to be the same for next to leading Fock state of the B meson. It was also established that the factorisability of this amplitude requires no

dependance on the transverse components of momenta of any of the particles. Furthermore, the distribution amplitudes of the mesons are defined with the z_- light cone component, the z_+ and $z_\perp$ (transverse) components being zero.

The tree level Hard Kernel at leading order in $1/m_b$ is found to be

$$T^{(0)}(\tilde{k}_+) = e^2 V_{ub} \frac{\not{\epsilon}^*_\gamma \not{q} \not{\epsilon}_W P_L}{2q.\tilde{k}} \tag{6.1}$$

An entire class of amplitudes where the light quark propagator is replaced with the b quark propagator is suppressed due to the heavy quark limit.
At one gluon loop order or g_s^2 the Hard Kernel receives contributions from six different amplitudes.

$$T^{(1)} = T_{em} + T_{u\bar{u}} + T_u + T_b + T_{wk} + T_{Box} \tag{6.2}$$

At leading order in $1/m_b$ the Hard Kernels are :

$$T_{em}(\tilde{k_+}) = \frac{\alpha}{4\pi} C_F T^{(0)}(\tilde{k}_+) \left[\ln\left(\frac{-2q.\tilde{k}\mu_R^2}{\mu_F^4} \right) - 4 \right] \tag{6.3}$$

$$T_{u\bar{u}}(\tilde{k}_+) = \frac{\alpha}{4\pi} C_F T^{(0)}(\tilde{k}_+) \left[-1 + \ln\left(\frac{-2q.\tilde{k}}{\mu_R^2} \right) \right] \tag{6.4}$$

$$T_u(\tilde{k}_+) = \frac{\alpha}{4\pi} C_F T^{(0)}(\tilde{k}_+) \left[\frac{1}{2} \ln\left(\frac{\mu_F^2}{\mu_R^2} \right) \right] \tag{6.5}$$

$$T_b(\tilde{k}_+) = \frac{\alpha}{4\pi} C_F T^{(0)}(\tilde{k}_+) \left[-2 + \ln\left(\frac{m_b^3}{\mu_F^2 \mu_R} \right) \right] \tag{6.6}$$

$$\begin{aligned} T_{wk}(\tilde{k}_+) = {} & \frac{\alpha}{4\pi} C_F T^{(0)}(\tilde{k}_+) \\ & \times \left[\frac{2m_b^2 + 3a}{a + m_b^2} \ln\left(\frac{-a}{m_b^2} \right) - 2\ln\left(1 + \frac{m_b^2}{a} \right) \ln\left(\frac{-m_b^2}{a} \right) \right. \\ & - 2\mathrm{Li}_2\left(\frac{-m_b^2}{a} \right) + \ln\left(\frac{a}{2q.\tilde{k}} \right) \ln\left(\frac{a\ 2q.\tilde{k}}{m_b^4} \right) - 2i\pi \ln\left(\frac{a}{2q.\tilde{k}} \right) \\ & \left. + 2\ln\left(\frac{-\mu_R\ m_b}{2q.\tilde{k}} \right) - 2\ln^2\left(\frac{a}{2q.\tilde{k}} \right) - \frac{17\pi^2}{12} - 2\ln^2\left(\frac{\mu_F}{\sqrt{2}\tilde{k}_+} \right) \right] \end{aligned} \tag{6.7}$$

The contribution from the box amplitude is suppressed due to it being next to leading order in $1/m_b$

$$T_{Box} = 0 \tag{6.8}$$

The above expressions illustrate how the renormalisation scale cancels in the NLO Hard Kernel, something which is a necessary condition for factorisation to hold at an order. T_{em} and $T_{\bar{u}u}$ cancel their μ_R's mutually. T_u and T_b give a μ_R which cancels with the one in T_{wk}.

Lorentz decomposition of the amplitudes gives an expression in terms of two scalar form factors F_V and F_A and those were shown to be equal. They can be computed from the Hard Kernel and a model dependant expression of the Φ_+ component of the LCDA.

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