Some fixed point theorems for single-valued and multi-valued contraction mappings

by

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A dissertation submitted in partial fulfillment of the requirements

for the degree of Master of Philosophy in Mathematics

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2013

Introduction

Banach showed that every contraction on a complete metric space (U, d) has a unique fixed point. A number of authors introduced various contractive type conditions in order to generalize Banach fixed point theorem. Rhoades has given a comparison of various contractive type conditions, considered by different authors. Rhoades pointed out that some contractive conditions are equivalent, some depend upon others whereas some are independent.

Nadler extended Banach contraction principle to multi-valued mappings by generalizing the definition of contraction from single-valued to multi-valued mappings. Since then, many authors extended contractive type conditions from single-valued to multi-valued mappings.

In this dissertation, we define some new contractive type conditions in order to generalize and unify some existing contractive type conditions and prove some fixed point theorems. In Chapter one, basic definitions and results are given, which are needed for subsequent chapters. In Chapter two, we obtain some fixed point theorems for single-valued as well as multi-valued mappings by using new contractive conditions. In Chapter three, we extend some results of Chapter two to partially ordered metric space, whereas in Chapter four, we define another contractive condition and obtain some fixed point theorems for pre-ordered metric spaces.

Contents

Chapter 1

Preliminaries

In this chapter, we recall some definitions and results which are needed in the sequel. In the last section of this chapter, we define a new contractive condition. We use this contractive condition in the subsequent chapters.

Throughout this dissertation, It is assume that (U, d) be a metric space, $N(U)$ denotes the set of each nonempty subsets of U, $B(U)$ denotes the set of each nonempty bounded subsets of U and $CB(U)$ denotes the set of each nonempty closed and bounded subsets of U.

1.1 Some basic definitions and results

Definition 1.1.1. Let U be a nonempty set. A relation $\preceq$ on U is called partial ordering if it is reflexive, antisymmetric and transitive, i.e., for each $u_1, u_2, u_3 \in U$, we have

(i) $u_1 \preceq u_1$;

(ii) $u_1 \preceq u_2$ and $u_2 \preceq u_1$, then $u_1 = u_2$;

(iii) $u_1 \preceq u_2$ and $u_2 \preceq u_3$, then $u_1 \preceq u_3$.

A set together with a partial order is a partially ordered set. A relation $\preceq$ on U is called pre-ordering, if it is reflexive and transitive.

Example 1.1.2. Let $U \neq \emptyset$ and $P(U)$ is the power set of U. Then the set inclusion $\subseteq$ is a partial ordering on $P(U)$.

Example 1.1.3. Let $U = \mathbb{R}^2$. Define $\preceq$ on $\mathbb{R}^2$ by $(u_1, v_1) \preceq (u_2, v_2)$ if and only if $u_1 \leq u_2$ and $v_1 \leq v_2$. Then $\preceq$ is a partial ordering on $\mathbb{R}^2$.

Definition 1.1.4. For a set $A \subseteq (U, d)$, the distance of $x \in U$ from A is defined by $D(x, A) = \inf\{d(x, a) : a \in A\}$.

Example 1.1.5. Let $U = [0, 1]$ be endowed with the usual metric d and $A = U \cap \mathbb{Q}$. Then $D(u, A) = 0$ for each $u \in U$.

Definition 1.1.6. A set $B \subseteq (U, d)$ is called bounded if and only if $\delta(B) = \sup\{d(a, b) : a, b \in B\}$ is finite. $\delta(B)$ is called diameter of the set B.

Example 1.1.7. Let $U = [0, 1]$ be endowed with the usual metric d and $A = U \cap \mathbb{Q}$. Then $\delta(A) = 1$.

Definition 1.1.8. For $A, B \subseteq (U, d)$, we define

$$\delta(A, B) = \sup\{d(a, b) : a \in A,\ b \in B\}, \tag{1.1.1}$$

and

$$H(A, B) = \max\{\sup_{a \in A} D(a, B), \sup_{b \in B} D(b, A)\}. \tag{1.1.2}$$

Theorem 1.1.9. $H(A, B)$ is a metric on $CB(U)$.

Remark 1.1.10. Note that $(CB(U), H)$ is complete if and only if (U, d) is complete.

Definition 1.1.11. Let $T : U \to U$ be a mapping. Fixed point of T is a point $u \in U$ when $u = Tu$. Coincidence point of $T, S : U \to U$ is a point $u \in U$ when $Su = Tu$. Moreover, if $u = Su = Tu$ then u is a common fixed point of T and S.

Example 1.1.12. Let $U = [0, \infty)$. Define $T_i : U \to U$ for $i = 1, 2, 3$ as follows.

- $T_1 u = u + 1$ for each $u \in U$. Then T_1 has no fixed point.
- $T_2 u = \frac{u}{2}$ for each $u \in U$. Then T_2 has a unique fixed point, i.e., $u = 0$.
- $T_3 u = u$ for each $u \in U$. Then each point of U is a fixed point of T_3, i.e., T_3 has infinitely many fixed points.

The following theorem, usually know as Banach fixed point theorem, was first appeared in Banach's doctoral thesis [1].

Theorem 1.1.13 (Banach [1])**.** Let $T : U \to U$ be a mapping and d be complete. Assume that there exists $\alpha \in [0, 1)$ such that

$$d(Tu, Tv) \leq \alpha d(u, v), \ \forall \ u, v \in U. \tag{1.1.3}$$

Then T has a unique fixed point.

Nadler [11] extended Banach fixed point theorem to multi-valued mappings in the following way.

Theorem 1.1.14 (Nadler [11])**.** Let $T : U \to CB(U)$ be a mapping and d be a complete. Assume that their exists $\alpha \in [0, 1)$ such that

$$H(Tu, Tv) \leq \alpha d(u, v), \ \forall \ u, v \in U. \tag{1.1.4}$$

Then T has a fixed point.

Definition 1.1.15 (Berinde et al. [2])**.** We say that two self mappings S and T on (U, d) are weakly compatible if they commute at their coincidence points, i.e., $TSu = STu$ whenever $Su = Tu$.

Definition 1.1.16 (Kamran [8])**.** Let $T : U \to B(U)$ be a mapping. Then a mapping $f : U \to U$ is said to be T-weakly commuting at $u \in U$ if $ffu \in Tfu$.

1.2 Contractive type mappings

In this section we consider various contractive type conditions defined by different authors. Throughout this section T is a mapping from U into U.

Banach used the contractive condition given in (1.1.3) on the mapping T.

Following condition for contraction was introduced by Kannan [9]. There exists a real number $\alpha \in [0, \frac{1}{2})$ such that

$$d(Tu, Tv) \leq \alpha(d(u, Tu) + d(v, Tv)), \ \forall \ u, v \in U. \tag{1.2.1}$$

Another condition is due to Bainchini [3] which states that there exists a real number $\alpha \in [0, 1)$ such that

$$d(Tu, Tv) \leq \alpha \max\{d(u, Tu), d(v, Tv))\}, \ \forall \ u, v \in U. \tag{1.2.2}$$

Following condition was given by Reich [14]. There exist nonnegative real numbers a, b, c satisfying $a + b + c < 1$ such that

$$d(Tu, Tv) \leq ad(u, Tu) + bd(v, Tv) + cd(u, v), \ \forall \ u, v \in U. \tag{1.2.3}$$

Chatterjea [4] introduced that there exists a real number $\alpha \in [0, \frac{1}{2})$ such that

$$d(Tu, Tv) \leq \alpha(d(u, Tv) + d(v, Tu)), \ \forall \ u, v \in U. \tag{1.2.4}$$

Following condition is due to Hardy and Rogers [7]. There exist nonnegative real numbers a, b, c satisfying $a + 2b + 2c < 1$ such that

$$d(Tu, Tv) \leq ad(u, v) + b(d(u, Tu) + d(v, Tv)) + c(d(v, Tu) + d(u, Tv)), \ \forall \ u, v \in U. \tag{1.2.5}$$

Following conditions are mentioned by Rhoades [15]. There exists a real number $\alpha \in [0, 1)$ such that

$$d(Tu, Tv) \leq \alpha \max\left\{d(u, v), \frac{d(u, Tu) + d(v, Tv)}{2}, \frac{d(v, Tu) + d(u, Tv)}{2}\right\}, \ \forall \ u, v \in U. \tag{1.2.6}$$

There exists a real number $\alpha \in [0, 1)$ such that

$$d(Tu, Tv) \leq \alpha \max\left\{d(u, v), d(u, Tu), d(v, Tv)), \frac{d(v, Tu) + d(u, Tv)}{2}\right\}, \ \forall \ u, v \in U. \tag{1.2.7}$$

Rhoades [15] gave a comparison of above contraction conditions along with other contractive conditions. Now looking at the above contractive conditions, we introduce the following.

Definition 1.2.1. Let $\mathbb{R}_+ = [0, \infty)$. A mapping $T : U \to U$ is said to be a ϕ-contraction, if there exists a mapping $\phi : {\mathbb{R}_+}^4 \to \mathbb{R}_+$ such that

$$d(Tu, Tv) \leq \phi\left(d(u, v), d(u, Tu), d(v, Tv), \frac{d(v, Tu) + d(u, Tv)}{2}\right), \ \forall \ u, v \in U. \tag{1.2.8}$$

Note that the contractive conditions (1.1.3) and (1.2.2) $-$ (1.2.7) can be obtained from (1.2.8) for different choices of function ϕ. For example, (1.2.5) follows from (1.2.8) by taking $\phi(u_1, u_2, u_3, u_4) = au_1 + b(u_2 + u_3) + cu_4$, where a, b, c are nonnegative real numbers such that $a + 2b + c \in [0, 1)$.

Chapter 2

Fixed point theorems for ϕ-contractions

Banach contraction principle [1] says that every contraction T on a complete metric space $U = (U, d)$ has a unique fixed point. Also, starting from any $u_0 \in U$ the sequence of iterates $T^n u_0$ converges to unique fixed point of T. A number of authors generalized Banach contraction principle by introducing various contractive conditions. In Section 2.1 of this chapter, we prove a fixed point theorem for ϕ-contraction whereas in Section 2.2, we extend the result of section one to multi-valued mappings.

2.1 Single-valued ϕ-contractions

Let $\psi : [0, \infty) \to [0, \infty)$ be a nondecreasing mapping such that $\sum_{n=1}^{\infty} \psi^n(t) < \infty$ for all $t \geq 0$ and $\psi(t) < t$ for all $t > 0$. By Φ we denote the family of functions $\phi : {\mathbb{R}_+}^4 \to \mathbb{R}_+ = [0, \infty)$ satisfying the following conditions:

(i) ϕ is continuous and nondecreasing in each coordinate;

(ii) let $u_1, u_2 \in \mathbb{R}_+$ such that if $u_1 < u_2$ and $u_1 \leq \phi(u_2, u_2, u_1, u_2)$ then $u_1 \leq \psi(u_2)$. Further if $u_1 \geq u_2$ and $u_1 \leq \phi(u_1, u_2, u_1, u_1)$ then $u_1 = 0$;

(iii) if $u \in \mathbb{R}_+$ such that $u \leq \phi(0, 0, u, \frac{1}{2}u)$ or $u \leq \phi(0, u, 0, \frac{1}{2}u)$ or $u \leq \phi(u, 0, 0, u)$ then $u = 0$.

Example 2.1.1. Let $\phi_1(u_1, u_2, u_3, u_4) = \alpha \max\{u_1, u_2, u_3, u_4\}$, where $\alpha \in [0, 1)$. Clearly ϕ is continuous and nondecreasing in each coordinate. If $u_1 < u_2$ and $u_1 \leq \phi(u_2, u_2, u_1, u_2)$, then $u_1 \leq \alpha u_2$. If $u_1 \geq u_2$ and $u_1 \leq \phi(u_1, u_2, u_1, u_1)$, then $u_1 \leq \alpha u_1$ which implies $u_1 = 0$. If $u \leq \phi(0, 0, u, \frac{1}{2}u)$, then $u \leq \alpha u$ which implies $u = 0$. Similarly, for $u \leq \phi(0, u, 0, \frac{1}{2}u)$ and $u \leq \phi(u, 0, 0, u)$, we have $u = 0$. Taking $\psi(t) = \alpha t$ implies $\phi_1 \in \Phi$.

Example 2.1.2. Let $\phi_2(u_1, u_2, u_3, u_4) = \alpha u_4$, where $\alpha \in [0, 1)$. Clearly ϕ is continuous and nondecreasing in each coordinate. If $u_1 < u_2$ and $u_1 \leq \phi(u_2, u_2, u_1, u_2)$, then $u_1 \leq \alpha u_2$. If $u_1 \geq u_2$ and $u_1 \leq \phi(u_1, u_2, u_1, u_1)$, then $u_1 \leq \alpha u_1$ which implies $u_1 = 0$. If $u \leq \phi(0, 0, u, \frac{1}{2}u)$, then $u \leq (\alpha/2)u$ which implies $u = 0$. Similarly, for $u \leq \phi(0, u, 0, \frac{1}{2}u)$ and $u \leq \phi(u, 0, 0, u)$, we have $u = 0$. Taking $\psi(t) = \alpha t$ implies $\phi_2 \in \Phi$.

Example 2.1.3. Let $\phi_3(u_1, u_2, u_3, u_4) = \alpha \max\{u_1, u_2, u_3\}$, where $\alpha \in [0, 1)$. Clearly ϕ is continuous and nondecreasing in each coordinate. If $u_1 < u_2$ and $u_1 \leq \phi(u_2, u_2, u_1, u_2)$, then $u_1 \leq \alpha u_2$. If $u_1 \geq u_2$ and $u_1 \leq \phi(u_1, u_2, u_1, u_1)$, then $u_1 \leq \alpha u_1$ which implies $u_1 = 0$. If $u \leq \phi(0, 0, u, \frac{1}{2}u)$, then $u \leq \alpha u$ which implies $u = 0$. Similarly, for $u \leq \phi(0, u, 0, \frac{1}{2}u)$ and $u \leq \phi(u, 0, 0, u)$, we have $u = 0$. Taking $\psi(t) = \alpha t$ implies $\phi_3 \in \Phi$.

Example 2.1.4. Let $\phi_4(u_1, u_2, u_3, u_4) = \alpha \max\{u_2, u_3\}$, where $\alpha \in [0, 1)$. Clearly ϕ is continuous and nondecreasing in each coordinate. If $u_1 < u_2$ and $u_1 \leq \phi(u_2, u_2, u_1, u_2)$, then $u_1 \leq \alpha u_2$. If $u_1 \geq u_2$ and $u_1 \leq \phi(u_1, u_2, u_1, u_1)$, then $u_1 \leq \alpha u_1$ which implies $u_1 = 0$. If $u \leq \phi(0, 0, u, \frac{1}{2}u)$, then $u \leq \alpha u$ which implies $u = 0$. Similarly, for $u \leq \phi(0, u, 0, \frac{1}{2}u)$ and $u \leq \phi(u, 0, 0, u)$, we have $u = 0$. Taking $\psi(t) = \alpha t$ implies $\phi_4 \in \Phi$.

Example 2.1.5. Let $\phi_5(u_1, u_2, u_3, u_4) = \alpha u_1$, where $\alpha \in [0, 1)$. Clearly ϕ is continuous and nondecreasing in each coordinate. If $u_1 < u_2$ and $u_1 \leq \phi(u_2, u_2, u_1, u_2)$, then $u_1 \leq \alpha u_2$. If $u_1 \geq u_2$ and $u_1 \leq \phi(u_1, u_2, u_1, u_1)$, then $u_1 \leq \alpha u_1$ which implies $u_1 = 0$. If $u \leq \phi(u, 0, 0, u)$, then $u \leq \alpha u$, which implies $u = 0$. Similarly, for $u \leq \phi(0, u, 0, \frac{1}{2}u)$ and $u \leq \phi(0, 0, u, \frac{1}{2}u)$, we have $u = 0$. Taking $\psi(t) = \alpha t$ implies $\phi_5 \in \Phi$.

Example 2.1.6. Let $\phi_6(u_1, u_2, u_3, u_4) = \frac{\alpha}{2}(u_2 + u_3)$, where $\alpha \in [0, 1)$. Clearly ϕ is continuous and nondecreasing in each coordinate. If $u_1 < u_2$ and $u_1 \leq \phi(u_2, u_2, u_1, u_2)$, then $u_1 \leq \frac{\alpha}{2}(u_2 + u_1) < \alpha u_2$. If $u_1 \geq u_2$ and $u_1 \leq \phi(u_1, u_2, u_1, u_1)$, then $u_1 \leq \frac{\alpha}{2}(u_2 + u_1) \leq \alpha u_1$ which implies $u_1 = 0$. If $u \leq \phi(0, 0, u, \frac{1}{2}u)$, then $u \leq \frac{\alpha}{2}u$ which implies $u = 0$. Similarly, for $u \leq \phi(0, u, 0, \frac{1}{2}u)$ and $u \leq \phi(u, 0, 0, u)$, we have $u = 0$. Taking $\psi(t) = \alpha t$ implies $\phi_6 \in \Phi$.

Example 2.1.7. Let $\phi_7(u_1, u_2, u_3, u_4) = \alpha \max\{u_1, \frac{1}{2}(u_2 + u_3), u_4\}$, where $\alpha \in [0, 1)$. Clearly ϕ is continuous and nondecreasing in each coordinate. If $u_1 < u_2$ and $u_1 \leq \phi(u_2, u_2, u_1, u_2)$, then $u_1 \leq \alpha u_2$ (note that since $u_1 < u_2$ so $u_1+u_2 < 2u_2 \implies \frac{1}{2}(u_1+u_2) < u_2$). If $u_1 \geq u_2$ and $u_1 \leq \phi(u_1, u_2, u_1, u_1)$, then $u_1 \leq \alpha u_1$ (note that since $u_1 \geq u_2$ so $2u_1 \geq u_1 + u_2 \implies u_1 \geq \frac{1}{2}(u_1 + u_2)$) which implies $u_1 = 0$. If $u \leq \phi(0, 0, u, \frac{1}{2}u)$, then $u \leq \frac{\alpha}{2}u$ which implies $u = 0$. Similarly, for $u \leq \phi(0, u, 0, \frac{1}{2}u)$ and $u \leq \phi(u, 0, 0, u)$, we have $u = 0$. Taking $\psi(t) = \alpha t$ implies $\phi_7 \in \Phi$.

Example 2.1.8. Let $\phi_8(u_1, u_2, u_3, u_4) = au_1 + b(u_2 + u_3) + cu_4$, where a, b, c are non-negative real numbers such that $a + 2b + c \in [0, 1)$. Clearly ϕ is continuous and nondecreasing in each coordinate. If $u_1 < u_2$ and $u_1 \leq \phi(u_2, u_2, u_1, u_2)$, then $u_1 \leq au_2 + b(u_2 + u_1) + cu_2 < (a + 2b + c)u_2$. If $u_1 \geq u_2$ and $u_1 \leq \phi(u_1, u_2, u_1, u_1)$, then $u_1 \leq au_1+b(u_2+u_1)+cu_1 \leq (a+2b+c)u_1$ which implies $u_1 = 0$. If $u \leq \phi(0, 0, u, \frac{1}{2}u)$, then $u \leq (b + \frac{c}{2})u$ which implies $u = 0$. Similarly, for $u \leq \phi(0, u, 0, \frac{1}{2}u)$ and $u \leq \phi(u, 0, 0, u)$, we have $u = 0$. Taking $\psi(t) = (a + 2b + c)t$ implies $\phi_8 \in \Phi$.

Example 2.1.9. Let $\phi_9(u_1, u_2, u_3, u_4) = au_2+bu_3+cu_1$, where a, b, c are nonnegative real numbers such that $a + b + c \in [0, 1)$. Clearly ϕ is continuous and nondecreasing in each coordinate. If $u_1 < u_2$ and $u_1 \leq \phi(u_2, u_2, u_1, u_2)$, then $u_1 \leq au_2+bu_1+cu_2 < (a+b+c)u_2$. If $u_1 \geq u_2$ and $u_1 \leq \phi(u_1, u_2, u_1, u_1)$, then $u_1 \leq au_2+bu_1+cu_1 \leq (a+b+c)u_1$ which implies $u_1 = 0$. If $u \leq \phi(0, 0, u, \frac{1}{2}u)$, then $u \leq a(0) + b(u) + c(0)$ which implies $u = 0$. Similarly, for $u \leq \phi(0, u, 0, \frac{1}{2}u)$ and $u \leq \phi(u, 0, 0, u)$, we have $u = 0$. Taking $\psi(t) = (a + b + c)t$ implies $\phi_9 \in \Phi$.

We are now in a position to state and prove our first result.

Theorem 2.1.10. Let $T : U \to U$ be a ϕ-contraction and d be complete. Then T has a unique fixed point whenever $\phi \in \Phi$.

Proof. Let $\{u_n\}$ be a sequence in U such that $u_{n+1} = Tu_n$ for each $n \in \mathbb{N} \cup \{0\}$. If $u_{N+1} = u_N$ for some $N \in \mathbb{N} \cup \{0\}$. Then u_N is a fixed point. Suppose $u_{n+1} \neq u_n$ for each

$n \in \mathbb{N} \cup \{0\}$. From (1.2.8), we have

$$\begin{aligned} d(u_{n+1}, u_{n+2}) &= d(Tu_n, Tu_{n+1}) \\ &\leq \phi\Big(d(u_n, u_{n+1}), d(u_n, Tu_n), d(u_{n+1}, Tu_{n+1}), \\ &\quad \frac{d(u_{n+1}, Tu_n) + d(u_n, Tu_{n+1})}{2}\Big) \\ &= \phi\Big(d(u_n, u_{n+1}), d(u_n, u_{n+1}), d(u_{n+1}, u_{n+2}), \\ &\quad \frac{d(u_{n+1}, u_{n+1}) + d(u_n, u_{n+2})}{2}\Big). \end{aligned} \tag{2.1.1}$$

By triangular inequality, we have

$$d(u_n, u_{n+2}) \leq d(u_n, u_{n+1}) + d(u_{n+1}, u_{n+2}). \tag{2.1.2}$$

We claim that $d(u_{n+1}, u_{n+2}) < d(u_n, u_{n+1})$ for each $n \in \mathbb{N} \cup \{0\}$. Suppose on contrary that $d(u_{n+1}, u_{n+2}) \geq d(u_n, u_{n+1})$ for some $n \in \mathbb{N} \cup \{0\}$. From (2.1.2), we get $d(u_n, u_{n+2}) \leq 2d(u_{n+1}, u_{n+2})$. Since ϕ is nondecreasing, by using these in (2.1.1), we have

$$d(u_{n+1}, u_{n+2}) \leq \phi(d(u_{n+1}, u_{n+2}), d(u_n, u_{n+1}), d(u_{n+1}, u_{n+2}), d(u_{n+1}, u_{n+2})). \tag{2.1.3}$$

By (2.1.3) and property (ii) of Φ, we have

$$d(u_{n+1}, u_{n+2}) = 0.$$

A contradiction to our assumption that $u_{n+1} \neq u_n$ for each $n \in \mathbb{N} \cup \{0\}$. Thus $d(u_{n+1}, u_{n+2}) < d(u_n, u_{n+1})$ for each $n \in \mathbb{N} \cup \{0\}$. From (2.1.2), we have $d(u_n, u_{n+2}) < 2d(u_n, u_{n+1})$. Using it in (2.1.1), we have

$$d(u_{n+1}, u_{n+2}) \leq \phi(d(u_n, u_{n+1}), d(u_n, u_{n+1}), d(u_{n+1}, u_{n+2}), d(u_n, u_{n+1})). \tag{2.1.4}$$

By (2.1.4) and property (ii) of Φ, we have

$$d(u_{n+1}, u_{n+2}) \leq \psi(d(u_n, u_{n+1})), \ \forall \ n \in \mathbb{N} \cup \{0\}.$$

Continuing in the same way, we get

$$d(u_{n+1}, u_{n+2}) \leq \psi^{n+1}(d(u_0, u_1)), \ \forall \ n \in \mathbb{N} \cup \{0\}. \tag{2.1.5}$$

Let $n > m$. Then we have

$$\begin{aligned} d(u_m, u_n) &\leq d(u_m, u_{m+1}) + d(u_{m+1}, u_{m+2}) + \cdots + d(u_{n-1}, u_n) \\ &\leq \psi^m(d(u_0, u_1)) + \psi^{m+1}(d(u_0, u_1)) + \cdots + \psi^{n-1}(d(u_0, u_1)) \\ &= \sum_{i=m}^{n-1} \psi^i(d(u_0, u_1)). \end{aligned}$$

Therefore $\{u_n\}$ is a Cauchy sequence in U. By completeness of U, there exists $u^* \in U$ such that $u_n \to u^*$ as $n \to \infty$. From (1.2.8), we have

$$\begin{aligned} d(u_{n+1}, Tu^*) &= d(Tu_n, Tu^*) \\ &\leq \phi\Big(d(u_n, u^*), d(u_n, Tu_n), d(u^*, Tu^*), \\ &\qquad \frac{d(u^*, Tu_n) + d(u_n, Tu^*)}{2}\Big) \\ &= \phi\Big(d(u_n, u^*), d(u_n, u_{n+1}), d(u^*, Tu^*), \\ &\qquad \frac{d(u^*, u_{n+1}) + d(u_n, Tu^*)}{2}\Big). \end{aligned} \tag{2.1.6}$$

Letting $n \to \infty$ in (2.1.6), we have

$$d(u^*, Tu^*) \leq \phi\left(0, 0, d(u^*, Tu^*), \frac{0 + d(u^*, Tu^*)}{2}\right). \tag{2.1.7}$$

By property (iii) of Φ, we have $d(u^*, Tu^*) = 0$. Hence $Tu^* = u^*$. Suppose that u^* and v^* be two distinct fixed points of T. Then from (1.2.8), we have

$$d(Tu^*, Tv^*) \leq \phi\left(d(u^*, v^*), d(u^*, Tu^*), d(v^*, Tv^*), \frac{d(v^*, Tu^*) + d(u^*, Tv^*)}{2}\right).$$

It implies that

$$d(u^*, v^*) \leq \phi(d(u^*, v^*), 0, 0, d(u^*, v^*)). \tag{2.1.8}$$

By (2.1.8) and property (iii) of Φ, we have $d(u^*, v^*) = 0$. A contradiction to our assumption. Hence T has a unique fixed point. □

As a consequence of our result we have the following corollaries.

Corollary 2.1.11 (Banach [1]). Let $T : U \to U$ be a mapping and d be complete. Assume that there exists $\alpha \in [0, 1)$ such that

$$d(Tu, Tv) \leq \alpha d(u, v), \ \forall \ u, v \in U.$$

Then T has a unique fixed point.

Proof. Let $\phi(u_1,u_2,u_3,u_4)=\phi_5(u_1,u_2,u_3,u_4)=\alpha u_1$, where $\alpha\in[0,1)$. From (1.2.8), we have

$$d(Tu,Tv)\le \alpha d(u,v),\ \forall\ u,v\in U.$$

Therefore by Theorem 2.1.10, T has a unique fixed point. □

Corollary 2.1.12 (Kannan [9])**.** Let $T:U\to U$ be a mapping and d be complete. Assume that there exists $\alpha\in[0,1/2)$ such that

$$d(Tu,Tv)\le \alpha(d(u,Tu)+d(v,Tv)),\ \forall\ u,v\in U.$$

Then T has a unique fixed point.

Proof. Let $\phi(u_1,u_2,u_3,u_4)=\phi_6(u_1,u_2,u_3,u_4)=\frac{\beta}{2}(u_2+u_3)$, where $\beta\in[0,1)$. From (1.2.8), we have

$$d(Tu,Tv)\le \frac{\beta}{2}(d(u,Tu)+d(v,Tv))=\alpha(d(u,Tu)+d(v,Tv)),\ \forall\ u,v\in U.$$

Therefore by Theorem 2.1.10, T has a unique fixed point. □

Corollary 2.1.13 (Bainchini [3])**.** Let $T:U\to U$ be a mapping and d be complete. Assume that there exists $\alpha\in[0,1)$ such that

$$d(Tu,Tv)\le \alpha\max\{d(u,Tu),d(v,Tv)\},\ \forall\ u,v\in U.$$

Then T has a unique fixed point.

Proof. Let $\phi(u_1,u_2,u_3,u_4)=\phi_4(u_1,u_2,u_3,u_4)=\alpha\max\{u_2,u_3\}$, where $\alpha\in[0,1)$. From (1.2.8), we have

$$d(Tu,Tv)\le \alpha\max\{d(u,Tu),d(v,Tv))\},\ \forall\ u,v\in U.$$

Therefore by Theorem 2.1.10, T has a unique fixed point. □

Corollary 2.1.14 (Reich [14])**.** Let $T:U\to U$ be a mapping and d be complete. Assume that there exist nonnegative real numbers a,b,c, satisfying $a+b+c<1$ such that

$$d(Tu,Tv)\le ad(u,Tu)+bd(v,Tv)+cd(u,v),\ \forall\ u,v\in U.$$

Then T has a unique fixed point.

Proof. Let $\phi(u_1,u_2,u_3,u_4) = \phi_9(u_1,u_2,u_3,u_4) = au_2 + bu_3 + cu_1$, where a,b,c are nonnegative real numbers satisfying $a+b+c \in [0,1)$. From (1.2.8), we have

$$d(Tu,Tv) \le ad(u,Tu) + bd(v,Tv) + cd(u,v), \ \forall\ u,v \in U.$$

Therefore by Theorem 2.1.10, T has a unique fixed point. □

Corollary 2.1.15 (Chatterjea [4])**.** Let $T : U \to U$ be a mapping and d be complete. Assume that there exists $\alpha \in [0,1/2)$ such that

$$d(Tu,Tv) \le \alpha(d(v,Tu) + d(u,Tv)), \ \forall\ u,v \in U.$$

Then T has a unique fixed point.

Proof. Let $\phi(u_1,u_2,u_3,u_4) = \phi_2(u_1,u_2,u_3,u_4) = \beta u_4$, where $\beta \in [0,1)$. From (1.2.8), we have

$$d(Tu,Tv) \le \frac{\beta}{2}(d(v,Tu) + d(u,Tv)) = \alpha(d(v,Tu) + d(u,Tv)), \ \forall\ u,v \in U.$$

Therefore by Theorem 2.1.10, T has a unique fixed point. □

Corollary 2.1.16 (Handy [7])**.** Let $T : U \to U$ be a mapping and d be complete. Assume that there exist nonnegative real numbers a_0, b_0, c_0, satisfying $a_0 + 2b_0 + 2c_0 \in [0,1)$ such that

$$d(Tu,Tv) \le a_0 d(u,v) + b_0(d(u,Tu) + d(v,Tv)) + c_0(d(v,Tu) + d(u,Tv)), \ \forall\ u,v \in U.$$

Then T has a unique fixed point.

Proof. Let $\phi(u_1,u_2,u_3,u_4) = \phi_8(u_1,u_2,u_3,u_4) = au_1 + b(u_2+u_3) + cu_4$, where a,b,c are nonnegative real numbers satisfying $a+2b+c \in [0,1)$. From (1.2.8), we have

$$d(Tu,Tv) \le ad(u,v) + b(d(u,Tu) + d(v,Tv)) + \frac{c}{2}(d(v,Tu) + d(u,Tv)), \ \forall\ u,v \in U.$$

Let $c = 2c_1$, then

$$d(Tu,Tv) \le ad(u,v) + b(d(u,Tu) + d(v,Tv)) + c_1(d(v,Tu) + d(u,Tv)), \ \forall\ u,v \in U.$$

Therefore by Theorem 2.1.10, T has a unique fixed point. □

Corollary 2.1.17. Let $T : U \to U$ be a mapping and d be complete. Assume that there exists $\alpha \in [0, 1)$ such that

$$d(Tu, Tv) \leq \alpha \max\left\{d(u, v), \frac{d(u, Tu) + d(v, Tv)}{2}, \frac{d(v, Tu) + d(u, Tv)}{2}\right\}, \ \forall \ u, v \in U.$$

Then T has a unique fixed point.

Proof. Let $\phi(u_1, u_2, u_3, u_4) = \phi_7(u_1, u_2, u_3, u_4) = \alpha \max\{u_1, \frac{1}{2}(u_2 + u_3), u_4\}$, where $\alpha \in [0, 1)$. From (1.2.8), we have

$$d(Tu, Tv) \leq \alpha \max\left\{d(u, v), \frac{d(u, Tu) + d(v, Tv)}{2}, \frac{d(v, Tu) + d(u, Tv)}{2}\right\}, \ \forall \ u, v \in U.$$

Therefore by Theorem 2.1.10, T has a unique fixed point. □

Corollary 2.1.18. Let $T : U \to U$ be a mapping and d be complete. Assume that there exists $\alpha \in [0, 1)$ such that

$$d(Tu, Tv) \leq \alpha \max\left\{d(u, v), d(u, Tu), d(v, Tv), \frac{d(v, Tu) + d(u, Tv)}{2}\right\}, \ \forall \ u, v \in U.$$

Then T has a unique fixed point.

Proof. Let $\phi(u_1, u_2, u_3, u_4) = \phi_1(u_1, u_2, u_3, u_4) = \alpha \max\{u_1, u_2, u_3, u_4\}$, where $\alpha \in [0, 1)$. From (1.2.8), we have

$$d(Tu, Tv) \leq \alpha \max\left\{d(u, v), d(u, Tu), d(v, Tv), \frac{d(v, Tu) + d(u, Tv)}{2}\right\}, \ \forall \ u, v \in U.$$

Therefore by Theorem 2.1.10, T has a unique fixed point. □

Corollary 2.1.19. Let $T : U \to U$ be a mapping and d be complete. Assume that there exists $\alpha \in [0, 1)$ such that

$$d(Tu, Tv) \leq \alpha \max\{d(u, v), d(u, Tu), d(v, Tv))\}, \ \forall \ u, v \in U.$$

Then T has a unique fixed point.

Proof. Let $\phi(u_1, u_2, u_3, u_4) = \phi_3(u_1, u_2, u_3, u_4) = \alpha \max\{u_1, u_2, u_3\}$, where $\alpha \in [0, 1)$. From (1.2.8), we have

$$d(Tu, Tv) \leq \alpha \max\{d(u, v), d(u, Tu), d(v, Tv))\}, \ \forall \ u, v \in U.$$

Therefore by Theorem 2.1.10, T has a unique fixed point. □

Example 2.1.20. Let $U = [1, \infty)$ be endowed with the usual metric d. Define $T : U \to U$ by $Tu = \sqrt{u}$ for each $u \in U$. Consider $\phi(u_1, u_2, u_3, u_4) = \frac{1}{2}u_1$. Clearly $\phi \in \Phi$. Now for each $u, v \in U$, we have

$$|\sqrt{u} - \sqrt{v}| = |(\sqrt{u} - \sqrt{v}) \times \frac{\sqrt{u} + \sqrt{v}}{\sqrt{u} + \sqrt{v}}| = \frac{|u - v|}{\sqrt{u} + \sqrt{v}} \leq \frac{1}{2}|u - v|.$$

Therefore by Theorem 2.1.10, T has a unique fixed point.

Theorem 2.1.21. Let $T, S : U \to U$ are mappings such that $TU \subseteq SU$. Assume that SU is complete subspace of U and there exists $\phi \in \Phi$ such that

$$d(Tu, Tv) \leq \phi\left(d(Su, Sv), d(Su, Tu), d(Sv, Tv), \frac{d(Sv, Tu) + d(Su, Tv)}{2}\right), \ \forall\ u, v \in U. \tag{2.1.9}$$

Then T and S have a coincidence point. Moreover, if T and S are weakly compatible, then T and S have a unique common fixed point.

Proof. Let $u_0 \in U$. As $TU \subseteq SU$, we can choose a sequence $\{Tu_n\}$ with initial point u_0 such that $Su_{n+1} = Tu_n$ for each $n \in \mathbb{N} \cup \{0\}$. Suppose that $Su_n \neq Su_{n+1}$ for each $n \in \mathbb{N} \cup \{0\}$, for otherwise, u_n is a coincidence point of T and S. From (2.1.9), we have

$$\begin{aligned} d(Su_{n+1}, Su_{n+2}) &= d(Tu_n, Tu_{n+1}) \\ &\leq \phi\Big(d(Su_n, Su_{n+1}), d(Su_n, Tu_n), d(Su_{n+1}, Tu_{n+1}), \\ &\quad \frac{d(Su_{n+1}, Tu_n) + d(Su_n, Tu_{n+1})}{2}\Big) \\ &= \phi\Big(d(Su_n, Su_{n+1}), d(Su_n, Su_{n+1}), d(Su_{n+1}, Su_{n+2}), \\ &\quad \frac{d(Su_{n+1}, Su_{n+1}) + d(Su_n, Su_{n+2})}{2}\Big). \end{aligned} \tag{2.1.10}$$

By triangular inequality, we have

$$d(Su_n, Su_{n+2}) \leq d(Su_n, Su_{n+1}) + d(Su_{n+1}, Su_{n+2}). \tag{2.1.11}$$

We claim that $d(Su_{n+1}, Su_{n+2}) < d(Su_n, Su_{n+1})$ for each $n \in \mathbb{N} \cup \{0\}$. Suppose on contrary that $d(Su_{n+1}, Su_{n+2}) \geq d(Su_n, Su_{n+1})$ for some $n \in \mathbb{N} \cup \{0\}$. From (2.1.11), we get $d(Su_n, Su_{n+2}) \leq 2d(Su_{n+1}, Su_{n+2})$. Since ϕ is nondecreasing, by using these in (2.1.10), we have

$$d(Su_{n+1}, Su_{n+2}) \leq \phi(d(Su_{n+1}, Su_{n+2}), d(Su_n, Su_{n+1}), d(Su_{n+1}, Su_{n+2}), d(Su_{n+1}, Su_{n+2})). \tag{2.1.12}$$

By (2.1.12) and property (ii) of Φ, we have

$$d(Su_{n+1}, Su_{n+2}) = 0.$$

A contradiction to our assumption that $Su_{n+1} \neq Su_n$ for each $n \in \mathbb{N} \cup \{0\}$. Thus $d(Su_{n+1}, Su_{n+2}) < d(Su_n, Su_{n+1})$ for each $n \in \mathbb{N} \cup \{0\}$. From (2.1.11), we have $d(Su_n, Su_{n+2}) < 2d(Su_n, Su_{n+1})$. Using it in (2.1.10), we have

$$d(Su_{n+1}, Su_{n+2}) \leq \phi(d(Su_n, Su_{n+1}), d(Su_n, Su_{n+1}), d(Su_{n+1}, Su_{n+2}), d(Su_n, Su_{n+1})). \tag{2.1.13}$$

By (2.1.13) and property (ii) of Φ, we have

$$d(Su_{n+1}, Su_{n+2}) \leq \psi(d(Su_n, Su_{n+1})), \ \forall \ n \in \mathbb{N} \cup \{0\}.$$

Continuing in the same way, we get

$$d(Su_{n+1}, Su_{n+2}) \leq \psi^{n+1}(d(Su_0, Su_1)), \ \forall \ n \in \mathbb{N} \cup \{0\}. \tag{2.1.14}$$

Let $n > m$. Then we have

$$\begin{aligned} d(Su_m, Su_n) &\leq d(Su_m, Su_{m+1}) + d(Su_{m+1}, Su_{m+2}) + \cdots + d(Su_{n-1}, Su_n) \\ &\leq \psi^m(d(Su_0, Su_1)) + \psi^{m+1}(d(Su_0, Su_1)) + \cdots + \psi^{n-1}(d(Su_0, Su_1)) \\ &= \sum_{i=m}^{n-1} \psi^i(d(Su_0, Su_1)). \end{aligned}$$

Therefore $\{Su_n\}$ is a Cauchy sequence in SU. By completeness of SU, there exist $u^*, z^* \in U$ such that $z^* = Su^*$ and

$$\lim_{n\to\infty} Tu_n = \lim_{n\to\infty} Su_{n+1} = Su^* = z^*.$$

From (2.1.9), we have

$$\begin{aligned} d(Su_{n+1}, Tu^*) &= d(Tu_n, Tu^*) \\ &\leq \phi\Big(d(Su_n, Su^*), d(Su_n, Tu_n), d(Su^*, Tu^*), \\ &\quad \frac{d(Su^*, Tu_n) + d(Su_n, Tu^*)}{2}\Big) \\ &= \phi\Big(d(Su_n, Su^*), d(Su_n, Su_{n+1}), d(Su^*, Tu^*), \\ &\quad \frac{d(Su^*, Su_{n+1}) + d(Su_n, Tu^*)}{2}\Big). \end{aligned} \tag{2.1.15}$$

Letting $n \to \infty$ in (2.1.15), we have

$$d(Su^*, Tu^*) \leq \phi\left(0, 0, d(Su^*, Tu^*), \frac{0 + d(Su^*, Tu^*)}{2}\right). \tag{2.1.16}$$

By property (iii) of Φ, we have $d(Su^*, Tu^*) = 0$ which implies $Tu^* = Su^*$. Therefore u^* is a coincidence point of T and S. Now we assume that T and S are weakly compatible. Then $Tz^* = TSu^* = STu^* = Sz^*$. From (2.1.9), we have

$$\begin{aligned} d(z^*, Tz^*) &= d(Tu^*, Tz^*) \\ &\leq \phi\Big(d(Su^*, Sz^*), d(Su^*, Tu^*), d(Sz^*, Tz^*), \\ &\quad \frac{d(Sz^*, Tu^*) + d(Su^*, Tz^*)}{2}\Big) \\ &= \phi(d(z^*, Tz^*), 0, 0, d(z^*, Tz^*)). \end{aligned} \tag{2.1.17}$$

By property (iii) of Φ, we have $d(z^*, Tz^*) = 0$ this implies $z^* = Tz^*$. Hence $z^* = Tz^* = Sz^*$. So T and S have a common fixed point. Suppose that z^* and v^* be two distinct common fixed points of T and S. Then from (2.1.9), we have

$$d(Tz^*, Tv^*) \leq \phi\left(d(Sz^*, Sv^*), d(Sz^*, Tz^*), d(Sv^*, Tv^*), \frac{d(Sv^*, Tz^*) + d(Sz^*, Tv^*)}{2}\right).$$

It implies that

$$d(z^*, v^*) \leq \phi\left(d(z^*, v^*), 0, 0, d(z^*, v^*)\right). \tag{2.1.18}$$

By property (iii) of Φ, we have $d(z^*, v^*) = 0$. A contradiction to our assumption. Hence T and S have a unique common fixed point. □

The following corollary can be obtained by the above Theorem.

Corollary 2.1.22. Let $T, S : U \to U$ are mappings with $TU \subseteq SU$. Assume that SU is a complete subspace of U and for each $u, v \in U$, there exists $\alpha \in [0, 1)$ such that

$$d(Tu, Tv) \leq \alpha \max\left\{d(Su, Sv), d(Su, Tu), d(Sv, Tv), \frac{d(Sv, Tu) + d(Su, Tv)}{2}\right\}.$$

Then T and S have a coincidence point. Moreover, if T and S are weakly compatible, then T and S have a unique common fixed point.

Proof. Let $\phi(u_1, u_2, u_3, u_4) = \phi_1(u_1, u_2, u_3, u_4) = \alpha \max\{u_1, u_2, u_3, u_4\}$, where $\alpha \in [0, 1)$. From (2.1.9), we have

$$d(Tu, Sv) \leq \alpha \max\left\{d(u, v), d(u, Tu), d(v, Sv), \frac{d(v, Tu) + d(u, Sv)}{2}\right\},$$

for each $u, v \in U$, where $\alpha \in [0, 1)$. Therefore by Theorem 2.1.21, T and S have a coincidence point. Moreover, if T and S are weakly compatible then T and S have a unique common fixed point. □

2.2 Multi-valued ϕ-contractions

Nadler [11] extended Banach contraction principle to multi-valued mappings. Consequentially, many authors extended contractive conditions (1.1.3) and (1.2.2) $-$ (1.2.7) from single-valued to multi-valued mappings. In this section, we extend Theorems 2.1.10 and 2.1.21 to multi-valued mappings.

By Φ_B we denote the family of functions $\phi : {\mathbb{R}_+}^4 \to \mathbb{R}_+ = [0, \infty)$ satisfying the following conditions:

(i) ϕ is continuous and nondecreasing in each coordinate;

(ii) let $u_1, u_2 \in \mathbb{R}_+$ such that if $u_1 < u_2$ and $u_1 \leq \phi(u_2, u_2, u_1, u_2)$ then $u_1 \leq \psi(u_2)$. Further if $u_1 \geq u_2$ and $u_1 \leq \phi(u_1, u_2, u_1, u_1)$ then $u_1 = 0$;

(iii) if $u \in \mathbb{R}_+$ such that $u \leq \phi(0, 0, u, \frac{1}{2}u)$ or $u \leq \phi(0, u, 0, \frac{1}{2}u)$ then $u = 0$.

Note that the functions ϕ_1 to ϕ_9 given in Section 2.1, belong to the family of functions Φ_B.

Theorem 2.2.1. Let $T : U \to B(U)$ be a mapping and d be complete. Assume that there exists $\phi \in \Phi_B$ such that

$$\delta(Tu, Tv) \leq \phi\left(d(u, v), D(u, Tu), D(v, Tv), \frac{D(v, Tu) + D(u, Tv)}{2}\right), \forall\, u, v \in U. \quad (2.2.1)$$

Then T has a fixed point.

Proof. Let $\{u_n\}$ be a sequence in U such that $u_{n+1} \in Tu_n$ for each $n \in \mathbb{N} \cup \{0\}$. If $u_{N+1} = u_N$ for some $N \in \mathbb{N} \cup \{0\}$. Then u_N is a fixed point. Suppose $u_{n+1} \neq u_n$ for each

$n \in \mathbb{N} \cup \{0\}$. From (2.2.1), we have

$$\begin{aligned} d(u_{n+1}, u_{n+2}) &\leq \delta(Tu_n, Tu_{n+1}) \\ &\leq \phi\Big(d(u_n, u_{n+1}), D(u_n, Tu_n), D(u_{n+1}, Tu_{n+1}), \\ &\quad \frac{D(u_{n+1}, Tu_n) + D(u_n, Tu_{n+1})}{2}\Big) \\ &\leq \phi\Big(d(u_n, u_{n+1}), d(u_n, u_{n+1}), d(u_{n+1}, u_{n+2}), \\ &\quad \frac{d(u_{n+1}, u_{n+1}) + d(u_n, u_{n+2})}{2}\Big). \end{aligned} \tag{2.2.2}$$

By triangular inequality, we have

$$d(u_n, u_{n+2}) \leq d(u_n, u_{n+1}) + d(u_{n+1}, u_{n+2}). \tag{2.2.3}$$

We claim that $d(u_{n+1}, u_{n+2}) < d(u_n, u_{n+1})$ for each $n \in \mathbb{N} \cup \{0\}$. Suppose on contrary that $d(u_{n+1}, u_{n+2}) \geq d(u_n, u_{n+1})$ for some $n \in \mathbb{N} \cup \{0\}$. From (2.2.3), we get $d(u_n, u_{n+2}) \leq 2d(u_{n+1}, u_{n+2})$. Since ϕ is nondecreasing, by using these in (2.2.2), we have

$$d(u_{n+1}, u_{n+2}) \leq \phi(d(u_{n+1}, u_{n+2}), d(u_n, u_{n+1}), d(u_{n+1}, u_{n+2}), d(u_{n+1}, u_{n+2})). \tag{2.2.4}$$

By (2.2.4) and property (ii) of Φ_B, we have

$$d(u_{n+1}, u_{n+2}) = 0.$$

A contradiction to our assumption that $u_{n+1} \neq u_n$ for each $n \in \mathbb{N} \cup \{0\}$. Thus $d(u_{n+1}, u_{n+2}) < d(u_n, u_{n+1})$ for each $n \in \mathbb{N} \cup \{0\}$. From (2.2.3), we have $d(u_n, u_{n+2}) < 2d(u_n, u_{n+1})$. Using it in (2.2.2), we have

$$d(u_{n+1}, u_{n+2}) \leq \phi(d(u_n, u_{n+1}), d(u_n, u_{n+1}), d(u_{n+1}, u_{n+2}), d(u_n, u_{n+1})). \tag{2.2.5}$$

By (2.2.5) and property (ii) of Φ_B, we have

$$d(u_{n+1}, u_{n+2}) \leq \psi(d(u_n, u_{n+1})), \ \forall \ n \in \mathbb{N} \cup \{0\}.$$

Continuing in the same way, we get

$$d(u_{n+1}, u_{n+2}) \leq \psi^{n+1}(d(u_0, u_1)), \ \forall \ n \in \mathbb{N} \cup \{0\}. \tag{2.2.6}$$

Let $n > m$. Then we have

$$\begin{aligned} d(u_m, u_n) &\leq d(u_m, u_{m+1}) + d(u_{m+1}, u_{m+2}) + \cdots + d(u_{n-1}, u_n) \\ &\leq \psi^m(d(u_0, u_1)) + \psi^{m+1}(d(u_0, u_1)) + \cdots + \psi^{n-1}(d(u_0, u_1)) \\ &= \sum_{i=m}^{n-1} \psi^i(d(u_0, u_1)). \end{aligned}$$

Therefore $\{u_n\}$ is a Cauchy sequence in U. By completeness of U, there exists $u^* \in U$ such that $u_n \to u^*$ as $n \to \infty$. From (2.2.1), we have

$$\begin{aligned} \delta(u_{n+1}, Tu^*) &\leq \delta(Tu_n, Tu^*) \\ &\leq \phi\Big(d(u_n, u^*), D(u_n, Tu_n), D(u^*, Tu^*), \\ &\qquad \frac{D(u^*, Tu_n) + D(u_n, Tu^*)}{2}\Big) \\ &\leq \phi\Big(d(u_n, u^*), d(u_n, u_{n+1}), D(u^*, Tu^*), \\ &\qquad \frac{d(u^*, u_{n+1}) + D(u_n, Tu^*)}{2}\Big). \end{aligned} \tag{2.2.7}$$

Letting $n \to \infty$ in (2.2.7), we have

$$\begin{aligned} \delta(u^*, Tu^*) &\leq \phi\left(0, 0, D(u^*, Tu^*), \frac{0 + D(u^*, Tu^*)}{2}\right) \\ &\leq \phi\left(0, 0, \delta(u^*, Tu^*), \frac{\delta(u^*, Tu^*)}{2}\right). \end{aligned} \tag{2.2.8}$$

By (2.2.8) and property (iii) of Φ_B, we have $\delta(u^*, Tu^*) = 0$. Hence $Tu^* = \{u^*\}$. Moreover, u^* is a fixed point. □

The following corollaries reduce from our result.

Corollary 2.2.2. Let $T : U \to B(U)$ be a mapping and d be complete. Assume that there exists $\alpha \in [0, 1/2)$ such that

$$\delta(Tu, Tv) \leq \alpha(D(v, Tu) + D(u, Tv)), \forall\ u, v \in U.$$

Then T has a fixed point.

Proof. Let $\phi(u_1, u_2, u_3, u_4) = \phi_2(u_1, u_2, u_3, u_4) = \beta u_4$, where $\beta \in [0, 1)$. From (2.2.1), we have

$$\delta(Tu, Tv) \leq \frac{\beta}{2}(D(v, Tu) + D(u, Tv)) = \alpha(D(v, Tu) + D(u, Tv)), \forall\ u, v \in U.$$

Therefore by Theorem 2.2.1, T has a fixed point. □

Corollary 2.2.3. Let $T : U \to B(U)$ be a mapping and d be complete. Assume that there exists $\alpha \in [0,1)$ such that

$$\delta(Tu, Tv) \le \alpha \max\left\{d(u,v), D(u,Tu), D(v,Tv), \frac{D(v,Tu)+D(u,Tv)}{2}\right\}, \forall\ u,v \in U.$$

Then T has a fixed point.

Proof. Let $\phi(u_1,u_2,u_3,u_4) = \phi_1(u_1,u_2,u_3,u_4) = \alpha \max\{u_1,u_2,u_3,u_4\}$, where $\alpha \in [0,1)$. From (2.2.1), we have

$$\delta(Tu, Tv) \le \alpha \max\left\{d(u,v), D(u,Tu), D(v,Tv), \frac{D(v,Tu)+D(u,Tv)}{2}\right\}, \forall\ u,v \in U.$$

Therefore by Theorem 2.2.1, T has a fixed point. □

Example 2.2.4. Let $U = \{(0,0),(0,1),(-\frac{1}{4},\frac{2}{3})\}$ be endowed with metric d define by $d(u,v) = \max\{|u_1 - v_1|, |u_2 - v_2|\}$ for each $u, v \in U$. Define $T : U \to B(U)$ by

$$Tu = \begin{cases} \{(0,1)\} & \text{if } u \neq (0,0); \\ \{(0,1),(-\frac{1}{4},\frac{2}{3})\} & \text{if } u = (0,0). \end{cases} \tag{2.2.9}$$

Consider $\phi(u_1,u_2,u_3,u_4) = \frac{2}{3}\max\{u_1,u_2\}$. Now we discuss (2.2.1) by following cases:

(i) Consider $u = v = (0,1)$. Then $Tu = Tv = \{(0,1)\}$. Also, we have $\delta(Tu,Tv) = 0, d(u,v) = 0, D(u,Tu) = 0$. Thus (2.2.1) is satisfied because $0 = \frac{2}{3}(0)$.

(ii) Consider $u = v = (0,0)$. Then $Tu = Tv = \{(0,1),(-\frac{1}{4},\frac{2}{3})\}$. Also, we have $\delta(Tu,Tv) = 1/3$, $d(u,v) = 0$, $D(u,Tu) = 2/3$. Thus (2.2.1) is satisfied because $\frac{1}{3} < \frac{2}{3}(\frac{2}{3})$.

(iii) Consider $u = v = (-\frac{1}{4},\frac{2}{3})$. Then $Tu = Tv = \{(0,1)\}$. Also, we have $\delta(Tu,Tv) = 0$, $d(u,v) = 0$, $D(u,Tu) = 1/3$. Thus (2.2.1) is satisfied because $0 < \frac{2}{3}(\frac{1}{3})$.

(iv) Consider $u = (-\frac{1}{4},\frac{2}{3})$ and $v = (0,1)$. Then $Tu = Tv = \{(0,1)\}$. Also, we have $\delta(Tu,Tv) = 0$, $d(u,v) = 1/3$, $D(u,Tu) = 1/3$. Thus (2.2.1) is satisfied because $0 < \frac{2}{3}(\frac{1}{3})$.

(v) Consider $u = (0,0)$ and $v = (-\frac{1}{4},\frac{2}{3})$. Then $Tu = \{(0,1),(-\frac{1}{4},\frac{2}{3})\}$ and $Tv = \{(0,1)\}$. Also, we have $\delta(Tu,Tv) = 1/3$, $d(u,v) = 2/3$, $D(u,Tu) = 2/3$. Thus (2.2.1) is satisfied because $\frac{1}{3} < \frac{2}{3}(\frac{2}{3})$.

(vi) Consider $u = (0, 0)$ and $v = (0, 1)$. Then $Tu = \{(0, 1), (-\frac{1}{4}, \frac{2}{3})\}$ and $Tv = \{(0, 1)\}$. Also, we have $\delta(Tu, Tv) = 1/3$, $d(u, v) = 1$, $D(u, Tu) = 2/3$. Thus (2.2.1) is satisfied because $\frac{1}{3} < \frac{2}{3}(1)$.

Hence by Theorem (2.2.1), T has a fixed point.

Theorem 2.2.5. Let $f : U \to U$ be a single-valued mapping and $T : U \to B(U)$ be a multi-valued mapping such that $TU \subseteq fU$. Assume that fU is a complete subspace of U and there exists $\phi \in \Phi_B$ such that

$$\delta(Tu, Tv) \leq \phi\left(d(fu, fv), D(fu, Tu), D(fv, Tv), \frac{D(fv, Tu) + D(fu, Tv)}{2}\right), \forall\ u, v \in U. \tag{2.2.10}$$

Then T and f have a coincidence point, say w. Moreover, if f is T-weakly commuting at w and $ffw = fw$, then T and f have a common fixed point.

Proof. Let $u_0 \in U$. As $TU \subseteq fU$, we can choose a sequence $\{Tu_n\}$ with initial point u_0 such that $fu_{n+1} \in Tu_n$ for each $n \in \mathbb{N} \cup \{0\}$. Suppose that $fu_n \neq fu_{n+1}$ for each $n \in \mathbb{N} \cup \{0\}$, for otherwise, u_n is a coincidence point of T and f. From (2.2.10), we have

$$\begin{aligned} d(fu_{n+1}, fu_{n+2}) &= \delta(Tu_n, Tu_{n+1}) \\ &\leq \phi\Big(d(fu_n, fu_{n+1}), D(fu_n, Tu_n), D(fu_{n+1}, Tu_{n+1}), \\ &\quad \frac{D(fu_{n+1}, Tu_n) + D(fu_n, Tu_{n+1})}{2}\Big) \\ &\leq \phi\Big(d(fu_n, fu_{n+1}), d(fu_n, fu_{n+1}), d(fu_{n+1}, fu_{n+2}), \\ &\quad \frac{d(fu_{n+1}, fu_{n+1}) + d(fu_n, fu_{n+2})}{2}\Big). \end{aligned} \tag{2.2.11}$$

By triangular inequality, we have

$$d(fu_n, fu_{n+2}) \leq d(fu_n, fu_{n+1}) + d(fu_{n+1}, fu_{n+2}). \tag{2.2.12}$$

We claim that $d(fu_{n+1}, fu_{n+2}) < d(fu_n, fu_{n+1})$ for each $n \in \mathbb{N} \cup \{0\}$. Suppose on contrary that $d(fu_{n+1}, fu_{n+2}) \geq d(fu_n, fu_{n+1})$ for some $n \in \mathbb{N} \cup \{0\}$. From (2.2.12), we get $d(fu_n, fu_{n+2}) \leq 2d(fu_{n+1}, fu_{n+2})$. Since ϕ is nondecreasing, by using these in (2.2.11), we have

$$d(fu_{n+1}, fu_{n+2}) \leq \phi(d(fu_{n+1}, fu_{n+2}), d(fu_n, fu_{n+1}), d(fu_{n+1}, fu_{n+2}), d(fu_{n+1}, fu_{n+2})). \tag{2.2.13}$$

By (2.2.13) and property (ii) of Φ_B, we have

$$d(fu_{n+1}, fu_{n+2}) = 0.$$

A contradiction to our assumption that $fu_{n+1} \neq fu_n$ for each $n \in \mathbb{N} \cup \{0\}$. Thus $d(fu_{n+1}, fu_{n+2}) < d(fu_n, fu_{n+1})$ for each $n \in \mathbb{N} \cup \{0\}$. From (2.2.12), we have $d(fu_n, fu_{n+2}) < 2d(fu_n, fu_{n+1})$. Using it in (2.2.11), we have

$$d(fu_{n+1}, fu_{n+2}) \leq \phi(d(fu_n, fu_{n+1}), d(fu_n, fu_{n+1}), d(fu_{n+1}, fu_{n+2}), d(fu_n, fu_{n+1})). \tag{2.2.14}$$

By (2.2.14) and property (ii) of Φ_B, we have

$$d(fu_{n+1}, fu_{n+2}) \leq \psi(d(fu_n, fu_{n+1})), \ \forall \ n \in \mathbb{N} \cup \{0\}.$$

Continuing in the same way, we get

$$d(fu_{n+1}, fu_{n+2}) \leq \psi^{n+1}(d(fu_0, fu_1)), \ \forall \ n \in \mathbb{N} \cup \{0\}. \tag{2.2.15}$$

Let $n > m$. Then we have

$$\begin{aligned} d(fu_m, fu_n) &\leq d(fu_m, fu_{m+1}) + d(fu_{m+1}, fu_{m+2}) + \cdots + d(fu_{n-1}, fu_n) \\ &\leq \psi^m(d(fu_0, fu_1)) + \psi^{m+1}(d(fu_0, fu_1)) + \cdots + \psi^{n-1}(d(fu_0, fu_1)) \\ &= \sum_{i=m}^{n-1} \psi^i(d(fu_0, fu_1)). \end{aligned}$$

Therefore $\{fu_n\}$ is a Cauchy sequence in fU. By completeness of fU, there exist $u^*, z^* \in U$ such that $z^* = fu^*$ and

$$\lim_{n\to\infty} fu_{n+1} = fu^* = z^*.$$

From (2.2.10), we have

$$\begin{aligned} \delta(fu_{n+1}, Tu^*) &\leq \delta(Tu_n, Tu^*) \\ &\leq \phi\Big(d(fu_n, fu^*), D(fu_n, Tu_n), D(fu^*, Tu^*), \\ &\quad \frac{D(fu^*, Tu_n) + D(fu_n, Tu^*)}{2}\Big) \\ &\leq \phi\Big(d(fu_n, fu^*), d(fu_n, fu_{n+1}), D(fu^*, Tu^*), \\ &\quad \frac{d(fu^*, fu_{n+1}) + D(fu_n, Tu^*)}{2}\Big). \end{aligned} \tag{2.2.16}$$

Letting $n \to \infty$ in (2.2.16), we have

$$\begin{aligned}\delta(fu^*, Tu^*) &\leq \phi\left(0, 0, D(fu^*, Tu^*), \frac{D(fu^*, Tu^*)}{2}\right) \\ &\leq \phi\left(0, 0, \delta(fu^*, Tu^*), \frac{\delta(fu^*, Tu^*)}{2}\right). \qquad (2.2.17)\end{aligned}$$

By property (iii) of Φ_B, we have $\delta(fu^*, Tu^*) = 0$. This implies $Tu^* = \{fu^*\}$. Therefore u^* is a coincidence point of T and f. By assumption we have $fu^* = ffu^* \in Tfu^*$ which implies that $z^* = fz^* \in Tz^*$. □

The following corollary can be obtained by the above Theorem.

Corollary 2.2.6. Let $f : U \to U$ and $T : U \to B(U)$ be mappings. Assume that $TU \subseteq fU$ and there exists $\alpha \in [0, 1/2)$ such that

$$\delta(Tu, Tv) \leq \alpha(D(fu, Tu) + D(fv, Tv)), \ \forall\ u, v \in U.$$

If fU is a complete subspace of U, then T and f have a coincidence point, say u. Moreover, if f is T-weakly commuting at u and $ffu = fu$, then T and f have a common fixed point.

Proof. Let $\phi(u_1, u_2, u_3, u_4) = \phi_6(u_1, u_2, u_3, u_4) = \frac{\beta}{2}(u_2 + u_3)$, where $\beta \in [0, 1)$. From (2.2.10), we have

$$\delta(Tu, Tv) \leq \frac{\beta}{2}(D(fu, Tu) + D(fv, Tv)) = \alpha(D(fu, Tu) + D(fv, Tv)),$$

for each $u, v \in U$, where $\alpha \in [0, \frac{1}{2})$. Therefore by Theorem 2.2.5, T and f have a coincidence point, say u. Moreover, if f is T-weakly commuting at u and $ffu = fu$, then T and f have a common fixed point. □

By Φ_{CB} we denote the family of functions $\phi : {\mathbb{R}_+}^4 \to \mathbb{R}_+ = [0, \infty)$ satisfying the following conditions:

(i) ϕ is continuous and nondecreasing in each coordinate;

(ii) let $u_1, u_2, w \in \mathbb{R}_+$ such that if $u_1 < u_2$, $u_1 < qw$, where $q > 1$ and $w \leq \phi(u_2, u_2, u_1, u_2)$ then $w \leq \psi(u_2)$. Further if $u_1 \geq u_2$, $u_1 \leq qw$, where $q > 1$ and $w \leq \phi(u_1, u_2, u_1, u_1)$ then $w = 0$;

(iii) if $u \in \mathbb{R}_+$ such that $u \leq \phi(0, 0, u, \frac{1}{2}u)$ or $u \leq \phi(0, u, 0, \frac{1}{2}u)$ then $u = 0$.

The functions ϕ_1 to ϕ_9 given in Section 2.1, are contained in the family of Φ_{CB}.

We need following lemma to prove our next result.

Lemma 2.2.7. Let $B \in CB(U)$. Then for each $u \in U$ with $D(u, B) > 0$ and $p > 1$ there exists an element $b \in B$ such that

$$d(u, b) < pD(u, B). \tag{2.2.18}$$

Proof. It is given that $D(u, B) > 0$. Choose

$$\epsilon = (p - 1)D(u, B).$$

Then by using definition of $D(u, B)$, it follows that there exists $b \in B$ such that

$$d(u, b) < D(u, B) + \epsilon = pD(u, B).$$

□

Theorem 2.2.8. Let $T : U \to CB(U)$ be a mapping and d be complete. Assume that there exists $\phi \in \Phi_{CB}$ such that

$$H(Tu, Tv) \leq \phi\left(d(u, v), D(u, Tu), D(v, Tv), \frac{D(v, Tu) + D(u, Tv)}{2}\right), \ \forall\ u, v \in U. \tag{2.2.19}$$

Then T has a fixed point.

Proof. Let $u_0 \in U$. Since $Tu_0 \neq \emptyset$, there exists $u_1 \in Tu_0$. Suppose $u_0 \neq u_1$, for otherwise, u_0 is a fixed point. From (2.2.19), we have

$$H(Tu_0, Tu_1) \leq \phi\left(d(u_0, u_1), D(u_0, Tu_0), D(u_1, Tu_1), \frac{D(u_1, Tu_0) + D(u_0, Tu_1)}{2}\right). \tag{2.2.20}$$

For $p > 1$, there exists $u_2 \in Tu_1$. Suppose $u_1 \neq u_2$, for otherwise, u_1 is a fixed point. Then we have

$$0 < d(u_1, u_2) < pH(Tu_0, Tu_1). \tag{2.2.21}$$

From (2.2.20), we have

$$H(Tu_0, Tu_1) \leq \phi\left(d(u_0, u_1), d(u_0, u_1), d(u_1, u_2), \frac{0 + d(u_0, u_2)}{2}\right). \tag{2.2.22}$$

By triangular inequality, we have

$$d(u_0, u_2) \leq d(u_0, u_1) + d(u_1, u_2). \tag{2.2.23}$$

We claim that $d(u_1, u_2) < d(u_0, u_1)$. Suppose on contrary that $d(u_1, u_2) \geq d(u_0, u_1)$. Then from (2.2.23), we have $d(u_0, u_2) \leq 2d(u_1, u_2)$. Using these in (2.2.22), we have

$$H(Tu_0, Tu_1) \leq \phi(d(u_1, u_2), d(u_0, u_1), d(u_1, u_2), d(u_1, u_2)). \tag{2.2.24}$$

By property (ii) of Φ_{CB}, we have $H(Tu_0, Tu_1) = 0$. A contradiction to (2.2.21). Hence $d(u_1, u_2) < d(u_0, u_1)$. From (2.2.22), we have

$$H(Tu_0, Tu_1) \leq \phi(d(u_0, u_1), d(u_0, u_1), d(u_1, u_2), d(u_0, u_1)). \tag{2.2.25}$$

By property (ii) of Φ_{CB}, we have

$$H(Tu_0, Tu_1) \leq \psi(d(u_0, u_1)). \tag{2.2.26}$$

From (2.2.21) and (2.2.26), we have

$$d(u_1, u_2) < p\psi(d(u_0, u_1)). \tag{2.2.27}$$

Since ψ is strictly increasing, we have

$$\psi(d(u_1, u_2)) < \psi(p\psi(d(u_0, u_1))). \tag{2.2.28}$$

Thus we have $p_1 = \frac{\psi(p\psi(d(u_0,u_1)))}{\psi(d(u_1,u_2))}$. Again from (2.2.19), we have

$$H(Tu_1, Tu_2) \leq \phi\left(d(u_1, u_2), D(u_1, Tu_1), D(u_2, Tu_2), \frac{D(u_2, Tu_1) + D(u_1, Tu_2)}{2}\right). \tag{2.2.29}$$

For $p_1 > 1$, there exists $u_3 \in Tu_2$. Suppose $u_2 \neq u_3$, for otherwise, u_2 is a fixed point. Then we have

$$0 < d(u_2, u_3) < p_1 H(Tu_1, Tu_2). \tag{2.2.30}$$

From (2.2.29), we have

$$H(Tu_1, Tu_2) \leq \phi\left(d(u_1, u_2), d(u_1, u_2), d(u_2, u_3), \frac{0 + d(u_1, u_3)}{2}\right). \tag{2.2.31}$$

By triangular inequality, we have

$$d(u_1, u_3) \leq d(u_1, u_2) + d(u_2, u_3). \tag{2.2.32}$$

We claim that $d(u_2, u_3) < d(u_1, u_2)$. Suppose on contrary that $d(u_2, u_3) \geq d(u_1, u_2)$. Then from (2.2.32), we have $d(u_1, u_3) \leq 2d(u_2, u_3)$. Using these in (2.2.31), we have

$$H(Tu_1, Tu_2) \leq \phi(d(u_2, u_3), d(u_1, u_2), d(u_2, u_3), d(u_2, u_3)). \tag{2.2.33}$$

By property (ii) of Φ_{CB}, we have $H(Tu_1, Tu_2) = 0$. A contradiction to (2.2.30). Hence $d(u_2, u_3) < d(u_1, u_2)$. From (2.2.31), we have

$$H(Tu_1, Tu_2) \leq \phi(d(u_1, u_2), d(u_1, u_2), d(u_2, u_3), d(u_1, u_2)). \tag{2.2.34}$$

By property (ii) of Φ_{CB}, we have

$$H(Tu_1, Tu_2) \leq \psi(d(u_1, u_2)). \tag{2.2.35}$$

From (2.2.30) and (2.2.35), we have

$$d(u_2, u_3) < p_1\psi(d(u_1, u_2)) = \psi(q\psi(d(u_0, u_1))). \tag{2.2.36}$$

Since ψ is strictly increasing, we have

$$\psi(d(u_2, u_3)) < \psi^2(p\psi(d(u_0, u_1))). \tag{2.2.37}$$

Thus we have $p_2 = \frac{\psi^2(p\psi(d(u_0,u_1)))}{\psi(d(u_2,u_3))}$. Continuing in this way, we get a sequence $\{u_n\}$ in U such that $u_{n+1} \in Tu_n$, $u_n \neq u_{n+1}$ and $d(u_{n+1}, u_{n+2}) < d(u_n, u_{n+1})$ for each $n \in \mathbb{N} \cup \{0\}$. Further, we have

$$d(u_n, u_{n+1}) < \psi^{n-1}(p\psi(d(u_0, u_1))), \ \forall \ n \in \mathbb{N}. \tag{2.2.38}$$

Let $n > m$. Then we have

$$\begin{aligned} d(u_m, u_n) &\leq d(u_m, u_{m+1}) + d(u_{m+1}, u_{m+2}) + \cdots + d(u_{n-1}, u_n) \\ &< \psi^{m-1}(d(u_0, u_1)) + \psi^m(d(u_0, u_1)) + \cdots + \psi^{n-2}(d(u_0, u_1)) \\ &= \sum_{i=m-1}^{n-2} \psi^i(d(u_0, u_1)). \end{aligned}$$

Therefore $\{u_n\}$ is a Cauchy sequence in U. By completeness of U, there exists $u^* \in U$

such that $u_n \to u^*$ as $n \to \infty$. From (2.2.19), we have

$$\begin{aligned} D(u_{n+1}, Tu^*) &\leq H(Tu_n, Tu^*) \\ &\leq \phi\Big(d(u_n, u^*), D(u_n, Tu_n), D(u^*, Tu^*), \\ &\quad \frac{D(u^*, Tu_n) + D(u_n, Tu^*)}{2}\Big) \\ &\leq \phi\Big(d(u_n, u^*), d(u_n, u_{n+1}), D(u^*, Tu^*), \\ &\quad \frac{d(u^*, u_{n+1}) + D(u_n, Tu^*)}{2}\Big). \end{aligned} \tag{2.2.39}$$

Letting $n \to \infty$ in (2.2.39), we have

$$D(u^*, Tu^*) \leq \phi(0, 0, D(u^*, Tu^*), D(u^*, Tu^*)). \tag{2.2.40}$$

By property (iii) of Φ_{CB}, we have $D(u^*, Tu^*) = 0$. By closedness of U, we have $u^* \in Tu^*$. □

The following corollaries are immediately follow form our above result.

Corollary 2.2.9 (Nadler [11]). Let $T : U \to CB(U)$ be a mapping and d be complete. Assume that their exists $\alpha \in [0, 1)$ such that

$$H(Tu, Tv) \leq \alpha d(u, v), \ \forall \ u, v \in U.$$

Then T has a fixed point.

Proof. Let $\phi(u_1, u_2, u_3, u_4) = \phi_5(u_1, u_2, u_3, u_4) = \alpha u_1$, where $\alpha \in [0, 1)$. From (2.2.19), we have

$$H(Tu, Tv) \leq \alpha d(u, v), \ \forall \ u, v \in U.$$

Therefore by Theorem 2.2.8, T has a fixed point. □

Corollary 2.2.10. Let $T : U \to CB(U)$ be a mapping and d be complete. Assume that their exists $\alpha \in [0, 1/2)$ such that

$$H(Tu, Tv) \leq \alpha(D(u, Tu) + D(v, Tv)), \ \forall \ u, v \in U.$$

Then T has a fixed point.

Proof. Let $\phi(u_1, u_2, u_3, u_4) = \phi_6(u_1, u_2, u_3, u_4) = \frac{\beta}{2}(u_2 + u_3)$, where $\beta \in [0, 1)$. From (2.2.19), we have

$$H(Tu, Tv) \leq \frac{\beta}{2}(D(u, Tu) + D(v, Tv)) = \alpha(D(u, Tu) + D(v, Tv)), \ \forall \ u, v \in U.$$

Therefore by Theorem 2.2.8, T has a fixed point. □

Corollary 2.2.11 (Gordji [6]). Let $T : U \to CB(U)$ be a mapping and d be complete. Assume that their exist nonnegative real numbers a_0, b_0, c_0, satisfying $a_0+2b_0+2c_0 \in [0, 1)$ such that

$$H(Tu, Tv) \leq a_0 d(u, v) + b_0(D(u, Tu) + D(v, Tv)) + c_0(D(v, Tu) + D(u, Tv)), \ \forall \ u, v \in U.$$

Then T has a fixed point.

Proof. Let $\phi(u_1, u_2, u_3, u_4) = \phi_8(u_1, u_2, u_3, u_4) = au_1 + b(u_2 + u_3) + cu_4$, where a, b, c are nonnegative real numbers such that $a + 2b + c \in [0, 1)$. From (2.2.19), we have

$$H(Tu, Tv) \leq ad(u, v) + b(D(u, Tu) + D(v, Tv)) + \frac{c}{2}(D(v, Tu) + D(u, Tv)), \ \forall \ u, v \in U.$$

Let $c = 2c_1$, then

$$H(Tu, Tv) \leq ad(u, v) + b(D(u, Tu) + D(v, Tv)) + c_1(D(v, Tu) + D(u, Tv)), \ \forall \ u, v \in U.$$

Therefore by Theorem 2.2.8, T has a fixed point. □

Corollary 2.2.12 (Gordji [6]). Let $T : U \to CB(U)$ be a mapping and d be complete. Assume that their exists $\alpha \in [0, 1/2)$ such that

$$H(Tu, Tv) \leq \alpha(D(v, Tu) + D(u, Tv)), \ \forall \ u, v \in U.$$

Then T has a fixed point.

Proof. Let $\phi(u_1, u_2, u_3, u_4) = \phi_2(u_1, u_2, u_3, u_4) = \beta u_4$, where $\beta \in [0, 1)$. From (2.2.19), we have

$$H(Tu, Tv) \leq \frac{\beta}{2}(D(v, Tu) + D(u, Tv)) = \alpha(D(v, Tu) + D(u, Tv)), \ \forall \ u, v \in U.$$

Therefore by Theorem 2.2.8, T has a fixed point. □

Example 2.2.13. Let $U = [1, \infty)$ be endowed with the usual metric d. Define $T : U \to CB(U)$ by $Tu = [1, \sqrt{u}]$ for each $u \in U$. Consider $\phi(u_1, u_2, u_3, u_4) = \frac{1}{2}u_1$. Clearly $\phi \in \Phi$. Now for each $u, v \in U$, we have

$$H(Tu, Tv) = |\sqrt{u} - \sqrt{v}| = |(\sqrt{u} - \sqrt{v}) \times \frac{\sqrt{u} + \sqrt{v}}{\sqrt{u} + \sqrt{v}}| = \frac{|u - v|}{\sqrt{u} + \sqrt{v}} \leq \frac{1}{2}|u - v| = \frac{1}{2}d(u, v).$$

Therefore by Theorem 2.2.8, T has a fixed point.

Theorem 2.2.14. Let $f : U \to U$ be a single-valued mapping and $T : U \to CB(U)$ be a multi-valued mapping such that $TU \subseteq fU$. Assume that fU is a complete subspace of U and there exists $\phi \in \Phi_{CB}$ such that

$$H(Tu, Tv) \leq \phi\left(d(fu, fv), D(fu, Tu), D(fv, Tv), \frac{D(fv, Tu) + D(fu, Tv)}{2}\right), \ \forall\, u, v \in U. \tag{2.2.41}$$

Then T and f have a coincidence point, say w. Moreover, if f is T-weakly commuting at w and $ffw = fw$, then T and f have a common fixed point.

Proof. Let $u_0 \in U$. Since $Tu_0 \neq \emptyset$ and $Tu_0 \subseteq fU$. Then there exists $u_1 \in U$ such that $fu_1 \in Tu_0$. Suppose $fu_0 \neq fu_1$, for otherwise, u_0 is a coincidence point of T and f. From (2.2.41), we have

$$H(Tu_0, Tu_1) \leq \phi\left(d(fu_0, fu_1), D(fu_0, Tu_0), D(fu_1, Tu_1), \frac{D(fu_1, Tu_0) + D(fu_0, Tu_1)}{2}\right). \tag{2.2.42}$$

For $p > 1$, there exists $fu_2 \in Tu_1$. Suppose $fu_1 \neq fu_2$, for otherwise, u_1 is a coincidence point of T and f. Then we have

$$0 < d(fu_1, fu_2) < pH(Tu_0, Tu_1). \tag{2.2.43}$$

From (2.2.42), we have

$$H(Tu_0, Tu_1) \leq \phi\left(d(fu_0, fu_1), d(fu_0, fu_1), d(fu_1, fu_2), \frac{0 + d(fu_0, fu_2)}{2}\right). \tag{2.2.44}$$

By triangular inequality, we have

$$d(fu_0, fu_2) \leq d(fu_0, fu_1) + d(fu_1, fu_2). \tag{2.2.45}$$

We claim that $d(fu_1, fu_2) < d(fu_0, fu_1)$. Suppose on contrary that $d(fu_1, fu_2) \geq d(fu_0, fu_1)$. Then from (2.2.45), we have $d(fu_0, fu_2) \leq 2d(fu_1, fu_2)$. Using these in

(2.2.44), we have

$$H(Tu_0, Tu_1) \leq \phi(d(fu_1, fu_2), d(fu_0, fu_1), d(fu_1, fu_2), d(fu_1, fu_2)). \tag{2.2.46}$$

By property (ii) of Φ_{CB}, we have $H(Tu_0, Tu_1) = 0$. A contradiction to (2.2.43). Hence $d(fu_1, fu_2) < d(fu_0, fu_1)$. From (2.2.44), we have

$$H(Tu_0, Tu_1) \leq \phi(d(fu_0, fu_1), d(fu_0, fu_1), d(fu_1, fu_2), d(fu_0, fu_1)). \tag{2.2.47}$$

By property (ii) of Φ_{CB}, we have

$$H(Tu_0, Tu_1) \leq \psi(d(fu_0, fu_1)). \tag{2.2.48}$$

From (2.2.43) and (2.2.48), we have

$$d(fu_1, fu_2) < p\psi(d(fu_0, fu_1)). \tag{2.2.49}$$

Since ψ is strictly increasing, we have

$$\psi(d(fu_1, fu_2)) < \psi(p\psi(d(fu_0, fu_1))). \tag{2.2.50}$$

Thus we have $p_1 = \frac{\psi(p\psi(d(fu_0, fu_1)))}{\psi(d(fu_1, fu_2))}$. Again from (2.2.41), we have

$$H(Tu_1, Tu_2) \leq \phi\left(d(fu_1, fu_2), D(fu_1, Tu_1), D(fu_2, Tu_2), \frac{D(fu_2, Tu_1) + D(fu_1, Tu_2)}{2}\right). \tag{2.2.51}$$

For $p_1 > 1$, there exists $fu_3 \in Tu_2$. Suppose $fu_2 \neq fu_3$, for otherwise, u_2 is a coincidence point of T and f. Then we have

$$0 < d(fu_2, fu_3) < p_1 H(Tu_1, Tu_2). \tag{2.2.52}$$

From (2.2.51), we have

$$H(Tu_1, Tu_2) \leq \phi\left(d(fu_1, fu_2), d(fu_1, fu_2), d(fu_2, fu_3), \frac{0 + d(fu_1, fu_3)}{2}\right). \tag{2.2.53}$$

By triangular inequality, we have

$$d(fu_1, fu_3) \leq d(fu_1, fu_2) + d(fu_2, fu_3). \tag{2.2.54}$$

We claim that $d(fu_2, fu_3) < d(fu_1, fu_2)$. Suppose on contrary that $d(fu_2, fu_3) \geq d(fu_1, fu_2)$. Then from (2.2.54), we have $d(fu_1, fu_3) \leq 2d(fu_2, fu_3)$. Using these in (2.2.53), we have

$$H(Tu_1, Tu_2) \leq \phi(d(fu_2, fu_3), d(fu_1, fu_2), d(fu_2, fu_3), d(fu_2, fu_3)). \tag{2.2.55}$$

By property (ii) of Φ_{CB}, we have $H(Tu_1, Tu_2) = 0$. A contradiction to (2.2.52). Hence $d(fu_2, fu_3) < d(fu_1, fu_2)$. From (2.2.53), we have

$$H(Tu_1, Tu_2) \leq \phi(d(fu_1, fu_2), d(fu_1, fu_2), d(fu_2, fu_3), d(fu_1, fu_2)). \tag{2.2.56}$$

By property (ii) of Φ_{CB}, we have

$$H(Tu_1, Tu_2) \leq \psi(d(fu_1, fu_2)). \tag{2.2.57}$$

From (2.2.52) and (2.2.57), we have

$$d(fu_2, fu_3) < p_1\psi(d(fu_1, fu_2)) = \psi(q\psi(d(fu_0, fu_1))). \tag{2.2.58}$$

Since ψ is strictly increasing, we have

$$\psi(d(fu_2, fu_3)) < \psi^2(p\psi(d(fu_0, fu_1))). \tag{2.2.59}$$

Thus we have $p_2 = \frac{\psi^2(p\psi(d(fu_0,fu_1)))}{\psi(d(fu_2,fu_3))}$. Continuing in this way, we get a sequence $\{Tu_n\}$ in $CB(U)$ such that $fu_{n+1} \in Tu_n$, $fu_n \neq fu_{n+1}$ and $d(fu_{n+1}, fu_{n+2}) < d(fu_n, fu_{n+1})$ for each $n \in \mathbb{N} \cup \{0\}$. Further, we have

$$d(fu_n, fu_{n+1}) < \psi^{n-1}(p\psi(d(fu_0, fu_1))), \ \forall\ n \in \mathbb{N}. \tag{2.2.60}$$

Let $n > m$. Then we have

$$\begin{aligned}
d(fu_m, fu_n) &\leq d(fu_m, fu_{m+1}) + d(fu_{m+1}, fu_{m+2}) + \cdots + d(fu_{n-1}, fu_n) \\
&< \psi^{m-1}(d(fu_0, fu_1)) + \psi^m(d(fu_0, fu_1)) + \cdots + \psi^{n-2}(d(fu_0, fu_1)) \\
&= \sum_{i=m-1}^{n-2} \psi^i(d(fu_0, fu_1)).
\end{aligned}$$

Therefore $\{fu_n\}$ is a Cauchy sequence in fU. By completeness of fU, there exist $u^*, z^* \in U$ such that

$$\lim_{n\to\infty} fu_n = fu^* = z^*.$$

From (2.2.41), we have

$$\begin{aligned}
D(fu_{n+1}, Tu^*) &\leq H(Tu_n, Tu^*) \\
&\leq \phi\Big(d(fu_n, fu^*), D(fu_n, Tu_n), D(fu^*, Tu^*), \\
&\qquad \frac{D(fu^*, Tu_n) + D(fu_n, Tu^*)}{2}\Big) \\
&\leq \phi\Big(d(fu_n, fu^*), d(fu_n, fu_{n+1}), D(fu^*, Tu^*), \\
&\qquad \frac{d(fu^*, fu_{n+1}) + D(fu_n, Tu^*)}{2}\Big). \qquad (2.2.61)
\end{aligned}$$

Letting $n \to \infty$ in (2.2.61), we have

$$D(fu^*, Tu^*) \leq \phi(0, 0, D(fu^*, Tu^*), D(fu^*, Tu^*)). \qquad (2.2.62)$$

By property (iii) of Φ_{CB}, we have $D(fu^*, Tu^*) = 0$. By closedness of T, we have $fu^* \in Tu^*$. Therefore u^* is a coincidence point of T and f. By assumption we have $fu^* = ffu^* \in Tfu^*$ which implies that $z^* = fz^* \in Tz^*$. □

Example 2.2.15. Let $U = [1, \infty)$ be endowed with the usual metric d. Define $f : U \to U$ by $fu = 2\sqrt{u} - 1$ for each $u \in U$ and $T : U \to CB(U)$ by $Tu = [1, \sqrt{u}]$ for each $u \in U$. Consider $\phi(u_1, u_2, u_3, u_4) = \frac{1}{2}u_1$. Clearly $\phi \in \Phi$. Now for each $u, v \in U$, we have

$$H(Tu, Tv) = |\sqrt{u} - \sqrt{v}| = \frac{2|\sqrt{u} - \sqrt{v}|}{2} = \frac{1}{2}|fu - fv| = \frac{1}{2}d(fu, fv).$$

Therefore by Theorem 2.2.14, T and f have a coincidence point. Moreover, $ff1 \in Tf1$ and $ff1 = f1$.

Chapter 3

Fixed point theorems in partially ordered metric spaces

Ran et al. [13] initiated the study of fixed points for mappings of partially ordered metric spaces. A number of authors obtained many interesting results in this direction. In this chapter we generalize some results of Chapter one. Although, we have considered same contractive conditions as introduced in Chapter one but we require that these conditions hold for those pair of points from the metric space that are related to each other. Throughout this chapter, U is a nonempty set endowed with a complete metric d and a partial ordering $\preceq$.

3.1 Fixed point theorems for single-valued mappings

In this section, we establish some fixed point theorems for ϕ-contraction mappings in partially ordered metric spaces. We can obtain some results of Nieto et al. [12] as consequences of our results.

Theorem 3.1.1. Let $T : U \to U$ be a continuous and nondecreasing mapping. Assume that there exists $\phi \in \Phi$ such that

$$d(Tu, Tv) \leq \phi \left(d(u, v), d(u, Tu), d(v, Tv), \frac{d(v, Tu) + d(u, Tv)}{2} \right) \tag{3.1.1}$$

for each $u, v \in U$ with $u \preceq v$ and there exists $u_0 \in U$ such that $u_0 \preceq Tu_0$. Then T has a fixed point.

Proof. Let $u_0 \in U$ such that $u_0 \preceq Tu_0$. Since T is nondecreasing, by induction we have

$$u_0 \preceq Tu_0 \preceq T^2u_0 \preceq T^3u_0 \preceq \cdots \preceq T^nu_0 \preceq T^{n+1}u_0 \preceq \cdots . \tag{3.1.2}$$

Put $u_{n+1} = Tu_n = T^{n+1}u_0$ for each $n \in \mathbb{N} \cup \{0\}$. If $u_{N+1} = u_N$ for some $N \in \mathbb{N} \cup \{0\}$. Then u_N is a fixed point. Suppose $u_{n+1} \neq u_n$ for each $n \in \mathbb{N} \cup \{0\}$. As $u_n \preceq u_{n+1}$ for each $n \in \mathbb{N} \cup \{0\}$. Then from (3.1.1), we have

$$\begin{aligned} d(u_{n+1}, u_{n+2}) &= d(Tu_n, Tu_{n+1}) \\ &\leq \phi\Big(d(u_n, u_{n+1}), d(u_n, Tu_n), d(u_{n+1}, Tu_{n+1}), \\ &\quad \frac{d(u_{n+1}, Tu_n) + d(u_n, Tu_{n+1})}{2}\Big) \\ &= \phi\Big(d(u_n, u_{n+1}), d(u_n, u_{n+1}), d(u_{n+1}, u_{n+2}), \\ &\quad \frac{d(u_{n+1}, u_{n+1}) + d(u_n, u_{n+2})}{2}\Big). \end{aligned} \tag{3.1.3}$$

By triangular inequality, we have

$$d(u_n, u_{n+2}) \leq d(u_n, u_{n+1}) + d(u_{n+1}, u_{n+2}). \tag{3.1.4}$$

We claim that $d(u_{n+1}, u_{n+2}) < d(u_n, u_{n+1})$ for each $n \in \mathbb{N} \cup \{0\}$. Suppose on contrary that $d(u_{n+1}, u_{n+2}) \geq d(u_n, u_{n+1})$ for some $n \in \mathbb{N} \cup \{0\}$. From (3.1.4), we get $d(u_n, u_{n+2}) \leq 2d(u_{n+1}, u_{n+2})$. Since ϕ is nondecreasing, by using these in (3.1.3), we have

$$d(u_{n+1}, u_{n+2}) \leq \phi(d(u_{n+1}, u_{n+2}), d(u_n, u_{n+1}), d(u_{n+1}, u_{n+2}), d(u_{n+1}, u_{n+2})). \tag{3.1.5}$$

By (3.1.5) and property (ii) of Φ, we have

$$d(u_{n+1}, u_{n+2}) = 0.$$

A contradiction to our assumption that $u_{n+1} \neq u_n$ for each $n \in \mathbb{N} \cup \{0\}$. Thus $d(u_{n+1}, u_{n+2}) < d(u_n, u_{n+1})$ for each $n \in \mathbb{N} \cup \{0\}$. From (3.1.4), we have $d(u_n, u_{n+2}) < 2d(u_n, u_{n+1})$. Using it in (3.1.3), we have

$$d(u_{n+1}, u_{n+2}) \leq \phi(d(u_n, u_{n+1}), d(u_n, u_{n+1}), d(u_{n+1}, u_{n+2}), d(u_n, u_{n+1})). \tag{3.1.6}$$

By (3.1.6) and property (ii) of Φ, we have

$$d(u_{n+1}, u_{n+2}) \leq \psi(d(u_n, u_{n+1})), \ \forall \ n \in \mathbb{N} \cup \{0\}.$$

Continuing in the same way, we get

$$d(u_{n+1}, u_{n+2}) \leq \psi^{n+1}(d(u_0, u_1)), \ \forall \ n \in \mathbb{N} \cup \{0\}. \tag{3.1.7}$$

Let $n > m$. Then we have

$$\begin{aligned} d(u_m, u_n) &\leq d(u_m, u_{m+1}) + d(u_{m+1}, u_{m+2}) + \cdots + d(u_{n-1}, u_n) \\ &\leq \psi^m(d(u_0, u_1)) + \psi^{m+1}(d(u_0, u_1)) + \cdots + \psi^{n-1}(d(u_0, u_1)) \\ &= \sum_{i=m}^{n-1} \psi^i(d(u_0, u_1)). \end{aligned}$$

Therefore $\{u_n\}$ is a Cauchy sequence in U. By completeness of U, there exists $u^* \in U$ such that $u_n \to u^*$ as $n \to \infty$. Since T is continuous, then $u_{n+1} = Tu_n \to Tu^*$ as $n \to \infty$. By the uniqueness of limit, we have $u^* = Tu^*$. □

Corollary 3.1.2 (Nieto et al. [12]). Let $T : U \to U$ be a continuous and nondecreasing mapping. Assume that there exists $\alpha \in [0, 1)$ such that $d(Tu, Tv) \leq \alpha d(u, v)$ for all $u, v \in U$ with $u \preceq v$. If there exists $u_0 \in U$ such that $u_0 \preceq Tu_0$, then T has a fixed point.

Proof. Let $\phi(u_1, u_2, u_3, u_4) = \phi_5(u_1, u_2, u_3, u_4) = \alpha u_1$, where $\alpha \in [0, 1)$. From (3.1.1), we have

$$d(Tu, Tv) \leq \alpha d(u, v),$$

for each $u, v \in U$ with $u \preceq v$. Therefore by Theorem 3.1.1, T has a fixed point. □

Corollary 3.1.3. Let $T : U \to U$ be a continuous and nondecreasing mapping. Assume that there exists $\alpha \in [0, 1/2)$ such that $d(Tu, Tv) \leq \alpha(d(v, Tu)+d(u, Tv))$ for all $u, v \in U$ with $u \preceq v$. If there exists $u_0 \in U$ such that $u_0 \preceq Tu_0$, then T has a fixed point.

Proof. Let $\phi(u_1, u_2, u_3, u_4) = \phi_2(u_1, u_2, u_3, u_4) = \beta u_4$, where $\beta \in [0, 1)$. From (3.1.1), we have

$$d(Tu, Tv) \leq \frac{\beta}{2}(d(v, Tu) + d(u, Tv)) = \alpha(d(v, Tu) + d(u, Tv)),$$

for each $u, v \in U$ with $u \preceq v$. Therefore by Theorem 3.1.1, T has a fixed point. □

Corollary 3.1.4. Let $T : U \to U$ be a continuous and nondecreasing mapping. Assume that there exists $\alpha \in [0, 1)$ such that $d(Tu, Tv) \leq \alpha \max\{d(u, Tu), d(v, Tv)\}$ for all $u, v \in U$ with $u \preceq v$. If there exists $u_0 \in U$ such that $u_0 \preceq Tu_0$, then T has a fixed point.

Proof. Let $\phi(u_1, u_2, u_3, u_4) = \phi_4(u_1, u_2, u_3, u_4) = \alpha \max\{u_2, u_3\}$, where $\alpha \in [0, 1)$. From (3.1.1), we have

$$d(Tu, Tv) \leq \alpha \max\{d(u, Tu), d(v, Tv)\},$$

for each $u, v \in U$ with $u \preceq v$. Therefore by Theorem 3.1.1, T has a fixed point. □

Theorem 3.1.5. Let $T : U \to U$ be a nondecreasing mapping and there exists $\phi \in \Phi$ such that

$$d(Tu, Tv) \leq \phi\left(d(u, v), d(u, Tu), d(v, Tv), \frac{d(v, Tu) + d(u, Tv)}{2}\right) \tag{3.1.8}$$

for each $u, v \in U$ with $u \preceq v$. Assume that $u_n \preceq u$ for each $n \in \mathbb{N} \cup \{0\}$, whenever $\{u_n\}$ is a nondecreasing sequence in U such that $u_n \to u$. If there exists $u_0 \in U$ such that $u_0 \preceq Tu_0$, then T has a fixed point.

Proof. Following the proof of Theorem 3.1.1, we know that $\{u_n\}$ is a Cauchy sequence in U. Since U is complete, there exists $u^* \in U$ such that $u_n \to u^*$ as $n \to \infty$. As $u_n \preceq u^*$ for each $n \in \mathbb{N} \cup \{0\}$. From (3.1.8), we have

$$\begin{aligned} d(u_{n+1}, Tu^*) &= d(Tu_n, Tu^*) \\ &\leq \phi\Big(d(u_n, u^*), d(u_n, Tu_n), d(u^*, Tu^*), \\ &\qquad \frac{d(u^*, Tu_n) + d(u_n, Tu^*)}{2}\Big) \\ &= \phi\Big(d(u_n, u^*), d(u_n, u_{n+1}), d(u^*, Tu^*), \\ &\qquad \frac{d(u^*, u_{n+1}) + d(u_n, Tu^*)}{2}\Big). \end{aligned} \tag{3.1.9}$$

Letting $n \to \infty$ in (3.1.9), we have

$$d(u^*, Tu^*) \leq \phi\left(0, 0, d(u^*, Tu^*), \frac{0 + d(u^*, Tu^*)}{2}\right). \tag{3.1.10}$$

By property (iii) of Φ, we have $d(u^*, Tu^*) = 0$. Hence $Tu^* = u^*$. □

Corollary 3.1.6 (Nieto et al. [12])**.** Let $T : U \to U$ be a nondecreasing mapping. Assume that there exists $\alpha \in [0, 1)$ such that $d(Tu, Tv) \leq \alpha d(u, v)$ for all $u, v \in U$ with $u \preceq v$. Also, assume that $u_n \preceq u$ for each $n \in \mathbb{N} \cup \{0\}$, whenever $\{u_n\}$ is a nondecreasing sequence in U such that $u_n \to u$. If there exists $u_0 \in U$ such that $u_0 \preceq Tu_0$, then T has a fixed point.

Proof. Let $\phi(u_1, u_2, u_3, u_4) = \phi_5(u_1, u_2, u_3, u_4) = \alpha u_1$, where $\alpha \in [0, 1)$. From (3.1.8), we have

$$d(Tu, Tv) \leq \alpha d(u, v),$$

for each $u, v \in U$ with $u \preceq v$. Therefore by Theorem 3.1.5, T has a fixed point. $\square$

Corollary 3.1.7. Let $T : U \to U$ be a nondecreasing mapping. Assume that there exists $\alpha \in [0, 1/2)$ such that $d(Tu, Tv) \leq \alpha(d(u, Tu) + d(v, Tv))$ for all $u, v \in U$ with $u \preceq v$. Also, assume that $u_n \preceq u$ for each $n \in \mathbb{N} \cup \{0\}$, whenever $\{u_n\}$ is a nondecreasing sequence in U such that $u_n \to u$. If there exists $u_0 \in U$ such that $u_0 \preceq Tu_0$, then T has a fixed point.

Proof. Let $\phi(u_1, u_2, u_3, u_4) = \phi_6(u_1, u_2, u_3, u_4) = \frac{\beta}{2}(u_2 + u_3)$, where $\beta \in [0, 1)$. From (3.1.8), we have

$$d(Tu, Tv) \leq \frac{\beta}{2}(d(u, Tu) + d(v, Tv)) = \alpha(d(u, Tu) + d(v, Tv)),$$

for each $u, v \in U$ with $u \preceq v$. Therefore by Theorem 3.1.5, T has a fixed point. $\square$

Corollary 3.1.8. Let $T : U \to U$ be a nondecreasing mapping. Assume that there exists $\alpha \in [0, 1)$ such that $d(Tu, Tv) \leq \alpha \max\{d(u, Tu), d(v, Tv)\}$ for all $u, v \in U$ with $u \preceq v$. Also, assume that $u_n \preceq u$ for each $n \in \mathbb{N} \cup \{0\}$, whenever $\{u_n\}$ is a nondecreasing sequence in U such that $u_n \to u$. If there exists $u_0 \in U$ such that $u_0 \preceq Tu_0$, then T has a fixed point.

Proof. Let $\phi(u_1, u_2, u_3, u_4) = \phi_4(u_1, u_2, u_3, u_4) = \alpha \max\{u_2, u_3\}$, where $\alpha \in [0, 1)$. From (3.1.8), we have

$$d(Tu, Tv) \leq \alpha \max\{d(u, Tu), d(v, Tv)\},$$

for each $u, v \in U$ with $u \preceq v$. Therefore by Theorem 3.1.5, T has a fixed point. $\square$

To ensure the uniqueness of fixed point, we will consider the following condition:
(C): For each $u, v \in U$, there exists $z \in U$ such that $u \preceq z$ and $v \preceq z$.

Theorem 3.1.9. Adding condition (C) to the hypothesis of Theorem 3.1.1 (resp. Theorem 3.1.5), we obtain the uniqueness of fixed point of T.

Proof. Suppose u^* and v^* be two distinct fixed points of T. By using (C), for $u^*, v^* \in U$, we have $z \in U$ such that $u^* \preceq z$ and $v^* \preceq z$. Since T is nondecreasing, we have $u^* \preceq T^n z$ for each $n \in \mathbb{N}$. Put $z_n = T^n z$ for each $n \in \mathbb{N}$. Moreover, it is easy to show that $\{z_n\}$ is a Cauchy sequence in U, thus there exists $z^* \in U$ such that $z_n \to z^*$ as $n \to \infty$. From (3.1.1), we have

$$\begin{aligned} d(u^*, Tz_n) &= d(Tu^*, Tz_n) \\ &\leq \phi\left(d(u^*, z_n), d(u^*, Tu^*), d(z_n, Tz_n), \frac{d(z_n, Tu^*) + d(u^*, Tz_n)}{2}\right) \\ &= \phi\left(d(u^*, z_n), 0, d(z_n, z_{n+1}), \frac{d(z_n, u^*) + d(u^*, z_{n+1})}{2}\right). \end{aligned} \tag{3.1.11}$$

Letting $n \to \infty$ in (3.1.11), we have

$$d(u^*, z^*) \leq \phi\left(d(u^*, z^*), 0, 0, d(u^*, z^*)\right). \tag{3.1.12}$$

By (3.1.12) and property (iii) of Φ, we have $d(u^*, z^*) = 0$, i.e., $u^* = z^*$. Similarly, we have $v^* = z^*$. By the uniqueness of limit we have $u^* = v^*$. A contradiction to our assumption. Hence T has a unique fixed point in U. □

Corollary 3.1.10. Let $T : U \to U$ be a continuous and nondecreasing mapping. Assume that there exists $\alpha \in [0, 1)$ such that $d(Tu, Tv) \leq \alpha d(u, v)$ for all $u, v \in U$ with $u \preceq v$. Also, assume that U holds (C). If there exists $u_0 \in U$ such that $u_0 \preceq Tu_0$, then T has a unique fixed point.

Proof. Let $\phi(u_1, u_2, u_3, u_4) = \phi_5(u_1, u_2, u_3, u_4) = \alpha u_1$, where $\alpha \in [0, 1)$. From (3.1.1), we have

$$d(Tu, Tv) \leq \alpha d(u, v),$$

for each $u, v \in U$ with $u \preceq v$. Therefore by Theorem 3.1.9, T has a unique fixed point. □

Example 3.1.11. Let $U = [0, \infty) \times [0, \infty)$ be a partially ordered set with relation $\preceq$, where $(u_1, u_2) \preceq (v_1, v_2)$ if and only if $u_1 \leq v_1$ and $u_2 \leq v_2$. Define a metric d on U such that for each $u = (u_1, u_2), v = (v_1, v_2) \in U$, we have $d(u, v) = \max\{|u_1 - v_1|, |u_2 - v_2|\}$ and $T : U \to U$ is given by $T(u_1, u_2) = (\xi, \frac{u_2}{2})$ for each $u \in U$ and for some $\xi \in U$. Consider $\phi(u_1, u_2, u_3, u_4) = \frac{1}{2}u_1$. Clearly $\phi \in \Phi$. For each $u, v \in U$ with $u \preceq v$, we have

$$d(Tu, Tv) = \frac{1}{2}|u_2 - v_2| \leq \frac{1}{2}(\max\{|u_1 - v_1|, |u_2 - v_2|\}) = \frac{1}{2}d(u, v).$$

Therefore by Theorem 3.1.9, T has a unique fixed point.

3.2 Fixed point theorems for multi-valued mappings

For $A, B \in N(U)$, we have following relations:

- $A \prec_1 B$, if for each $a \in A$ we have $b \in B$ with $a \preceq b$.
- $A \prec_2 B$, if for each $b \in B$ we have $a \in A$ with $a \preceq b$.

Theorem 3.2.1. Let $T : U \to B(U)$ be a mapping satisfying the following conditions:

(i) there exists $u_0 \in U$ such that $\{u_0\} \prec_1 Tu_0$;

(ii) for $u, v \in U$, $u \preceq v$ implies $Tu \prec_1 Tv$;

(iii) if $\{u_n\}$ is a nondecreasing sequence in U such that $u_n \to u$, then $u_n \preceq u$ for each $n \in \mathbb{N} \cup \{0\}$;

(iv) there exists $\phi \in \Phi_B$ such that

$$\delta(Tu, Tv) \leq \phi\left(d(u,v), D(u,Tu), D(v,Tv), \frac{D(v,Tu) + D(u,Tv)}{2}\right) \tag{3.2.1}$$

for each $u, v \in U$ with $u \preceq v$.

Then T has a fixed point.

Proof. By (i) there exists $u_0 \in U$ such that $\{u_0\} \prec_1 Tu_0$. Then there exists $u_1 \in Tu_0$ such that $u_0 \preceq u_1$. By (ii) we have $Tu_0 \prec_1 Tu_1$. Then for $u_1 \in Tu_0$, there exists $u_2 \in Tu_1$ such that $u_1 \preceq u_2$. Continuing in this way, we get a sequence $\{u_n\}$ in U such that $u_{n+1} \in Tu_n$ for each $n \in \mathbb{N} \cup \{0\}$ and

$$u_0 \preceq u_1 \preceq u_1 \preceq u_2 \preceq \cdots \preceq u_n \preceq u_{n+1} \preceq \cdots . \tag{3.2.2}$$

If there exists some $N \in \mathbb{N} \cup \{0\}$ such that $u_N = u_{N+1}$, then u_N is a fixed point. Suppose $u_n \neq u_{n+1}$ for each $n \in \mathbb{N} \cup \{0\}$. As $u_n \preceq u_{n+1}$ for each $n \in \mathbb{N} \cup \{0\}$. Then from (3.2.1), we have

$$\begin{aligned} d(u_{n+1}, u_{n+2}) &\leq \delta(Tu_n, Tu_{n+1}) \\ &\leq \phi\Big(d(u_n, u_{n+1}), D(u_n, Tu_n), D(u_{n+1}, Tu_{n+1}), \\ &\qquad \frac{D(u_{n+1}, Tu_n) + D(u_n, Tu_{n+1})}{2}\Big) \\ &\leq \phi\Big(d(u_n, u_{n+1}), d(u_n, u_{n+1}), d(u_{n+1}, u_{n+2}), \\ &\qquad \frac{d(u_{n+1}, u_{n+1}) + d(u_n, u_{n+2})}{2}\Big). \end{aligned} \tag{3.2.3}$$

By triangular inequality, we have

$$d(u_n, u_{n+2}) \leq d(u_n, u_{n+1}) + d(u_{n+1}, u_{n+2}). \tag{3.2.4}$$

We claim that $d(u_{n+1}, u_{n+2}) < d(u_n, u_{n+1})$ for each $n \in \mathbb{N} \cup \{0\}$. Suppose on contrary that $d(u_{n+1}, u_{n+2}) \geq d(u_n, u_{n+1})$ for some $n \in \mathbb{N} \cup \{0\}$. From (3.2.4), we have $d(u_n, u_{n+2}) \leq 2d(u_{n+1}, u_{n+2})$. Since ϕ is nondecreasing, by using these in (3.2.3), we have

$$d(u_{n+1}, u_{n+2}) \leq \phi(d(u_{n+1}, u_{n+2}), d(u_n, u_{n+1}), d(u_{n+1}, u_{n+2}), d(u_{n+1}, u_{n+2})). \tag{3.2.5}$$

By (3.2.5) and property (ii) of Φ_B, we have $d(u_{n+1}, u_{n+2}) = 0$, i.e., $u_{n+1} = u_{n+2}$. A contradiction to assumption that $u_n \neq u_{n+1}$ for each $n \in \mathbb{N} \cup \{0\}$. Thus $d(u_{n+1}, u_{n+2}) < d(u_n, u_{n+1})$ for each $n \in \mathbb{N} \cup \{0\}$. From (3.2.4), we have $d(u_n, u_{n+2}) < 2d(u_n, u_{n+1})$. Using it in (3.2.3), we have

$$d(u_{n+1}, u_{n+2}) \leq \phi(d(u_n, u_{n+1}), d(u_n, u_{n+1}), d(u_{n+1}, u_{n+2}), d(u_n, u_{n+1})). \tag{3.2.6}$$

By (3.2.6) and property (ii) of Φ_B, we have

$$d(u_{n+1}, u_{n+2}) \leq \psi(d(u_n, u_{n+1})), \ \forall \ n \in \mathbb{N} \cup \{0\}.$$

Continuing in the same way, we get

$$d(u_{n+1}, u_{n+2}) \leq \psi^{n+1}(d(u_0, u_1)), \ \forall \ n \in \mathbb{N} \cup \{0\}. \tag{3.2.7}$$

Let $n > m$. Then we have

$$\begin{aligned} d(u_m, u_n) &\leq d(u_m, u_{m+1}) + d(u_{m+1}, u_{m+2}) + \cdots + d(u_{n-1}, u_n) \\ &\leq \psi^m(d(u_0, u_1)) + \psi^{m+1}(d(u_0, u_1)) + \cdots + \psi^{n-1}(d(u_0, u_1)) \\ &= \sum_{i=m}^{n-1} \psi^i(d(u_0, u_1)). \end{aligned}$$

Therefore $\{u_n\}$ is a Cauchy sequence in U. By completeness of U, there exists $u^* \in U$ such that $u_n \to u^*$ as $n \to \infty$. As $u_n \preceq u^*$ for each $n \in \mathbb{N} \cup \{0\}$. From (3.2.1), we have

$$\begin{aligned} \delta(u_{n+1}, Tu^*) &\leq \delta(Tu_n, Tu^*) \\ &\leq \phi\Big(d(u_n, u^*), D(u_n, Tu_n), D(u^*, Tu^*), \\ &\qquad \frac{D(u^*, Tu_n) + D(u_n, Tu^*)}{2}\Big) \\ &\leq \phi\Big(d(u_n, u^*), d(u_n, u_{n+1}), D(u^*, Tu^*), \\ &\qquad \frac{d(u^*, u_{n+1}) + D(u_n, Tu^*)}{2}\Big). \end{aligned} \tag{3.2.8}$$

Letting $n \to \infty$ in (3.2.8), we have

$$\begin{aligned} \delta(u^*, Tu^*) &\leq \phi\left(0, 0, D(u^*, Tu^*), \frac{D(u^*, Tu^*)}{2}\right) \\ &\leq \phi\left(0, 0, \delta(u^*, Tu^*), \frac{\delta(u^*, Tu^*)}{2}\right). \end{aligned} \tag{3.2.9}$$

By (3.2.9) and property (ii) of Φ_B, we have $\delta(u^*, Tu^*) = 0$. Hence $Tu^* = \{u^*\}$. Moreover, u^* is a fixed point. □

Corollary 3.2.2. Let $T : U \to B(U)$ be a mapping satisfying the following conditions:

(i) there exists $u_0 \in U$ such that $\{u_0\} \prec_1 Tu_0$;

(ii) for $u, v \in U$, $u \preceq v$ implies $Tu \prec_1 Tv$;

(iii) if $\{u_n\}$ is a nondecreasing sequence in U such that $u_n \to u$, then $u_n \preceq u$ for each $n \in \mathbb{N} \cup \{0\}$;

(iv) there exists $\alpha \in [0, 1)$ such that

$$\delta(Tu, Tv) \leq \alpha \max\left\{d(u, v), \frac{D(u, Tu) + D(v, Tv)}{2}, \frac{D(v, Tu) + D(u, Tv)}{2}\right\}$$

for all $u, v \in U$ with $u \preceq v$.

Then T has a fixed point.

Proof. Let $\phi(u_1, u_2, u_3, u_4) = \phi_7(u_1, u_2, u_3, u_4) = \alpha \max\{u_1, \frac{1}{2}(u_2 + u_3), u_4\}$, where $\alpha \in [0, 1)$. From (3.2.1), we have

$$\delta(Tu, Tv) \leq \alpha \max\left\{d(u, v), \frac{D(u, Tu) + D(v, Tv)}{2}, \frac{D(v, Tu) + D(u, Tv)}{2}\right\}$$

for all $u, v \in U$ with $u \preceq v$. Therefore by Theorem 3.2.1, T has a fixed point. □

Corollary 3.2.3 (Choudhury et al. [5])**.** Let $T : U \to B(U)$ be a mapping satisfying the following conditions:

(i) there exists $u_0 \in U$ such that $\{u_0\} \prec_1 Tu_0$;

(ii) for $u, v \in U$, $u \preceq v$ implies $Tu \prec_1 Tv$;

(iii) if $\{u_n\}$ is a nondecreasing sequence in U such that $u_n \to u$, then $u_n \preceq u$ for each $n \in \mathbb{N} \cup \{0\}$;

(iv) there exists $\alpha \in [0, 1)$ such that

$$\delta(Tu, Tv) \leq \alpha \max \left\{ d(u, v), D(u, Tu), D(v, Tv), \frac{D(v, Tu) + D(u, Tv)}{2} \right\}$$

for all $u, v \in U$ with $u \preceq v$.

Then T has a fixed point.

Proof. Let $\phi(u_1, u_2, u_3, u_4) = \phi_1(u_1, u_2, u_3, u_4) = \alpha \max\{u_1, u_2, u_3, u_4\}$, where $\alpha \in [0, 1)$. From (3.2.1), we have

$$\delta(Tu, Tv) \leq \alpha \max \left\{ d(u, v), D(u, Tu), D(v, Tv), \frac{D(v, Tu) + D(u, Tv)}{2} \right\}$$

for all $u, v \in U$ with $u \preceq v$. Therefore by Theorem 3.2.1, T has a fixed point. □

Example 3.2.4. Let $U = \{(0, 0), (0, 1), (-\frac{1}{4}, \frac{2}{3})\}$ be a partially ordered set with relation $\preceq$. Where $(u, v) \preceq (u, v)$ if and only if $u \leq u$, $v \leq v$. Define a metric d on U such that $d(u, v) = \max\{|u_1 - v_1|, |u_2 - v_2|\}$ for $u, v \in U$ and $T : U \to B(U)$ by

$$Tu = \begin{cases} \{(0, 1)\} & \text{if } u \neq (0, 0); \\ \{(0, 1), (-\frac{1}{4}, \frac{2}{3})\} & \text{if } u = (0, 0). \end{cases} \tag{3.2.10}$$

Consider $\phi(u_1, u_2, u_3, u_4) = \frac{2}{3} \max\{u_1, u_2\}$. Comparable elements are: $(0, 0) \preceq (0, 0)$, $(0, 1) \preceq (0, 1)$, $(-\frac{1}{4}, \frac{2}{3}) \preceq (-\frac{1}{4}, \frac{2}{3})$, $(0, 0) \preceq (0, 1)$, $(-\frac{1}{4}, \frac{2}{3}) \preceq (0, 1)$. Let $u_0 = (-\frac{1}{4}, \frac{2}{3})$, we have $\{u_0\} \prec_1 Tu_0$. If $u \preceq v$, then clearly we have $Tu \prec_1 Tv$. Now we discuss condition (iv) by following cases:

(i) Consider $u = v = (0, 0)$. Then $Tu = Tv = \{(0, 1), (-\frac{1}{4}, \frac{2}{3})\}$. Also, we have $\delta(Tu, Tv) = 1/3$, $d(u, v) = 0$, $D(u, Tu) = 2/3$. Thus condition (iv) of Theorem (3.2.1) is satisfied because $\frac{1}{3} < \frac{2}{3}(\frac{2}{3})$.

(ii) Consider $u = v = (0, 1)$. Then $Tu = Tv = \{(0, 1)\}$. Also, we have $\delta(Tu, Tv) = 0, d(u, v) = 0, D(u, Tu) = 0$. Thus condition (iv) of Theorem (3.2.1) is satisfied because $0 = \frac{2}{3}(0)$.

(iii) Consider $u = v = (-\frac{1}{4}, \frac{2}{3})$. Then $Tu = Tv = \{(0,1)\}$. Also, we have $\delta(Tu, Tv) = 0$, $d(u,v) = 0$, $D(u, Tu) = 1/3$. Thus condition (iv) of Theorem (3.2.1) is satisfied because $0 < \frac{2}{3}(\frac{1}{3})$.

(iv) Consider $u = (0,0) \preceq v = (0,1)$. Then $Tu = \{(0,1), (-\frac{1}{4}, \frac{2}{3})\}$ and $Tv = \{(0,1)\}$. Also, we have $\delta(Tu, Tv) = 1/3$, $d(u,v) = 1$, $D(u, Tu) = 2/3$. Thus condition (iv) of Theorem (3.2.1) is satisfied because $\frac{1}{3} < \frac{2}{3}(1)$.

(v) Consider $u = (-\frac{1}{4}, \frac{2}{3}) \preceq v = (0,1)$. Then $Tu = Tv = \{(0,1)\}$. Also, we have $\delta(Tu, Tv) = 0$, $d(u,v) = 1/3$, $D(u, Tu) = 1/3$. Thus condition (iv) of Theorem (3.2.1) is satisfied because $0 < \frac{2}{3}(\frac{1}{3})$.

Moreover, if $\{u_n\}$ is a nondecreasing sequence in U such that $u_n \to u$, then $u_n \preceq u$ for each $n \in \mathbb{N} \cup \{0\}$. Since all conditions of Theorem 3.2.1 are satisfied. Therefore T has a fixed point.

Theorem 3.2.5. Let $T : U \to B(U)$ be a mapping satisfying the following conditions:

(i) there exists $u_0 \in U$ such that $Tu_0 \prec_2 \{u_0\}$;

(ii) for $u, v \in U$, $u \preceq v$ implies $Tu \prec_2 Tv$;

(iii) if $\{u_n\}$ is a nonincreasing sequence in U such that $u_n \to u$, then $u_n \succeq u$ for each $n \in \mathbb{N} \cup \{0\}$;

(iv) there exists $\phi \in \Phi_B$ such that

$$\delta(Tu, Tv) \leq \phi\left(d(u,v), D(u,Tu), D(v,Tv), \frac{D(v,Tu) + D(u,Tv)}{2}\right)$$

for each $u, v \in U$ with $u \succeq v$;

Then T has a fixed point.

Proof. The proof follows on the same lines as in Theorem 3.2.1. □

Chapter 4

Fixed point theorems in pre-ordered metric spaces

In this chapter, we define another contractive type condition and obtain some fixed point theorems. We show that some well known results can be obtained as a special case of our results. Throughout this chapter, we denote by U a nonempty set endowed with a metric d and pre-ordering $\preceq$, by Ψ the class of altering distance functions and by Φ the class of continuous functions from $[0, \infty)$ into $[0, \infty)$.

Definition 4.0.6 (Khan et al. [10])**.** An altering distance function is a function $\psi : [0, \infty) \to [0, \infty)$ which satisfies the following conditions: (i) ψ is continuous and nondecreasing; (ii) $\psi(t) = 0$ if and only if $t = 0$.

Definition 4.0.7. Let $\beta : U \times U \to [0, \infty)$ be a mapping. A mapping $T : U \to U$ is said to be (β, ψ, ϕ)-contraction on $S \subset U$ if there exist two functions $\psi \in \Psi$ and $\phi \in \Phi$ such that

$$\beta(u, v)\psi(d(Tu, Tv)) \leq \phi(d(u, v)), \ \forall \ u, v \in S. \tag{4.0.1}$$

Definition 4.0.8. Let $\beta : U \times U \to [0, \infty)$ be a mapping. A mapping $T : U \to U$ is said to β-subadmissible if

(i) for $u, v \in U$, $\beta(u, v) \geq 1 \Rightarrow \beta(Tu, Tv) \geq 1$;

(ii) for $u \in U$, $\beta(T^{n+1}u, T^n u) \geq 1$ for each $n \in \mathbb{N} \cup \{0\}$ implies $\beta(T^m u, T^n u) \geq 1$ for each $m, n \in \mathbb{N} \cup \{0\}$ with $m > n$.

Remark 4.0.9. Note that $T^0 u = u \; \forall \; u \in U$.

Example 4.0.10. Let $U = [0, \infty)$. Define mappings $T : U \to U$ and $\beta : U \times U \to [0, \infty)$ by

$$Tu = u + 1 \; \forall \; u \in U,$$

and

$$\beta(u, v) = u + v \; \forall \; u, v \in U.$$

Then T is β-subadmissible.

Lemma 4.0.11 (Yan et al. [16])**.** If ψ is an alternating distance function and $\phi : [0, \infty) \to [0, \infty)$ is a continuous function with condition $\psi(t) > \phi(t)$ for all $t > 0$, then $\phi(0) = 0$.

Theorem 4.0.12. Let $S = \{(u, v) \in U \times U : u \preceq v \;\; or \;\; u \succeq v\}$ and $\beta : S \to [0, \infty)$ be a mapping such that

$$\beta(u, v) = \begin{cases} 0 & \text{if } u \prec v; \\ > 0 & \text{if } u \succeq v. \end{cases} \tag{4.0.2}$$

Assume that d is complete and $T : U \to U$ be a mapping such that:

(i) T is (β, ψ, ϕ)-contraction on S;

(ii) T is β-subadmissible on S;

(iii) there exists $u_0 \in U$ such that $\beta(Tu_0, u_0) \geq 1$;

(iv) T is continuous;

(v) $\psi(t) > \phi(t)$ for all $t > 0$.

Then T has a fixed point.

Proof. It follows from (iii) that there exists $u_0 \in U$ such that $\beta(Tu_0, u_0) \geq 1$. Let $u_1 = Tu_0$. Assume that $u_1 \neq u_0$, for otherwise, u_0 is a fixed point of T. From (4.0.2), we get $u_0 \preceq u_1$. Let $u_2 = Tu_1$. Assume that $u_2 \neq u_1$. As T is β-subadmissible on S, we have $\beta(Tu_1, Tu_0) \geq 1$. It follows from (4.0.2) that $u_2 \succeq u_1$. Further, using (4.0.1)

$$\begin{aligned} \psi(d(u_2, u_1)) &= \psi(d(Tu_1, Tu_0)) \\ &\leq \beta(u_1, u_0)\psi(d(Tu_1, Tu_0)) \\ &\leq \phi(d(u_1, u_0)). \end{aligned} \tag{4.0.3}$$

Continuing in this manner, we get a nondecreasing sequence $\{u_n\} = \{Tu_{n-1}\} = \{T^n u_0\}$ in U such that

$$\beta(u_{n+1}, u_n) \geq 1 \ \forall\ n \in \mathbb{N} \cup \{0\} \tag{4.0.4}$$

and

$$\psi(d(u_{n+1}, u_n)) \leq \phi(d(u_n, u_{n-1})). \tag{4.0.5}$$

Assume that $u_n \neq u_{n-1}$, for otherwise, $u_{n-1} = Tu_{n-1}$. It follows form (v) and (4.0.5) that

$$d(u_{n+1}, u_n) < d(u_n, u_{n-1}). \tag{4.0.6}$$

Thus $\{d(u_{n+1}, u_n)\}$ is a decreasing sequence of real numbers, which is bounded below by 0. Therefore, there exists $r \geq 0$ such that

$$d(u_{n+1}, u_n) \to r. \tag{4.0.7}$$

letting $n \to \infty$ in (4.0.5), we get

$$\psi(r) \leq \phi(r). \tag{4.0.8}$$

Thus

$$d(u_{n+1}, u_n) \to 0. \tag{4.0.9}$$

We claim that $\{u_n\}$ is a Cauchy sequence in U. For otherwise, there exists an $\epsilon > 0$ and a subsequence $\{u_{n_k}\}$ with $n_k > m_k > k$ such that

$$d(u_{n_k}, u_{m_k}) \geq \epsilon \ \forall k \geq 1. \tag{4.0.10}$$

Let n_k is the smallest positive integer greater than m_k satisfying (4.0.10), then

$$d(u_{n_{k-1}}, u_{m_k}) < \epsilon. \tag{4.0.11}$$

From (4.0.10) and (4.0.11), we get

$$\begin{aligned} \epsilon \leq d(u_{n_k}, u_{m_k}) &\leq d(u_{n_k}, u_{n_{k-1}}) + d(u_{n_{k-1}}, u_{m_k}) \\ &< d(u_{n_k}, u_{n_{k-1}}) + \epsilon. \end{aligned} \tag{4.0.12}$$

Letting $k \to \infty$ in (4.0.12) and using (4.0.9), we get

$$\lim_{n \to \infty} d(u_{n_k}, u_{m_k}) = \epsilon. \tag{4.0.13}$$

By using triangular inequality, we get

$$d(u_{n_k}, u_{m_k}) \leq d(u_{n_k}, u_{n_{k-1}}) + d(u_{n_{k-1}}, u_{m_{k-1}}) + d(u_{m_{k-1}}, u_{m_k}). \tag{4.0.14}$$

$$d(u_{n_{k-1}}, u_{m_{k-1}}) \leq d(u_{n_{k-1}}, u_{n_k}) + d(u_{n_k}, u_{m_k}) + d(u_{m_k}, u_{m_{k-1}}). \tag{4.0.15}$$

Letting $k \to \infty$ in (4.0.14), (4.0.15) and using (4.0.9), (4.0.13), we get

$$\lim_{k\to\infty} d(u_{n_{k-1}}, u_{m_{k-1}}) = \epsilon. \tag{4.0.16}$$

As T is β-subadmissible, by using (4.0.4), we have

$$\begin{aligned} \psi(d(u_{n_k}, u_{m_k})) &\leq \beta(u_{n_{k-1}}, u_{m_{k-1}})\psi(d(Tu_{n_{k-1}}, Tu_{m_{k-1}})) \\ &\leq \phi(d(u_{n_{k-1}}, u_{m_{k-1}})). \end{aligned} \tag{4.0.17}$$

Letting $k \to \infty$ in (4.0.17) and using (4.0.13), (4.0.16), we get

$$\psi(\epsilon) \leq \phi(\epsilon). \tag{4.0.18}$$

This implies that $\epsilon = 0$, a contradiction. This shows that $\{u_n\}$ is a Cauchy sequence in U. Therefore, there exists $u^* \in U$ such that $u_n \to u^*$ and continuity of T implies that

$$u^* = \lim_{n\to\infty} u_{n+1} = \lim_{n\to\infty} Tu_n = Tu^*.$$

□

Corollary 4.0.13 (Nieto et al. [12]). Let $(U, \preceq)$ be a partially ordered set and suppose that (U, d) be a complete metric space. Let $T : U \to U$ be a mapping satisfying the following assumptions:

(i) There exists $k \in [0, 1)$ such that $d(Tu, Tv) \leq kd(u, v)$ for all $u, v \in U$ with $u \succeq v$;

(ii) there exists $u_0 \in U$ such that $u_0 \preceq Tu_0$;

(iii) T is continuous and nondecreasing with respect to $\preceq$.

Then T has a fixed point.

Proof. Choose $\psi(t) = t$, $\phi(t) = kt$ and define $\beta : S \to [0, \infty)$ by

$$\beta(u, v) = \begin{cases} 1 & \text{if } u \succeq v; \\ 0 & \text{otherwise.} \end{cases}$$

It follows that $\beta(Tu_0, u_0) = 1$, $\psi(t) = t > kt = \phi(t)$ and $\beta(u, v)d(Tu, Tv) \leq kd(u, v)$ for all $(u, v) \in S$. Suppose $(u, v) \in S$ be such that $\beta(u, v) = 1$. Then $u \succeq v$ which in turns implies that $\beta(Tu, Tv) = 1$ since T is nondecreasing. Also, for each $u \in U$ with $\beta(T^{n+1}u, T^n u) \geq 1$ for each $n \in \mathbb{N} \cup \{0\}$, we have $\beta(T^m u, T^n u) \geq 1$ for each $m, n \in \mathbb{N} \cup \{0\}$ with $m > n$, since T nondecreasing. Therefore, all conditions of Theorem 4.0.12 are satisfied and T has a fixed point. □

Corollary 4.0.14 (Yan et al. [16])**.** Let $(U, \preceq)$ be a partially ordered set and suppose that (U, d) be a complete metric space. Let $T : U \to U$ be a continuous and nondecreasing mapping such that

$$\psi(d(Tu, Tv)) \leq \phi(d(u, v)), \quad \forall \;\; u \succeq v \tag{4.0.19}$$

where ψ is alternating distance function and $\phi : [0, \infty) \to [0, \infty)$ is continuous function with condition $\psi(t) > \phi(t)$ for all $t > 0$. If there exists $u_0 \in U$ such that $u_0 \preceq Tu_0$, then T has a fixed point.

Proof. Taking $\beta : S \to [0, \infty)$ by

$$\beta(u, v) = \begin{cases} 1 & \text{if } u \succeq v; \\ 0 & \text{otherwise.} \end{cases}$$

Then it is easy to see that all conditions of Theorem 4.0.12 are satisfied. □

Next two examples show that Theorem 4.0.12 is proper generalization of Yan et al. [16].

Example 4.0.15. Let $U = [0, \infty)$ be endowed with the usual metric $d(u, v) = |u - v|$ and $u \preceq v$ if and only if $u \leq v$. Define $\psi : [0, \infty) \to [0, \infty)$ by $\psi(t) = \frac{t}{2}$, $\phi : [0, \infty) \to [0, \infty)$ by $\phi(t) = \frac{t}{4}$ and $\beta : S \to [0, \infty)$ by

$$\beta(u, v) = \begin{cases} \frac{1}{2(u+v)} & \text{if} \quad u \succeq v \quad and \quad u \neq 0; \\ 1 \quad \text{if} & u = v = 0; \\ 0 \quad \text{otherwise.} \end{cases} \tag{4.0.20}$$

Let $T : U \to U$ be a mapping such that $Tu = u^2$ for all $u \in U$. Note that ψ is an altering distance function, ϕ and T are continuous functions, $\beta(T0, 0) = 1$ and $\psi(t) > \phi(t)$ for all $t > 0$. Further, if $u \succeq v$ then $\beta(u, v)\psi(d(Tu, Tv)) = \frac{1}{4}|u - v| = \phi(d(u, v))$, for otherwise $\beta(u, v)\psi(d(Tu, Tv)) = 0 \leq \phi(d(u, v))$. Furthermore, $\beta(u, v) \geq 1$ when $u = v = 0$ or $u, v \in (0, \frac{1}{4}]$ and in each case $\beta(u, v) \geq 1$ implies $\beta(Tu, Tv) \geq 1$. Clearly, for each $u \in U$ with $\beta(T^{n+1}u, T^n u) \geq 1$ for each $n \in \mathbb{N} \cup \{0\}$, we have $\beta(T^m u, T^n u) \geq 1$ for each $m, n \in \mathbb{N} \cup \{0\}$ with $m > n$. Therefore, all conditions of Theorem 4.0.12 are satisfied and $0, 1$ are fixed points of T. Note that Corollary 4.0.14 is not applicable here since $\psi(d(T5, T2)) > \phi(d(5, 2))$. Also, note that T is not a Banach contraction.

Example 4.0.16. Let $U = [1, \infty)$ be endowed with the usual metric $d(u, v) = |u-v|$ and $u \preceq v$ if and only if $u \leq v$. Define $\psi : [0, \infty) \to [0, \infty)$ by $\psi(t) = t$, $\phi : [0, \infty) \to [0, \infty)$ by $\phi(t) = \frac{t}{2}$ and $\beta : S \to [0, \infty)$ by

$$\beta(u, v) = \begin{cases} 1 & \text{if} \quad u = v = 1; \\ \frac{1}{4} & \text{if} \quad u \geq v, \quad \text{and} \quad u > 1; \\ 0 & \text{otherwise.} \end{cases} \tag{4.0.21}$$

Let $T : U \to U$ be a mapping such that $Tu = 2u - 1$ for all $u \in U$. Note that ψ is an altering distance function, ϕ and T are continuous functions, $\beta(T1, 1) = 1$ and $\psi(t) > \phi(t)$ for all $t > 0$. Further, T is (β, ψ, ϕ)-contraction on S and T is β-subadmissible on S. Therefore, all conditions of Theorem 4.0.12 are satisfied and 1 is a fixed point of T. Note that Corollary 4.0.14 is not applicable here since $\psi(d(T4, T1)) > \phi(d(4, 1))$. Also, note that T is not a Banach contraction.

Example 4.0.17. Let $U = \{(u_1, u_2) : \ u_1, u_2 \in \mathbb{R}\}$ be endowed with the metric $d(u, v) = |u_1 - v_1| + |u_2 - v_2|$ and $(u_1, u_2) \preceq (v_1, v_2)$ if and only if $u_1 \leq v_1$ and $u_2 \leq v_2$. Define $\psi : [0, \infty) \to [0, \infty)$ by $\psi(t) = t$, $\phi : [0, \infty) \to [0, \infty)$ by $\phi(t) = \frac{t}{4}$ and $\beta : S \to [0, \infty)$ by

$$\beta(u, v) = \begin{cases} \frac{1}{8} & \text{if} \quad u \geq v; \\ 1 & \text{if} \quad u = v; \\ 0 & \text{otherwise.} \end{cases} \tag{4.0.22}$$

Let $T : U \to U$ be a mapping such that $T(u_1, u_2) = (2u_1, 2u_2)$ for all $u = (u_1, u_2) \in U$. Note that ψ is an altering distance function, ϕ and T are continuous functions,

$\beta(Tu_0, u_0) = 1$ for $u_0 = (0,0)$ and $\psi(t) > \phi(t)$ for all $t > 0$. Further, T is (β, ψ, ϕ)-contraction on S and T is β-subadmissible on S. Therefore, all conditions of Theorem 4.0.12 are satisfied and $(0,0)$ is a fixed point of T.

Theorem 4.0.18. Let $S = \{(u,v) \in U \times U : u \preceq v \ \ or \ \ u \succeq v\}$ and $\beta : S \to [0,\infty)$ be a mapping such that

$$\beta(u,v) = \begin{cases} 0 & \text{if } u \prec v; \\ > 0 & \text{if } u \succeq v. \end{cases} \tag{4.0.23}$$

Assume that d is complete and $T : U \to U$ be a mapping such that:

(i) T is (β, ψ, ϕ)-contraction on S;

(ii) T is β-subadmissible on S;

(iii) there exists $u_0 \in U$ such that $\beta(Tu_0, u_0) \geq 1$;

(iv) if $\{u_n\}$ is a sequence in U such that $u_n \to u$ and $\beta(u_{n+1}, u_n) \geq 1$ for each $n \in \mathbb{N} \cup \{0\}$, then $\beta(u, u_n) \geq 1$ for each $n \in \mathbb{N} \cup \{0\}$;

(v) $\psi(t) > \phi(t)$ for all $t > 0$.

Then T has a fixed point.

Proof. Following the proof of Theorem 4.0.12, we construct a Cauchy sequence $\{u_n\}$ in U such that $u_n \to u^* \in U$. Now using (4.0.4) and condition (iv), we get $\beta(u^*, u_n) \geq 1$ for each $n \in \mathbb{N} \cup \{0\}$. Further, we have

$$\begin{aligned} \psi(d(Tu^*, u_{n+1})) &= \psi(d(Tu^*, Tu_n)) \\ &\leq \beta(u^*, u_n)\psi(d(Tu^*, Tu_n)) \\ &\leq \phi(d(u^*, u_n)). \end{aligned} \tag{4.0.24}$$

Letting $n \to \infty$ in (4.0.24), using the properties of ϕ, ψ and condition (v), it follows that $Tu^* = u^*$. □

Corollary 4.0.19 (Nieto et al. [12])**.** Let $(U, \preceq)$ be a partially ordered set and suppose that (U, d) be a complete metric space. Let $T : U \to U$ be a mapping satisfying the following assumptions:

(i) There exists $k \in [0,1)$ such that $d(Tu,Tv) \leq kd(u,v)$ for all $u,v \in U$ with $u \succeq v$;

(ii) there exists $u_0 \in U$ such that $u_0 \preceq Tu_0$;

(iii) if $\{u_n\}$ is a nondecreasing sequence in U such that $u_n \to u$ as $n \to \infty$, then $u_n \preceq u$ for each $n \in \mathbb{N} \cup \{0\}$.

Then T has a fixed point.

Proof. Define $\beta : S \to [0,\infty)$ by

$$\beta(u,v) = \begin{cases} 1 & \text{if } u \succeq v; \\ 0 & \text{otherwise.} \end{cases}$$

Note that if $\{u_n\}$ is a sequence in U such that $\beta(u_{n+1}, u_n) = 1$ for each $n \in \mathbb{N} \cup \{0\}$ and $u_n \to u^*$, then $u_n \preceq u_{n+1}$ for each $n \in \mathbb{N} \cup \{0\}$. Moreover, it follows by assumption (iii) that $\beta(u^*, u_n) = 1$ for each $n \in \mathbb{N} \cup \{0\}$. Choosing $\psi(t) = t$ and $\phi(t) = kt$, it is easy to see that all conditions of Theorem 4.0.18 hold. □

Corollary 4.0.20 (Yan et al. [16])**.** Let $(U, \preceq)$ be a partially ordered set and suppose that (U,d) be a complete metric space. Assume that if $\{u_n\}$ is a nondecreasing sequence in U such that $u_n \to u$, then $u_n \preceq u$ for each $n \in \mathbb{N} \cup \{0\}$. Let $T : U \to U$ be a nondecreasing mapping such that

$$\psi(d(Tu,Tv)) \leq \phi(d(u,v)), \quad \forall \;\; u \succeq v \tag{4.0.25}$$

where ψ is alternating distance function and $\phi : [0,\infty) \to [0,\infty)$ is continuous function with condition $\psi(t) > \phi(t)$ for all $t > 0$. If there exists $u_0 \in U$ such that $u_0 \preceq Tu_0$, then T has a fixed point.

Proof. Taking $\beta : S \to [0,\infty)$, by

$$\beta(u,v) = \begin{cases} 1 & \text{if } u \succeq v; \\ 0 & \text{otherwise.} \end{cases}$$

Then it is easy to see that all conditions of Theorem 4.0.18 are satisfied. □

Following example shows that Theorem 4.0.18 is a proper generalization of Yan et al. [16].

Example 4.0.21. Let $U = \{0, 1, 2, 3, \cdots\}$ be endowed with metric d define by

$$d(u, v) = \begin{cases} u + v & \text{if } \; u \neq v; \\ 0 & \text{if } \; u = v, \end{cases} \tag{4.0.26}$$

and $u \preceq v$ if and only if $\frac{u-v}{2} \in \mathbb{Z}$. Here $\preceq$ is a pre-order which is not a partial order. Define $\psi : [0, \infty) \to [0, \infty)$ by $\psi(t) = t$, $\phi : [0, \infty) \to [0, \infty)$ by $\phi(t) = \frac{t}{2}$ and $\beta : S \to [0, \infty)$ by

$$\beta(u, v) = \begin{cases} \frac{1}{2(u+v)} & \text{if } \; u \succeq v \;\; and \;\; u \neq 0; \\ 1 & \text{if } \; u = v = 0; \\ 0 & \text{otherwise.} \end{cases} \tag{4.0.27}$$

Let $T : U \to U$ be a mapping such that

$$Tu = \begin{cases} 0 & \text{if } \; u = 0, 1; \\ u^2 & \text{otherwise.} \end{cases} \tag{4.0.28}$$

Note that ψ is an altering distance function, ϕ is continuous functions, $\beta(T0, 0) = 1$ and $\psi(t) > \phi(t)$ for all $t > 0$. Further, If $u \succeq v$ then clearly $\beta(u, v)\psi(d(Tu, Tv)) = \beta(u, v)(Tu+Tv) \leq \frac{u+v}{2} = \phi(d(u, v))$, for otherwise, $\beta(u, v)\psi(d(Tu, Tv)) = 0 \leq \phi(d(u, v))$. Furthermore, If $u = v = 0$ then $\beta(u, v) = 1 \Rightarrow \beta(Tu, Tv) \geq 1$. Also, for $u \in U$ we have $\beta(T^{n+1}u, T^n u) \geq 1$ for each $n \in \mathbb{N} \cup \{0\}$ implies $\beta(T^m u, T^n u) = 1$ for each $m, n \in \mathbb{N} \cup \{0\}$ with $m > n$. Moreover, if $\{u_n\}$ is a sequence in U such that $u_n \to u$ and $\beta(u_{n+1}, u_n) \geq 1$ for each $n \in \mathbb{N} \cup \{0\}$, then $\beta(u, u_n) \geq 1$ for each $n \in \mathbb{N} \cup \{0\}$. Therefore, all conditions of Theorem 4.0.18 are satisfied and 0 is a fixed points of T. Note that Corollary 4.0.20 is not applicable here since $\psi(d(T6, T2)) > \phi(d(6, 2))$. Also, note that T is not a Banach contraction.

Theorem 4.0.22. In addition to the hypotheses of Theorem 4.0.12 (resp. Theorem 4.0.18), if for all $u, v \in U$, there exists $z \in U$ such that either $\beta(u, z) \geq 1$ and $\beta(v, z) \geq 1$ or $\beta(z, u) \geq 1$ and $\beta(z, v) \geq 1$, then T has a unique fixed point.

Proof. Let u^* and v^* be the two distinct fixed points of T. Suppose that there exists $z \in U$ such that $\beta(u^*, z) \geq 1$ and $\beta(v^*, z) \geq 1$. Since T is β-subadmissible, we have

$\beta(u^*, T^n z) \geq 1$ for each $n \in \mathbb{N}$. Moreover, we have

$$\begin{aligned} \psi(d(u^*, T^{n+1}z)) &\leq \beta(u^*, T^n z)\psi(d(Tu^*, T^{n+1}z)) \\ &\leq \phi(d(u^*, T^n z)). \end{aligned} \tag{4.0.29}$$

Since ψ is nondecreasing and $\psi(t) > \phi(t)$ (for $t > 0$), (4.0.29) implies that $d(u^*, T^{n+1}z) < d(u^*, T^n z)$. Therefore, $\{d(u^*, T^n z)\}$ is a decreasing sequence of real numbers, which is bounded below by 0. Thus $d(u^*, T^n z) \to r \geq 0$. Letting $n \to \infty$, from (4.0.29) we get

$$\psi(r) \leq \phi(r). \tag{4.0.30}$$

It follows that $r = 0$, by using Lemma 4.0.11 and properties of ψ. Hence $\lim_{n\to\infty} T^n z = u^*$. Similarly, we have $\lim_{n\to\infty} T^n z = v^*$. Thus $u^* = v^*$ by uniqueness of limit. The proof of other case runs on same lines. □

Corollary 4.0.23 (Yan et al. [16]). Adding the following condition in the hypotheses of Theorem 4.0.14 (resp.Theorem 4.0.20), we obtain the uniqueness of the fixed point of T. For $u, v \in U$, there exists $z \in U$ which is comparable to u and v.

Proof. Taking $\beta : S \to [0, \infty)$, by

$$\beta(u, v) = \begin{cases} 1 & \text{if } u \succeq v; \\ 0 & \text{otherwise.} \end{cases}$$

Then it is easy to see that all conditions of Theorem 4.0.12 are satisfied. □

Example 4.0.24. Let $U = [0, \frac{1}{4}]$ be endowed with the usual metric $d(u, v) = |u - v|$ and $u \preceq v$ if and only if $u \leq v$. Define $\psi : [0, \infty) \to [0, \infty)$ by $\psi(t) = \frac{t}{2}$, $\phi : [0, \infty) \to [0, \infty)$ by $\phi(t) = \frac{t}{4}$ and $\beta : S \to [0, \infty)$ by

$$\beta(u, v) = \begin{cases} \frac{1}{2(u+v)} & \text{if} \quad u \succeq v \quad \textit{and} \quad u \neq 0; \\ 1 & \text{if} \quad u = v = 0; \\ 0 & \text{otherwise.} \end{cases} \tag{4.0.31}$$

Let $T : U \to U$ be a mapping such that $Tu = u^2$ for all $u \in U$. Note that ψ is an altering distance function, ϕ and T are continuous functions, $\beta(T0, 0) = 1$ and $\psi(t) > \phi(t)$ for all

$t > 0$. Further, If $u \succeq v$ then $\beta(u,v)\psi(d(Tu,Tv)) = \frac{1}{4}|u-v| = \phi(d(u,v))$, for otherwise $\beta(u,v)\psi(d(Tu,Tv)) = 0 \leq \phi(d(u,v))$. Furthermore, $\beta(u,v) \geq 1$ when $u = v = 0$ or $u,v \in (0,\frac{1}{4}]$ and in each case $\beta(u,v) \geq 1$ implies $\beta(Tu,Tv) \geq 1$. Clearly for each $u \in U$ with $\beta(T^{n+1}u, T^n u) \geq 1$ for each $n \in \mathbb{N} \cup \{0\}$, we have $\beta(T^m u, T^n u) \geq 1$ for each $m,n \in \mathbb{N} \cup \{0\}$ with $m > n$. Note that for $u,v \in U$, there exists $z = \min\{u,v\}$ in U such that $\beta(u,z) \geq 1$ and $\beta(v,z) \geq 1$. Therefore, all conditions of Theorem (4.0.22) are satisfied and 0 is a unique fixed point of T.

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