Exact Solution of Manakov System From Group Foliation Approach

by

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Abstract

Sophus Lie introduced a powerful method based on the Lie algebra of vector fields to overcome the problems of finding the exact solutions of nonlinear ordinary as well as partial differential equations. In this thesis we find the Lie algebra of Lie point symmetries of a class of nonlinear partial differential equations, in particular, Manakov system and a system of coupled nonlinear Schrödinger equations. Using similarity transformations of the admitted Lie point symmetries of these equations, we apply group reduction to obtain their exact solutions. Furthermore, these solutions are classified into distinct conjugate classes using optimal subalgebras. Besides, we apply the group foliation approach based on the invariants and differential invariants of the Lie symmetries to determine larger classes of exact solutions which have important physical characteristics. The group foliation approach provides better results as compared to the symmetry reduction approach in the sense that the former gives similarity as well as nonsimilarity solutions while the later only yields similarity solutions.

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To my teachers, my family

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Chapter 1

Fundamentals of Symmetry Analysis

1.1 Introduction

In our daily life, we come across several scientific real world phenomena, like motion of a pendulum, heat flow across a metal rod, waves generated on the sea level etc. Depending on the nature of phenomena, we investigate and formulate models, in terms of ordinary differential equations (ODEs) and partial differential equations (PDEs). Mathematicians are often interested in finding analytic or numerical solutions, whereas engineers, physicists and scientists compare them with experimental results. Till now, many techniques have been developed to determine analytic solutions of nonlinear ODEs and PDEs, but it is very difficult to solve them generally. To overcome problems of finding analytic solutions, Sophus Lie [5] introduced a powerful method to circumvent the difficulties in solving nonlinear ODEs and PDEs.

Sophus Lie was inspired by Sylows lecture on the use of group theory to obtain solvability of algebraic equations. In 1869-70, Lie discovered far reaching technique intended to solve large variety of ODEs. Lie observed invariance under continuous deformations of the dependent and independent variables. Further he introduced groups of continuous transformations (Lie groups) and developed several methods of change of variables. The idea of Lie groups led him to construct roots for new branch of mathematics, which is known as *Symmetry Analysis*. In this chapter, we briefly explain basics of symmetry analysis namely manifold, vector field, flow, prolongation, Lie group, orbit, Lie algebra, adjoint representation and group foliation for differential equations.

1.2 Manifold

The basic idea of a manifold is to introduce local objects, that will support a differential process and then to patch these objects together smoothly. Manifold plays a key role in the study of topology, symmetry analysis, geometry and so on. It is an object which locally looks like open subset of Euclidean space $\mathbb{R}^n$, but its global topology is quite complicated. Several manifolds are realized as a subset of Euclidean space so in general it is an abstraction [1].

Definition 1.2.1. An n-dimensional manifold is a set M, together with a countable collection of subsets $\mathbf{U}_\alpha \subset$ M, called coordinate charts, and one-to-one functions $\chi_\alpha : \mathbf{U}_\alpha \to \mathbf{V}_\alpha$ onto connected open subsets $\mathbf{V}_\alpha \subset \mathbb{R}^n$, called local coordinate maps, which satisfy the following properties

(1) The coordinate charts cover M

$$\cup_\alpha | \mathbf{U}_\alpha = M. \tag{1.1}$$

(2) On the overlap of any pair of coordinate charts $\mathbf{U}_\alpha \cap \mathbf{U}_\beta$ the composite map

$$\chi_\beta \bullet \chi_\alpha^{-1} : \chi_\alpha(\mathbf{U}_\alpha \cap \mathbf{U}_\beta) \to \chi_\beta(\mathbf{U}_\alpha \cap \mathbf{U}_\beta), \tag{1.2}$$

is a smooth (infinitely differentiable) function.

(3) If $x \in \mathbf{U}_\alpha, \tilde{x} \in \mathbf{U}_\beta$ are distinct points of M, then there exist open subsets $W \subset \mathbf{V}_\alpha$, $\tilde{W} \subset \mathbf{V}_\beta$ with $\chi_\alpha(x) \in W, \chi_\beta(\tilde{x}) \in \tilde{W}$, satisfying $\chi_\alpha^{-1}(W) \cap \chi_\beta^{-1}(\tilde{W}) = \varnothing$.

Example 1.2.1. The basic n-dimensional manifold is Euclidean space $\mathbb{R}^n$ itself and any subset $\mathbb{R}^m$ of Euclidean space is also m-dimensional manifold which is covered by a single coordinate chart $\mathbf{U} = \mathbb{R}^n$, with a local coordinate map such as $\varphi = \mathbf{1} : \mathbb{R}^n \to \mathbb{R}^n$.

Example 1.2.2. The special orthogonal group $SO(2)$, is a collection of 2×2 invertible matrices, whose determinant is 1. It is a group under matrix multiplication. It also carries the structure of a manifold whose dimension is that of the dimension of a real line $\mathbb{R}$.

1.3 Vector Field and Flow

Tangent vector fields are fundamental in the study of geometric objects. Prior to seeing the concept of a vector field we look at curve on the manifold. A curve C on a smooth manifold M is parametrized by a smooth map $\chi : I \to M$, where I is a sub-interval of real numbers. In local coordinates $\mathbf{x} = (x^1, x^2, \ldots, x^m)$, a curve is defined by m smooth functions $\psi(\zeta) = (\psi^1(\zeta), \psi^2(\zeta), ..., \psi^m(\zeta))$ where ζ is real variable, the term smooth means that we have tangent defined at every point which varies continuously from point to point.

Example 1.3.1. If $M = \mathbb{R}^2$,

$$\sigma(\zeta) = (\zeta, e^\zeta), \tag{1.3}$$

therefore $\sigma(\zeta) : I \to M$ is a smooth curve on M, where $I = \mathbb{R}$ and

$$\dot{\sigma}(\zeta) = (1, e^\zeta), \tag{1.4}$$

the dot over σ shows derivative w.r.t ζ and $\dot{\sigma}$ is not equal to zero at any $\zeta \in \mathbb{R}$.

Tangent Vector: A tangent vector to a manifold M on a parametrized curve C is the derivative at each point $x = \psi(\zeta)$ of the curve which is denoted by

$$\mathbf{v} |_x = \frac{d\psi}{d\zeta} = \dot{\psi}^1(\zeta)\frac{\partial}{\partial x^1} + \dot{\psi}^2(\zeta)\frac{\partial}{\partial x^2} + ... + \dot{\psi}^m(\zeta)\frac{\partial}{\partial x^m}. \tag{1.5}$$

Example 1.3.2. Tangent vector to a circular helix

$$\psi(\zeta) = (\cos\zeta, \sin\zeta, \zeta), \tag{1.6}$$

in $\mathbb{R}^3$, with coordinates (x, y, z), is

$$\begin{aligned}\dot{\psi}(\zeta) &= -\sin\zeta\frac{\partial}{\partial x} + \cos\zeta\frac{\partial}{\partial y} + \frac{\partial}{\partial z},\\ &= -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y} + \frac{\partial}{\partial z},\end{aligned}$$

at the point $(x, y, z) = \psi(\zeta)$.

The collection of all feasible curves passing through a point x and all collection of tangent vectors to these curves is called a *tangent space* to M at point x, where it is denoted by $\mathrm{TM}|_x$. If m is dimension of the manifold M then the tangent vector space is also m-dimensional. Generally if these tangent spaces glued together then they form a tangent bundle. The collection of all the tangent spaces to all the points $x \in M$, is called tangent bundles which is represented as

$$TM = \cup_\alpha TM\mid_x . \tag{1.7}$$

The tangent spaces join together in a smooth manner, so that if $\psi(t)$ is a smooth curve then the tangent vectors $\dot{\psi}(t) \in TM\mid_{\psi(t)}$, will vary from point to point (means smooth). This makes tangent bundle TM, a $2m$-dimensional manifold.

Definition 1.3.1. If at each point $x \in M$ of a manifold there assigned a tangent vector $\mathbf{v}\mid_x \in TM\mid_x$, then this field is called a vector field. In local coordinates $(x^1, x^2, ..., x^n)$, a vector field is represented as

$$\mathbf{v}\mid_x = \sum \varrho^i(x)\frac{\partial}{\partial x^i}, \qquad i = 1, 2, ..., n, \tag{1.8}$$

each $\varrho^i(x)$ for $i = 1, ..., n$ is a smooth function of x.

Example 1.3.3. Consider the velocity field of a steady fluid flow in an open subset $M \subset \mathbb{R}^3$. Then the vector $\mathbf{v}\mid_{(x,y,z)}$ at each point $(x, y, z) \in M$, be the velocity of the particles passing through the point (x, y, z).

Integral Curve: An *integral curve* is defined as a parametrized curve $\psi : I \to M$ of a vector field $\mathbf{v}$ whose tangent vector at any point coexist with the value of $\mathbf{v}$ at the same point

$$\dot{\psi}(\zeta) = \mathbf{v}\mid_{\psi(\zeta)}, \qquad \forall \quad \zeta \in I \subset \mathbb{R}. \tag{1.9}$$

A flow is basically generated by a vector field. Where in local coordinates,$x = \psi(\zeta) = (\psi^1(\zeta), \psi^2(\zeta), \cdots, \psi^n(\zeta))$ must be the solution to the autonomous system of ODEs

$$\frac{dx^i}{d\zeta} = \varrho^i(x), \quad i = 1, 2, \ldots n, \tag{1.10}$$

where $\varrho^i(x)$ are the coefficients of $\mathbf{v}$ at x. Also a maximal integral curve is a curve that is not contained in any longer curve.

Definition 1.3.2. If $\mathbf{v}$ is a vector field and $\varphi(\zeta, x)$ is a parametrized maximal integral curve passing through x in M, then we call φ flow generated by v.

The flow of a vector field has the following properties.

(a) $$\varphi(\delta, (\varphi(\zeta, x)) = \varphi((\delta + \zeta), x),$$
where $\delta, \zeta \in \mathbb{R}$ in a sense that both the sides are defined.

(b) $$\varphi(0, x) = x,$$
where $\varphi(0, x)$ gives the initial conditions for the integral curves.

(c) $$\frac{d}{d\zeta}\varphi(\zeta, x) = \mathbf{v} \mid_{\varphi(\zeta,x)}, \quad \forall \quad \zeta. \tag{1.11}$$

This property simply shows that $\mathbf{v}$ is a tangent to the curve $\varphi(\zeta, x)$ for some fixed x. The vector field is called an ***infinitesimal generator*** of the action, since by Taylor series in local coordinates

$$\varphi(\zeta, x) = x + \zeta\varrho(x) + O(\varrho^2), \tag{1.12}$$

here $\varrho = (\varrho^{(1)}, \varrho^{(2)}, \ldots, \varrho^{i})$ are the coefficients of $\mathbf{v}$. We represent infinitesimal generator by $\mathbf{X}$. If $\varphi(\zeta, x)$ is one parameter group of transformation, its infinitesimal generator is obtained by specializing equation (1.11) at $\zeta = 0$

$$\mathbf{v} \mid_x = \frac{d}{d\zeta} \mid_{\zeta=0} \varphi(\zeta, x). \tag{1.13}$$

For the computation of one parameter group which is generated by a given vector field or flow, we use the exponentiation of $\mathbf{v}$. The notation is given by

$$\exp(\zeta\mathbf{v})x \equiv \varphi(\zeta, x), \tag{1.14}$$

where

$$\exp(0\mathbf{v})x = x. \tag{1.15}$$

Example 1.3.4. The group of rotation in a plane is defined as

$$\varphi\big(\zeta, (x, y)\big) = \big(x\cos\zeta - y\sin\zeta, x\sin\zeta + y\cos\zeta\big). \tag{1.16}$$

Here we have a 2-diminsional manifold and $\zeta \in \mathbb{R}$. Then by using exponential notation (1.14)

$$\begin{aligned}
\varphi\big(\delta, \varphi(\zeta, (x, y))\big) &= \big((x\cos\zeta - y\sin\zeta)(\cos\delta) - (x\sin\zeta + y\cos\zeta)(\sin\delta),\\
&\qquad (x\cos\zeta - y\sin\zeta)(\sin\delta) + (x\sin\zeta + y\cos\zeta)(\cos\delta)\big), && (1.17)\\
\varphi\big(\delta, \varphi(\zeta, (x, y))\big) &= \big(x\cos(\zeta + \delta) - y\sin(\zeta + \delta), x\sin(\zeta + \delta) + y\cos(\zeta + \delta)\big), && (1.18)\\
\varphi\big(\delta, \varphi(\zeta, (x, y))\big) &= \varphi((\delta + \zeta), (x, y)), && (1.19)
\end{aligned}$$

where $\zeta,\ \delta \in \mathbb{R}$. In above equation both sides are defined for all values of ζ and δ, where $(\zeta + \delta)$ belongs to $\mathbb{R}$

$$\varphi\big(0, (x, y)\big) = (x, y), \tag{1.20}$$

the last relation show initial condition for the integral curve

$$\frac{d}{d\zeta}\varphi\big(\zeta, (x, y)\big) = \big(- x \sin\zeta - y \cos\zeta, x \cos\zeta - y \sin\zeta\big), \tag{1.21}$$

an infinitesimal generator $\mathbf{X} = \mathbf{v}\ |_x$ for (1.16) is

$$\boldsymbol{X} = \varrho(x, y)\partial_x + \phi(x, y)\partial_y, \tag{1.22}$$

here $\partial_x = \frac{\partial}{\partial x}$, now to onward we use subscript for partial derivatives, by using exponential notation (1.14) the equation (1.11) takes the form

$$\frac{d}{d\zeta}(\exp(\zeta\boldsymbol{v})x) = \mathbf{v}\ |_{\exp(\zeta\boldsymbol{v})x}, \tag{1.23}$$

at $\zeta = 0$, by applying (1.13) we get

$$\varrho(x, y) = -y \quad and \quad \phi(x, y) = x, \tag{1.24}$$

which satisfies the conditions of flow. The Fig. 1.1 shows the rotational flow in x, y plane.

Example 1.3.5. A particular case of a translation group in $\mathbb{R}^2$ is given as

$$\varphi\left(\zeta, (x, y)\right) = (x + \zeta, y + \zeta). \tag{1.25}$$

The flow is shown in the Fig. 1.2. Note that the corresponding vector field is

$$\boldsymbol{X} = \partial_x + \partial_y, \tag{1.26}$$

indicating a translational flow at 45° to the horizontal axis.

Example 1.3.6. Consider a group of scaling in a plane

$$\varphi\left(\zeta, (x, y)\right) = (e^{\zeta}x, e^{\zeta}y). \tag{1.27}$$

Its infinitesimal generator is

$$\boldsymbol{X} = \varrho(x, y)\partial_x + \phi(x, y)\partial_y, \tag{1.28}$$

according to equation (1.13).

$$\varrho(x, y) = \frac{d}{d\zeta}\ |_{\zeta=0}\ (e^{\zeta}x) = x, \tag{1.29}$$

$$\phi(x, y) = \frac{d}{d\zeta}|_{\zeta=0}(e^{\zeta}y) = y. \tag{1.30}$$

Thus we get

$$\boldsymbol{X} = x\partial_x + y\partial_y, \tag{1.31}$$

which is an infinitesimal generator of the given group of transformation representing a uniform scaling.

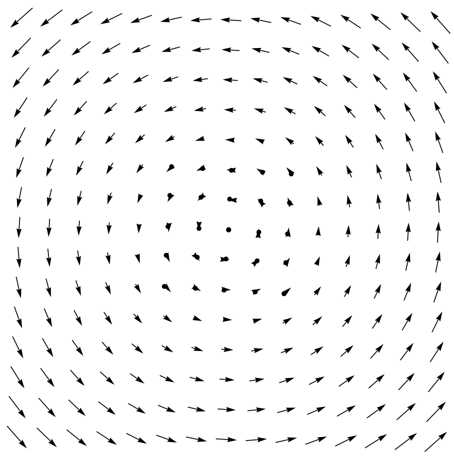

Figure 1.1: Rotational flow around a line perpendicular to plane.

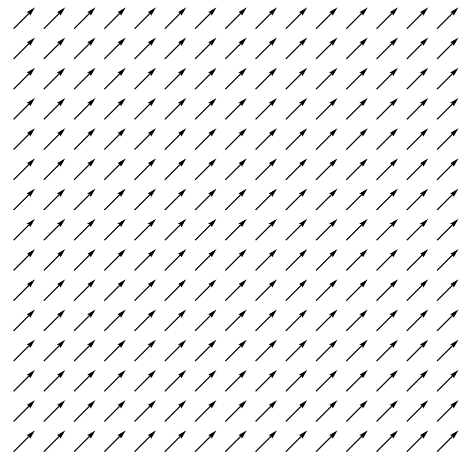

Figure 1.2: Translational flow along $y = x$ line.

Lie Brackets: There is an important operation among vector fields which is to take their commutator or Lie brackets. Basically a Lie bracket is an operator that will assign, to any two vectors or infinitesimals generator $\mathbf{X}_1$ and $\mathbf{X}_2$ on a manifold M, a third vector or infinitesimals generator. It is denoted as

$$[\mathbf{X}_1, \mathbf{X}_2] = \mathbf{X}_1\mathbf{X}_2 - \mathbf{X}_2\mathbf{X}_1, \tag{1.32}$$

where $[\mathbf{X}_2, \mathbf{X}_1] = -[\mathbf{X}_1, \mathbf{X}_2]$.

Example 1.3.7. Consider two infinitesimal generators in $\mathbb{R}^3$

$$\boldsymbol{X}_1 = \partial_t, \qquad \boldsymbol{X}_2 = t\partial_x + \partial_u, \tag{1.33}$$

where the first vector field shows a translational flow in t direction and $\mathbf{X}_2$ is a combination of rotation and translation known as Galelian boost. Then the Lie bracket is

$$[\boldsymbol{X}_1, \boldsymbol{X}_2] = \boldsymbol{X}_1\boldsymbol{X}_2 - \boldsymbol{X}_2\boldsymbol{X}_1, \tag{1.34}$$
$$= \partial_t(t\partial_x + \partial_u) - (t\partial_x + \partial_u)(\partial_t), \tag{1.35}$$
$$= \partial_x. \tag{1.36}$$

The above vector field shows the translational flow in x direction.

1.4 Prolongation

For finding the symmetries of differential equations, we have to prolong our basic space $X \times V$, which is a space of independent and dependent variables [1]. Our basic infinitesimal generator contains only derivatives of independent and dependent variables (not the derivatives of dependent variables w.r.t independent variables), as we are interested in higher order differential equations so we have to prolong our infinitesimal generator. If we have an mth order differential equations, for the purpose of finding symmetries, we must prolong our infinitesimal generator upto order m, so that it contains derivatives of dependent variables upto order m. To understand the prolongation, we start with the prolongation of a real valued function $g(x)$. If $g(x)$ is a smooth function of $\mathbf{x} = (x^1, x^2, ..., x^p)$, where p is the number of independent variables, then the kth order partial derivatives of g are

$$p_k = \frac{(p+k-1)!}{(p-1)!k!}, \tag{1.37}$$

where p_k shows how many number of kth order derivative are there, for instance if we have $p = 2$ means two independent variables and we need 2nd order derivative. So we get three 2nd order derivatives. We use following multi-index notation for these derivatives

$$\partial_L g(x) = \frac{\partial^k g(x)}{\partial x^{l_1} \partial x^{l_2} \cdots \partial x^{l_k}}, \qquad L = (l_1, l_2 \cdots, l_k), \tag{1.38}$$

where L is unordered k tuple of integers and the entries $1 \leq l_k \leq p$ indicate which derivatives are being taken. The order of multi index is denoted by $*L \equiv k$, shows how many derivatives are being taken. Further, if $g : X \to V$ is a function from $X \backsimeq \mathbb{R}^p$ to $V \backsimeq \mathbb{R}^q$, so $u = g(x) = (g(x)^1, g(x)^2, \cdots, g(x)^q)$ there are $q{\cdot}_k$ numbers $u^\beta_L = \partial_L g^\beta(x)$ needed to represent all the different kth order derivatives of elements of g at x. Let we have $V_k \equiv \mathbb{R}^{q \cdot p_k}$ be the Euclidean space of this dimension, for instance, suppose $p, k = 2$ and $q = 1$, so $q \cdot p_k = 3$ where p_k is obtained from (1.37), thus V_2 is equivalent to $\mathbb{R}^3$ dimensional Euclidean space, provided with the coordinates u^β_L corresponding to $\beta = 1, 2, \cdots, q$. Furthermore, set $V^{(m)} = V \times V_1 \cdots \times V_m$ to be Cartesian product space. Where $V \times V_1 \cdots \times V_m$ represents all the derivatives of the functions $u = g(x)$ of all orders from 0 to m.

Example 1.4.1. If $p = 2, q = 2$, then $X \backsimeq \mathbb{R}^2$ has coordinates $(x^1, x^2) = (x, y)$ and $V \backsimeq \mathbb{R}^2$ has the coordinates $(u^1, u^2) = (u, v)$. The space V_1 is isomorphic to $\mathbb{R}^4$ with coordinates (u_x, u_y, v_x, v_y) since these represents all the first order derivatives of u, v with respect to x and y. Similarly, $V_2 \backsimeq \mathbb{R}^6$ has coordinates $(u_{xx}, u_{xy}, u_{yy}, v_{xx}, v_{xy}, v_{yy})$ represents the second derivatives of u, v. Finally, the space $V^{(2)} = V \times V_1 \times V_2 \backsimeq \mathbb{R}^{12}$.

If we have a smooth function $u = g(x)$, such that $g : X \to V$, then there is an induced function $u^{(n)} = pr^{(n)} g(x)$, called the nth prolongation of g, which is given by equation

$$u^\beta_L = \partial_L g^\beta(x).$$

A nth order jet space, is a total space $X \times V^n$ whose coordinates shows independent and dependent variables which also includes the derivatives of dependent variables upto mth order, where mth order jet space is

$$M^m = M \times V^1 \times V^2 \times ...V^m,$$

of M, where $M \subset X \times V$ because we are often interested in some open subset M of the space, instead of the differential equations defined over all of $X \times V$. As we can prolong a function, we can also prolong its infinitesimal generator. The general formula for the prolongation of infinitesimal generator can be obtained by using the following Theorem [1].

Theorem 1.4.1. *Let*

$$\boldsymbol{X} = \sum_{i=1}^{p} \varrho^i(\boldsymbol{r}, \boldsymbol{s}) \frac{\partial}{\partial r^i} + \sum_{\beta=1}^{q} \eta^\beta(\boldsymbol{r}, \boldsymbol{s}) \frac{\partial}{\partial s^\beta},$$

be an infinitesimal defined on an open subset $M \subset X \times V$, the m-th prolongation of $\boldsymbol{X}$ in

$$\mathbb{P}r^{(m)}\boldsymbol{X} = \boldsymbol{X} + \sum_{\beta=1}^{q} \sum_{L} \eta^\beta_{[L]}(\boldsymbol{r}, \boldsymbol{s}^{(m)}) \frac{\partial}{\partial s^\beta_L}, \tag{1.39}$$

defined on the corresponding jet space $M^{(m)} \subset X \times V^{(m)}$, the second summation being over all (unordered) multi-indices $L = (l_1, l_2, \cdots, l_k)$, with $1 \leqslant l_k \leqslant p$, $1 \leqslant k \leqslant m$. The coefficients functions $\phi^\beta_{[L]}$ of $\mathbb{P}r^{(m)}$ are given by the following formula

$$\eta^\beta_{[L]}(\boldsymbol{r}, \boldsymbol{s}^{(m)}) = D_L \left(\eta^\beta - \sum_{i=1}^{p} \varrho^i s^\beta_i \right) + \sum_{i=1}^{p} \varrho^i s^\beta_{L,i}, \tag{1.40}$$

here $s^\beta_i = \partial s^\beta / \partial r^i$, and $s^\beta_{J,i} = \partial s^\beta_i / \partial r^i$

Here D_L is total derivative operator, it is given by the formula

$$D_i = \frac{\partial}{\partial r^i} + \sum_{\beta=1}^{q} \sum_{L} s^\beta_{L,i} \frac{\partial}{\partial s^\beta_L}, \tag{1.41}$$

where $L = (l_1, l_2, ..., l_k)$ and

$$s^\beta_{L,i} = \frac{\partial s^\beta_L}{\partial r^i} = \frac{\partial^k s^\beta}{\partial r^i \partial x^{l_1} .... \partial x^{l_k}}. \tag{1.42}$$

In (1.41) the sum is over all L's of order $0 \leq *L \leq m$, m is the highest derivative. The explicit form of total derivative for higher order, in multi index notation for $L = (l_1, l_2, ..., l_k)$ is kth order, with $0 \leq L_k \leq p$ for each k, then Lth total derivative is

$$D_L = D_{l_1} D_{l_2} \cdots D_{l_k}.$$

For $m = 1$ the equation (1.40) is valid, now for finding coefficients of higher order prolongation, we use the following relation

$$\eta^\beta_{[Lk]} = D_k(\eta^\beta_{[L]}) - \sum_{i=1}^{p} \varrho^i u^\beta_{L,i}, \tag{1.43}$$

$$= D_k D_L(\eta^\beta - \sum_{i=1}^{p} \varrho^i u^\beta_i) + \sum_{i=1}^{p} \varrho^i u^\beta_{L,ik}, \tag{1.44}$$

here $u^\beta_{L,ik} = \dfrac{\partial^2 u^\beta_L}{\partial x^i \partial x^k}$.

Example 1.4.2. If we have two independent variables and one dependent variable, i.e., $p = 2$ and $q = 1$, with coordinates (t, x, u), there are two total derivatives D_t, D_x, with

$$\begin{aligned} D_t =& \partial_t + u_t\partial_u + u_{tt}\partial_{u_t} + u_{tx}\partial_{u_x} + u_{ttt}\partial_{u_{tt}} + \cdots , \\ D_x =& \partial_x + u_x\partial_u + u_{tx}\partial_{u_t} + u_{xx}\partial_{u_x} + u_{xxx}\partial_{u_{xx}} + \cdots . \end{aligned}$$

Example 1.4.3. Suppose we have $p = 2$, $q = 2$, in the prolongation formula, if we consider a PDE involving functions $u = f(x, t)$ and $v = g(x, t)$, an infinitesimal generator on $X \times V \backsimeq \mathbb{R}^2 \times \mathbb{R}^2$ takes the form

$$\boldsymbol{X} = \varrho(t, x, u, v)\partial_t + \tau(t, x, u, v)\partial_x + \eta^1(t, x, u, v)\partial_u + \eta^2(t, x, u, v)\partial_v. \tag{1.45}$$

By using equation (1.39) we can find first prolongation of $\mathbf{X}$, which is

$$\mathbb{P}r^{(1)}\boldsymbol{X} = \boldsymbol{X} + \eta^1_{[t]}(t, x, u, v)\partial_{u_t} + \eta^1_{[x]}(t, x, u, v)\partial_{u_x} + \eta^2_{[t]}(t, x, u, v)\partial_{v_t} + \eta^2_{[x]}(t, x, u, v)\partial_{v_x}, \tag{1.46}$$

for finding the expressions for $\eta^{\beta}_{[L]}$ we use equation (1.40)

$$\begin{aligned} \eta^1_{[t]} =& D_t(\eta^1 - \varrho u_t - \tau u_x) + \varrho u_{tt} + \tau u_{tx}, \\ &= (\partial_t + u_t\partial_u + v_t\partial_v + u_{tt}\partial_{u_t} + u_{tx}\partial_{u_x} + v_{tt}\partial_{v_t} + v_{tx}\partial_{u_x})(\eta^1 - \varrho u_t - \tau u_x) + \varrho u_{tt} + \tau u_{tx}, \\ &= \eta^1_t + u_t(\eta^1_u - \varrho_t) + v_t\eta^1_v - u_t^2\varrho_u - u_t v_t\varrho_v - u_x\tau_t + u_x v_t\tau_v - u_x u_t\tau_u. \qquad (1.47) \\ \eta^1_{[x]} =& D_x(\eta^1 - \varrho u_t - \tau u_x) + \varrho u_{tx} + \tau u_{xx}, \\ &= (\partial_x + u_x\partial_u + v_x\partial_v + u_{tx}\partial_{u_t} + u_{xx}\partial_{u_x} + v_{tx}\partial_{v_t} + v_{xx}\partial_{v_x})(\eta^1 - \varrho u_t - \tau u_x) + \varrho u_{xt} + \tau u_{xx}, \\ &= \eta^1_x + u_x(\eta^1_u - \varrho_x) + v_x\eta^1_v - u_x^2\varrho_u - u_x v_x\varrho_v - u_x\tau_x + u_x v_x\tau_v - u_x u_t\tau_u, \qquad (1.48) \end{aligned}$$

where $\eta^1_x = \frac{\partial\eta^1}{\partial x}$. Similarly we can get $\eta^2_{[t]}$, $\eta^2_{[x]}$ in the same manner. Now we can prolong our $\mathbf{X}$ to the second order, which is given by

$$\mathbb{P}r^{(2)}\boldsymbol{X} = \boldsymbol{X} + \mathbb{P}r^{(1)}\boldsymbol{X} + \eta^1_{[tt]}\partial_{u_{tt}} + \eta^1_{[tx]}\partial_{u_{tx}} + \eta^1_{[xx]}\partial_{u_{xx}} + \eta^2_{[tt]}\partial_{v_{tt}} + \eta^2_{[tx]}\partial_{v_{tx}} + \eta^2_{[xx]}\partial_{v_{xx}}. \tag{1.49}$$

We can use equation (1.40) to find $\eta^1_{[tt]}, \eta^1_{[tx]}, \eta^1_{[xx]}, \eta^2_{[tt]}, \eta^2_{[tx]}, \eta^2_{[xx]}$. The explicit expression for $\eta^1_{[tt]}$ is given as

$$\begin{aligned} \eta^1_{[tt]} =& D_t(\eta^1_{[t]} - \varrho u_{tt} - \tau u_{tx}) + \varrho u_{ttt} + \tau u_{ttx}, \\ &= \eta^1_{tt} + u_t(2\eta^1_{tu} - \varrho_{tt}) + 2v_t\eta^1_{tv} + u_t^2(\eta^1_{uu} - 2\varrho_{tu}) - u_t v_t(2\eta^1_{uv} - 2\varrho_{t,v}) - 2u_t u_x\tau_{tu} \\ &- 2u_x v_t\tau_{tv} - u_t^3\varrho_{uu} - 2u_t^2 v_t\varrho_{uv} - u_t^2 v_t\varrho_{uv} - u_t^2 u_x\tau_{uu} - 2u_t u_x v_t\tau_{uv} - u_x\tau_{tt} + v_t^2\eta^1_{vv} \\ &- u_t v_t^2\varrho_{vv} - u_x v_t^2\tau_{vv} + u_{tt}(\eta_u - 2\varrho_t) - 3u_{tt}u_t\varrho_u - 2u_{tt}v_t\varrho_v - u_{tt}u_x\tau_u - 2u_{tx}\tau_t - 2u_{tx}u_t\tau_u \\ &- 2u_{tx}v_t\tau_v + v_{tt}(\eta_v - u_t\varrho_v - u_x\tau_v). \qquad (1.50) \end{aligned}$$

In above expression $\eta^1_{tt} = \dfrac{\partial^2\eta^1}{\partial t\partial t}$. We get other expressions for $\eta^1_{[tx]}, \eta^1_{[xx]}, \eta^2_{[tt]}, \eta^2_{[tx]}, \eta^2_{[xx]}$ in the same way.

1.5 Lie Group

A group G is an algebraic structure which consists of a set of elements equipped with some operations that will combine two elements to form a third element. These operations satisfy some conditions that are called group axioms, which are

a. closure property.

b. associative property.

c. identity element exists.

d. inverse element exists.

If a group also satisfies an additional property

e. commutative property, i.e., A+B = B+A, or AB=BA,

then the group is said to be an abelian group. An n parameter Lie group is a group G, which carries the structure of an n-dimensional smooth manifold in such a way that both the group operation

$$m : G \times G \to G, \quad m(g, h) = g * h, \quad g, h \in G, \tag{1.51}$$

and the inversion

$$\iota : G \to G, \quad \iota(g) = g^{-1}, \quad g \in G, \tag{1.52}$$

are the smooth maps between manifolds.

Example 1. A simple example of Lie group is $G = \mathbb{R}$, with obvious manifold structure and the group operation is addition $m(a, b) = a + b$ and $inverse(a) = -a$ for all $a, b \in \mathbb{R}$. Then G is an abelian Lie group.

1.6 Orbit

An orbit of a group is a minimal nonempty group invariant subset of manifold M. Simply if O ⊂ M is an orbit provided it satisfies the following conditions.

(a) If $x \in O$, $g \in G$ and $g \cdot x$ is defined, then $g \cdot x \in O$.

(b) If $\tilde{O} \subset O$ and $\tilde{O}$ satisfies (a) then either $\tilde{O} = O$ or $\tilde{O}$ is empty.

We now explain the definition of group orbits by considering a few examples

Example 1.6.1. Consider group of translations in $\mathbb{R}$.
Let $G = \mathbb{R}$, defined as

$$\varphi(\zeta, x) = x + \zeta, \quad x \in \mathbb{R}, \quad \zeta \in \mathbb{R}. \tag{1.53}$$

This is a global group action. The orbits of above transformation are straight lines.

Example 1.6.2. Consider a group of rotation in $\mathbb{R}^2$, defined as

$$\varphi(\zeta, (x, y)) = (x \cos\zeta - y \sin\zeta, y \cos\zeta + x \sin\zeta). \tag{1.54}$$

The orbits are circles which are invariant subset of our manifold shown in the Figure (1.1)

1.7 Lie Algebra

A Lie algebra is basically a tangent space associated to a Lie group. Prior to define Lie algebra we introduce some basic concepts of Lie group. Let G be a Lie group then for any element $a \in G$, the left multiplication map is

$$L_a : G \longrightarrow G, \tag{1.55}$$

defined as

$$L_a(b) = a \cdot b, \quad b \in G, \tag{1.56}$$

is a diffeomorphism, with inverse

$$L_{a^{-1}} = (L_a)^{-1}. \tag{1.57}$$

A vector field $\mathbf{v}$ on G is called left invariant if it is unaffected by the left multiplication of any group element, i.e.,

$$dL_a(\mathbf{v}_{|b}) = \mathbf{v} \mid_{L_a(b)} = \mathbf{v} \mid_{a \cdot b}, \quad \forall a, b \in G. \tag{1.58}$$

The right invariant is also given in the same manner $dR_a(\mathbf{v}_{|b}) = \mathbf{v} \mid_{b \cdot a}$.

Definition 1.7.1. The Lie algebra of a Lie group is denoted by $\mathfrak{g}$ and it is defined as the vector space of all the left invariant vector fields on G.

Any left (right) invariant vector field at identity is uniquely determined.

Definition 1.7.2. A Lie algebra $\mathfrak{g}$ is a vector space equipped with the bracket operation

$$[.,.] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$$

satisfying the following axioms

(a) which is bilinear

$$[c\mathbf{v} + c'\mathbf{v}', \mathbf{w}] = c[\mathbf{v}, \mathbf{w}] + c'[\mathbf{v}', \mathbf{w}], \tag{1.59}$$

$$[\mathbf{v}, c\mathbf{w} + c'\mathbf{w}'] = c[\mathbf{v}, \mathbf{w}] + c'[\mathbf{v}, \mathbf{w}'], \qquad \text{for any constant } c, c' \in \mathbb{R}.$$

(b) antisymmetric

$$[\mathbf{v}, \mathbf{w}] = -[\mathbf{w}, \mathbf{v}]. \tag{1.60}$$

(c) Jacobi identity

$$[\mathbf{u}, [\mathbf{v}, \mathbf{w}] + [\mathbf{v}, [\mathbf{w}, \mathbf{u}] + [\mathbf{w}, [\mathbf{u}, \mathbf{v}] = 0, \tag{1.61}$$

for all $\mathbf{u}, \mathbf{v}, \mathbf{w}, \mathbf{v}', \mathbf{w}'$ in $\mathfrak{g}$.

Example 1.7.1. Consider the two parameter group $A(1)$ of the affine transformation $x \to ax + b$ on the line $x \in R$. The group multiplication law is given by $(a, b).(c, d) = (ac, ad + b)$ and the identity element is $\mathbf{e} = (1, 0)$. The right translation is $R_{(a,b)}(c, d) = (c, d).(a, b) = (ac, bc + d)$. A basis for the right Lie algebra $\mathfrak{a}(1)_R$ corresponding to the coordinate basis $\frac{\partial}{\partial a}|_e, \frac{\partial}{\partial b}|_e$ of $TA(1)|_e$ is therefore

$$\mathbf{v}_1 = dR_{(a,b)}[\frac{\partial}{\partial a}|_e] = a\frac{\partial}{\partial a} + b\frac{\partial}{\partial b},$$
$$\mathbf{v}_2 = dR_{(a,b)}[\frac{\partial}{\partial b}|_e] = \frac{\partial}{\partial b},$$

where their commutation relation is $[\mathbf{v}_1, \mathbf{v}_2] = -\mathbf{v}_2$.

The appropriate way to show the structure of a given Lie algebra is to display it in its tabular form. If we have m dimensional Lie algebra $\mathfrak{g}$, and $\mathbf{X}_1, \mathbf{X}_2, \ldots, \mathbf{X}_m$ form the basis for $\mathfrak{g}$, then there commutator table will be the $m \times m$ table, where the (i, j)-th entry of the table shows the Lie brackets $[\mathbf{X}_i, \mathbf{X}_j]$.

Example 1.7.2. Suppose we have a four dimensional Lie algebra

$$\boldsymbol{X}_1 = \partial_x, \qquad \boldsymbol{X}_2 = \partial_t, \tag{1.62}$$
$$\boldsymbol{X}_3 = t\partial_x + \partial_u, \qquad \boldsymbol{X}_4 = x\partial_x + 3t\partial_t - 2u\partial_u. \tag{1.63}$$

The procedure to find the Lie brackets is given in Section (1.3), where the non zero Lie brackets are

$$[\boldsymbol{X}_1, \boldsymbol{X}_4] = \boldsymbol{X}_1, \quad [\boldsymbol{X}_2, \boldsymbol{X}_3] = \boldsymbol{X}_1,$$
$$[\boldsymbol{X}_2, \boldsymbol{X}_4] = 3\boldsymbol{X}_2, \quad [\boldsymbol{X}_3, \boldsymbol{X}_4] = -2\boldsymbol{X}_3.$$

The remaining commutator relations are

$$[\boldsymbol{X}_1, \boldsymbol{X}_1] = [\boldsymbol{X}_1, \boldsymbol{X}_2] = [\boldsymbol{X}_1, \boldsymbol{X}_3] = 0,$$
$$[\boldsymbol{X}_2, \boldsymbol{X}_1] = [\boldsymbol{X}_2, \boldsymbol{X}_2] = [\boldsymbol{X}_3, \boldsymbol{X}_3] = 0,$$

the commutator table is given in Table (1.1)

$\mathbf{X}_i$	$\mathbf{X}_1$	$\mathbf{X}_2$	$\mathbf{X}_3$	$\mathbf{X}_4$
$\mathbf{X}_1$	0	0	0	$\mathbf{X}_1$
$\mathbf{X}_2$	0	0	$\mathbf{X}_1$	$3\mathbf{X}_2$
$\mathbf{X}_3$	0	$-\mathbf{X}_1$	0	$-2\mathbf{X}_3$
$\mathbf{X}_4$	$-\mathbf{X}_1$	$-3\mathbf{X}_2$	$2\mathbf{X}_3$	0

Table 1.1: Commutator Table

1.7.1 Solvable Lie Algebras

Definition 1.7.3. A $\mathfrak{g}^k$ is a k dimensional solvable algebra if there exist a chain of subalgebras

$$\mathfrak{g}^{(1)} \subset \mathfrak{g}^{(2)} \subset ......\mathfrak{g}^{(k-1)} \subset \mathfrak{g}^{(k)} = \mathfrak{g}^{(k)}, \tag{1.64}$$

such that $\mathfrak{g}^{(a)}$ is a dimensional Lie algebra and $\mathfrak{g}^{(a-1)}$ is a normal subalgebra of $\mathfrak{g}^{(a)}$, $a = 1, 2, ..., k$.

Definition 1.7.4. A subalgebra $\mathfrak{l} \subset \mathfrak{g}$ is called normal subalgebra or ideal of $\mathfrak{g}$ if $[\mathbf{X}, \mathbf{Y}] \in \mathfrak{l}$ for all $\mathbf{X} \in \mathfrak{l}$, $\mathbf{Y} \in \mathfrak{g}$.

If an algebra $\mathfrak{g}$ satisfies $[\mathbf{X}_a, \mathbf{X}_b] = 0$, for all $\mathbf{X}_a, \mathbf{X}_b \in \mathfrak{g}$. Then $\mathfrak{g}$ is called Abelian algebra.

Example 1.7.3. Suppose we have a four dimensional Lie algebra

$$\begin{aligned} &\boldsymbol{X}_1 = -y\partial_x + x\partial_y, \quad \boldsymbol{X}_2 = \partial_x, \\ &\boldsymbol{X}_3 = \partial_y, \quad \boldsymbol{X}_4 = x\partial_x + y\partial_y. \end{aligned}$$

The commutator relations are given as

$$[\boldsymbol{X}_1, \boldsymbol{X}_2] = 0, \quad [\boldsymbol{X}_1, \boldsymbol{X}_4] = 0, \quad [\boldsymbol{X}_2, \boldsymbol{X}_3] = 0, \tag{1.65}$$

$$[\boldsymbol{X}_1, \boldsymbol{X}_2] = -\boldsymbol{X}_3, \quad [\boldsymbol{X}_1, \boldsymbol{X}_3] = \boldsymbol{X}_2, \quad [\boldsymbol{X}_2, \boldsymbol{X}_4] = \boldsymbol{X}_2, \quad [\boldsymbol{X}_3, \boldsymbol{X}_4] = \boldsymbol{X}_3, \tag{1.66}$$

the Table (1.3) shows all the Lie brackets of the Lie algebra.

	$\mathbf{X}_1$	$\mathbf{X}_2$	$\mathbf{X}_3$	$\mathbf{X}_4$
$\mathbf{X}_1$	0	$-\mathbf{X}_3$	$\mathbf{X}_2$	0
$\mathbf{X}_2$	$\mathbf{X}_3$	0	0	$\mathbf{X}_2$
$\mathbf{X}_3$	$-\mathbf{X}_2$	0	0	$\mathbf{X}_3$
$\mathbf{X}_4$	0	$-\mathbf{X}_2$	$-\mathbf{X}_3$	0

Table 1.2: Commutator Table

The four dimensional subalgebra is $\mathfrak{g}^{(4)}$ it self, next we have to choose a three dimensional subalgebra which is normal, if we choose

$$\mathfrak{g}^{(3)} = \{\boldsymbol{X}_1, \boldsymbol{X}_2, \boldsymbol{X}_3\} \tag{1.67}$$

then we have to check weather it is normal subalgebra or not, from the Definition (1.7.4)

$$\{\boldsymbol{X}_1, \boldsymbol{X}_2, \boldsymbol{X}_3\} \in \mathfrak{l} \subset \mathfrak{g}^{(4)}, \tag{1.68}$$

$$[\boldsymbol{X}_1, \boldsymbol{X}_2] = -\boldsymbol{X}_3 \in \mathfrak{l}, \quad [\boldsymbol{X}_1, \boldsymbol{X}_3] = \boldsymbol{X}_2 \in \mathfrak{l}, \quad \text{where} \quad \{\boldsymbol{X}_1, \boldsymbol{X}_2, \boldsymbol{X}_3, \boldsymbol{X}_4\} \in \mathfrak{g}^{(4)}, \tag{1.69}$$

hence $\mathfrak{g}^{(3)}$ is normal subalgebra of $\mathfrak{g}^{(4)}$. Similarly

$$[\boldsymbol{X}_2, \boldsymbol{X}_1] = \boldsymbol{X}_3 \in \mathfrak{l}, \quad [\boldsymbol{X}_2, \boldsymbol{X}_4] = \boldsymbol{X}_2 \in \mathfrak{l}, \tag{1.70}$$

so $\mathfrak{g}^{(2)} = \{\mathbf{X}_2, \mathbf{X}_3\}$ is a two dimensional normal subalgebra, also $\mathfrak{g}^{(1)} = \{\mathbf{X}_1\}$ is one dimensional normal subalgebra

$$\mathfrak{g}^{(4)} = \{\boldsymbol{X}_1, \boldsymbol{X}_2, \boldsymbol{X}_3, \boldsymbol{X}_4\}, \quad \mathfrak{g}^{(3)} = \{\boldsymbol{X}_1, \boldsymbol{X}_2, \boldsymbol{X}_3\}, \quad \mathfrak{g}^{(2)} = \{\boldsymbol{X}_2, \boldsymbol{X}_3\}, \quad \mathfrak{g}^{(1)} = \{\boldsymbol{X}_2\},$$

therefore, there exist a chain of subalgebras

$$\mathfrak{g}^{(1)} \subset \mathfrak{g}^{(2)} \subset \mathfrak{g}^{(3)} \subset \mathfrak{g}^{(4)} = \mathfrak{g}^{(4)},$$

as all the subalgebras are the normal, so this is a solvable algebra.

Example 1.7.4. Suppose we have a three dimensional Lie algebra

$$\boldsymbol{X}_1 = (1 + x^2)\partial_x + xy\partial_y, \quad \boldsymbol{X}_2 = xy\partial_x + (1 + y^2)\partial_y, \quad \boldsymbol{X}_3 = y\partial_x - x\partial_y.$$

In this algebra we have

$$\mathfrak{g}^{(3)} = [\boldsymbol{X}_1, \boldsymbol{X}_2, \boldsymbol{X}_3],$$

only three dimension normal subalgebra but no normal subalgebra of two-dimension. Therefore no chain of algebra exist, so it is not a solvable algebra.

In the next chapters, we shall discuss the group invariant solutions of the system of differential equations (DEs). Therefore we need systematic and effective means for the classification of these solutions. So we need adjoint representation which leads to the optimal system of group invariant solutions from which every other such solution can be derived.

1.8 Adjoint Representation

If we have a full symmetry group G of a system of DEs, then for each r parameter subgroup H of the group G there will be a family of group invariant solutions. There are almost an infinite number of such subgroups and normally it is not feasible to list all possible group invariant solutions. Therefore to classify the group invariant solutions we need to classify the subgroups of the symmetry group G under conjugation, thus we can study the conjugacy map $a \to gag^{-1}$ on the Lie group.

Let G be a Lie group. For every $g \in G$, group conjugation is $K_g(a) \equiv gag^{-1}$, where $a \in G$ determines the diffeomorphism on G, besides this $K_g \cdot K_{g'} = K_g K_{g'}$ and $K_e = \mathbf{1}$, so K_g determines a global action on G itself, with each conjugacy map $h \longmapsto ghg^{-1}$ being a group homomorphism. The differential dK_g without difficulty is seen to preserve the right invariance and hence detemines a linear map on Lie algebra of Lie group G, called the adjoint representation.

On Lie algebra of a Lie group its adjoint representation is often reconstructed from its infinitesimal generators. If $\mathbf{X}$ generates one parameter subgroup $\exp(\epsilon\mathbf{X})$, then we suppose $Ad\ \mathbf{X}$ is the vector field on $\mathfrak{g}$ generating a one parameter group of adjoint representation

$$Ad\mathbf{X}|_Y \equiv \quad \frac{d}{d\epsilon}|_{\epsilon=0} \quad Ad((\exp(\epsilon\mathbf{X})), \quad \mathbf{X} \in \mathfrak{g}. \tag{1.71}$$

The basic property is that the infinitesimal adjoint action agrees with the Lie bracket on $\mathfrak{g}$. So

$$Ad(\exp(\epsilon\mathbf{X}))Y = \sum_{m=0}^{\infty} \frac{\epsilon^m}{m!}(ad\mathbf{X})^m Y, \tag{1.72}$$

$$= Y - \epsilon[\mathbf{X}, Y] + \frac{\epsilon^2}{2}[\mathbf{X}, [\mathbf{X}, Y]] - \cdots . \tag{1.73}$$

we now illustrate it using an example.

Example 1.8.1. The Lie algebra spanned by

$$\boldsymbol{X}_1 = \partial_x, \qquad \boldsymbol{X}_2 = \partial_t, \tag{1.74}$$

$$\boldsymbol{X}_3 = t\partial_x + \partial_u, \qquad \boldsymbol{X}_4 = x\partial_x + 3t\partial_t - 2u\partial_u, \tag{1.75}$$

generates the symmetry group of the Korteweg-de Vares equation. The adjoint representation is computed by using the condition (1.72). For instance we take $Ad(\exp(\epsilon\mathbf{X}_2))\mathbf{X}_4$, therefore we use (1.72)

$$Ad(\exp(\epsilon\boldsymbol{X}_2))\boldsymbol{X}_4 = \boldsymbol{X}_4 - \epsilon[\boldsymbol{X}_2, \boldsymbol{X}_4] + \frac{\epsilon^2}{2}[\boldsymbol{X}_2, [\boldsymbol{X}_2, \boldsymbol{X}_4]] - \cdots , \tag{1.76}$$

where the commutator relation of $\mathbf{X}_2$ and $\mathbf{X}_4$ in the above equation is given below

$$[\boldsymbol{X}_2, \boldsymbol{X}_4] = 3\boldsymbol{X}_2, \quad [\boldsymbol{X}_2, [\boldsymbol{X}_2, \boldsymbol{X}_4]] = 0, \tag{1.77}$$

so by substituting the above relations in equation (1.76) the adjoint representation is

$$Ad(\exp(\epsilon\boldsymbol{X}_2))\boldsymbol{X}_4 = \boldsymbol{X}_4 - 3\epsilon\boldsymbol{X}_2. \tag{1.78}$$

Similarly we can find $Ad(\exp(\epsilon\mathbf{X}_4))\mathbf{X}_2$ by employing the condition (1.72), which is given as

$$Ad(\exp(\epsilon\boldsymbol{X}_4))\boldsymbol{X}_2 = \boldsymbol{X}_2 - \epsilon[\boldsymbol{X}_4, \boldsymbol{X}_2] + \frac{\epsilon^2}{2}[\boldsymbol{X}_4, [\boldsymbol{X}_4, \boldsymbol{X}_2]] - \cdots , \tag{1.79}$$

the commutator relations in above equation is given below

$$[\boldsymbol{X}_4, \boldsymbol{X}_2] = -3\boldsymbol{X}_2, \qquad [\boldsymbol{X}_4, [\boldsymbol{X}_4, \boldsymbol{X}_2]] = 9\boldsymbol{X}_2, \tag{1.80}$$

when we put (1.80) in the equation (1.79) the equation (1.79) gets the following form

$$Ad(\exp(\epsilon\boldsymbol{X}_4))\boldsymbol{X}_2 = \boldsymbol{X}_2 + 3\epsilon\boldsymbol{X}_2 + \frac{(\epsilon 3)^2}{2}\boldsymbol{X}_2 + \cdots , \tag{1.81}$$

$$= \boldsymbol{X}_2(1 + 3\epsilon + (3\epsilon)^2/2! + (3\epsilon)^3/3! + \cdots , \tag{1.82}$$

$$= e^{3\epsilon}\boldsymbol{X}_2 \tag{1.83}$$

In this manner, we constructed the Table (2) of adjoint representation

$\text{Ad}\mathbf{X}_i$	$\mathbf{X}_1$	$\mathbf{X}_2$	$\mathbf{X}_3$	$\mathbf{X}_4$
$\mathbf{X}_1$	$\mathbf{X}_1$	$\mathbf{X}_2$	$\mathbf{X}_3$	$\mathbf{X}_4 - \epsilon\mathbf{X}_1$
$\mathbf{X}_2$	$\mathbf{X}_1$	$\mathbf{X}_2$	$\mathbf{X}_3 - \epsilon\mathbf{X}_1$	$\mathbf{X}_4 - 3\epsilon\mathbf{X}_2$
$\mathbf{X}_3$	$\mathbf{X}_1$	$\mathbf{X}_2 + \epsilon\mathbf{X}_1$	$\mathbf{X}_3$	$\mathbf{X}_4 + 2\epsilon\mathbf{X}_3$
$\mathbf{X}_4$	$e^{\epsilon}\mathbf{X}_1$	$e^{3\epsilon}\mathbf{X}_2$	$e^{-2\epsilon}\mathbf{X}_3$	$\mathbf{X}_4$

Table 1.3: Adjoint representation

Definition 1.8.1. Let G be a Lie group. An optimal system of r parameter subgroups is a list of conjugacy in-equivalent r parameter subgroups with the property that any subgroup is conjugate to precisely one subgroup in the list. Similarly, a list of r parameter subalgebars forms an optimal system if ever r parameter subalgebras of $\mathfrak{g}$ is equivalent to a unique member of the list under some element of the adjoint representation.

So the classification of one dimensional subalgebra is essentially the same as classfiying the orbits of the adjoint representation. Although some techniques are available to find optimal system but we follow as given in [1]. We take a general element $\mathbf{X}$ in $\mathfrak{g}$, subjecting it to various adjoint representation transformations and simplify it as much as possible. Where each one dimensional subalgebra is determined by a non zero vector in $\mathfrak{g}$.

1.9 Group Foliation for PDE

The group foliation approach is a powerful technique to solve nonlinear PDEs in an algorithmic manner. The concept of group foliation first brought up by Sophus Lie [5] developed by Vessiot [11] and L.V. Ovsiannikov [6] amelioration in its modern form. If we have second order nonlinear partial differential equation which admits Lie algebra. One can construct group foliation by using any admitted Lie point symmetry. We use finite dimensional algebra for group foliation. The main idea of foliation is that we can transform the original PDE into a system of first order PDEs, then by using an ansatz this system is reduced to an over determined algebraic equations and by solving them, we can get the exact solutions of original PDE. The group foliation approach contains following few algorithmic steps:

1. Find all Lie piont symmetries of PDE.
2. Find functionally independent invariants and differential invariants.
3. Construct group resolving equations and solve.
4. Solve parametric ODEs and get exact solutions.

Briefly these steps are performed as follows
Step 1: Suppose we have a second order nonlinear PDE with two independent variables and one dependent variable

$$\tilde{\Delta}(t, x, u, u_t, u_x, u_{tt}, u_{tx}, u_{xx}) = 0. \tag{1.84}$$

Then for finding the symmetries of $\tilde{\Delta}$ we need infinitesimal generator $\mathbf{X}$. The explicit form of infinitesimal generator is given by

$$\mathbf{X} = \varrho^1(t, x, u)\partial_t + \varrho^2(t, x, u)\partial_x + \eta(t, x, u)\partial_u, \tag{1.85}$$

Then by applying Theorem (2.1.1) we will obtain the symmetries of the PDE $\tilde{\Delta}$

$$\mathbb{P}r^m \mathbf{X}\tilde{\Delta}(t, x, u, u_t, u_x, u_{tt}, u_{tx}, u_{xx}) = 0, \tag{1.86}$$

Suppose the dimension of this symmetry group G is finite, i.e., dim $G < \infty$. So the infinitesimal generators $\mathbf{X}_i$, $i = 1, 2, \ldots, n$, are finite. Moreover by using any of the admitted Lie point symmetry of the PDE $\tilde{\Delta}$ we can construct group foliation approach.

Step 2: Now we dig out the invariants which are obtained by using the following condition

$$\mathbf{X}\varphi = 0, \tag{1.87}$$

where φ is function of t, r, u. The condition (1.87) gives us first order PDE and by solving the PDE we will obtain the invariant. Our next task is to find differential invariants, therefore we first prolong the infinitesimal generator, for one dependent variable the first order prolongation is given below

$$\mathbb{P}r^{(1)}\mathbf{X} = \mathbf{X} + \eta_{[t]}\partial_t + \eta_{[x]}\partial_x, \tag{1.88}$$

where $\eta_{[t]}, \eta_{[x]}$ are calculated by using the equation (1.42). We shall obtain differential invariants by using the following condition

$$\mathbb{P}r^{(1)}\mathbf{X}\psi = 0, \tag{1.89}$$

here ψ is function of t, r, u, u_t and u_r. The invariants and differential invariants are basically new independent and dependent variable respectively.

Step 3: Now by using these new variables, we can transform our original PDE $\tilde{\Delta}$ into system of first order PDEs, which is known as group resolving equations. This system of first order PDE is relatively easy to solve.

Step 4: The solution of group resolving equations lead us to a set of parametic ODEs. By solving these parametric ODEs we will obtain the exact solution of the original PDE $\tilde{\Delta}$. In chapter 4 we will see group foliation in detail and find exact solutions of some nonlinear PDEs.

Chapter 2

Symmetry Reduction

2.1 Infinitesimal Invariance

The power of Lie group lies in the crucial observation that one can replace the complicated, nonlinear conditions for invariance of a subset or function under the group transformations themselves by an equivalent linear condition of infinitesimal invariance under the corresponding infinitesimal generators of group action. Let $\tilde{\Delta}$ represents a system of m-th order differential equations in p, q, independent variables and dependent variables, respectively

$$\tilde{\Delta}_l(\mathbf{r}, \mathbf{s}^{(m)}) = 0, \qquad l = 1, 2, ..., n, \tag{2.1}$$

involving $\mathbf{r} = (r^1, r^2, ..., r^p)$, $\mathbf{s} = (s^1, s^2, ..., s^q)$ and derivatives of $\mathbf{s}$ with respect to $\mathbf{r}$ up to order m. The functions $\tilde{\Delta}(\mathbf{r}, \mathbf{s}^{(m)}) = (\tilde{\Delta_1}(\mathbf{r}, \mathbf{s}^{(m)}), \tilde{\Delta_2}(\mathbf{r}, \mathbf{s}^{(m)}), ..., \tilde{\Delta_l}(\mathbf{r}, \mathbf{s}^{(m)}))$ are assumed to be smooth, so $\tilde{\Delta}(\mathbf{r}, \mathbf{s}^{(m)})$ seen to be a smooth map from space $X \times V^m$ to l dimension Euclidean space

$$\tilde{\Delta} : X \times V^m \to \mathbb{R}^l.$$

The differential equation itself tells where the map $\tilde{\Delta}$ vanishes on $X \times V^m$, and thus determines a subvariety

$$\Im = ((\mathbf{r}, \mathbf{s}^{(m)}) : \tilde{\Delta}(\mathbf{r}, \mathbf{s}^{(m)}) = 0) \subset X \times V^m, \tag{2.2}$$

of the jet space [1].

Definition 2.1.1. Suppose we have a system of DE $\tilde{\Delta}_l$ of order m, then a local group of transformation G which is acting on some open subset $M \subset X \times V$ of the space of the independent and dependent variables for the system of DE $\tilde{\Delta}_l$, carries the property that whenever $u = g(x)$ is a solution of the system of DE $(\tilde{\Delta}_l)$, and whenever $g(x) \cdot f$ is defined for $f \in G$, then $u = f \cdot g(x)$ is also a solution of the system of DE then this group is known as symmetry group.

In other words symmetry group can transform solution of a system of DE into other solution. For instance we take a simple example, suppose we have a ordinary differential equation

$$y'' = 0, \tag{2.3}$$

then the group translation

$$(x, y) \longrightarrow (x + \zeta, y), \quad \zeta \in \mathbb{R}, \tag{2.4}$$

is a symmetry group. Since $y = f(x+\zeta, y)$ is a solution to the ordinary differential equation (2.3) whenever $y = f(x, y)$. For finding the infinitesimal condition for a symmetry group G to be a symmetry group of a given system of DEs, we need a maximal rank condition for the system of DEs. To apply the latter theorem for finding the infinitesimal condition for a symmetry group G, we need a maximal rank condition for the system of DEs.

Definition 2.1.2. Suppose

$$\tilde{\Delta}_l(\mathbf{r}, \mathbf{s}^{(m)}) = 0, \qquad l = 1, 2, \ldots, n, \tag{2.5}$$

be a system of DEs. The system is said to be of maximal rank if the $l \times (p + qp^m)$ where $qp^{(m)} = q + qp_1 + qp_2 + \cdots + qp_k$, Jacobian matrix

$$J_{\tilde{\Delta}_l}(\mathbf{r}, \mathbf{s}^{(m)}) = \left(\frac{\partial \tilde{\Delta}_l}{\partial r^p}, \frac{\partial \tilde{\Delta}_l}{\partial s^q_L} \right), \tag{2.6}$$

of $\tilde{\Delta}$ w.r.t all the variables $(\mathbf{r}, \mathbf{s}^{(m)})$ is of rank l whenever $\tilde{\Delta}(\mathbf{r}, \mathbf{s}^{(m)}) = 0$.

Example 2.1.1. Suppose a first order DE

$$\tilde{\Delta}(x, y, y') = (y - x)y' + y + x = 0, \tag{2.7}$$

the Jacobian matrix is

$$J_{\tilde{\Delta}} = \left(\frac{\partial \tilde{\Delta}}{\partial x}, \frac{\partial \tilde{\Delta}}{\partial y}, \frac{\partial \tilde{\Delta}}{\partial y'} \right), \tag{2.8}$$

$$=(1 + y', 1 + y', y - x), \tag{2.9}$$

which is of rank 1 every where.

Theorem 2.1.1. *Suppose*

$$\tilde{\Delta}_l(\boldsymbol{r}, \boldsymbol{s}^{(m)}) = 0 \quad l = 1, 2, ..., n, \tag{2.10}$$

is a system of differential equations of maximal rank defined over $M \subset X \times V$. If G is a local group of transformations acting on M, and

$$\mathbb{P}r^m \boldsymbol{X}[\tilde{\Delta}_l(\boldsymbol{r}, \boldsymbol{s}^{(m)})] = 0, \quad l = 1, 2, ..., n, \qquad \text{whenever} \quad \tilde{\Delta}_l(\boldsymbol{r}, \boldsymbol{s}^{(m)}) = 0, \tag{2.11}$$

for every infinitesimal generator $\boldsymbol{X}$ of G, then G is a symmetry group of system.

Suppose we have an ODE or PDE, first we can identify a linear subvariety in $X \times V^m$ determined by vanishing of $\tilde{\Delta}_l(\mathbf{r}, \mathbf{s}^{(m)})$. Then we have

$$\mathbf{X} = \sum_{i=1}^{p} \varrho^i(\mathbf{r}, \mathbf{s}) \frac{\partial}{\partial r^i} + \sum_{\beta=1}^{q} \eta^\beta(\mathbf{r}, \mathbf{s}) \frac{\partial}{\partial s^\beta}, \tag{2.12}$$

be an infinitesimal generator on $M \subset X \times V$. Now calculate all possible coefficient functions of ϱ^i and ϕ^β, so that the corresponding one parameter group is a symmetry group of the ODE or PDE. By using the Theorem (1.4.1), we should prolong our infinitesimal generator to the order of equation

$$\mathbb{P}r^{(m)}\mathbf{X} = \mathbf{X} + \sum_{\beta=1}^{q} \sum_{L} \eta^\beta_{[L]}(\mathbf{r}, \mathbf{s}^{(m)}) \frac{\partial}{\partial s^\beta_L}, \tag{2.13}$$

by employing the prolongation in Theorem (2.1.1) and comparing the monomials, we get an over determined system of linear PDEs. Then we can obtain the values of ϱ^i and η^β. Let us elaborate it briefly with an example.

Example 2.1.2. Consider we have system of ODEs

$$u_{xx} = -(u^2 + v^2)u, \tag{2.14}$$
$$v_{xx} = -(u^2 + v^2)v, \tag{2.15}$$

the above system contains only one independent variable x and two dependent variables u and v. If p, q, represents number of independent and dependent variables, respectively, here $p = 1$ and $q = 2$. The system of ODEs (2.14)-(2.15) are of second order, i.e., $m = 2$, thus it can be identified with subvariety in $X \times V^2$, determined by vanishing of $\tilde{\Delta}(t, u^{(2)}, v^{(2)})_1 = u_{xx} + (u^2 + v^2)u = 0$, $\tilde{\Delta}(t, u^{(2)}, v^{(2)})_2 = v_{xx} + (u^2 + v^2)v = 0$. The Jacobian matrix

$$J_{\tilde{\Delta}_1} = \left(\frac{\partial \tilde{\Delta}_1}{\partial x}, \frac{\partial \tilde{\Delta}_1}{\partial u}, \frac{\partial \tilde{\Delta}_1}{\partial v}; \frac{\partial \tilde{\Delta}_1}{\partial u_x}, \frac{\partial \tilde{\Delta}_1}{\partial v_x}; \frac{\partial \tilde{\Delta}_1}{\partial u_{xx}}, \frac{\partial \tilde{\Delta}_1}{\partial v_{xx}} \right), \tag{2.16}$$

$$J_{\tilde{\Delta}_1} = (0, 3u^2 + v^2, 2uv; 0, 0, 1, 1). \tag{2.17}$$

Similarly the Jacobian matrix for $J_{\tilde{\Delta}_2}$ is

$$J_{\tilde{\Delta}_2} = (0, 2uv, 3v^2 + u^2; 0, 0, 1, 1). \tag{2.18}$$

from (2.17) and (2.18), it is clear that the system (2.14),(2.15) has rank 2. So it satisfies maximal rank condition. The infinitesimal generator for the system of ODEs (2.14),(2.15) is

$$\boldsymbol{X} = \varrho(x, u, v)\partial_x + \eta^1(x, u, v)\partial_u + \eta^2(x, u, v)\partial_v, \tag{2.19}$$

on $M \subset X \times V$. Further we want to find all possible coefficient functions ϱ, η^1 and η^2 such that it correspond to symmetry group of the given system of equations. Then for applying infinitesimal criteria, we have to find second prolongation of infinitesimal generator $\mathbf{X}$

$$\mathbb{P}r^{(2)}\boldsymbol{X} = \boldsymbol{X} + \eta^1_{[x]}\partial_{u_x} + \eta^2_{[x]}\partial_{v_x} + \eta^1_{[xx]}\partial_{u_{xx}} + \eta^2_{[xx]}\partial_{v_{xx}}, \tag{2.20}$$

the coefficients $\eta^1_{[x]}$ and $\eta^2_{[x]}$ are calculated by using the equation (1.40). The explicit form $\eta^1_{[x]}$ is given below

$$\begin{aligned}
\eta^1_{[x]} &= D_x(\eta^1 - \varrho u_x) + \varrho u_{xx}, && (2.21)\\
&= D_x\eta^1 - u_x D_x\varrho, && (2.22)\\
&= \eta^1_x + (\eta^1_u - \varrho_x)u_x + \eta^1_v v_x - \varrho_v u_x v_x - \varrho_u u_x^2, && (2.23)
\end{aligned}$$

similarly expression for $\eta^2_{[x]}$ is

$$\eta^2_{[x]} = \eta^2_x + (\eta^2_v - \varrho_x)v_x + \eta^2_u u_x - \varrho_u u_x v_x - \varrho_v v_x^2, \tag{2.24}$$

we can use equation (1.43) to calculate $\eta^1_{[xx]}$ and $\eta^2_{[xx]}$, where the explicit form is given as

$$\begin{aligned}
\eta^1_{[xx]} &= D_x^2(\eta^1 - \varrho u_x) + \varrho u_{xxx}, && (2.25)\\
&= \eta^1_{xx} + (2\eta^1_{xu} - \varrho_{xx})u_x + (\eta^1_{uu} - 2\varrho_{xu})u_x^2 - \varrho_{uu}u_x^3 - 2\varrho_{uv}u_x^2 v_x\\
&\quad - \varrho_{vv}u_x v_x^2 + (2\eta^1_{uv} - 2\varrho_{xv})u_x v_x + \eta^1_{vv}v_x^2 - 3\varrho_u u_x u_{xx} - \varrho_v u_x v_{xx}\\
&\quad - 2\varrho_v u_{xx} v_x + 2\eta^1_{xv}v_x + \eta^1_u u_{xx} + \eta^1_v v_{xx} - 2\varrho_x u_{xx}. && (2.26)
\end{aligned}$$

In the same manner we can obtain the expression for $\eta^2_{[xx]}$. Now we will compute symmetry group of the system of equations (2.14)-(2.15). by using Theorem (2.1.1). Further, we can first apply the infinitesimal generator (2.20) on the system ODEs (2.14)-(2.15). If we apply infinitesimal generator (2.20) on the equation (2.14), the term $\phi^1_{[xx]}\partial_{u_{xx}}$ when applies on u_{xx} gives us $\phi^1_{[xx]}$ and the terms $\eta^1\partial_u, \eta^2\partial_v$ gives $3u^2\eta^1, v^2\eta^1$ and $2uv\eta^2$, respectively. Similarly we obtain the following equations

$$\begin{aligned}
\eta^1_{[xx]} &= -3u^2\eta^1 - v^2\eta^1 - 2uv\eta^2,\\
\eta^2_{[xx]} &= -2uv\eta^1 - u^2\eta^2 - 3v^2\eta^2, && (2.27)
\end{aligned}$$

which is satisfied whenever $u_{xx} = -(u^2 + v^2)u$, $v_{xx} = -(u^2 + v^2)v$. When we substitute the formula (2.23), (2.24), (2.26) into (2.27), this will lead us into a number of linear PDEs for the coefficient functions ϱ, η^1, η^2 of infinitesimal generator called the determining equations. Following are the determining equations for the system of ODEs (2.14)-(2.15), which we obtain by using MAPLE

$$\begin{aligned}
\eta^1_u &= -\varrho_x, && (2.28)\\
\eta^1_v &= (\varrho_x u + \eta^1)/v, && (2.29)\\
\eta^1_x &= 0, \quad \varrho_u = 0, \quad \varrho_v = 0, && (2.30)\\
\varrho_{xx} &= 0, && (2.31)\\
\eta^2_v &= (\varrho_x u^2 - \varrho_x v^2 - \eta^1 u)/v. && (2.32)
\end{aligned}$$

If we look at equations (2.30.2) and (2.30.3), they require that ϱ be just a function of x, where equation (2.31) shows that ϱ is linear in x. Next equation (2.30.1) shows that η^1 does not depend on x and by equation (2.28)

$$\eta^1_u(u, v) = -c_1, \qquad \eta^1(u, v) = -c_1 u + f(v),$$

for certain function f. We get the value of function f by substituting ϕ^1 in equation (2.29). Now by substituting values of ϱ and η^1 in the remaining equation (2.32), we get the coefficient functions of our required system of ordinary differential equations

$$\varrho = c_1 x + c_2, \tag{2.33}$$
$$\eta^1 = -c_1 u + c_3 v, \tag{2.34}$$
$$\eta^2 = -c_1 v - c_3 u, \tag{2.35}$$

where c_1, c_2, c_3 are arbitrary constants. Thus the infinitesimal symmetries of the system are

$$\boldsymbol{X}_1 = \partial_x, \quad \boldsymbol{X}_2 = x\partial_x - u\partial_u - v\partial_v, \quad \boldsymbol{X}_3 = v\partial_u - u\partial_v. \tag{2.36}$$

2.2 Symmetry Reduction

Often we are interested in the solutions of nonlinear ODE and PDEs, therefore, we use many analytical techniques to solve nonlinear ODEs and PDEs, like inverse scattering transform method, Lax pair formalism, bi Hamiltonian approach, conservation laws, non-local method, Fourier transformation and Ad-hoc methods ($\tan h, \ldots$ etc.). It is convenient to deal with lower order differential equations as compared to higher order. It is important to reduce the order of differential equations. If an ODE is invariant under one parameter group of transformations, then the order of the ODE can be reduced by one. For reduction of order of an m-th ODE, one need a *solvable algebra*. If we have ζ parameter Lie group of transformations, where $\zeta = 2$ and $m \geq 2$, we can reduce order of ODE by two. In general, it is not necessarily true that the order can be reduced by more than one. However, an ODE is reduced by order k, if we have ζ-dimensional Lie algebra of infinitesimals of an admitted ζ parameter group that has a k dimensional solvable subalgebra. The knowledge of a group of symmetries of a higher order ordinary differential equation has much the same consequences as the knowledge of a similar symmetry group of a system of first order ODEs. Where for the system of first order ODE, if we have one parameter group of symmetry, we can reduce it to one fewer equation. The invariance of an m-th order under one parameter group of symmetry, allows to reduce the order of one of the equations by one in the system.

In PDEs the group reduction follows through the reduction of independent variables. If a PDE admits one-parameter group of symmetry, then we can reduce its one independent variable. Also the system of PDEs is reduced into one fewer independent variable. For multi-parameter group of symmetry, we need a solvable Lie algebra. We can reduce the order of PDE by more then one (here order means the number of independent variables), but there is no general algorithm at present. As the reduced equations arise from invariants and differential invariants of the infinitesimal, so the solutions of the reduced equations are the solution of given PDE. For the reduction of PDE, we must have $p - s$ number of functionally independent invariants, where s is the dimension of the orbits, whereas, we can always replace a higher order system by an equivalent first order system. In group reduction we can first find out the point transformation by using the Theorem (2.1.1). Then by using these transformations we can reduce order of given ODE. There are two ways to find the invariant transformations, by using canonical coordinate and by differential invariants. We explain both procedures below

2.2.1 Reduction through canonical coordinates

Suppose we have a nonlinear ODE

$$y^{(m)} = F(x, y, y', ..., y^{m-1}). \tag{2.37}$$

Let an infinitesimal generator is

$$\mathbf{X} = \varrho(x, y)\partial_x + \eta(x, y)\partial_y, \tag{2.38}$$

where ϱ and ϕ are functions of x and y, then we can calculate canonical coordinates $u(x, y)$ and $v(x, y)$ satisfying the condition

$$\mathbf{X}u = 0, \quad \mathbf{X}v = 1.$$

Following theorem explains how the reduction is carried out in a consistent way.

Theorem 2.2.1. *Suppose a non trivial one-parameter Lie group of transformation, with infinitesimal generator (2.38), is admitted by an mth-order ODE (2.37), $m \geq 2$. Let $u(x, y)$ and $v(x, y)$ be corresponding canonical coordinates satisfying $\boldsymbol{X}u = 0, \boldsymbol{X}v = 1$. Then the nth-order ODE (2.37) reduces to an $(m - 1)$th order ODE*

$$\frac{d^{m-1}z}{dr^{m-1}} = F(r, z, \frac{dz}{dr}, \dots, \frac{d^{m-2}z}{dr^{m-2}}),$$

where

$$\frac{du}{dt} = z.$$

For second order ODE, we have

$$y' = \frac{u_x - t_x \dfrac{du}{dt}}{t_y \dfrac{du}{dt} - u_y}, \tag{2.39}$$

$$y'' = \frac{d^2u}{dt^2}G + H, \tag{2.40}$$

where G and H are function of $u, t, \dfrac{du}{dt}$ and $G = \dfrac{1}{j}$ and $H = -\dfrac{k}{j}$, where the values of j and k is given below

$$j = \frac{u_y t_x - u_x t_y}{(t_x + t_y y')^3}, \tag{2.41}$$

$$k = \frac{1}{(t_x + t_y y')^3}[(y')^3(t_y u_{yy} - u_y t_{yy}) + (y')^2(2t_y u_{xy} + t_x u_{yy} - 2u_y t_{xy} - u_x t_{yy}) + \tag{2.42}$$

$$(y')(2t_x u_{xy} + t_y u_{xx} - 2u_x t_{xy} - u_y t_{xx}) + (t_x s_{xx} - u_x t_{xx})]. \tag{2.43}$$

Since t and u are canonical coordinates, so it follows that $u_y t_x - u_x t_y \neq 0$, therefore $j \neq 0$. We can substitute above relations into any given second order ODE, this will reduce order of the equation.

Example 2.2.1.

$$y'' = xyy', \tag{2.44}$$

The infinitesimal generator of equation (2.44) is

$$\boldsymbol{X} = -\frac{1}{2}x\partial_x + y\partial_y. \tag{2.45}$$

The canonical coordinates corresponding to (2.45) are given by

$$t(x,y) = -2ln(x), \quad u(x,y) = yx^2, \tag{2.46}$$

by using (2.39) and (2.40), we obtain

$$y' = \frac{-(2(x^2y + z))}{x^3} - \frac{2}{x}\frac{du}{dt}, \tag{2.47}$$

$$y'' = \left(\frac{4}{x^4}\right)\frac{d^2u}{dt^2} - \frac{1}{2x}\left[10y' + \frac{8y}{x}\right]. \tag{2.48}$$

Consequently

$$\frac{du}{dt} = v, \quad \frac{d^2u}{dt^2} = v'. \tag{2.49}$$

By putting the above relations in equation (2.44), we obtain the following reduced equation

$$v' + v(4 + v) = 0. \tag{2.50}$$

2.2.2 Reduction through invariants and differential invariants

An mth order ordinary differential equation (2.37) is represented by the surface

$$F(x, y, y', y'', ..., y^m) = y_m - f(x, y, y', y'', ..., y^{m-1}) = 0, \tag{2.51}$$

admits one parameter group of point transformation iff the surface is invariant

$$\mathbb{P}r^{(m)}\mathbf{X}F = 0, \quad when \quad F = 0, \tag{2.52}$$

where $\mathbb{P}r^{(m)}\mathbf{X}$ denoted the mth prolongation of our infinitesimal generator. It follows that $F(x, y, y', y'', ..., y^m)$ is some function of group's invariant

$$\phi(x, y), \psi_1(x, y, y'), ..., \psi_k(x, y, y', ..., y^m). \tag{2.53}$$

Now by applying the infinitesimal generator $\mathbf{X}$ upon ϕ we get

$$\mathbf{X}\phi(x, y) = 0, \tag{2.54}$$

which leads us to first order PDE by solving this we get invariants.

$$\mathbb{P}r^{(k)}\mathbf{X}\psi_l(x, y, y', ..., y^l) = 0, \quad k = 1, 2, 3, ..., n, \quad l = 1, 2, 3, ..., n, \tag{2.55}$$

above condition will give differential invariants with $\dfrac{\partial v_l}{\partial y^l} \neq 0$. Here $\psi^l(x, y, y', ..., y^l)$ is integration constant of characteristic equations

$$\frac{dx}{\varrho(x, y)} = \frac{dy}{\phi(x, y)} = ... = \frac{dy^l}{\phi^k(x, y, y', ..., y^l)}, \tag{2.56}$$

where ϕ^l is given in Section (1.6). By using any set of invariants (2.53), the equation (2.51) becomes

$$H(\phi, \psi_1, \psi_2, ..., \psi_n) = 0, \tag{2.57}$$

from a feasible choice of invariants and differential invariants we can reduce order of ODE. Whereas differential invariants are calculated by any choice of invariants $\phi(x, y)$ and $\psi(x, y, y')$

$$\frac{d\psi}{d\phi} = \frac{\frac{\partial\psi}{\partial x} + y'\frac{\partial\psi}{\partial y} + y''\frac{\partial\psi}{\partial y'}}{\frac{\partial\phi}{\partial x} + y'\frac{\partial\phi}{\partial y}}, \tag{2.58}$$

then constructively the reduced equation (2.57), in terms of differential invariants is an $(n-1)$th order ODE

$$\frac{d^{n-1}\psi}{d\phi^{n-1}} = I\left(\phi, \psi, \frac{d\psi}{d\phi}, ..., \frac{d^{n-2}\psi}{d\phi^{n-2}}\right), \tag{2.59}$$

for some function I of $\phi, \psi, d\psi/d\phi, ..., d^{n-2}\psi/d\phi^{n-2}$.

Example 2.2.2. Suppose that

$$y'' = xyy', \tag{2.60}$$

which admints a Lie point symmetry

$$\boldsymbol{X} = -\frac{1}{2}x\partial_x + y\partial_y.$$

The invariants and differential invariants are

$$\phi(x, y) = x^2y,$$
$$\psi(x, y) = x^3y',$$

by using relation (2.59), we obtain

$$\frac{d\psi}{d\phi} = \frac{3x^2y' + x^3y''}{2xy + x^2y'}. \tag{2.61}$$

By substituting above relation in equation (2.44), we get the following reduced equation

$$3\psi' - \psi - 3 = 0, \tag{2.62}$$

here prime shows the derivative w.r.t ϕ.

2.2.3 Korteweg de Vries equation

In the start of this chapter, we see infinitesimal criteria, find symmetries of ODEs and apply reduction procedure. Now we will apply the same criteria on *Korteweg de Vries* equation, which is a nonlinear partial differential equation. We will find out the symmetries of this equation, after that we see how the reduction take place in PDE.

Example 2.2.3. Consider the *Korteweg de Vries* equation

$$u_t + u_{xxx} + uu_x = 0, \tag{2.63}$$

which is a 3rd order PDE. The infinitesimal generators of *Korteweg de Vries* equation is following

$$\boldsymbol{X} = \varrho(t, x, u)\partial_t + \tau(t, x, u)\partial_x + \eta(t, x, u)\partial_u. \tag{2.64}$$

Its first order prolongation is

$$\mathbb{P}r^{(1)}\boldsymbol{X} = \boldsymbol{X} + \eta_{[t]}(t, x, u)\partial_{u_t} + \eta_{[x]}(t, x, u)\partial_{u_x}. \tag{2.65}$$

Since the *Korteweg-de Vries* is 3rd order PDE, so we take 3rd order prolongation

$$\mathbb{P}r^{(3)}\boldsymbol{X} = \boldsymbol{X} + \mathbb{P}r^{(1)}\boldsymbol{X} + \mathbb{P}r^{(2)}\boldsymbol{X} + \eta_{[ttt]}\partial_{u_{ttt}} + \eta_{[ttx]}\partial_{u_{ttx}} + \eta_{[txx]}\partial_{u_{txx}} + \eta_{[xxx]}\partial_{u_{xxx}}. \tag{2.66}$$

When we apply the infinitesimal generator (2.66) on equation (2.63), then by Theorem (2.1.1) we get

$$\eta_{[t]} = -\eta_{[xxx]} - \eta u_x - \eta_{[x]}u, \tag{2.67}$$

from equations (1.40) and (1.43), we can get the relations $\eta_{[t]}, \eta_{[x]}$ and $\eta_{[xxx]}$. By using MAPLE we get determining equations which are

$$\varrho_u = 0, \tag{2.68}$$

$$\varrho_x = 0, \tag{2.69}$$

$$\varrho_{tt} = 0, \tag{2.70}$$

$$\tau_u = 0, \tag{2.71}$$

$$\tau_{tt} = 0, \tag{2.72}$$

$$\tau_x = \frac{1}{3}\varrho_t, \tag{2.73}$$

$$\eta = \frac{-2}{3}\varrho_t u + \tau_t. \tag{2.74}$$

Equation (2.69) show that ϱ is neither function of u nor function of x, also it is linear in t which is seen in equation (2.70). The equation (2.71) specifies that τ is not function of u, according to equation (2.73)

$$\tau(t,x) = \frac{1}{3}c_1 x + f(t), \tag{2.75}$$

by substituting the value of $\tau(t,x)$ in the equation (2.72) we get following relation

$$f(t)_{,tt} = 0, \quad f(t) = c_3 t + c_4, \tag{2.76}$$

this relation gives $\tau(t,x)$. Now when we put above relations in equation (2.74), we get $\phi(t,x,u)$. We get the following coefficient functions for *Korteweg de Vries* equation

$$\varrho(t,x,u) = c_1 t + c_2, \tag{2.77}$$

$$\tau(t,x,u) = \frac{1}{3}c_1 x + c_3 t + c_4, \tag{2.78}$$

$$\eta(t,x,u) = -\frac{2}{3}c_1 u + c_3, \tag{2.79}$$

for some arbitrary constants c_1, c_2, c_3, c_4. By comparing constants we get the following symmetries

$$\boldsymbol{X}_1 = \partial_x, \quad \boldsymbol{X}_2 = t\partial_x + \partial_u, \quad \boldsymbol{X}_3 = \partial_t, \quad \boldsymbol{X}_4 = x\partial_x + 3t\partial_t - 2u\partial_u. \tag{2.80}$$

In order to reduce *Korteweg de Vries* equation, we use any of the admitted symmetry. For reduction we need appropriate variables, which are functionally independent invariant and differential invariant. Now we use one of the symmetry and reduce our PDE.
Lets start with the infinitesimal generator $\mathbf{X}_1$, which is space translation. By using the condition (2.54), we obtain the following invariant

$$r = t, \quad \psi(r) = u(x,t),$$

by using above relation, *Korteweg de Vries* equation completely converts in to simple 1st order ODE which is

$$\psi' = 0, \quad \text{where} \quad \psi' = \frac{d\psi}{dr}, \tag{2.81}$$

from above equation we see that the *Korteweg de Vries* is reduced into one fewer variable and obtained a simple ODE, by solving it we get $u(t,x) = c$, which is a trivial solution of *Korteweg de Vries* equation. The infinitesimal generator $\mathbf{X}_2$, which is a Galilean boost, holds the following invariant

$$r = t, \quad \psi(r) + \frac{x}{t} = u(x,t),$$

these invariants reduce *Korteweg de Vries* equation into a 1st order ODE

$$\psi'(r) + \frac{\psi(r)}{r} = 0, \qquad \psi'(r) = \frac{d\psi}{dr}, \tag{2.82}$$

we can solve the reduced equation easily and get $\psi = c_1/r$. The resulting solution of *Korteweg de Vries* equation is

$$u(t,x) = \frac{c_1 + x}{t}.$$

In the same way we can reduce *Korteweg de Vries* equation, by using other symmetries and obtain similarity solutions.

Chapter 3

Manakov System: Exact Solution by Group Reduction

3.1 Introduction

Rogue wave solutions are the solutions of nonlinear Schrödinger equation (NLSE) and its generalization in higher dimensions which are like waves that are generated abruptly and from nowhere that suggested their other names as monster waves, killer or freak waves. This area of research has flourished over the past few years with a search for new solutions of coupled nonlinear Schrödinger equations (CNSEs) as such behaviors arise in several branches of applied sciences for example study of plasmas, fiber optics, condensates, to name a few.

A Manakov system is a system of coupled nonlinear Schrödinger equations, it is investigated by S. V. Manakov [30]. E. V. Zakharov and E. I. Schulman [12] proved the integrability of CNSEs. Moreover O. C. Wright [21, 22] gives the elliptic solutions as well as modulation instability in a defocussing coupled nonlinear Schrödinger system. The periodic solutions and their modulations of the Manakov system is given by A. M. Kamchatnov [8]. D F Parker [18] obtained the dark and bright solitions. A. Sciarrino and P. Winternitz [7] found symmetries and solutions of the vector nonlinear Schrödinger equations (VNSEs). The characterization of breather and Rogue wave and modulation instability of CNSEs is given by N. Vishnu Priya, M. Senthivelan [18]. The Manakov system

$$\begin{aligned} iu_t + u_{xx} + 2(\sigma_1|u|^2 + \sigma_2|v|^2)u = 0, \\ iv_t + v_{xx} + 2(\sigma_1|u|^2 + \sigma_2|v|^2)v = 0, \end{aligned} \tag{3.1}$$

which is comprised of a system of two semi-linear PDEs with cubic nonlinearty. Here $u(t,x)$ and $v(t,x)$ carry the representation of two waves which are complex functions whose physical meaning depends on a particular context and $\sigma_i = \pm 1$, $(i = 1, 2)$. The Manakov system is completely integrable in the sense that it has infinitely many conservation laws besides it admits a Lax pair and a bi-Hamiltonian structure. The above system is well studied and its solutions are obtained from various approaches. The solutions of Manakov system include

two important types that include Rogue waves and breather solutions both of which arise in different experimental and physical settings.
We use symmetry reduction approach to obtain invariant solutions of the system (3.1). The symmetry analysis of the Manakov system is carried out in several papers where the algebra of Lie point symmetries is given. Our goal is to exploit full set of symmetries to obtain invaiant solutions of (3.1) by integrating the reduced ODEs corresponding to Lie point symmetries.

3.2 Symmetry Analysis

First we investigate the Lie point symmetries of the Manakov system (3.1).

$$\begin{aligned} &iu_t + u_{xx} + 2(\sigma_1|u|^2 + \sigma_2|v|^2)u = 0, \\ &iv_t + v_{xx} + 2(\sigma_1|u|^2 + \sigma_2|v|^2)v = 0, \end{aligned} \tag{3.2}$$

where $u(t,x)$ and $v(t,x)$ define two interacting waves represented by two complex functions. It is more convenient to work in polar coordinates, therefore we introduce

$$u = \rho(t,x)e^{i\phi(t,x)}, \tag{3.3}$$

$$v = \tau(t,x)e^{i\psi(t,x)}, \tag{3.4}$$

where $\rho = |u|$ and $\tau = |v|$, determine the strength or amplitude of the two waves, respectively. Similarly ϕ and ψ correspond to two phases of the waves. When we substitute (3.3), (3.4) in the system of equations (3.2), we get the following equations

$$e^{i\phi}(\rho_{xx} - \rho\phi_t - \rho\phi_x^2 + 2(\sigma_1\rho^2 + \sigma_2\tau^2)\rho) + i(\rho_t + 2\rho_x\phi_x + \rho\phi_{xx})e^{i\phi} = 0, \tag{3.5}$$

$$e^{i\psi}(\tau_{xx} - \tau\psi_t - \tau\psi_x^2 + 2(\sigma_1\rho^2 + \sigma_2\tau^2)\tau) + i(\tau_t + 2\tau_x\psi_x + \tau\psi_{xx})e^{i\psi} = 0. \tag{3.6}$$

The transformation gives rise to a system of four second-order PDEs by comparing real and imaginary parts

$$\begin{aligned} &\rho_{xx} - \rho\phi_t - \rho\phi_x^2 + 2(\sigma_1\rho^2 + \sigma_2\tau^2)\rho = 0, \\ &\tau_{xx} - \tau\psi_t - \tau\psi_x^2 + 2(\sigma_1\rho^2 + \sigma_2\tau^2)\tau = 0, \\ &\rho_t + 2\rho_x\phi_x + \rho\phi_{xx} = 0, \\ &\tau_t + 2\tau_x\psi_x + \tau\psi_{xx} = 0. \end{aligned} \tag{3.7}$$

The space $M \subset X \times V$ of all independent and dependent variables is a zero-order jet space, here t and x are the independent variables and ρ, ϕ, τ and ψ are the dependent variables. The infinitesimal generator for $p = 2$ and $q = 4$ where p, q represents number of independent and dependent variables respectively, is given as

$$\mathbf{X} = \varrho^1\partial_t + \varrho^2\partial_x + \eta^1\partial_\rho + \eta^2\partial_\tau + \eta^3\partial_\phi + \eta^4\partial_\psi, \tag{3.8}$$

where $\varrho^1, \varrho^2, \eta^1, \eta^2, \eta^3, \eta^4$ are functions of $t, x, \rho, \tau, \phi, \psi$. The first order prolongation of the infinitesimal generator $\mathbf{X}$ is

$$\mathbb{P}r^{(1)}\mathbf{X} = \mathbf{X} + \eta^1_{[x]}\partial_{\rho_x} + \eta^1_{[t]}\partial_{\rho_t} + \eta^2_{[x]}\partial_{\tau_x} + \eta^2_{[t]}\partial_{\tau_t} + \eta^3_{[x]}\partial_{\phi_x} + \eta^3_{[t]}\partial_{\phi_t} + \eta^4_{[x]}\partial_{\psi_x} + \eta^4_{[t]}\partial_{\psi_t}, \tag{3.9}$$

since the system (3.7) is of second order, so we require second order prolongation of infinitesimal generator given by

$$\mathbb{P}r^{(2)}\mathbf{X} = \mathbb{P}r^{(1)}\mathbf{X} + \eta^1_{[xx]}\partial_{\rho_{xx}} + \eta^1_{[xt]}\partial_{\rho_{xt}} + \eta^1_{[tt]}\partial_{\rho_{tt}} + \eta^2_{[xx]}\partial_{\tau_{xx}} + \eta^2_{[xt]}\partial_{\tau_{xt}} + \eta^2_{[tt]}\partial_{\tau_{tt}} + \eta^3_{[xx]}\partial_{\phi_{xx}} + \eta^3_{[xt]}\partial_{\phi_{xt}} + \eta^3_{[tt]}\partial_{\phi_{tt}} + \eta^4_{[xx]}\partial_{\psi_{xx}} + \eta^4_{[xt]}\partial_{\psi_{xt}} + \eta^4_{[tt]}\partial_{\psi_{tt}}, \tag{3.10}$$

next employ Theorem (2.1.1) on the reduced system (3.7)

$$\rho_{xx} - \rho\phi_t - \rho\phi_x^2 + 2(\sigma_1\rho^2 + \sigma_2\tau^2)\rho = 0, \tag{3.11}$$
$$\tau_{xx} - \tau\psi_t - \tau\psi_x^2 + 2(\sigma_1\rho^2 + \sigma_2\tau^2)\tau = 0, \tag{3.12}$$
$$\rho_t + 2\rho_x\phi_x + \rho\phi_{xx} = 0, \tag{3.13}$$
$$\tau_t + 2\tau_x\psi_x + \tau\psi_{xx} = 0. \tag{3.14}$$

The term $\eta^1_{[xx]}\partial_{\rho_{xx}}$ in (3.10) involves partial derivative w.r.t ρ_{xx}, when applies on equation (3.11), it will give $\eta^1_{[xx]}$, also when $\eta^3_{[x]}\partial_{\phi_x}, \eta^3_{[t]}\partial_{\phi_t}$, $\eta^1\partial_\rho$ applies on equation (3.11), it will give $\eta^1\phi_t$, $\rho\eta^3_{[t]}$, $3\sigma_1\rho^2\eta^1$, $\sigma_2(\eta^1\tau^2)$, $2\rho\phi_x\eta^3_{[x]}$ and $\eta^1\phi_x^2$. Similarly we get $2\rho\tau\eta^2$ by applying (3.10) on equation (3.11). In the same manner we applied (3.10) on equations (3.12), (3.13), (3.14) and got the following system of equations

$$\eta^1_{[xx]} - \eta^1\phi_t - \rho\eta^3_{[t]} - \eta^1\phi_x^2 - 2\rho\phi_x\eta^3_{[x]} + 2(3\sigma_1\rho^2\eta^1 + (\sigma_1\eta^1\tau^2 + 2\sigma_2\rho\tau\eta^2)) = 0, \tag{3.15}$$
$$\eta^2_{[xx]} - \eta^2\psi_t - \tau\eta^4_{[t]} - \eta^2\psi_x^2 - 2\tau\psi_x\eta^4_{[x]} + 2(\sigma_1(\eta^2\rho^2 + 2\rho\tau\eta^1) + 3\sigma_2\tau^2\eta^2) = 0, \tag{3.16}$$
$$\eta^1_{[t]} + 2\eta^1_{[x]}\phi_x + 2\eta^3_{[x]}\rho_x + \eta^1\phi_{xx} + \rho\eta^3_{[xx]} = 0, \tag{3.17}$$
$$\eta^2_t + 2\eta^2_{[x]}\psi_x + 2\eta^4_{[x]}\tau_x + \eta^2\psi_{xx} + \tau\eta^4_{[xx]} = 0. \tag{3.18}$$

The explicit form $\eta^1_{[t]}, \eta^1_{[x]}, \eta^2_{[t]}, \eta^1_{[x]}, \eta^3_{[t]}, \eta^1_{[x]}, \eta^4_{[t]}, \eta^4_{[x]}, \eta^1_{[xx]}, \eta^2_{[xx]}, \eta^3_{[xx]}$ and $\eta^4_{[xx]}$ can be obtained from equation (1.40) and (1.43). By comparing monomials we obtain the determining equations

$$\varrho^1_\phi = 0, \quad \varrho^1_\psi = 0, \quad \varrho^1_\rho = 0, \quad \varrho^1_\tau = 0, \quad \varrho^1_x = 0, \tag{3.19}$$
$$\varrho^1_{tt} = 0, \quad \varrho^2_\phi = 0, \quad \varrho^2_\psi = 0, \quad \varrho^2_\rho = 0, \quad \varrho^2_\tau = 0, \tag{3.20}$$
$$\varrho^2_{tt} = 0, \tag{3.21}$$
$$\varrho^2_x = \frac{1}{2}\varrho^1_t, \tag{3.22}$$
$$\eta^3_\phi = -\eta^3_\psi, \quad \eta^3_\rho = \frac{-\eta^3_\tau\tau}{\rho}, \quad \eta^3_t = 0, \quad \eta^3_x = \frac{1}{2}(\varrho^2_t), \quad \eta^3_{\psi\psi} = -\phi^3_\tau\tau, \tag{3.23}$$
$$\eta^3_{\psi\tau} = \frac{\eta^3_\psi}{\tau}, \quad \eta^3_{\tau\tau} = 0, \quad \eta^4_\phi = \frac{-\eta^3_\psi\rho^2\sigma_1}{\tau^2\sigma_2}, \tag{3.24}$$

$$\eta^4_\psi = (\eta^3_\psi \rho^2 \sigma_1)/(\tau^2 \sigma_2), \tag{3.25}$$
$$\eta^4_\rho = (\eta^3_\tau \rho \sigma_1)/(\tau \sigma_2), \tag{3.26}$$
$$\eta^4_\tau = (-\eta^3_\tau \rho^2 \sigma_1)/(\tau^2 \sigma_2), \quad \eta^4_t = 0, \quad \eta^4_x = \frac{1}{2}\varrho^2_t, \tag{3.27}$$
$$\eta^3_x = \frac{1}{2}\varrho^2_t, \tag{3.28}$$
$$\eta^4_x = \frac{1}{2}\varrho^2_t, \tag{3.29}$$
$$\eta^2 = (2\phi^3_\psi \rho^2 \sigma_1 - \varrho^1_t \tau^2 \sigma_2)/(2\tau\sigma_2), \tag{3.30}$$
$$\eta^1 = \frac{1}{2}\rho(2\phi^3_\psi - \varrho^1_t). \tag{3.31}$$

The process of finding the determining PDEs is very time taking, there are several software's to compute the determining PDEs. Here we use MAPLE to get these determining equations. By looking on equation (3.19), we conclude that ϱ^1 is not functions of ϕ, ψ, ρ, τ and x, now by using equation (3.20.1) we get

$$\varrho^1 = c_1 t + c_2. \tag{3.32}$$

Similarly from equation (3.20.2), we conclude that ϱ^2 is not functions of ϕ, ψ, ρ and τ so they are only function of t, x. If we observe equation (3.21), it will indicate that

$$\varrho^2 = f_1(x)t + f_2(x), \tag{3.33}$$

on substituting the above value in equation (3.22) we get

$$f_1 = c_3, \quad f_2 = \frac{1}{2}c_1 x + c_4, \tag{3.34}$$

thus we obtain

$$\varrho^2 = \frac{1}{2}c_1 x + c_3 t + c_4, \tag{3.35}$$

, by substituting the values of ϱ^1, ϱ^2 in the remaining determining PDEs, we get

$$\eta^1 = -\tau(c_5 \sin(\psi) - c_6 \cos(\psi))\cos(\phi) + \sin(\psi)\sin(\phi)c_6\tau + \cos(\psi)\sin(\phi)c_5\tau - (1/2)\rho c_1, \tag{3.36}$$

$$\eta^2 = \frac{1}{2}\frac{(-2\rho\sigma_1(c_6\cos(\psi) - c_5\sin(\psi))\cos(\phi) - 2\cos(\psi)\sin(\phi)c_5\rho\sigma_1 - 2\sin(\psi)\sin(\phi)c_6\rho\sigma_1 - c_1\tau\sigma_2)}{\sigma_2}, \tag{3.37}$$

$$\eta^3 = \frac{(\tau(\sin(\psi)c_6 + c_5\cos(\psi))\cos(\phi) - \cos(\psi)\sin(\phi)c_6\tau + \sin(\psi)\sin(\phi)c_5\tau + \rho(c_3 x + c_7))}{\rho}, \tag{3.38}$$

$$\eta^4 = c_3 x + \frac{\sigma_1\rho(-c_6\sin(-\psi + \phi) + c_5\cos(-\psi + \phi))}{(\tau\sigma_2)} + c_8, \tag{3.39}$$

where $c_1, c_2, c_3, c_4, c_5, c_6, c_7$ and c_8 are arbitrary constants. Therefore the Lie algebra of infinitesimal generators of the Manakov system (3.1) is spanned by eight vector fields

$$\mathbf{X}_1 = \partial_x, \quad \mathbf{X}_2 = \partial_t, \tag{3.40}$$
$$\mathbf{X}_3 = \partial_\phi, \quad \mathbf{X}_4 = \partial_\psi, \tag{3.41}$$
$$\mathbf{X}_5 = 2t\partial_t + x\partial_x - \rho\partial_\rho - \tau\partial_\tau, \tag{3.42}$$
$$\mathbf{X}_6 = 2t\partial_x + x\partial_\phi + x\partial_\psi, \tag{3.43}$$
$$\mathbf{X}_7 = \cos(\phi - \psi)(\tau\partial_\rho - \frac{\sigma_1}{\sigma_2}\rho\partial_\tau) + \sin(\phi - \psi)(\frac{\tau}{\rho}\partial_\phi - \frac{\sigma_1\rho}{\sigma_2\tau}\partial_\psi)\,, \tag{3.44}$$
$$\mathbf{X}_8 = -\sin(\phi - \psi)(\tau\partial_\rho - \frac{\sigma_1}{\sigma_2}\rho\partial_\tau) + \cos(\phi - \psi)(\frac{\tau}{\rho}\partial_\phi + \frac{\sigma_1\rho}{\sigma_2\tau}\partial_\psi). \tag{3.45}$$

The symmetry transformation generated by the infinitesimal generators: $\mathbf{X}_1 = \partial_x$, which is space translation ($\tilde{x} = x + \zeta$), $\mathbf{X}_2 = \partial_t$ is time translations ($\tilde{t} = t + \zeta$), where the infinitesimal generators (3.41.1), (3.41.2) are phase translation ($\tilde{\phi} = \phi + \zeta$), ($\tilde{\psi} = \psi + \zeta$) respectively, Galilean boost (3.42) with simultaneous phase transformations ($\tilde{x} = x + 2\zeta t, \tilde{\phi} = \phi + \zeta x, \tilde{\psi} = \psi + \zeta x$), where the infinitesimal generator (3.43) is inhomogeneous scaling with transformations ($\tilde{t} = e^{2\zeta}t, \tilde{x} = e^{\zeta}x, \tilde{\rho} = e^{-\zeta}\rho, \tilde{\tau} = e^{-\zeta}\tau$). where we have the following decomposition of Lie algebra $\mathcal{R}^4 \oplus d_1 \oplus K_1 \oplus U_2$, or equivalently four translations, an inhomongeneous scaling transformation, one dilation and two phase translations. The action of underlying Lie group for $\mathcal{R}^4 \oplus d_1$, on the solutions of Manakov system is given by

$$\begin{aligned}
\phi(t,x) &\longrightarrow \phi(e^{2\epsilon_5}t + \epsilon_1, e^{\epsilon_5}x + \epsilon_2) + \epsilon_3\,,\\
\psi(t,x) &\longrightarrow \psi(e^{2\epsilon_5}t + \epsilon_1, e^{\epsilon_5}x + \epsilon_2) + \epsilon_3\,,\\
\rho(t,x) &\longrightarrow e^{-\epsilon_5}\rho(e^{2\epsilon_5}t + \epsilon_1, e^{\epsilon_5}x + \epsilon_2)\,,\\
\tau(t,x) &\longrightarrow e^{-\epsilon_5}\tau(e^{2\epsilon_5}t + \epsilon_1, e^{\epsilon_5}x + \epsilon_2)\,.
\end{aligned}$$

The symmetry $\mathbf{X}_6$ corresponds to a Galilean boost transformation whose action is

$$\begin{aligned}
\phi(t,x) &\longrightarrow \phi(t, x + 2\epsilon_6 t) + \epsilon_6 x + \epsilon_6^2 t,\\
\psi(t,x) &\longrightarrow \psi(t, x + 2\epsilon_6 t) + \epsilon_6 x + \epsilon_6^2 t,\\
\rho(t,x) &\longrightarrow \rho(t, x + 2\epsilon_6 t),\\
\tau(t,x) &\longrightarrow \tau(t, x + 2\epsilon_6 t).
\end{aligned}$$

To investigate the structure of the Lie algebra we use the commutator table. The Lie algebra spanned by $\mathbf{X}_1 = \partial_x$, $\mathbf{X}_2 = \partial_t$, $\mathbf{X}_3 = \partial_\phi$, $\mathbf{X}_4 = \partial_\psi$, $\mathbf{X}_5 = 2t\partial_x + x\partial_\phi + x\partial_\psi$, $\mathbf{X}_6 = 2t\partial_t + x\partial_x - \rho\partial_\rho + \tau\partial_\tau$, generates the symmetry group of the polar reduced system (3.7). Our next task is to find the exact solution, so we perform group reduction.

3.3 Group Reduction of Manakov System

We discuss the group reduction of the polar reduced Manakov system

$$\begin{aligned}
&\rho_{xx} - \rho\phi_t - \rho\phi_x^2 + 2(\sigma_1\rho^2 + \sigma_2\tau^2)\rho = 0,\\
&\tau_{xx} - \tau\psi_t - \tau\psi_x^2 + 2(\sigma_1\rho^2 + \sigma_2\tau^2)\tau = 0,\\
&\rho_t + 2\rho_x\phi_x + \rho\phi_{xx} = 0,\\
&\tau_t + 2\tau_x\psi_x + \tau\psi_{xx} = 0,
\end{aligned} \tag{3.46}$$

to find the exact solutions. Therefore we use the admitted Lie point symmetries which are

$$\begin{aligned}
&\mathbf{X}_1 = \partial_x, \quad \mathbf{X}_2 = \partial_t, \quad \mathbf{X}_3 = \partial_\phi, \quad \mathbf{X}_4 = \partial_\psi,\\
&\mathbf{X}_5 = 2t\partial_t + x\partial_x - \rho\partial_\rho - \tau\partial_\tau, \quad \mathbf{X}_6 = 2t\partial_x + x\partial_\phi + x\partial_\psi,\\
&\mathbf{X}_7 = \cos(\phi-\psi)(\tau\partial_\rho - \frac{\sigma_1}{\sigma_2}\rho\partial_\tau) + \sin(\phi-\psi)(\frac{\tau}{\rho}\partial_\phi - \frac{\sigma_1\rho}{\sigma_2\tau}\partial_\psi)\,,\\
&\mathbf{X}_8 = -\sin(\phi-\psi)(\tau\partial_\rho - \frac{\sigma_1}{\sigma_2}\rho\partial_\tau) + \cos(\phi-\psi)(\frac{\tau}{\rho}\partial_\phi + \frac{\sigma_1\rho}{\sigma_2\tau}\partial_\psi).
\end{aligned} \tag{3.47}$$

1.Space Translation:
We start the reduction using the infinitesimal generator $\mathbf{X}_1 = \partial_x$ which is a space translation symmetry. Now we find the invariants by using the condition

$$\mathbf{X}\varphi = 0, \tag{3.48}$$

here $\mathbf{X} = \mathbf{X}_1$ and $\varphi(t, x, \rho, \tau, \phi, \psi)$, so we get

$$\varphi_x = 0. \tag{3.49}$$

The equation (3.49) is 1st order PDE which has infinite many solution, so on solving it we get

$$t, \quad \rho, \quad \tau, \quad \phi, \quad \psi, \tag{3.50}$$

which are invariants, by giving them new names we get

$$r = t,\ M_1(r) = \rho,\ M_2(r) = \phi,\ M_3(r) = \tau,\ M_4(r) = \psi, \tag{3.51}$$

these are basically similarity variables. Now by using the similarity variable and applying the chain rule the system (3.14), can be reduced to

$$M_1M_2' - 2(\sigma_1M_1^2 + \sigma_2M_3^2)M_1 = 0, \tag{3.52}$$
$$M_3M_4' - 2(\sigma_1M_1^2 + \sigma_2M_3^2)M_3 = 0, \tag{3.53}$$
$$M_1' = 0, \tag{3.54}$$
$$M_3' = 0, \tag{3.55}$$

the above system is 1st order ODEs, which is quite simple and easy to solve. So from the equations (3.54) and (3.55) of the above system, we conclude that both M_1 and M_3 are independent of time, they can be integrated easily and we get $M_1 = c_1, M_3 = c_2$. Furthermore on substituting the values of M_1, M_3 in the remaining equations (3.52) and (3.53) and by solving we obtain

$$M_2 = 2(\sigma_1 c_1^2 + \sigma_2 c_2^2)t + c_3, \tag{3.56}$$
$$M_4 = 2(\sigma_1 c_1^2 + \sigma_2 c_2^2)t + c_4, \tag{3.57}$$

where c_1, c_2, c_3, c_4 are arbitrary constants. Now by substituting the values of M_1, M_2, M_3, M_4 in the relation (3.51), we obtain

$$\rho = c_1, \quad \phi = 2(\sigma_1 c_1^2 + \sigma_2 c_2^2)t + c_3, \tag{3.58}$$
$$\tau = c_2, \quad \psi = 2(\sigma_1 c_1^2 + \sigma_2 c_2^2)t + c_4, \tag{3.59}$$

which is the solution of the polar reduced system (3.11)-(3.14). Where ρ, τ are amplitude of two interacting waves and ϕ, ψ are the phases of the waves. Now put the values of ρ, τ, ϕ, ψ in the relations (3.3), (3.4), we obtain

$$u(t,x) = c_1 e^{2i(\sigma_1 c_1^2 + \sigma_2 c_2^2)t + c_3},$$
$$v(t,x) = c_2 e^{2i(\sigma_1 c_1^2 + \sigma_2 c_2^2)t + c_4}, \tag{3.60}$$

which is exact solution of the Manakov system (3.14).

2. Time Translation:
Similarly for $t-$translation, the infinitesimal generator is $\mathbf{X}_2 = \partial_t$. Now again we will apply the condition (3.48) and we get the following invariants

$$x, \quad \rho, \quad \tau, \quad \phi, \quad \psi, \tag{3.61}$$

therefore the similarity variables are

$$r = x, \; M_1(r) = \rho, \; M_2(r) = \phi, \; M_3(r) = \tau, \; M_4(r) = \psi. \tag{3.62}$$

Now by using the similarity transformation the system of equations (3.11)-(3.14) reduces to

$$M_1'' - M_1 M_2' + 2(\sigma_1 M_1^2 + \sigma_2 M_3^2)M_1 = 0, \tag{3.63}$$
$$M_3'' - M_3 M_4' + 2(\sigma_1 M_1^2 + \sigma_2 M_3^2)M_3 = 0, \tag{3.64}$$
$$2M_1' M_2' + M_1 M_2'' = 0, \tag{3.65}$$
$$2M_3' M_4' + M_3 M_4'' = 0, \tag{3.66}$$

which is system of 2nd order ODEs and it has eight Lie point symmetries. We obtain time independent solutions from this system. The above system can be solved if we assume one of the function as constant, for example $M_1 = c_1$, in which case the last two equations of reduced system give

$$M_2(r) = c_2 r + c_3, \quad M_4(r) = \pm\, c_2 r + c_4, \tag{3.67}$$

by using above values in the remaining equations of the system (3.63)-(3.66), we get

$$M_3 = \pm\frac{\sqrt{-2\sigma_2(2c_1^2\sigma_1 - c_2^2)}}{2\sigma_2},$$

the resulting solution of Manakov system is

$$\begin{aligned} u(t,x) &= c_1 e^{(i(c_2x+c_3))}, \\ v(t,x) &= \pm\frac{\sqrt{-2\sigma_2(2c_1^2\sigma_1 - c_2^2)}}{2\sigma_2} e^{(\pm i(c_2x+c_4))}. \end{aligned} \tag{3.68}$$

Similarly the choice of $M_1(r) = c_1 r$ also led us to explicitly solve the system (3.63)-(3.66) and we obtain

$$M_3(r) = \mp\frac{\sqrt{-\sigma_2\sigma_1}rc_1}{\sigma_2}, \quad M_2(r) = c_2, \quad M_4(r) = c_3 \tag{3.69}$$

We obtain the following solution of Manakov system

$$\begin{aligned} u(t,x) &= c_1 x e^{ic_2}, \\ v(t,x) &= \pm\frac{\sqrt{-\sigma_2\sigma_1}xc_1}{\sigma_2} e^{ic_4}. \end{aligned} \tag{3.70}$$

where an extra care is required to interpret the above solutions.

Furthermore if we take $M_1(r) = c_1 r^2$ and solved the system (3.63)-(3.66), we obtained

$$M_4(r) = \arctan\left(c_1\sqrt{\sigma_1}r^3\right) + c_3,\ M_2(r) = c_2,\ M_3(x) = \pm\frac{\sqrt{-\sigma_2(c_1^2r^6\sigma_1 + 1)}}{\sigma_2 r}, \tag{3.71}$$

and

$$M_4(r) = -\arctan\left(c_1\sqrt{\sigma_1}r^3\right) + c_3,\ M_2(r) = c_2,\ M_3(x) = \pm\frac{\sqrt{-\sigma_2(c_1^2r^6\sigma_1 + 1)}}{\sigma_2 r}, \tag{3.72}$$

Therefore we yield two more exact solutions of Manakov system

$$u(t,x) = c_1x^2e^{ic_2}, \quad v(t,x) = \pm\frac{\sqrt{-\sigma_2(c_1^2x^6\sigma_1 + 1)}}{\sigma_2 x} e^{(i(\arctan(c_1\sqrt{\sigma_1}x^3)+c_3))}, \tag{3.73}$$

$$u(t,x) = c_1x^2e^{ic_2}, \quad v(t,x) = \pm\frac{\sqrt{-\sigma_2(c_1^2x^6\sigma_1 + 1)}}{\sigma_2 x} e^{(i(-\arctan(c_1\sqrt{\sigma_1}x^3)+c_3))}. \tag{3.74}$$

Another similar result is obtained for $M_1 = c_1/r$, we obtain the following result

$$M_4(r) = c_3,\ M_2(x) = c_2,\ M_3(x) = \pm\frac{\sqrt{-\sigma_2(c_1^2\sigma_1 + 1)}}{\sigma_2 r}, \tag{3.75}$$

by putting these results in the relations (3.3),(3.4), we obtain another solution of Manakov system.

$$u(t,x) = \frac{c_1}{x} e^{(ic_2)},$$
$$v(t,x) = \pm \frac{\sqrt{-\sigma_2(c_1^2\sigma_1 + 1)}}{\sigma_2 x} e^{(ic_2)}. \tag{3.76}$$

Next we use the scaling symmetry to get exact solution.

3. Scaling:

The scaling symmetry has more significance as it will give us nontrivial results. The interacting waves which are generated from the solutions of Manakov system, depends on the amplitude. So the scaling symmetry will give us such solutions in which ρ and τ both are functions of t and x. Now we find the similarity variables corresponding to the scaling symmetry, therefore we use the condition (3.48) and we get

$$\mathbf{X}_5\varphi = 0,$$
$$(2t\partial_t + x\partial_x - \rho\partial_\rho - \tau\partial_\tau)(\varphi) = 0,$$
$$2t\varphi_t + x\varphi_x - \rho\varphi_\rho - \tau\varphi_\tau = 0, \tag{3.77}$$

now we use characteristic method to solve the above first order PDE

$$\frac{dt}{2t} = \frac{dx}{x} = \frac{d\rho}{-\rho} = \frac{d\tau}{-\tau} = \frac{d\phi}{0} = \frac{d\psi}{0}, \tag{3.78}$$
$$\frac{dt}{2t} = \frac{dx}{x}, \frac{d\rho}{-\rho} = \frac{d\phi}{0}, \frac{d\tau}{-\tau} = \frac{d\phi}{0}, \tag{3.79}$$
$$\frac{1}{2}\ln t = \ln x, \tag{3.80}$$

so we get following invariants

$$\frac{x}{\sqrt{t}},\ \rho,\ \phi,\ \tau,\ \psi, \tag{3.81}$$

where the similarity variables are following

$$s = \frac{1}{2}\ln t,\ r = \frac{x}{\sqrt{t}},\ M_1(r) = \sqrt{t}\,\rho,\ M_2(r) = \phi,\ M_3(r) = \sqrt{t}\,\tau,\ M_4(r) = \psi, \tag{3.82}$$

similarly by using similarity transformation, the system of equations (3.11)-(3.14) reduces into ODEs

$$2M_1'' + M_1(rM_2' - 2M_2'^2 + 4\sigma_1 M_1^2 + 4\sigma_2 M_3^2) = 0, \tag{3.83}$$
$$2M_3'' + M_3(rM_4' - 2M_4'^2 + 4\sigma_1 M_1^2 + \sigma_2 4M_3^2) = 0, \tag{3.84}$$
$$2M_3M_4'' + (4M_4' - r)M_3' - M_3 = 0, \tag{3.85}$$
$$2M_1M_2'' + (4M_2' - r)M_1' - M_1 = 0, \tag{3.86}$$

where the derivative is with respect to similarity variable r. The above system has four Lie point symmetries.

In order to obtain other solutions we assume that either M_1 or M_3 is constant. Thus consider $M_1 = c_2$, by solving system (3.7) we get

$$M_4 = \frac{r^2}{4} + c_1,\ M_2 = \frac{r^2}{4} + c_3,\ M_3 = \pm\sqrt{\frac{-\sigma_1}{\sigma_2}}\, c_2\,, \tag{3.87}$$

which directly gives us a non-trivial solution

$$\rho = \frac{c_2}{\sqrt{t}}, \quad \phi = \frac{x^2}{4t} + c_3, \tag{3.88}$$

$$\tau = \pm\sqrt{\frac{-\sigma_1}{\sigma_2}}\,\frac{c_2}{\sqrt{t}}, \quad \psi = \frac{x^2}{4t} + c_1. \tag{3.89}$$

Thus the solution of Manakov system is

$$u(t,x) = \frac{c_2}{\sqrt{t}}\ e^{i(x^2/4t+c_3)},$$

$$v(t,x) = \pm\sqrt{\frac{-\sigma_1}{\sigma_2}}\,\frac{c_2}{\sqrt{t}}\ e^{i(x^2/4t+c_1)}\,, \tag{3.90}$$

where c_1, c_2, c_3 are arbitrary constants.

Interpretation.

Note that the solution (3.90) of Manakov system (3.7) is such that the waves are equivalent up to a phase difference, i.e.,

$$v(t,x) = \pm\sqrt{\frac{-\sigma_1}{\sigma_2}} e^{i(c_1-c_3)} u(t,x), \tag{3.91}$$

Therefore, if we assume that $c_1 = c_3$ the above solution becomes the solution of scalar non-linear Schrödinger equation as in the case $v = ku$, the first equation in the Manakov system converts into a single PDE while the other equation is vacuous. We now discuss the analytic property of this solution in both cases

Case 1. $(c_1 = c_3)$ in the case we obtain

$$u(t,x) = \frac{c_2}{\sqrt{t}}\ e^{i(x^2/4t+c_3)}, \tag{3.92}$$

which satisfies a cubically semi-linear Schröndiger equation which is given below

$$iu_t + u_{xx} + \mid u \mid = 0, \tag{3.93}$$

the above solution can be interpreted in two ways.

(a) If we assume that a wave is continuum of tiny localized particles whose locations are described by x, then the motion of particle at some fixed location $x = k_1$ can be described easily. In particular we obtain

$$u(t,k) = \frac{c_2}{\sqrt{t}}\ e^{i(k_1^2/4t+c_3)}, \tag{3.94}$$

which has two orthogonal modes corresponding to cos and sin whose graphs are shown in Figure (3.1), (c_3 is assumed zero with out loss of generality). The sinusoidal motion of a localized particle at $x = k_1$, starts abruptly while decaying uniformly as time progresses till it evaporates at infinity.

(b) We can also consider graph of function $u(x)$ at particular instant and simulate the corresponding function over an entire range of t, given by

$$u(x) = c_2 e^{i(x^2+c_3)}, \tag{3.95}$$

As before the above wave comprises of two modes we take the sin mode to describe the behavior of the wave. For instance we take four graphs corresponding to the values $t = 2, 3.5, 4.512$, indicating that the waves start abruptly while disperse away at the asymptotic limit $t \to \infty$ as depicted in Figure (3.2- 3.5), where we have assumed that $c_3 = 0$. The contribution of nonzero values of the constant c_3 is merely to translate the value

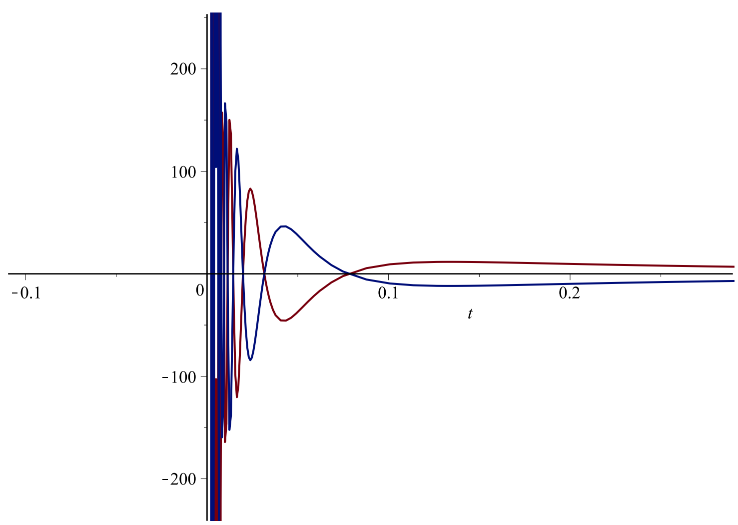

Figure 3.1: The motion of sin-wave particles in opposite phases. The sinusoidal motion start abruptly and decays as $t \to \infty$.

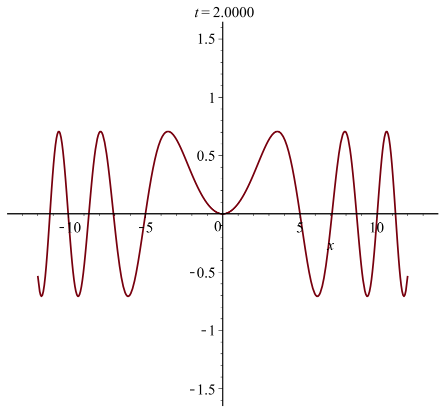

Figure 3.2: Wave at $t = 2$

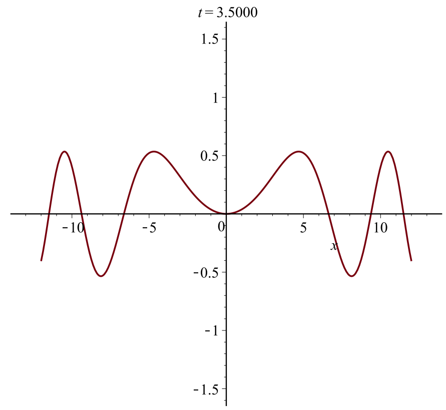

Figure 3.3: Wave at $t = 3.5$

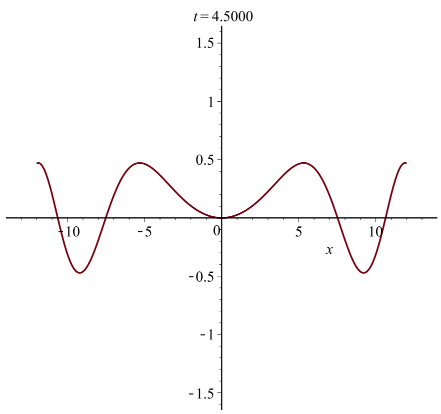

Figure 3.4: Wave at $t = 4.5$

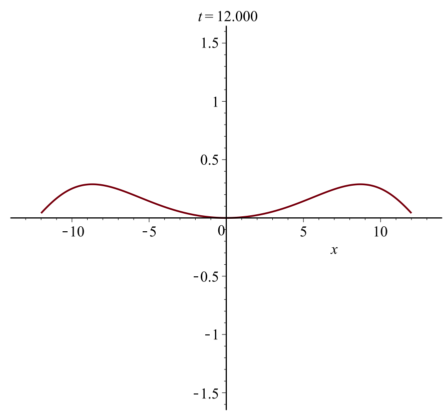

Figure 3.5: Wave at $t = 12$

Similarly if M_1 or M_3 is a multiple of r, the integration gives

$$M_1 = c_1 r, \quad M_2 = (1/6)r^2 + c_2, \tag{3.96}$$

$$M_3 = \frac{\sqrt{-\sigma_2(36c_1^2\sigma_1 + 1))}}{6\sigma_2} r, \quad M_4 = (1/6)r^2 + c_3, \tag{3.97}$$

where the $\rho = c_1 x/t$ and $\dfrac{\sqrt{-\sigma_2(36c_1^2\sigma_1 + 1))}x}{6\sigma_2 t}$ are the amplitudes of both the interacting waves which is nontrivial and also depends on both t and x. It is physically very important. The values of M_1, M_2, M_3, M_4 directly give us a non-trivial solution of system of equations (3.11)-(3.14), which is given below

$$\phi = \frac{x^2}{6t} + c_2, \quad \psi = \frac{x^2}{6t} + c_3, \tag{3.98}$$

where ϕ, ψ represents the phase of the waves. Thus the exact solution of Manakov system is

$$\begin{aligned} u(t,x) &= \frac{c_1 x}{t} e^{(i(\frac{x^2}{6t}+c_2))}, \\ v(t,x) &= \frac{\sqrt{-\sigma_2(36c_1^2\sigma_1+1))}x}{6\sigma_2 t} e^{(i(\frac{x^2}{6t}+c_3))}, \end{aligned} \tag{3.99}$$

which under the action of four translations give us a general solution

$$\begin{aligned} u(t,x) &= \frac{c_2(x+\tilde{c}_1)}{t+\tilde{c}_2}\ e^{(i((x+\tilde{c}_1)^2/6(t+\tilde{c}_2)+c_3))}, \\ v(t,x) &= \pm\frac{\sqrt{-\sigma_2(1+36c_2^2\sigma_1)}(x+\tilde{c}_1)}{6\sigma_2(t+\tilde{c}_2)}\ \frac{(x+\tilde{c}_1)}{(t+\tilde{c}_2)}\ e^{(i((x+\tilde{c}_1)^2/6(t+\tilde{c}_2)+c_1))}. \end{aligned} \tag{3.100}$$

The two interacting sin waves, i.e., $\sin((x+\tilde{c}_1)^2/6(t+\tilde{c}_2)+c_1)$ is given in Figure (3.1). For $\tilde{c}_1 = 1, \tilde{c}_2 = 1,\ c_3, c_2 = 1$ and $\sigma_1 = 1, \sigma_2 = -1$, the particle or waves gain stability from instability, here $\tilde{c}_1$ corresponds to the translation of x also $\tilde{c}_2$ is time translation factor. In order to integrate reduced system of ODEs we employ non-classical approaches.

4. Galileo Boost:

The symmetry $\mathbf{X}_6$ induces the similarity variables

$$s = \frac{x}{t},\ r = t,\ M_4(r) = \frac{4\psi t - x^2}{4t},\ M_1(r) = \rho,\ M_2(r) = \frac{4\phi t - x^2}{4t},\ M_3(r) = \tau\,, \tag{3.101}$$

which reduces system (3.14) into a system of four first order ODEs

$$M_2{}' - 2(M_1^2 + M_3^2) = 0, \tag{3.102}$$
$$M_4{}' - 2(M_1^2 - M_3^2) = 0, \tag{3.103}$$
$$2rM_1{}' + M_1 = 0, \tag{3.104}$$
$$2rM_3{}' + M_3 = 0, \tag{3.105}$$

where the derivative is with respect to similarity variable r. The above system can be integrated easily and we obtain a solution

$$M_4 = 2(\sigma_1 c_1^2 + \sigma_2 c_2^2)\ln(r) + c_3,\ M_1 = \frac{c_1}{r}, \tag{3.106}$$
$$M_2 = 2(\sigma_1 c_1^2 + \sigma_2 c_2^2)\ln(r) + c_4,\ M_3 = \frac{c_2}{r}, \tag{3.107}$$

which in terms of the original variables gives us

$$\phi(t,x) = \frac{1}{4t}\left((c_1^2+c_2^2)\,t\ln(t) + 4c_3 t + x^2\right),\ \rho(t,x) = \frac{c_1}{\sqrt{t}}, \tag{3.108}$$
$$\psi(t,x) = \frac{1}{4t}\left((c_1^2-c_2^2)\,t\ln(t) + 4c_4 t + x^2\right),\ \tau(t,x) = \frac{c_2}{\sqrt{t}}. \tag{3.109}$$

Thus the exact solution of Manakov system is

$$u(t,x) = \frac{c_1}{\sqrt{t}} e^{i\left(\frac{1}{4t}\left((c_1^2+c_2^2)\,t\ln(t)+4c_3t+x^2\right)\right)},$$
$$v(t,x) = \frac{c_2}{\sqrt{t}} e^{i\left(\frac{1}{4t}\left((c_1^2-c_2^2)\,t\ln(t)+4c_4t+x^2\right)\right)}. \tag{3.110}$$

If we take the combinations of the infinitesimal generators, for instance take $\mathbf{X}_1 + c\mathbf{X}_2$, which admits traveling wave structure, where c is any arbitrary constant. The invariants for $\partial_t + c\partial_x$ are

$$r = (x - c_1t), \quad M_1(r) = \rho, \quad M_2(r) = \phi, \tag{3.111}$$
$$M_3(r) = \tau, \quad M_4(r) = \psi, \tag{3.112}$$

here r is our new independent variable, M_1, M_2, M_3, M_4 are our similarity variables. Using these variables and applying chain rule, we get the following reduced system

$$M_1'' + cM_1M_2' - M_1M_2'^2 + 2(\sigma_1M_1^2 + \sigma_2M_3^2)M_1 = 0, \tag{3.113}$$
$$M_3'' + cM_3M_4' - M_3M_4'^2 + 2(\sigma_1M_1^2 + \sigma_2M_3^2)M_3 = 0, \tag{3.114}$$
$$-cM_1' + 2M_1'M_2' + M_1M_2'' = 0, \tag{3.115}$$
$$-cM_3' + 2M_3'M_4' + M_3M_4'' = 0. \tag{3.116}$$

Take $M_1 = c_1$ and put in the equations (3.115)-(3.114), then on integration, we get the following solution

$$M_2 = c_2(x-ct) + c_3, \quad M_3 = \frac{\sqrt{-2\sigma_2(2c_1^2\sigma_1 - c_2^2 + c_2c))}}{2\sigma_2}, \quad M_4 = c_2(x-ct) + c_4, \tag{3.117}$$

$c_i, i = (1,..,4)$ all are constants. The exact solution of Manakov system is obtained by substituting $M_i, i = (1,4)$ in the relations (3.111)-(3.112).

$$u(t,x) = c_1e^{(i(c_2(x-ct)+c_3))},$$
$$v(t,x) = \frac{\sqrt{-2\sigma_2(2c_1^2\sigma_1 - c_2^2 + c_2c))}}{2\sigma_2} e^{(i(c_2(x-ct)+c_4))} \tag{3.118}$$

If we choice M_1 as a function of r, we get the solution of the equations (3.115)-(3.112) which is given as

$$M_1 = c_1r, \quad M_2 = \frac{1}{2}cr + c_2, \tag{3.119}$$
$$M_3 = \pm\frac{\sqrt{-2\sigma_2(8c_1^2r^2\sigma_1 + c^2)}}{4\sigma_2}, \quad M_4 = \frac{1}{2}cr + \arctan\left(\frac{2c_1\sqrt{2\sigma_1}r}{c}\right) + c_3, \tag{3.120}$$

and

$$M_1 = c_1r, \quad M_2 = \frac{1}{2}cr + c_2, \tag{3.121}$$
$$M_3 = \pm\frac{\sqrt{-2\sigma_2(8c_1^2r^2\sigma_1 + c^2)}}{4\sigma_2}, \quad M_4 = \frac{1}{2}cr - \arctan\left(\frac{2c_1\sqrt{2\sigma_1}r}{c}\right) + c_3, \tag{3.122}$$

Thus the Solution of Manakov system is

$$u(t,x) = c_1(x-ct)e^{(i(\frac{1}{2}c(x-ct)+c_2))}, \tag{3.123}$$

$$v(t,x) = \pm\frac{\sqrt{-2\sigma_2(8c_1^2(x-ct)^2\sigma_1+c^2)}}{4\sigma_2}e^{(i(\frac{1}{2}c(x-ct)+\arctan(\frac{2c_1\sqrt{\sigma_1 2}(x-ct)}{c})+c_3))} \tag{3.124}$$

and

$$u(t,x) = c_1(x-ct)e^{(i(\frac{1}{2}c(x-ct)+c_2))},$$

$$v(t,x) = \pm\frac{\sqrt{-2\sigma_2(8c_1^2(x-ct)^2\sigma_1+c^2)}}{4\sigma_2}e^{(i(\frac{1}{2}c(x-ct)-\arctan(\frac{2c_1\sqrt{\sigma_1 2}(x-ct)}{c})+c_3))}. \tag{3.125}$$

3.4 Optimal Subalgebras of CNSEs

The propagation of pulses in birefringent fibers and fiber optics can also be described by the CNSEs. Where several versions of CNSEs show the propagation of nonlinear waves in optical fibers. A set of CNSEs is a basic mathematical model in different branches of physics. In previous section we discuss the importance of CNSEs, now we focus on a particular case of CNSEs

$$\begin{aligned} &iu_x + \frac{1}{2}u_{tt} + 2(|u|^2 + h|v|^2)u = 0,\\ &iv_x - \frac{1}{2}v_{tt} + 2(|u|^2 + h|v|^2)v = 0, \end{aligned} \tag{3.126}$$

where $u(t,x)$ and $v(t,x)$ are complex functions, also $h = 1, 2, \frac{2}{3}$. For $h = 2$ the CNSEs describes propagation of waves at different carrier wavelength in two mode optical fibers [30], also for $h = \frac{2}{3}$ two modes in fibers with strong birefringance [31]. For convenience we will work in polar coordinates therefore we introduce

$$u = \rho e^{i\phi}, \quad v = \tau e^{i\psi}, \tag{3.127}$$

where ρ, τ determines amplitude of waves and ϕ, ψ represents phase of the waves. When we put relation (3.127) in the CNSEs (3.126) we get

$$\begin{aligned} &\frac{1}{2}\rho_{tt} - \rho\phi_x - \frac{1}{2}\rho\phi_t^2 + (\rho^2 + h\tau^2)\rho = 0,\\ &-\frac{1}{2}\tau_{tt} - \tau\psi_x + \frac{1}{2}\tau\psi_t^2 + (\rho^2 + h\tau^2)\tau = 0,\\ &\rho_x + \rho_t\phi_t + \frac{1}{2}\rho\phi_{tt} = 0,\\ &\tau_x - \tau_t\psi_t - \frac{1}{2}\tau\psi_{tt} = 0. \end{aligned} \tag{3.128}$$

The CNSEs admits the following symmetries

$$\mathbf{X}_1 = \partial_t, \quad \mathbf{X}_2 = \partial_x, \tag{3.129}$$

$$\mathbf{X}_3 = \partial_\phi, \quad \mathbf{X}_4 = \partial_\psi, \tag{3.130}$$

$$\mathbf{X}_5 = x\partial_t + t\partial_\phi - t\partial_\psi, \tag{3.131}$$

$$\mathbf{X}_6 = t\partial_t + 2x\partial_x - \rho\partial_\rho - \tau\partial_\tau. \tag{3.132}$$

The optimal system gives us all the linear combinations of infinitesimal generators that will generate different conjugacy classes. For this purpose we first construct the commutator table, we follow the procedure as discuss in Section (1.7). We calculate the Lie brackets by using the relation (1.32), where all the non zero Lie brackets are given below

$$[\mathbf{X}_2, \mathbf{X}_5] = \mathbf{X}_1, \qquad [\mathbf{X}_2, \mathbf{X}_6] = 2\mathbf{X}_2, \qquad [\mathbf{X}_5, \mathbf{X}_6] = -\mathbf{X}_5,$$
$$[\mathbf{X}_1, \mathbf{X}_5] = \mathbf{X}_3 - \mathbf{X}_4, \qquad [\mathbf{X}_1, \mathbf{X}_6] = \mathbf{X}_2,$$

the commutator relations between the infinitesimal generators is shown in Table (3.1). Next

Commutator Table						
$\mathbf{X}_i$	$\mathbf{X}_1$	$\mathbf{X}_2$	$\mathbf{X}_3$	$\mathbf{X}_4$	$\mathbf{X}_5$	$\mathbf{X}_6$
$\mathbf{X}_1$	0	0	0	0	$\mathbf{X}_3 - \mathbf{X}_4$	$\mathbf{X}_1$
$\mathbf{X}_2$	0	0	0	0	$\mathbf{X}_1$	$2\mathbf{X}_2$
$\mathbf{X}_3$	0	0	0	0	0	0
$\mathbf{X}_4$	0	0	0	0	0	0
$\mathbf{X}_5$	$-\mathbf{X}_3 + \mathbf{X}_4$	$-\mathbf{X}_1$	0	0	0	$-\mathbf{X}_5$
$\mathbf{X}_6$	$-\mathbf{X}_1$	$-2\mathbf{X}_2$	0	0	$\mathbf{X}_5$	0

Table 3.1: Commutator table

we will find out the adjoint representation which is given in Section (1.8). To compute the adjoint representation, we use the relation (1.72) in conjunction with the commutator Table (3.1). For instance

$$Ad(\exp(\epsilon \mathbf{X}_2))\mathbf{X}_4 = \mathbf{X}_4 - \epsilon[\mathbf{X}_2, \mathbf{X}_4] + \frac{1}{2}(\epsilon)^2[\mathbf{X}_2, [\mathbf{X}_2, \mathbf{X}_4]] - \cdots, \tag{3.133}$$
$$= \mathbf{X}_4. \tag{3.134}$$

Similarly if we take $Ad(\exp(\epsilon \mathbf{X}_5))\mathbf{X}_2$

$$Ad(\exp(\epsilon \mathbf{X}_6))\mathbf{X}_2 = \mathbf{X}_2 + \epsilon[\mathbf{X}_6, \mathbf{X}_2] + \frac{1}{2}(\epsilon)^2[\mathbf{X}_6, [\mathbf{X}_6, \mathbf{X}_2]] - \cdots, \tag{3.135}$$

from the Table (3.1) we know that

$$[\mathbf{X}_6, \mathbf{X}_2] = -\mathbf{X}_1, \quad [\mathbf{X}_6, [\mathbf{X}_6, \mathbf{X}_2]] = \mathbf{X}_3 - \mathbf{X}_4, \tag{3.136}$$

remaining all other terms in the series (3.135) are zero, so we get

$$Ad(\exp(\epsilon \mathbf{X}_2))\mathbf{X}_4 = \mathbf{X}_4 - \epsilon[\mathbf{X}_2, \mathbf{X}_4] + \frac{1}{2}(\epsilon)^2[\mathbf{X}_2, [\mathbf{X}_2, \mathbf{X}_4]], \tag{3.137}$$
$$Ad(\exp(\epsilon \mathbf{X}_2))\mathbf{X}_4 = \mathbf{X}_4 + \epsilon \mathbf{X}_1 + \frac{\epsilon^2}{2}\mathbf{X}_3 - \mathbf{X}_4. \tag{3.138}$$

In the same manner, we construct the Table (3.2) with the (i, j)-th entry indicates $Ad(\exp(\epsilon \mathbf{X}_i))\mathbf{X}_j$. Now using the symmetry algebra of $\mathfrak{g}$ of the reduced system of equations (3.7), whose adjoint representation is given in Table (3.2), we can find an optimal system [1]. Here we are

Adjoint representation						
$Ad(\mathbf{X}_i)$	$\mathbf{X}_1$	$\mathbf{X}_2$	$\mathbf{X}_3$	$\mathbf{X}_4$	$\mathbf{X}_5$	$\mathbf{X}_6$
$\mathbf{X}_1$	$\mathbf{X}_1$	$\mathbf{X}_2$	$\mathbf{X}_3$	$\mathbf{X}_4$	$\mathbf{X}_5 - \epsilon(\mathbf{X}_3 - \mathbf{X}_4)$	$\mathbf{X}_6 - \epsilon\mathbf{X}_1$
$\mathbf{X}_2$	$\mathbf{X}_1$	$\mathbf{X}_2$	$\mathbf{X}_3$	$\mathbf{X}_4$	$\mathbf{X}_5 - \epsilon\mathbf{X}_1$	$\mathbf{X}_6 - 2\epsilon\mathbf{X}_2$
$\mathbf{X}_3$	$\mathbf{X}_1$	$\mathbf{X}_2$	$\mathbf{X}_3$	$\mathbf{X}_4$	$\mathbf{X}_5$	$\mathbf{X}_6$
$\mathbf{X}_4$	$\mathbf{X}_1$	$\mathbf{X}_2$	$\mathbf{X}_3$	$\mathbf{X}_4$	$\mathbf{X}_5$	$\mathbf{X}_6$
$\mathbf{X}_5$	$\mathbf{X}_1 + \epsilon(\mathbf{X}_3 - \mathbf{X}_4)$	$\mathbf{X}_2 + \epsilon\mathbf{X}_1 + \frac{\epsilon^2}{2}(\mathbf{X}_3 - \mathbf{X}_4)$	$\mathbf{X}_3$	$\mathbf{X}_4$	$\mathbf{X}_5$	$\mathbf{X}_6 + \epsilon\mathbf{X}_5$
$\mathbf{X}_6$	$e^{\epsilon}\mathbf{X}_1$	$e^{2\epsilon}\mathbf{X}_2$	$\mathbf{X}_3$	$\mathbf{X}_4$	$e^{-\epsilon}\mathbf{X}_5$	$\mathbf{X}_6$

Table 3.2: Adjoint representation table

considering the symmetry algebra of the system (3.7) whose adjoint representation is given in Table (3.2).

Case 1:

Suppose we have a non zero vector

$$\mathbf{X} = a_1\mathbf{X}_1 + a_2\mathbf{X}_2 + a_3\mathbf{X}_3 + a_4\mathbf{X}_4 + a_5\mathbf{X}_5 + a_6\mathbf{X}_6, \tag{3.139}$$

we have to simplify as many of the coefficients a_i as possible through sensible applications of adjoint maps to $\mathbf{X}$. Consider first that $a_6 \neq 0$,if necessary scale $\mathbf{X}$, take $a_6 = 1$

$$\mathbf{X} = a_1\partial_t + a_2\partial_x + a_3\partial_\phi + a_4\partial_\psi + a_5(x\partial_t + t(\partial_\phi - \partial_\psi)) + t\partial_t + 2x\partial_x - \rho\partial_\rho - \tau\partial_\tau, \tag{3.140}$$

using Table (3.2), if we act on $\mathbf{X}$ by $Ad(\exp(-a_5\mathbf{X}_5))$, the coefficient of $\mathbf{X}_5$ vanishes

$$\begin{aligned} \mathbf{X}' =& Ad(\exp(-a_5\mathbf{X}_5))\mathbf{X}, && (3.141)\\ =& \mathbf{X} + a_5[\mathbf{X}_5, \mathbf{X}] + \frac{a_5^2}{2}[\mathbf{X}_5, [\mathbf{X}_5, \mathbf{X}]] + \cdots, && (3.142) \end{aligned}$$

to compute Lie brackets we use the relation (1.32) and we get

$$\begin{aligned} [\mathbf{X}_5, \mathbf{X}] =& [x\partial_t + t\partial_\phi - t\partial_\psi, a_1\partial_t + a_2\partial_x + a_3\partial_\phi + a_4\partial_\psi + a_5(x\partial_t + t(\partial_\phi - \partial_\psi)) + && (3.143)\\ & t\partial_t + 2x\partial_x - \rho\partial_\rho - \tau\partial_\tau], && (3.144)\\ =& x(a_5\partial_\phi + a_5\partial_\psi) + x\partial_t - a_1\partial_t - a_2\partial_\phi + a_2\partial_\psi - t\partial_\phi + t\partial_\psi - 2x\partial_t && (3.145)\\ =& -x\partial_t - t\partial_\phi + t\partial_\psi - a_1\partial_t - a_2\partial_\phi + a_2\partial_\psi, && (3.146)\\ =& -\mathbf{X}_5 - a_1\mathbf{X}_1 - a_2\mathbf{X}_3 + a_2\mathbf{X}_4. && (3.147) \end{aligned}$$

the remaining Lie bracket relation in the series (3.141) is given below

$$[\mathbf{X}_5, [\mathbf{X}_5, \mathbf{X}]] = -a_1\mathbf{X}_3 + a_1\mathbf{X}_4, \quad [\mathbf{X}_5, [\mathbf{X}_5, [\mathbf{X}_5, \mathbf{X}]]] = 0, \tag{3.148}$$

now put the relations (3.147) and (3.148) in the equation (3.141), we get

$$\begin{aligned}
\mathbf{X}' =& \mathbf{X} + a_5[\mathbf{X}_5, \mathbf{X}] + \frac{(a_5)^2}{2}[\mathbf{X}_5, [\mathbf{X}_5, \mathbf{X}]], && (3.149)\\
=& a_1\partial_t + a_2\partial_x + a_3\partial_\phi + a_4\partial_\psi + a_5(x\partial_t + t(\partial_\phi - \partial_\psi)) + t\partial_t + 2x\partial_x - \rho\partial_\rho - \\
& \tau\partial_\tau + a_5(-x\partial_t - t\partial_\phi + t\partial_\psi - a_1\partial_t - a_2\partial_\phi + a_2\partial_\psi) + \frac{(a_5)^2}{2}(-a_1\partial_\phi + a_1\partial_\psi), && (3.150)\\
=& a_1\mathbf{X}_1 + a_2\mathbf{X}_2 + a_3\mathbf{X}_3 + a_4\mathbf{X}_4 + a_5\mathbf{X}_5 + \mathbf{X}_6 - a_5\mathbf{X}_5 - a_1a_2\mathbf{X}_2 - a_2a_5(\mathbf{X}_3 - && (3.151)\\
& \mathbf{X}_4) - a_1a_5^2/2(\mathbf{X}_3 - \mathbf{X}_4), && (3.152)\\
=& a_1\mathbf{X}_1 + a_2'\mathbf{X}_2 + a_3'\mathbf{X}_3 + a_4'\mathbf{X}_4 + \mathbf{X}_6, && (3.153)
\end{aligned}$$

here $(a_1 - a_1a_5) = a_3'$, $(a_3 - a_2a_5 - a_1a_2^2/2) = a_3'$ and $(a_4 - a_2a_5 - a_1a_2^2/2) = a_4'$ so

$$\mathbf{X}' = a_1\mathbf{X}_1 + a_2'\mathbf{X}_2 + a_3'\mathbf{X}_3 + a_4'\mathbf{X}_4 + \mathbf{X}_6, \tag{3.154}$$

for certain scalars a_3', a_4' depending on a_1, a_2, a_3, a_4. Now acting on $\mathbf{X}'$ by $Ad(\exp(a_1/2)\mathbf{X}_1)$ a_2 will make the coefficient of $\mathbf{X}_2$ vanish

$$\begin{aligned}
\mathbf{X}'' =& Ad(\exp(a_2/2)\mathbf{X}_1)\mathbf{X}', && (3.155)\\
=& \mathbf{X}' - \frac{a_2}{2}[\mathbf{X}_1, \mathbf{X}'] + \frac{(a_2/2)^2}{2!}[\mathbf{X}_1, [\mathbf{X}_1, \mathbf{X}']] - \cdots, && (3.156)
\end{aligned}$$

again use the relation (1.32) to calculate $[\mathbf{X}_2, \mathbf{X}']$

$$\begin{aligned}
[\mathbf{X}_1, \mathbf{X}'] =& [\partial_t, a_1\partial_t + a_2\partial_x + a_3'\partial_\phi + a_4'\partial_\psi + t\partial_t + 2x\partial_x - \rho\partial_\rho - \tau\partial_\tau], && (3.157)\\
=& \partial_t, && (3.158)\\
=& \mathbf{X}_1, && (3.159)\\
[\mathbf{X}_1, [\mathbf{X}_1, \mathbf{X}']] =& 0, && (3.160)
\end{aligned}$$

put the values of $[\mathbf{X}_1, \mathbf{X}'], [\mathbf{X}_1, [\mathbf{X}_1, \mathbf{X}']]$ in equation (3.155) and we get $\mathbf{X}''$

$$\begin{aligned}
\mathbf{X}'' =& \mathbf{X}' - a_1\mathbf{X}_1, && (3.161)\\
=& a_1\mathbf{X}_1 + a_3'\mathbf{X}_3 + a_4'\mathbf{X}_4 + \mathbf{X}_6. && (3.162)
\end{aligned}$$

Furthermore, we will cancel the remaining coefficient of $\mathbf{X}_1$, by act on $\mathbf{X}''$ by $Ad(\exp((a_1/2)\mathbf{X}_1))$,

$$\begin{aligned}
\mathbf{X}''' =& Ad(\exp((a_1/2)\mathbf{X}_1))\mathbf{X}'', && (3.163)\\
=& \mathbf{X}'' - \frac{a_1}{2}[\mathbf{X}_1, \mathbf{X}''] + \frac{(a_1/2)^2}{2!}[\mathbf{X}_1, [\mathbf{X}_1, \mathbf{X}'']] - \cdots, && (3.164)
\end{aligned}$$

where the Lie bracket $[\mathbf{X}_1, \mathbf{X}'']$ is calculated below

$$\begin{aligned}
[\mathbf{X}_1, \mathbf{X}''] =& [\partial_x, a_1\partial_x + a_3'\partial_\phi + a_4'\partial_\psi + t\partial_t + 2x\partial_x - \rho\partial_\rho - \tau\partial_\tau], && (3.165)\\
=& 2\partial_x, && (3.166)\\
=& 2\mathbf{X}_1, && (3.167)
\end{aligned}$$

similarly $[\mathbf{X}_1, [\mathbf{X}_1, \mathbf{X}'']] = 0$. Thus by using the values of $[\mathbf{X}_1, \mathbf{X}'']$ and $[\mathbf{X}_1, [\mathbf{X}_1, \mathbf{X}'']]$ in equation (3.163), we obtain

$$\mathbf{X}''' = \mathbf{X}'' - a_1/2[\mathbf{X}_1, \mathbf{X}''], \tag{3.168}$$

$$= a_3'\mathbf{X}_3 + a_4'\mathbf{X}_4 + \mathbf{X}_6. \tag{3.169}$$

So every one dimensional subalgebra generated by $\mathbf{X}$ with $a_6 \neq 0$ is equivalent to the subalgebra spanned by $a_3'\mathbf{X}_3 + a_4'\mathbf{X}_4 + \mathbf{X}_6$.

Case 2:
The remaining one dimensional subalgebras are spanned by vectors of the above form with $a_6 = 0$. Now take $a_5 \neq 0$

$$\mathbf{X} = a_1\mathbf{X}_1 + a_2\mathbf{X}_2 + a_3\mathbf{X}_3 + a_4\mathbf{X}_4 + a_5\mathbf{X}_5, \tag{3.170}$$

so we scale to make $a_5 = 1$

$$\mathbf{X} = a_1\partial_t + a_2\partial_x + a_3\partial_\phi + a_4\partial_\psi + x\partial_t + t(\partial_\phi - \partial_\psi), \tag{3.171}$$

next act on $\mathbf{X}$ by $Ad(\exp(a_1\mathbf{X}_2))$, so that the coefficient of $\mathbf{X}_1$ will vanish

$$\mathbf{X}' = Ad(\exp(a_1\mathbf{X}_2)\mathbf{X}, \tag{3.172}$$

$$= \mathbf{X} - a_1[\mathbf{X}_2, \mathbf{X}] + \frac{a_1^2}{2}[\mathbf{X}_2, [\mathbf{X}_2, \mathbf{X}]] + \cdots, \tag{3.173}$$

calculate the commutator relations $[\mathbf{X}_2, \mathbf{X}]$, $[\mathbf{X}_2, [\mathbf{X}_2, \mathbf{X}]]$ by using (1.32)

$$[\mathbf{X}_2, \mathbf{X}] = [\partial_x, a_1\partial_t + a_2\partial_x + a_3\partial_\phi + a_4\partial_\psi + x\partial_t + t(\partial_\phi - \partial_\psi, \tag{3.174}$$

$$= \partial_x, \tag{3.175}$$

$$= \mathbf{X}_1, \tag{3.176}$$

$$[\mathbf{X}_2, [\mathbf{X}_2, \mathbf{X}]] = 0, \tag{3.177}$$

substitute values of $[\mathbf{X}_2, \mathbf{X}]$, $[\mathbf{X}_2, [\mathbf{X}_2, \mathbf{X}]]$ in equation (3.172), thus we obtain

$$\mathbf{X}' = \mathbf{X} - a_1[\mathbf{X}_2, \mathbf{X}], \tag{3.178}$$

$$= a_1\partial_t + a_2\partial_x + a_3\partial_\phi + a_4\partial_\psi + x\partial_t + t(\partial_\phi - \partial_\psi) - a_1\partial_t, \tag{3.179}$$

$$= a_2\partial_x + a_3\partial_\phi + a_4\partial_\psi + x\partial_t + t(\partial_\phi - \partial_\psi), \tag{3.180}$$

$$= a_2\mathbf{X}_2 + a_3\mathbf{X}_3 + a_4\mathbf{X}_4 + \mathbf{X}_5, \tag{3.181}$$

where $\mathbf{X}$ is equivalent to $\mathbf{X}'$. We can further act on $\mathbf{X}'$ by $Ad(\exp(a_3\mathbf{X}_1))$to vanish the coefficient of $\mathbf{X}_3$

$$\mathbf{X}'' = Ad(\exp(a_3\mathbf{X}_1))\mathbf{X}', \tag{3.182}$$

$$= \mathbf{X}' - a_3[\mathbf{X}_1, \mathbf{X}'] + \frac{(a_3)^2}{2}[\mathbf{X}_1, [\mathbf{X}_1, \mathbf{X}']] + \cdots, \tag{3.183}$$

further calculate Lie brackets take place in equation (3.182) which is given below

$$[\mathbf{X}_1, \mathbf{X}'] =[\partial_t, a_2\partial_t + a_3\partial_\phi + a_4\partial_\psi + x\partial_t + t(\partial_\phi - \partial_\psi)], \tag{3.184}$$
$$=\partial_\phi - \partial_\psi, \tag{3.185}$$
$$=\mathbf{X}_3 - \mathbf{X}_4, \tag{3.186}$$

where $[\mathbf{X}_1, [\mathbf{X}_1, \mathbf{X}]] = 0$. Therefore relation (3.182) gives us

$$\mathbf{X}'' =\mathbf{X}' - a_3[\mathbf{X}_1, \mathbf{X}'], \tag{3.187}$$
$$=a_2\partial_x + a_3\partial_\phi + a_4\partial_\psi + x\partial_t + t(\partial_\phi - \partial_\psi) - a_3\partial_\phi + a_3\partial_\psi, \tag{3.188}$$
$$=a_2\partial_x + a_4'\partial_\psi + x\partial_t + t(\partial_\phi - \partial_\psi), \tag{3.189}$$

where $a_4' = (a_4 - a_3)$. So $\mathbf{X}_5 + a_2\mathbf{X}_2 + a_4'\mathbf{X}_4$ is the optimal subalgebra.

Case 3:
Next moving to column three and four of the Table (3.2), we see that there is no linear combination of any symmetry generator, so moving forward to column two of the (3.2), we have a combination of $\mathbf{X}_2 + \epsilon\mathbf{X}_1 + \epsilon^2/2(\mathbf{X}_3 - \mathbf{X}_4)$. Therefore we take $a_2 \neq 0$, so we assume $a_2 = 1$

$$\mathbf{X} = a_1\mathbf{X}_1 + a_2\mathbf{X}_2 + a_3\mathbf{X}_3 + a_4\mathbf{X}_4, \tag{3.190}$$

now act on $\mathbf{X}$ by $Ad(\exp((-a_1/a_2)\mathbf{X}_5))$, so that the coefficient of $\mathbf{X}_1$ will vanish and we obtain $\mathbf{X}_2 + a_3'\mathbf{X}_3 + a_4'\mathbf{X}_4$. Similarly from column one of Table (3.2), we get $\mathbf{X}_1 + \zeta\mathbf{X}_3$. The complete set of optimal system of one dimensional subalgebras to be those spanned by

$$\mathbf{X}_6 + \zeta\mathbf{X}_3 + \delta\mathbf{X}_4, \quad \zeta = 0, \delta = 0, \quad \zeta, \delta \in \mathbb{R}, \tag{3.191}$$
$$\mathbf{X}_5 + \zeta\mathbf{X}_2 + \delta\mathbf{X}_4, \quad \zeta = 0, \pm1, \quad \delta = 0, \pm1 \tag{3.192}$$
$$\mathbf{X}_2 + \zeta\mathbf{X}_3 + \delta\mathbf{X}_4, \quad \zeta = 0, \quad \zeta, \delta = \pm1, \tag{3.193}$$
$$\mathbf{X}_1 + \zeta\mathbf{X}_2 + \delta\mathbf{X}_3, \quad \zeta = 0, \pm1, \quad \delta = 0, \pm1. \tag{3.194}$$

Now we can use the optimal subalgebra to get exact solution of the system (3.126). Here we use $\mathbf{X}_5 + \mathbf{X}_4$, the invariants are

$$r = x, \quad M_1(r) = \rho, \quad M_2(r) = \phi - \frac{t^2}{2x}, \tag{3.195}$$
$$M_3(r) = \tau, \quad M_4(r) = \psi + \frac{t^2}{2x} - \frac{t}{x}, \tag{3.196}$$

by using the above similarity variables the CNSEs (3.126) reduces to a system of ODEs

$$2rM_1' + M_1 = 0, \tag{3.197}$$
$$2rM_3' + M_3 = 0, \tag{3.198}$$
$$M_1' - M_1^2 - 2M_3^2 = 0, \tag{3.199}$$
$$M_4' - M_1^2 - 2M_3^2 + \frac{1}{2r^2} = 0, \tag{3.200}$$

by solving the above system we get the following solution

$$M_1 = \sqrt{\frac{c_1}{r}}, \quad M_3 = (c_1 + 2c_2)\ln(r) + c_4, \tag{3.201}$$

$$M_2 = \sqrt{\frac{c_2}{r}}, \quad M_4 = (c_1 + 2c_2)\ln(r) - \frac{1}{r} + c_3, \tag{3.202}$$

by putting the above values in relation (3.195) we get

$$\rho = \sqrt{\frac{c_1}{x}}, \quad \phi = (c_1 + 2c_2)\ln(x) + c_4 + \frac{t^2}{2x}, \tag{3.203}$$

$$\tau = \sqrt{\frac{c_2}{x}}, \quad \psi = (c_1 + 2c_2)\ln(x) - \frac{(t-1)^2}{2x}, \tag{3.204}$$

so the exact solution of the CNSEs (3.126) is

$$u = \sqrt{\frac{c_1}{x}} e^{i((c_1+2c_2)\ln(x)+c_4+\frac{t^2}{2x})},$$

$$v = \sqrt{\frac{c_2}{x}} e^{i(c_1+2c_2)\ln(x)-\frac{(t-1)^2}{2x})}, \tag{3.205}$$

from the above solution we see that the amplitude of the waves decay as x increases, where we see that phase of these waves have nontrivial dependence on x and t.
In next chapter we will apply the group foliation approach to get solutions of Manakov system.

Chapter 4

Manakov System: Exact Solution by Group Foliation

4.1 Introduction

In this chapter we will construct the group foliation technique by using admitted finite dimensional Lie point symmetry group for PDEs. In finite dimensional symmetry group, i.e., $dim\ G < \infty$, S. C. Anco and S. Liu [27], S. C. Anco and S. Ali and T. Wolf [28], S. C. Anco and W. Feng and T. Wolf [29] made a lot of contributions. Where in infinite dimensional symmetry group, i.e., $dim\ G = \infty$, Y. Nutku, M.B. Sheftel [10], P. Winternitz [14] successfully developed the group foliation approach.

Generally a group foliation technique can transform a given nonlinear PDE into an equivalent system of PDEs. This system is so called group-resolving equations, where the independent variables consist of invariants and the dependent variables are differential invariants of a given one dimensional group of point transformations. As the group-resolving equations are obtained from the invariants and differential invariants so the solutions obtained from these equations are invariant. The splendid thing about foliation is, it gives us all similarity as well as non similarity solutions. Each solution of the group-resolving equations geometrically corresponds to an explicit one-parameter family of exact solutions of the original nonlinear PDE, such that the family is closed under the given one dimensional symmetry group acting in the solution space of the PDE.

Basically Group foliation is a geometrical generalization of symmetry reduction. In group foliation method, we can foliate the solution space of the equations into orbits, by choosing a finite-dimensional symmetry group of the foliation. We can determine each orbit by the automorphic system coupled to the original equations. The automorphic equations are obtained from the invariants and differential invariants. The automorphic property states that by an appropriate symmetry transformation each solution can be obtained from any other solution. The symmetry reduction is the standard method for obtaining exact solutions of PDEs in symmetry group analysis. This method can only give invariant solutions, while the drawback of this method is that many invariant solutions and noninvariant solutions are lost. In the applications of physics noninvariant solutions are very important. Where the similarity solutions provide insight into asymptotic behavior, critical dynamic and blow-up behavior. Now we restate group foliation approach in simple algorithmic steps.

4.2 Symmetries and Group Foliation

Following are the algorithmic steps of group foliation technique:

1. Find all Lie piont symmetries of PDE.
2. Find functionally independent invariants and differential invariants.
   (New independent and dependent variables.)
3. Construct group resolving equations and solve.
   (System of first order PDEs.)
4. Solve parametric ODEs and get solutions.

These are the four basic steps. Now we can construct group foliation method for a class of semi-linear reaction diffusion equations

$$u_t = u_{rr} + mr^{-1}u_r + (p - ku^q)u, \quad k = constant \neq 0, \quad p = constant, \tag{4.1}$$

where u depends on t and r. In equation (4.1), q is nonlinear power and it is neither 0 nor -1. Where m is the coefficient of spatial derivative. The number of independent and dependent variables in equation (4.1) are two and one respectively. The first task is to find symmetry structure of this group of equations. The infinitesimal generator for this class of equations is

$$\mathbf{X} = \varrho^1(t, r, u)\partial_t + \varrho^2(t, r, u)\partial_x + \eta(t, r, u)\partial_u, \tag{4.2}$$

The prolongation of infinitesimal equation is given in Section (1.4). Now we apply the Therorem (2.1.1) on equation (4.1), thus obtain the following three cases

Case 1:

If we select all values of m, q and p, following are the determining equations

$$\varrho^1_r = 0, \quad \varrho^1_t = 0, \quad \varrho^1_u = 0, \tag{4.3}$$

$$\varrho^2 = 0, \quad \eta = 0, \tag{4.4}$$

in above relations it is seen that η and ϱ^2 is zero and ϱ^1 is not function of t,r and u, so it is a constant. Thus the Infinitesimal generator is

$$\mathbf{X} = \partial_t. \tag{4.5}$$

Case 2:

For $m = 0$, the equation(4.1) becomes

$$u_t = u_{rr} + (p - ku^q)u. \tag{4.6}$$

The determining equations of (4.6) are

$$\varrho^1_r = 0, \quad \varrho^1_t = 0, \quad \varrho^1_u = 0, \tag{4.7}$$
$$\varrho^2_r = 0, \quad \varrho^2_t = 0, \quad \varrho^2_u = 0, \tag{4.8}$$
$$\eta = 0, \tag{4.9}$$

we use MAPLE to get determining equations. Now by simple observation we can say that both ϱ^1 and ϱ^2 are not function of t, r and u, they are arbitrary constants. Where ϕ is zero, so there arise two generators

$$\mathbf{X}_1 = \partial_t, \quad \mathbf{X}_2 = \partial_r. \tag{4.10}$$

Case 3:

If we have $p = 0$ only, the equation (4.1), gets the form

$$u_t = u_{rr} + mr^{-1}u_r - ku^{q+1}. \tag{4.11}$$

Similarly, we get the following generators

$$\mathbf{X}_1 = \partial_t, \quad \mathbf{X}_2 = 2t\partial_t + r\partial_r - (2/q)\partial_u, \tag{4.12}$$

moreover, now we explain how the group resolving equations are constructed in detail.

4.2.1 Group resolving equations and Integration

If we look at the infinitesimal generators of the reaction diffusion equation, i.e., $\mathbf{X}_i,\ i = 1, 2$, the time translation generator is obtained in all cases. While it is natural to use time translation infinitesimal generator for the construction of group foliation. Now by using time translation symmetry

$$\mathbf{X} = \partial_t, \tag{4.13}$$

we can set up the system of group resolving equations for the reaction diffusion equation (4.1). Therefore we first obtain the invariants of the infinitesimal generator (4.13) in terms of t,r and u

$$\mathbf{X}\varphi = 0, \quad where \quad \varphi(t, r, u), \tag{4.14}$$
$$\varphi_t = 0, \tag{4.15}$$

by solving the first order PDE (4.15) by characteristic method. We get two functional independent invariants

$$r, \quad u, \tag{4.16}$$

and by giving them new names

$$x = r, \quad v = u, \tag{4.17}$$

we get new independent variables x, v, also the new independent variables satisfy the following condition

$$\mathbf{X}_{trans} x = \mathbf{X}_{trans} v = 0. \tag{4.18}$$

To find the differential invariants, we have to prolong our infinitesimal generator (4.13) up to order one, i.e., $\mathbb{P}r^1$. The infinitesimal generator (4.13) is in its simple form, so

$$\mathbb{P}r^1 \mathbf{X} = \mathbf{X}, \tag{4.19}$$

to determine the differential invariants we use the following condition

$$\mathbb{P}r^{(1)} \mathbf{X}_{trans} \Xi = 0, \tag{4.20}$$

here $\Xi(t, r, u, u_t, u_r)$. The differential invariants for the infinitesimal generator (4.13) are

$$u_t, \quad u_r, \tag{4.21}$$

by giving these differential invariants new names we obtain new dependent variables that are

$$G = u_t, \quad H = u_r, \tag{4.22}$$

these differential invariants satify the condition

$$\mathbb{P}r^{(1)} \mathbf{X}_{trans} G = \mathbb{P}r^{(1)} \mathbf{X}_{trans} H = 0. \tag{4.23}$$

Also x and v are independent while G, H are related by equality of mixed derivatives (r, t) on u_t and u_r, which gives

$$D_t H = D_r G, \tag{4.24}$$

since D_t and D_r are total derivative operator with respect to t and r, which are

$$\begin{aligned} D_t &= \partial_t + u_t \partial_t, \\ D_r &= \partial_r + u_r \partial_r, \end{aligned} \tag{4.25}$$

where x and v are our new independent variables, also G and H are our new dependent variables which are functions of x and v. Now we substitute $G = G(x, v)$ and $H = H(x, v)$, so the relation (4.24) becomes

$$G H_v = G_x + H G_v. \tag{4.26}$$

The reaction-diffusion equation (4.1) is related by v, G and H

$$G = D_r H + m x^{-1} H + (p - k v^q) v, \tag{4.27}$$

we will reach at a system of first-order PDE, by using the relation (4.17), (4.22) into equation (4.24 and (4.27) combined with the chain rule

$$G_x + HG_v - GH_v = 0, \tag{4.28}$$

$$G - H_x - HH_v - \frac{mH}{x} = (p - kv^q)v, \tag{4.29}$$

This system of first-order PDE are so called *time translation group resolving system* for the reaction diffusion equation. The solution spaces of the system (4.28),(4.29) and equation (4.1) are related by a group invariant mapping that is defined through the invariants and differential invariants. The group resolving system is consistent to the original reaction diffusion equation in a way the degree of freedom of both are the same. The number of independent and dependent variable in reaction diffusion equation is three and there is only one equation so the degree of freedom is two. Where in group resolving system the number of independent and dependent variables are four and the number of equations are two so the degree of freedom is again two.

Now we have to find the exact solutions of the system of nonlinear PDEs (4.28),(4.29) for $(G(x,v), H(x,v))$, where it is well suited to solve it by separation of variable ansatz. First the terms $(p - ku^q)u$ in reaction diffusion equation (4.1) taking part only as an inhomogeneous term in equation (4.29). Although in system of PDEs (4.28) and (4.29), there are no derivatives involved with respect to v into the linear terms, i.e., G, G_x, H_x and mH_x/x. By careful observance, in the homogeneous equation (4.28), the nonlinear terms have skew symmetric form which is $HG_v - GH_v$. Where in the non homogeneous equation (4.29), the nonlinear term has the symmetric form $HH_v = (H^2/2)_v$. On the base of these properties, the group resolving system have solutions of the form

$$G = g_1(x)v + g_2(x)v^a, \quad H = h_1(x)v + h_2(x)v^a, \tag{4.30}$$

where a is not equal to zero. The ansatz (4.30) is *separation ansatz.* In the equations (4.28) and (4.29), the linear terms which are $G, G_x, H/x$ and H_x will contain the same powers v^a, v that appears both in G and H. Besides in homogeneous equation (4.28), due to identities the nonlinear terms $HG_v - GH_v$ will only produce the power v^a. Therefore the equation can be satisfied by having coefficients of v^a and v separately vanish. Similarly in the non homogeneous equation (4.29), the nonlinear term HH_v will induce the powers v^{2a-1}, v^a and v. As we know that $q \neq 0$ and $a \neq 0$, so the coefficients v^a and v of equation(4.29) be satisfied by separately vanish them in homogeneous term kv^{q+1} will be balance if we have the term containing v^{2a-1}. Therefore by comparing the powers of v, v^a and v^{2a-1}, we obtain following over determined system of five algebraic differential equations in the course of $g_1(x), g_2(x), h_1(x)$ and $h_2(x)$. By using the relation $a = (1 + \frac{q}{2})$, we can solve the system by careful analysis.

$$
\begin{aligned}
&g_1' = 0,\\
&h_1 g_2 a - h_2 g_1 a - h_1 g_2 + h_2 g_1 + g_2' = 0,\\
&h_2^2 a - k = 0,\\
&h_1 h_2 a + h_2 h_1 + h_2' + h_2 m - g_2 = 0,\\
&h_1^2 + h_1' + g_1 + p - m h_1 = 0.
\end{aligned} \tag{4.31}
$$

These over determined equations are reduced form of equation (4.28) and (4.29).
If we have $q \neq 0$, then the separation ansatz (4.30) gives us to the solutions of translation group resolving system (4.28) and (4.29).

Case 1:

If we have $m = 0$, $q = 2$, $p = 0$ and $k > 0$, then the system (4.31) gets the form

$$g_1' = 0, \tag{4.32}$$
$$h_1 g_2 - h_2 g_1 + g_2' = 0, \tag{4.33}$$
$$2h_2^2 - k = 0, \tag{4.34}$$
$$3h_1 h_2 + h_2' - g_2 = 0, \tag{4.35}$$
$$h_1^2 + h_1' + g_1 = 0, \tag{4.36}$$

from equation (4.34), we obtain $h_2 = \pm\sqrt{\frac{k}{2}}$. Put the value of h_2 in remaining equations and by applying simple integration techniques, we obtain

$$g_1 = 0, \qquad h_1 = \frac{1}{x + c_1}, \qquad g_2 = \pm\sqrt{\frac{k}{2}}\left(\frac{1}{x + c_1}\right), \tag{4.37}$$

Now substitute the values of $g_1(x), g_2(x), h_1(x)$ and $h_2(x)$ in (4.30), as a result we obtain the following solution of time translation group resolving equations

$$G(x, v) = \pm\sqrt{\frac{k}{2}}\left(\frac{1}{x + c}\right) v^2, \tag{4.38}$$

$$H(x, v) = \frac{v}{x + c} \pm \sqrt{\frac{k}{2}} v^2, \tag{4.39}$$

where c is arbitrary constant. We get two parametric ODEs by replacing G, H in equations (4.38) and (4.39)

$$u_t = \pm\sqrt{\frac{k}{2}}\left(\frac{1}{r + c}\right) u^2, \tag{4.40}$$

$$u_r = \frac{u}{r + c} \pm \sqrt{\frac{k}{2}} u^2, \tag{4.41}$$

the solution of the parametric ODEs give us

$$u(t,x) = \left(\pm\left(\sqrt{\frac{k}{2}}\left(\frac{r+c}{2} - \frac{(t+c_1)}{(r+c)}\right)\right)\right)^{-1}, \tag{4.42}$$

which is exact solution of the reaction diffusion equation (4.1).

Case 2:

If we have $m = 0$, $q = 2$, $p > 0$ and $k > 0$, then we have the following system of ODEs

$$g_1' = 0, \tag{4.43}$$

$$h_1 g_2 - h_2 g_1 + g_2' = 0, \tag{4.44}$$

$$2h_2^2 - k = 0, \tag{4.45}$$

$$3h_1 h_2 + h_2' - g_2 = 0, \tag{4.46}$$

$$h_1^2 + h_1' + g_1 + p = 0, \tag{4.47}$$

the equation (4.45) easily gives us value of $h_2 = \pm\sqrt{\frac{k}{2}}$, by substituting the value of h_2 in remaining equations, we obtain over determine equations and by simple applying integration techniques, we can get

$$g_1(x) = -(3/2)p, \quad h_1 = \sqrt{\frac{p}{2}}\tanh\left(\sqrt{\frac{p}{2}}(x+c)\right), \quad g_2 = \pm\frac{3}{2}\sqrt{pk}\tanh\left(\sqrt{\frac{p}{2}}(x+c)\right), \tag{4.48}$$

next by substitute values of h_1, h_2, g_1 and g_2 in relation (4.30), we obtain

$$G(x,v) = -\frac{3}{2}pv \pm \frac{3}{2}\sqrt{pk}\tanh\left(\sqrt{\frac{p}{2}}(x+c)\right)v^2,$$

$$H(x,v) = \sqrt{\frac{p}{2}}\tanh\left(\sqrt{\frac{p}{2}}(x+c)\right)v \pm \sqrt{\frac{k}{2}}v^2, \tag{4.49}$$

which are solutions of time translation group resolving equations. In order to get parametric ODEs we replace $G(x,v)$, $H(x,v)$ in above equations by the differential invariants as mention in equation (4.22), therefore we obtain

$$u_t = -\frac{3}{2}pu \pm \frac{3}{2}\sqrt{pk}\tanh\left(\sqrt{\frac{p}{2}}(r+c)\right)u^2, \tag{4.50}$$

$$u_r = \sqrt{\frac{p}{2}}\tanh\left(\sqrt{\frac{p}{2}}(x+c)\right)u \pm \sqrt{\frac{k}{2}}u^2, \tag{4.51}$$

thus we obtain the following exact solution of reaction diffusion equation by solving above parametric equations

$$u(t,r) = \frac{\sinh(\sqrt{\frac{p}{2}}(r+c_1))}{(\sqrt{\frac{p}{k}})(\cosh(\sqrt{\frac{p}{2}}(r+c_1))) \pm \exp\left(\frac{-3p(t+c_1)}{2}\right)}, \tag{4.52}$$

where c, c_1 are arbitrary constant.

4.3 Group Foliation of Manakov system

In this section we construct the group foliation approach to reduce Manakov system

$$
\begin{aligned}
&\rho_{xx} - \rho\phi_t - \rho\phi_x^2 + 2(\sigma_1\rho^2 + \sigma_2\tau^2)\rho = 0,\\
&\tau_{xx} - \tau\psi_t - \tau\psi_x^2 + 2(\sigma_1\rho^2 + \sigma_2\tau^2)\tau = 0,\\
&\rho_t + 2\rho_x\phi_x + \rho\phi_{xx} = 0,\\
&\tau_t + 2\tau_x\psi_x + \tau\psi_{xx} = 0.
\end{aligned} \tag{4.53}
$$

This reduced system inherits eight Lie point symmetries which are given in (3.47). We use time translation symmetry

$$\mathbf{X}_{trans} = \partial_t, \tag{4.54}$$

to construct group resolving system. For this purpose, we shall find the functionally independent invariants of the corresponding infinitesimal generator, which are obtained by using the condition

$$\mathbf{X}_{trans}\varphi = 0,$$

where φ is function of $(t, x, \rho, \tau, \phi, \psi)$, which leads us to first order PDE

$$\varphi_t = 0, \tag{4.55}$$

by solving above PDE, we obtain the invariants of the corresponding infinitesimal generator

$$x, \qquad \rho, \qquad \tau, \qquad \phi, \qquad \psi. \tag{4.56}$$

For the construction of group resolving system, we also need differential invariants. So we use the following condition

$$\mathbb{P}r^{(1)}\mathbf{X}_{trans}\Theta = 0, \tag{4.57}$$

here $\Theta(t, x, \rho, \tau, \phi, \psi, \rho_t, \rho_x, \tau_t, \tau_x, \phi_t, \phi_x, \psi_t, \psi_x)$. As our infinitesimal generator is in its normal form, so the first order prolongation $\mathbb{P}r^{(1)}\mathbf{X}_{trans} = \mathbf{X}_{trans}$, therefore we obtain the following differential invariants

$$\rho_t, \qquad \rho_x, \qquad \tau_t, \qquad \tau_x, \tag{4.58}$$

$$\phi_t, \qquad \phi_x, \qquad \psi_t, \qquad \psi_x, \tag{4.59}$$

by using (4.56), we obtain the following group foliation variables

$$r = x, \quad s = \rho, \tag{4.60}$$

where r and s are mutually independent, also they are our new independent variables. The group foliation variables r and s satisfy the condition

$$\mathbf{X}_{trans}r = \mathbf{X}_{trans}s = 0. \tag{4.61}$$

We will give the differential invariants new name G, H, J, K, L, M, P and Q, also these variables are our new dependent variables for group resolving equations

$$G = \rho_t, \quad H = \rho_x, \quad J = \tau_t, \quad K = \tau_x, \tag{4.62}$$
$$L = \phi_t, \quad M = \phi_x, \quad P = \psi_t, \quad Q = \psi_x, \tag{4.63}$$

these variables satisfy the condition

$$\mathbb{P}r^{(1)}\mathbf{X}_{trans}G = \mathbb{P}r^{(1)}\mathbf{X}_{trans}H = \mathbb{P}r^{(1)}\mathbf{X}_{trans}J = \mathbb{P}r^{(1)}\mathbf{X}_{trans}K = 0,$$
$$\mathbb{P}r^{(1)}\mathbf{X}_{trans}L = \mathbb{P}r^{(1)}\mathbf{X}_{trans}M = \mathbb{P}r^{(1)}\mathbf{X}_{trans}P = \mathbb{P}r^{(1)}\mathbf{X}_{trans}Q = 0, \tag{4.64}$$

where G, H, J, K, L, M, P and Q are related by equality of mixed derivatives of t, x, on ρ_t, ρ_x, τ_t, τ_x, ϕ_t, ϕ_x, ψ_t and ψ_x, from integrability condition we know that

$$\rho_{tx} = \rho_{xt}, \quad \tau_{tx} = \tau_{xt}, \quad \phi_{tx} = \phi_{xt}, \quad \psi_{tx} = \psi_{xt}, \tag{4.65}$$

which gives

$$D_xG = D_tH, \tag{4.66}$$
$$D_xJ = D_tK, \tag{4.67}$$
$$D_xL = D_tM, \tag{4.68}$$
$$D_xP = D_tQ, \tag{4.69}$$

where $(\rho_t, \rho_x) = (G, H)$, $(\tau_t, \tau_x) = (J, K)$, $(\phi_t, \phi_x) = (L, M)$ and $(\psi_t, \psi_x) = (P, Q)$. Here D_t, D_x are total derivative w.r.t t, x, which are given as

$$D_t = \partial_t + \rho_t\partial_\rho + \tau_t\partial_\tau + \phi_t\partial_\phi + \psi_t\partial_\psi, \tag{4.70}$$
$$D_x = \partial_x + \rho_x\partial_\rho + \tau_t\partial_\tau + \phi_t\partial_\phi + \psi_t\partial_\psi. \tag{4.71}$$

As we discuss in Section (1.8) that in group foliation, we will enlarge the space of independent and dependent variables. So in the sense

$$(t, x) \quad \rightarrow \quad (r, s), \tag{4.72}$$
$$(\rho, \tau) \quad \rightarrow \quad (G, H, J, K), \tag{4.73}$$
$$(\phi, \psi) \quad \rightarrow \quad (L, M, P, Q), \tag{4.74}$$

our space of independent and dependent variables are enlarged. Initially we have two independent and four dependent variables, after using group foliation variables we got two independent and eight dependent variables. Now we put $G(r, s), H(r, s), J(r, s), K(r, s), L(r, s)$, $M(r, s), P(r, s)$ and $Q(r, s)$ into equations (4.66), (4.67), (4.68), (4.69) and use the relation (4.60), combined by chain rule, we get four first order PDEs.

$$G_r + HG_s - GH_s = 0,$$
$$J_r + HJ_s - GK_s = 0,$$
$$L_r + HL_s - GM_s = 0,$$
$$P_r + HP_s - GQ_s = 0. \tag{4.75}$$

Furthermore the new variables $r, s, G, H, J, K, L, M, P$ and Q are related through the reduced system (4.53) by

$$\begin{aligned}
&D_r(H) - sL - sM^2 + 2s^3 + 2s\tau^2 = 0,\\
&D_r(K) - \tau P - \tau Q^2 + 2s^2\tau - 2\tau^3 = 0,\\
&G + 2HM + sD_r(M) = 0,\\
&L + 2KQ + \tau D_r(Q) = 0,
\end{aligned} \tag{4.76}$$

by applying total derivative, we arrive at a system of first order PDEs

$$G_r + HG_s - GH_s = 0, \tag{4.77}$$
$$J_r + HJ_s - GK_s = 0, \tag{4.78}$$
$$L_r + HL_s - GM_s = 0, \tag{4.79}$$
$$P_r + HP_s - GQ_s = 0, \tag{4.80}$$
$$H_r + HH_s - sL - sM^2 + 2s^3 + 2s\tau^2 = 0, \tag{4.81}$$
$$K_r + HK_s - \tau P - \tau Q^2 + 2s^2\tau - 2\tau^3 = 0, \tag{4.82}$$
$$G + 2HM + sM_r + sHM_s = 0, \tag{4.83}$$
$$L + 2KQ + \tau Q_r + \tau HQ_s = 0. \tag{4.84}$$

These differential equations are called time translation group resolving equation, with r, s are independent variables and G, H, J, K, L, M, P, Q are the dependent variables. In equations (4.81), (4.82), we see that the problematic nonlinear terms of the reduced system (4.53) becomes linear, also the second order system is transformed into first order. The time translation group resolving system inherits two symmetries

$$\mathbf{X}_1 = \partial_r, \qquad \mathbf{X}_2 = -2Q\partial_P + \partial_Q. \tag{4.85}$$

The solution of Manakov system consist of amplitudes and phases of waves which are obtained in the form ρ, τ and ϕ, ψ respectively, so there arises different cases. If we take amplitudes of the wave as constant and check the behavior of the phases. So for this purpose we treat τ as constant, therefore $J = 0$, $K = 0$. Then the equations (4.77)- (4.84), becomes

$$G_r + HG_s - GH_s = 0, \tag{4.86}$$
$$L_r + HL_s - GM_s = 0, \tag{4.87}$$
$$P_r + HP_s - GQ_s = 0, \tag{4.88}$$
$$H_r + HH_s - sL - sM^2 + 2s^3 + 2sc_1^2 = 0, \tag{4.89}$$
$$-c_1P - c_1Q^2 + 2s^2c_1 - 2c_1^3 = 0, \tag{4.90}$$
$$G + 2HM + sM_r + sHM_s = 0, \tag{4.91}$$
$$L + Q_r + c_1HQ_s = 0. \tag{4.92}$$

Equation (4.90), is an algebraic equation from which we obtain P, where it is some function of Q. Now substitute $P = Q^2 - 2s^2 + 2c_1^2$ in equations (4.86) - (4.92), we obtain

$$G_r + HG_s - GH_s = 0, \tag{4.93}$$
$$L_r + HL_s - GM_s = 0, \tag{4.94}$$
$$4sH - 2QQ_sH - 2QQ_r - GQ_s = 0, \tag{4.95}$$
$$H_r + HH_s - sL - sM^2 + 2s^3 + 2sc_1^2 = 0, \tag{4.96}$$
$$G + 2HM + sM_r + sHM_s = 0, \tag{4.97}$$
$$L + Q_r + c_1HQ_s = 0. \tag{4.98}$$

Case 1:

Now our next task is to find out the exact solution of the above system of nonlinear PDEs, it is suitable to use a simple separation of variable ansatz. The linear terms in equations (4.93), (4.94), (4.95), (4.96), (4.91),and (4.92) contain derivative w.r.t r only, i.e., $G_r, L_r, H_r, M_r, Q_r, Q, G$ and L. The nonlinear term $HG_s - GH_s$ in equation (4.86) shows that G and H have same behavior. So by investigating these properties, we suggest that the group resolving system have the solution of the form

$$G = sg_1(r) + g_2(r), \quad H = sh_1(r) + h_2(r), \tag{4.99}$$
$$L = sl_1(r) + l_2(r), \quad M = sm_1(r) + m_2(r), \tag{4.100}$$
$$Q = sq_1(r) + q_2(r). \tag{4.101}$$

The terms $G_r, L_r, H_r, M_r,$ and Q_r contain s^2, s, where the terms Q, G, L and M contain s^3, s^2, s. where the nonlinear term $HG_s - GH_s$ contain s^2, s, s^0, the nonlinear terms in equations (4.94), (4.95) and (4.96) also contains s^2, s, s^0. Therefore by comparing the powers of s^3, s^2, s, s^0, we obtain over determined system of algebraic differential equations

$$h_1g_2 - h_2g_1 - g_2' = 0, \tag{4.102}$$
$$g_1' = 0, \tag{4.103}$$
$$m_1g_2 - l_2' - l_1h_2 = 0, \tag{4.104}$$
$$l_1' + l_1h_1 - m_1g_1 = 0, \tag{4.105}$$
$$2q_1q_2h_2 + q_1g_2 + 2q_2q_2' = 0, \tag{4.106}$$
$$2q_2q_1' + 2q_1^2h_2 + 2q_1q_2h_1 - 4h_2 + 2q_1q_2' + q_1g_1 = 0, \tag{4.107}$$
$$2q_1^2h_1 - 4h_1 + 2q_1q_1' = 0, \tag{4.108}$$
$$h_1h_2 + h_2' = 0, \tag{4.109}$$
$$h_1^2 + h_1' - l_2 + 2c_1^2 - m_2^2 = 0, \tag{4.110}$$
$$m_1^2 - 2 = 0, \tag{4.111}$$
$$l_1 + 2m_2m_1 = 0, \tag{4.112}$$
$$2m_2h_2 + g_2 = 0, \tag{4.113}$$

$$m_2' + 2m_2h_1 + g_1 + 3m_1h_2 = 0, \tag{4.114}$$
$$3m_1h_1 + m_1' = 0, \tag{4.115}$$
$$2q_2 + q_1h_2 + q_2' = 0, \tag{4.116}$$
$$2q_1 + q_1' + q_1h_1 = 0. \tag{4.117}$$

By solving these equations we get

$$\begin{aligned}
&g_1(r) = 0, \qquad g_2(r) = 0,\\
&h_1(r) = 0, \qquad h_2(r) = 0,\\
&l_1(r) = -2c_2c_3, \qquad l_2(r) = 2c_1^2 - c_2^2,\\
&m_1(r) = c_3, \qquad m_2(r) = c_2,\\
&q_1(r) = 0, q_2(r) = 0.
\end{aligned} \tag{4.118}$$

Now when we put these values in equations (4.99), (4.100) and (4.101) we obtained

$$\begin{aligned}
&G = 0, \qquad H = 0,\\
&L = s(-2c_2c_3) + 2c_1^2 - c_2^2, \qquad M = sc_3 + c_2,\\
&Q = 0, \qquad P = -2c_1^2 + 2s^2.
\end{aligned} \tag{4.119}$$

Where G, H, L, M, P, and Q are the solution of time translation group resolving system. Furthermore, when we substitute the values of G, H, L, M, Q in equations (4.62), (4.63), obtain six parametric ODEs

$$\rho_t = 0, \qquad \rho_x = 0, \tag{4.120}$$
$$\phi_t = \rho(-2c_2c_3) + 2c_1^2 - c_2^2, \qquad \phi_x = \rho c_3 + c_2, \tag{4.121}$$
$$\psi_t = -2c_1^2 + 2\rho^2, \qquad \psi_x = 0, \tag{4.122}$$

These parametric ODEs are easily solved by simple integration techniques, by solving the parametric ODEs, we obtain

$$\rho = c_4, \qquad \phi = -tc_2^2 + (-2c_3C_4t + x)c_2 + 2tc_1^2 + c_3c_4x + c_6, \tag{4.123}$$
$$\tau = c_1, \qquad \psi = -2c_1^2x + 2c_4^2x + c_5, \tag{4.124}$$

where the values of ρ, τ, ϕ and ψ satisfies the reduced system (4.53). The exact solution of Manakov system is given as

$$\begin{aligned}
&u(t, x) = c_4e^{i(-tc_2^2+(-2c_3c_4t+x)c_2+2tc_1^2+c_3c_4x+c_6)},\\
&v(t, x) = c_1e^{i(-2c_1^2x+2c_4^2x+c_5)}.
\end{aligned} \tag{4.125}$$

Case 2:

If we look at the system of nonlinear differential equations (4.93) - (4.97), the linear terms in equations (4.93), (4.94), (4.95), (4.96), (4.91) and (4.92) contain derivative w.r.t r only, i.e., $G_r, L_r, H_r, M_r, Q_r, Q, G$ and L. The nonlinear term $HG_s - GH_s$ in equation (4.86) shows that G and H have same behavior. So by investigating these properties, it is suitable to use a simple separation of variable ansatz

$$G = rg_1(s) + g_2(s), \quad H = rh_1(s) + h_2(s), \tag{4.126}$$
$$L = rl_1(s) + l_2(s), \quad M = rm_1(s) + m_2(s), \tag{4.127}$$
$$Q = rq_1(s) + q_2(s). \tag{4.128}$$

The linear terms $G_r, L_r, H_r, M_r, Q_r, Q, G$ and L contain r^2, r, r^0, where the nonlinear term $2QQ_sH$ in equation (4.95) contains r^3, the other nonlinear terms in equations (4.94), (4.95) and (4.96) also contains r^2, r, r^0. Therefore by comparing the powers of r^3, r^2, r, r^0, we obtain over determined system of algebraic differential equations

$$g_1 + h_2 g_2' - g_2 h_2' = 0, \tag{4.129}$$
$$g_2 h_1' - h_2 g_1' + g_1 h_2' - h_1 g_2' = 0, \tag{4.130}$$
$$h_1 g_1' - g_1 h_1' = 0, \tag{4.131}$$
$$l_1 + l_2' h_2 - m_2' g_2 = 0, \tag{4.132}$$
$$m_2' g_1 + m_1' g_2 - l_1' h_2 - l_1' h_1 = 0, \tag{4.133}$$
$$m_1' g_1 - l_1' h_1 = 0, \tag{4.134}$$
$$2q_1 q_2 + q_2' g_2 - 4h_2 s + 2q_2 q_2' h_2 = 0, \tag{4.135}$$
$$4h_1 s - q_2' g_1 - 2q_1 q_2' h_2 - 2q_2 q_2' h_1 - 2q_1^2 - 2q_2 q_1' h_2 - q_1' g_2 = 0, \tag{4.136}$$
$$2q_1 q_1' h_1 = 0, \tag{4.137}$$
$$2q_2 q_1' h_1 + q_1' g_1 + 2q_1 q_1' h_2 + 2q_1 q_2' h_1 = 0, \tag{4.138}$$
$$2s^3 - l_2 s + h_2 h_2' - m_2^2 s + 2c_1^2 s + h_1 = 0, \tag{4.139}$$
$$h_1 h_2' - l_1 s - 2m_1 m_2 s + h_2 h_1' = 0, \tag{4.140}$$
$$m_1^2 s - h_1 h_1' = 0, \tag{4.141}$$
$$2m_2 h_2 + m_2' h_2 s + s m_1 + g_2 = 0, \tag{4.142}$$
$$2m_2 h_1 + 2m_1 h_2 + g_1 + m_2' h_1 s + m_1' h_2 s = 0, \tag{4.143}$$
$$s m_1' h_1 + 2m_1 h_1 = 0, \tag{4.144}$$
$$q_2' h_2 + q_1 + 2q_2 = 0, \tag{4.145}$$
$$2q_1 + q_1' h_2 + q_2' h_1 = 0, \tag{4.146}$$
$$q_1' h_1 = 0. \tag{4.147}$$

by solving above ODEs, we get the following

$$\begin{aligned} g_1(s) &= 0, & g_2(s) &= 0, \\ h_1(s) &= 0, & h_2(s) &= 0, \\ l_1(s) &= 0, & l_2(s) &= 2c_1^2 + s^2, \\ m_1(s) &= 0, & m_2(s) &= s, \\ q_1(s) &= 0, & q_2(s) &= 0. \end{aligned} \tag{4.148}$$

Now when we put these values in equations (4.126), (4.127) and (4.128) we obtained

$$G = 0, \qquad H = 0, \qquad L = 2c_1^2 + s^2, \tag{4.149}$$
$$M = s, P = -2c_1^2 + 2s^2, \qquad Q = 0. \tag{4.150}$$

Now substitute the values of G, H, L, M, Q in equations (4.62), (4.63), we obtain six parametric ODEs

$$\rho_t = 0, \qquad \rho_x = 0, \tag{4.151}$$
$$\phi_t = 2c_1^2 + \rho^2, \qquad \phi_x = \rho, \tag{4.152}$$
$$\psi_t = -2c_1^2 + 2\rho^2, \qquad \psi_x = 0, \tag{4.153}$$

on solving these parametric ODEs, we will obtain the solution of reduced system (4.53)

$$\rho = c_2, \quad \phi(t,s) = (2c_1^2 + c_2^2)t + c_2x + c_4, \tag{4.154}$$
$$\tau = c_1, \quad \psi(t,s) = -2c_1^2t + 2c_2^2t + c_33, \tag{4.155}$$

where ϕ and ψ are the phase of two different waves. The exact solution of Manakov system (3.1) is

$$u(t,x) = c_2e^{i((2c_1^2+c_2^2)t+c_2x+c_4)},$$
$$v(t,x) = c_1e^{i(-2c_1^2t+2c_2^2t+c_3)}. \tag{4.156}$$

Case 3:

Solving the system (4.93)-(4.98), we obtain

$$\rho_t = c_2\rho^3, \qquad \rho_x = 0, \tag{4.157}$$
$$\tau_t = 0, \qquad \tau_x = 0, \tag{4.158}$$
$$\phi_t = 2c_7^2 - \rho^4(c_2x - c_3)^2 + 2\rho^2, \qquad \phi_x = -\rho^2(c_2x - c_3), \tag{4.159}$$
$$\psi_t = -c_1^2 - 2c^2 + 2\rho^2, \qquad \psi_x = c_1, \tag{4.160}$$

where by integrating (4.158), we get $\tau(t,x) = c_7$. Here c_7 is an arbitrary constant, we get the following results by solving remaining equations

$$\rho = \frac{1}{\sqrt{(-2c_2t + c_4)}}, \tag{4.161}$$
$$\phi = \frac{(-c_2x^2 + 2c_3x)}{(-4c_2t + 2c_4) + 2c_7^2t} - \frac{(1/2)c_3^2}{(c_2(-2c_2t + c_4))} - \frac{\ln(2c_2t - c_4)}{c_2} + c_6, \tag{4.162}$$
$$\psi = c_1x - 2c_7^2t - c_1^2t - \frac{\ln(2c_2t - c_4)}{c_2} + c_5. \tag{4.163}$$

Therefore we found an other exact solution of Manakov system

$$u(t,x) = \frac{1}{\sqrt{(-2c_2t + c_4)}}e^{\left(i\left(\frac{(-c_2x^2 + 2c_3x)}{(-4c_2t + 2c_4) + 2c_7^2t} - \frac{(1/2)c_3^2}{(c_2(-2c_2t + c_4))} - \frac{\ln(2c_2t - c_4)}{c_2} + c_6\right)\right)},$$
$$v(t,x) = c_7e^{\left(i\left(c_1x - 2c_7^2t - c_1^2t - \frac{\ln(2c_2t - c_4)}{c_2} + c_5\right)\right)}. \tag{4.164}$$

Note that in the above solution the first wave has time dependent amplitude while other wave has fixed amplitude. We have applied group foliation approach to successfully obtain exact solution of Manakov system in a few cases described above.

Chapter 5

Conclusion

Lie group analysis is one of the feasible way of providing a possibility for various classes of exact solutions or exact solutions to be specified. The classical method of finding the exact solution of differential equations is symmetry reduction. The symmetry reduction gives us reduced equations of a given PDEs which may be intractable to solve. While the exact solutions which are obtained from symmetry reduction has limited class, a large class of solutions are lost when the space of independent and dependent variables are reduced.

Group foliation is a power full technique to unveil the solution of nonlinear PDEs. It involves algorithmic steps of construction of group resolving equation by means of using group foliation variables, solving the parametric ODEs. The main aim of the group foliation is to find all invariant and noninvariant solutions of PDEs. It is a better approach as compared to the symmetry reductions because in it we enlarge the space (independent and dependent variables), therefore we get more information.

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