On Metric Dimension of Certain Extensions of Regular Graphs

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Dedicated to

My Loving Parents

&

My Husband

for supporting me all the way.

Abstract

A very important and useful parameter in graph theory is the metric dimension of a graph that has many applications in image processing, pattern recognition, networking, chemistry, mastermind games, robot navigation and combinatorial optimization, to name a few. In this thesis, we study certain graph classes with respect to their metric dimension.

The purpose of this research is two fold. First, we want to know that how the value of the metric dimension is affected by making a change in the graph? Secondly, we want to construct certain extension of graphs such that the metric dimension of the resulting graph is not changed or there is a minimum change in the metric dimension of resulting graphs. This construction is useful to expand the existing network with minimum cost.

We investigate the metric dimension of s-crown graph. It can be seen that an s-crown graph denoted by Cr_s constitutes a family of graph with constant metric dimension and $\beta(Cr_s) = 2$. We determine the upper bounds for the metric dimensions of certain extensions of antiprism and Möbius ladder, denoted by $A(n, 2)$ and M'_n respectively. The upper bounds are also found for the metric dimension of barycentric subdivision and corona product of Möbius ladder with complete graph K_1. We find that all the classes of graph we have studied, have bounded metric dimensions.

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Introduction

The concept of resolving set for graphs was first introduced by Slater in [31] with different terminology. He referred the resolving set as *locating set*, minimum resolving set as *reference set* and cardinality of reference set as *location number*. Harary and Melter [14] introduced this concept independently and used the term metric dimension instead of location number. It has wide applications in different areas of research. Its application in chemistry were discussed in [8]. These concepts are also used in image processing and in pattern recognition problems. The problem of finding metric dimension in graphs is NP-complete [12].

This thesis is devoted to the study of some classes of graphs with respect to the nature of their metric dimension. It consists of five chapters. The first chapter is devoted to some basic concepts and introduction to the terminologies which will be used in subsequent chapters. Some common classes of graph and basic operations on graphs are also introduced in this chapter.

Distance is the basic concept in understanding the resolvability in graphs. The second chapter deals with the study about distances in graphs and its different invariants. Some useful and known results are also presented in this chapter.

The third chapter includes the concept and a brief history of resolvability in graphs and metric dimension. Some known results on metric dimension are also included in this chapter. Further, few famous classes of graphs are discussed under the parameter of metric dimension.

The fourth chapter contains the main results of this thesis. We investigate the metric dimension of s-crown graph. We also determine the upper bounds for the metric dimensions of certain extensions of antiprism and Möbius ladder, denoted by $A(n, 2)$ and M'_n respectively. The upper bounds are also found for the metric dimension of barycentric subdivision of antiprism and corona product of Möbius ladder with complete graph K_1. We shall find that all the classes of graph that are discussed in this chapter have bounded metric dimension. The fifth chapter is the conclusion and contains some open problems that are arising from the text for the future research in this direction.

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Chapter 1

Preliminaries and basic concepts

In this chapter, we discuss the basic graph theoretic terminologies with examples and very basic results on graphs which are needed for the subsequent chapters. It also consists of some common classes of graphs and operation on graphs. We refer to Chartrand [10] for graph theoretic terminologies used in this chapter.
In mathematics, graph theory is the study of graphs, which are mathematical structures, used to model pair-wise relations between objects. Graphs can be used to model many types of relations and processes in physical, biological, social and information systems. In computer science, graphs are used to represent networks of communication, data organization, computational devices and the flow of computation etc. Graph theory is also used to study molecules in chemistry and physics. Since it has a very close connection with other sciences, therefore it is categorized in different branches. Chemical graph theory, topological graph theory, spectral graph theory to name a few that are to be discussed in this section.

- *Chemical graph theory* is the topology branch of mathematical chemistry which applies graph theory to mathematical modeling of chemical phenomena.
- *Spectral graph theory* is the study and exploration of graphs through the characteristic polynomials, eigenvalues and eigenvectors of matrices associated with the graph, such as its adjacency matrix or Laplacian matrix.

- *Topological graph theory* is the study of embedding of graphs in surfaces, graphs as topological spaces and the methods of embedding graphs into topological spaces so that certain graph-theoretic properties are maintained.

- *Metric graph theory* is a branch of graph theory in which we study the properties of graphs with respect to the distance related parameters.

- *Algebraic graph theory* is the application of abstract algebra (sometimes associated with matrix groups) to graph theory. Matrices and other algebraic properties can be used to prove many interesting results about graphs.

- *Extremal graph theory* studies extremal (maximal or minimal) graphs which satisfy a certain property. Extremality can be taken with respect to different graph invariants, such as order, size or girth to name a few. It also studies how global properties of a graph influence local substructures of the graph.

1.1 Basic Concepts

A *graph* G is a triple consisting of a vertex set $V(G)$, an edge set $E(G)$, and a relation that associates with each edge two vertices (not necessarily distinct) called its endpoints. The number of vertices in a graph is called *order* of G, denoted by n, and the number of edges in a graph is called *size* of G, denoted by m. If a vertex set and edge set of a graph are finite then the graph is called *finite graph*, otherwise *infinite graph*.

An edge whose endpoints are same is called a *loop* while the edges having the same pair of endpoints are called *multiple* or *parallel edges*. A graph consists of exactly one vertex is called a *trivial graph*. A graph having no loops and multiple edges is called a *simple graph*. Two vertices u and v are said to be *adjacent* or *neighbors* if there is an edge between u and v, written as $uv \in E(G)$ and set of all neighbours of a vertex v is called neighbourhood of v in G. In Fig. 1.1, we can see that the neighbourhood of v_5 is: $N_{v_5} = \{v_2, v_4, v_6\}$. Similarly, two edges are said to be *adjacent* if they have a common endpoint. If there is an edge e between two vertices u and v then we say that e is incident to its endpoints u and v.

The number of edges incident to a vertex v is known as *degree* of v, except that each loop at v counts twice. The degree of a vertex is denoted by $d(v)$. The sum of degrees of every vertex of a graph is called the *total degree* of a graph. A vertex having 0 degree is called an *isolated vertex* and a vertex of degree 1 is called a *pendant vertex* or a *leaf.* The minimum degree among the vertices of a graph G is called *minimum degree* of G, denoted by $\delta(G)$, and the maximum degree among the vertices of G is called *maximum degree* of G, denoted by $\Delta(G)$. Mathematically, we have

$$\delta(G) = \min\{d(v) : v \in V(G)\}$$

$$\Delta(G) = \max\{d(v) : v \in V(G)\}$$

We illustrate the above concepts by an example. Consider the graph in Fig. 1.1.

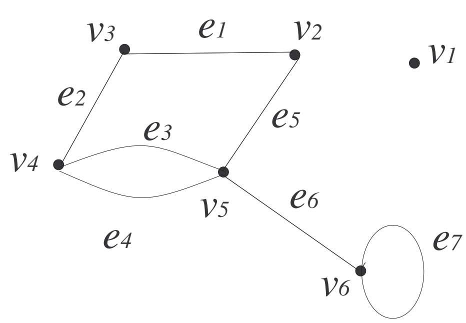

Figure 1.1: A graph G

Here G is a multiple graph of order 6 and size 7. In this graph, v_1 is an isolated vertex, e_7 is a loop while e_3 and e_4 are multiple edges. The maximum and minimum degree of G are 4 and 0 respectively.

A graph G is *regular* if every vertex of G has same degree. In other words, G is regular if and only if $\Delta(G) = \delta(G)$. A graph is called *k-regular* if the degree of its each vertex is k. A 3-regular graph is called a *cubic* graph. A 2-regular and a 3-regular graphs are shown in Fig.1.2.

One of the useful result in counting problems is the "First Theorem of Graph

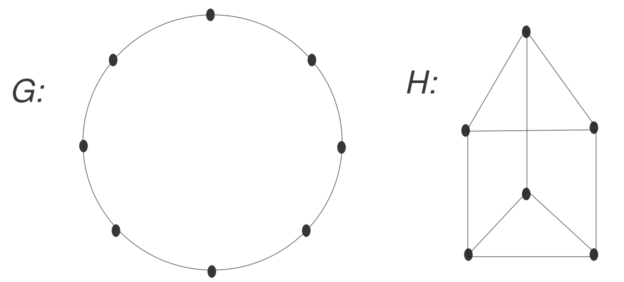

Figure 1.2: A 2-regular and a 3-regular graph

Theory" or the "Handshaking Lemma". This theorem shows the relation between the size of a graph and the total degree of a graph.

Theorem 1.1.1. [10] *Let G be a graph of size m, then*

$$\sum_{v \in V(G)} d(v) = 2m$$

The following results are immediate consequences of Theorem 1.1.1.

Corollary 1.1.2. [10] *Every graph contains an even number of vertices of odd degree.*

Corollary 1.1.3. [11] *Let G be a k-regular graph with n vertices, then G has $\frac{nk}{2}$ edges.*

A *walk* W in G is a sequence of vertices in G such that consecutive vertices in the sequence are adjacent. A $u-v$ walk is a walk that starts from a vertex u and ends at a vertex v. We can express W as $W : u = v_0, v_1, ..., v_k = v$, where $k \geq 0$.
A walk with no repeated edge is called a *trail*. A $u-v$ walk or trail is *open* if $u \neq v$ and it is *closed* if $u = v$. A $u-v$ walk with no repeated vertex is called a $u-v$ *path*. On the other hand, a simple graph G whose vertices can be listed in a sequence such that two vertices are consecutive in a list if and only if they are adjacent in G and no vertex repeats itself in the sequence, such a graph is called *path*. A path consists of n vertices is denoted by P_n and it has $n-1$ edges. The length of walk, trail or

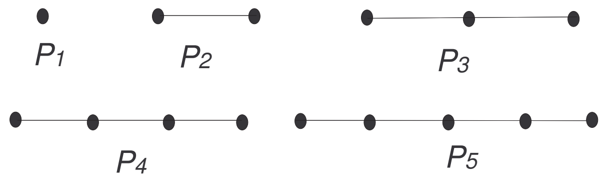

Figure 1.3: Paths $P_i\,(1 \leq i \leq 5)$

path is the number of edges in it.

A *circuit* is a closed trail of length three or more. Hence no edge is repeated in a circuit but vertices can be repeated. A *cycle* is a circuit with no repeated vertex, except for the first and last. We can also say that cycle is a closed path. The length of cycle is the number of edges in it. A cycle of odd length is called *odd cycle* while a cycle of even length is called *even cycle*. A graph with no cycle is called *acyclic graph* while a connected graph having exactly one cycle is *unicyclic graph*.

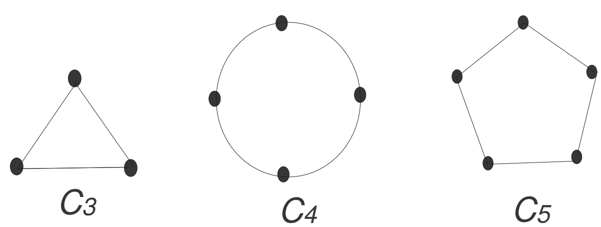

Figure 1.4: Cycles C_3, C_4 and C_5

Let G and H be two graphs such that $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$, then H is a *subgraph* of G written as $H \subseteq G$. If $V(H) \subset V(G)$ or $E(H) \subset E(G)$, then H is *proper subgraph* of G. A subgraph H is a *spanning subgraph* of G if H has same vertex set as G. A subgraph H is *induced subgraph* of G if for any $u, v \in V(H)$ and $uv \in E(G)$, there exist $uv \in E(H)$. A list of subgraphs of a graph G such that each edge of G appears in exactly one subgraph in the list is called *decomposition* of a graph G.

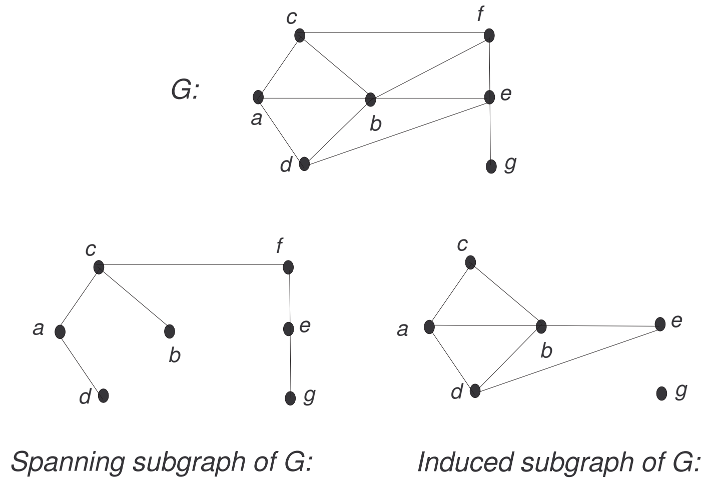

Figure 1.5: Subgraphs of a graph G

A graph G is *connected* if for every pair of distinct vertices, say u and v, G contains a $u-v$ path otherwise G is *disconnected*. Let G be a connected graph and $u, v \in V(G)$, the *distance* from u to v is the length of shortest path from u to v, written as $d(u, v)$. The *diameter* of G that is denoted by $diam(G)$ is $\max\limits_{u,v \in V(G)} d(u, v)$.
Two graphs G_1 and G_2 are said to be *isomorphic*, denoted by $G_1 \cong G_2$, if there exists a bijection f between their vertex sets such that two vertices are adjacent in G_1 if and only if their images under f are adjacent in G_2. In other words, $G_1 \cong G_2$ if there exists a bijection $f : V(G_1) \to V(G_2)$ such that an edge $xy \in E(G_1)$ if and only if $f(x)f(y) \in E(G_2)$.
The order and size of two isomorphic graphs are always same.

Theorem 1.1.4. [10] *If two graph G_1 and G_2 are isomorphic, then the degrees of the vertices of G_1 are the same as the degrees of the vertices of G_2.*

Let G be a simple graph, the *complement* $\overline{G}$ of G is a simple graph with vertex set $V(G)$ such that $uv \in E(\overline{G})$ if and only if $uv \notin E(G)$. In other words, a graph H is said to be a complement of G if H has same vertex set as G and two distinct vertices of H are adjacent if and only if they are not adjacent in G. If a graph G and its complement $\overline{G}$ are isomorphic to each other, then G is *self-complementary*. A set of pairwise adjacent vertices is called a *clique* in a graph while a set of pairwise nonadjacent vertices is called an *independent set* or *stable set*.

Theorem 1.1.5. [11] *If a graph G is disconnected, then its complement $\overline{G}$ is connected.*

If two graphs are isomorphic under a function f, then their complements are also isomorphic.

Theorem 1.1.6. [10] *Two graph G_1 and G_2 are isomorphic if and only if their complements $\overline{G_1}$ and $\overline{G_2}$ are isomorphic.*

1.2 Some common graph classes

A *complete* graph with n vertices denoted by K_n is a simple graph in which each pair of distinct vertices is adjacent. The size of K_n is $\binom{n}{2}$. Every complete graph with n vertices is $(n-1)$-regular.

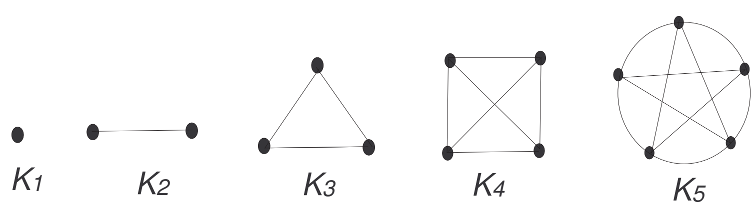

Figure 1.6: Complete graphs $K_i\,(1 \le i \le 5)$

A graph with no edge is said to be an *empty graph.* The complement of K_n denoted by $\overline{K_n}$ is an empty graph with n vertices.
A simple graph whose vertex set $V(G)$ can be partitioned into two sets X and Y, called partite sets, such that every edge of G has one endpoint in X and other endpoint in Y, is called a *bipartite graph.*
The next theorem gives a useful characterization of bipartite graph.

Theorem 1.2.1. [11] *A graph G is bipartite if and only if G has no odd cycle.*

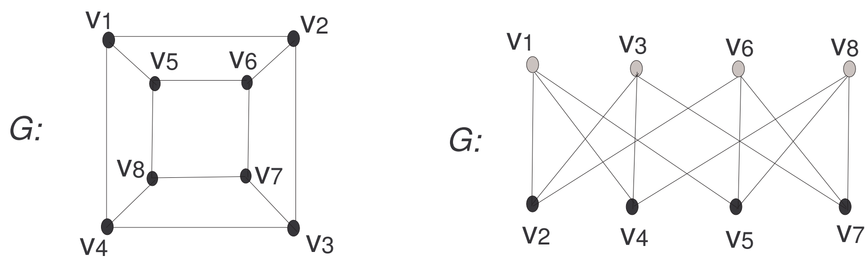

Figure 1.7: A graph G and its bipartition

A bipartite graph with partite set X and Y is called *complete bipartite* if every vertex of X is adjacent to every vertex of Y. Sometimes it is also referred as *biclique* and is denoted by $K_{r,s}$, where r and s are the cardinalities of X and Y. If either r or s is 1, then it is referred as a *star*.

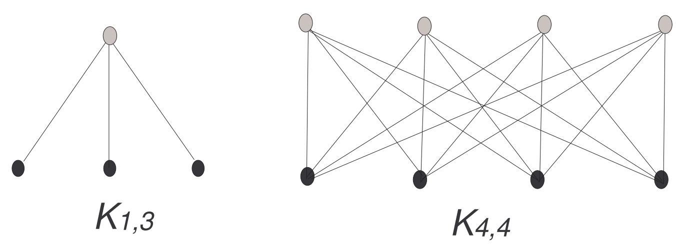

Figure 1.8: Complete bipartite graphs

An acyclic graph is called a *forest*. An acyclic connected graph is called a *tree*.
A caterpillar is a tree in which the removal of its pendant vertices produces a path called the spine of the caterpillar. In Fig. 1.9, T_1 and T_3 are caterpillars. It can be seen that every path and star is a caterpillar.
In the sequel, we describe the important properties of trees.

Theorem 1.2.2. [10] *A graph G is a tree if and only if there is a unique path between every two vertices of G.*

Since trees are acyclic, therefore trees have at least one pendant vertex. The next two theorems use the acyclic property of a tree.

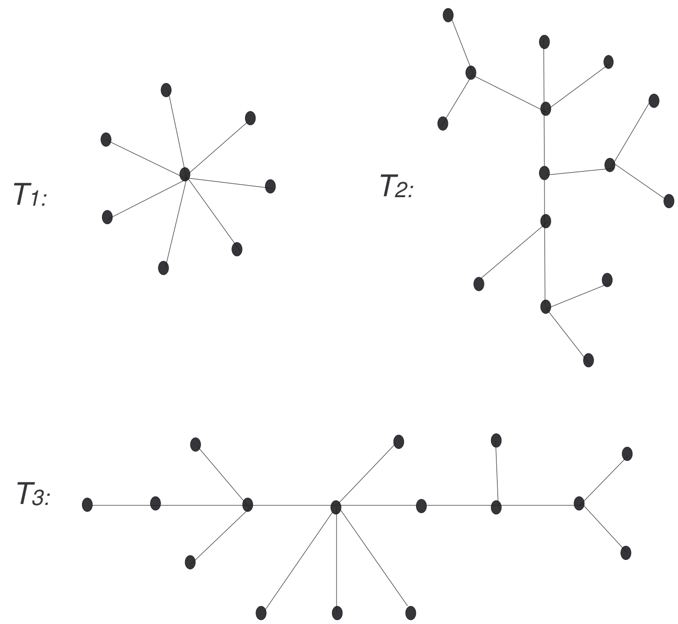

Figure 1.9: Trees

Theorem 1.2.3. [11] *Every tree is bipartite.*

Theorem 1.2.4. [10] *Every non trivial tree has at least two pendant vertices.*

We have observed that number of edges in a tree is one less than the number of its vertices. We can see in Fig. 1.9, T_1 has 8 vertices and 7 edges, T_2 has 15 vertices and 14 edges and T_3 has 16 vertices and 15 edges.

Theorem 1.2.5. [10] *Every tree with n vertices has $n-1$ edges.*

1.3 Graph operations

A new graph can be produced by performing different operations on graphs. Graph operations are widely classified into unary and binary operations. *Unary operation* produces a new graph from one graph. For example, addition or deletion of a vertex or an edge in the existing graph creates a new graph. On the other hand, *binary operations* produce a new graph from two initial graphs. Some commonly used

graph operations will be discussed in this section.
One of the basic operation in graph theory is edge contraction. A *contraction* of edge e with endpoints u and v is a replacement of vertices u and v with a single

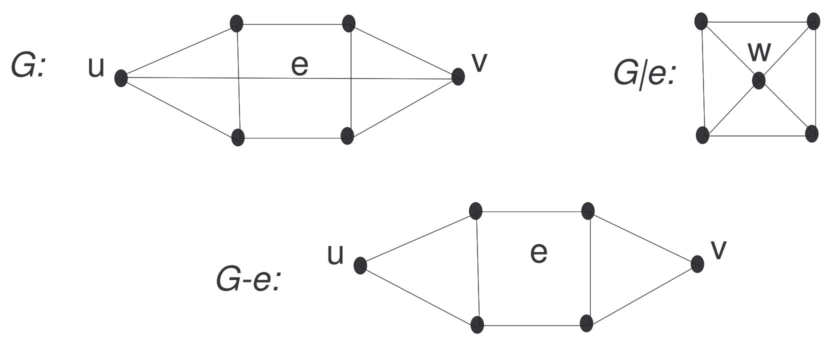

Figure 1.10: Subgraphs produced by contraction and deletion of edge e in G

vertex w. All the edges that were incident to u and v are now incident to w except the edge e. The resulting graph, denoted by $G|e$, has one less edge than the original graph. The *deletion* of an edge e in a graph G produces a subgraph $G - e$ which consists of all the vertices and edges of G except the edge e. The contraction and deletion of edge e in G are shown in Fig. 1.10. Similarly, the *deletion* of a vertex v in a graph G produces a subgraph $G - v$ which consists of all the vertices and edges of G except the vertex v and the edges that are incident to v.
For a graph G, the *k-th* power G^k of G is a graph that has vertex set $V(G)$ and every two vertices that are at a distance k in G are adjacent in G^k. We can define a *square of a graph* G, denoted by G^2, is a graph with vertex set $V(G)$ and every two vertices that are at a distance 2 in G are adjacent in G^2. For a graph G with diameter k, G^k is a complete graph. In Fig.1.11, we can see that a graph H has diameter 2 and its square H^2 is a complete graph K_7.
The *disjoint union* of two graphs G and H, denoted by $G \cup H$ is a graph obtained by taking the union of graphs G and H with disjoint vertex sets $V(G)$ and $V(H)$. The union of two graphs is always a disconnected graph. While *join* of two graphs G and H denoted by $G + H$ consists of $G \cup H$ and all the edges joining every vertex of G to every vertex of H. The join of two graph is shown in Fig. 1.12.

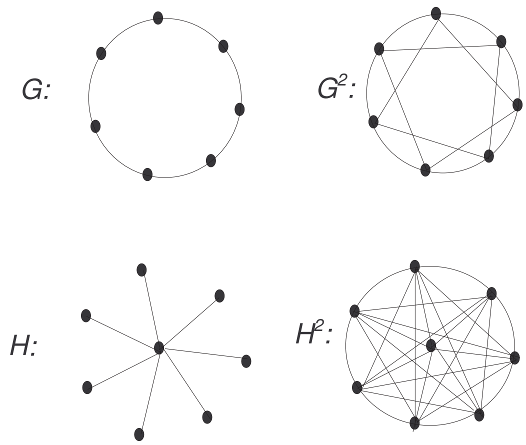

Figure 1.11: Square of graphs

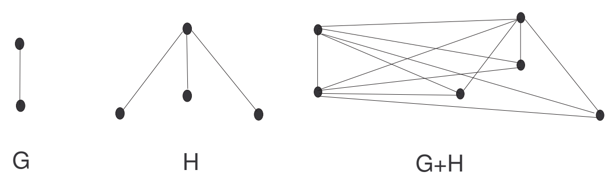

Figure 1.12: Join of two graphs $G + H$

For a graph G, the *line graph* of G, written $L(G)$ is a graph in which the edges of G represents its vertices and two vertices in $L(G)$ are adjacent if there corresponding edges have a common endpoint in G.

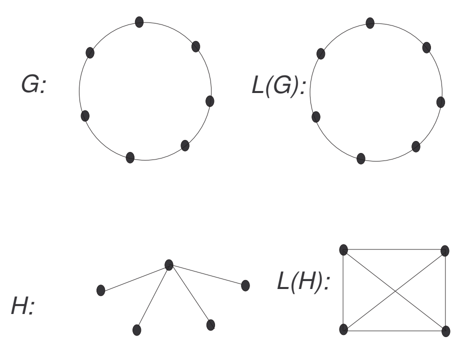

Figure 1.13: Graphs and their corresponding line graphs

The *Cartesian product* of two graphs G and H is a graph,written $G\Box H$ whose vertex set is the cartesian product $V(G)\times V(H)$, and two vertices (u,u') and (v,v') in $G\Box H$

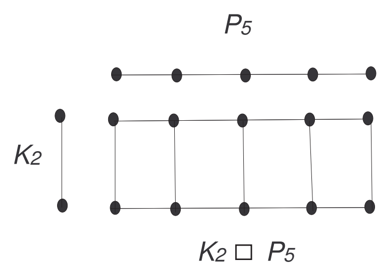

Figure 1.14: Cartesian product of K_2 and a path

are adjacent if either $u=v$ and u' is adjacent to v' in H or $u'=v'$ and u is adjacent to v in G. The cartesian product obeys commutative and associative law.
The Cartesian product of K_2 and a path is a *ladder graph.* The Cartesian product of two paths is called a *grid graph.*

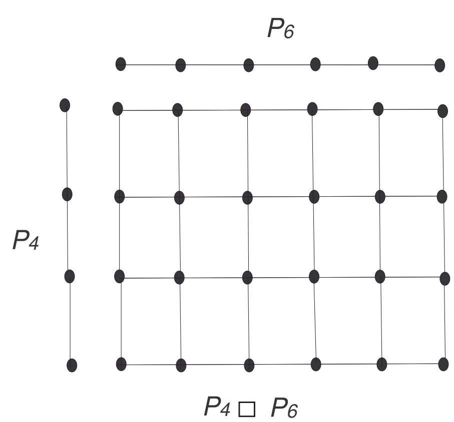

Figure 1.15: Cartesian product of two paths $P_4\Box P_6$

The *corona product* of two graphs G and H is a graph,written $G\odot H$ which is the disjoint union of one copy of G and $|V(G)|$ copies of $H(|V(G)|$ is the order of $G)$ such that each vertex of the copy of G is adjacent to all vertices of a separate copy of H. It is neither commutative nor associative.

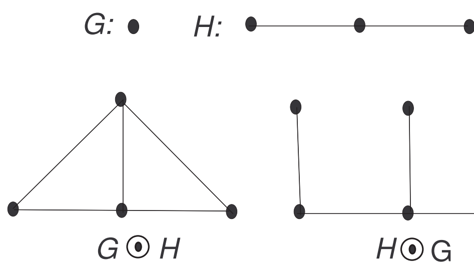

Figure 1.16: Corona product of graphs

The operation of subdivision is used to generate a simple graph from any graph. The *barycentric subdivision* of a graph is a subdivision in which a new vertex of degree two is added to the interior of each edge.

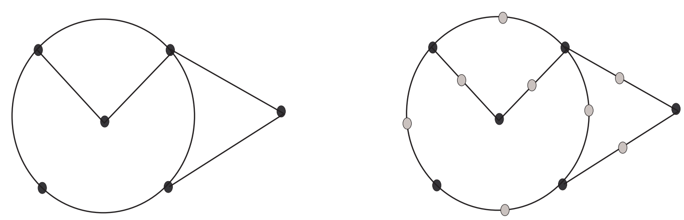

Figure 1.17: Barycentric subdivision of a graph

Following are the properties of barycentric subdivision of a graph.

- The barycentric subdivision is used to transform a multigraph into a simple graph.
- The barycentric subdivision of any graph is a bipartite graph.
- The barycentric subdivision of any graph yields a loopless graph.
- The barycentric subdivision of any loopless graph yields a simple graph.

1.4 Connectivity in graphs

In a connected graph G, a vertex v is said to be *cut-vertex* of G if $G - v$ is disconnected. Consider a graph in Fig.1.18, the vertices b, c and f are the cut-vertices of

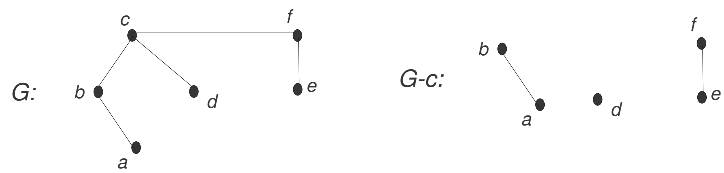

Figure 1.18: The graph G and $G - c$

G but there is no cut vertex in $G - c$. Similarly, an edge e is said to be *cut-edge* or *bridge* in G if $G - e$ is disconnected. A connected graph having bridge contains cut-vertices as well. The following facts about cut-vertices were established in the following theorem.

Theorem 1.4.1. [10] *Let G be a connected graph and a bridge is incident to a vertex v. Then v is a cut-vertex of G if and only if $deg(v) \geq 2$.*

The following is an immediate consequence of Theorem 1.4.1.

Corollary 1.4.2. [10] *Let T be a nontrivial tree and v is not the pendant vertex in T, then v is a cut-vertex of T.*

Another consequence of Theorem 1.4.1 is stated below.

Corollary 1.4.3. [10] *Let a graph G of order three or more is connected and G contains a bridge, then G contains a cut-vertex.*

A nontrivial connected graph is said to be *non-separable* if it contains no cut-vertices. For example, K_2 is a non-separable graph.

Let we have a set U of vertices of a graph G such that $G - U$ is disconnected, then U is called a *vertex-cut* in a graph G. A vertex-cut of minimum cardinality in a graph is called *minimum vertex-cut*.

The *vertex-connectivity* or *connectivity* of a graph G, denoted by $\kappa(G)$ is the minimum cardinality among all the subsets U of the vertex set of G such that $G - U$ is either disconnected or trivial. For an n-vertex graph G,

$$0 \leq \kappa(G) \leq n - 1.$$

A set E of edges of a graph G such that $G - E$ is disconnected, then E is called an *edge-cut* in a graph G. An edge-cut of minimum cardinality in a graph is called *minimum edge-cut*. An edge-cut E of a connected graph G is *minimal* if no proper subset of E is an edge-cut of G. If G is a connected graph and E is a minimal edge-cut of G, then $G - E$ contains exactly two components H_1 and H_2. This means that E contains all those edges of G joining H_1 and H_2.
The *edge-connectivity* of a graph G, denoted by $\lambda(G)$ is the minimum cardinality among all the subsets E of the edge set of G such that $G - E$ is either disconnected or trivial. For an n-vertex graph G,

$$0 \leq \lambda(G) \leq n - 1.$$

The relationship between connectivity, edge-connectivity and minimum degree of a graph is given in the following theorem.

Theorem 1.4.4. [10] *Let G be a graph, then we have*

$$\kappa(G) \leq \lambda(G) \leq \delta(G).$$

A property of a cubic graph in term of connectivity is stated in the next theorem.

Theorem 1.4.5. [10] *Let G be a cubic graph, then we have $\kappa(G) = \lambda(G)$.*

1.5 Planarity in graphs

Let G be a graph such that G can be drawn in a plane without crossing any pair of its edges, then G is called a *planar graph*. If a graph is not planar, then it is called a *non-planar graph*. A graph that is drawn in the plane in which no two edges of a graph cross each other is called a *plane graph*. A complete graph K_4 is a planar

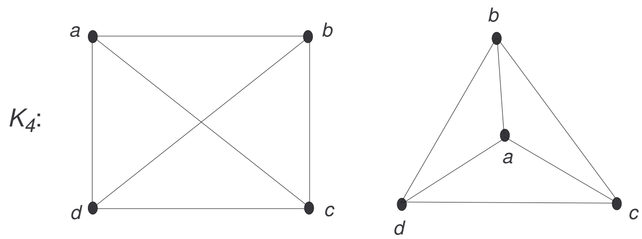

Figure 1.19: K_4 and its planar drawing

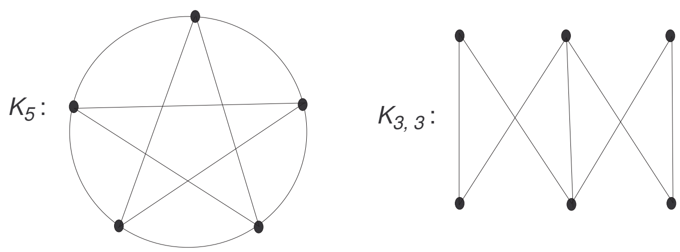

Figure 1.20: Non-planar graphs K_5 and $K_{3,3}$

graph. K_4 and its plane graph are shown in Fig. 1.19.
Path, cycle, star and tree are some well-known classes that are planar but there are some common classes of graphs that are nonplanar that are discussed in the following theorems.

Theorem 1.5.1. [10] *The complete graph K_5 is nonplanar.*

Theorem 1.5.2. [10] *The complete bipartite graph $K_{3,3}$ is nonplanar.*

A plane graph divides the plane into connected pieces called *regions*. In Fig. 1.21, a plane graph H divides the plane into four regions R_1, R_2, R_3 and R_4. The region which is unbounded in a plane graph is called the *exterior region*. R_4 is the exterior region in Fig. 1.21. In a plane graph, a subgraph induced by the edges and vertices that are incident to a region R is called the *boundary* of R. Leonhard Euler presented a very useful result, known as *Euler Identity*, and it is stated as:

Theorem 1.5.3. [10](The Euler Identity) *Let G be a connected plane graph having order n, size m and region r, then*

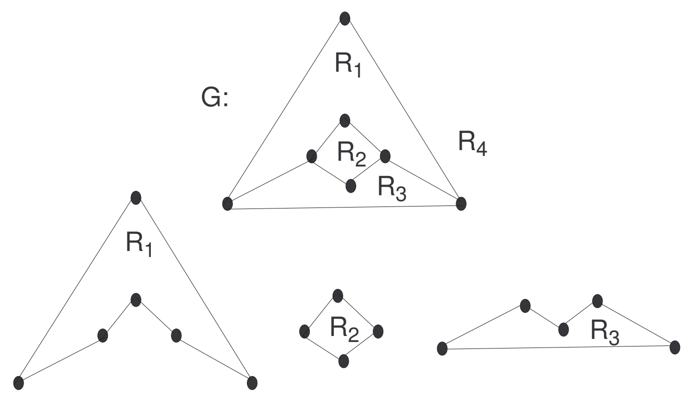

Figure 1.21: A plane graph and its regions

$$n - m + r = 2.$$

The Euler Identity has many interesting consequences which are to be discussed in the following theorems.

Theorem 1.5.4. [10] *Let G be a planar graph having order $n \geq 3$ and size m, then*

$$m \leq 3n - 6.$$

The following theorem characterizes the planar graphs in terms of its forbidden subgraphs.

Theorem 1.5.5. [10] *A graph is planar if and only if it does not contain K_5 or $K_{3,3}$ as a subgraph.*

Chapter 2

Distance in Graphs

In this chapter, we discuss the distance in graphs, its variants with examples and their illustration through different graphs-theoretic parameters. Some important properties of graphs related to different distance related parameters are also presented in this chapter. All the graphs studies in this thesis are simple, finite and connected.

2.1 Distance and its related parameter

Distance and its related parameters such as radius, eccentricity and diameter etc. are most commonly used invariants of a graph. Let G be a connected graph and $u, v \in V(G)$, the *distance* from u to v is the length of shortest path from u to v, written as $d(u, v)$. The distance $d(u, v)$ is also called a $u - v$ *geodesic*.

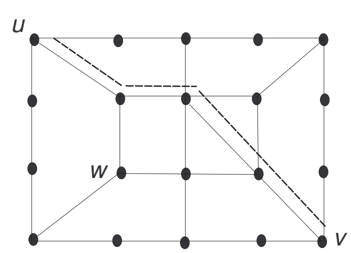

Figure 2.1: Distance between vertices in a graph

In Fig. 2.1, the distance between u and v denoted by $d(u, v)$ is 4 which is shown in dotted line and the distance between the vertices u and w is 2.
The distance function d satisfies the properties of a metric over a vertex set $V(G)$ and satisfies the following axioms:
For any $u, v, w \in V(G)$ of a connected graph G,

- $d(u, v) \geq 0$ for all $u, v \in V(G)$.
- $d(u, v) = 0$ if and only if $u = v$.
- $d(u, v) = d(v, u)$ for all $u, v \in V(G)$.
- $d(u, w) \leq d(u, v) + d(v, w)$ for all $u, v, w \in V(G)$.

Thus $(V(G), d)$ is a metric space.
The distance from v to a vertex u which is farthest from v in G is called the *eccentricity* of v written $e(v)$, that is,

$$e(v) = max\{d(u, v) : u \in V(G)\}.$$

The maximum eccentricity among the vertices of G is the *diameter* of G and the minimum eccentricity is its *radius*, written diam(G) and rad(G), respectively. Mathematically,

$$diam(G) = max\{e(v) : v \in V(G)\},$$
$$rad(G) = min\{e(v) : v \in V(G)\}.$$

The graph shown in Fig. 2.2 has diameter 6 and its radius is 3.
In graphs theory, we can not write diameter in terms of radius and vice versa in general. But somehow we can relate the radius and diameter which is stated in the following theorem.

Theorem 2.1.1. [10] *Let G be a nontrivial connected graph, then we have*

$$rad(G) \leq diam(G) \leq 2rad(G).$$

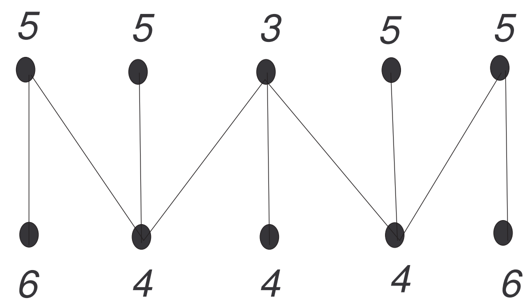

Figure 2.2: The eccentricities of the vertices of a graph

From the Fig. 2.2, we can also observe that the eccentricities of adjacent vertices differ by at most 1. This fact is elaborated in the following theorem and it is true for every connected graph G.

Theorem 2.1.2. [10] *Let G be a connected graph and $uv \in G$, then we have*

$$|e(u) - e(v)| \leq 1.$$

The following result is an immediate consequence of Theorem 2.1.2.

Theorem 2.1.3. [10] *Let G be a connected graph and the vertices u and v are adjacent in G, then for every vertex w of G*

$$|d(u, w) - d(v, w)| \leq 1.$$

2.2 Center and periphery of a graph

In this section, we discuss the center and periphery of a graph and some known results about them.

For a vertex v in G, if $e(v) = rad(G)$, then v is called *central vertex* of G. While the subgraph induced by the central vertices of G is referred as *center* of G denoted by $Cen(G)$. A graph G whose every vertex is a central vertex is called *self-centered* and we have $Cen(G) \cong G$.

For $n \geq 3$, if $G \cong C_n$, then G is called self-centered [10].

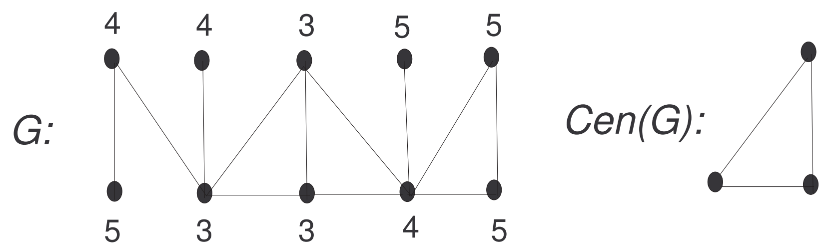

Figure 2.3: A graph and its center

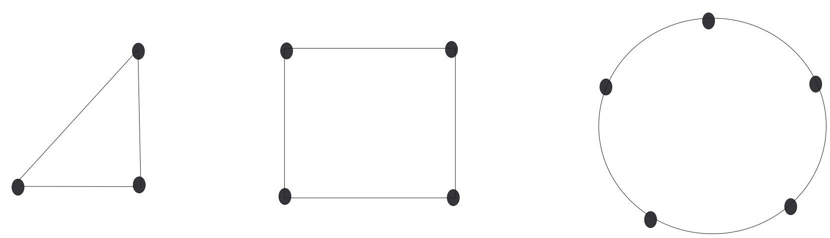

Figure 2.4: Self-centered graphs C_3, C_4 and C_5

Stephen Hedetniemi[10] gave a result regarding the center of a graph which is stated in the following theorem.

Theorem 2.2.1. [10] *Every graph is the center of some graph.*

A vertex v in a connected graph G whose eccentricity is equal to the diameter of G is referred as *peripheral vertex*. A subgraph induced by the peripheral vertices of G is called periphery of G, written as $Per(G)$.

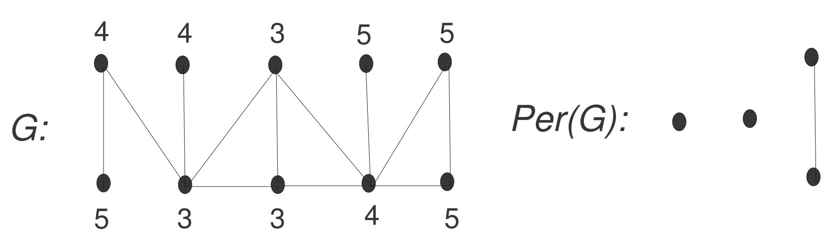

Figure 2.5: A graph and its periphery

The next theorem gives the characterization of periphery graphs.

Theorem 2.2.2. [10] *A nontrivial graph G is the periphery of some graph if and only if every vertex of G has eccentricity* 1 *or no vertex of G has eccentricity* 1*.*

According to Theorem 2.2.2, we can give the following remark.

Remark 2.2.1. [10] *A star of order* 3 *or more is not the periphery of any graph.*

Let u, v be two vertices in a connected graph G such that $d(u, v) = e(v)$, then u is called the *eccentric vertex* of v. Further, if a vertex w in a graph G is an eccentric vertex of some vertex of G, then w is an eccentric vertex of a graph G.

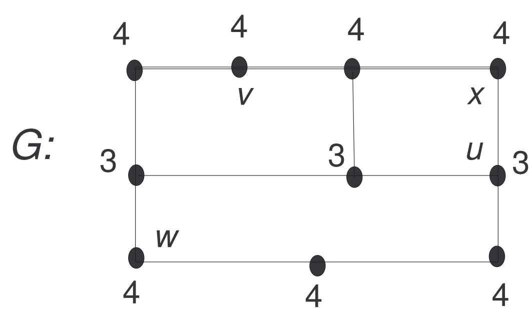

Figure 2.6: An eccentric vertex in a graph

Consider a graph in Fig. 2.6 in which all the vertices of G are labelled with their eccentricities. Here $e(u) = 3$ and we can see that $d(u, v) = 3$ and also $d(u, w) = 3$, it follows that v and w are the eccentric vertices of u in G. Further, we can observe that $e(x) = 4 = diam(G)$, so x is the peripheral vertex in G. Since $e(w) = 4$ and $d(w, x) = 4$, it follows that x is an eccentric vertex of w. From this observation, we establish a result in the following remark.

Remark 2.2.2. [10] *Every peripheral vertex of a graph is an eccentric vertex.*

Let G be a connected graph, if every vertex of G is an eccentric vertex then G is referred as an *eccentric graph*. An eccentric graph is shown in Fig. 2.7.
A subgraph of a graph G induced by the set of eccentric vertices of G is called the *eccentric subgraph* of G, denoted by $Ecc(G)$. A graph with its eccentric subgraph is

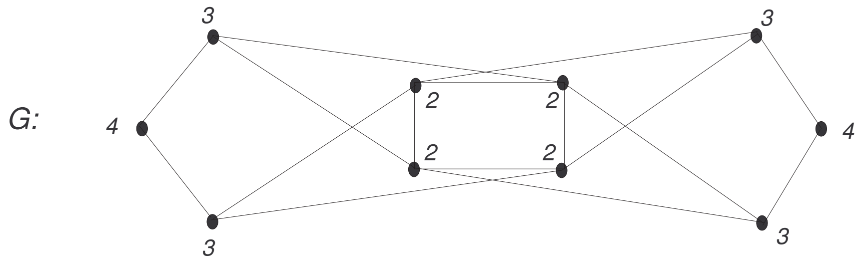

Figure 2.7: An eccentric graph

shown in Fig. 2.8. If every vertex of G is an eccentric vertex, then $Ecc(G) \cong G$. For example, in Fig. 2.7, $Ecc(G) \cong G$. Is every graph an eccentric subgraph of some

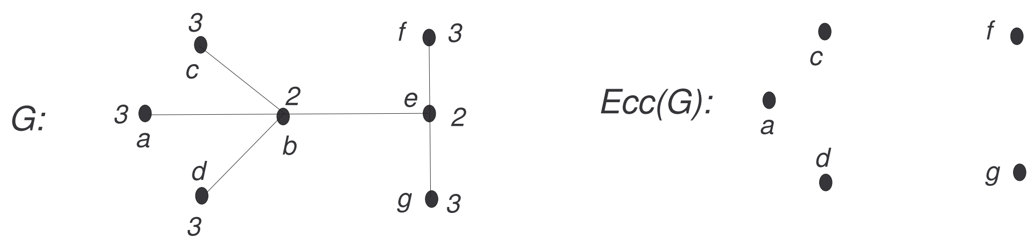

Figure 2.8: A graph and its eccentric subgraph

graph? The answer of this question is given in the following theorem.

Theorem 2.2.3. [10] *A nontrivial graph G is the eccentric subgraph of some graph if and only if every vertex of G has eccentricity* 1 *or no vertex of G has eccentricity* 1.

The concept of status was introduced by Harary[30]. Let G be a connected graph, the sum of the distances from vertex v to every other vertex of G is called the *status* of a vertex v, denoted by $s(v)$.

The *minimum status* of a graph G is the value of the minimum status among the vertices of G, denoted by $ms(G)$ and the *total status*, denoted by $ts(G)$ of a graph G is the sum of all the status values. The set of vertices having minimum status in G is called *median*, denoted by $M(G)$.

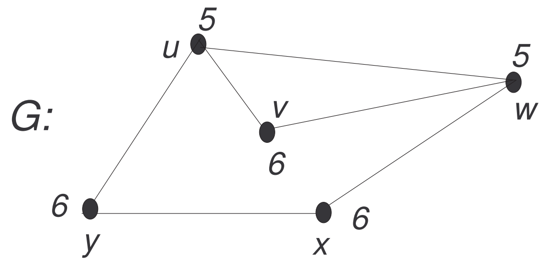

Figure 2.9: A graph whose vertices are labelled with their respective status

Consider the graph G shown in Fig. 2.9, the vertices of G are labelled with their respective status. The minimum status of G is 5 while the total status of G is 28. The set $M(G) = \{u, w\}$ is the median of G.

2.3 Boundary vertex and interior vertex of a graph

Let G be a connected graph and $u, v \in G$. A vertex v is a *boundary vertex* of a vertex u if $d(u, w) \leq d(u, v)$ for each neighbour w of v. If a vertex v in a graph G is a boundary vertex of some vertex of G, then v is a boundary vertex of G.

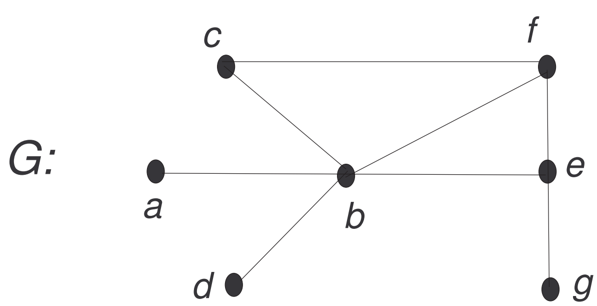

Figure 2.10: A graph G

In Fig. 2.10, the neighbours of vertex f are b, c and e. We calculate the distance from a to f and to its neighbours that are $d(a, f) = 2, d(a, b) = 1, d(a, c) = 2$ and $d(a, e) = 2$. Since the distances from a to all neighbours of f are less or equal to 2,

so f is a boundary of a vertex a.
Some useful properties of boundary vertices are given in the following theorems.

Theorem 2.3.1. [10] *Let v be a cut-vertex in a connected graph G, then v is not a boundary vertex of G.*

Theorem 2.3.2. [10] *Let v be a vertex in a nontrivial connected graph G. Every vertex which is distinct from v is a boundary vertex of v if and only if $e(v) = 1$.*

In a graph G, a vertex v is called *complete vertex (extreme* or *simplicial vertex)* if a subgraph of G induced by the neighbours of v is complete. Every pendant vertex in a graph is complete. In a graph G shown in Fig. 2.11, v is a complete vertex in G since the neighbours of v induced a subgraph K_3.

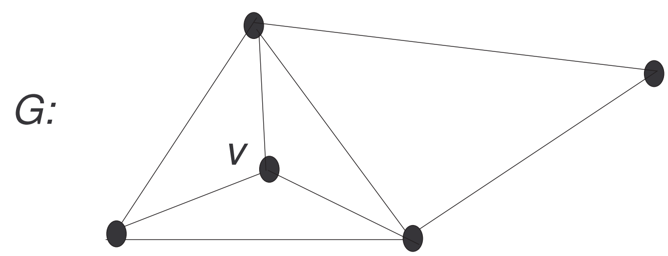

Figure 2.11: A complete vertex v in G

The next theorem gives a characterization of a vertex v to be complete.

Theorem 2.3.3. [10] *In a connected graph G, a vertex v is a boundary vertex of every vertex distinct from v if and only if v is a complete vertex of G.*

Let u, v, w be the distinct vertices in a graph G. A vertex w is said to lie between u and v if $d(u, v) = d(u, w) + d(w, v)$. A vertex v is an *interior vertex* of G if there exist two distinct vertices x and y, other than v such that v lies between x and y. A subgraph induced by the set of interior vertices of G is called *interior* of G, denoted by $Int(G)$.
An interior vertex of a graph G is not a boundary vertex of G and vice versa. This observation is true for any connected graph and is stated in the following theorem.

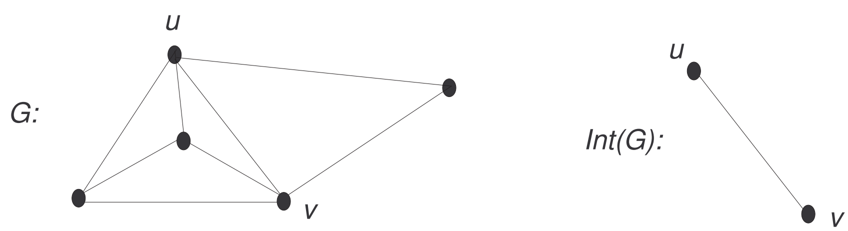

Figure 2.12: A graph and its interior graph

Theorem 2.3.4. [10] *Let v be a vertex of a connected graph G, then v is an interior vertex of G if and only if it is not a boundary vertex of G.*

Chapter 3

Resolvability in graphs and some known results

This chapter includes the discussion about resolvability in graphs and its known results.
Slater introduced the concept of resolving sets and metric dimension in [31] after getting motivation from the problem of uniquely determining the position of an intruder in a network. This concept was further studied independently by Harary and Melter in [14]. Slater represented the metric dimension of a graph as its location number. He used this concept to the placement of a smallest number of loran or sonar detecting devices in a network in order to uniquely determine the location of each vertex in the network in terms of its distances to the devices in the set.

3.1 Resovability in graphs

Consider a connected graph G and a vertex $v \in G$. Let $W = \{w_1, w_2, ..., w_k\}$ be an ordered subset of vertices of G, then the representation of v with respect to W is the k-vector $r(v|W) = (d(v, w_1), d(v, w_2), ..., d(v, w_k))$. For any pair of vertices $u, v \in G$, if $r(u|W) = r(v|W)$ implies $u = v$, then W is called a *resolving set* or *locating set* for G. In other words, W is called a *resolving set* for G if the representation of any pair of vertices of G with respect to W is distinct.

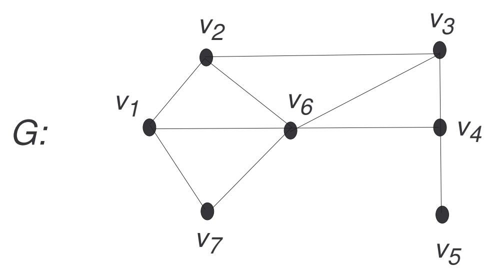

Figure 3.1: A graph G

Consider the graph G shown in Fig. 3.1. If we choose $W = \{v_1, v_6\}$, then $r(v_2|W) = r(v_7|W) = (1, 1)$ and $r(v_3|W) = r(v_4|W) = (2, 1)$. Thus W is not a resolving set for the vertices of G. But if we choose $W' = \{v_1, v_3, v_6\}$, then the representations of vertices of G with respect to W' are

$r(v_1|W') = (0, 2, 1)$, $r(v_2|W') = (1, 1, 1)$, $r(v_3|W') = (2, 0, 1)$,
$r(v_4|W') = (2, 1, 1)$, $r(v_5|W') = (3, 2, 2)$, $r(v_6|W') = (1, 1, 0)$,
$r(v_7|W') = (1, 2, 1)$.

We can see that the representation of every vertex of G with respect to W' is distinct, thus W' is a resolving set for the vertices of G.

A resolving set of minimum cardinality for a graph G is called *metric basis* for G. The cardinality of a metric basis is referred as *metric dimension* or *location number* denoted by $dim(G)$ or $\beta(G)$.

Again consider the graph G in 3.1, if we choose $W'' = \{v_5, v_7\}$, then the representations of vertices of G with respect to W'' are

$r(v_1|W'') = (3, 1)$, $r(v_2|W'') = (3, 2)$, $r(v_3|W'') = (2, 2)$,
$r(v_4|W'') = (1, 2)$, $r(v_5|W'') = (0, 3)$, $r(v_6|W'') = (2, 1)$,
$r(v_7|W'') = (3, 0)$.

Since W'' is the minimum resolving set for the vertices of G, therefore W'' is the metric basis for G and $\beta(G) = 2$.

Let $\mathcal{G}$ be a family of simple connected graphs and $\mathcal{G} = (G_n)_{n\geq 1}$ depending on n as follows: the order $|V(G_n)| = \varphi(n)$ and $\lim\limits_{n\to\infty} \varphi(n) = \infty$. We say that $\mathcal{G}$ has *bounded* *metric dimension* if there exists a constant $C > 0$ such that $\beta(G_n) \leq C$ for every $n \geq 1$; the metric dimension of $\mathcal{G}$ is *unbounded* otherwise. If all graphs in $\mathcal{G}$ have the same metric dimension (which does not depend on n), $\mathcal{G}$ is called a family of graph with *constant metric dimension* [21]. A graph is said to be *k-dimensional* if its metric dimension is k [5].

3.2 Some known results on metric dimension

Let G be a connected graph of order two or more, we can think a resolving set of G as a set $W \subseteq V(G)$ so that each vertex in G is uniquely determined by its distances to the vertices of W. For every ordered set $W \subseteq V(G)$, the only vertex of G whose representation with respect to W is 0 in its $i-th$ coordinate is w_i. So the vertices of W necessarily have distinct representations, therefore we need to examine the vertices of $V(G) \setminus W$. This implies that the metric dimension of G is at most $n-1$. It follows that

$$1 \leq \beta(G) \leq n-1.$$

One of the important and very useful property in finding metric dimension is given in the following theorem.

Theorem 3.2.1. [16] *Let G be a connected graph and W be a resolving set for G. For any $u, v \in V(G)$, if $d(u,w) = d(v,w)$ for every vertex $w \in V(G) \setminus \{u, v\}$, then $\{u, v\} \cap W \neq \emptyset$.*

The following theorems give a complete characterization of graphs having metric dimension 1, $n-1$ and $n-2$, respectively.

Theorem 3.2.2. [8] *Let G be a connected graph of order n, $\beta(G) = 1$ if and only if $G \cong P_n$.*

Theorem 3.2.3. [8] *For a connected graph G of order two or more, $\beta(G) = n-1$ if and only if $G \cong K_n$.*

Theorem 3.2.4. [8] *Let G be a connected simple graph of order $n \geq 4$, $\beta(G) = n-2$ if and only if $G \cong K_{r,s}, r, s \geq 1, G \cong K_r + \overline{K_s}, r \geq 1, s \geq 2$ or $G \cong K_r + (K_1 \cup K_s), r, s \geq 1$.*

The properties of simple connected graphs with metric dimension 2 are presented in next two theorems.

Theorem 3.2.5. [26] *Let G be a simple connected graph with metric dimension 2 and the metric basis in G be $\{v_1, v_2\} \subseteq V(G)$, then the degrees of v_1 and v_2 are at most 3 and there exists a unique path between v_1 and v_2.*

Theorem 3.2.6. [26] *Let G be a simple connected graph with metric dimension 2, then G can not have the following*:

- K_5 *as a subgraph.*
- $K_5 - e$ *as a subgraph, where e is an edge.*
- $K_{3,3}$ *as a subgraph.*
- *The Petersen graph as a subgraph.*

Moreover, we also know the diameter and maximum degree of G, thus bounds for the metric dimension can be deduced.

Theorem 3.2.7. [10] *Let G be a nontrivial connected graph of order two or more, then*

$$\lceil \log_3(\Delta + 1) \rceil \leq \beta(G) \leq n - \mathrm{diam}(G).$$

Cycles with $n(\geq 3)$ vertices constitute a family of graphs with metric dimension 2. The prisms denoted by D_n are the trivalent plane graphs obtained by the Cartesian product of the path P_2 with a cycle C_n; they also constitute a family of 3-regular graphs with constant metric dimension. The metric dimension for the Cartesian product of path and cycle was determined in [6]. It was proved that [6]

$$\beta(P_m \square C_n) = \begin{cases} 2, & \text{if } n \text{ is odd;} \\ 3, & \text{otherwise.} \end{cases}$$

This implies that

$$\beta(D_n) = \begin{cases} 2, & \text{if } n \text{ is odd;} \\ 3, & \text{otherwise.} \end{cases}$$

The antiprism denoted by A_n $(n \geq 3)$ is a plane graph, consists of an outer n-cycle $y_1, y_2, ..., y_n$, an inner n-cycle $x_1, x_2, ..., x_n$, and a set of n spokes $x_i y_i$ and $x_{i+1} y_i$, where $n+i$ is taken modulo n. In [21], Javaid et al. proved antiprisms A_n constitutes a family of regular graphs with constant metric dimension for every $n \geq 3$. This result is established in the following theorem.

Theorem 3.2.8. [21] *Let A_n $(n \geq 3)$ be the antiprism, then*

$$\beta(A_n) = 3$$

In [27], Ali et al. studied the extensions of antiprism. These extensions of antiprism are denoted by H_n and R_n. The graph H_n can be produced from the graph A_n by deleting the edges $y_i y_{i+1}$ from A_n. For each $i = 1, 2, ..., n$, we introduce new vertices c_i and d_i for y_i and x_i, respectively. For each $i = 1, 2, ..., n$, introduce new edges $x_i c_i, y_i d_i, c_i d_i$ and $x_i c_i$, where $n+i$ is taken modulo n.

Theorem 3.2.9. [27] *Let H_n be a graph. For $n \geq 6$, we have*

$$\beta(H_n) = 3.$$

The graph R_n can be produced from the graph A_n by deleting the edges $y_i y_{i+1}$ from A_n. For each $i = 1, 2, ..., n$, we introduce new vertices c_i and d_i for y_i and x_i respectively. For each $i = 1, 2, ..., n$, introduce new edges $x_i c_i, y_i d_i, c_i d_i, d_i d_{i+1}$ and $x_i c_i$, where $n+i$ is taken modulo n.

Theorem 3.2.10. [27] *Let R_n be a graph. For $n \geq 6$, we have*

$$\beta(R_n) = 3.$$

Another antiprism related graph which is known as generalized antiprism was studied by Naeem *et al.* [29]. A generalized antiprism A_n^m can be obtained by the generlized prism $C_m \square P_n$ by adding edges $\{v_{i,j+1} v_{i+1,j} : 1 \leq i \leq m-1, 1 \leq j \leq n-1\} \cup \{v_{m,j+1} v_{1,j} : 1 \leq j \leq n-1\}$. Let $V(A_n^m) = V(C_m \square P_n)$ and $E(A_n^m) = E(C_m \square P_n) \cup \{v_{i,j+1} v_{i+1,j} : 1 \leq i \leq m-1, 1 \leq j \leq n-1\} \cup \{v_{m,j+1} v_{1,j} : 1 \leq j \leq n-1\}$ be the edge set of A_n^m, where i is taken modulo m. The metric dimension of generalized antiprism is given in the following theorem.

Theorem 3.2.11. [29] *Let A_n^m be the generalized antiprism with $n \geq 6$, we have*

$$\beta(A_n^m)=\begin{cases} 3, & \text{if } 2 \leq m \leq 5; \\ \geq 4, & \text{if } m \geq 6. \end{cases}$$

The flower snark J_n is a cubic graph with $6n$ edges and its order is $4n$. It can be constructed in the following way:

- First, build n copies of the star $K_{1,3}$. Denote the central vertex of each star by b_i and the outer vertices by a_i, c_i and d_i. This results in a disconnected graph on $4n$ vertices with $3n$ edges (b_ia_i, b_ic_i and b_id_i for $1 \leq i \leq n$).
- Construct the n-cycle $(a_1a_2 \ldots a_na_1)$. This adds n edges.
- Finally construct the $2n$-cycle $(c_1c_2 \ldots c_nd_1d_2 \ldots d_nc_1)$. This adds $2n$ edges.

It was proved in [15] that the flower snarks constitute a family of cubic graphs with constant metric dimension 3.
The *quasi flower snark* denoted by G_n is a nontrivial simple connected cubic graph, where $V(G_n) = \{a_i, b_i, c_i, d_i : 0 \leq i \leq n-1\}$ and $E(G_n) = \{a_ia_{i+1}, b_ib_{i+1}, c_ic_{i+1}, a_id_i, b_id_i, c_id_i : 0 \leq i \leq n-1\}$, the indices are taken modulo n. The metric dimension of some rotationally-symmetric graphs are given in [29], that are stated in the next two theorem.

Theorem 3.2.12. [29] *Suppose G_n be the quasi flower snark with $n \geq 4$, then we have*

$$\beta(G_n)=\begin{cases} 3, & \text{if } n \text{ is odd}; \\ \leq 4, & \text{otherwise}. \end{cases}$$

Theorem 3.2.13. [29] *Suppose $C_n^2 \square P_t$ denotes the Cartesian product of square cycle and path. Then for every $n \geq 5$,*

$$\beta(C_n^2 \square P_t)=\begin{cases} 3, & \text{when } n \equiv 0, 2, 3 \pmod 4; \\ \leq 4, & \text{otherwise}. \end{cases}$$

The metric dimensions of a connected graph G and its Cartesian product with K_2 are related in the following way.

Theorem 3.2.14. [8] *Suppose G be a nontrivial simple connected graph, then*

$$\beta(G) \leq \beta(G \square K_2) \leq \beta(G) + 1.$$

The Möbius ladder M_n is a cubic circulant graph. It is obtained from even cycle by adding edges connecting opposite pair of vertices in the cycle. Imran *et al.* investigated the metric dimension of Möbius ladders in [1] and it has been shown that $\beta(M_n) = 3$ except when $n \equiv 2 \pmod 8$. The metric dimension of barycentric subdivision of Möbius ladder SM_n has been studied in [30] that is presented in this theorem.

Theorem 3.2.15. [30] *Let SM_n be the barycentric subdivision of Möbius ladder, then for every positive even integer $n \geq 8$, we have*

$$\beta(SM_n) = 3$$

Chapter 4

Metric dimension of certain extensions of regular graphs

A fundamental and most widely studied question in graph theory is the following; How the value of a graph-theoratic parameter is affected by making a small change in a graph?

To answer this question and to know whether is it possible to expand the existing graphs (networks) with minimum possible change in their metric dimension, we study the metric dimension of certain extension of graphs.

It is well-known fact that if an edge is added to a graph G, then the metric dimension of the new graph $G' = G + e$ is given by

$$\beta(G) \leq \beta(G + e) \leq \beta(G) + 1.$$

However, when a vertex is added to a graph, then the metric dimension of the resulting graph may remain the same or can decrease or increase significantly.

In this thesis, we study the effect of adding set of vertices or set of edges in the graph on the metric dimension of graph. In this chapter, we determine the metric dimension of s-crown graph which is an extension of the cycle. We also determine the upper bounds for the metric dimension of extension of antiprism, barycentric subdivision of antiprism, extension of Möbius ladders and corona product of Möbius ladders with complete graph K_1.

4.1 Metric dimension of s-crown Cr_s

The class of s-crown graphs was introduced and constructed in [13]. An s-crown denoted by Cr_s is a connected, undirected and plane graph with no loop. The vertex set $V(Cr_s)$ of s-crown is the disjoint union of the sets $O = \{o_j\}$ (outer vertices), $I = \{i_j\}$ (inner vertices) and $M = \{m_{jk}\}$ (vertices along the cycle or mid vertices). The edge set $E(Cr_s) = \{i_j m_{j-1,2}, i_j m_{j1}, o_j m_{j1}, o_j m_{j2}, m_{j1} m_{j-1,2}, m_{j1} m_{j,2} : j = 1 \ldots s\}$, the vertex indices are taken modulo s. The s-crown graph denoted by Cr_s is a certain extension of the cycle graph C_n.

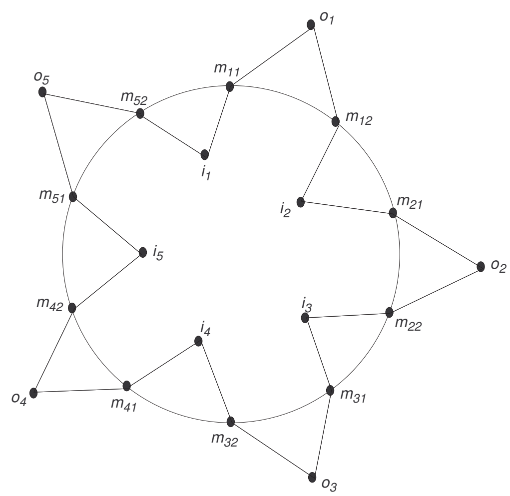

Figure 4.1: 5-Crown Cr_5

Theorem 4.1.1. *Let Cr_s be an s-crown graph. Then for any positive integer $s \geq 2$, we have*

$$\beta(Cr_s) = 2.$$

Proof. We discuss the representations of $V(Cr_s)$ in the following cases.

Case(1). We denote $W = \{i_1, o_{\frac{s}{2}}\} \subseteq V(Cr_s)$ be the resolving set for the case when $s \equiv 0 \pmod 2$. Then we have $s = 2k\,(k \geq 1)$.

The representations of the outer vertices are:

$$r(o_j|W)=\begin{cases} (2j, s-2j+1), & 1 \leq j \leq \frac{s}{2}-1; \\ (s, 0), & j = \frac{s}{2}; \\ (2s-2j+2, 2j-s+1), & \frac{s}{2}+1 \leq j \leq s. \end{cases}$$

The representations of the inner vertices are:

$$r(i_1|W) = (0, s).$$

$$r(i_j|W)=\begin{cases} (2j-1, s-2j+2), & 2 \leq j \leq \frac{s}{2}; \\ (2s-2j+3, 2j-s), & \frac{s}{2}+1 \leq j \leq s. \end{cases}$$

The representations of the mid vertices are:

$$r(m_{j1}|W)=\begin{cases} (2j-1, s-2j+1), & 1 \leq j \leq \frac{s}{2}; \\ (s, 2), & j = \frac{s}{2}+1; \\ (2s-2j+2, 2j-s), & \frac{s}{2}+2 \leq j \leq s. \end{cases}$$

The representations of the mid vertices are:

$$r(m_{j2}|W)=\begin{cases} (2j, s-2j), & 1 \leq j \leq \frac{s}{2}-1; \\ (s, 1), & j = \frac{s}{2}; \\ (2s-2j+1, 2j-s+1), & \frac{s}{2}+1 \leq j \leq s-1; \\ (1, s), & j = s. \end{cases}$$

Case(2). We denote $W = \{i_1, i_{\frac{s+1}{2}}\} \subseteq V(Cr_s)$ be the resolving set for the case when $s \equiv 1 \pmod 2$. Then we have $n = 2k+1\,(k \geq 1)$.

The representations of the outer vertices are:

$$r(o_j|W)=\begin{cases} (2j, s-2j+1), & 1 \leq j \leq \frac{s+1}{2}-1; \\ (2s-2j+2, 2j-s+1), & \frac{s+1}{2} \leq j \leq s. \end{cases}$$

The representations of the inner vertices are:

$$r(i_1|W) = (0, s,)$$

.

$$r(i_j|W)=\begin{cases} (2j-1, s-2j+2), & 2 \le j \le \frac{s+1}{2}-1; \\ (s,0), & j=\frac{s+1}{2}; \\ (2s-2j+3, 2j-s), & \frac{s+1}{2}+1 \le j \le s. \end{cases}$$

The representations of the mid vertices are:

$$r(m_{j1}|W)=\begin{cases} (2j-1, s-2j+1), & 1 \le j \le \frac{s+1}{2}-1; \\ (s,1), & j=\frac{s+1}{2}; \\ (2s-2j+2, 2j-s), & \frac{s+1}{2}+1 \le j \le s. \end{cases}$$

The representations of the mid vertices are:

$$r(m_{j2}|W)=\begin{cases} (2j, s-2j), & 1 \le j \le \frac{s+1}{2}-1; \\ (2s-2j+1, 2j-s+1), & \frac{s+1}{2} \le j \le s-1; \\ (1,s), & j=s. \end{cases}$$

It can be seen that every vertex has distinct representation with respect to the set W in both cases implying that $\beta(Cr_s) \le 2$.

Conversely, we show that $\beta(Cr_s) \ge 2$. On contrary, suppose $\beta(C_s) = 1$. Then by Theorem 3.2.2, we get a contradiction. Thus we have $\beta(Cr_s) \ge 2$. Hence $\beta(Cr_s) = 2$, which completes the proof. □

4.2 Metric dimension of certain extension of antiprism

The antiprism denoted by A_n is a 6-regular graph. For $n \ge 3$, it consists of an outer cycle $y_1, y_2, \ldots, y_n$ and an inner cycle $x_1, x_2, \ldots, x_n$ and a set of n spokes x_iy_i and $x_{i+1}y_i$ $i = 1, 2, \ldots, n$ with indices taken modulo n. The metric dimension of antiprism A_n has been determined by Javaid *et al.* in [21]. We extend the antiprism by adding edges y_iy_{i+2}, x_ix_{i+2} in the edge set $E(A_n)$. We denote such extension of antiprism by $A(n, 2)$. The vertex set of $A(n, 2)$ will be the vertex set of A_n and the edge set of $A(n, 2)$ will be $E(A_n \cup \{y_iy_{i+2},\ x_ix_{i+2}\}$.

In the next theorem, we prove that four vertices are sufficient to resolve all the vertices of $A(n, 2)$.

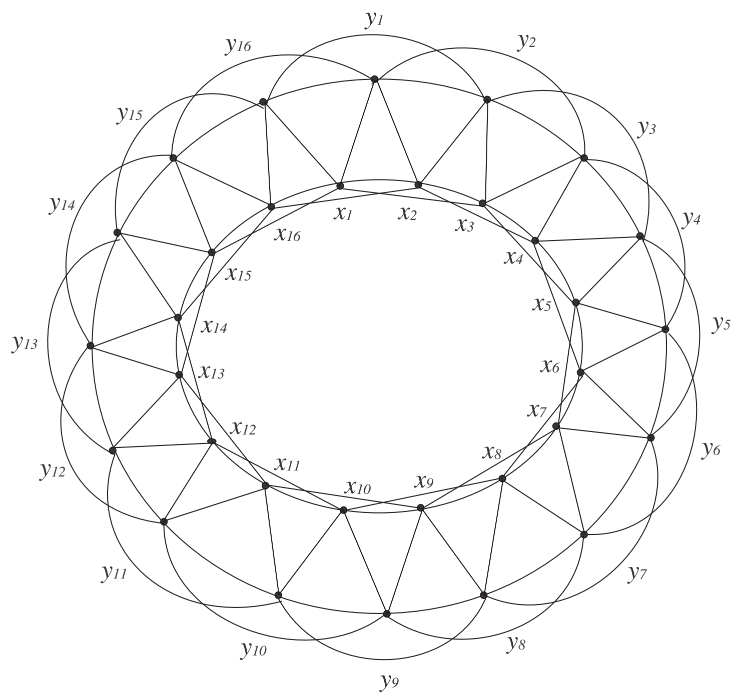

Figure 4.2: An extension of antiprism $A(16, 2)$

Theorem 4.2.1. *Let $A(n, 2)$ be a graph. Then for $n \geq 13$*

$$\beta(A(n, 2)) \leq 4.$$

Proof. We denote $W = \{y_1, y_2, y_3, y_{l+1}\}$ be a resolving set for $n \equiv 1 \pmod 2$ and $W = \{y_1, y_2, y_{l+2}, y_{l+\lceil \frac{l}{2} \rceil+2}\}$ be a resolving set for $n \equiv 0 \pmod 2$.
The representation of $V(A(n, 2))$ can be given in the following cases.
Case(1). When $n \equiv 1 \pmod 2$ which can also be written as $n = 2l + 1$, where l is a positive integer and $l \geq 6$. The representations of the vertices on the outer cycle are:

$$r(y_1|W) = (0, 1, 1, \lfloor \frac{l}{2} \rfloor), \qquad r(y_2|W) = (1, 0, 1, \lceil \frac{l}{2} \rceil).$$

$$r(y_{2i+1}|W)=\begin{cases}(i,i,i-1,\lceil\frac{l}{2}\rceil-i+1), & 1\le i\le\lceil\frac{l}{2}\rceil;\\ (l-\lceil\frac{l}{2}\rceil,l-\lceil\frac{l}{2}\rceil,\lceil\frac{l}{2}\rceil,2\lceil\frac{l}{2}\rceil-l), & i=\lceil\frac{l}{2}\rceil+1;\\ (n-l-i,n-l-i,n-l-i+1,i-\lceil\frac{l}{2}\rceil-1), & \lceil\frac{l}{2}\rceil+2\le i\le l.\end{cases}$$

$$r(y_{2i}|W)=\begin{cases}(i,i-1,i-1,\lceil\frac{l}{2}\rceil-i+2), & 2\le i\le\lceil\frac{l}{2}\rceil;\\ (i-1,i-1,i-1,i-\lceil\frac{l}{2}\rceil), & i=\lceil\frac{l}{2}\rceil+1;\\ (n-l-i,n-l-i,n-l-i+1,i-\lceil\frac{l}{2}\rceil-1), & \lceil\frac{l}{2}\rceil+2\le i\le l.\end{cases}$$

The representations of vertices on the inner cycle are:

$$r(x_{2i+1}|W)=\begin{cases}(i+1,i,i,\lceil\frac{l}{2}\rceil-i+2), & 1\le i\le\lceil\frac{l}{2}\rceil;\\ (l-i+2,l-i+2,\lceil\frac{l}{2}\rceil+1,1), & i=\lceil\frac{l}{2}\rceil+1;\\ (n-l-i+1,n-l-i+1,n-l-i+2,i-\lceil\frac{l}{2}\rceil), & \\ \lceil\frac{l}{2}\rceil+2\le i\le l. & \end{cases}$$

$$r(x_{2i}|W)=\begin{cases}(i+2,i+2,i,\lceil\frac{l}{2}\rceil-i+3), & 2\le i\le\lceil\frac{l}{2}\rceil+1;\\ (n-l-i+1,n-l-i+2,n-l-i+2,i-\lceil\frac{l}{2}\rceil-1), & \\ \lceil\frac{l}{2}\rceil+2\le i\le l. & \end{cases}$$

It can be seen that every vertex has distinct representation with respect to W giving that $\beta(A(n,2))\le 4$.

Case(2). When $n\equiv 0 \pmod 2$ which can also be written as $n=2l$ where l is a positive integer and $l\ge 7$. The representations of the vertices on the outer cycle are:

$$r(y_1|W)=(0,1,\lfloor\frac{l}{2}\rfloor,\lfloor\frac{l}{4}\rfloor),\qquad r(x_1|W)=(1,2,\lceil\frac{l}{2}\rceil,\lceil\frac{1}{2}\lfloor\frac{l}{2}\rfloor\rceil),$$
$$r(x_2|W)=(1,1,\lceil\frac{l+1}{2}\rceil,\lceil\frac{l+1}{4}\rceil).$$

$$r(y_{2i+1}|W)=\begin{cases}(i,i,\lceil\frac{l}{2}\rceil-i+1,i+3), & 1\le i\le\lceil\frac{1}{2}\lfloor\frac{l+1}{2}\rfloor\rceil;\\ (i,i,\lceil\frac{l}{2}\rceil-i+1,l-i-2), & \lceil\frac{1}{2}\lfloor\frac{l+1}{2}\rfloor\rceil+1\le i\le\lceil\frac{l}{2}\rceil;\\ (l-i,l-i+1,i-\lceil\frac{l}{2}\rceil,l-i-2), & \lceil\frac{l}{2}\rceil+1\le i\le l-3;\\ (\lceil\frac{l}{2}\rceil-i+6,\lceil\frac{l}{2}\rceil-i+7,i-\lceil\frac{l}{2}\rceil,i-\lceil\frac{l}{2}\rceil-3), & \\ i=l-1,l-2. & \end{cases}$$

$$r(y_{2i}|W)=\begin{cases} (i, i-1, l-\lceil\frac{l}{2}\rceil-i+1, i+2), & 1\leq i\leq \lceil\frac{1}{2}\lfloor\frac{l+1}{2}\rfloor\rceil;\\ (i, i-1, l-\lceil\frac{l}{2}\rceil-i+1, l-i-2), & \lceil\frac{1}{2}\lfloor\frac{l+1}{2}\rfloor\rceil+1\leq i\leq \lceil\frac{l}{2}\rceil;\\ (l-i+1, l-i+1, i-\lceil\frac{l+1}{2}\rceil, l-i-2), & \lceil\frac{l}{2}\rceil+1\leq i\leq l-2;\\ (l-i+1, l-i+1, i-\lceil\frac{l}{2}\rceil, i-l+2), & i=l-1, l. \end{cases}$$

$$r(x_{2i+1}|W)=\begin{cases} (i+1, i, \lfloor\frac{l}{2}\rfloor-i+2, i+3), & 1\leq i\leq \lceil\frac{l+1}{4}\rceil;\\ (i+1, i, \lfloor\frac{l}{2}\rfloor-i+2, l-i-1), & \lceil\frac{l+1}{4}\rceil+1\leq i\leq \lfloor\frac{l}{2}\rfloor;\\ (l-i+1, l-i+1, i-\lfloor\frac{l}{2}\rfloor, l-i-3), & i=\lceil\frac{l}{2}\rceil, l\equiv 1 \pmod 2;\\ (l-i+1, l-i+2, i-\lfloor\frac{l}{2}\rfloor, l-i-1), & \lceil\frac{l}{2}\rceil+1\leq i\leq l-2;\\ (l-i+1, l-i+2, i-\lfloor\frac{l}{2}\rfloor, l-i+1), & i=l-1. \end{cases}$$

$$r(x_{2i}|W)=\begin{cases} (i, i, l-\lceil\frac{l+1}{2}\rceil-i+3, i+3), & 2\leq i\leq \lceil\frac{l+1}{4}\rceil;\\ (i, i, l-\lceil\frac{l+1}{2}\rceil-i+3, l-i-1), & \lceil\frac{l+1}{4}\rceil+1\leq i\leq \lceil\frac{l+1}{2}\rceil;\\ (l-i+2, l-i+2, i-\lceil\frac{l}{2}\rceil, l-i-1), & \lceil\frac{l+1}{2}\rceil+1\leq i\leq l-2;\\ (l-i+2, l-i+2, i-\lceil\frac{l}{2}\rceil, l-i+3), & i=l-1, l. \end{cases}$$

Again all the vertices have distinct representation, which implies that $\beta(A(n,2))\leq 4$. □

4.3 Metric dimension of barycentric subdivision of antiprism

The barycentric subdivision of antiprism denoted by $S(A_n)$ can be obtained by inserting a new vertex in the interior of each edge of antiprism. The vertex set of $S(A_n)$ is $V(S(A_n))=\{u_i, v_i, w_i, x_i, y_i, z_i\}=V(A_n)\cup\{w_i, x_i, y_i, z_i; 1\leq i\leq n\}$ and the edge set of $S(A_n)$ is $E(S(A_n))=\{v_iw_i, w_iv_{i+1}, u_iz_i, z_iu_{i+1}, v_ix_i, x_iu_i, u_iy_i, y_iv_{i+1}; 1\leq i\leq n\}$, the vertex indices are taken modulo n.

Theorem 4.3.1. *Let $S(A_n)$ be a graph. Then for $n\geq 11$, we have*

$$\beta(S(A_n))\leq 4.$$

Proof. Let $W=\{x_1, y_1, x_{\lceil\frac{n}{2}\rceil}, y_{\lceil\frac{n}{2}\rceil}\}$ be a resolving set for $n\geq 11$. We give the representations of V($S(A_n)$) in the following way.

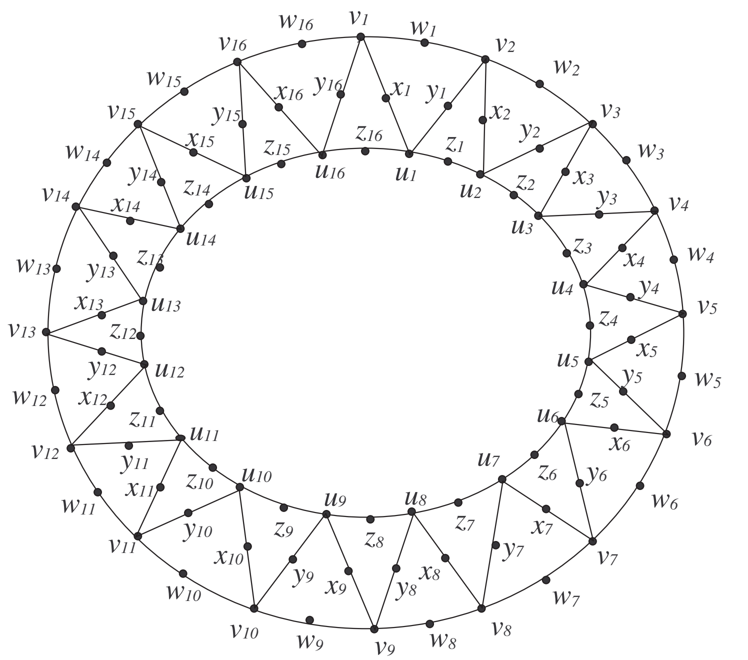

Figure 4.3: Barycentric subdivision of antiprism $S(A_{16})$

The representations of the vertices on the outer cycle are:

$$r(v_1|W) = (1, 3, 2\lceil\frac{n}{2}\rceil - 1, 2\lfloor\frac{n}{2}\rfloor + 1).$$

$$r(v_i|W)=\begin{cases} (2i-1, 2i-3, n-2i+2, n-2i+4), & 2 \le i \le \lceil\frac{n}{2}\rceil; \\ (n, n, 3, 1), & i = \lceil\frac{n}{2}\rceil + 1, \quad \lceil\frac{n}{2}\rceil \ne \lfloor\frac{n}{2}\rfloor; \\ (2n-2i+3, 2n-2i+5, 2i-2\lceil\frac{n}{2}\rceil+1, 2i-2\lceil\frac{n}{2}\rceil-1), \\ \begin{cases} \lceil\frac{n}{2}\rceil + 2 \le i \le n, & \lceil\frac{n}{2}\rceil \ne \lfloor\frac{n}{2}\rfloor; \\ \lceil\frac{n}{2}\rceil + 1 \le i \le n, & \lceil\frac{n}{2}\rceil = \lfloor\frac{n}{2}\rfloor. \end{cases} \end{cases}$$

and

$$r(w_1|W) = (2, 2, 2\lceil\frac{n}{2}\rceil - 2, 2\lceil\frac{n}{2}\rceil).$$

$$r(w_i|W)=\begin{cases} (2i, 2i-2, n-2i+1, n-2i+3), & 2 \leq i \leq \lfloor \frac{n}{2} \rfloor; \\ (2\lceil \frac{n}{2} \rceil, 2\lceil \frac{n}{2} \rceil - 2, 2, 2), & i = \lfloor \frac{n}{2} \rfloor + 1, \quad \lceil \frac{n}{2} \rceil \neq \lfloor \frac{n}{2} \rfloor; \\ (2n-2i+2, 2n-2i+2, 2i-2\lceil \frac{n}{2} \rceil - 2, 2i - 2\lceil \frac{n}{2} \rceil), \\ i = \frac{n}{2}+1 \quad where \quad \frac{n}{2} = \lceil \frac{n}{2} \rceil = \lfloor \frac{n}{2} \rfloor; \\ (2n-2i+2, 2n-2i+4, 2i-2\lceil \frac{n}{2} \rceil - 2, 2i - 2\lceil \frac{n}{2} \rceil), \\ \lfloor \frac{n}{2} \rfloor + 2 \leq i \leq n. \end{cases}$$

The representations of the vertices on the inner cycle are:

$$r(u_i|W)=\begin{cases} (2i-1, 2i-1, 2\lceil \frac{n}{2} \rceil - 2i+1, 2\lceil \frac{n}{2} \rceil - 2i+1), \\ 1 \leq i \leq \lceil \frac{n}{2} \rceil; \\ (2n-2i+3, 2n-2i+3, 2i-2\lceil \frac{n}{2} \rceil + 1, 2i-2\lceil \frac{n}{2} \rceil + 1), \\ \lceil \frac{n}{2} \rceil + 1 \leq i \leq n. \end{cases}$$

$$r(z_i|W)=\begin{cases} (2i, 2i, 2\lceil \frac{n}{2} \rceil - 2i, 2\lceil \frac{n}{2} \rceil - 2i), & 1 \leq i \leq \lceil \frac{n}{2} \rceil - 1; \\ (2i, 2i, 2i-2\lceil \frac{n}{2} \rceil + 2, 2i - 2\lceil \frac{n}{2} \rceil + 2), & i = \lceil \frac{n}{2} \rceil = \lfloor \frac{n}{2} \rfloor; \\ (2n-2i+2, 2n-2i+2, 2i-2\lceil \frac{n}{2} \rceil + 2, 2i-2\lceil \frac{n}{2} \rceil + 2), \\ \begin{cases} \lceil \frac{n}{2} \rceil \leq i \leq n, & \lceil \frac{n}{2} \rceil \neq \lfloor \frac{n}{2} \rfloor; \\ \lceil \frac{n}{2} \rceil + 1 \leq i \leq n, & \lceil \frac{n}{2} \rceil = \lfloor \frac{n}{2} \rfloor. \end{cases} \end{cases}$$

The representations of the vertices $x_i\,'s\ and\ y_i\,'s$ are:

$$r(x_1|W) = (0, 2, 2\lceil \frac{n}{2} \rceil, 2\lceil \frac{n}{2} \rceil).$$

$$r(x_i|W)=\begin{cases} (2i, 2i-2, n-2i+3, n-2i+3), & 2 \leq i \leq \lceil \frac{n}{2} \rceil - 1; \\ (2\lceil \frac{n}{2} \rceil, 2\lceil \frac{n}{2} \rceil - 2, 0, 2), & i = \lceil \frac{n}{2} \rceil; \\ (2n-2i+4, 2n-2i+4, 2i-2\lfloor \frac{n}{2} \rfloor, 2i-2\lceil \frac{n}{2} \rceil), & \lceil \frac{n}{2} \rceil + 1 \leq i \leq n. \end{cases}$$

$$r(y_1|W) = (2, 0, 2\lceil \frac{n}{2} \rceil - 2, 2\lceil \frac{n}{2} \rceil).$$

$$r(y_i|W)=\begin{cases} (2i, 2i, n-2i+1, n-2i+3), & 2 \leq i \leq \lceil \frac{n}{2} \rceil - 1; \\ (2\lceil \frac{n}{2} \rceil, 2\lceil \frac{n}{2} \rceil, 2, 0), & i = \lceil \frac{n}{2} \rceil; \\ (2i-2\lfloor \frac{n}{2} \rfloor, 2i-2\lceil \frac{n}{2} \rceil, 2i-2\lfloor \frac{n}{2} \rfloor, 2i-2\lfloor \frac{n}{2} \rfloor), & \lceil \frac{n}{2} \rceil + 1 \leq i \leq n. \end{cases}$$

There is no pair of vertices having same representation with respect to W. Therefore we have, $\beta(S(A_n)) \leq 4$. □

4.4 Metric dimension of corona product of Möbius ladder and complete graph K_1

The Möbius ladder denoted by M_n is a cubic circulant graph with an even number of vertices. It consists of an n-cycle and the edges (called rungs) connecting opposite pair of vertices in the cycle. In [1], Imran *et al.* proved that $\beta(M_n) = 3$ when $n \not\equiv 2 \pmod 8$ and $3 \leq \beta(M_n) \leq 4$ otherwise. The corona product of Möbius ladders with complete graph K_1 denoted by $M_n \odot K_1$ can be obtained by attaching a pendant vertex to every vertex of Möbius ladder M_n.

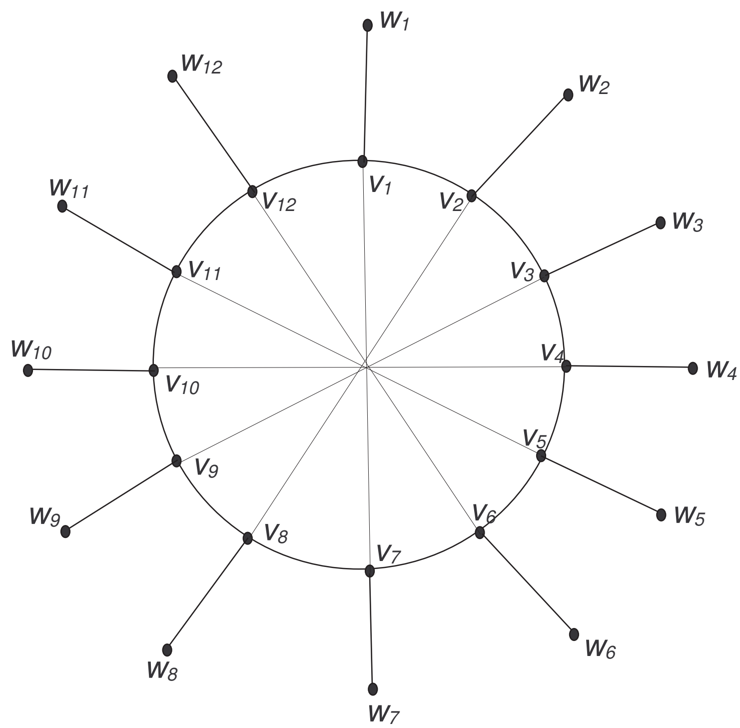

Figure 4.4: Graph of $M_{12} \odot K_1$

Theorem 4.4.1. *Let $M_n \odot K_1$ be the corona product of Möbius ladder M_n with* *complete graph K_1. Then for every positive even integer $n \geq 8$, we have*

$$\beta(M_n \odot K_1) \leq 4.$$

Proof. We denote $W = \{v_1, v_2, v_{l-\lfloor \frac{l}{2} \rfloor+1}, v_{l+1}\} \subseteq V(M_n \odot K_1)$ to be the resolving set. Then the representations of $V(M_n \odot K_1)$ can be given in the following cases.

Case(1). When $n \equiv 0 \pmod 4$. Then we have $n = 4k (k \geq 2)$.

The representations of the vertices on the cycle are:

$$r(v_1|W) = (0, 1, \lfloor \frac{l}{2} \rfloor, 1).$$

$$r(v_i|W)=\begin{cases} (i-1, i-2, \lfloor \frac{l}{2} \rfloor - i + 1, i), & 2 \leq i \leq \lfloor \frac{l}{2} \rfloor; \\ (i-1, i-2, 0, i-1), & i = \lfloor \frac{l}{2} \rfloor + 1; \\ (\lfloor \frac{l}{2} \rfloor, \lfloor \frac{l}{2} \rfloor, 1, l-i+1), & i = \lfloor \frac{l}{2} \rfloor + 2; \\ (l-i+2, l-i+3, i - \lfloor \frac{l}{2} \rfloor - 1, l-i+1), & \lfloor \frac{l}{2} \rfloor + 3 \leq i \leq l+1; \\ (i-l, i-l-1, l + \lfloor \frac{l}{2} \rfloor - i + 2, i-l-1), & l+2 \leq i \leq l + \lfloor \frac{l}{2} \rfloor; \\ (\lfloor \frac{l}{2} \rfloor, \lfloor \frac{l}{2} \rfloor, 1, \lfloor \frac{l}{2} \rfloor), & i = l + \lfloor \frac{l}{2} \rfloor + 1; \\ (n-i+1, n-i+2, i-l-\lfloor \frac{l}{2} \rfloor, n-i+2), & l + \lfloor \frac{l}{2} \rfloor + 2 \leq i \leq n. \end{cases}$$

The representations of the pendant vertices are:

$$r(w_1|W) = (1, 2, \lfloor \frac{l}{2} \rfloor + 1, 2).$$

$$r(w_i|W)=\begin{cases} (i, i-1, \lfloor \frac{l}{2} \rfloor - i + 2, i+1), & 2 \leq i \leq \lfloor \frac{l}{2} \rfloor; \\ (i, i-1, 1, i), & i = \lfloor \frac{l}{2} \rfloor + 1; \\ (\lfloor \frac{l}{2} \rfloor + 1, \lfloor \frac{l}{2} \rfloor + 1, 2, l-i+2), & i = \lfloor \frac{l}{2} \rfloor + 2; \\ (l-i+3, l-i+4, i - \lfloor \frac{l}{2} \rfloor, l-i+2), & \lfloor \frac{l}{2} \rfloor + 3 \leq i \leq l+1; \\ (i-l+1, i-l, l + \lfloor \frac{l}{2} \rfloor - i + 3, i-l), & l+2 \leq i \leq l + \lfloor \frac{l}{2} \rfloor; \\ (\lfloor \frac{l}{2} \rfloor + 1, \lfloor \frac{l}{2} \rfloor + 1, 2, \lfloor \frac{l}{2} \rfloor + 1), & i = l + \lfloor \frac{l}{2} \rfloor + 1; \\ (n-i+2, n-i+3, i-l-\lfloor \frac{l}{2} \rfloor + 1, n-i+3), & l + \lfloor \frac{l}{2} \rfloor + 2 \leq i \leq n. \end{cases}$$

Case(2). When $n \equiv 2 \pmod 4$. Then we have $n = 4k + 2\,(k \geq 2)$.

The representations of the vertices on the cycle are:

$$r(v_1|W) = (0, 1, \lfloor\frac{l}{2}\rfloor + 1, 1).$$

$$r(v_i|W) = \begin{cases} (i-1, i-2, \lfloor\frac{l}{2}\rfloor - i + 2, i), & 2 \le i \le \lfloor\frac{l}{2}\rfloor + 1; \\ (i-1, i-2, 0, i-2), & i = \lfloor\frac{l}{2}\rfloor + 2; \\ (l-i+2, l-i+3, i-\lfloor\frac{l}{2}\rfloor - 2, l-i+1), & \lfloor\frac{l}{2}\rfloor + 3 \le i \le l+1; \\ (i-l, i-l-1, l-i+6, i-l-1), & \\ l+2 \le i \le l + \lfloor\frac{l}{2}\rfloor + 1; & \\ (n-i+1, n-i+2, i-l-\lfloor\frac{l}{2}\rfloor - 1, n-i+2), & l + \lfloor\frac{l}{2}\rfloor + 2 \le i \le n. \end{cases}$$

The representations of the pendant vertices are:

$$r(w_1|W) = (1, 2, \lfloor\frac{l}{2}\rfloor + 2, 2).$$

$$r(w_i|W) = \begin{cases} (i, i-1, \lfloor\frac{l}{2}\rfloor - i + 3, i+1), & 2 \le i \le \lfloor\frac{l}{2}\rfloor + 1; \\ (i, i-1, 1, i-1), & i = \lfloor\frac{l}{2}\rfloor + 2; \\ (l-i+3, l-i+4, i-\lfloor\frac{l}{2}\rfloor - 1, l-i+2), & \lfloor\frac{l}{2}\rfloor + 3 \le i \le l+1; \\ (i-l+1, i-l, l-i+7, i-l), & l+2 \le i \le l + \lfloor\frac{l}{2}\rfloor + 1; \\ (n-i+2, n-i+3, i-l-\lfloor\frac{l}{2}\rfloor, n-i+3), & l + \lfloor\frac{l}{2}\rfloor + 2 \le i \le n. \end{cases}$$

It can be seen that every vertex has distinct representation with respect to the set W implying that $\beta(M_n \odot K_1) \le 4$. □

4.5 Metric dimension of a certain extension of Möbius ladders

This extension of Möbius ladder can be obtained by joining the vertices of a Möbius ladders at a distance two along the principal cycle, either clockwise or counter clockwise. We do not join u and v if a rung is counted in $d(u, v) = 2$. We denote such graphs by M'_n. Let u and v be two vertices in M_n and $d(u, v) = 2$ along a principal cycle then these two vertices u and v are adjacent in M'_n which is shown in Fig. 4.5.

In the next theorem, we determine an upper bound of the metric dimension of such graphs.

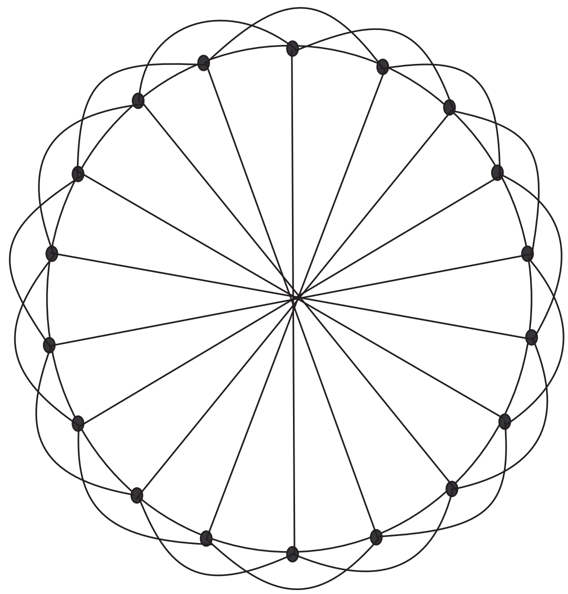

Figure 4.5: The extension of Möbius ladders M'_{18}

Theorem 4.5.1. *Let M'_n be a certain extension of Möbius ladder defined above. Then for every positive even integer $n > 16$, we have*

$$\beta(M'_n) \leq \begin{cases} 4, & \text{if } n \equiv 0 \pmod 8; \\ 5, & \text{otherwise.} \end{cases}$$

Proof. We denote $W = \{v_1, v_2, v_{l+1}, v_{l+3}\}$ to be a resolving set for $n \equiv 0 \pmod 8$. We can write $n = 8l$ where $l > 2$. Suppose $W = \{v_1, v_2, v_{\lceil \frac{l}{2} \rceil}, v_{l+1}, v_{l+3}\}$ be a resolving set for $n \equiv 2, 4, 6 \pmod 8$, which can also be written as $n = 8l + 2, n = 8l + 4, n = 8l + 6$, respectively where $l \geq 2$. The representation of $V(M'_n)$ can be given in the following cases.

Case(1). When $n \equiv 0 \pmod 8$. Then we have $n = 8l\,(l > 2)$. The representation of the vertices are the followings.

$$r(v_1|W) = (0, 1, 1, 2).$$

$$r(v_{2i+1}|W)=\begin{cases} (i, i, i+1, i), & 1 \le i \le \frac{l}{4} - 1; \\ (i, i, i, i), & i = \frac{l}{4}; \\ (\frac{l}{2} - i + 1, \frac{l}{2} - i + 2, \frac{l}{2} - i, \frac{l}{2} - i + 1), & \frac{l}{4} + 1 \le i \le \frac{l}{2}; \\ (i - \frac{l}{2} + 1, i\frac{l}{2} + 1, i - \frac{l}{2}, i - \frac{l}{2} - l), & \frac{l}{2} + 1 \le i \le \lfloor l + \frac{l}{2} - 1 \rfloor; \\ (l - i, l - i + 1, l - i, i - \frac{l}{2} - 1), & i = \lfloor l + \frac{l}{2} - 1 \rfloor + 1; \\ (l - i, l - i + 1, l - i + 1, l - i + 1), & i = \lfloor l + \frac{l}{2} - 1 \rfloor + 2; \\ (l - i, l - i + 1, l - i + 1, l - i + 2), & \lfloor l + \frac{l}{2} - 1 \rfloor + 3 \le i \le l - 1. \end{cases}$$

$$r(v_2|W) = (1, 0, 2, 2).$$

$$r(v_{2i}|W)=\begin{cases} (i, i - 1, i + 1, i), & 2 \le i \le \frac{l}{4}; \\ (i, i - 1, i - 1, i), & i = \frac{l}{4} + 1; \\ (\frac{l}{2} - i + 2, \frac{l}{2} - i + 2, \frac{l}{2} - i + 1, \frac{l}{2} - i + 2), & \frac{l}{4} + 2 \le i \le \frac{l}{2}; \\ (2, 1, 1, 1), & i = \frac{l}{2} + 1; \\ (i - \frac{l}{2} + 1, i\frac{l}{2}, i - \frac{l}{2}, i - \frac{l}{2} - l), & \frac{l}{2} + 2 \le i \le \frac{1}{2}\lceil l + \frac{l}{2} \rceil; \\ (l - i + 1, l - i + 1, l - i + 2, i - \frac{l}{2} - 1), & \\ i = \frac{1}{2}\lceil l + \frac{l}{2} \rceil + 1, \frac{1}{2}\lceil l + \frac{l}{2} \rceil + 2; & \\ (l - i + 1, l - i + 1, l - i + 2, l - i + 3), & \frac{1}{2}\lceil l + \frac{l}{2} \rceil + 3 \le i \le l. \end{cases}$$

Case(2). When $n \equiv 2 \pmod 8$. Then we have $n = 8l + 2\,(l \ge 2)$.

$$r(v_1|W) = (0, 1, \lfloor \frac{l}{4} \rfloor, 1, 2).$$

$$r(v_{2i+1}|W)=\begin{cases} (i,i,\lfloor\frac{l}{4}\rfloor-i+2,i+1,i), & 1\leq i\leq\lfloor\frac{l}{4}\rfloor; \\ (i,i,1,i-1,i), & i=\lfloor\frac{l}{4}\rfloor+1; \\ (\lceil\frac{l}{2}\rceil-i+1,\lceil\frac{l}{2}\rceil-i+1,i-\lfloor\frac{l}{4}\rfloor-1,\lceil\frac{l}{2}\rceil-i,\lceil\frac{l}{2}\rceil-i+1), & \\ \lfloor\frac{l}{4}\rfloor+2\leq i\leq\rceil\frac{l}{2}\lceil-1; & \\ (2,1,\lfloor\frac{l}{4}\rfloor,1,1), & i=\lceil\frac{l}{2}\rceil; \\ (i-\lceil\frac{l}{2}\rceil+2,i-\lceil\frac{l}{2}\rceil+1,l-i-3,i-\lceil\frac{l}{2}\rceil+1,i-\lceil\frac{l}{2}\rceil), & \\ \lceil\frac{l}{2}\rceil+1\leq i\leq\lceil\frac{l}{2}\rceil+\lfloor\frac{l}{4}\rfloor-1; & \\ (\lfloor\frac{l}{4}\rfloor,\lfloor\frac{l}{4}\rfloor+1,2,\lfloor\frac{l}{4}\rfloor+1,\lfloor\frac{l}{4}\rfloor), & i=\lceil\frac{l}{2}\rceil+\lfloor\frac{l}{4}\rfloor; \\ (l-i,l-i+1,i-\lceil\frac{l}{2}\rceil-\lfloor\frac{l}{4}\rfloor-1,l-i+1,l-i+2), & \\ \lceil\frac{l}{2}\rceil+\lfloor\frac{l}{4}\rfloor+1\leq i\leq l-1. & \end{cases}$$

$$r(v_2|W)=(1,0,\lfloor\frac{l}{4}\rfloor+1,2,2).$$

$$r(v_{2i}|W)=\begin{cases} (i,i-1,\lfloor\frac{l}{4}\rfloor-i+2,i+1,i), & 2\leq i\leq\lfloor\frac{l}{4}\rfloor; \\ (i,i-1,1,i-1,i), & i=\lfloor\frac{l}{4}\rfloor+1; \\ (\lceil\frac{l}{2}\rceil-i+1,\lceil\frac{l}{2}\rceil-i+2,i-\lfloor\frac{l}{4}\rfloor-2,\lceil\frac{l}{2}\rceil-i,\lceil\frac{l}{2}\rceil-i+1), & \\ \lfloor\frac{l}{4}\rfloor+2\leq i\leq\rceil\frac{l}{2}\lceil; & \\ (2,2,\lfloor\frac{l}{4}\rfloor,1,0), & i=\lceil\frac{l}{2}\rceil+1; \\ (i-\lceil\frac{l}{2}\rceil+1,i-\lceil\frac{l}{2}\rceil+1,l-i-2,i-\lceil\frac{l}{2}\rceil,i-\lceil\frac{l}{2}\rceil-1), & \\ \lceil\frac{l}{2}\rceil+2\leq i\leq\lceil\frac{l}{2}\rceil+\lfloor\frac{l}{4}\rfloor; & \\ (\lfloor\frac{l}{4}\rfloor,\lfloor\frac{l}{4}\rfloor,2,\lfloor\frac{l}{4}\rfloor+1,\lfloor\frac{l}{4}\rfloor), & i=\lceil\frac{l}{2}\rceil+\lfloor\frac{l}{4}\rfloor+1; \\ (l-i+1,l-i+1,i-\lceil\frac{l}{2}\rceil-\lfloor\frac{l}{4}\rfloor,l-i+2,l-i+3), & \\ \lceil\frac{l}{2}\rceil+\lfloor\frac{l}{4}\rfloor+2\leq i\leq l. & \end{cases}$$

Case(3). When $n\equiv 4 \pmod 8$. Then we have $n=8l+4\,(l\geq 2)$.

$$r(v_1|W)=(0,1,\lfloor\frac{l}{4}\rfloor,1,2).$$

$$r(v_{2i+1}|W)=\begin{cases}(i,i,\lfloor\frac{l}{4}\rfloor-i,i+1,i), & 1\le i\le\lfloor\frac{l}{4}\rfloor;\\ (i,i,1,i-1,i), & i=\lfloor\frac{l}{4}\rfloor+1;\\ (\frac{l}{2}-i+1,\frac{l}{2}-i+2,i-\lfloor\frac{l}{4}\rfloor,\frac{l}{2}-i,\frac{l}{2}-i+1), & \\ \lfloor\frac{l}{4}\rfloor+2\le i\le\frac{l}{2}; & \\ (i-\frac{l}{2}+1,i-\frac{l}{2}+1,l-i-\lfloor\frac{l}{4}\rfloor,i-\frac{l}{2},i-\frac{l}{2}-1), & \\ \frac{l}{2}+1\le i\le\frac{l}{2}+\lfloor\frac{l}{4}\rfloor; & \\ (\lfloor\frac{l}{4}\rfloor,\lfloor\frac{l}{4}\rfloor+1,i-\frac{l}{2}-\lfloor\frac{l}{4}\rfloor+1,\lfloor\frac{l}{4}\rfloor+1,\lfloor\frac{l}{4}\rfloor), & \\ i=\frac{l}{2}+\lfloor\frac{l}{4}\rfloor+1; & \\ (l-i,l-i+1,i-\frac{l}{2}-\lfloor\frac{l}{4}\rfloor+2,l-i+1,l-i+2), & \\ \lceil\frac{l}{2}\rceil+\lfloor\frac{l}{4}\rfloor+2\le i\le l-1. & \end{cases}$$

$$r(v_2|W)=(1,0,\lfloor\frac{l}{4}\rfloor,2,2).$$

$$r(v_{2i}|W)=\begin{cases}(i,i-1,\lfloor\frac{l}{4}\rfloor-i+1,i+1,i), & 2\le i\le\lfloor\frac{l}{4}\rfloor;\\ (i,i-1,1,i,i), & i=\lfloor\frac{l}{4}\rfloor+1;\\ (\frac{l}{2}-i+2,\frac{l}{2}-i+2,i-\lfloor\frac{l}{4}\rfloor,\frac{l}{2}-i+1,\frac{l}{2}-i+2), & \\ \lfloor\frac{l}{4}\rfloor+2\le i\le\frac{l}{2}; & \\ (2,1,\lfloor\frac{l}{4}\rfloor+1,1,1), & i=\frac{l}{2}+1;\\ (i-\frac{l}{2}+1,i-\frac{l}{2},l-i-3,i-\frac{l}{2},i-\frac{l}{2}-1), & \\ \frac{l}{2}+2\le i\le\frac{l}{2}+\lfloor\frac{l}{4}\rfloor; & \\ (\lfloor\frac{l}{4}\rfloor+1,\lfloor\frac{l}{4}\rfloor+1,2,\lfloor\frac{l}{4}\rfloor+1,\lfloor\frac{l}{4}\rfloor), & i=\frac{l}{2}+\lfloor\frac{l}{4}\rfloor+1;\\ (\lfloor\frac{l}{4}\rfloor,\lfloor\frac{l}{4}\rfloor,3,\lfloor\frac{l}{4}\rfloor+1,\lfloor\frac{l}{4}\rfloor+1), & i=\frac{l}{2}+\lfloor\frac{l}{4}\rfloor+2;\\ (l-i+1,l-i+1,i-\frac{l}{2}-\lfloor\frac{l}{4}\rfloor+1,l-i+2,l-i+3), & \\ \frac{l}{2}+\lfloor\frac{l}{4}\rfloor+3\le i\le l-1; & \\ (1,1,\lfloor\frac{l}{4}\rfloor+1,2,3), & i=l.\end{cases}$$

Case(4). When $n\equiv 6 \pmod 8$. Then we have $n=8l+6\,(l\ge 2)$.

$$r(v_1|W)=(0,1,\frac{l+1}{4},1,2).$$

$$r(v_{2i+1}|W)=\begin{cases}(i,i,\frac{l+1}{4}-i,i-1,i), & 1\leq i\leq \frac{l+1}{4}-1;\\ (\frac{l+1}{4},\frac{l+1}{4},1,\frac{l+1}{4},\frac{l+1}{4}), & i=\frac{l+1}{4};\\ (\lceil\frac{l}{2}\rceil-i+1,\lceil\frac{l}{2}\rceil-i+1,i-\frac{l+1}{4}+1,\lceil\frac{l}{2}\rceil-i,\lceil\frac{l}{2}\rceil-i+1), \\ \frac{l+1}{4}+1\leq i\leq\rceil\frac{l}{2}\lceil-1; \\ (2,1,\frac{l+1}{4},1,1), & i=\lceil\frac{l}{2}\rceil;\\ (i-\lceil\frac{l}{2}\rceil+2,i-\lceil\frac{l}{2}\rceil+1,\lceil\frac{l}{2}\rceil+\frac{l+1}{4}-i,i-\lceil\frac{l}{2}\rceil+1,i-\lceil\frac{l}{2}\rceil), \\ \lceil\frac{l}{2}\rceil+1\leq i\leq\lceil\frac{l}{2}\rceil+\frac{l+1}{4}-2; \\ (\frac{l+1}{4},\frac{l+1}{4},1,\frac{l+1}{4},\frac{l+1}{4}-1), & i=\lceil\frac{l}{2}\rceil+\frac{l+1}{4}-1;\\ (\frac{l+1}{4}-1,\frac{l+1}{4},2,\frac{l+1}{4},\frac{l+1}{4}), & i=\lceil\frac{l}{2}\rceil+\frac{l+1}{4};\\ (l-i,l-i+1,i-\lceil\frac{l}{2}\rceil-\frac{l+1}{4}+2,l-i+1,l-i+2), \\ \lceil\frac{l}{2}\rceil+\frac{l+1}{4}+1\leq i\leq l-1.\end{cases}$$

$$r(v_2|W)=(1,0,\frac{l+1}{4}-1,2,2).$$

$$r(v_{2i}|W)=\begin{cases}(i,i-1,\frac{l+1}{4}-i,i+1,i), & 1\leq i\leq\frac{l+1}{4}-1;\\ (\frac{l+1}{4},\frac{l+1}{4}-1,0,\frac{l+1}{4},\frac{l+1}{4}), & i=\frac{l+1}{4};\\ (\frac{l+1}{4},\frac{l+1}{4},1,\frac{l+1}{4}-1,\frac{l+1}{4}), & i=\frac{l+1}{4}+1;\\ (\lceil\frac{l}{2}\rceil-i+1,\lceil\frac{l}{2}\rceil-i+2,i-\frac{l+1}{4},\lceil\frac{l}{2}\rceil-i,\lceil\frac{l}{2}\rceil-i+1), \\ \frac{l+1}{4}+2\leq i\leq\rceil\frac{l}{2}\lceil; \\ (i-\lceil\frac{l}{2}\rceil+1,i-\lceil\frac{l}{2}\rceil+1,\lceil\frac{l}{2}\rceil+\frac{l+1}{4}-i+1,i-\lceil\frac{l}{2}\rceil,i-\lceil\frac{l}{2}\rceil-1), \\ \lceil\frac{l}{2}\rceil+1\leq i\leq\lceil\frac{l}{2}\rceil+\frac{l+1}{4}-1; \\ (\frac{l+1}{4},\frac{l+1}{4},2,\frac{l+1}{4},\frac{l+1}{4}-1), & i=\lceil\frac{l}{2}\rceil+\frac{l+1}{4};\\ (l-i+1,l-i+1,i-\lceil\frac{l}{2}\rceil-\frac{l+1}{4}+2,l-i+2,\frac{l+1}{4}), \\ i=\lceil\frac{l}{2}\rceil+\frac{l+1}{4}+1; \\ (l-i+1,l-i+1,i-\lceil\frac{l}{2}\rceil-\frac{l+1}{4}+2,l-i+2,l-i+3), \\ \lceil\frac{l}{2}\rceil+\frac{l+1}{4}+2\leq i\leq l-1; \\ (1,1,\frac{l+1}{4},2,3), & i=l.\end{cases}$$

It can be seen that no two vertices have same representation. This implies that $\beta(M_n')\leq 4$ if $n\equiv 0 \pmod 8$ and $\beta(M_n')\leq 5$ if $n\equiv 2,4,6 \pmod 8$.

□

Chapter 5

Conclusion and Open Problems

A fundamental and most studied question in graph theory is the following: How the value of a parameter is affected by making a small change in the graph?

In this thesis, we have discussed the metric dimension of s-crowns, certain extensions of antiprism and Möbius ladder. We have proved that the s-crowns Cr_s constitutes a family of graph with constant metric dimension and we have $\beta(C_n) = \beta(Cr_s) = 2$. We have studied the upper bounds for the metric dimension of extension of antiprism denoted by $A(n, 2)$ is 4 for $n \geq 13$. We have also determined that for $n > 16$, the the upper bounds for the metric dimension of extension of Möbius ladder denoted by M'_n is 4 for $n \equiv 0 \pmod 8$ and 5 for $n \equiv 2, 4, 6 \pmod 8$. We further studied the barycentric subdivision of antiprism denoted by $S(A_n)$, for $n \geq 11$ and deduced that the upper bound for the metric dimension of these graphs is 4. We have also studied the corona product of Möbius ladder and complete graph K_1 denoted by $M_n \odot K_1$ and concluded that $\beta(M_n \odot K_1) \leq 4$ for $n \geq 4$. It can be noted that these graphs have bounded metric dimensions. Also the metric dimensions of these graphs do not depend upon the number of vertices they have.

Although we were able to find the upper bounds in most cases, however the results obtained show that we can expand the existing networks with minimum cost if we use the extensions describe above and there will be no significant change in the metric dimension of extension of regular graphs.

We close the discussion by raising questions that naturally arise from the text.

Open Problem 1: *Determine the exact value for the metric dimension of extension of antiprism* $A(n,2)$ *and extension of Möbius ladder* $M_n{}'$.

Open Problem 2:*Determine the exact value for the metric dimension of barycentric subdivision of antiprism* $S(A_n)$.

Open Problem 3: *Determine the exact value for the metric dimension of corona product of Mòbius ladder and complete graph* K_1 *denoted by* $M_n \odot K_1$.

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