Discrete Symmetries of First Order Differential Equations

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Dedication

To my Parents, Sisters, Brother

and

Uncle Latif Ullah.

Acknowledgments

First of all I am thankful to **Allah** Almighty who blessed me with good health and give me the strength to complete this work.

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Contents

Chapter 1

Introduction

After the invention of calculus by I. Newton (1642-1727) and G.W. Leibnitz (1646-1716), differential equations came into existence. Differential equations are one of the most important discipline of mathematics and many real-world problems are expressed in the form of differential equations (ordinary differential equations (ODEs) and partial differential equations (PDEs)). In 17th century Newton write down some basic laws of nature in term of differential equations. There are also many other well-known mathematicians who worked alot in the field of differential equations, some of them are Cauchy, Liouvillie, Gronwall, Lyapunov, Birkoff etc. [1, 2].

The hard but necessary aspect of differential equations is to find out their solutions. There are different methods for finding solutions of differential equations but these methods did not work for all type of differential equations (i.e. they are bounded to some class of ODEs and PDEs). Also the study of algebraic properties of system of differential equations are very difficult task. In 19th century Marius Sophus Lie (A Norwegian Mathematician) was the first who invented an outstanding technique to find solutions of differential equations. The analysis of Lie unified three main branches of mathematics (Analysis, Geometry and Algebra). According to Lie a unified approach must be constructed that contains all the classical methods for solving differential equations [3, 4, 5, 6, 7, 8].

Lie acquired the idea of transformations groups from the Galois theory which relates permutations groups with solutions of algebraic equations. Lie used his concept on solutions of differential equations and proved that there are transformations groups related to solutions of differential equations. Further to this, Lie devised group of continuous transformations (called Lie group, which are mathematical tools for solution of differential equations), and proved that if differential equation is invariant under one-parameter Lie group of point transformation then its order can be decreased by one. Positive aspect of Lie technique is that it is used for all kind of differential equations. Later on Lie's techniques are used for linearization process, to find out the invariants corresponding to differential equations and group classification [2]. These observations made by Lie were very useful and thus established a new method for solving differential equations called "symmetry methods for differential equations". A symmetry of system of differential equations is transformation group (depends on continuous parameters, real or complex), that maps solutions into solutions. Translation, rotation and scaling are the basic example of Lie groups [1, 7].

Those symmetries which are non-continuous are known as "discrete symmetries". Discrete point symmetry has a lot of importance in mathematics and physics. It can be use in computational methods to increase their efficiency, also discrete point symmetry is used for some numerical methods to improve their accuracy, and from the existing known solutions we can generate new solutions with the help of discrete point symmetries [9, 10, 11, 12]. Many methods have been developed for finding these symmetries, in which Peter E. Hydon technique for finding discrete symmetries of differential equations, having a finite dimensional Lie algebra of infinitesimal generators of its Lie group of point symmetries are perfect [9, 10, 11, 13, 14]. With the help of this technique all the discrete symmetries of differential equation could be found. His method is based on the observation that every point symmetry yields an automorphism of the Lie algebra of the Lie point symmetry.

In Chapter 2, we give some basic definitions, theorems and methods necessary for finding Lie point symmetries of differential equations. In Chapter 3, method for finding discrete symmetries of differential equations is explained, and discrete symmetries of some first order differential equations are obtained. In last chapter there is brief summary of the thesis.

Chapter 2

Symmetry Methods for Differential Equations

We begin this chapter with some basic definitions and theorems. A discussion on some principles which are helpful while finding the Lie point symmetries for differential equations is also presented in this chapter.

2.1 One Parameter Point Transformation Group

While solving a differential equation one may simplify the equation by changing the variables,

$$\check{x} = \check{x}(x, v), \qquad \check{v} = \check{v}(x, v), \tag{2.1}$$

where $\check{x}$ and $\check{v}$ are continuous functions.

The transformation (2.1) maps a point (x, v) into point $(\check{x}, \check{v})$, so it is called a point transformation [15].

In symmetry, we take invertible point transformation that must depend on at least one parameter α.

$$\check{x} = \check{x}(x, v, \alpha), \qquad \check{v} = \check{v}(x, v, \alpha), \tag{2.2}$$

where ($\check{x}$ and $\check{v}$) are infinitely differentiable w.r.t x and v.

Consider the point transformation (2.2) defined for $\check{x}$ and $\check{v}$ lies on region $D \subset \mathbb{R}^2$ that depends on $\alpha \in S \subset R$ with composition $\phi(\alpha, \delta)$.

We call such transformations a one-parameter group of transformations if the following holds:

1. $\check{x}(x, v, \alpha) = x \quad$ and $\quad \check{v}(x, v, \alpha) = v \qquad$ at $\alpha = 0$.
2. $\check{\check{x}} = \check{\check{x}}(\check{x}, \check{v}, \delta) = \check{\check{x}}(x, v, \phi(\alpha, \delta))$, similarly $\check{\check{v}} = \check{\check{v}}(x, v, \phi(\alpha, \delta))$.

Now we define one-parameter Lie transformation group and their infinitesimal transformations [4, 8].

2.1.1 One-Paramter Lie Transformation Group

A transformation

$$\check{x} = \check{x}(x, v, \alpha), \qquad \check{v} = \check{v}(x, v, \alpha),$$

is one-parameter Lie transformation group if holds the above conditions given for one-parameter group as well as the following :

1. For a continuous parameter α, $\alpha = 0$ yields to an identity element e.
2. $\check{x}$ and $\check{v}$ are analytic in α and are continuously differentiable $w.r.t$ x and v in S $(S \subset \mathbb{R})$.
3. The composition function $\phi(\alpha, \delta)$ is analytic in α and δ.

2.1.2 Infinitesimal Transformation

Consider one-parameter Lie group of transformations

$$\check{\boldsymbol{v}} = \boldsymbol{z}(\boldsymbol{v}, \alpha). \tag{2.3}$$

Using Tylor expansion near $\alpha = 0$, we can write eqn.(2.3) as

$$\check{\boldsymbol{v}} = \boldsymbol{v} + \alpha \frac{\partial \boldsymbol{z}(\boldsymbol{v}, \alpha)}{\partial \alpha} |_{\alpha=0} + \frac{\alpha^2}{2} \frac{\partial^2 \boldsymbol{z}(\boldsymbol{v}, \alpha)}{\partial \alpha^2} |_{\alpha=0} + \cdots,$$

let us consider

$$\frac{\partial \boldsymbol{z}(\boldsymbol{v}, \alpha)}{\partial \alpha} |_{\alpha=0} = \xi(\boldsymbol{v}). \tag{2.4}$$

The transformation $\check{\boldsymbol{v}} = \boldsymbol{v} + \alpha\xi(\boldsymbol{v})$ is the infinitesimal transformation of Lie group. This relation defined in eqn.(2.4) is used in "Lie First Fundamental Theorem" which describes a method for re-parametrization of one-parameter group of transformation that are in standard forms.

2.1.3 First Fundamental Theorem of Lie

There exists a parametrization $\tau(\alpha)$ such that the Lie group of transformations (2.3) is equal to the solution of an IVP for a system of first-order ODE stated by

$$\frac{d\check{\boldsymbol{v}}}{d\tau} = \xi(\boldsymbol{v}), \tag{2.5}$$

with $\check{v} = v$ when $\tau = 0$ [4]. In particular

$$\tau(\alpha) \ = \ \int_0^\alpha \gamma(\alpha^{'})d\alpha^{'}, \tag{2.6}$$

where

$$\gamma(\alpha) = \left.\frac{\partial\phi(a,b)}{\partial b}\right|_{(a,b)=(\alpha,\alpha^{-1})}, \quad \gamma(0) = 1. \tag{2.7}$$

2.1.4 Infinitesimal Generators

In light of the above Lie's theorem, we define "Infinitesimal Generator also called Lie symmetry vector field" as [4, 16]:

Definition 2.1.1. *The infinitesimal generator of one-parameter Lie group of transformation* (2.3) *is given by*

$$\boldsymbol{X} = \xi(\boldsymbol{v}) \cdot \nabla = \sum_{i=1}^{n} \xi_i(\boldsymbol{v}) \cdot \frac{\partial}{\partial v_i}, \tag{2.8}$$

where ∇ *represent gradient.*

$$\nabla = (\frac{\partial}{\partial v_1}, \frac{\partial}{\partial v_2}, ..., \frac{\partial}{\partial v_n})^t \qquad and \qquad \xi(\boldsymbol{v}) = (\xi_1, \xi_2, ..., \xi_n),$$

For differential equation

$$G(\mathbf{x}) = G(x_1, x_2, ..., x_n), \tag{2.9}$$

we write

$$\mathbf{X}G(\mathbf{x}) = \xi(\boldsymbol{v}).\nabla G(\mathbf{x}) = \sum_{i=1}^{n} \xi_i(\boldsymbol{v}).\frac{\partial G(\mathbf{x})}{\partial v_i}. \tag{2.10}$$

With the help of Theorem 2.1.1 [4, 16], we can define a relation between one-parameter Lie group of transformations and its generators.

Theorem 2.1.1. *The one-parameter Lie group of transformation* (2.3) *can be expressed as*

$$\check{v} = e^{\alpha \boldsymbol{X}} v = v + \alpha \boldsymbol{X} v + \frac{\alpha^2}{2}\boldsymbol{X}^2 v + \cdots = [1 + \alpha \boldsymbol{X} + \frac{\alpha^2}{2}\boldsymbol{X}^2 + ...]v.$$

This implies

$$\check{v} = \sum_{m=0}^{\infty} \frac{\alpha^m}{m!}\boldsymbol{X}^m v. \tag{2.11}$$

where $\boldsymbol{X}$ *is defined by eqn.*(2.8) *and* $\boldsymbol{X}^m = \boldsymbol{X}\boldsymbol{X}^{m-1}, \quad m = 1, 2, \cdots.$

2.2 Extension or Prolongation of Lie Group of Point Transformation and Their Generators

Using eqn.(2.2) and eqn.(2.4) we can write

$$\xi = \frac{\partial \check{x}}{\partial \alpha}\,|_{\alpha=0} \quad \textit{and} \quad \eta = \frac{\partial \check{v}}{\partial \alpha}\,|_{\alpha=0}, \tag{2.12}$$

and

$$\mathbf{X} = \xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial v}. \tag{2.13}$$

If we want to apply eqn.(2.2) to the differential equation [15],

$$\mathbf{G}(x, v, v', v'', ..., v^n) = 0, \quad v' = \frac{dv}{dx}, \tag{2.14}$$

and want to know how transformation be prolonged to derivatives, we must transform first the derivatives by defining

$$\check{v}^{'} = \frac{d\check{v}(x,v,\alpha)}{d\check{x}(x,v,\alpha)} = \frac{v^{'}(\frac{\partial \check{v}}{\partial v}) + \frac{\partial \check{v}}{\partial x}}{v^{'}(\frac{\partial \check{x}}{\partial v}) + \frac{\partial \check{x}}{\partial x}} = \check{v}^{'}(x,v,v^{'},\alpha).$$

Similarly, we can find

$$\check{v}^{''} = \frac{d\check{v}^{'}(x,v,v^{'},\alpha)}{d\check{x}(x,v,\alpha)} = \check{v}^{''}(x,v,v^{'},v^{''},\alpha),$$

and so on

$$\check{v}^{(n)} = \frac{d\check{v}^{n-1}(x,v,v^{'},v^{''},\cdots,v^{n-1},\alpha)}{d\check{x}(x,v,\alpha)} = \check{v}^{(n)}(x,v,v^{'},v^{''},\cdots,\check{v}^{(n)},\alpha). \quad (2.15)$$

These are the transformed derivatives with respect to transformed variables.

From the above relations, we can write

$$\begin{aligned} \check{x} &= x + \alpha\xi(x,v) + \cdots = x + \alpha\mathbf{X}x + \cdots . \\ \check{v} &= v + \alpha\eta(x,v) + \cdots = v + \alpha\mathbf{X}v + \cdots . \\ \check{v}^{'} &= v^{'} + \alpha\eta^{'}(x,v) + \cdots = v^{'} + \alpha\mathbf{X}v^{'} + \cdots . \\ &\vdots \\ \check{v}^{(n)} &= v^{(n)} + \alpha\eta^{(n)}(x,v) + \cdots = v^{(n)} + \alpha\mathbf{X}v^{(n)} + \cdots , \end{aligned} \quad (2.16)$$

where $\eta, \eta^{'}, \eta^{''}, \cdots, \eta^{(n)}$ are defined by

$$\eta = \frac{d\check{v}}{d\alpha}, \quad \eta^{'} = \frac{d\check{v}^{'}}{d\alpha}, \quad \eta^{''} = \frac{d\check{v}^{''}}{d\alpha}, \cdots, \eta^{(n)} = \frac{d\check{v}^{n}}{d\alpha}, \ at \ \ \alpha = 0. \quad (2.17)$$

By using eqn.(2.15) and eqn.(2.16), we have

$$\check{v}^{(n)} = v^{(n)} + \alpha(\frac{d\eta^{(n-1)}}{dx} - v^{(n)}\frac{d\xi}{dx}), \quad (2.18)$$

where

$$\frac{d}{dx} = \frac{\partial}{\partial x} + v^{'}\frac{\partial}{\partial v} + v^{''}\frac{\partial}{\partial v^{'}} + \cdots + v^{(n)}\frac{\partial}{\partial v^{(n-1)}}.$$

Similarly, from eqn.(2.16) and eqn.(2.18), we have

$$\eta^{(n)} = \frac{d\eta^{(n-1)}}{dx} - v^{(n)}\frac{d\xi}{dx}. \tag{2.19}$$

From the above eqn.(2.19), we can compute the components of extended generators of any order, i.e.

$$\begin{aligned} \eta^{'} &= \frac{d\eta^{0}}{dx} - v^{'}\frac{d\xi}{dx} \\ &= \eta_x + v^{'}(\eta_v - \xi_x) - v^{'2}\xi_v. \end{aligned} \tag{2.20}$$

Similarly,

$$\begin{aligned} \eta^{''} &= \frac{d\eta^{'}}{dx} - v^{''}\frac{d\xi}{dx} \\ &= \eta_{xx} + (2\eta_{xv} - \xi_{xx})v^{'} - (\eta_{vv} - 2\xi_{xv})(v^{'})^2 \\ &\quad -\xi_{vv}(v^{'})^3 + (\eta_v - 2\xi_x - 3\xi_v v^{'})v^{''}, \end{aligned} \tag{2.21}$$

and so on.

Eqn.(2.13) is a point transformation generator and prolongation of generator **X** is given by

$$\mathbf{X}^{[n]} = \xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial v} + \eta^{'}\frac{\partial}{\partial v^{'}} + \cdots + \eta^{(n)}\frac{\partial}{\partial v^{(n)}}. \tag{2.22}$$

2.3 Multi-Parameter Group of Transformation and Their Infinitesimal Generators

Let the group of transformations [1, 4, 15]

$$\check{x} = \check{x}(x, v, \alpha_M), \quad \check{v} = \check{v}(x, v, \alpha_M), \ \ M = 1, 2, \cdots, r, \tag{2.23}$$

depends on more than one parameter say r-parameters α_M, satisfying all the properties of a group, then its general infinitesimal transformation can be written in the form

$$\mathbf{X}_M = \xi_M(x, v)\frac{\partial}{\partial x} + \eta_M(x, v)\frac{\partial}{\partial v}, \tag{2.24}$$

where

$$\xi_M(x,v) = \frac{\partial \check{x}}{\partial \alpha_M}|_{\alpha=0},\ \eta_M(x,v) = \frac{\partial \check{v}}{\partial \alpha_M}|_{\alpha=0}. \tag{2.25}$$

If $\mathbf{X}_l$ and $\mathbf{X}_m$ be two generators, then their commutator is defined by

$$[\mathbf{X}_l, \mathbf{X}_m] = \mathbf{X}_l\mathbf{X}_m - \mathbf{X}_m\mathbf{X}_l. \tag{2.26}$$

The commutator relation defined by eqn.(2.24) satisfies the properties [9, 15] :

1. $[\mathbf{X}_l, \mathbf{X}_m] = -[\mathbf{X}_m, \mathbf{X}_l]$,

2. $[a\mathbf{X}_l + b\mathbf{X}_m, \mathbf{X}_n] = a[\mathbf{X}_l, \mathbf{X}_n] + b[\mathbf{X}_m, \mathbf{X}_n]$, a,b are scalers,

3. $[[\mathbf{X}_l, \mathbf{X}_m], \mathbf{X}_n] + [[\mathbf{X}_m, \mathbf{X}_n], \mathbf{X}_l] + [[\mathbf{X}_n, \mathbf{X}_l], \mathbf{X}_m] = 0$.

2.3.1 Lie Second Fundamental Theorem

The commutator of any two infinitesimal generators of an r-parameter Lie group of point transformations is also an infinitesimal generator.
In particular

$$[\mathbf{X}_l, \mathbf{X}_m] = \sum_{p=1}^{r} C^p_{lm}\mathbf{X}_p, \tag{2.27}$$

where the coefficient C^p_{lm}, $l, m = 1, \cdots, r$ are constants, known as "Structure constants" [4].

2.3.2 Lie Third Fundamental Theorem

The structure constants defined in eqn.(2.27) satisfy the relation

$$C^p_{lm} = -C^p_{ml}, \tag{2.28}$$

$$\sum_{\gamma=1}^{r}[C^{\gamma}_{lm}C^{\delta}_{\gamma n} + C^{\gamma}_{mn}C^{\delta}_{\gamma l} + C^{\gamma}_{nl}C^{\delta}_{\gamma m}]. \tag{2.29}$$

2.4 Lie Algebra

A Lie algebra $\mathcal{L}$ is vector space over some field on which relation [,] is defined having following features :

1. $[\mathbf{X}_l, \mathbf{X}_m] \in \mathcal{L}, \ \forall \ \mathbf{X}_l, \mathbf{X}_m \in \mathcal{L}$.

2. $[\mathbf{X}_l, \mathbf{X}_m] = -[\mathbf{X}_m, \mathbf{X}_l], \ \forall \ \mathbf{X}_l, \mathbf{X}_m \in \mathcal{L}$.

3. $[a\mathbf{X}_l + b\mathbf{X}_m, \mathbf{X}_n] = a[\mathbf{X}_l, \mathbf{X}_n] + b[\mathbf{X}_m, \mathbf{X}_n], \ , \forall \ \mathbf{X}_l, \mathbf{X}_m, \mathbf{X}_n \in \mathcal{L}$ and a,b are scalars.

4. $[[\mathbf{X}_l, \mathbf{X}_m], \mathbf{X}_n] + [[\mathbf{X}_m, \mathbf{X}_n], \mathbf{X}_l] + [[\mathbf{X}_n, \mathbf{X}_l], \mathbf{X}_m] = 0, \ \forall \ \mathbf{X}_l, \mathbf{X}_m, \mathbf{X}_n \in \mathcal{L}$.

Lie algebra is said to be real/complex if the field is of real/complex number, and the vector field $\mathbf{X}_l$, $\mathbf{X}_m$ are generators of Lie algebra. Lie algebra is called finite dimensional if its generators are finite otherwise infinite dimensional [9, 15].

2.5 Lie Point Symmetries of ODEs

We have defined important definitions and theorems necessary for defining an important theorem [7, 15] and use it to find Lie point symmetries of differential equations.

Theorem 2.5.1. *An ODE*

$$G(x, v, v^{'}, v^{''}, \cdots, v^{(n)}) = 0,$$

admits a group of symmetries with generator $\mathbf{X}$ *iff*

$$\mathbf{X}^{[\mathbf{n}]}\mathbf{G} \mid_{\boldsymbol{G}=0} \equiv 0. \tag{2.30}$$

We write $\mathbf{X}^{[\mathbf{n}]}\mathbf{G} \equiv 0$ rather than $\mathbf{X}^{[\mathbf{n}]}\mathbf{G} = 0$ because we focus on that with respect to all variables $(x, v, v^{'}, v^{''}, \cdots, v^{(n-1)})$ this equation has to be satisfied. At $\mathbf{G} = 0$ shows that the maximum derivative $v^{(n)}$ is to replaced with $(x, v, v^{'}, v^{''}, \cdots, v^{(n-1)})$.

Example 2.5.1.

$$\boldsymbol{G(x, v, v', v'') = v'' + v = 0}. \tag{2.31}$$

Solution: Applying the condition given in eqn.(2.30) on the given differential equation (2.31),

$$\mathbf{X^{[2]}}(v'' + v) \mid_{v''+v=0} = 0. \tag{2.32}$$

From eqn.(2.19) putting value of η'' and then substituting $v'' = -v$ and simplify, we get

$$\begin{aligned} \eta_{xx} + (2\eta_{xv} - \xi_{xx})v' + (\eta_{vv} - 2\xi_{xv})v'^2 - \xi_{vv}v'^3 \\ +(\eta_v - 2\xi_x - 3\xi_v v')(-v) + \eta = 0. \end{aligned} \tag{2.33}$$

Now comparing the coefficients of v'^0, v'^1, v'^2, v'^3.

$$v'^0; \qquad \eta_{xx} + (2\xi_x - \eta_v)v + \eta = 0, \tag{2.34}$$

$$v'^1; \qquad 2\eta_{xv} - \xi_{xx} + 3\xi_v v = 0, \tag{2.35}$$

$$v'^2; \qquad \eta_{vv} - 2\xi_{xv} = 0, \tag{2.36}$$

$$v'^3; \qquad \xi_{vv} = 0. \tag{2.37}$$

Eqn.(2.37) gives

$$\xi = a_1(x)v + a_2(x). \tag{2.38}$$

From eqn.(2.36)

$$\eta = a_1'(x)v^2 + a_3(x)v + a_4(x). \tag{2.39}$$

Eqn.(2.35) implies that

$$2(2a_1^{''}(x)v + a_3^{'}(x)) - a_1^{''}(x)v - a_2^{''}(x) + 3a_1(x)v = 0. \tag{2.40}$$

Now comparing the coefficients of v^0, v^1.

$$v^0; \qquad 2a_3^{'}(x) - a_2^{''}(x) = 0, \tag{2.41}$$

$$v^1; \qquad a_1^{''}(x) + a_1(x) = 0. \tag{2.42}$$

Eqn.(2.41) implies

$$a_1(x) = c_1 \cos x + c_2 \sin x. \tag{2.43}$$

Eqn.(2.34) implies

$$\begin{aligned} a_1^{'''}(x)v^2 + a_3^{''}(x)v + a_4^{''}(x) + (-2a_1^{'}(x)v - a_3(x) + 2a_1^{'}(x)v \\ +2a_2^{'}(x)v + a_1^{'}(x)v^2 + a_3(x) + a_4(x)) = 0 \end{aligned} \tag{2.44}$$

Comparing coefficients of v^0, v^1, v^2, we have

$$v^0; \qquad a_4^{''}(x) + a_4(x) = 0, \tag{2.45}$$

$$v^1; \qquad a_3^{''}(x) + 2a_2^{'}(x) = 0, \tag{2.46}$$

$$v^2; \qquad a_1^{'''}(x) + a_1^{'}(x) = 0. \tag{2.47}$$

Simplifying eqns.(2.45), (2.46) and (2.47), we get

$$a_4(x) = c_3 \cos x + c_4 \sin x, \tag{2.48}$$

$$a_2(x) = c_5 \cos 2x + c_6 \sin 2x + c_7, \tag{2.49}$$

$$a_3(x) = c_6 \cos 2x - c_5 \sin 2x + c_8. \tag{2.50}$$

So ξ, η and $\mathbf{X}$ becomes

$$\xi = (c_1 \cos x + c_2 \sin x)v + c_5 \cos 2x + c_6 \sin 2x + c_7, \tag{2.51}$$

$$\begin{aligned} \eta \;=\;& (c_2 \cos x - c_1 \sin x)v^2 + (c_6 \cos 2x - c_5 \sin 2x + c_8)v \\ & + c_3 \cos x + c_4 \sin x, \end{aligned} \tag{2.52}$$

and

$$\begin{aligned} \mathbf{X} = \quad & ((c_1 \cos x + c_2 \sin x)v + c_5 \cos 2x + c_6 \sin 2x + c_7)\frac{\partial}{\partial x} \\ & +((c_2 \cos x - c_1 \sin x)v^2 + (c_6 \cos 2x - c_5 \sin 2x + c_8)v \\ & + c_3 \cos x + c_4 \sin x)\frac{\partial}{\partial v}, \end{aligned} \tag{2.53}$$

where c_i are constants.
Consider $c_1 = 1$ and $c_i = 0$, $i \neq 1$, we have

$$\mathbf{X}_1 = v \cos x \frac{\partial}{\partial x} - v^2 \sin x \frac{\partial}{\partial v},$$

Similarly, consider $c_2 = 1$, $c_i = 0$, $i \neq 2$, we get

$$\mathbf{X}_2 = v \sin x \frac{\partial}{\partial x} + v^2 \cos x \frac{\partial}{\partial v},$$

similarly, we can find

$$\begin{aligned} \mathbf{X}_3 = \cos x \frac{\partial}{\partial v},\; \mathbf{X}_4 &= \sin x \frac{\partial}{\partial v}, \\ \mathbf{X}_5 = \cos 2x \frac{\partial}{\partial x} &- v \sin 2x \frac{\partial}{\partial v}, \\ \mathbf{X}_6 = \sin 2x \frac{\partial}{\partial x} &+ v \cos 2x \frac{\partial}{\partial v}, \\ \mathbf{X}_7 = \frac{\partial}{\partial x},\;\; \mathbf{X}_8 &= v \frac{\partial}{\partial v}. \end{aligned}$$

Lie algebras corresponding to above generators are :

$$[\mathbf{X}_1, \mathbf{X}_3] = -\frac{1}{2}[\mathbf{X}_5 + \mathbf{X}_7],\ \ [\mathbf{X}_1, \mathbf{X}_4] = -\frac{1}{2}[3\mathbf{X}_8 + \mathbf{X}_6],\ \ [\mathbf{X}_1, \mathbf{X}_5] = -\mathbf{X}_2,\ \ [\mathbf{X}_1, \mathbf{X}_6] = \mathbf{X}_1,$$

$$[\mathbf{X}_1, \mathbf{X}_7] = -\mathbf{X}_2,\ \ [\mathbf{X}_1, \mathbf{X}_8] = \mathbf{X}_1,\ \ [\mathbf{X}_2, \mathbf{X}_3] = -\frac{1}{2}[\mathbf{X}_6 + 3\mathbf{X}_8],\ \ [\mathbf{X}_2, \mathbf{X}_4] = \frac{1}{2}[\mathbf{X}_5 - \mathbf{X}_7],$$

$$[\mathbf{X}_2, \mathbf{X}_5] = -\mathbf{X}_1,\ \ [\mathbf{X}_2, \mathbf{X}_6] = -\mathbf{X}_2,\ \ [\mathbf{X}_2, \mathbf{X}_7] = -\mathbf{X}_1,\ \ [\mathbf{X}_2, \mathbf{X}_8] = -\mathbf{X}_2,\ \ [\mathbf{X}_3, \mathbf{X}_5] = -\mathbf{X}_4,$$

$$[\mathbf{X}_3, \mathbf{X}_6] = \mathbf{X}_3,\ \ [\mathbf{X}_3, \mathbf{X}_7] = \mathbf{X}_4,\ \ [\mathbf{X}_3, \mathbf{X}_8] = \mathbf{X}_3,\ \ [\mathbf{X}_4, \mathbf{X}_5] = -\mathbf{X}_3,\ \ [\mathbf{X}_4, \mathbf{X}_6] = -\mathbf{X}_3,$$

$$[\mathbf{X}_4, \mathbf{X}_7] = \mathbf{X}_3,\ \ [\mathbf{X}_4, \mathbf{X}_8] = \mathbf{X}_4,\ \ [\mathbf{X}_5, \mathbf{X}_6] = \mathbf{X}_7,\ \ [\mathbf{X}_5, \mathbf{X}_7] = 2\mathbf{X}_6,\ \ [\mathbf{X}_5, \mathbf{X}_8] = \mathbf{X}_3,$$

$$[\mathbf{X}_6, \mathbf{X}_7] = -2\mathbf{X}_5,$$

and others $[\mathbf{X}_l, \mathbf{X}_m] = 0$.

Chapter 3

Discrete Symmetries of First Order Differential Equations

Introduction

Discrete symmetries are those point symmetries of a differential equation which are not continuous [9, 10, 11].

In this chapter, we discuss and explain method in detail which help in finding discrete symmetries of differential equations. Finding discrete symmetries of differential equations was not an easy task. Peter E. Hydon was the first to developed an indirect method with the help of which discrete symmetries of the differential equations having finite dimensional Lie algebra of infinitesimal generators of its Lie group of point symmetries could be found. So far his method is used to find the discrete symmetries of second order or higher order differential equations [9, 10, 11].

Our main aim is to find the discrete symmetries of first order differential equations using Peter E. Hydon method. We know that the first order differential equations have infinite many symmetries. Therefore, we consider a specific subgroup of symmetries from these infinite symmetries and find out the discrete symmetries corresponding to that subgroup.

3.1 Determining Equations for Discrete Symmetries

We start this section by recalling some important definitions and theorems [9, 10, 11]. Let us consider an ODE

$$v^{(n)} = w(x, v, v^{'}, \cdots, v^{(n-1)}). \tag{3.1}$$

A one-parameter Lie group of point symmetry is denoted by

$$\lambda : (x, v) \longmapsto (\check{x}(x, v), \check{v}(x, v)), \tag{3.2}$$

and its infinitesimal generator is given by

$$\mathbf{X} = \xi(x, v)\frac{\partial}{\partial x} + \eta(x, v)\frac{\partial}{\partial v}. \tag{3.3}$$

If $\{\mathbf{X}_l\}_{l=1}^{r}$ is the basis for Lie algebra $\mathcal{L}$ of all infinitesimal generators of one-parameter Lie group of point symmetries of ODE (3.1), then we have

$$\lambda_l : (x, v) \longmapsto (e^{\alpha\mathbf{X}_l}x, e^{\alpha\mathbf{X}_l}v), \tag{3.4}$$

to represent the one-parameter Lie group.

Theorem 3.1.1. *If $\boldsymbol{G}(x, v)$ is an infinitely differentiable function, then for one-parameter Lie group of point symmetries $\lambda(\alpha)$ with generator (3.3), we have*

$$\boldsymbol{G}(\check{x}, \check{v}) = \boldsymbol{G}(e^{\alpha\boldsymbol{X}_l}x, e^{\alpha\boldsymbol{X}_l}v) = e^{\alpha\boldsymbol{X}_l}\boldsymbol{G}(x, v) = \lambda_l\boldsymbol{G}(x, v). \tag{3.5}$$

Theorem 3.1.2. *Suppose that the Lie algebra $\mathcal{L}$ of infinitesimal generators of one-parameter Lie group of point symmetries of the differential equation (3.1) has a basis $\{\boldsymbol{X}_l\}_{l=1}^{r}$. If the point transformation (3.2) is a symmetry of ODE (3.1), then the point transformation*

$$\check{\lambda}_l(\alpha) = \lambda\lambda_l\lambda^{-1}, \tag{3.6}$$

is also symmetry for each α. Also, if $\{\boldsymbol{X}_l\}_{l=1}^{r}$ is a basis for $\mathcal{L}$, then $\{\lambda\boldsymbol{X}_l\lambda^{-1}\}_{l=1}^{r}$ is also a basis for $\mathcal{L}$ for arbitrary λ.

From the above theorems we know that both the generators $\{\mathbf{X}_l\}_{l=1}^r$ and $\{\check{\mathbf{X}}_l\}_{l=1}^r$ are basis of some Lie algebra $\mathcal{L}$, so we can write each $\mathbf{X}_l$ as a linear combination of $\check{\mathbf{X}}_l$, and the structure constants are defined by the transformation

$$\mathbf{X}_l \longmapsto \check{\mathbf{X}}_l.$$

If

$$[\mathbf{X}_l, \mathbf{X}_m] = C^p_{lm}\mathbf{X}_p, \tag{3.7}$$

then,

$$[\check{\mathbf{X}}_l, \check{\mathbf{X}}_m] = C^p_{lm}\check{\mathbf{X}}_p. \tag{3.8}$$

We can generalize the above results with the help of following lemma as [9, 10, 11]:

Lemma 3.1.1. *Every point symmetry of a differential equation (3.1) induces an automorphism of the Lie algebra $\mathcal{L}$ of generator of one-parameter Lie group of point symmetries of the differential equation.*

For each λ, there exists a constant $N \times N$ non-singular matrix (b^p_l) such that

$$\mathbf{X}_l = b^p_l \check{\mathbf{X}}_p. \tag{3.9}$$

All the structure constants are preserved by this automorphism.

Finding Lie Groups of a Differential Equation Corresponding to Some Generators

Consider the first order differential equation $v' = f(x)$ with generators

$$\mathbf{X}_1 = \frac{\partial}{\partial v}, \;\; \mathbf{X}_2 = \frac{1}{f(x)}\frac{\partial}{\partial x}.$$

Using eqns.(2.12), one can find the corresponding Lie groups.
First we find the Lie group corresponding to $\mathbf{X}_1 = \frac{\partial}{\partial v}$.

Here $\xi = 0$ and $\eta = 1$.
As

$$\xi = \frac{\partial \check{x}}{\partial \alpha} \mid_{\alpha=0}$$
$$\Rightarrow \check{x} = x, \;\; as \;\; \check{x}(0) = x.$$

Similarly

$$\eta = \frac{\partial \check{v}}{\partial \alpha} \mid_{\alpha=0} = 1$$
$$\Rightarrow \check{v} = v + \alpha \;\; as \;\; \check{v}(0) = v.$$

$$\therefore \qquad \lambda_1(\alpha); \quad (\check{x}, \check{v}) = (x, v + \alpha).$$

Similarly, for

$$\mathbf{X}_2 = \frac{1}{f(x)} \frac{\partial}{\partial x},$$

we have

$$\lambda_2(\alpha); \;\; (\check{x}, \check{v}) = \left(x + \frac{1}{f(x)} \alpha, v \right).$$

3.2 Peter E. Hydon for Finding Discrete Symmetries

In this method there are two steps. In the first step, let us apply the Lemma 3.1.1 to obtain the following first-order PDEs which every point symmetry (3.2) of the ODE (3.1) should satisfy [13, 14]:

$$\begin{aligned} \mathbf{X}_l \check{x} &= b_l^p \check{\mathbf{X}}_p \check{x}, \qquad l = 1, 2, \cdots, R \\ &= b_l^p \lambda \mathbf{X}_p \lambda^{-1} \check{x}, \;\; \because \check{\mathbf{X}} = \lambda \mathbf{X} \lambda^{-1} \\ &= b_l^p \lambda \mathbf{X}_p x \\ &= b_l^p \lambda \xi_p(x, v) \\ &= b_l^p \xi_p(\check{x}, \check{v}). \end{aligned}$$

This implies that

$$\mathbf{X}_l \check{x} = b_l^p \check{\xi}_p. \tag{3.10}$$

Similarly, we can write

$$\mathbf{X}_l \check{v} = b_l^p \check{\eta}_p. \tag{3.11}$$

Combining eqn.(3.10) and eqn.(3.11) make a system of equations (first-order PDEs). This system can be solved by using characteristic equations to get $(\check{x}, \check{v})$ in terms of x, v, b_l^p and some other constants or functions (unknown). All the point symmetries should satisfy the above system. However there are also certain solutions of the system which are not point symmetries. Note that, the solution of this system always contains the trivial symmetry $(\check{x}, \check{v}) = (x, v)$ corresponding to $b_l^p = \delta_l^p$.

In the second step, we check that which of these solutions are point symmetries of the ODE, and so we can separate point symmetry and non-point symmetry solution of the general solution of the system of PDEs.

With the help of this two-step method we can find all the point symmetries of differential equation (3.1). Also we know about the Lie-point symmetries, so any symmetries other than Lie-point symmetries are known as discrete symmetries of eqn.(3.1). Now, writing eqn.(3.10) in matrix form as:

$$\begin{bmatrix} \mathbf{X}_1 \check{x} \\ \\ \mathbf{X}_2 \check{x} \\ \vdots \\ \mathbf{X}_n \check{x} \end{bmatrix} = \begin{bmatrix} b_1^1 & b_1^2 & \cdots & b_1^n \\ \\ b_2^1 & b_2^2 & \cdots & b_2^n \\ \vdots & \vdots & \ddots & \vdots \\ b_n^1 & b_n^2 & \cdots & b_n^n \end{bmatrix} \begin{bmatrix} \check{\xi}_1 \\ \\ \check{\xi}_2 \\ \vdots \\ \check{\xi}_n \end{bmatrix}. \tag{3.12}$$

Similarly, eqn.(3.11) can be written as

$$\begin{bmatrix} \mathbf{X}_1 \check{v} \\ \\ \mathbf{X}_2 \check{v} \\ \vdots \\ \mathbf{X}_n \check{v} \end{bmatrix} = \begin{bmatrix} b_1^1 & b_1^2 & \cdots & b_1^n \\ \\ b_2^1 & b_2^2 & \cdots & b_2^n \\ \vdots & \vdots & \ddots & \vdots \\ b_n^1 & b_n^2 & \cdots & b_n^n \end{bmatrix} \begin{bmatrix} \check{\eta}_1 \\ \\ \check{\eta}_2 \\ \vdots \\ \check{\eta}_n \end{bmatrix}. \tag{3.13}$$

From eqn.(3.12) and eqn.(3.13), we can write

$$\begin{bmatrix} \mathbf{X}_1\check{x} & \mathbf{X}_1\check{v} \\ \\ \mathbf{X}_2\check{x} & \mathbf{X}_2\check{v} \\ \vdots & \vdots \\ \mathbf{X}_n\check{x} & \mathbf{X}_n\check{v} \end{bmatrix} = \begin{bmatrix} b_1^1 & b_1^2 & \cdots & b_1^n \\ \\ b_2^1 & b_2^2 & \cdots & b_2^n \\ \vdots & \vdots & \ddots & \vdots \\ b_n^1 & b_n^2 & \cdots & b_n^n \end{bmatrix} \begin{bmatrix} \check{\xi}_1 & \check{\eta}_1 \\ \\ \check{\xi}_2 & \check{\eta}_2 \\ \vdots & \vdots \\ \check{\xi}_n & \check{\eta}_n \end{bmatrix}. \tag{3.14}$$

Eqn.(3.14) is known as system of determining equations. Note that eqn.(3.14) gives a set of first order linear and un-coupled system of PDEs. Further, if there is a complex valued parameter in the symmetry condition, then this method gives all the complex discrete symmetries of the given differential equation.

3.3 Modification in Method

In this section, we explain the modification in the method that is to be made.

3.3.1 Abelian Lie Algebra

For abelian Lie algebra it is easy to work in canonical coordinates system. This is specifically effective when the dimension of Lie algebra is low. Because, as the dimension increases many unknown constants b_l^p are to be considered there, due to which it can be difficult to solve the determining equations [10].

Use of Canonical Coordinates

Canonical coordinates $s(x,v)$ and $t(x,v)$ satisfy

$$\mathbf{X}_{1s} = 0, \qquad \mathbf{X}_{1t} = 1, \tag{3.15}$$

so that

$$\mathbf{X}_1 = \frac{\partial}{\partial s} = \partial_s.$$

Since, $\check{s} = s(\check{x}, \check{v})$ and $\check{t} = t(\check{x}, \check{v})$, if $dim(\mathcal{L}) = 1$, then from eqn.(3.14) we have

$$[\mathbf{X}_1\check{s}, \mathbf{X}_1\check{t}] = [b_1^1][1, 0], \tag{3.16}$$

$$\Rightarrow \mathbf{X}_1\check{s} = b^1 \neq 0, \qquad \mathbf{X}_1\check{t} = 0,$$
$$\Rightarrow \frac{\partial \check{s}}{\partial s} = b_1^1. \qquad \Rightarrow \frac{\partial \check{t}}{\partial s} = 0.$$

$$\Rightarrow \check{s} = b_1^1 s + l(t), \qquad \check{t} = m(t), \tag{3.17}$$

is the general solution for some function l and m. Symmetry condition on the transformation decide that which functions l, m and constant $(b_1^1\ )$ are permissible.

3.3.2 Non-abelian Lie Algebra

If $\mathcal{L}$ is not abelian, then at least some of the equation

$$[\mathbf{X}_l, \mathbf{X}_m] = \mathbf{X}_l\mathbf{X}_m - \mathbf{X}_m\mathbf{X}_l = C_{lm}^p\mathbf{X}_p,$$

are non-trivial. Also, since the generator $\lambda\mathbf{X}_l\lambda^{-1}$ satisfy the same commutator relation as the generator $\mathbf{X}_l$. That is

$$[\check{\mathbf{X}}_l, \check{\mathbf{X}}_m] = C_{lm}^p\check{\mathbf{X}}_p.$$

From eqn.(3.7), eqn.(3.8) and eqn.(3.9), we have

$$C_{ij}^n b_l^i b_m^j = C_{lm}^p b_p^n, \tag{3.18}$$

where the indices are from 1 to $dim(\mathcal{L})$. Since, we know that the structure constants are anti-symmetric in the lower indices, that is

$$C_{lm}^p = -C_{ml}^p.$$

Therefore,

$$C_{ll}^p = 0.$$

If we have $dim(\mathcal{L}) = r$, then there will be r^3 equations, but since $C^p_{lm} = -C^p_{ml}$ and $C^P_{ll} = 0$. Therefore, we have only $\frac{r^2(r-1)}{2}$ equations. So we focus on the eqn.(3.18) with $l < m$.

Before we solve the system given by eqn.(3.14), we can simplify it using these constraints on the matrix B, which makes the system easier to solve [10].

Note that computer algebra is to be used when the number of equations is too large.

Let us consider an example in detail to understand the above process.

Example 3.3.1. *Consider the differential equation*

$$v^{''} = 0. \tag{3.19}$$

with

$$\mathbf{X}_1 = \frac{\partial}{\partial x}, \tag{3.20}$$

$$\mathbf{X}_2 = \frac{\partial}{\partial v}, \tag{3.21}$$

$$\mathbf{X}_3 = x\frac{\partial}{\partial v}. \tag{3.22}$$

Here

$$[\mathbf{X}_1, \mathbf{X}_2] = 0 \Rightarrow C^p_{12} = 0. \tag{3.23}$$

$$[\mathbf{X}_1, \mathbf{X}_3] = \mathbf{X}_2 \Rightarrow C^2_{13} = 1, C^2_{31} = -1. \tag{3.24}$$

$$[\mathbf{X}_2, \mathbf{X}_3] = 0 \Rightarrow C^p_{23} = 0. \tag{3.25}$$

So here the non-zero structure constants are $C^2_{13} = 1, C^2_{31} = -1$.

Now we solve the equation

$$C^n_{ij} b^i_l b^j_m = C^p_{lm} b^n_p, \ \ where \ \ i, j, l, m, n = 1, 2, 3. \tag{3.26}$$

Since $l < m$ to get distinct equations, here $(l, m) = (1, 2), (1, 3), (2, 3)$.

Taking $n = 3$, so

$$C^3_{ij} = 0, \ \ i, j = 1, 2, 3. \tag{3.27}$$

So eqn.(3.26) reduced to

$$C^p_{lm} b^3_p = 0, \tag{3.28}$$

$$C^1_{lm} b^3_1 + C^2_{lm} b^3_2 + C^3_{lm} b^3_3 = 0. \tag{3.29}$$

When $(l, m) = (1, 2)$, then eqn.(3.29) is satisfied.

For $(l, m) = (1, 3)$, eqn.(3.29) becomes

$$C^1_{13} b^3_1 + C^2_{13} b^3_2 + C^3_{13} b^3_3 = 0, \tag{3.30}$$

$$\Rightarrow b^3_2 = 0 \quad \because C^2_{13} = 1,\ C^1_{13} = 0 = C^3_{13}. \tag{3.31}$$

For $(l, m) = (2, 3)$ eqn.(3.29) is true.

Now fix $n = 1$, then

$$C^1_{ij} = 0. \tag{3.32}$$

The constraints eqn.(3.26) reduced to,

$$C^p_{lm} b^1_p = 0, \tag{3.33}$$

$$C^1_{lm} b^1_1 + C^2_{lm} b^1_2 + C^3_{lm} b^1_3 = 0. \tag{3.34}$$

Eqn.(3.34) is true for $(l, m) = (1, 2)$.

Putting $(l, m) = (1, 3)$ in eqn.(3.34),

$$C^1_{13} b^1_1 + C^2_{13} b^1_2 + C^3_{13} b^1_2 = 0, \tag{3.35}$$

$$\Rightarrow b^1_2 = 0 \quad \because C^1_{13} = 0 = C^3_{13},\ C^2_{13} = 1. \tag{3.36}$$

Similarly, eqn.(3.34) is true when $(l, m) = (2, 3)$.

Consider $n = 2$, then

$$C^1_{ij} = 0, \quad (i, j) \neq (1, 3), (3, 1). \tag{3.37}$$

Therefore eqn.(3.26) is reduced to no-linear equations

$$C^2_{13} b^1_l b^3_m + C^2_{31} b^3_l b^1_m = C^p_{lm} b^2_p, \tag{3.38}$$

$$(1) b^1_l b^3_m + (-1) b^3_l b^1_m = C^1_{lm} b^2_1 + C^2_{lm} b^2_2 + C^3_{lm} b^2_3, \tag{3.39}$$

$$b^1_l b^3_m - b^3_l b^1_m = C^1_{lm} b^2_1 + C^2_{lm} b^2_2 + C^3_{lm} b^2_3. \tag{3.40}$$

Eqn.(3.40) is true for $(l,m) = (1,2)$.

By putting $(l,m) = (1,3)$, in eqn.(3.40), we have

$$b_1^1 b_3^3 - b_1^3 b_3^1 = C_{13}^1 b_1^2 + C_{13}^2 b_2^2 + C_{13}^3 b_3^2, \tag{3.41}$$

$$\Rightarrow b_1^1 b_3^3 - b_1^3 b_3^1 = b_2^2. \tag{3.42}$$

Similarly, eqn.(3.40) is true for $(l,m) = (2,3)$.

Now we are able to write $B = (b_l^p)$ as

$$B = \begin{bmatrix} b_1^1 & b_1^2 & b_1^3 \\ b_2^1 & b_2^2 & b_2^3 \\ b_3^1 & b_3^2 & b_3^3 \end{bmatrix} = \begin{bmatrix} b_1^1 & b_1^2 & b_1^3 \\ 0 & b_2^2 & 0 \\ b_3^1 & b_3^2 & b_3^3 \end{bmatrix}, \tag{3.43}$$

with condition

$$b_1^1 b_3^3 - b_1^3 b_3^1 = b_2^2, \tag{3.44}$$

because B is non-singular.

3.3.3 For Finding Inequivalent Discrete Symmetries

By equivalent symmetries of the differential equation, we mean that two point symmetries λ and $\check{\lambda}$ of given differential equation is equivalent if there exist an $\mathbf{X} \in \mathcal{L}$ such that $\check{\lambda} = e^{\alpha \mathbf{X}} \lambda$ [9, 13].

Note that, if the Lie algebra is abelian (i.e. all the structural constants are zero), then there are no constraints, also majority of the Lie algebra exists are of non-linear type and by finding the matrices $B = b_l^p$ that satisfy the constraints, we can simplify the problem of finding all symmetries.

In this section, our main focus is to obtain set of in-equivalent discrete symmetries, so that we can find out one-parameter Lie groups. Let us define some important matrices notation and Lemmas that are used in next section [13].

$$(C(m))|_l^p = C_{lm}^p, \tag{3.45}$$

and

$$A(m,\alpha) = \sum_{n=0}^{\infty} \frac{\alpha^n}{n!}(C(m))^n = exp(\alpha C(m)). \tag{3.46}$$

Lemma 3.3.2.

1. *If the automorphism of Lie algebra $\mathcal{L}$ induced by λ is given by $\lambda = e^{\alpha \boldsymbol{X}_l}$ for a given α, then $B = A(m,\alpha)$, where $\boldsymbol{X}_l$ is basis element of $\mathcal{L}$.*

2. *If the induced automorphism of the point symmetries λ and $\check{\lambda}$ are denoted by matrices B and $\check{B}$ respectively, then the induced automorphism of the point symmetry $\check{\lambda} \circ \lambda$ is denoted by $\check{B}B$.*

3. *If λ and $\check{\lambda} = e^{\alpha \boldsymbol{X}}\lambda$ induces automorphism with matrices B and $\check{B}$, then*

$$\check{B} = A(1,\alpha_1)A(2,\alpha_2), \cdots, A(R,\alpha_R)B, \tag{3.47}$$

for some parameters α_n and R is the $dim(\mathcal{L})$.

We can find out the in-equivalent discrete symmetries by simplifying the system (3.14) for in-equivalent matrices. If $C(m)$ is non-zero for some α, then we can find Lie point symmetries generated by $\mathbf{X}_l$ as, first of all replace the matrix B by $A(m,\alpha)B$ or $BA(m,\alpha)$ and then assign a value to each α in the new matrix to simplify the entry, that is to create zeros in the matrix B. This helps in simplifying the determining equations and non-linear constraints. Also note that, each matrix $A(m,\alpha)$ is used only once. Conversely, if $C(m) = 0$ for some m, then $A(m,\alpha)$ is the identity matrix for all α. Such symmetries induces a non-trivial automorphism of $\mathcal{L}$, and can be find out if the determining equations have been solved.

3.3.4 Algorithm for Finding Discrete Symmetries of Differential Equations

Once we know the Lie algebra of point symmetries of a differential equation, we can compute discrete symmetries by using the following steps:

1. Specify Lie point symmetries of the given differential equations.
2. Simplify the non-linear constraints.
3. Each $A(j, \alpha)$ should be simplified when Lie algebra is non-abelian.
4. Write down the simplified matrix B.
5. Solve determining equations to obtain general solution.
6. Factor out discrete symmetries from general solution.

3.4 Problems

Now we consider some problems of first order differential equations and try to obtain its discrete symmetries using the above method.

3.4.1 Problem 1

Consider the first order differential equation

$$v^{'} = f(x), \tag{3.48}$$

with

$$\mathbf{X}_1 = \frac{\partial}{\partial v},\ \mathbf{X}_2 = \frac{1}{f(x)}\frac{\partial}{\partial x}.$$

Here

$$[\mathbf{X}_1, \mathbf{X}_2] = 0,$$

so the system of determining eqn.(3.14) becomes,

$$\begin{bmatrix} \mathbf{X}_1\check{x} & \mathbf{X}_1\check{v} \\ \mathbf{X}_2\check{x} & \mathbf{X}_2\check{v} \end{bmatrix} = \begin{bmatrix} b_1^1 & b_1^2 \\ b_2^1 & b_2^2 \end{bmatrix} \begin{bmatrix} \check{\xi}_1 & \check{\eta}_1 \\ \check{\xi}_2 & \check{\eta}_2 \end{bmatrix}, \tag{3.49}$$

$$\Rightarrow \begin{bmatrix} \mathbf{X}_1\check{x} & \mathbf{X}_1\check{v} \\ \mathbf{X}_2\check{x} & \mathbf{X}_2\check{v} \end{bmatrix} = \begin{bmatrix} b_1^1 & b_1^2 \\ b_2^1 & b_2^2 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ \frac{1}{f(x)} & 0 \end{bmatrix} = \begin{bmatrix} \frac{b_1^2}{f(x)} & b_1^1 \\ \frac{b_2^2}{f(x)} & b_2^1 \end{bmatrix}. \tag{3.50}$$

This implies that

$$\mathbf{X}_1\breve{x} = \tfrac{b_1^2}{f(x)}, \tag{3.51}$$

$$\mathbf{X}_1\breve{v} = b_1^1, \tag{3.52}$$

$$\mathbf{X}_2\breve{x} = \tfrac{b_2^2}{f(x)}, \tag{3.53}$$

$$\mathbf{X}_2\breve{v} = b_2^1. \tag{3.54}$$

Solving eqn.(3.51) and eqn.(3.53), we get

$$\breve{x} = b_2^2 x + \frac{b_1^2}{f(x)} v + c_1, \tag{3.55}$$

similarly, using eqn.(3.52) and eqn.(3.54), we have

$$\breve{v} = b_1^1 v + \int b_2^1 f(x) dx + c_2. \tag{3.56}$$

$$\therefore \ (\breve{x}, \breve{v}) = \left(b_2^2 x + \frac{b_1^2}{f(x)} v + c_1, b_1^1 v + \int b_2^1 f(x) dx + c_2 \right), \tag{3.57}$$

is the general solution of determining equation (3.50).
By definition eqn.(3.57) is the symmetry of eqn.(3.48) if and only if

$$\breve{v}' = f(\breve{x}). \tag{3.58}$$

Now

$$\begin{aligned}
\breve{v}' &= \frac{d\breve{v}}{d\breve{x}} \\
&= \frac{d(b_1^1 v + \int b_2^1 f(x) dx + c_2)}{d(b_2^2 x + \frac{b_1^2}{f(x)} v + c_1)} \\
&= \frac{b_1^1 v' + b_2^1 f(x)}{b_2^2 + b_1^2 \frac{d}{dx}(\frac{v}{f(x)})} \\
&= \frac{b_1^1 f(x) + b_2^1 f(x)}{b_2^2} \\
&= \frac{f(x)(b_1^1 + b_2^1)}{b_2^2 + b_1^2 \frac{d}{dx}(\frac{v}{f(x)})}.
\end{aligned}$$

So, eqn.(3.58) becomes

$$\frac{f(x)(b_1^1+b_2^1)}{b_2^2+b_1^2\frac{d}{dx}(\frac{v}{f(x)})}=f(b_2^2x+\frac{b_1^2}{f(x)}v+c_1), \tag{3.59}$$

is the symmetry condition.
The symmetry condition eqn.(3.59) is satisfied if $c_1=0=b_2^1=b_1^2, b_2^2=1=b_1^1$.
Therefore, the discrete symmetry of the given differential equation is

$$\lambda(\check{x},\check{v})=(x,v+c_2) \tag{3.60}$$

This result is exhaustive. Any other discrete symmetry $\check{\lambda}(\check{x},\check{v})$ of the given differential equation can be obtained by using the discrete symmetry $\lambda(\check{x},\check{v})$ as:

$$\check{\lambda}=e^{\alpha\mathbf{X}}\lambda,$$

where $\mathbf{X}$ is the infinitesimal generator from Lie algebra of infinitesimal generators of one-parameter Lie groups of point symmetries of the equation.

3.4.2 Problem 2

Taking first order differential equation

$$v^{'}=f(v), \tag{3.61}$$

with

$$\mathbf{X}_1=\frac{\partial}{\partial x},\ \ \mathbf{X}_2=f(v)\frac{\partial}{\partial v}. \tag{3.62}$$

Here

$$[\mathbf{X}_1,\mathbf{X}_2]=0.$$

So the system of determining equation (3.14) becomes

$$\begin{bmatrix}\mathbf{X}_1\check{x} & \mathbf{X}_1\check{v}\\ \mathbf{X}_2\check{x} & \mathbf{X}_2\check{v}\end{bmatrix}=\begin{bmatrix}b_1^1 & b_1^2\\ b_2^1 & b_2^2\end{bmatrix}\begin{bmatrix}1 & 0\\ 0 & f(v)\end{bmatrix}, \tag{3.63}$$

$$\Rightarrow \begin{bmatrix} \mathbf{X}_1\check{x} & \mathbf{X}_1\check{v} \\ \mathbf{X}_2\check{x} & \mathbf{X}_2\check{v} \end{bmatrix} = \begin{bmatrix} b_1^1 & b_1^2 f(v) \\ b_2^1 f(x) & b_2^2 f(v) \end{bmatrix}. \tag{3.64}$$

Eqn.(3.64) gives

$$\mathbf{X}_1\check{x} = b_1^1, \tag{3.65}$$

$$\mathbf{X}_1\check{v} = b_1^2 f(v), \tag{3.66}$$

$$\mathbf{X}_2\check{x} = b_2^1, \tag{3.67}$$

$$\mathbf{X}_2\check{v} = b_2^2 f(v). \tag{3.68}$$

Solving eqn.(3.65) and eqn.(3.67), we get

$$\check{x} = b_1^1 x + \int \frac{b_2^1}{f(v)} dv + c_1, \tag{3.69}$$

similarly, from eqn.(3.66) and eqn.(3.68), we have

$$\check{v} = b_2^2 v + b_1^2 f(v) x + c_2. \tag{3.70}$$

Therefore, the general solution of eqn.(3.64) is given by

$$(\check{x}, \check{v}) = \left(b_1^1 x + \int \frac{b_2^1}{f(v)} dv + c_1, b_2^2 v + b_1^2 f(v) x + c_2 \right). \tag{3.71}$$

Eqn.(3.71) is the symmetry of eqn.(3.61) if and only if

$$\check{v}' = f(\check{v}). \tag{3.72}$$

Now

$$\check{v}' = \frac{d\check{v}}{d\check{x}}$$

$$= \frac{d(b_2^2 v + b_1^2 f(v) x + c_2)}{d(b_1^1 x + \int \frac{b_2^1}{f(v)} dv + c_1)}$$

$$\Rightarrow \check{v}' = \frac{f(v)(b_2^2 + b_1^2)}{b_1^1 + b_2^1}.$$

Also since $\check{v}' = f(\check{v})$. So eqn.(3.72) becomes

$$\begin{aligned} f(\check{v}) &= \tfrac{f(v)(b_2^2+b_1^2)}{b_1^1+b_2^1}, \\ \Rightarrow f(b_2^2 v + b_1^2 f(v)x + c_2) &= \tfrac{f(v)(b_2^2+b_1^2)}{b_1^1+b_2^1}, \end{aligned}$$

is the symmetry condition, and it is satisfied only if $b_2^2 = 1 = b_1^1, b_1^2 = 0 = b_2^1 = c_2$. Therefore,

$$\lambda(\check{x}, \check{v}) = (x, v), \tag{3.73}$$

is the only discrete symmetry of the differential equation (3.61) up to equivalence. This result is exhaustive and by using the result $\check{\lambda} = e^{\alpha \mathbf{X}}\lambda$, we can find out other discrete symmetries $\check{\lambda}(\check{x}, \check{v})$ of the differential equation, where $\mathbf{X}$ is the infinitesimal generator from Lie algebra of infinitesimal generators of one-parameter Lie groups of point symmetries.

3.4.3 Problem 3

Consider the first order differential equation

$$v' = xv, \tag{3.74}$$

with

$$\mathbf{X}_1 = \frac{1}{x}\frac{\partial}{\partial x}, \;\; \mathbf{X}_2 = e^{\frac{x^2}{2}}\frac{\partial}{\partial v}. \tag{3.75}$$

Here

$$\begin{aligned} &[\mathbf{X}_1, \mathbf{X}_2] = \mathbf{X}_2, \\ &\Rightarrow C_{12}^2 = 1 \;\; and \;\; C_{21}^2 = -1, \end{aligned}$$

are the only non-zero structure constants (i.e., the algebra is non-abelian).
Now we solve

$$C_{ij}^n b_l^i b_m^j = C_{lm}^k b_k^n, \quad i, j, k, l, m, n = 1, 2. \tag{3.76}$$

Consider $n = 1$, we have

$$C^1_{ij} = 0,$$

so eqn.(3.76) becomes

$$C^k_{lm} b^1_k = 0, \tag{3.77}$$

$$\Rightarrow C^1_{lm} b^1_1 + C^2_{lm} b^1_2 = 0. \tag{3.78}$$

Taking $(l, m) = (1, 2)$, so that eqn.(3.78) implies that

$$C^1_{12} b^1_1 + C^2_{12} b^1_2 = 0, \tag{3.79}$$

$$\Rightarrow b^1_2 = 0 \quad \because C^1_{12} = 0, C^2_{12} = 1. \tag{3.80}$$

When $n = 2$, then

$$C^2_{ij} = 0, \quad (i, j) \neq (1, 2), (2, 1). \tag{3.81}$$

Eqn.(3.76) becomes

$$C^2_{12} b^1_l b^2_m + C^2_{21} b^2_l b^1_m = C^k_{lm} b^2_k, \tag{3.82}$$

$$\Rightarrow b^1_l b^2_m - b^2_l b^1_m = C^1_{lm} b^2_1 + C^2_{lm} b^2_2, \ \because C^2_{12} = 1, C^2_{21} = -1. \tag{3.83}$$

For $(l, m) = (1, 2)$, eqn.(3.83) becomes

$$b^1_1 b^2_2 = b^2_2, \ \because b^2_1 = 0 = C^1_{12}, \ C^2_{12} = 1, \tag{3.84}$$

$$\Rightarrow b^1_1 = 1, \quad b^2_2 \neq 0. \tag{3.85}$$

Putting value of b^1_1 and b^1_2 in matrix B, so it becomes

$$B = \begin{bmatrix} b^1_1 & b^2_1 \\ b^1_2 & b^2_2 \end{bmatrix} = \begin{bmatrix} 1 & b^2_1 \\ 0 & b^2_2 \end{bmatrix}. \tag{3.86}$$

Now we have to find the inequivalent matrix. First we find $C(m)$ and $A(m, \alpha)$, as by definition

$$C(m)^l_p = C^p_{lm}, \ \ and \ \ A(m, \alpha) = e^{\alpha C(m)}. \tag{3.87}$$

So,

$$C(1) = \begin{bmatrix} C^1_{11} & C^2_{11} \\ C^1_{21} & C^2_{21} \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix}. \tag{3.88}$$

And

$$A(1,\alpha) = exp(\alpha C(1)) = exp\left(\begin{bmatrix} 0 & 0 \\ 0 & -\alpha \end{bmatrix} \right),$$

$$\Rightarrow A(1,\alpha) = \begin{bmatrix} 1 & 0 \\ 0 & 1-\alpha+\frac{\alpha^2}{2} \end{bmatrix}. \quad (3.89)$$

Similarly,

$$C(2) = \begin{bmatrix} C^1_{12} & C^2_{12} \\ C^1_{22} & C^2_{22} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad (3.90)$$

and

$$A(2,\alpha) = \begin{bmatrix} 1 & \alpha \\ 0 & 1 \end{bmatrix}. \quad (3.91)$$

Now

$$\begin{aligned} A(1,\alpha)B &= \begin{bmatrix} 1 & 0 \\ 0 & 1-\alpha+\frac{\alpha^2}{2} \end{bmatrix} \begin{bmatrix} 1 & b^2_1 \\ 0 & b^2_2 \end{bmatrix} \\ &= \begin{bmatrix} 1 & b^2_1 \\ 0 & b^2_2(1-\alpha+\frac{\alpha^2}{2}) \end{bmatrix}, \end{aligned} \quad (3.92)$$

choosing $\alpha = \alpha_1 = 1 \pm i$, so that $1-\alpha+\frac{\alpha^2}{2} = 0$.

$$\therefore \qquad A(1,\alpha_1) = \begin{bmatrix} 1 & b^2_1 \\ 0 & 0 \end{bmatrix}. \quad (3.93)$$

$$\begin{aligned} A(2,\alpha)A(1,\alpha_1)B &= \begin{bmatrix} 1 & \alpha \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & b^2_1 \\ 0 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 1 & b^2_1 \\ 0 & 0 \end{bmatrix}. \end{aligned} \quad (3.94)$$

$A(2,\alpha)A(1,\alpha_1)B = \check{B}$ is the required inequivalent matrix. Now our next goal is to find the solution by using determining equations, i.e.

$$\begin{bmatrix} X_1\check{x} & X_1\check{v} \\ X_2\check{x} & X_2\check{v} \end{bmatrix} = \begin{bmatrix} 1 & b^2_1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{x} & 0 \\ 0 & e^{\frac{x^2}{2}} \end{bmatrix},$$

$$\Rightarrow \begin{bmatrix} X_1\check{x} & X_1\check{v} \\ X_2\check{x} & X_2\check{v} \end{bmatrix} = \begin{bmatrix} \frac{1}{x} & b^2_1 e^{\frac{x^2}{2}} \\ 0 & 0 \end{bmatrix}. \quad (3.95)$$

Eqn.(3.95) implies that

$$X_1\check{x} = \frac{1}{x}, \tag{3.96}$$

$$X_1\check{v} = b_1^2 e^{\frac{x^2}{2}}, \tag{3.97}$$

$$X_2\check{x} = 0, \tag{3.98}$$

$$X_2\check{v} = 0. \tag{3.99}$$

Simplifying eqn.(3.96) and eqn.(3.98), we get

$$\check{x} = x + c_1, \tag{3.100}$$

similarly using eqn.(3.97) and eqn.(3.99), we have

$$\check{v} = b_1^2 e^{\frac{x^2}{2}} + c_2. \tag{3.101}$$

$$\therefore \qquad (\check{x}, \check{v}) = (x + c_1, b_1^2 e^{\frac{x^2}{2}} + c_2), \tag{3.102}$$

is the general solution of eqn.(3.95).

For the transformations given by eqn.(3.102) to be a symmetry of the eqn.(3.74), we must have

$$\check{v'} = \check{x}\check{v}. \tag{3.103}$$

Now,

$$\begin{aligned} \check{v'} &= \frac{d\check{v}}{d\check{x}} \\ &= \frac{d(b_1^2 e^{\frac{x^2}{2}} + c_2)}{d(x + c_1)} \\ &= b_1^2 x e^{\frac{x^2}{2}}. \end{aligned} \tag{3.104}$$

So, eqn.(3.103) becomes

$$\begin{aligned} b_1^2 x e^{\frac{x^2}{2}} &= (x+c_1)(b_1^2 e^{\frac{x^2}{2}} + c_2) \\ b_1^2 x e^{\frac{x^2}{2}} &= b_1^2 x e^{\frac{x^2}{2}} + x c_2 + c_1 e^{\frac{x^2}{2}} + c_1 c_2, \end{aligned} \tag{3.105}$$

is the symmetry condition.

The symmetry condition given by eqn.(3.105) is satisfied if $b_1^2 = 1$ and $c_1 = 0 = c_2$.

So, discrete symmetry of differential equation (3.74) is

$$(\check{x}, \check{v}) = (x, e^{\frac{x^2}{2}}). \tag{3.106}$$

Chapter 4

Summary

Peter E. Hydon's method is used to find the discrete symmetries of the differential equations having order≥ 2, but here we use it for differential equations of first order. Basically this method consists of two major steps, i.e. finding solutions of the determining equations and apply the symmetry condition on that solution. His method is available for abelian and non-abelian Lie algebra. For abelian Lie algebra one can use canonical coordinates while in other case one can use the non-linear constraints equations and then solve the system by using in-equivalent matrices. As first order differential equations have infinite dimensional Lie algebra, we considered a subgroup from these infinite dimensional Lie algebra and find out its discrete symmetries. With the help of his method we can not only reduce the equations to simple forms but also can find all the discrete symmetries of the given differential equations.

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