Fluid Structure Interaction Model for Cerebral Aneurysm in the Presence of Magnetic Field

by

Muhammad Shoaib

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Supervised by

Dr. Asim Aziz

Dr. Mazhar Iqbal

Centre for Advanced Mathematics and Physics
National University of Sciences and Technology
Islamabad, Pakistan

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Examination Committee Members

1. Name: Prof. Faiz Ahmad — Signature: ____________
2. Name: Dr. Moniba Shams — Signature: ____________
3. Name: Dr. Noreen Sher Akbar — Signature: ____________
4. Name: Dr. Mazhar Iqbal — Signature: ____________
5. Name: Dr. Sohail Nadeem — Signature: ____________

Supervisor's Name: Dr. Asim Aziz — Signature: ____________

Head of Department — Date: 20-02-2013

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Dean/Principal

Dedicated to

To My Parents

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Abstract

Aneurysm is the term that refers to the abnormal enlargement or bulging of an artery wall. This condition is the result of a weakness or thinning of the blood vessels. When the size of an aneurysm is enlarged, there is a significant risk of rupture, resulting in subarachnoid hemorrhage, stroke and other complications or death. Aneurysms can occur in any type of blood vessel, but they usually occur in the arteries of abdomen and the brain. Surgery or surgical clipping and invasive endovascular coiling techniques are two techniques currently practiced for the treatment of cerebral aneurysms. Cerebral aneurysms are undergoing spacious research however the processes that cause an aneurysm to form and consequently rupture are still unknown. It is hypothesized that the reason of the brain aneurysm expansion and rupture is due to the fact that dynamic behavior of the arterial wall is unstable because the flow of blood is pulsatile

In the present research work a one dimensional coupled model of a cerebral aneurysm is considered that combines the interaction between the arterial wall structure, the pressure of blood and the cerebral spinal fluid that is around the aneurysm in the presence of external magnetic field. The mathematical equations are solved algebraically. The displacement of the arterial wall is plotted to visualize the wall movement. The results show the inclusion of the MHD stabilizes the movement of the arterial wall and in turn prevent the rupture of the cerebral aneurysm.

Contents

Chapter 1

Introduction

Aneurysm is the term that refers to the abnormal enlargement or bulging of an artery wall. This condition is the result of a weakness or thinning of the blood vessels. Aneurysms can be found in any type of blood vessels, but they usually occur in the arteries of abdomen and the brain. These are known as abdominal aortic aneurysms (AAA) and cerebral aneurysms (CA) respectively. Abdominal aortic aneurysms (AAAs) are known as silent killers because in most cases, there are no symptoms associated with the pathology. When an AAA ruptures, it will result in life-threatening blood loss. Most AAAs are diagnosed during a physical exam when the person is undergoing a checkup for some other health concern. Cerebral aneurysm (CA) are known as intracranial or brain aneurysms occur most commonly in the anterior cerebral artery, which is part of the circle of Willis [1]. When the size of an aneurysm increases there is a significant risk of rupture, resulting in subarachnoid hemorrhage, stroke and other complications or death.

Surgery or surgical clipping and invasive endovascular coiling techniques are two techniques currently practiced for treatment of cerebral aneurysms. Surgery involves opening the skull and placing a titanium clip across the neck of an aneurysm to exclude it from the circulation [2]. Surgery is a successful treatment of cerebral aneurysms, however, surgery is limited to aneurysm location and is extremely risky. Endovascular techniques are a form of treatment that enters the cerebral aneurysm via micro-catheters through arteries. They include delivery of balloons, liquid embolics, coils, stents, or a combination of them in order to promote thrombosis within the aneurysm sac, thus occluding it from the blood flow and prevent rupture. It is most important to note that not all the aneurysms are treated at the time of diagnosis or are susceptible to both forms of treatment.

There has been much research on the behavior of arterial walls in aneurysms and predicting their rupture rate [3, 4]. Cerebral aneurysms are undergoing extensive research by those in the medical field in collaboration with numerical analysts and engineers, the processes that cause an aneurysm to form and subsequently rupture are still unknown [5]. It is hypothesized that the reason of the brain aneurysm expansion and rupture is due to the fact that dynamic behavior of the arterial wall is unstable because the flow of blood is pulsatile [6]. To examine this hypothesis, SM Venuti [7] build a one dimensional coupled model of an cerebral aneurysm, that combines the interaction between the arterial wall structure, the pressure of blood and the CSF that is around the aneurysm.

During the last decades, great research has been done on the dynamical importance of the biological fluid in the presence of external magnetic field with significance in medical technology and bio-engineering. The most popular and important applications in MHD includes targeted drug delivery system in which magnetic particles used as a drug carriers, cell separation, magnetic wound or cancer treatment and control of blood loss during surgeries [3, 4].

In the present research work we extend the work of SM Venuti [7] to include the magnetohydrodynamics (MHD) effect on fluid structure interaction (FSI) model. The focus in on the fact that when the patient is in the critical situation, i.e. aneurysmic region is going to rupture. Under this situation it is not possible to give the patient a surgical treatment. Therefore, an external magnetic field may be used to reduce the movement of the arterial wall and prevent rupture. The plan of dissertation is as follows: Chapter 2, contains some basic definitions of fluid dynamics. In chapter 3, we review the work of SM Venuti [7] and reproduce the results. MHD effect on the FSI model for cerebral aneurysms are studied in chapter 4. Finally in chapter 5, we extend our model a step closer to the real situation by including the effect of viscosity in the presence of external magnetic field.

Chapter 2

Preliminaries

This chapter contains some basic notions of the mathematical theory of fluid dynamics. An introduction to fluid flow, their types and the concept of stress and strain are discussed. Sections (2.7) and (2.8) define basic equations which are applicable to any fluid. Finally in the last two sections we define Magnetohydrodynamics (MHD) and write down the basic equations of MHD fluid flow.

2.1 Fluid

A fluid is a substance that deforms continuously under the influence of a shear or tangential stress, it does not matter how small the stress is [8]. Fluids are the phases of matter and include liquids, gases, plasmas and to some extent plastic solids.

2.2 Unsteady and Steady Flows

A flow in which fluid properties at each point in the flow field do not depend upon time is called *steady flow.* For such flow

$$\frac{\partial \lambda}{\partial t} = 0, \tag{2.1}$$

where $\partial/\partial t$ is the partial derivative with respect to time t and λ, represents any fluid flow property. A flow in which fluid properties depend upon time is called *unsteady flow.* For such flow

$$\frac{\partial \lambda}{\partial t} \neq 0. \tag{2.2}$$

2.3 Laminar and Turbulent Flows

Laminar flow is one in which each fluid particle has a definite path. In such flow, the course lines of fluid particles do not intersect each other. In turbulent flow the paths of fluid particles may crosses each other.

2.4 Compressible and Incompressible Flows

A fluid flow in which density within the flow is considered constant is known as *incompressible flow*. A fluid flow in which density varies is said to be *compressible flow* [8].

The mathematical equation that defines the incompressibility of the fluid is given by

$$\frac{D\rho}{Dt} = 0, \tag{2.3}$$

where ρ is density of the fluid and D/Dt is the material time derivative given by

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + \mathbf{V}.\nabla. \tag{2.4}$$

In equation (2.4) $\mathbf{V}$ represents the velocity of the flow and ∇ is differential operator. In Cartesian coordinate system ∇ is given by

$$\nabla = \hat{\mathbf{i}}\frac{\partial}{\partial \mathbf{x}} + \hat{\mathbf{j}}\frac{\partial}{\partial \mathbf{y}} + \hat{\mathbf{k}}\frac{\partial}{\partial \mathbf{z}}$$

where $(\hat{\mathbf{i}}, \hat{\mathbf{j}}, \hat{\mathbf{k}})$ are the unit vectors in their respective directions.

2.5 Viscosity

Viscosity measures the resistance of a fluid which is deformed by either tangential or a shear stress. So, we can say that it is the thickness or internal friction of the fluid. Simply, we can say that if the fluid is less viscous then it will move easier [9]. The coefficient of viscosity is denoted by the symbol μ.

2.6 Newtonian and non-Newtonian Fluids

A fluid whose stress versus strain rate curve is linear and passes through the origin is called *Newtonian fluid* [8]. A simple equation to describe Newtonian fluid behavior is

$$\tau = \mu\frac{du}{dy}. \tag{2.5}$$

In equation (3.1) τ is the shear stress exerted by the fluid, μ is viscosity-constant of proportionality, and du/dy is the rate of deformation.

A fluid whose flow properties differ in any respect from those of Newtonian fluids is called *non-Newtonian fluid.* In non-Newtonian fluid stress versus strain rate curve is not linear one. For non-Newtonian fluid behavior

$$\tau = k(\frac{du}{dy})^n. \tag{2.6}$$

The above equation can be written as

$$\tau = k(\frac{du}{dy})^{n-1}(\frac{du}{dy}) = \eta(\frac{du}{dy}). \tag{2.7}$$

In equation (2.7) $\eta = k(\frac{du}{dy})^{n-1}$ is known as the apparent viscosity [8].

2.7 The Continuity Equation

The continuity equation of fluid flow is based on the law conservation of mass. Which states that mass can neither be created nor destroyed inside a control volume region. If we consider a differential control volume system encased by a surface fixed in space, then the mass inside the fixed control system will not change. For such a system, equation of continuity can be expressed as

$$\frac{\partial \rho}{\partial t} + \nabla.(\rho \mathbf{V}) = \mathbf{0}. \tag{2.8}$$

If density ρ is temporally constant and spatially uniform, then the equation (2.8) becomes

$$\nabla.\mathbf{V} = \mathbf{0}. \tag{2.9}$$

Equation (2.9) is the equation of continuity for incompressible fluids.

2.8 The Momentum Equation

The equation of linear momentum is obtained by applying Newton's second law of motion on a fluid particle. Which states that, "the net force acting on a fluid particle is equal to the time rate of change of linear momentum".

Consider the mass in a system defined by control surface of infinitesimally small dimensions dx, dy and dz. The mass of the system is constant, therefore Newton's second law can be written as

$$m\frac{D\mathbf{V}}{Dt} = F. \tag{2.10}$$

The differential equation of motion (2.10) describing the flow of a fluid is

$$\rho\frac{D\mathbf{V}}{Dt} = \rho\mathbf{b} + \text{div}\mathbf{T}. \tag{2.11}$$

In equation (2.11) D/Dt is defined in section (2.4) $\rho\mathbf{b}$ are body forces per unit mass and div$\mathbf{T}$ are the surface forces and $\mathbf{T}$ is the Cauchy stress tensor which is given by

$$\mathbf{T} = \begin{bmatrix} \sigma_{xx} & \tau_{yx} & \tau_{zx} \\ \tau_{xy} & \sigma_{yy} & \tau_{zy} \\ \tau_{xz} & \tau_{yz} & \sigma_{xx} \end{bmatrix} \tag{2.12}$$

In which σ_{ii} $(i = \text{x}, \text{y}, \text{z})$ and τ_{ij} $(j = \text{x}, \text{y}, \text{z})$ are the normal and shear stresses respectively. The scalar form of the momentum equation (2.11) in terms of velocity components (u, v, w) in x, y and z directions are given by

$$\rho(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}) = \rho b_x + \frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{xz}}{\partial z}, \tag{2.13}$$

$$\rho(\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z}) = \rho b_y + \frac{\partial \tau_{yx}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \tau_{yz}}{\partial z}, \tag{2.14}$$

$$\rho(\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z}) = \rho b_z + \frac{\partial \tau_{zx}}{\partial x} + \frac{\partial \tau_{zy}}{\partial y} + \frac{\partial \sigma_{zz}}{\partial z}. \tag{2.15}$$

2.9 Constitutive Equations for Newtonian Fluid

A constitutive equation describes relation between stress and properties of fluid. For a fluid, which is at rest the stress is given only by the static pressure. Although when a fluid is in relative motion, the relation between the fluid properties and stress is complicated. In the neighbourhood of the system, some changes should be made so that the stress will only depend on instantaneous distribution of fluid velocity. This distribution can be expressed in the form of velocity gradient component. Newtonian incompressible fluid stresses can be expressed in the form of some fluid properties and velocity gradient as follows

$$\sigma_{xx} = -P + 2\mu\frac{\partial \text{v}_x}{\partial x}, \tag{2.16}$$

$$\sigma_{yy} = -P + 2\mu\frac{\partial \text{v}_y}{\partial y}, \tag{2.17}$$

$$\sigma_{zz} = -P + 2\mu\frac{\partial \text{v}_z}{\partial z}, \tag{2.18}$$

$$\tau_{xy} = \mu(\frac{\partial \text{v}_x}{\partial y} + \frac{\partial \text{v}_y}{\partial x}) = \tau_{yx}, \tag{2.19}$$

$$\tau_{yz} = \mu(\partial \frac{\mathrm{v}_y}{\partial z} + \frac{\partial \mathrm{v}_z}{\partial y}) = \tau_{zy}, \tag{2.20}$$

$$\tau_{zx} = \mu(\partial \frac{\mathrm{v}_x}{\partial z} + \frac{\partial \mathrm{v}_z}{\partial x}) = \tau_{xz}. \tag{2.21}$$

Substituting these stresses in equations (2.13)-(2.15) we get

$$\rho(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}) = \rho \mathrm{b}_x - \frac{\partial P}{\partial x} + \mu\frac{\partial^2 \mathrm{u}}{\partial x^2}, \tag{2.22}$$

$$\rho(\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z}) = \rho \mathrm{b}_y - \frac{\partial P}{\partial y} + \mu\frac{\partial^2 \mathrm{v}}{\partial y^2}, \tag{2.23}$$

$$\rho(\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z}) = \rho \mathrm{b}_z - \frac{\partial P}{\partial z} + \mu\frac{\partial^2 \mathrm{w}}{\partial z^2}. \tag{2.24}$$

In vector form equations (2.22)-(2.24) can be written as:

$$\rho\frac{D\mathbf{V}}{Dt} = \rho\mathbf{b} - \nabla \mathrm{P} + \mu\nabla^{\mathbf{2}}\mathbf{V}. \tag{2.25}$$

2.10 Magnetohydrodynamics

Magnetohydrodynamics is a branch of engineering in which the behaviors of magnetic field in electrically conducting fluids are discussed. Examples of such fluids include plasmas, liquid metals, and salt water or electrolytes. The word magnetohydrodynamics (MHD) is derived from magneto- meaning magnetic field, and hydro- meaning liquid and dynamics- meaning movement.

2.10.1 Current Density

The ratio of the intensity of current to the area, which is perpendicular to the direction of current, trough which the current is flowing, is known as the current density. Mathematically it is defined as

$$I = \int \mathbf{J}.\mathbf{dS}. \tag{2.26}$$

Here **J** denotes the density of current at the area element $d\mathbf{S}$ and I denotes the total current passing through area.

2.10.2 Electrical Conductivity

Electrical conductivity is a measure of how a material accommodates the transport of electric charge. It is represented by the symbol σ.

$$\sigma = \frac{1}{\rho}, \tag{2.27}$$

where ρ is the resistivity. For a body of length L, resistance R and cross sectional area A, the electrical conductivity is given by

$$\sigma = \frac{L}{RA}. \tag{2.28}$$

2.11 Basic Equations of MHD

The basic concept of MHD is that magnetic fields can induce current in a moving conductive fluid, which create forces on the fluid and the magnetic field itself. The set of equations which represent MHD are a combination of the equations of motion of fluid dynamics and Maxwell's equations of electromagnetism.

To develop substantial foreknowledge on the subject of MHD, it is very Important to understand its two major key effects that appear in this subject (Fig. 2.1). its very first effect arise as soon as a conducting material is introduced in a magnetic field. Such that according to Lenz's law induced current in the conductor creates its own magnetic field. Finally, the induced field tries to cancel out the actual field. On the other hand, when the conductor is accessed by the magnetic field, the conductor escapes the field and thus the induced field fortify the applied field. Now if the conductor is supposed to be a fluid having complex variation, the consequential magnetic field results into a purely complex distribution, and the electric current will increase until it is balanced by Ohmic dissipation. The second fundamental cause arises when current is induced due to the moving conducting fluid through a magnetic field. Now, a force will transform its motion and this force is also known as a Lorentz (or $\mathbf{J} \times \mathbf{B}$) force. However in MHD, the motion changes the field and in result the field reacts back and changes the motion. Here the theory will become highly non-linear.

The fluid dynamical attitude of MHD are handled by adding an electromagnetic force term to the Navier-Stokes equation or equation of motion. Lorentz force is the only body force which is acting on the fluid. When this force is substituted to the Navier-Stokes equation then for incompressible fluid flow, equation (2.25) will take the following form

$$\rho \frac{D\mathbf{V}}{Dt} = \rho(\mathbf{J} \times \mathbf{B}) - \nabla \mathrm{P} + \mu \nabla^{\mathbf{2}} \mathbf{V}. \tag{2.29}$$

Figure 2.1: The two basic physical effects taking place in MHD. A moving conductor which modifies the magnetic field by dragging the field lines with it. In the case of infinite conductivity, the field lines looks frozen into the moving conductor (Left). When electric current crosses magnetic field lines there will be a Lorentz force, which will cause acceleration in the fluid (Right).

In equation (2.29) $\mathbf{J}$ is the current density and $\mathbf{B} = \mathbf{B_o} + \mathbf{B_1}$, is total magnetic field where $\mathbf{B_1}$ is the induced magnetic field considered to be negligible in comparison with the external magnetic field which is justified for MHD flow at small magnetic Reynolds number. By Ohm's law, we have

$$\mathbf{J} = \sigma(\mathbf{E} + \mathbf{V} \times \mathbf{B}), \tag{2.30}$$

where σ is the electrical conductivity and $\mathbf{E}$ is the electric field. The imposed and induced electrical fields are assumed to be negligible. The force $\mathbf{J} \times \mathbf{B}$ can be simplified to

$$\mathbf{J} \times \mathbf{B} = -\sigma \mathbf{B_o^2} \mathbf{V}. \tag{2.31}$$

It is also assumed that the electric field due to polarization of charges is also negligible. Thus, the equation (2.29), will get the following form

$$\rho \frac{D\mathbf{V}}{Dt} = -\rho\sigma \mathbf{B_o^2} \mathbf{V} - \nabla \mathrm{P} + \mu \nabla^2 \mathbf{V}. \tag{2.32}$$

Chapter 3

Fluid Structure Interaction Model for Cerebral Aneurysm

This chapter mathematically models a cerebral artery, specifically in application to cerebral aneurysms. A spring-mass system is used to model the arterial wall. Fourier series is used to model the pressure of the blood, which is acting through the inside of the arterial wall. The surrounding cerebral spinal fluid (CSF) is taken to be a Newtonian fluid. Section (3.4) shows how the CSF and the blood pressure affect the movement of the arterial wall for several model parameters. These include the density of the CSF, elasticity of the wall and influence of the pulsatile frequency of the blood in order to assess their affect on the movement on the wall.

3.1 Mathematical Models

The problem we consider models three components of the cerebral aneurysm-the blood pressure which is acting on the inside of the arterial wall, CSF that surrounds the aneurysm and the arterial wall structure (see Figure 3.1). To derive the one dimensional model, we consider a line that runs from the aneurysm to the arterial wall and out into the CSF, as shown in Figure 3.1. The point x = 0 is to be where the outside wall of the artery and the CSF meet, for x > 0 we move into the CSF i.e, away from the wall, and for x < 0 we move into the aneurysm through the wall.

3.1.1 Model of the Cerebral Spinal Fluid

CSF is taken to be a Newtonian fluid. The velocity v is in the x direction, The one dimensional Navier-Stokes equation (2.22) will be as follows

$$\rho v_t + \rho v v_x + P_x - \mu v_{xx} = \rho b. \tag{3.1}$$

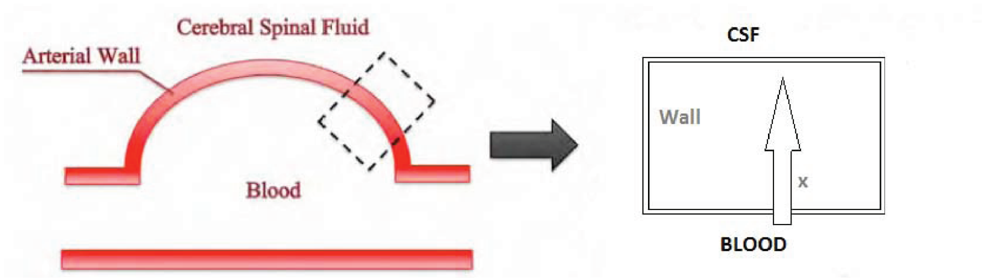

Figure 3.1: Aneurysm with direction x shown

Here ρ denotes density, $P(x,t)$ implies the pressure of the fluid, $v(x,t)$ is used for velocity, μ express the viscosity of the fluid and b denotes the body forces on fluid. It is considered that the no external forces are acting on the fluid and CSF is slightly compressible and inviscid. This simplifies equation (3.1) to

$$\rho v_t + \rho v v_x + P_x = 0. \tag{3.2}$$

Furthermore we assume nonlinear effects are negligible. Therefore, equation (3.2) reduce to the following form

$$\rho v_t + P_x = 0. \tag{3.3}$$

The equation (3.3) includes two unknowns, the pressure and the velocity.

The state equation for a slightly compressible mixture which is given by SM Venuti [7]:

$$\rho_P = \rho e^{\gamma^{-1}P(x,t)}. \tag{3.4}$$

Here $\gamma = \rho c^2$, c is the speed of sound through the fluid. Taking the derivative of ρ_P with respect to P gives

$$\frac{d\rho_P}{dP} = (\frac{\rho}{\gamma})e^{\gamma^{-1}P(x,t)}, \tag{3.5}$$

Substituting equation (3.4), into equation (3.5), we get

$$\frac{d\rho_P}{dP} = \frac{\rho_P}{\gamma}. \tag{3.6}$$

The Law of Conservation of Mass gives

$$\frac{\partial \rho_P}{\partial t} = -\frac{\partial}{\partial x}(\rho_P v) = -\rho_P\frac{\partial v}{\partial x} - v\frac{\partial \rho_P}{\partial x}, \tag{3.7}$$

$$\frac{d\rho_P}{dP}\frac{\partial P}{\partial t} = -\rho_P\frac{\partial v}{\partial x} - v\frac{d\rho_P}{dP}\frac{\partial P}{\partial x}, \tag{3.8}$$

by using the product rule and chain rule. Substituting equation (3.6) into equation (3.8) we get

$$\frac{\rho_P}{\gamma}\frac{\partial P}{\partial t} = -\rho_P\frac{\partial v}{\partial x} - v\frac{\rho_P}{\gamma}\frac{\partial P}{\partial x}. \tag{3.9}$$

As γ is very large, the second term on the right hand side is very small compared to the first, we can assume it is negligible. By integrating with respect to time, equation (3.9), reduces to

$$\frac{\rho_P}{\gamma}\frac{\partial P}{\partial t} = -\rho_P\frac{\partial v}{\partial x}. \tag{3.10}$$

For the displacement of fluid we introduce a new variable $u(x,t)$. It is related to the velocity of the fluid through

$$u(x,t) = \int_0^t v(x,s)ds. \tag{3.11}$$

Using equation (3.11), equation (3.10) simplifies to

$$\frac{\rho_P}{\gamma}\frac{\partial P}{\partial t} = -\rho_P\frac{\partial u_t}{\partial x}. \tag{3.12}$$

Hence equation (3.10) reduce to

$$P = -\rho c^2 u_x. \tag{3.13}$$

Equation (3.13) gives a relationship between the pressure and displacement. Using this relationship and equation (3.11) simplifies (3.3) to

$$v_t = c^2 u_{xx}, \tag{3.14}$$

$$u_t = v. \tag{3.15}$$

Here the interest is on the movement of the wall, which is the same as the movement of the fluid at the point x = 0. Thus by finding the solution to equations (3.14) and (3.15) at the point x = 0 for time $t \geq 0$. We will know the movement of the wall for all time after $t = 0$.

3.1.2 Model for the Arterial Wall

The arterial wall is modeled by a spring and mass system, with the spring constant k and mass m. Figure 3.2 illustrates this coupled system between the inner wall and outer wall. The force generated by this system is given by $k(x_{outerwall} - x_{innerwall})$, where $x_{outerwall}$ and $x_{innerwall}$ are the respective displacements of the outer and inner wall from equilibrium.

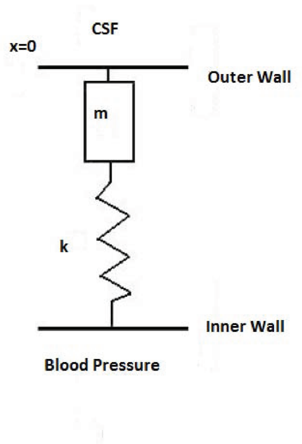

Figure 3.2: Spring and mass system

3.1.3 Model for the Blood Pressure

As the pressure of blood is considered to be pulsatile, it can be modeled by the Fourier series [10, 11, 12].

$$P_{BLOOD}(t) = P_m + \sum_{n=1}^{N} (AA_n \cos(n\omega t) + BB_n \sin(n\omega t)), \tag{3.16}$$

here P_m denotes the mean pressure of the blood, and for N harmonics AA_n and BB_n denotes the Fourier coefficients , and ω is used ad the fundamental circular frequency.

3.1.4 Governing Equation of Motion

To solve the partial differential equations (3.14) and (3.15), we need two boundary conditions and two initial conditions. For initial conditions, it is considered that the CSF starts from rest and has no initial velocity, giving us the following initial condition

$$u(x, 0) = v(x, 0) = 0. \tag{3.17}$$

The first boundary condition, at the point x = 0, is derived by writing a force balance equation at the point x = 0, where

$$F_{TOTAL} = F_{FLUID} + F_{SPRING}. \tag{3.18}$$

In equation (3.18) F_{TOTAL} is the total force, F_{FLUID} is the force of the fluid and F_{SPRING} is the force from the spring. The total force can be given by

$$F_{TOTAL} = mv_t(0, t). \tag{3.19}$$

The force of the fluid for slightly compressible fluid is given as

$$F_{FLUID} = \rho c^2 u_x(0,t)a, \tag{3.20}$$

where a is the cross-sectional area. The force from the spring is given by

$$F_{SPRING} = k(x_{outerwall} - x_{innerwall}) \tag{3.21}$$

Displacement of the outer wall is $u(0,t)$. For the interior wall, as it is affected only by the blood pressure, its displacement is proportional to the blood pressure. So equation (3.21) becomes

$$F_{SPRING} = ku(0,t) - aP_{BLOOD}(t). \tag{3.22}$$

Using equations (3.16), (3.19), (3.20) and (3.22) into equation (3.18) gives us the following boundary condition

$$mv_t(0,t) = aP_m - ku(0,t) + \rho c^2 a u_x(0,t) + \sum_{n=1}^{N}(aAA_n \cos(n\omega t) + aBB_n \sin(n\omega t)). \tag{3.23}$$

For the second boundary condition, we use the plane wave approximation, which says that the waves from the wall will die down some long distance away from the wall, which we call the point x = L. This is given by

$$v(L,t) = -cu_x(L,t). \tag{3.24}$$

Summarizing the various models developed in sections (3.3.1)-(3.3.4), we obtain the following coupled fluid-structure interaction (FSI) problem:

$$v_t = c^2 u_{xx}, \tag{3.25}$$

$$u_t = v, \tag{3.26}$$

$$v(x,0) = u(x,0) = 0, \tag{3.27}$$

$$mv_t(0,t) = \rho c^2 a u_x(0,t) + aP_{BLOOD}(t) - ku(0,t), \tag{3.28}$$

$$v(L,t) = -cu_x(L,t), \tag{3.29}$$

where P_{BLOOD} is given by equation (3.16).
We will present an analytical solution of the above system in the next section.

3.2 Mathematical Solution

In order to simplify our work, we will rewrite equations (3.25)-(3.29) in terms of only $u(x,t)$, which gives us

$$u_{tt} = c^2 u_{xx}, \tag{3.30}$$

$$u(x,0) = u_t(x,0) = 0, \tag{3.31}$$

$$m u_{tt}(0,t) = \rho c^2 a u_x(0,t) + a P_{BLOOD}(t) - k u(0,t), \tag{3.32}$$

$$u_t(L,t) = -c u_x(L,t), \tag{3.33}$$

where P_{BLOOD} is given by equation (3.16). Taking the Laplace Transform (see Appendix A) of (3.30) gives

$$s^2 U(x,s) = c^2 U_{xx}(x,s), \tag{3.34}$$

where $U(x,s)$ is the transformed displacement and s is the Laplace parameter. Equation (3.34) has the solution

$$U(x,s) = c_1 \cosh\left(\frac{s}{c}x\right) + c_2 \sinh\left(\frac{s}{c}x\right), \tag{3.35}$$

where c_1 and c_2 are arbitrary constants. To find them we first take the Laplace Transform of the boundary condition (3.33) at the point x = L which gives

$$sU(L,s) = -cU_x(L,s). \tag{3.36}$$

Substituting equation (3.35) into equation (3.36) gives

$$c_2 = -c_1. \tag{3.37}$$

Next take the Laplace Transform of the boundary condition (3.32) at the point x = 0 which gives

$$\begin{aligned} ms^2 U(0,s) &= \frac{aPm}{s} - kU(0,s) + \rho c^2 a U_x(0,s) \\ &\quad + \sum_{n=1}^{N} \left(aAA_n\left(\frac{s}{s^2 + (n\omega)^2}\right) + aBB_n\left(\frac{n\omega}{s^2 + (n\omega)^2}\right)\right). \end{aligned} \tag{3.38}$$

Now substitution of equation (3.35) into equation (3.38) gives

$$U(0,s) = \frac{\frac{aP_m}{s} + \sum_{n=1}^{N}\left(aAA_n\left(\frac{s}{s^2+(n\omega)^2}\right) + aBB_n\left(\frac{n\omega}{s^2+(n\omega)^2}\right)\right)}{ms^2 + \rho c a s + k}. \tag{3.39}$$

Taking the inverse Laplace Transform of equation (3.39)

$$\begin{aligned} u(0,t) &= A + Be^{rr_1 t} + Ce^{rr_2 t} + \\ &+ \sum_{n=1}^{N} (D_n \cos(n\omega t) + \frac{E_n}{n\omega} \sin(n\omega t) + F_n e^{rr_1 t} + G_n e^{rr_2 t}). \end{aligned} \tag{3.40}$$

In equation (3.40)

$$rr_1 = \frac{-\rho ca + \sqrt{(\rho ca)^2 - 4mk}}{2m}, \tag{3.41}$$

$$rr_2 = \frac{-\rho ca - \sqrt{(\rho ca)^2 - 4mk}}{2m}, \tag{3.42}$$

$$A = \frac{aP_m}{mrr_1 rr_2}, \tag{3.43}$$

$$B = -\frac{aP_m}{rr_1(rr_2 - rr_1)m}, \tag{3.44}$$

$$C = \frac{aP_m}{rr_2(rr_2 - rr_1)m}, \tag{3.45}$$

$$D_n = -F_n - G_n, \tag{3.46}$$

$$E_n = -r_1 F_n - r_2 G_n, \tag{3.47}$$

$$F_n = \frac{aAA_n - mG_n(rr_2^2 + n^2\omega^2)}{m(rr_1^2 + n^2\omega^2)}, \tag{3.48}$$

$$G_n = \frac{a(rr_2 AA_n + n\omega BB_n)}{m(rr_2 - rr_1)(rr_2^2 + n^2\omega^2)}. \tag{3.49}$$

3.3 Analysis of Solution

Looking at the solution for $u(0,t)$, some observations can be made about the behavior of this equation. Equation (3.40) is a sum of periodic terms, exponential terms and a constant. As both the periodic terms and the constant term are bounded, a finite sum of these terms is also bounded. Therefore to understand what this function looks like as t increases towards infinity we need to understand the contribution of exponential terms.

Note that all of the exponential terms are of the form ye^{rt} where y is some constant and $r = rr_1$ or rr_2. For rr_1 and rr_2 to be real, there is a restriction on our values for ρ, c, a, m and k i.e.

$$(\rho ca)^2 \geq 4mk. \tag{3.50}$$

For the bounded solution rr_1 and rr_2 must be negative. All the constants in rr_1 and rr_2 (ρ, c, a, m, k) are all positive values. So from equation (3.50) we can write

$$\frac{\rho ca + \sqrt{(\rho ca)^2 - 4mk}}{2m} > 0. \tag{3.51}$$

Comparison of equations (3.51) and (3.42) gives $rr_2 < 0$.

For rr_1 consider the following fact

$$\sqrt{(\rho ca)^2 - 4mk} < \rho ca, \tag{3.52}$$

which makes $rr_1 < 0$.

So all the exponential terms will go to zero as t becomes very large. The value of the exponential terms at the point $t = 0$ will depend on the constant y. Thus for large values of t only the bounded periodic terms and the constant will affect the graph of $u(0, t)$. So we expect the function to start at $u(0, 0) = 0$ and become periodic over time.

3.4 Results and Discussion

This section shows the plot of the displacement of the wall $u(0, t)$ versus time t and discuss the influence of different parameters on the displacement of the wall $u(0, t)$.

Consider the following values for our experiments. For the CSF, $\rho = 1000$ [13] and $c = 1500$. For the model of blood pressure, $P_m = 65.7$, $\omega = 1$, and for the few harmonics $AA_1 = -7.13$, $BB_1 = 4.64$, $AA_2 = -3.08$, $BB_2 = -1.18$, $AA_3 = -0.130$, $BB_3 = -0.564$,$AA_4 = -0.205$, $BB_4 = -0.346$, $AA_5 = 0.0662$, $BB_5 = -0.120$, all are in $mmHg$ [12]. Finally for the wall $a = .01\ m^2$, $k = 8000\ N/m$ and $m = .001\ kg$. These values are according to our restriction from the analytical solution given in equation (3.50).

The displacement of the wall $u(0, t)$ versus time t is plotted in Figure 3.3. It starts from zero and becomes stable after a few seconds. As the analytical solution predicted earlier, initially there is a rapid increase in the displacement because of the exponential terms. Afterwards since rr_1 and rr_2 are negative, so for larger values of time t, exponential terms will becomes negligible and have no effect and stabilizes into a bounded periodic motion.

3.4.1 Influence of the Spring-Constant

The stiffness of the arterial wall physically models by the spring constant. Therefore, as the spring constant is decreased, the wall becomes more flexible, which causes the wall to move further out and to take longer to stabilize. Figure (3.4) illustrates this. In Figure (3.4) the solution is plotted for decreasing values of the spring constant

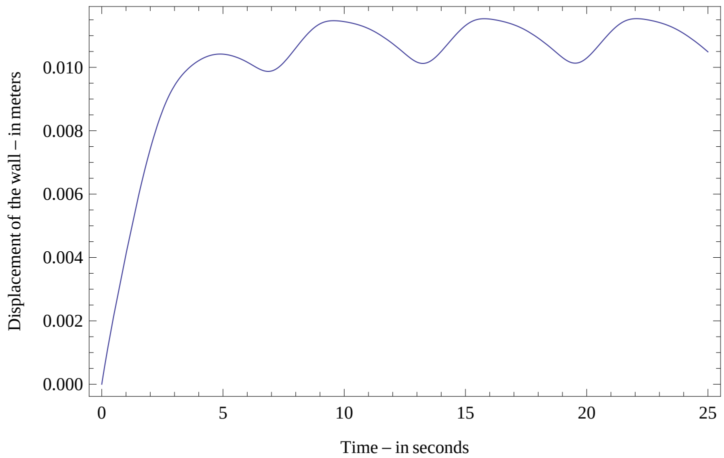

Figure 3.3: Analytical solution.

from $k = 8000\ N/m$ to $k = 3000\ N/m$. As k decreases the displacement is greater and it takes longer for the wall to settle into a steady periodic motion. However, note that the value of the spring constant has no affect on the amplitude of the periodic movement of the wall.

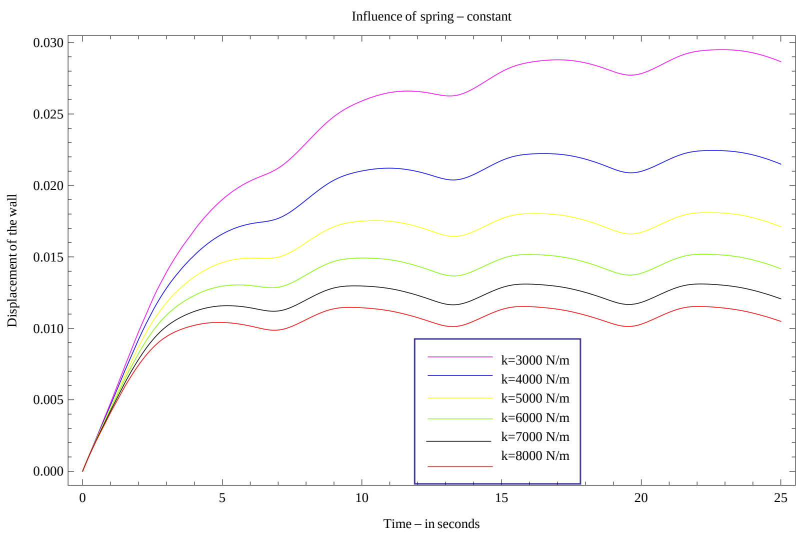

Figure 3.4: Influence of wall stiffness.

3.4.2 Influence of the CSF's Density

As the CSF becomes more dense, it resists the movement of the arterial wall, so the amplitude of the periodic movement of the wall is expected to become much less. Figure (3.5) shows this. Figure (3.5) illustrates the motion of the wall for increasing values of the density from $\rho = 1000\ kg/m^3$ to $\rho = 6000\ kg/m^3$ of the CSF. Note that this graph is at a later time period than Figure (3.4), it is after the movement of the wall has stabilized. Also note that the maximum and minimum points of the walls movement shift to a later time as the CSF becomes more dense, showing that it takes longer for the wall to push the CSF aside.

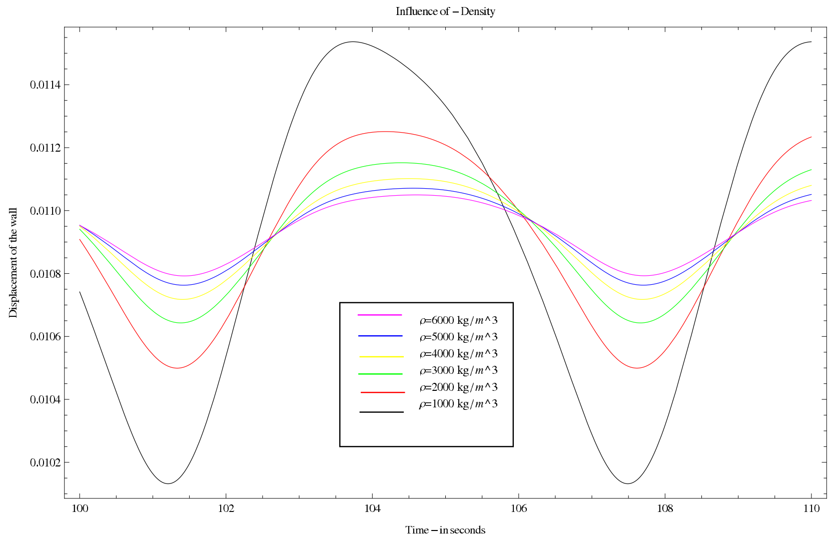

Figure 3.5: Influence of density of CSF.

3.4.3 Influence of Pulsatile Frequency of the Blood

As the frequency is increased, the period of the periodic movement of the arterial wall is expected to decrease, as is the amplitude. To verify this, the motion of the wall was investigated for different values of the pulsatile frequency. In Figure (3.6), the pulsatile frequency of the blood pressure (ω) is varied from $\omega = 0.5$ to $\omega = 1.5$. By looking at equation (3.40) we know that the frequency of the periodic motion depends only on ω so it makes sense that the period decreases as frequency increases. The fact that increasing the frequency results in the amplitude of the wall decreases means that the outer wall has less time to react to the pressure from the blood pushing before it switches directions, thus the lower amplitude.

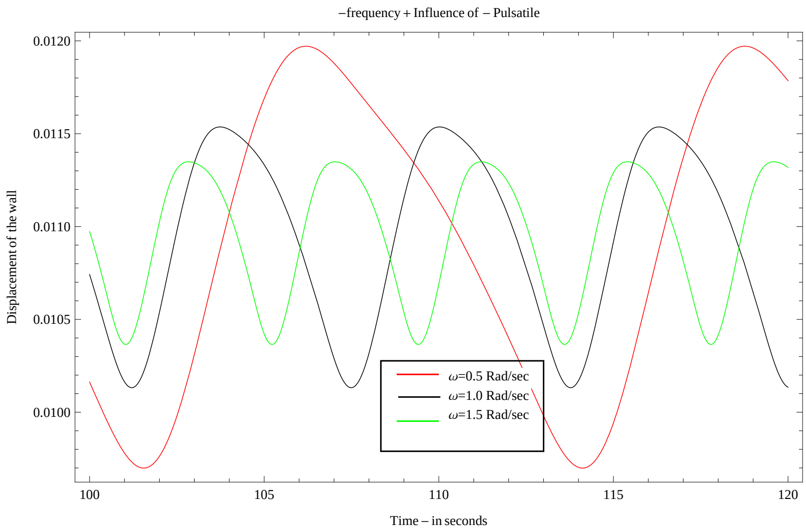

Figure 3.6: Influence of pulsatile frequency of the blood.

Chapter 4

MHD Model for Fluid Structure Interaction

In this chapter fluid structure interaction model is studied in the influence of external magnetic field. The effect of elasticity of the wall, density of the CSF and influence of pulsatile frequency of the blood on the movement of the arterial wall are analyzed.

4.1 Mathematical Model

Equation of motion (2.32) for CSF in the presence of external magnetic field becomes

$$\rho v_t + \rho v v_x + P_x - \mu v_{xx} = -\rho\sigma B_o^2 v. \tag{4.1}$$

Here the CSF is assumed inviscid and the nonlinear effects are negligible. These assumptions simplifies equation (4.1) to

$$\rho v_t + P_x = -\rho\sigma B_o^2 v. \tag{4.2}$$

Moreover we assume that the CSF is slightly compressible. Therefore, substitution of equation (3.13) into equation (4.2) gives

$$v_t - c^2 u_{xx} = -\sigma B_o^2 v, \tag{4.3}$$

using equation (3.15) in equation (4.3) we get

$$u_{tt} + \sigma B_o^2 u_t = c^2 u_{xx}. \tag{4.4}$$

The boundary conditions and initial conditions are same as given in equations (3.17), (3.23) and (3.24). Also the models for arterial wall and blood pressure are considered to be the same as described in the previous chapter.

4.2 Mathematical Solution

In order to simplify our work, we will rewrite all equations in terms of only $u(x,t)$, we obtain following fluid-structure interaction (FSI) problem

$$u_{tt} + \sigma B_o^2 u_t = c^2 u_{xx}, \tag{4.5}$$

$$u(x,0) = u_t(x,0) = 0, \tag{4.6}$$

$$mu_{tt}(0,t) = aP_{BLOOD}(t) - ku(0,t) + \rho c^2 a u_x(0,t), \tag{4.7}$$

$$u_t(L,t) = -cu_x(L,t), \tag{4.8}$$

where P_{BLOOD} is given by equation (3.16). Taking the Laplace Transform (see Appendix A) of equation (4.5)

$$s^2 U(x,s) + \sigma s B_o^2 U(x,s) = c^2 U_{xx}(x,s). \tag{4.9}$$

Equation (4.9) has the has the following solution

$$U(x,s) = c_1 \cosh(\frac{\sqrt{s^2 + \sigma s B_o^2}}{c} x) + c_2 \sinh(\frac{\sqrt{s^2 + \sigma s B_o^2}}{c} x), \tag{4.10}$$

where c_1 and c_2 are arbitrary constants. To find them take the Laplace transform of the boundary condition (4.8) at the point $x = L$, which gives

$$sU(L,s) = -cU_x(L,s). \tag{4.11}$$

Substitution of equation (4.10) into equation (4.11) gives

$$c_2 = -c_1 \frac{\sqrt{s^2 + \sigma s B_o^2}}{s}. \tag{4.12}$$

Next, take the Laplace transform of the boundary condition (4.7) at the point $x = 0$, which gives

$$\begin{aligned} ms^2 U(0,s) &= \frac{aPm}{s} - kU(0,s) + \rho c^2 a U_x(0,s) \\ &\quad + \sum_{n=1}^{N} (aA_n(\frac{s}{s^2 + (n\omega)^2}) + aB_n(\frac{n\omega}{s^2 + (n\omega)^2})). \end{aligned} \tag{4.13}$$

Substituting equation (4.10) into equation (4.13) we get

$$U(0,s) = \frac{\frac{aP_m}{s} + \sum_{n=1}^{N} (aA_n(\frac{s}{s^2+(n\omega)^2}) + aB_n(\frac{n\omega}{s^2+(n\omega)^2}))}{ms^2 + \rho ca(s + \sigma B_o^2) + k}. \tag{4.14}$$

Taking the inverse Laplace Transform of equation (4.14), we find that

$$\begin{aligned} u(0,t) &= F + Ge^{r_{11}t} + He^{r_{22}t} + \\ &\sum_{n=1}^{N}(M_n \cos(n\omega t) + \frac{O_n}{n\omega}\sin(n\omega t) + Q_n e^{r_{11}t} + R_n e^{r_{22}t}). \end{aligned} \tag{4.15}$$

In equation (4.15)

$$r_{11} = \frac{-\rho ca + \sqrt{\rho ca(\rho ca - 4m\sigma B_o^2) - 4mk}}{2m}, \tag{4.16}$$

$$r_{22} = \frac{-\rho ca - \sqrt{\rho ca(\rho ca - 4m\sigma B_o^2) - 4mk}}{2m}, \tag{4.17}$$

$$F = \frac{aP_m}{mr_{11}r_{22}}, \tag{4.18}$$

$$G = -\frac{aP_m}{r_{11}(r_{22} - r_{11})m}, \tag{4.19}$$

$$H = \frac{aP_m}{r_{22}(r_{22} - r_{11})m}, \tag{4.20}$$

$$M_n = -Q_n - R_n, \tag{4.21}$$

$$O_n = -r_{11}Q_n - r_{22}R_n, \tag{4.22}$$

$$Q_n = \frac{aA_n - mR_n(r_{22}^2 + n^2\omega^2)}{m(r_{11}^2 + n^2\omega^2)}, \tag{4.23}$$

$$R_n = \frac{a(r_{22}A_n + n\omega B_n)}{m(r_{22} - r_{11})(r_{22}^2 + n^2\omega^2)}. \tag{4.24}$$

Equation (4.15), describes the solution for the displacement of the CSF at the point $x = 0$. But the outer wall meets the CSF at the point $x = 0$, equation (4.15), is the equation that describes the movement of the outer wall for all time $t \geq 0$.

4.3 Analysis of Solution

Equation (4.15) is a sum of periodic terms, exponential terms and a constant. As both the periodic terms and the constant term are bounded, a finite sum of these terms is also bounded. Therefore to understand what this function looks like as t increases towards infinity we need to understand the contribution of exponential terms.

Note that all of the exponential terms are of the form ze^{st} where z is some constant and $s = r_{11}$ or r_{22}. For $s = r_{11}$ and r_{22} to be real, there is a restriction on our values for ρ, c, a, m, σ, B_o and k i.e.

$$\rho ca(\rho ca - 4m\sigma B_o^2) \geq 4mk. \tag{4.25}$$

For the bounded solution r_{11} and r_{22} must be negative. All the constants in r_{11} and r_{22} $(\rho, c, a, m, k, \sigma, B_o)$ are all positive values. So from equation (4.25) we can write

$$\frac{\rho ca + \sqrt{\rho ca(\rho ca - 4m\sigma B_o^2) - 4mk}}{2m} > 0. \tag{4.26}$$

Comparison of equations (4.26) and (4.17) gives $r_{22} < 0$.

For r_{11}, we consider the fact that

$$\sqrt{\rho ca(\rho ca - 4m\sigma B_o^2) - 4mk} < \rho ca. \tag{4.27}$$

Which makes $r_{11} < 0$.

So all the exponential terms will go to zero as t becomes very large. The value of the exponential terms at the point $t = 0$, will depend on the constant z. Thus for large values of t only the bounded periodic terms and the constant will affect the graph of $u(0, t)$. So we expect the function to start at $u(0, 0) = 0$ and become periodic over time.

4.4 Result and Discussion

In this section, we plot the displacement of the wall $u(0, t)$ versus time t and discuss the influence of different parameters on the displacement of the wall $u(0, t)$.

We consider the values for constants as we have used in section (3.4). Moreover we have use $\sigma = 1.79\ S/m$ [14] and $B_o = 0.2\ T$. Note that these values meet our restriction from the analytical solution given in equation (4.25).

The displacement of the wall $u(0, t)$ versus time t is plotted in Figure 4.1. It starts from zero and stabilizes after a few seconds. Note that, the analytical solution already predicted, initially there is a rapid increase in the displacement because of the exponential terms. Afterwards since r_1 and r_2 are negative, so for larger values of time t, exponential terms will becomes negligible and have no effect and stabilizes into a bounded periodic motion. Comparison of Figure 4.1 and 3.3 it is seen that the displacement of the arterial wall from origin is less in the presence of the external magnetic field. The results show the inclusion of the MHD stabilizes the movement of the arterial wall and in turn prevent the rupture of the cerebral aneurysm.

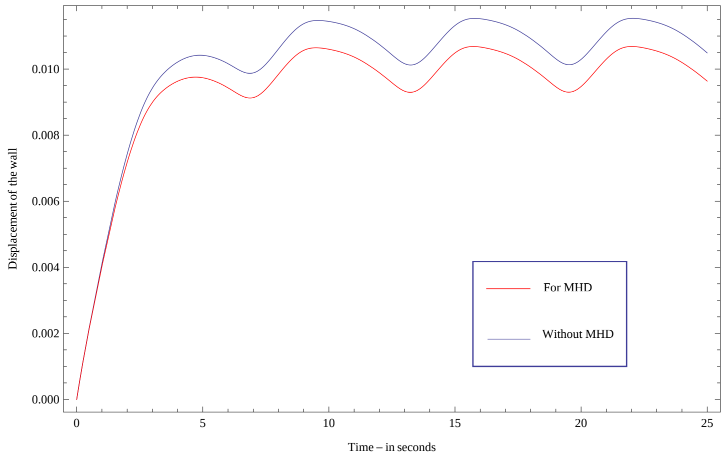

Figure 4.1: Analytical Solution under MHD affect.

4.4.1 Influence of the Spring-Constant

The stiffness of the arterial wall physically models by the spring constant. Therefore, as the spring constant is decreased, the wall becomes more flexible, which causes the wall to move further out and take longer to stabilize. Figure 4.2 illustrates this. In Figure 4.2 we plot the solution for decreasing values of the spring constant from $k = 8000\ N/m$ to $k = 3000\ N/m$. We can see that as k decreases the displacement is greater and it takes longer for the wall to settle into a steady periodic motion.

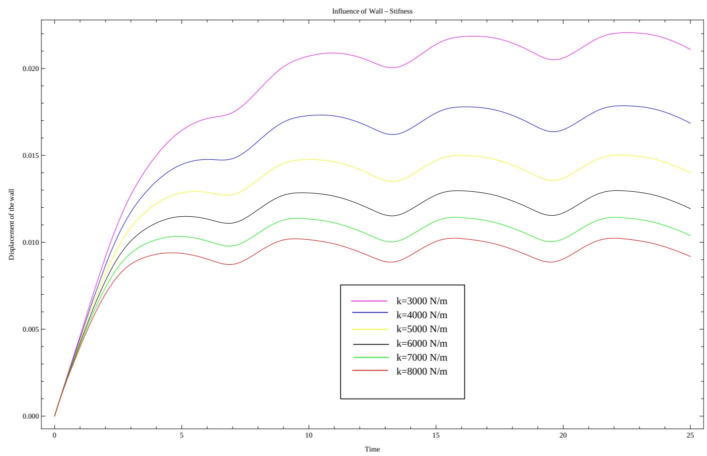

Figure 4.2: Influence of wall stiffness.

4.4.2 Influence of the CSF's Density

As the CSF becomes more dense, it resists the movement of the wall, so the amplitude of the periodic movement of the wall is expected to become much less. Figure 4.3 illustrates this fact. In Figure 4.3 we plot the motion of the wall for increasing values of the density from $\rho = 1000\ kg/m^3$ to $\rho = 6000\ kg/m^3$ of the CSF. Note that this graph is at a later time period than Figure 4.2, it is after the movement of the wall has stabilized. Also note that the maximum and minimum points of the walls movement shift to a later time as the CSF becomes more dense, showing that it takes longer for the wall to push the CSF aside.

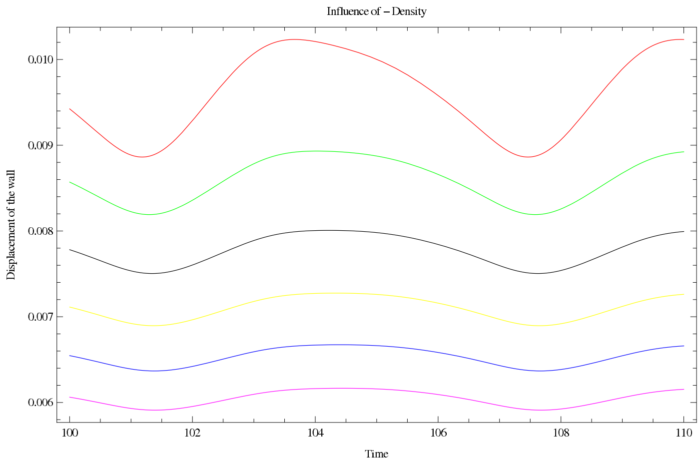

Figure 4.3: Influence of density of CSF.

4.4.3 Influence of Pulsatile Frequency of the Blood

With the increase in the the frequency, the period of the periodic movement of the wall is expected to decrease, as is the amplitude. To verify this, the motion of the wall was investigated for different values of the pulsatile frequency. In Figure 4.4 the pulsatile frequency of the blood pressure (ω) is varied from $\omega = 0.5$ to $\omega = 1.5$. By looking at equation (4.15) we know that the frequency of the periodic motion depends only on ω so it makes sense that the period decreases as frequency increases. The fact that increasing the frequency results in the amplitude of the wall decreases means that the outer wall has less time to react to the pressure from the blood pushing before it switches directions, thus the lower amplitude.

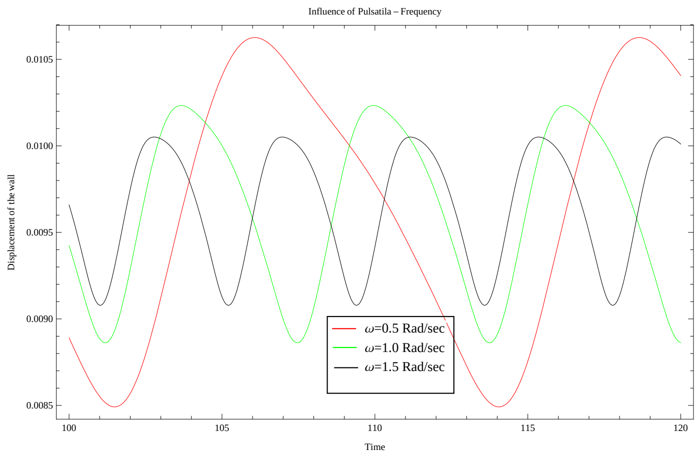

Figure 4.4: Influence of pulsatile frequency of the blood.

Chapter 5

FSI Model of a MHD Newtonian Fluid with Viscosity

In this chapter we extend our model one step closer to real situation by including the effect of viscosity in the presence of magnetic field.

5.1 Mathematical Model

When we consider the CSF to be a real fluid, that is we have taken viscosity of the CSF in account. In order to solve equation (4.1) for real CSF it is assumed CSF is slightly compressible and the non-linear effects are negligible. Now under these considerations equation (4.1) will reduced to the following one

$$v_t + c^2 u_{xx} - \frac{\mu}{\rho} v_{xx} = -\sigma B_o^2 v, \tag{5.1}$$

now using equation (3.15) in equation (5.1), we get

$$u_{tt} + \sigma B_o^2 u_t - \frac{\mu}{\rho} u_{txx} = c^2 u_{xx}. \tag{5.2}$$

5.2 Mathematical Solution

In order to simplify our work, we will rewrite all equations in terms of only $u(x, t)$, we obtain the following fluid-structure interaction (FSI) problem

$$u_{tt} + \sigma B_o^2 u_t - \frac{\mu}{\rho} u_{txx} = c^2 u_{xx}, \tag{5.3}$$

$$u(x, 0) = u_t(x, 0) = 0, \tag{5.4}$$

$$m u_{tt}(0, t) = a P_{BLOOD}(t) - k u(0, t) + \rho c^2 a u_x(0, t), \tag{5.5}$$

$$u_t(L,t) = -cu_x(L,t), \tag{5.6}$$

where P_{BLOOD} is given by equation (3.16). The Laplace transform of equation (5.3) is as follows

$$s^2U(x,s) + \sigma s B_o^2 U(x,s) = c^2 U_{xx}(x,s) + \frac{\mu}{\rho} s U_{xx}. \tag{5.7}$$

Equation (5.7) has the has the following solution

$$U(x,s) = c_1 \cosh(\sqrt{\frac{\rho(s^2 + \sigma s B_o^2)}{\mu s + \rho c^2}} x) + c_2 \sinh(\sqrt{\frac{\rho(s^2 + \sigma s B_o^2)}{\mu s + \rho c^2}} x), \tag{5.8}$$

where c_1 and c_2 are arbitrary constants. To find them take the Laplace transform of the boundary condition (5.6) at the point $x = L$, which gives

$$sU(L,s) = -cU_x(L,s). \tag{5.9}$$

Substitution of equation (5.8) into equation (5.9) gives

$$c_2 = -\frac{c}{s}\sqrt{\frac{\rho(s^2 + \sigma s B_o^2)}{\mu s + \rho c^2}} c_1. \tag{5.10}$$

Next, take the Laplace Transform of the boundary condition (5.5) at the point $x = 0$, which gives

$$\begin{aligned} ms^2U(0,s) &= \frac{aP_m}{s} - kU(0,s) + \rho c^2 a U_x(0,s) \\ &+ \sum_{n=1}^{N} (aA_n(\frac{s}{s^2 + (n\omega)^2}) + aB_n(\frac{n\omega}{s^2 + (n\omega)^2})). \end{aligned} \tag{5.11}$$

Substituting equation (5.8) into equation (5.11) we get

$$U(0,s) = \frac{\frac{aP_m}{s} + \sum_{n=1}^{N}(aA_n(\frac{s}{s^2+(n\omega)^2}) + aB_n(\frac{n\omega}{s^2+(n\omega)^2}))}{ms^2 + \rho^2 c^3 a \frac{(s+\sigma B_o^2)}{\mu s + \rho c^2} + k}. \tag{5.12}$$

The inverse Laplace Transform using MATHEMATICA of equation (5.12), will give us the displacement of the wall at $x = 0$.

5.3 Results and Discussion

This section shows the plot of the displacement of the wall $u(0,t)$ versus time t. We consider the values for constants as we have used in sections (3.4) and (4.3). Moreover we have used $\mu = 1.0$ i.e viscosity of the CSF [15]. Figure 5.1 represents the analytical solution for the displacement of the outer wall $u(0,t)$. Because of initial conditions the displacement starts at zero stabilizes into a bounded periodic motion. It is seen that for viscosity $0.7 - 1.0$ [15] will have negligible effect on the movement of the arterial wall at $x = 0$.

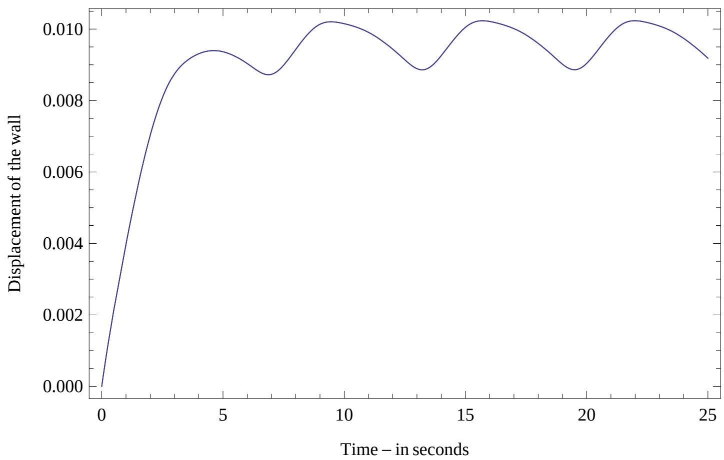

Figure 5.1: Analytical solution in the Presence of Viscosity.

Chapter 6

Conclusions and Future Work

6.1 Conclusions

In this dissertation we have build a one dimensional coupled model of a cerebral aneurysm, that incorporates the interaction between the blood pressure, the wall structure and the cerebral spinal fluid that surrounds the aneurysm. The blood pressure is considered to be pulsatile and modeled by the Fourier series. The cerebral spinal fluid is taken as a slightly compressible inviscid Newtonian fluid. Furthermore we assume the non-linear effects are negligible. To model the arterial wall we use spring and mass system. The governing mathematical equations are solved mathematically using Laplace Transform method. The arterial wall displacement is calculated at $x = 0$ i.e., at the point where the CSF meet with the arterial wall.

It is shown that the displacement of the wall starts from zero and stabilizes after a few seconds. The analytical solution predicted, initially there is a rapid increase in the displacement which then stabilizes into a bounded periodic motion. The graphs are the plotted for the displacement of the wall versus time. Influence of the various parameters, stiffness of the wall, density of the CSF and the pulsatile frequency of the blood are studied. We have further extended our fluid structure interaction model to include the effect of the external magnetic field on the movement of the wall. The governing equations are solved analytically and results are analyzed. It is shown that in the presence of the magnetic field the displacement of the wall is less as compared to the model without MHD. It is concluded MHD reduces the wall movement and therefore reduce the risk of rupture of the cerebral aneurysm. The influence of the various parameters, stiffness of the wall, density of the CSF and the pulsatile frequency of the blood are studied in the presence of the external magnetic field.

6.2 Future Work

Future work would be carried out with the aim of a better understanding of the fluid structure interaction model for the cerebral aneurysm. In particular the non-linear effects can be included and more realistic non-Newtonian fluid model for example, Power Law fluid model for CSF can be considered.

Appendix A

Laplace Transform

The Laplace transform which is frequently used for the initial value problems, with time t as the independent variable is defined as

$$\overline{f}(p) = \mathcal{L}[f](p) = \int_{-0}^{\infty} f(t)e^{-pt}dt, \tag{A.1}$$

where p is the Laplace transform parameter. Let the integral (A.1) converges for some $p = p_1$, then it also converges for any value of p satisfying $\mathrm{Re}(p) > \mathrm{Re}(p_1)$. The function $\overline{f}(p)$ is a regular function of the complex variable p for $\mathrm{Re}(p) > \mathrm{Re}(p_1)$. The inverse Laplace transform is given by

$$f(t) = \mathcal{L}^{-1}[\overline{f}](t) = \frac{1}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} \overline{f}(p)e^{pt}dp, \tag{A.2}$$

where $\mathrm{Re}(\gamma) > \mathrm{Re}(p_1)$. Thus the path of integration in Eq. (A.2) can be any vertical line to the right of all singularities of $\overline{f}(p)$

The Laplace transforms of the derivative $f'(t)$ of a function $f(t)$ can be obtained by integration by parts

$$\mathcal{L}[f'](p) = p\overline{f}(p) - f(-0), \tag{A.3}$$

$$\mathcal{L}[f''](p) = p^2\overline{f}(p) - pf(-0) - f'(-0). \tag{A.4}$$

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