Existence of Solutions of Four-Point Boundary Value Problems

By

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IN THE NAME OF

ALLAH

THE MOST BENEFICIENT

THE MOST MERCIFUL

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Dedicated To

My Parents

Having no substitute, always there

Where there is no one

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ABSTRACT

This dissertation is concerned with the existence and multiplicity results of some four-point boundary value problems (BVPs) of the type

$$x''(t) = f(t, x, x'),\ t \in I = [a, b]$$
$$x(a) = x(c),\ x(b) = x(d),$$

Where $a, b, c, d \in \mathbb{R}$, are parameters such that $a < c \le d < b$ and $f : I \times \mathbb{R}^2 \to \mathbb{R}$ is a continuous function which may be non-linear. These BVPs can arise in the bending of a beam where conditions may be imposed at the ends of the beam as well as at interior points to improve stability or for other reasons; or in the study of the steady states of a heated bar with a thermostat where controllers at $t = a$ and $t = b$ adds or dissipates heat according to the temperature detected by sensors at $t = c$ and d .

In **Chapter 1**, some basic definitions and notions concerning the existence of solutions of four-point BVPs are presented. They include the method of upper and lower solutions, modification of the nonlinearty, Arzela-Ascoli theorem, Schauder's fixed point theorem, Green's function, degree theory and properties of the degree theory. Moreover, some known results about the existence of at least one solution of the above BVP in the presence of constant lower and upper solutions are also discussed.

In **Chapter 2**, existence of at least one solution of the BVP in the presence of lower and upper solutions which are not necessarily constants is studied. In this chapter, the types of the lower and upper solutions admissible are more general whereas in chapter one, the upper and lower solutions are considered as constant functions. Moreover, existence of solutions, when the growth of the nonlinearity of $f(t, x, x')$ is allowed to be quadratic with respect to x′, is studied. Further, the existence theory under different combinations of sign conditions on f without requiring the Nagumo type growth condition is presented.

In **Chapter 3**, we investigate the new results on the existence of at least three solutions of the four point BVPs in the presence of two lower solutions α, α_1 and two upper solutions $\beta,\ \beta_1$ such that $\alpha \le \alpha_1$ and $\beta \ge \beta_1$ are investigated. It is assumed that the upper and lower solutions are to be continuous functions and some topological degree arguments are used to get multiplicity results. Moreover, the growth of $f(t, x, x')$ with respect to x' is allowed to be quadratic.

Contents

Chapter 1

Preliminaries

1.1 Introduction

In this chapter, some basic definitions and notions concerning the existence of solutions of four-point BVPs of the type

$$\begin{aligned} x''(t) &= f(t, x, x'),\ t \in I = [a, b], \\ x(a) &= x(c),\ x(b) = x(d), \end{aligned} \tag{1.1}$$

are discussed, where $a, b, c, d \in \mathbb{R}$, are parameters such that $a < c \leq d < b$ and $f : I \times \mathbb{R}^2 \to \mathbb{R}$ is a continuous function that may be non-linear. The method of upper and lower solutions is developed to establish existence of solutions of the BVPs (1.1). The basic idea of the method of upper and lower solutions is to modify the given problem suitably with respect to upper and lower solutions and then employ Leray-Schauder theory or known existence results of the modified problem together with the theory of differential and integral inequalities to establish existence of solutions of the given problems. The nonlinear function f is modified in such a way that solutions of the modified problems lie in a region where f is unmodified and hence are solutions of the original nonlinear BVPs. The method of upper and lower solutions for ordinary differential equations has introduced by E. Picard in 1893 and has further developed by Scorza Dragoni in 1931. Such methods allow us to ensure the existence of at least one solution of a given problem in the region bounded by well ordered upper and lower solutions, that is, the lower solution α and the upper solution β are such that $\alpha(x) \leq \beta(x)$, $x \in \mathbb{R}$.

1.2 Some basic definitions and notions

For the four point BVP (1.1), the concept of lower and upper solutions is introduced as follows

Upper and lower solutions

$\alpha \in C^2[a,b]$, is said to be a *lower solution* of (1.1), if α satisfies the following inequalities

$$\begin{aligned} \alpha''(t) &\geq f(t, \alpha(t), \alpha'(t)),\ t \in I, \\ \alpha(a) - \alpha(c) &\leq 0,\ \alpha(b) - \alpha(d) \leq 0. \end{aligned}$$

Similarly, $\beta \in C^2[a,b]$ is an *upper solution* of (1.1), if β satisfies the following inequalities

$$\begin{aligned} \beta''(t) &\leq f(t, \beta(t), \beta'(t)),\ t \in I, \\ \beta(a) - \beta(c) &\geq 0,\ \beta(b) - \beta(d) \geq 0. \end{aligned}$$

In order to understand the concept of upper and lower solutions for the BVPs (1.1), let us consider an example of two point BVP

$$\begin{aligned} x'' &= x' - te^{-x},\ t \in [0,1], \\ x(0) &= 0,\ x(1) = 0. \end{aligned}$$

Take $\alpha = 0$ on $[0,1]$, then $\alpha' = 0$ and $\alpha'' = 0$.

Clearly, α satisfies the following relations $\alpha(0) = 0$, $\alpha(1) = 0$ and

$$f(t, \alpha(t), \alpha'(t) = 0 - te^0 = -t \leq 0 = \alpha''(t),$$

that is,

$$\alpha''(t) \geq f(t, \alpha(t), \alpha'(t)),\ t \in [0,1].$$

Take $\beta = t$ on $[0,1]$, then $\beta' = 1$, and $\beta'' = 0$ on $[0,1]$.

Moreover, $\beta(0) = 0$, $\beta(1) = 1 > 0$ and

$$f(t, \beta(t), \beta'(t)) = 1 - te^{-t} = 1 - \frac{t}{e^t} \geq 0 = \beta''(t),$$

that is,

$$\beta''(t) \leq f(t, \beta(t), \beta'(t)), t \in [0, 1],$$

the functions $\alpha = 0$ and $\beta = t$ are known as lower and upper solutions of the above BVP.

The following definitions are useful for our later work.

Nagumo function

A continuous function $\omega : [0, \infty) \to [0, \infty)$ is called a *Nagumo function* if the integral $\int\limits_{\lambda}^{\infty} \frac{sds}{\omega(s)}$ diverges for any $\lambda \geq 0$, that is,

$$\int\limits_{\lambda}^{\infty} \frac{sds}{\omega(s)} = +\infty, \ \forall\ \lambda \geq 0.$$

A function $f : I \times \mathbb{R}^2 \to \mathbb{R}$ satisfies a Bernstein-Nagumo condition relative to the pair α, β, if there exists a *Nagumo function* ω such that

$$f(t, x, y) sgn(y) \leq \omega(|y|) \text{ on } I \times [\alpha, \beta] \times \mathbb{R}, \tag{1.2}$$

$$f(t, x, y) sgn(y) \geq -\omega(|y|) \text{ on } I \times [\alpha, \beta] \times \mathbb{R}. \tag{1.3}$$

Consider the following second order differential equation

$$x''(t) = f(t, x, x'), \ t \in I, \tag{1.4}$$

where $f : I \times \mathbb{R}^2 \to \mathbb{R}$ is continuous and may be nonlinear. Assume that f satisfies a Nagumo condition, that is, the nonlinearity of f, grows not faster than quadratically with respect to x'. In other word, f is chosen to be such that

$$\left|f(t, x, x')\right| \leq c(x)(1 + \left|x'\right|^2), \tag{1.5}$$

where $c = c(x)$. Let x be a solution of (1.4). Note that on any interval of monotonicity of $x(t)$, take $t = t(x)$. Denote x' by $p(x)$. Then, by the chain rule

$$x'' = \frac{d}{dt}(x') = \frac{dp}{dx}\frac{dx}{dt} = pp'.$$

Let x_0 be the extremum point of $x(t)$ which is closest to t and let $x(t_0) = x_0$. Setting $q = p^2$ and equations (1.4) and (1.5), imply

$$\begin{aligned} \frac{dq}{dx} &= 2p\frac{dp}{dx} = 2pp', \\ \frac{1}{2}\frac{dq}{dx} &= pp' = x'' = f(t, x, x') \leq c(x)(1+q). \end{aligned}$$

Integrating

$$\log(1+q) \leq 2\int c(x)dx + D,$$

implies that

$$q = Ae^{2\int c(x)dx} - 1,$$

where $A = e^D$. From here the boundedness of $q(x)$ (and hence of $p(x)$) follows, provided x lies in a bounded interval. Further, notice that this argument is independent of boundary conditions (BCs). If on the other hand, the Nagumo condition is violated, then $|x'|$ need not be bounded. For example, the problem

$$\begin{aligned} x'' + (1+x'^2)^{\frac{3}{2}} &= 0 \text{ on } (0,2), \\ x(0) &= x(2) = 0, \end{aligned}$$

has as a solution the upper half of the circle, $(t-1)^2 + x^2 = 1$, with infinite derivatives at $x = 0$ and $x = 2$.

Remark:

The Nagumo condition implies boundedness of the derivative x' of a solution x, provided x is bounded.

Uniformly bounded [5]

A family $\{f(x)\}$ of functions defined on a closed interval I is said to be *uniformly bounded,* if there exist a number $M \geq 0$ such that $|f(x)| < M$ for all $x \in I$ and for all f belonging to the given family.

Equicontinuous

A family of functions $\{f(x)\}$ is said to be *equicontinuous*, if for given $\epsilon > 0$, there is a $\delta > 0$ such that

$$|f(x_1) - f(x_2)| < \epsilon, \text{ whenever } |x_1 - x_2| < \delta, \ x_1, \ x_2 \in \mathbb{R}.$$

Compact set

A set M in a metric space R is said to be *compact,* if every sequence of elements in M contains a subsequence that converges to some $x \in R$**.**

Linear Operator

$T : C^1 \to C^1$ is said to be a linear operator, if

$$T(x + y) = Tx + Ty$$

and

$$T(\alpha x) = \alpha Tx, \ \forall \ \alpha \in F \text{ and } x, y \in \mathbb{R}.$$

Compact operator

Suppose that M is a subset of a Banach space B. An operator $A : M \to B$ is said to be *compact operator* (or completely continuous) if and only if it is continuous and it maps every bounded subset of M into a relatively compact set.

An $\in -net$

Let M be any set in a metric space R and let $\epsilon > 0$. The set $A \subset R$ is said to be an $\epsilon - net$ with respect to M, if for every point $x \in M$ at least one point $a \in A$ can be found such that $\rho(a, x) < \epsilon$.

Homotopy

Let $f, g \in C(\Omega)$ and $H : \Omega \times [0, 1] \to R$. Then H is a *homotopy* between f and g if H is continuous on $\bar{\Omega} \times [0, 1]$ and $H(x, 0) = f(x)$ and $H(x, 1) = g(x)$ for every $x \in \bar{\Omega}$.

Dirac delta function

The Dirac delta function often referred to as the unit impulse function and introduced by the British theoretical physicist Paul Dirac, can usually be informally thought of as a function $\delta(x)$ that has value infinity for $x = x_i$ and value zero for $x \neq x_i$, that is

$$\delta(x - x_i) = \begin{cases} 0, & \text{if } x \neq x_i, \\ \infty, & \text{if } x = x_i, \end{cases}$$

and integral of the Dirac delta from any negative limit to any positive limit is 1, that is

$$\int_{-\infty}^{\infty} \delta(x)dx = 1.$$

Heaviside unit step function

The heaviside unit step function is denoted by $H(x - x_i)$, and is defined by

$$H(x - x_i) = \begin{cases} 0, & \text{if } x < x_i, \\ 1, & \text{if } x > x_i. \end{cases}$$

Note that the Dirac delta function $\delta(x - x_i)$ can be defined (among others) as the derivative of the Heaviside unit step function $H(x - x_i)$, that is

$$\delta(x - x_i) = \frac{d}{dx} H(x - x_i).$$

The following results are useful for our later work.

Theorem 1 (*Arzela-Ascoli theorem*) [9] *A necessary and sufficient condition that a family of continuous functions defined on a closed interval* $[a, b]$ *be compact in* $C[a, b]$ *is that this family be uniformly bounded and equicontinuous.*

Proof. Let the set F be compact in $C[a, b]$. Then it is totally bounded, that is, for each $\epsilon > 0$ there exist a finite $(\frac{\epsilon}{3}) - net$, $f_1, f_2, ..., f_k$ in F. Each of the functions f_i, being a continuous function on a closed interval is bounded, that is, choose M_i such that $|f_i| \leq M_i$. Set

$$M = \max\left\{M_i + \frac{\epsilon}{3}\right\}.$$

Then by the definition of an $(\frac{\epsilon}{3}) - net$, for every $f \in F$, there is at least one f_i, such that

$$\rho(f, f_i) = \max |f(x) - f_i(x)| < \frac{\epsilon}{3}.$$

Consequently

$$|f| < |f_i| + \frac{\epsilon}{3} < M_i + \frac{\epsilon}{3} < M.$$

Thus, F is uniformly bounded. Further, Since each of the functions f_i is uniformly continuous and consequently uniformly continuous on $[a, b]$, hence for a given $\frac{\epsilon}{3}$, there exist a δ_i such that

$$|f_i(x_1) - f_i(x_2)| < \frac{\epsilon}{3}, \text{ if } |x_1 - x_2| < \delta_i.$$

Choose $\delta = \min\{\delta_i\}$. Then, for $|x_1 - x_2| < \delta$ and for any $f \in F$, taking f_i so that $\rho(f, f_i) < \frac{\epsilon}{3}$, implies

$$\begin{aligned} |f(x_1) - f(x_2)| &= |f(x_1) - f_i(x_1) + f_i(x_1) - f_i(x_2) + f_i(x_2) - f(x_2)| \\ &\leq |f(x_1) - f_i(x_1)| + |f_i(x_1) - f_i(x_2)| + |f_i(x_2) - f(x_2)| \\ &< \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon. \end{aligned}$$

Thus the equicontinuity of F is proved.

Conversely, let F be a uniformly bounded and equicontinuous family of functions. As, the necessary and sufficient condition for a subset M of a complete metric space R to be compact is that M be totally bounded. In order to prove its compactness in $C[a, b]$, it is sufficient to show that for an arbitrary $\epsilon > 0$, there exists a finite $\epsilon - net$ in $C[a, b]$. For this purpose, let

$$|f| \leq M, \text{ for all } f \in F$$

and let $\delta > 0$ be chosen so that

$$|f(x_1) - f(x_2)| < \frac{\epsilon}{5} \text{ for } |x_1 - x_2| < \delta \text{ and for all } f \in F, \text{ where } x_1, x_2 \in I.$$

Subdivide $[a, b]$ by means of the points $x_0 = a, x_1, x_2, \quad , x_n = b$ into intervals of length less than

δ and construct vertical lines at these points of subdivision. Subdivide the segment $[-M, M]$ on the $y - axis$ by means of the points

$$y_0 = -M,\ y_1, y_2, ..., y_m = M,$$

into intervals of length $\frac{\epsilon}{5}$ and construct horizontal lines at these points of subdivisions. Thus subdivide the rectangle

$$a \leq x \leq b,\ \ -M \leq y \leq M,$$

into cells with horizontal sides of length less than δ and vertical sides of length $\frac{\epsilon}{5}$. Now assign to every function $f \in F$ a polygonal arc $g(x)$ with vertices at the points (x_k, y_i). This polygonal arc $g(x)$ has a vertices on the points of the constructed net and deviates at points x_k from the function f by less than $\frac{\epsilon}{5}$ (the existence of such a polygonal arc is obvious), since, by the construction

$$\begin{aligned} |f(x_k) - g(x_k)| &< \frac{\epsilon}{5}, \\ |f(x_{k+1}) - g(x_{k+1})| &< \frac{\epsilon}{5} \end{aligned}$$

and

$$\left|f(x_k) - f(x_{_{k+1}})\right| < \frac{\epsilon}{5}.$$

This implies

$$|g(x_k) - g(x_{k+1})| < \frac{3\epsilon}{5}.$$

Since the function $g(x)$ is linear between the points x_k and x_{k+1}, hence

$$|g(x_k) - g(x)| < \frac{3\epsilon}{5} \text{ for all } x,\ x_k \leq x \leq x_{k+1}.$$

Now, let x be an arbitrary point of the closed interval $[a, b]$ and let x_k be the subdivision point chosen which is closest to x from the left. Therefore

$$|f(x) - g(x)| \leq |f(x) - f(x_k)| + |f(x_k) - g(x_k)| + |g(x_k) - g(x)| < \epsilon.$$

Consequently, the polygonal arcs $g(x)$ form an $\epsilon - net$ with respect to F. Their number is finite (since only a finite number of polygonal arcs can be drawn through a finite number of points). Hence, F is totally bounded. This proves the theorem completely. ■

Theorem 2 (*Schauder's fixed point*) [5] *If M is a bounded, closed and convex set in a Banach space X and $f : M \mapsto M$ is a compact mapping, then f has a fixed point.*

Proof. Since the continuous image of a compact set is compact, therefore $f(M)$ is compact and hence totally bounded. Therefore for each $n \in N$, there exists a finite subset

$$\{y_{1,}y_{2}, \ ..., y_{Nn}\} \subseteq f(M) \subseteq M,$$

such that for any element $x \in M$

$$\min_{1 \leq i \leq N_n} \|f(x) - y_i\| < \frac{1}{n}.$$

For each i, define

$$a_i(x) = \max\left\{\frac{1}{n} - \|f(x) - y_i\|, 0\right\}.$$

Then, for each $x \in M$, there must exists at least one $i \in \{1, 2, ...N_n\}$ such that $a_i(x) \neq 0$. Define an operator $p_n : M \to N$ by

$$p_n(x) = \frac{\sum_{i=1}^{Nn} \alpha_i(x) y_i}{\sum_{i=1}^{Nn} \alpha_i(x)}.$$

Note that $p_n(x) \in M$, because $p_n(x)$ is a convex combination of the elements $\{y_1, y_2, \ ..., y_{Nn}\} \subseteq M$. Also, since f is continuous, a_i is continuous as well. Therefore p_n is continuous. Now let

$$K_n = conv\left(\{y_{1,}y_2, ..., y_n\}\right),$$

which implies that

$$K_n \subset conv(f(M)) \subset M.$$

Therefore, K_n is a bounded, closed and convex subset of a finite dimensional Banach space which is spanned by vectors $\{y_1, y_2, \ ..., y_n\}$ and $p_n : K_n \to K_n$. Brouwer's fixed point theorem implies that each of the mappings p_n has a fixed point $x_n \in K_n$. Since K is compact, therefore the sequence $\{x_n\}$ has a subsequence $\{x_{n_k}\}$ which converges, say $x_{n_k} \to x \in K$. Now, observe that for any n

$$\begin{aligned} p_n(x) &= \left\| \frac{\sum_{i=1}^{Nn} \alpha_i(x) y_i}{\sum_{i=1}^{Nn} \alpha_i(x)} - f(x) \right\| \leq \frac{\sum_{i=1}^{Nn} \alpha_i(x) \|y_i - f(x)\|}{\sum_{i=1}^{Nn} \alpha_i(x)} \\ &\leq \frac{\sum_{i=1}^{Nn} \alpha_i(x) \frac{1}{n}}{\sum_{i=1}^{Nn} \alpha_i(x)} = \frac{1}{n}. \end{aligned}$$

It follows that x is a fixed point of f. Since

$$\|x_{n_k} - f(x)\| \leq \|p_{n_k}(x_{n_k}) - f(x_{n_k})\| + \|f(x_{n_k}) - f(x)\|$$

and the right-hand side tends to 0 as $n \to \infty$, where the left-hand side tends to $\|x - f(x)\|$. This proves the theorem. ■

An equivalent version of the above theorem without proof is,

Theorem 3 (*Schauder's fixed point*) *Let M be a closed, convex and bounded subset of a Banach space X. If $F : M \to M$ is compact, then F has at least one fixed point.*

1.3 Green's functions [3]

Let L be a linear operator and consider a nonhomogeneous problem

$$L(u) = f(x),$$

subject to some two point homogeneous BCs. The solution of the linear problem by [3] is

$$u(x) = \int_a^b f(x_0) G(x, x_0) dx_0,$$

where the function $G(x, x_0)$ can be regarded as the influence function for the source $f(x)$ and is known as *Green's function.*

1.3.1 Properties of Green's function

The Green's function has the following properties:

1. Symmetry: $G(x_0, x) = G(x, x_0)$;

2. Continuity: $G(x_0-, x) = G(x_0+, x)$;

3. Jump discontinuity of the derivative: $\frac{dG(x_0+,x)}{dx} - \frac{dG(x_0-,x)}{dx} = -1$.

1.4 Degree theory [12]

Let Ω be an open bounded subset of $\mathbb{R}^n$ and $f : \bar{\Omega} \to \mathbb{R}^n$ is a continuous map, $y \in \mathbb{R}^n$ and $y \notin f(\partial\Omega)$, define an integer, $deg(f, \Omega, y)$ which corresponds to the number of solutions $x \in \Omega$ of the equation $y = f(x)$. If f is a smooth function and y is not a critical value for f, then the degree is given by

$$deg(f, \Omega, y) = \sum_{x \in f^{-1}(y)} J_f(x),$$

where $J_f(x) = det f'(x)$ and $\bar{\Omega}$ denotes the closure of Ω.

1.4.1 Properties of the Brouwer degree

Some of properties of the Brouwer degree are listed here. Let Ω be an open bounded subset in $\mathbb{R}^n$, $f \in C(\bar{\Omega})$ and $y \notin f(\partial\Omega)$ (y is not an image of the boundary of Ω). Then the Brouwer degree has the following properties [13] :

1. Normalization: For the identity map I, $deg(I, \Omega, y) = 1$ for $y \in \Omega$;

2. Additivity: For disjoint open subsets Ω_1 and Ω_2 of Ω with $y \notin \overline{\Omega} \backslash \Omega_1 \cup \Omega_2$,

$$deg(f, \Omega, y) = deg(f, \Omega_1, y) + deg(f, \Omega_2, y);$$

3. Homotopy: The degree $deg(h(t, .), \Omega, y(t))$ is independent of t, whenever $h : [0, 1] \times \Omega \to \mathbb{R}^n$ and $y : [0, 1] \to \mathbb{R}^n$ are continuous and $y(t) \notin h(t, \partial\Omega)$ for every $t \in [0, 1]$;

4. Existence: $deg(f, \Omega, y) \neq 0$ implies $f^{(-1)}(y) \neq \varnothing$;

5. Boundary dependence: $deg(f, \Omega, y) = deg(g, \Omega, y)$,whenever $f\,|_{\partial\Omega} = g\,|_{\partial\Omega}$;

6. Excision Property: For every open set $\Omega_1 \subset \Omega$ such that $y \notin f(\bar{\Omega}/\Omega_1)$,

$$deg(f, \Omega, y) = deg(f, \Omega_1, y).$$

The Leray-Schauder degree is an extension of the Brouwer degree to the case of infinite dimensional spaces, in the particular case of maps of the form $T = I - C$, where I is the identity map and C is a compact map. The above properties also hold for the Leray-Schauder degree.

1.5 Upper and lower solutions method

In this section, the method of upper and lower solutions is discussed. Since the method involves modification of the nonlinearity, it is useful to introduce this concept first.

1.5.1 Modified function

Let $\alpha, \beta \in C^2(I)$ be lower and upper solutions of a differential equation

$$x'' = f(t, x, x')$$

such that $\alpha \leq \beta$ on I. Let $C > 0$ be such that

$$C > \max\{\left|\alpha'(t)\right|, \left|\beta'(t)\right| : t \in I\}.$$

Define the functions

$$F^*(t, x, x') = \begin{cases} f(t, x, C), & \text{for } x' > C, \\ f(t, x, x'), & \text{for } |x'| \leq C, \\ f(t, x, -C), & \text{for } x' < -C \end{cases}$$

and

$$F(t, x, x') = \begin{cases} F^*(t, \beta(t), x') + \frac{x - \beta(t)}{1 + x^2}, & \text{for } x > \beta(t), \\ F^*(t, x, x'), & \text{for } \alpha(t) \leq x \leq \beta(t), \\ F^*(t, \alpha(t), x') + \frac{x - \alpha(t)}{1 + x^2}, & \text{for } x < \alpha(t). \end{cases}$$

The function $F(t,x,x')$ is called a modification of $f(t,x,x')$ associated with the triple α, β, C. It is clear that the modified function $F(t,x,x')$ is continuous and bounded on $I \times \mathbb{R}^2$. Hence there exists a constant $M > 0$, such that

$$\left|F(t,x,x')\right| \leq M \text{ on } I \times \mathbb{R}^2,$$

where $M = M_0 + 1$, and

$$M_0 = \max\{\left|f(t,x,x')\right| : t \in I, \alpha(t) \leq x \leq \beta(t), \ \left|x'(t)\right| \leq C\} + \max_{t \in I} |\alpha(t)| + \max_{t \in I} |\beta(t)| \, .$$

As an example, let us consider

$$f\,(x,y,z) = a_1 x + b_1 y + c_1 z,$$

where a_1, b_1, c_1 are real constants, choose $C > 0$. Take $\alpha = -a, \beta = a$, then

$$F^*(x,y,z) = \begin{cases} a_1 x + b_1 y + c_1 C, & \text{if } z \geq C, \\ a_1 x + b_1 y + c_1 z, & \text{if } |z| < C, \\ a_1 x + b_1 y - c_1 C, & \text{if } z \leq -C \end{cases}$$

and

$$F(x,y,z) = \begin{cases} F^*(x,a,z), & \text{if } y \geq a, \\ F^*(x,y,z), & \text{if } -a \leq y \leq a, \\ F^*(x,-a,z), & \text{if } y < -a, \end{cases}$$

$$= \begin{cases} \begin{pmatrix} a_1x + b_1a + c_1C, & \text{if } z \geq C, \text{ if } y \geq a \\ a_1x + b_1a + c_1z, & \text{if } |z| < C, \text{ if } y \geq a \\ a_1x + b_1a - c_1C, & \text{if } z \leq -C, \text{ if } y \geq a \end{pmatrix} \\ \begin{pmatrix} a_1x + b_1y + c_1C, & \text{if } z \geq C, \text{ if } |y| \leq a \\ a_1x + b_1y + c_1z, & \text{if } |z| < C, \text{ if } |y| \leq a \\ a_1x + b_1y - c_1C, & \text{if } z \leq -C, \text{ if } |y| \leq a \end{pmatrix} \\ \begin{pmatrix} a_1x - b_1a + c_1C, & \text{if } z \geq C, \text{ if } y \leq -a \\ a_1x - b_1a + c_1z, & \text{if } |z| < C, \text{ if } y \leq -a \\ a_1x - b_1a - c_1C, & \text{if } z \leq -C, \text{ if } y \leq -a, \end{pmatrix} \end{cases}$$

is a modification of f with respect to $-a, a$. Clearly, F is continuous and bounded on $I \times \mathbb{R}^2$.

Now, to explain the basic idea of the method of upper and lower solutions, consider the following BVP

$$\begin{aligned} x''(t) &= f(t, x, x'), \ t \in I \\ x(a) &= c, \ x(b) = d \end{aligned} \tag{1.6}$$

and the corresponding modified problem

$$\begin{aligned} x''(t) &= F(t, x, x'), \ t \in I \\ x(a) &= c, \ x(b) = d. \end{aligned} \tag{1.7}$$

Solutions to the modified problems lie in a region where f is unmodified and hence are solutions of the original problems. Recall some results, from [1].

Theorem 4 *Assume that $\alpha, \beta \in C^2(I)$ are lower and upper solutions of (1.6). If $f : I \times \mathbb{R}^2 \to \mathbb{R}$ is continuous and $f(t, x, x')$ is strictly increasing in x for each $(t, x') \in I \times \mathbb{R}$, then $\alpha(t) \leq \beta(t)$, $t \in I$. In particular, there is at most one solution.*

Proof. Define

$$m(t) = \alpha(t) - \beta(t), \ t \in I,$$

then, $m \in C^2(I)$ and the BCs imply that

$$m(a) \leq 0, \ m(b) \leq 0.$$

It is required to show that $m(t) \leq 0$ on I. If not, then $m\ (t)$ has a positive maximum at some $t_0 \in I$. From the BCs, it is clear that $t_0 \in (a, b)$, and hence

$$m(t_0) > 0, \ m'(t_0) = 0, \ \text{and } m''(t_0) \leq 0.$$

However, using the increasing property of the function f in x and the definition of lower and upper solutions, we have

$$m''(t_0) = \alpha''(t_0) - \beta''(t_0) \geq f(t_0, \alpha(t_0), \alpha'(t_0)) - f(t_0, \beta(t_0), \beta'(t_0)) > 0,$$

which is a contradiction. Hence $\alpha(t) \leq \beta(t)$, $t \in I$. ■

The next theorem, taken from [1], is concerned with the existence of at least one solution of the BVP (1.6) with the assumption that the nonlinearity f be bounded.

Theorem 5 *Let $f \in C(I \times \mathbb{R}^2)$ be bounded on $I \times \mathbb{R}^2$. Then the BVP (1.6) with BCs $x(a) = 0 = x(b)$, has a solution.*

Proof. Choose M to be the bound for f on $I \times \mathbb{R}^2$ and define a mapping $T : E \to E$ by

$$Tx(t) = \int_a^b G(t, s) f(s, x(s), x'(s)) ds,$$

where the Banach space is $B = C^{(1)}(I)$, with the norm defined by

$$\|x\|_E = \max_{t \in I} |x(t)| + \max_{t \in I} |x'(t)|,$$

and $G(t, s)$ is a Green's function whose value is given by

$$G(t, s) = \frac{-1}{b - a} \begin{cases} (b - t)(s - a), & \text{if } a \leq s \leq t \leq b, \\ (b - s)(t - a), & \text{if } a \leq t \leq s \leq b. \end{cases}$$

Note that the derivative $G_t(t,s)$ of the Green's function has a bounded jump discontinuity, that is

$$G_t(s+,s) - G_t(s-,s) = -1.$$

It follows that

$$|G_t(t,s)| \le 1 \text{ on } I \times I.$$

Choose

$$N = \max_{s,t\in I} |G(t,s)(b-a)|$$

and

$$N_1 = \max_{s,t\in I} |G_t(t,s)(b-a)|.$$

Then

$$|(Tx)(t)| \le NM$$

and

$$\left|(Tx)'(t)\right| \le N_1 M.$$

Hence, T maps the closed, bounded and convex set

$$\Omega = \left\{x \in C^1 : |x(t)| \le NM, \ \left|x'(t)\right| \le N_1 M\right\},$$

into itself. Moreover, since

$$\left|(Tx)''(t)\right| \le M,$$

the relation

$$(Tx)'(t) = (Tx)'(a) + \int_s^t (Tx)''(\tau)d\tau,$$

this implies that

$$\left|(Tx)'(t) - (Tx)'(s)\right| = \int_s^t (Tx)''(\tau)d\tau \le M\,|t-s|,$$

for any $s, t \in I (s \le t)$. Thus $\{(Tx)\}$ is uniformly bounded and equicontinuous and hence by the Arzela-Ascoli theorem, T is completely continuous. Schauder's fixed point theorem then yields the fixed point of T which is a solution of the BVP (1.6). ■

In the following theorem, f is not required to be bounded on $I \times \mathbb{R}^2$ and existence of a solution of the BVP (1.6) with the help of the modified problem (1.7) is established, that is the basic idea of the method of upper and lower solutions.

Theorem 6 *Let $\alpha, \beta \in C^2(I)$ be lower and upper solutions of* (1.6) *such that $\alpha \leq \beta$ on I. Then the BVP* (1.6) *has a solution $x \in C^2(I)$ such that $\alpha(t) \leq x \leq \beta(t)$ on I.*

Proof. Since F is continuous and bounded on $I \times \mathbb{R}^2$, hence by Theorem (5), the modified problem (1.7) has a solution. Moreover, note that any solution of the modified problem which satisfy

$$\alpha(t) \leq x(t) \leq \beta(t), \ \ \left|x'(t)\right| \leq C \text{ on } I, \tag{1.8}$$

is a solution of (1.6). The only need is to show that (1.8) holds. Therefore, we only discuss the validity of $x(t) \leq \beta(t)$ on I. The arguments are essentially the same for the case $\alpha(t) \leq x(t)$.

Suppose, if possible, $x(t) > \beta(t)$ for some $t \in I$. Then $x(t) - \beta(t)$ has a positive maximum at some $t_0 \in (a, b)$. Hence it follows that

$$x'(t_0) = \beta'(t_0), \ \ \left|x'(t_0)\right| < C$$

and

$$\begin{aligned} x''(t_0) &= F(t_0, x(t_0), x'(t_0)) \\ &= f(t_0, x(t_0), x'(t_0)) + \frac{x(t_0) - \beta(t_0)}{1 + x^2(t_0)}. \end{aligned}$$

Since β is an upper solution, therefore

$$\beta''(t_0) \leq f(t_0, \beta(t_0), \beta'(t_0))$$

and this implies

$$x''(t_0) - \beta''(t_0) \geq \frac{x(t_0) - \beta(t_0)}{1 + x^2(t_0)} > 0,$$

which is impossible at a maximum of $x(t) - \beta(t)$.

Hence, $x(t) \leq \beta(t)$ on I. ■

Now, we turn our attention to the four point BVP (1.1) and discuss existence of a solution in the presence of constant lower and upper solutions. The following results are due to I. Rachunkova [14], who established existence of at least one solution in the presence of constant lower and upper solutions and under various combinations of sign conditions on the nonlinearity f. We state the following lemma without proof, the proof is given in [14].

Lemma 7 *Let there exist an open bounded set $\Omega \in C^1$ such that*
(a) for any $\lambda \in (0,1)$, each solution x of the problem

$$\begin{aligned} x'' &= \lambda f^*(t,x,x',\lambda), \ t \in I, \\ x(a) &= x(c), \ x(b) = x(d), \end{aligned} \tag{1.9}$$

satisfies $x \notin \partial\Omega$;
(b) for any root $x_0 \in \mathbb{R}$ of the equation $f_0(x) = 0$, where

$$f_0(x) = \frac{1}{b-d}\int_d^b \int_a^s f^*(t,x,0,0)dtds - \frac{1}{c-a}\int_a^c \int_a^s f^*(t,x,0,0)dtds, \tag{1.10}$$

the condition $x_0 \notin \partial\Omega$ is fulfilled, where x_0 is considered as a constant function on I;
(c) the Brouwer degree $d[f_0, D, 0] \neq 0$, where $D \subset \mathbb{R}$ is the set of constant c such that the function $x(t) = c$ belongs to Ω.
Then problem (1.1) has at least one solution in $\overline{\Omega}$.

Theorem 8 *Let there exist $r_1, r_2 \in \mathbb{R}$, $K \in (0,\infty)$ such that $r_1 \leq r_2$ and for a.e. $t \in I$, the inequalities*

$$f(t,r_1,0) \leq 0, \ f(t,r_2,0) \geq 0$$

and

$$\int_a^b |f(t,x,y)|\, dt \leq K, \ \forall x \in [r_1,r_2], \ y \in \mathbb{R}, \tag{1.11}$$

are satisfied. Then the problem (1.1) has at least one solution x with the property

$$r_1 \leq x(t) \leq r_2. \tag{1.12}$$

Proof. Choose an arbitrary fixed number $m \in \mathbb{N}$, where $m > 1$, and for $(t,x,y) \in I \times \mathbb{R}^2$, define the modification of f as follows

$$f_m(t,x,y)=\begin{cases} f(t,r_2,0), & \text{for } x\geq r_2+\frac{1}{m},\\ f(t,r_2,y)+[f(t,r_2,0)-f(t,r_2,y)]\,m(x-r_2), & \text{for } r_2<x<r_2+\frac{1}{m},\\ f(t,x,y), & \text{for } r_1\leq x\leq r_2,\\ f(t,r_1,y)+[f(t,r_1,0)-f(t,r_1,y)]\,m(x-r), & \text{for } r_1-\frac{1}{m}<x<r_1,\\ f(t,r_1,0), & \text{for } x\leq r_1-\frac{1}{m}.\end{cases}$$

Consider the system (1.9), where

$$f^*(t,x,y,\lambda)=\lambda f_m(t,x,y)+(1-\lambda)\left[\frac{x-r_1}{r_2-r_1+1}\right].$$

Note that $f^*(t,x,y,1)=f(t,x,y)$ for $r_1\leq x\leq r_2$. Put $r=1+\max\{|r_1|,|r_2|\}$, and define a set

$$\Omega=\left\{x\in C^1:\|x\|<r,\ \ \|x'\|<K+(b-a)\right\}.$$

(a) Let x be a solution of the problem (1.9) for some $\lambda\in(0,1)$. Put

$$\nu(t)=x(t)-r_2-\frac{1}{m}$$

and suppose that

$$\max\{\nu(t):t\in I\}=\nu(t_0)>0.$$

Since

$$\nu(a)=\nu(c)\text{ and }\nu(b)=\nu(d),$$

choose $t_0\in(a,b)$. Then there exists an interval $(\alpha,\beta)\subset(a,b)$ containing t_0 such that $\nu(t)\geq 0$, for each $t\in(\alpha,\beta)$ and $\nu'(\alpha)\geq 0$, $\nu'(\beta)\leq 0$. Hence for $a.e.$ $t\in(\alpha,\beta)$, it follows

$$\begin{aligned}\nu''(t) &= x''(t)=\lambda\left[\lambda f_m(t,x,x')+(1-\lambda)\left(\frac{x-r_1}{r_2-r_1+1}\right)\right]\\ &= \lambda\left[\lambda f(t,r_2,0)+(1-\lambda)\left(\frac{x-r_1}{r_2-r_1+1}\right)\right]>0.\end{aligned}$$

Integration of the last inequality from α to β yields

$$0 \geq \nu'(\beta) - \nu'(\alpha) > 0,$$

a contradiction.

Thus, $\nu(t) \leq 0$ on I, which means that

$$x(t) \leq r_2 + \frac{1}{m}, \forall\, t \in I.$$

By an analogous argument it can proved that

$$x(t) \geq r_1 - \frac{1}{m}, \forall\, t \in I.$$

The BCs of (1.9) guarantee the existence of at least one zero of x' on I. Integrating (1.9) and using (1.11), it follows that

$$\|x'\| < K + (b - a).$$

Therefore $x \notin \partial\Omega$. Hence, for any $\lambda \in (0, 1)$ no solution of (1.9) belongs to $\partial\Omega$.

(b) Since

$$f^*(t, x, 0, 0) = \left[\frac{x - r_1}{r_2 - r_1 + 1}\right],$$

using (1.10),it follows that

$$f_0(x) = \frac{b + d - a - c}{2} . \frac{x - r_1}{r_2 - r_1 + 1},$$

which implies that the equation $f_0(t) = 0$ has a unique root $x_0 = r_1$, and the constant function $x_0(t) = r_1$ does not belong to $\partial\Omega$.

(c) Choose $D = (-r, r)$. Since $f_0(-r) < 0$ and $f_0(r) > 0$, it follows that the Brouwer degree $d[f_0, D, 0] \neq 0$. Thus all the conditions of Lemma 7 are satisfied. Hence the problem

$$\begin{aligned} x'' &= f_m(t, x, x'),\ t \in I, \\ x(a) &= x(c),\ x(b) = x(d), \end{aligned} \tag{1.13}$$

has at least one solution in $\overline{\Omega}$.

Repeating this argument for each $m \in \mathbb{N}$, a sequence $\{x_n\}_1^\infty$ of solutions of the problems (1.13) is obtained. Obviously, the sequence is bounded and equicontinuous in X and hence by the Arzela-Ascoli Theorem, it is possible to choose a subsequence converging in X to a function x_0. Since

$$r_1 - \frac{1}{m} \leq x_m(t) \leq r_2 + \frac{1}{m}, \text{ for all } m \geq 1,$$

passing to the limit, it follows

$$r_1 \leq x_0 \leq r_1$$

and consequently x_0 is a solution of (1.1). ■

1.6 Existence of solution under sign changing nonlinearity

In the following theorem, sign conditions on f are imposed without requiring f to be a bounded function.

Theorem 9 *Let there exist real numbers* R_1, R_2, R_3, R_4, r_1, r_2 *such that* $r_1 \leq r_2$, $R_1 \neq R_3$, $R_2 \neq R_4$, $R_1 \leq 0 \leq R_2$, $R_3 \leq 0 \leq R_4$ *and for a.e.* $t \in I$, *let*

$$f(t, r_1, 0) \leq 0, \;\; f(t, r_2, 0) \geq 0, \tag{1.14}$$

$$f(t, x, R_2) \geq 0, \;\; f(t, x, R_1) \leq 0, \;\; \forall\; x \in [r_1, r_2].$$

Further, for a.e. $t \in [d, b]$ *and all* $x \in [r_1, r_2]$, *let*

$$f(t, x, R_3) \geq 0, \;\; f(t, x, R_4) \leq 0. \tag{1.15}$$

Then the BVP (1.1), *has at least one solution* x *which for all* $t \in I$ *fulfils the inequalities* (1.12) *and*

$$\min\{R_1, R_3\} \leq x'(t) \leq \max\{R_2, R_4\}. \tag{1.16}$$

Proof. Since $R_1 \neq R_3$ and $R_2 \neq R_4$, choose $R_3 < R_1$ and $R_4 > R_2$. Then there exist $n_0 \in \mathbb{N}$ such that for all $n \geq n_0\,(n \in N)$, the inequalities $R_2 + \frac{2}{n} < R_4$ and $R_1 - \frac{2}{n} > R_3$ hold.

For $n \geq n_0$, define

$$h_n(t,x,y) = \begin{cases} f(t,x,R_4), & \text{for } R_4 < y, \\ f(t,x,y), & \text{for } R_2 + \frac{2}{n} \leq y \leq R_4, \\ f(t,x,R_2+\frac{2}{n}) + \omega_2, & \text{for } \frac{1}{n} + R_2 < y < R_2 + \frac{2}{n}, \\ f(t,x,R_2), & \text{for } R_2 < y \leq R_2 + \frac{1}{n}, \\ f(t,x,y), & \text{for } R_1 \leq y \leq R_2, \\ f(t,x,R_1), & \text{for } R_1 - \frac{1}{n} \leq y < R_1, \\ f(t,x,R_1-\frac{2}{n}) - \omega_1, & \text{for } R_1 - \frac{2}{n} < y < R_1 - \frac{1}{n}, \\ f(t,x,y), & \text{for } R_3 \leq y \leq R_1 - \frac{2}{n}, \\ f(t,x,R_3), & \text{for } R_3 > y, \end{cases}$$

where ω_1 and ω_2 are given by

$$\begin{aligned} \omega_1 &= \left[f(t,x,R_1 - \frac{2}{n}) - f(t,x,R_1)\right] n \left(y - R_1 + \frac{2}{n}\right), \\ \omega_2 &= \left[f(t,x,R_2 + \frac{2}{n}) - f(t,x,R_2)\right] n \left(y - R_2 - \frac{2}{n}\right). \end{aligned}$$

It is not difficult to check that h_n fulfils the condition (1.11), with the value K given by

$$K = \int_a^b (\sup\{|h_n(t,x,y)| : x \in [r_1, r_2],\ y \in [R_3, R_4]\})dt.$$

Since h_n fulfils the conditions of Theorem 8, hence by Theorem 8, the problem

$$\begin{aligned} x''(t) &= h_n(t,x,x'), \\ x(a) &= x(c),\ x(b) = x(d). \end{aligned} \tag{1.17}$$

has a solution x_n satisfying the condition (1.12).

The priori estimates for x'_n independent of n can be obtained as follows. From the BCs of the BVP (1.1), it follows that there exist points $a_0 \in (a,c)$, $b_0 \in (d,b)$ such that $x'_n(a_0) = 0 = x'_n(b_0)$. Suppose that

$$\max\left\{x'_n(t) : t \in [a, b_0]\right\} = x'_n(z_0) > R_2 + \frac{1}{n}.$$

Then, $z_0 \neq b_0$ and an interval $(\alpha, \beta) \subset (a, b_0)$ can be found such that $x_n'(\beta) = R_2$, $x_n'(\alpha) = R_2 + \frac{1}{n}$ and

$$R_2 \leq x_n'(t) \leq R_2 + \frac{1}{n},\ \forall\ t \in (\alpha, \beta).$$

Thus, it follows

$$0 > \int_\alpha^\beta x_n''(t)dt = \int_\alpha^\beta f(t, x_n, R_2)dt \geq 0,$$

which is a contradiction.

A similar contradiction occurs if

$$\min\left\{x_n'(t)\ : t \in [a, b_0]\right\} < R_1 - \frac{1}{n}.$$

Thus, it has been proved the estimate on $[a, b_0]$. Now, suppose that

$$\max\left\{x_n'(t)\ : t \in [b_0, b]\right\} = x_n'(z_1) > R_4 + \frac{1}{n}.$$

Then $z_1 \in (b_0, b]$ and there exists $(\alpha, \beta) \subset (b_0, b)$ such that $x_n'(\alpha) = R_4$, $x_n'(\beta) = R_4 + \frac{1}{n}$ and

$$R_4 \leq x_n'(t) \leq R_4 + \frac{1}{n},\ \forall\ t \in (\alpha, \beta).$$

Thus, it follows

$$0 < \int_\alpha^\beta x_n''(t)dt = \int_\alpha^\beta f(t, x_n, R_4)dt \leq 0,$$

which is a contradiction. Similarly for

$$\min\left\{x_n'(t)\ : t \in [b_0, b]\right\} < R_3 - \frac{1}{n}.$$

So, it has been proved the estimate on $[b_0, b]$, and therefore

$$R_3 - \frac{1}{n} \leq x_n'(t) \leq R_4 + \frac{1}{n},\ \forall\ t \in I. \tag{1.18}$$

From (1.12) and (1.18) it follows that the sequence of solutions $(x_n)_{n_0}^\infty$ of the BVPs (1.17), is bounded and equicontinuous in X and thus by a limiting process, a function x can be found which is a solution of the problem

$$x''(t) = h(t,x,x'),\ t \in I, \quad (1.19)$$
$$x(a) = x(c),\ x(b) = x(d),$$

where $h(t,x,x')$ is given by

$$h(t,x,x') = \begin{cases} f(t,x,R_4), & \text{for } y > R_4, \\ f(t,x,y), & \text{for } R_3 \le y \le R_4, \\ f(t,x,R_3), & \text{for } y < R_3. \end{cases}$$

By (1.17), x fulfils the inequality

$$R_3 \le x_n'(t) \le R_4,\ \forall\ t \in I$$

and thus it is a solution of (1.1) with the properties (1.12) and (1.16).

In case of $R_3 > R_1$, $R_2 < R_4$ replace R_1 by R_3 in the formula for h_n , one can prove the existence of a solution x by the same argument. Similarly, the existence of solution x in the case of $R_4 < R_2$, can be proved. Thus it has been proved that under the assumptions of (1.14) and (1.15) , the four point BVP (1.1) has least one solution x such that

$$r_1 < x < r_2$$

and

$$\min\{R_1, R_3\} \le x' \le \max\{R_2, R_4\} \text{ on } I.$$

■

Chapter 2

Existence of at least one solution

This chapter deals with the existence of at least one solution of the four point BVP (1.1) in the presence of upper and lower solutions that are not necessarily constant functions but some continuous functions defined on I which may be nonlinear. This chapter is in fact a generalization of chapter 1 and almost all of the material of this chapter is taken from the recent work presented in [6] , [7] and [8] . The outline of this chapter is as follows: First, the nonlinearity is assumed to be bounded on $I \times \mathbb{R}^2$ and existence of at least one solution in the presence of upper and lower solutions is established. Second, the problem is studied under much weaker conditions by removing the boundedness assumption on f and imposing the Nagumo type condition on f. Finally, the existence of solutions is studied by assuming sign conditions on f without requiring f to satisfy the Nagumo type growth condition.

For our purpose, the method of upper and lower solutions is developed to prove the existence of a solution of the BVP (1.1) in a region bounded by lower and upper solutions. Considering approximate problem, a sequence of solutions of the approximated problems is obtained. It will show that the sequence converges to a solution of the original problem (1.1). In this chapter, the type of lower and upper solutions of the BVP (1.1) are more general, whereas in Theorems $(6, 7)$ of chapter 1, the upper and lower solutions are assumed to be constant functions. Moreover, here, the existence of solutions is studied when the growth of the nonlinearity $f(t, x, x')$ is allowed to be quadratic with respect to x'. Existence theory is also studied under the sign conditions of f .

2.1 Existence results for bounded nonlinearity

Theorem 10 *Suppose that $\alpha, \beta \in C^2(I)$ are respectively lower and upper solutions of the BVP* (1.1) *such that $\alpha \le \beta$ on I. Assume that there exists $K \in \mathbb{R}$, $K > 0$ such that*

$$\int_a^b |f(t, x, y)|\, dt \le K, \;\; \forall\, x \in [\alpha(t), \beta(t)], \;\; y \in \mathbb{R}.$$

Then, the BVP (1.1) *has at least one solution x verifying that $\alpha(t) \le x(t) \le \beta(t)$, $t \in I$.*

Proof. Choose $m \in \mathbb{N}$, where $m > 1$ and is fixed. Define the modification $f_m(t, x, y)$ of f as follows

$$f_m(t,x,y) = \begin{cases} f(t, \beta(t), \beta\prime(t)) + \frac{x-\beta(t)}{1+|x-\beta(t)|}, & \text{if } \beta(t) + \frac{1}{m} \le x, \\ f(t, \beta(t), y) + \Big[f(t, \beta(t), \beta\prime(t)) - f(t, \beta(t), y) + \frac{x-\beta(t)}{1+|x-\beta(t)|}\Big] & \\ m(x - \beta(t)), & \text{if } \beta(t) \le x < \beta(t) + \frac{1}{m}, \\ f(t, x, y), & \text{if } \alpha(t) \le x \le \beta(t), \\ f(t, \alpha(t), y) - [f(t, \alpha(t), \alpha'(t)) - f(t, \alpha(t), y) + \frac{x-\alpha(t)}{1+|x-\alpha(t)|}] & \\ m(x - \alpha(t)), & \text{if } \alpha(t) - \frac{1}{m} < x \le \alpha(t), \\ f(t, \alpha(t), \alpha'(t)) + \frac{x-\alpha(t)}{1+|x-\alpha(t)|}, & \text{if } x \le \alpha(t) - \frac{1}{m}. \end{cases}$$

Note that f_m is continuous on $I \times \mathbb{R}^2$. Consider the system of four point BVPs

$$\begin{cases} x'' = f_m(t, x(t), x'(t)), \;\; t \in I, \\ x(a) = x(c), \;\; x(b) = x(d). \end{cases} \tag{2.1}$$

From the definition of f_m, it is clear that if x is a solution of (2.1) such that

$$\alpha(t) \le x(t) \le \beta(t), \;\; t \in I,$$

then, x is a solution of (1.1).

Since $\alpha \le \beta$ on I. Choose $r = \max_I \beta = \min_I \alpha$. If $r = 0$, then

$$\max_I \beta = \min_I \alpha,$$

which implies that $\alpha = \beta$, is a constant solution of (1.1). Therefore, assume that $r > 0$ and define $\sigma = \frac{r}{1+r}$. Clearly $\sigma > 0$. Choose $R \geq 0$ such that

$$\left|f(t, \alpha(t), \alpha'(t))\right| \leq R, \ \left|f(t, \beta(t), \beta'(t))\right| \leq R, \ t \in I,$$

and take $M_1 > 0$ large enough such that $0 \leq R \leq M_1\sigma$. Then

$$-M_1\sigma \leq -R \leq f(t, \alpha(t), \alpha'(t)), \ f(t, \beta(t), \beta'(t)) \leq R \leq M_1\sigma, \ t \in I. \tag{2.2}$$

Define bounded continuous functions K_1 and K_2 as follows

$$K_1(t,x) = \begin{cases} \frac{R}{M_1\sigma}, & \text{if } x \geq \alpha(t) + \frac{R}{mM_1\sigma}, \\ m(x - \alpha(t)), & \text{if } \alpha(t) \leq x \leq \alpha(t) + \frac{R}{mM_1\sigma}, \\ 0, & \text{if } x \leq \alpha(t) \end{cases}$$

and

$$K_2(t,x) = \begin{cases} 0, & \text{if } x \geq \beta(t), \\ m(x - \beta(t)), & \text{if } \beta(t) - \frac{R}{mM_1\sigma} \leq x \leq \beta(t), \\ -\frac{R}{M_1\sigma}, & \text{if } x \leq \beta(t) - \frac{R}{mM_1\sigma}. \end{cases}$$

Moreover, define

$$\phi(t,x) = [K_1(t,x) + K_2(t,x)]M_1\sigma, \ (t,x) \in I \times \mathbb{R}.$$

Note that ϕ is continuous and bounded on $I \times \mathbb{R}$ and in view of (2.2), ϕ satisfies the following relation

$$-R \leq \phi(t,x) \leq R \text{ on } I \times \mathbb{R}.$$

Let

$$p(t,x) = \max\{\alpha(t), \min\{x, \beta(t)\}\}.$$

Consider the system of BVP to prove existence of (2.1)

$$\begin{cases} x''(t) = \overline{f}_m(t, x(t), x'(t), \lambda), \ t \in I, \\ x(a) = x(c), \ x(b) = x(d), \end{cases} \tag{2.3}$$

where,

$$\overline{f}(t,x,y,\lambda) = \lambda f_m(t,x,y,\lambda) + (1-\lambda)\{\phi(t,x) - p(t,x) + x\}$$

and $\lambda \in [0,1]$.

For $\lambda = 1$, $\overline{f} = f_m$ and consequently (2.3) reduces to (2.1). Moreover, for $\lambda = 1$, if x is a solution of (2.3) such that

$$\alpha(t) \le x(t) \le \beta(t), t \in I,$$

then

$$x''(t) = f_m(t, x(t), x'(t)) = f(t, x(t), x'(t)),\ t \in I,$$

and hence is a solution of (1.1).

If $\lambda = 0$, then (2.3) reduces to the BVP

$$\begin{aligned} x''(t) &= \phi(t, x(t)) - p(t, x(t)) + x,\ t \in I, \\ x(a) &= x(c),\ x(b) = x(d). \end{aligned} \tag{2.4}$$

Since $\phi(t,x) - p(t,x)$ is continuous and bounded and the linear problem

$$\begin{cases} x''(t) = x(t), & t \in I, \\ x(a) = x(c),\ x(b) = x(d), \end{cases}$$

has only the trivial solution, therefore the problem (2.4) has a solution. In other words, the problem (2.3) has a solution for $\lambda = 0$.

For $\lambda \in [0,1]$, any solution x of problem (2.3) satisfies

$$\alpha(t) - \frac{1}{m} \le x(t) \le \beta(t) + \frac{1}{m},\ t \in I.$$

For this, define a function

$$v(t) = x(t) - \beta(t) - \frac{1}{m},\ t \in I,$$

and suppose that

$$\max\{v(t) : t \in I\} = v(t_0) > 0.$$

Using the BCs

$$\begin{aligned} v(a) &= x(a) - \beta(a) - \frac{1}{m} \leq x(c) - \beta(c) - \frac{1}{m} = v(c), \\ v(d) &\geq v(b). \end{aligned}$$

Consequently, choose $t_0 \in (a, b)$. Then

$$v(t_0) > 0,\ v'(t_0) = 0,\ v''(t_0) \leq 0,$$

which implies that

$$x(t_0) > \beta(t_0) + \frac{1}{m},\ x'(t_0) = \beta'(t_0).$$

Hence

$$x(t_0) > \beta(t_0) + \frac{1}{m} \geq \alpha(t_0) + \frac{1}{m} \geq \alpha(t_0) + \frac{R}{mM_1\sigma}. \tag{2.5}$$

Using (2.5) and by the definition of K_1 and K_2, it follows that

$$K_1(t_0, x(t_0)) = \frac{R}{M_1\sigma},\ K_2(t_0, x(t_0) = 0.$$

The definition of ϕ and the hypotheses on β then leads to

$$\begin{aligned} v''(t_0) &= x''(t_0) - \beta''(t_0) \\ &= \lambda f_m(t_0, x(t_0), x'(t_0)) + (1-\lambda)\{\phi(t_0, x(t_0)) - p(t_0, x(t_0)) \\ &\quad +x(t_0)\} - \beta''(t_0), \\ &\geq \lambda f_m(t_0, x(t_0), x'(t_0)) + (1-\lambda)\{\phi(t_0, x(t_0)) - p(t_0, x(t_0)) \\ &\quad +x(t_0)\} - f(t_0, \beta(t_0), \beta'(t_0)) \\ &= \lambda \left[f(t_0, \beta(t_0), \beta'(t_0)) + \frac{x(t_0) - \beta(t_0)}{1 + |x(t_0) - \beta(t_0)|} \right] \\ &\quad +(1-\lambda)[R - \beta(t_0) + x(t_0)] - f(t_0, \beta(t_0), \beta'(t_0)) \\ &= (1-\lambda)\left[R - f(t_0, \beta(t_0), \beta'(t_0))\right] + \\ &\quad \left[\frac{\lambda}{1 + |x(t_0) - \beta(t_0)|} + (1-\lambda)\right](x(t_0) - \beta(t_0)) \\ &> (1-\lambda)[R - f(t_0, \beta(t_0), \beta'(t_0))] \geq 0, \end{aligned}$$

which is a contradiction. This proves that $v \leq 0$ on I, which implies that

$$x(t) \leq \beta(t) + \frac{1}{m},\ t \in I.$$

Now, to prove

$$x(t) \geq \alpha(t) - \frac{1}{m},\ \forall\, t \in I,$$

define

$$v(t) = x(t) - \alpha(t) + \frac{1}{m},\ t \in I.$$

Let

$$\min\{v(t) : t \in I\} = v(t_1) < 0.$$

The BCs

$$v(a) \geq v(c) \text{ and } v(d) \leq v(b),$$

imply that t_1 is an interior point, that is, $t_1 \in (a, b)$. Consequently,

$$v(t_1) < 0,\ v'(t_1) = 0 \text{ and } v''(t_1) \geq 0.$$

Hence

$$x(t_1) \leq \alpha(t_1) - \frac{1}{m} \text{ and } x'(t_1) = \alpha'(t_1).$$

Since

$$x(t_1) < \alpha(t_1) - \frac{1}{m} \leq \beta(t_1) - \frac{1}{m} \leq \beta(t_1) - \frac{R}{mM_1\sigma},$$

it follows that

$$K_1(t_1, x(t_1)) = 0,\ K_2(t_1, x(t_1)) = -\frac{R}{M_1\sigma}.$$

By the definition of ϕ and the hypotheses on α,

$$
\begin{aligned}
v''(t_0) &= x''(t_1) - \alpha''(t_1) \\
&\leq \lambda f_m(t_1, x(t_1), x'(t_1)) + (1-\lambda)\{\phi(t_1, x(t_1)) - \rho(t_1, x(t_1)) \\
&\quad + x(t_1)\} - f(t_1, \alpha(t_1), \alpha'(t_1)) \\
&= \lambda \left[f(t_1, \alpha(t_1), \alpha'(t_1)) + \frac{x(t_1) - \alpha(t_1)}{1 + |x(t_1) - \alpha(t_1)|} \right] \\
&\quad + (1-\lambda)[-R - \alpha(t_1) + x(t_1)] - f(t_1, \alpha(t_1), \alpha'(t_1)) \\
&= -(1-\lambda)[R + f(t_1, \alpha(t_1), \alpha'(t_1))] + \\
&\quad \left[\frac{\lambda}{1 + |x(t_1) - \alpha(t_1)|} + (1-\lambda) \right] (x(t_1) - \alpha(t_1)) \\
&< -(1-\lambda)[R + f(t_1, \alpha(t_1), \alpha'(t_1))] \leq 0,
\end{aligned}
$$

which is a contradiction. Hence

$$x(t) \geq \alpha(t) - \frac{1}{m}, \ \forall \ t \in I.$$

Thus,

$$\alpha(t) - \frac{1}{m} \leq x(t) \leq \beta(t) + \frac{1}{m}, \ t \in I.$$

Now, to find estimate for the derivative x' of the solution x.

The BCs imply that there exist $\widetilde{t} \in (a, b)$ such that $x'(\widetilde{t}) = 0$. Integrating the equation (2.3) from $\widetilde{t}$ to t,

$$
\begin{aligned}
x'(t) &= x'(\widetilde{t}) = \int_{\widetilde{t}}^{t} x''(s)ds, \ t \geq \widetilde{t}, \\
x'(t) &= \int_{\widetilde{t}}^{t} [\lambda f_m(s, x(s), x'(s)) + (1-\lambda)\{\phi(s, x(s)) - p(s, x(s)) + x(s)]ds \text{ for } t \geq \widetilde{t}.
\end{aligned}
$$

Hence

$$\begin{aligned}\left|x'(t)\right| &\leq \int\limits_{\widetilde{t}}^{t} \left|\lambda f_m(s,x(s),x'(s)) + (1-\lambda)\{\phi(s,x(s)) - p(s,x(s)) + x(s)\right| ds \\ &\leq \int\limits_{a}^{b} \left|\lambda f_m(s,x(s),x'(s)) + (1-\lambda)\{\phi(s,x(s)) - p(s,x(s)) + x(s)\right| ds.\end{aligned}$$

A similar estimate can be obtained for $t < \widetilde{t}$, so that for all $t \in I$, it follows that

$$\left|x'(t)\right| \leq \int\limits_{a}^{b} \left|\lambda f_m(s,x(s),x'(s))\right| ds + \int\limits_{a}^{b} \left|\phi(s,x(s)) - p(s,x(s)) + x(s)\right| ds. \tag{2.6}$$

Since

$$\alpha(t) - \frac{1}{m} \leq x(t) \leq \beta(t) + \frac{1}{m} \text{ for } t \in I \text{ and } -R \leq \phi(t,x(t)) \leq R, \text{ for } t \in I,$$

hence

$$-R - \beta(s) + \alpha(s) - \frac{1}{m} \leq \phi(s,x(s)) - p(s,x(s)) + x(s) \leq R - \alpha(s) + \beta(s) + \frac{1}{m} \text{ for } s \in I$$

and substituting

$$\left|\phi(s,x(s)) - p(s,x(s)) + x(s)\right| \leq R + \beta(s) - \alpha(s) + \frac{1}{m} \text{ for } s \in I,$$

in (2.6), it follows

$$\left|x'(t)\right| \leq \int\limits_{a}^{b} \left|f_m(s,x(s),x'(s))\right| ds + \int\limits_{a}^{b} \left[R + \beta(s) - \alpha(s) + \frac{1}{m}\right] ds, \; t \in I.$$

The expression

$$\alpha(t) - \frac{1}{m} \leq x(t) \leq \beta(t) + \frac{1}{m},$$

for $t \in I$, also provides that $f_m(t, x(t), x'(t))$ can be equal to

$$\begin{cases} f(s,\beta(s),x'(s)) + \left[f(s,\beta(s),\beta'(s)) - f(s,\beta(s),x'(s)) + \frac{x(s)-\beta(s)}{1+|x(s)-\beta(s)|} \right] \\ m(x(s)-\beta(s)), & \text{if } \beta(s) \le x(s) \le \beta(s) + \frac{1}{m}, \\ f(s,x(s),x'(s)), & \text{if } \alpha(s) \le x(s) \le \beta(s), \\ f(s,\alpha(s),x'(s)) - \left[f(s,\alpha(s),\alpha'(s)) - f(s,\alpha(s),x'(s)) + \frac{x(s)-\alpha(s)}{1+|x(s)-\alpha(s)|} \right] \\ m(x(s)-\alpha(s)), & \text{if } \alpha(s) - \frac{1}{m} \le x(s) \le \alpha(s). \end{cases}$$

For $\beta(s) \le x(s) \le \beta(s) + \frac{1}{m}$

$$\begin{aligned} & \left| f_m(s,x(s),x'(s)) \right| \\ \le\ & \left| f(s,\beta(s),x'(s)) \right| + \left| f(s,\beta(s),\beta'(s)) - f(s,\beta(s),x'(s)) + \frac{x(s)-\beta(s)}{1+|x(s)-\beta(s)|} \right| \\ & m(u(s)-\beta(s)) \\ \le\ & \left| f(s,\beta(s),x'(s)) \right| + \left| f(s,\beta(s),\beta'(s)) - f(s,\beta(s),x'(s)) + \frac{x(s)-\beta(s)}{1+|x(s)-\beta(s)|} \right| m\frac{1}{m} \\ \le\ & \left| f(s,\beta(s),x'(s)) \right| + \left| f(s,\beta(s),\beta'(s)) \right| + \left| f(s,\beta(s),x'(s)) \right| + \frac{x(s)-\beta(s)}{1+x(s)-\beta(s)} \\ <\ & \left| f(s,\beta(s),x'(s)) \right| + \left| f(s,\beta(s),\beta'(s)) \right| + \left| f(s,\beta(s),x'(s)) \right| + \frac{1}{m} \end{aligned}$$

and for $\alpha(s) - \frac{1}{m} \le x(s) \le \alpha(s)$

$$\begin{aligned} & \left| f_m(s,x(s),x'(s)) \right| \\ \le\ & \left| f(s,\alpha(s),x'(s)) \right| + \left| f(s,\alpha(s),\alpha'(s)) - f(s,\alpha(s),x'(s)) + \frac{x(s)-\alpha(s)}{1+|x(s)-\alpha(s)|} \right| \\ & m\,|x(s)-\alpha(s)| \\ \le\ & \left| f(s,\alpha(s),x'(s)) \right| + \left| f(s,\alpha(s),\alpha'(s)) - f(s,\alpha(s),x'(s)) + \frac{x(s)-\alpha(s)}{1+|x(s)-\alpha(s)|} \right| m\frac{1}{m} \\ \le\ & \left| f(s,\alpha(s),x'(s)) \right| + \left| f(s,\alpha(s),\alpha'(s)) \right| + \left| f(s,\alpha(s),x'(s)) \right| + \left| \frac{x(s)-\alpha(s)}{1+|x(s)-\alpha(s)|} \right| \\ <\ & \left| f(s,\alpha(s),x'(s)) \right| + \left| f(s,\alpha(s),\alpha'(s)) \right| + \left| f(s,\alpha(s),x'(s)) \right| + \frac{1}{m}. \end{aligned}$$

Hence

$$\int_a^b \left| f_m(s,x(s),x'(s)) \right| ds \le 3K + \frac{1}{m}(b-a),$$

which implies that

$$\begin{aligned} \|x'\| &\le 3K + \frac{1}{m}(b-a) + \int_a^b \left[R + \beta(s) - \alpha(s) + \frac{1}{m}\right] ds \\ &\le 3K + (b-a) + \int_a^b [R + \beta(s) - \alpha(s) + 1]\, ds = \gamma, \end{aligned}$$

where γ is independent of m, which provide estimate for x' on I.

Thus for $\lambda = 1$, the modified BVP (2.3) has a solution x_m such that

$$\alpha(t) - \frac{1}{m} \le x_m(t) \le \beta(t) + \frac{1}{m},\ t \in I$$

and

$$\|x'\| \le \gamma \text{ on } I.$$

Repeating this procedure for all $m \in \mathbb{N}$ $(m > 1)$, a sequence $\{x_m\}_{m=2}^{\infty}$ is obtained, where x_m is a solution of the problem

$$\begin{cases} x''(t) = f_m(t, x(t), x'(t)),\ t \in I, \\ x(a) = x(c),\ x(b) = x(d) \end{cases}$$

and satisfy the relations

$$\|x_m\| \le \max\{\|\alpha\|, \|\beta\|\} + 1,\ \ \|x_m'\| \le \gamma.$$

Thus, the sequence $\{x_m\}$ is bounded in $C^1(I)$ and since

$$\|x_m''(t)\| = \|f_m(t, x_m(t), x_m'(t))\|,$$

is bounded, hence $\{x_m\}$ is equicontinuous in $C^1(I)$. Thus, Arzela-Ascoli Theorem implies the existence of a subsequence of $\{x_m\}$ convergent in $C^1(I)$ to a function x. Since

$$\alpha(t) - \frac{1}{m} \le x_m(t) \le \beta(t) + \frac{1}{m},\ \text{for } t \in I,$$

then

$$\alpha(t) \leq x(t) \leq \beta(t), t \in I.$$

Moreover, $\|x'\| \leq \gamma$, or more specifically

$$\left\|x'\right\| \leq 3K + \int_a^b \left[R + \beta(s) - \alpha(s)\right] ds.$$

The function x is a solution to problem (1.1). ■

2.2 Existence under Nagumo type growth condition

In the following theorem, f is not required to be bounded and it is proved that the conclusion is still valid. The growth of f with respect to x' is allowed to be quadratic.

Theorem 11 *Assume that α and β are respectively lower and upper solutions of the BVP* (1.1) *such that $\alpha(t) \leq \beta(t)$, $t \in I$. If $f : I \times \mathbb{R}^2 \to \mathbb{R}$ is continuous and satisfies the Bernstein-Nagumo condition, then there exists a solution $x(t)$ of the BVP* (1.1) *such that*

$$\alpha(t) \leq x(t) \leq \beta(t),\ t \in I.$$

Moreover, there exists a constant C which depends on α, β and ω such that $|x'(t)| < C$ on I, where ω is a Nagumo function.

Proof. Let $r = \max_{t\in I} \beta(t) - \min_{t\in I} \alpha(t)$. Since f satisfies a Nagumo condition relative to α, β, there exists an $N > 0$, such that $\int_0^N \frac{sds}{\omega(s)} > r$. Let

$$C \geq \max\{N, \left\|\alpha'\right\|, \left\|\beta'\right\|\}$$

and define $q : \mathbb{R} \to \mathbb{R}$ by

$$q(x) = \begin{cases} C, & \text{if } x \geq C, \\ x, & \text{if } |x| \leq C, \\ -C, & \text{if } x \leq -C. \end{cases}$$

Note that q is continuous and bounded. Since $\int_0^x \frac{sds}{\omega(s)}$ is continuous and an increasing function of x and $C \geq N$, so

$$\int_0^C \frac{sds}{\omega(s)} \geq \int_0^N \frac{sds}{\omega(s)} > r. \tag{2.7}$$

Consider the modification F of f with respect to α, β as follows

$$F(t,x,x') = \begin{cases} f(t,\beta(t),\beta\prime(t)) + \frac{x-\beta(t)}{1+|x-\beta(t)|}, & \text{if } x \geq \beta(t)+\epsilon, \\ f(t,\beta(t),x') + [f(t,\beta(t),\beta\prime(t)) - f(t,\beta(t),x') + \frac{x-\beta(t)}{1+[x-\beta(t)}] \\ \frac{x-\beta(t)}{\epsilon}, & \text{if } \beta(t \leq x < \beta(t)+\epsilon, \\ f(t,x,x'), & \text{if } \alpha(t) \leq x \leq \beta(t), \\ f(t,\alpha(t),x') - [f(t,\alpha(t),\alpha'(t)) - f(t,\alpha(t),x') + \frac{x-\alpha(t)}{1+|x-\alpha(t)|}] \\ \frac{x-\alpha(t)}{\epsilon}, & \text{if } \alpha(t)- \in < x \leq \alpha(t), \\ f(t,\alpha(t),\alpha'(t)) + \frac{x-\alpha(t)}{1+|x-\alpha(t)|}, & \text{if } x \leq \alpha(t)-\epsilon, \end{cases}$$

where $\epsilon > 0$ is a small fixed number.

Consider the modified BVP

$$\begin{aligned} x''(t) &= F(t,x,x'),\ t \in I, \\ x(a) &= x(c),\ x(b) = x(d). \end{aligned} \tag{2.8}$$

Where $F(t,x,x')$ is continuous on $I \times \mathbb{R}^2$. It is to be noted that any solution x of the BVP (2.8) which satisfies the relation

$$\alpha(t) \leq x(t) \leq \beta(t), t \in I,$$

is solution of the BVP (1.1).

For the existence of solution of the BVP (2.8), consider the following system of BVP

$$\begin{aligned} x'' &= \lambda F(t,x,x') + (1-\lambda)(\sigma(t,x,x') + x),\ t \in I, \\ x(a) &= x(c),\ x(b) = x(d), \end{aligned} \tag{2.9}$$

where $\lambda \in [0,1]$, is a parameter and $\sigma(t,x,x')$ is bounded and continuous function defined by

$$\sigma(t,x,x') = f(t,p(t,x),q(x')) - p(t,x),$$

where

$$p(t,x) = \max\{\alpha(t), \min\{x, \beta(t)\}\}.$$

For $\lambda = 0$, the system reduces to

$$\begin{aligned} x''(t) &= x(t) + \sigma(t,x,x'),\ t \in I, \\ x(a) &= x(c),\ x(b) = x(d), \end{aligned} \tag{2.10}$$

and for $\lambda = 1$, it is equation (2.8). Since $\sigma(t,x,x')$ is continuous and bounded and the linear BVP

$$\begin{aligned} x''(t) &= x(t),\ t \in I, \\ x(a) &= x(c),\ x(b) = x(d), \end{aligned}$$

has a trivial solution only, it follows that the BVP (2.10) has a solution. That is the BVP (2.9) has a solution for $\lambda = 0$.

To show that for every $\lambda \in [0,1]$, any solution x of the BVP (2.9) satisfies the inequality

$$\alpha(t) \le x(t) \le \beta(t),\ t \in I.$$

Set

$$v(t) = x(t) - \beta(t),\ t \in I.$$

It is required to show that $v(t) \le 0$ for every $t \in I$.

For this, suppose that this is not true, then $v(t) > 0$ for some $t \in I$ and consequently, $v(t)$ has a positive maximum at some $t = t_0 \in I$. The BCs imply that

$$v(a) \le v(c),\ v(b) \le v(d),$$

hence $t_0 \in (a,b)$. It follows that

$$v(t_0) > 0, \;\; v'(t_0) = 0 \text{ and } v''(t_0) \le 0.$$

However, if

$$\beta(t_0) < x(t_0) \le \beta(t_0) + \epsilon,$$

then by the definition of F and that of the upper solution, we obtain

$$\begin{aligned}
v''(t_0) &= x''(t) - \beta''(t_0) \\
&\ge \lambda\left[f(t_0,\beta(t_0),\beta'(t_0)) + \frac{x(t_0)-\beta(t_0)}{1+|x(t_0)-\beta(t_0)|}\frac{x(t_0)-\beta(t_0)}{\varepsilon}\right] \\
&\quad +(1-\lambda)[f(t_0,\beta(t_0),\beta'(t_0)) - \beta(t_0) + x(t_0)] - f(t_0,\beta(t_0),\beta'(t_0)) \\
&= \lambda f(t_0,\beta(t_0),\beta'(t_0)) + \frac{\lambda(v(t_0))^2}{(1+v(t_0))} + f(t_0,\beta(t_0),\beta'(t_0) + \\
&\quad (1-\lambda)v(t_0) - \lambda f(t_0,\beta(t_0),\beta'(t_0)) - f(t_0,\beta(t_0),\beta'(t_0) \\
v''(t_0) &= \lambda\frac{(v(t_0))^2}{(1+v(t_0))\varepsilon} + (1-\lambda)v(t_0) > 0,
\end{aligned}$$

which is a contradiction.

If $x(t_0) > \beta(t_0) + \epsilon$, then

$$\begin{aligned}
v''(t_0) &= x''(t) - \beta''(t_0) \ge \lambda[f(t_0,\beta(t_0),\beta'(t_0)) + \frac{x(t_0)-\beta(t_0)}{1+|x(t_0)-\beta(t_0)|} + \\
&\quad (1-\lambda)[f(t_0,\beta(t_0),\beta'(t_0)) - \beta(t_0) + x(t_0)] - f(t_0,\beta(t_0),\beta'(t_0)) \\
&= \lambda f(t_0,\beta(t_0),\beta'(t_0)) + \frac{\lambda(v(t_0))}{1+v(t_0)} + f(t_0,\beta(t_0),\beta'(t_0) + \\
&\quad (1-\lambda)v(t_0) - \lambda f(t_0,\beta(t_0),\beta'(t_0)) - f(t_0,\beta(t_0),\beta'(t_0) \\
v''(t_0) &= \lambda\frac{v(t_0)}{1+v(t_0)} + (1-\lambda)v(t_0) > 0,
\end{aligned}$$

which is again a contradiction. Hence, it follows that

$$x(t) \le \beta(t) \text{ for every } t \in I.$$

Now, to show that

$$x(t) \ge \alpha(t) \text{ for every } t \in I,$$

set

$$u(t) = x(t) - \alpha(t),\ t \in I$$

and suppose that $u(t)$ has a negative minium at some $t_0 \in I$. The BCs

$$\upsilon(a) \geq \upsilon(c),\ \upsilon(b) \geq \upsilon(d),$$

imply that $t_0 \in (a, b)$. Consequently,

$$u(t_0) < 0,\ u'(t_0) = 0 \text{ and } u''(t_0) \geq 0.$$

If $x(t_0) < \alpha(t_0) - \epsilon$, then by definition of F and the lower solution

$$\begin{aligned} u''(t_0) &= x''(t_0) - \alpha''(t_0) \leq \lambda[f(t_0, \alpha(t_0), \alpha'(t_0)) + \frac{x(t_0) - \alpha(t_0)}{1 + |x(t_0) - \alpha(t_0)|} + \\ &\quad (1 - \lambda)[f(t_0, \alpha(t_0), \alpha'(t_0)) - \alpha(t_0) + x(t_0)] - f(t_0, \alpha(t_0), \alpha'(t_0)), \\ &= \lambda f(t_0, \alpha(t_0), \alpha'(t_0)) + \lambda \frac{u(t_0)}{1 + u(t_0)} + f(t_0, \alpha(t_0), \alpha'(t_0) + \\ &\quad (1 - \lambda) u(t_0) - \lambda f(t_0, \alpha(t_0), \alpha'(t_0)) - f(t_0, \alpha(t_0), \alpha'(t_0), \\ u''(t_0) &= \lambda \frac{u(t_0)}{1 + u(t_0)} + (1 - \lambda) u(t_0) < 0, \end{aligned}$$

a contradiction. Hence

$$x(t) \geq \alpha(t) \text{ for every } t \in I$$

and so x is a solution of (1.1).

Now, it remains to show that $|x'(t)| < C$ on I. The BCs imply the existence of $a_1 \in (a, c)$ and $b_1 \in (d, b)$ such that

$$x'(a_1) = 0,\ x'(b_1) = 0.$$

Suppose that there exist $t_0 \in (a_1, b_1)$ such that $x'(t_0) \geq C$. Let $[t_1, t_2] \subset [a_1, b_1]$ be the maximal interval containing t_0 such that $x'(t) > 0$ for every $t \in (t_1,\ t_2)$. Let

$$\max\{x'(t) : t \in [t_1, t_2]\} = x'(t^*) = C_1,$$

then $C_1 \geq C$, and hence, in view of (2.7) and the increasing property of function $\int_0^x \frac{sds}{\omega(s)}$ in x, it follows that

$$\int_0^{C_1} \frac{sds}{\omega(s)} \geq \int_0^{C} \frac{sds}{\omega(s)} > r. \tag{2.11}$$

Since, for each $t \in (t_1, t_2)$, and $x \in [\alpha(t), \beta(t)]$,

$$x''(t) = f(t, x, x') \leq \omega(x').$$

It follows that

$$\frac{x'(t)x''(t)}{\omega(x')} \leq x'(t), \ t \in (t_1,t_2)$$

and by integration,

$$\int_0^{C_1} \frac{sds}{\omega(s)} \leq x(t^*) - x(t_1) \leq \max_{t\in(a,b)} \beta(t) - \min_{t\in(a,b)} \alpha(t) = r, \tag{2.12}$$

which contradicts (2.11).

Now, let $t_0 \in [a, a_1)$ and $[t_1, t_2] \subset [a, a_1]$ be the maximal interval containing t_0 such that $x'(t_0) > 0$ for $t \in (t_1,t_2)$, where t_1, t_2 are such that,

$$t_2 \leq a_1, \ t_1 \geq a \text{ with } x'(t_2) = 0 \text{ and } x'(t_1) \geq 0.$$

Let

$$\max\{x'(t) : t \in [t_1,t_2]\} = x'(t^*) = C_1,$$

then $C_1 \geq C$, and $t^* < t_2$. Since $x \in [\alpha, \beta]$ for $t \in (t_1,t_2)$, by (2.8), it follows that

$$x''(t) = f(t, x, x') \geq -\omega(x'), \ t \in (t_1,t_2),$$

which implies that

$$-\frac{x'(t)x''(t)}{\omega(x')} \leq x'(t), \ t \in (t_1,t_2).$$

Integrating from t^* to t_2,

$$\int_0^{C_1} \frac{sds}{\omega(s)} \leq x(t_2) - x(t^*) \leq r, \tag{2.13}$$

which is again a contradiction.

If $t_0 \in (b_1, b]$ and $[t_1, t_2]$ be the maximal interval in $[b_1, b]$ containing t_0 such that

$$x'(t) > 0, \text{ for } t \in (t_1, t_2),$$

where $t_1 \geq b_1$ and $t_2 \leq b$ are such that $x'(t_1) = 0$ and $x'(t_2) \geq 0$. Then $t^* > t_1$, where $t^* \in (t_1, t_2]$ is the point such that

$$x'(t^*) = \max\{x'(t) : \ t \in [t_1, t_2]\}.$$

Since $x \in [\alpha, \beta]$, by (1.2), the following holds

$$\frac{x'(t)x''(t)}{\omega(x')} \leq x'(t), \ t \in (t_1, t_2).$$

Integration from t_1 to t^* leads to

$$\int_0^{C_1} \frac{sds}{\omega(s)} \leq x(t^*) - x(t_1) \leq r,$$

again is a contradiction. Hence

$$x'(t) < C, t \in I.$$

Now, it remains to show that

$$x'(t) > -C, t \in I.$$

Assume that there exists $t_0 \in (a_1, b_1)$ such that $x'(t_0) \leq -C$. Let $[t_1, t_2] \subset [a_1, b_1]$ be the maximal interval containing t_0 such that $x'(t) < 0$ for every $t \in (t_1, t_2)$. Let

$$\min\{x'(t) : t \in [t_1, t_2]\} = x'(t^*) = -C_2, \text{ then } C_2 \geq C$$

and in view of (2.7),

$$\int_0^{C_2} \frac{sds}{\omega(s)} > r.$$

For $t \in (t_1, t_2)$, since $x \in [\alpha, \beta]$, so

$$x''(t) = f(t, x, x') \geq -\omega(|x'|).$$

It follows that

$$\frac{x'(t)x''(t)}{\omega(|x'|)} \leq -x'(t)$$

and hence

$$\int_0^{C_2} \frac{sds}{\omega(s)} \leq x(t_1) - x(t^*) \leq \max_{t\in[a,b]} \beta(t) - \min_{t\in[a,b]} \alpha(t) = r, \tag{2.14}$$

which is a contradiction.

If $t_0 \in [a, a_1)$ or $(b_1, b]$, a contradiction occurs in the same way as above. Hence $|x'(t)| < C$, $t \in I$. ■

Let us consider the following example [7] which shows that the hypotheses of the above theorem can easily be verified and also, demonstrate that the above theorem is more general than one studied in chapter 1.

Example: Consider the BVP

$$\begin{aligned} x''(t) &= -x'(t) + g(x)\phi(t), \ t \in [0,1], \\ x(0) &= x(\eta), \ x(\delta) = x(1), \end{aligned} \tag{2.15}$$

where, $0 < \eta \leq \delta < 1$, $g : \mathbb{R} \longrightarrow \mathbb{R}$ is continuous and $\phi \in C^2[0,1]$ satisfies the linear problem with constant coefficients

$$\begin{aligned} \phi''(t) + \phi'(t) - \phi(t) &= -\rho(t), \ t \in [0,1], \\ \phi(0) &= \phi(\eta), \ \phi(\delta) = \phi(1), \end{aligned} \tag{2.16}$$

where $\rho(t) \in C[0,1]$ and $\rho(t) \geq 0$ on $[0,1]$. Assume that $g(0) \leq 0$, and g is increasing and there exist $a \geq 0$, such that $g(a) \geq 0$. Since the homogeneous problem

$$\begin{aligned} \phi''(t) + \phi'(t) - \phi(t) &= 0, \\ \phi(0) &= \phi(\eta), \ \phi(\delta) = \phi(1), \end{aligned} \tag{2.17}$$

has only the trivial solution and $\rho(t)$ is continuous and bounded on $[0,1]$, it follows that the BVP (2.16) has a solution.

Claim that any solution $\phi(t)$ of (2.16) satisfies $\phi(t) \geq 0$ on $[0,1]$. If not, then $\phi(t)$ has a negative minimum at some $t_0 \in [0,1]$. The BCs imply that $t_0 \in (0,1)$ and hence

$$\phi(t_0) < 0, \ \phi'(t_0) = 0, \ \phi''(t_0) \geq 0.$$

However, from (2.16), it is clear that

$$\phi''(t_0) = -\phi'(t_0) + \phi(t_0) - \rho(t_0) < 0,$$

is a contradiction. Thus $\phi(t) \geq 0$ on $[0,1]$ and the claim is verified.

Since $[0,1]$ is compact, then there exists $L \geq 0$ such that

$$0 \leq \phi(t) \leq L \text{ on } [0,1].$$

Now, take $\alpha(t) = 0$. Since $g(0) \leq 0$, therefore

$$\alpha''(t) + \alpha'(t) - g(t)\phi(t) = -g(0)\phi(t) \geq 0,$$

which implies that α is a lower solution of the problem (2.15).

Take $\beta(t) = a + g(a)\phi(t)$. As $\beta \geq \alpha$ and g is increasing, so $g(\beta) \geq g(\alpha)$. Moreover,

$$\begin{aligned}
\beta''(t) + \beta'(t) - g(\beta)\phi(t) &\leq g(a)\phi''(t) + g(a)\phi'(t) - g(a)\phi(t) \\
&= g(a)[\phi''(t) + \phi'(t) - \phi(t)] \leq 0 \text{ and} \\
\beta(0) &= a + g(a)\phi(0) = a + g(a)\phi(\eta) = \beta(\eta), \\
\beta(\delta) &= a + g(a)\phi(\eta) = a + g(a)\phi(1) = \beta(1).
\end{aligned}$$

Hence β is an upper solution of the problem (2.15). Clearly,

$$\alpha(t) \leq \beta(t) \text{ on } [0,1].$$

Now for $t \in [0,1]$ and $x \in [\min \alpha, \max \beta]$, we have

$$g(a + g(a)L) \geq g(\beta) \geq g(x).$$

Let

$$C_3 = \max\{|g(0)|, g(a + g(a)L)\},$$

then for $t \in [0,1]$ and $x \in [\min \alpha, \max \beta]$, it follows that

$$\left|-x'(t) + g(x)\phi(t)\right| \leq \left|x'(t)\right| + LC_3 = h(\left|x'(t)\right|),$$

where

$$h(s) = s + LC_3 \text{ for } s \geq 0.$$

Moreover

$$\int_0^\infty \frac{sds}{h(s)} = \int_0^\infty \frac{sds}{s + LC_3} = \infty.$$

Thus the Nagumo condition is satisfied.

Hence by Theorem (10), there exists a solution x of the BVP (2.15), such that

$$\alpha \leq x \leq \beta.$$

Note that for $x \in [0, a + g(a)\phi(t)]$, we have

$$\begin{aligned} f(t,x,R) &= -R + g(x)\phi(t), \\ f(t,x,-R) &= R + g(x)\phi(t). \end{aligned}$$

Thus, by the assumption, it has been proved that

$$\begin{aligned} f(t,0,R) &= -R + g(0)\phi(t) \leq 0 \text{ for } R \geq 0, \\ f(t,a,-R) &= R + g(a)\phi(t) \geq 0 \text{ for } -R \leq 0. \end{aligned}$$

If $R > 0$, or $g(0) < 0$ and $\phi(t) > 0$ somewhere in $[0, 1]$, or $g(a) > 0$ and $\phi(t) > 0$ somewhere in $[0, 1]$. Then it is not possible to apply Theorem (9) of chapter 1.

2.3 Existence results under sign conditions

In the following theorem, f is not required to satisfy the Nagumo type growth condition but some sign conditions on f are imposed.

Theorem 12 *If in Theorem* (11)*, the Berstein-Nagumo condition is replaced by the following sign conditions*

$$\begin{cases} f(t, x, R) \geq 0, \; f(t, x, -R) \leq 0 \text{ for } t \in [a, b], \\ f(t, x, R) \leq 0, \; f(t, x, -R) \geq 0 \text{ for } t \in (d, b], \end{cases} \tag{2.18}$$

where $x \in [\min \alpha(t), \max \beta(t)]$, $R \geq \max\{\|\alpha'\|, \|\beta'\|\}$*. Then the conclusion of Theorem* (11) *is valid, taking* $C = R$*.*

Proof. The proof is closely related to that of Theorem (9), chapter 1. In this theorem, condition (2.18) implies

$$f(t, x, R) = f(t, x, -R) = 0, \text{ for } t \in (d, b] \text{ and } x \in [\min \alpha(t), \max \beta(t)].$$

Let $m \in \mathbb{N}$ and consider the problem

$$\begin{aligned} x''(t) &= f_m(t, x, x'), \; t \in I, \\ x(a) &= x(c), \; x(b) = x(d), \end{aligned} \tag{2.19}$$

where

$$f_m(t,x,x') = \begin{cases} f(t,\beta(t),\beta\prime(t)) + \frac{x-\beta(t)}{1+|x-\beta(t)|}, & \text{if } x \geq \beta(t) + \frac{1}{m}, \\ f(t,\beta(t),q(x')) + [f(t,\beta(t),\beta\prime(t)) - f(t,\beta(t),q(x')) + \frac{x-\beta(t)}{1+|x-\beta(t)|}] \\ m(x-\beta(t)), & \text{if } \beta(t) \leq x < \beta(t) + \frac{1}{m}, \\ f(t,x,q(x')), & \text{if } \alpha(t) \leq x \leq \beta(t), \\ f(t,\alpha(t),q(x')) - [f(t,\alpha(t),\alpha'(t)) - f(t,\alpha(t),q(x')) + \frac{x-\alpha(t)}{1+|x-\alpha(t)|}] \\ m(x-\alpha(t)), & \text{if } \alpha(t) - \frac{1}{m} < x \leq \alpha(t), \\ f(t,\alpha(t),\alpha'(t)) + \frac{x-\alpha(t)}{1+|x-\alpha(t)|}, & \text{if } x \leq \alpha(t) - \frac{1}{m}, \end{cases}$$

and $q(x')$ is defined by

$$q(x') = \max\{-R, \min\{x', R\}\}.$$

Note that $sgn(q(y)) = sgn(y)$ and $q(y) = y$ for $|y| \leq R$. Further, note that $f_m(t,x,x')$ is continuous and bounded on $I \times \mathbb{R}^2$ and any solution $x(t)$ of the BVP (2.19) which satisfies the relations

$$\alpha(t) \leq x \leq \beta(t) \text{ and } \left|x'(t)\right| \leq R,\ t \in I,$$

is a solution of the BVP (1.1).

For the existence of solution of the BVP (2.19), consider the homotopy

$$\begin{aligned} x''(t) &= \lambda f_m(t,x,x') + (1-\lambda)(\sigma(t,x,x') + \frac{x}{m}),\ t \in I, \\ x(a) &= x(c),\ x(b) = x(d), \end{aligned} \tag{2.20}$$

where $\lambda \in [0,1]$, and $\sigma_m(t,x,x')$ is defined by

$$\begin{aligned} \sigma_m(t,x,x') &= f(t,p(t,x),q(x')) - \frac{p(t,x)}{m}, \\ p(t,x) &= \max\{\alpha(t), \min\{x, \beta(t)\}\}. \end{aligned}$$

For $\lambda = 0$, (2.20) has a solution. For $\lambda \in [0,1]$ any solution x_m of (2.20) satisfies the inequality

$$\alpha(t) - \frac{1}{m} \leq x_m(t) \leq \beta(t) + \frac{1}{m},\ t \in I. \tag{2.21}$$

For this, set

$$v(t) = x_m(t) - \beta(t) - \frac{I}{m}, t \in I,$$

and suppose that $v(t)$ has a positive maximum at some $t = t_0 \in I$. The BCs imply that

$$v(a) \le v(c), v(b) \le v(d),$$

choose $t_0 \in (a, b)$. It follows that

$$v(t_0) > 0,\ v'(t_0) = 0,\ v''(t_0) \le 0.$$

On the other hand

$$\begin{aligned} v''(t_0) &= x_m''(t_0) - \beta''(t_0) \\ &\ge \lambda \left[f(t, \beta(t_0), \beta'(t_0)) + \frac{v(t_0) + \frac{1}{m}}{1 + v(t_0) + \frac{1}{m}} \right] + (1-\lambda)[f(t, \beta(t_0), \beta'(t_0)) \\ &\quad - \frac{\beta(t_0)}{m} + \frac{x_m(t_0)}{m}] - f(t, \beta(t_0), \beta'(t_0)) \\ &= \left(v(t_0) + \frac{1}{m} \right) \left(\frac{\lambda}{1 + v(t_0) + \frac{1}{m}} + \frac{1-\lambda}{m} \right) > 0, \end{aligned}$$

which is a contradiction. Hence it follows that

$$x_m(t) \le \beta(t) + \frac{1}{m} \text{ for every } t \in I.$$

Similarly, it can be shown

$$x_m(t) \ge \alpha(t) - \frac{I}{m} \text{ for every } t \in I.$$

Thus, a sequence $\{x_m\}$ of solution the BVP (2.20) is obtained which satisfies the relation

$$\alpha(t) - \frac{I}{m} \le x_m(t) \le \beta(t) + \frac{1}{m},\ t \in I. \tag{2.22}$$

Moreover, the BCs guarantee the existence of at least one point $t_1 \in (a,b)$ such that $x'_m(t_1) = 0$. Integrating (2.20) from t_1 to t

$$\left|x'_m(t)\right| \leq \int_a^b \left|\lambda f_m(t, x_m(t), x'_m(t)) + (1-\lambda)(\sigma_m(t, x_m(t), x'_m(t)) + \frac{x_m(t)}{m}\right| dt,$$

which implies that $\{x'_m\}$ is uniformly bounded on I. Thus, the sequence $\{x_m\}$ is bounded and equicontinuous in $C^1(I)$ and by the Arzela-Ascoli theorem it is possible to choose a subsequence converging in $C^1(I)$ to a function $x \in C^1(I)$. Since (2.22) holds for every $m \in \mathbb{N}$ and every $t \in I$, it follows that

$$\alpha(t) \leq x(t) \leq \beta(t), \ t \in I$$

and hence x is solution of the problem

$$\begin{aligned} x''(t) &= f(t, x, q(x')), \ t \in I \\ x(a) &= x(c), \ x(b) = x(d). \end{aligned} \tag{2.23}$$

Now, to show that

$$\left|x'(t)\right| \leq R, \ t \in I,$$

the BCs

$$x(a) = x(c), x(b) = x(d),$$

imply that there exist $a_1 \in (a,c)$ and $b_1 \in (d,b)$ such that

$$x'(a_1) = 0, \ x'(b_1) = 0.$$

Let

$$\max\{x'(t) : t \in I\} = x'(t_0) \geq R + \frac{1}{m}.$$

Then, $t_0 \neq a_{1,}\ b_1$. If $t_0 \in [a, a_1)$, then there exist $t_1 \geq t_0$ and $t_2 \leq a_1$ with $t_1 < t_2$ such that

$$\begin{aligned} x'(t_1) &= R + \frac{1}{m},\ x'(t_2) = R \text{ and} \\ R &\leq x'(t) \leq R + \frac{1}{m},\ \text{for } t \in [t_1, t_2]. \end{aligned} \tag{2.24}$$

Integrating (2.23) from t_1 to t_2, using (2.24) and (2.18), it follows that

$$0 > \int_{t_1}^{t_2} x''(t)dt = \int_{t_1}^{t_2} f(t, x(t), R)dt \geq 0,$$

which is a contradiction. If $t_0 \in (a_1, b_1]$, then choose $t_1,\ t_2 \in (a_1, b_1]$ with $t_1 < t_2$, $t_1 \geq t_0$ and $t_2 \leq b_1$ such that

$$x'(t_1) = R + \frac{1}{m},\ x'(t_2) = R \tag{2.25}$$

and

$$R \leq x'(t) \leq R + \frac{1}{m},\ t \in [t_1, t_2].$$

Integrating (2.23) from t_1 to t_2, using (2.25) and (2.18), it follows that

$$0 > \int_{t_1}^{t_2} x''(t)dt = \int_{t_1}^{t_2} f(t, x(t), R)dt \geq 0,$$

again a contradiction.

Now if $t_0 \in (b_1, b]$, then there exists $t_1 \geq b_1$, $t_2 \leq t_0$ with $t_1 < t_2$ such that

$$\begin{aligned} x'(t_1) &= R,\ x'(t_2) = R + \frac{1}{m} \text{ and} \\ R &\leq x'(t) \leq R + \frac{1}{m},\ t \in [t_1, t_2]. \end{aligned} \tag{2.26}$$

Integrating (2.23) from t_1 to t_2, using (2.26) and conditions (2.18),

$$0 < \int_{t_1}^{t_2} x''(t)dt = \int_{t_1}^{t_2} f(t, x(t), R)dt = 0,$$

a contradiction. Hence,

$$x'(t) \leq R, \ t \in I.$$

Similarly, using the conditions

$$f(t, x, -R) \leq 0 \text{ for } t \in [a, b] \text{ and } f(t, x, -R) \geq 0 \text{ for } t \in (d, b],$$

it can also be shown that

$$x'(t) \geq -R, t \in I.$$

Consequently, $x(t)$ is a solution of the BVP (1.1). ∎

Chapter 3

Existence of at least three solutions

In this chapter, we investigate new results dealing with the existence of at least three solutions of the BVP (1.1) in the presence of two lower solutions and two upper solutions.

Existence of at least two solutions for some other BVPs has already been studied, e.g. [2], [13] and [14]. In [14], existence of at least two solutions in the presence of constant lower and upper solutions for some four point BVP has been studied, while in [2], the author studied existence of at least two solutions for two point Drichlet BVPs. An abstract result giving the existence of three solutions of nonlinear equations has given by Leggett and Williams [11]. They have developed a fixed point theorem in an ordered Banach space that guarantees the existence of three fixed points.

Motivated by the work of J. Henderson and H. B. Thompson [4] , R. A. Khan and M. Rafique [8], for the existence of at least three solutions of some nonlinear second order differential equations subject to two point BCs [4] and nonlinear three point BCs [8] , in this chapter, we study the existence of at least three solutions for the four point BVP (1.1) in the presence of two lower solutions α, α_1 and two upper solutions β, β_1 such that $\alpha \leq \alpha_1$,and $\beta \geq \beta_1$. It is assumed the existence of upper and lower solutions that are not necessarily constant and allow the growth of $f(t, x, y)$ with respect to y to be quadratic. Some topological degree theory arguments are used to get multiplicity results.

3.1 Explicit form of Green's Function

We use properties of a Green's function to construct its explicit form. Since the solution of the homogeneous differential equation

$$x''(t) + x(t) = 0,$$

is given by

$$x(t) = A\cos t + B\sin t,$$

where A and B are constants. Take

$$G(t,s) = \begin{cases} A\cos t + B\sin t, & a \le t \le s \le b, \\ C\cos t + D\sin t, & a \le s \le t \le b, \end{cases} \tag{3.1}$$

where A, B, C and D are constant to be determined. Using the BC, $G(a,s) = G(c,s)$,we obtain

$$B = A\tan(\frac{a+c}{2}), \text{ where } \frac{a+c}{2} \neq \frac{\pi}{2}$$

and the BC, $G(b,s) = G(d,s)$, leads to

$$D = C\tan(\frac{b+d}{2}), \text{ where } \frac{b+d}{2} \neq \frac{\pi}{2}.$$

Putting the values of B and D in (3.1), we get

$$G(t,s) = \begin{cases} A\cos t + A\tan(\frac{a+c}{2})\sin t, & a \le t \le s \le b, \\ C\cos t + C\tan(\frac{b+d}{2})\sin t, & a \le s \le t \le b. \end{cases} \tag{3.2}$$

Using the continuity of a Green's function at s, that is,

$$G(s-,s) = G(s+,s),$$

we have

$$A = \frac{C(\cos s + \sin s \tan\frac{b+d}{2})}{\cos s + \sin s \tan\frac{a+c}{2}},$$

where $\frac{a+c}{2} \neq \frac{\pi}{2}$ and $\frac{b+d}{2} \neq \frac{\pi}{2}$.

Consequently, $G(t,s)$ takes the form

$$G(t,s)=\begin{cases}\frac{C(\cos s+\sin s\tan\frac{b+d}{2})(\cos t+\sin t\tan\frac{a+c}{2})}{\cos s+\sin s\tan\frac{a+c}{2}} & a\le t\le s\le b,\\ C(\cos t+\tan(\frac{b+d}{2})\sin t), & a\le s\le t\le b.\end{cases} \tag{3.3}$$

Differentiating (3.3) with respect to t, we get

$$\frac{\partial}{\partial t}G(t,s)=\begin{cases}\frac{C(\cos s+\sin s\tan(\frac{b+d}{2}))(\cos t\tan(\frac{a+c}{2})-\sin t)}{\cos s+\sin s\tan\frac{a+c}{2}} & a\le t\le s\le b,\\ C(\cos t\tan(\frac{b+d}{2})-\sin t), & a\le s\le t\le b.\end{cases}$$

The property that the derivative of G has a jump-discontinuity that is given by

$$\frac{\partial}{\partial t}G(s+,s)-\frac{\partial}{\partial t}G(s-,s)=-1,$$

implies

$$C=-\frac{1+\tan s\tan\frac{a+c}{2}}{(1+\tan^2 s)(\tan\frac{b+d}{2}-\tan\frac{a+c}{2})},$$

where $\frac{a+c}{2}\neq\frac{\pi}{2}$ and $\frac{b+d}{2}\neq\frac{\pi}{2}$.

Hence

$$G(t,s)=\begin{cases}-\frac{(1+\tan s\tan\frac{b+d}{2})(\cos t+\sin t\tan\frac{a+c}{2})}{(1+\tan^2 s)(\tan\frac{b+d}{2}-\tan\frac{a+c}{2})}, & a\le t\le s\le b,\\ -\frac{(1+\tan s\tan\frac{a+c}{2})(\cos t+\sin t\tan\frac{b+d}{2})}{(1+\tan^2 s)(\tan\frac{b+d}{2}-\tan\frac{a+c}{2})}, & a\le s\le t\le b,\end{cases}$$

is a Green's function of the homogenous four point BVP.

3.2 Existence of at least three solutions

In this section, we study existence of at least three solutions of the BVP (1.1). We use the method of upper and lower solutions and some degree theory arguments to establish multiplicity results.

Theorem 13 *Assume that*

(A_1) α, $\alpha_1\in C^2(I)$ *are two lower solutions and* β, $\beta_1\in C^2(I)$ *are two upper solutions of the BVP* (1.1) *such that*

$$\alpha\le\alpha_1\le\beta,\ \alpha\le\beta_1\le\beta \text{ and } \alpha_1\not\le\beta_1 \text{ on } I.$$

(A_2) *$f : I \times \mathbb{R}^2 \to \mathbb{R}$ is continuous and satisfies a Nagumo condition relative to α, β.*

(A_3) *α_1, β_1 are strict lower and upper solutions of the BVP* (1.1) *.*

Then the BVP (1.1) *has at least three solutions x_i, $i = 1, 2, 3$ such that*

$$\alpha \leq x_1 \leq \beta_1,\ \alpha_1 \leq x_2 \leq \beta,\ x_3 \nleq \beta_1 \text{ and } x_3 \ngeq \alpha_1 \text{ on } I.$$

Proof. Define $g : I \times \mathbb{R}^2 \to \mathbb{R}$ by

$$g(t, x, x') = f(t, x, x') + x,$$

then g is continuous and the BVP (1.1) can be written as follows

$$\begin{aligned} x''(t) + x(t) &= g(t, x(t), x'(t)),\ t \in I, \\ x(a) &= x(c),\ x(b) = x(d). \end{aligned} \tag{3.4}$$

Define the modification $g^*_{\alpha\beta}$ of g with respect to α, β as follows

$$g^*_{\alpha\beta}(t, x, x') = \begin{cases} g(t, \beta(t), \beta'(t)) + x - \frac{x-\beta(t)}{1+|x-\beta(t)|}, & \text{if } x \geq \beta(t) + \epsilon, \\ g(t, \beta(t), q(x')) + [g(t, \beta(t), \beta'(t) - g(t, \beta(t), q(x')) + \frac{x-\beta(t)}{1+|x-\beta(t)}] & \\ \frac{x-\beta(t)}{\epsilon}, & \text{if } \beta(t) \leq x < \beta(t) + \epsilon, \\ g(t, x, q(x')), & \text{if } \alpha(t) \leq x \leq \beta(t), \\ g(t, \alpha(t), q(x')) - [g(t, \alpha(t), \alpha'(t)) - g(t, \alpha(t), q(x')) + \frac{x-\alpha(t)}{1+|x-\alpha(t)|}] & \\ \frac{x-\alpha(t)}{\epsilon}, & \text{if } \alpha(t) - \epsilon < x \leq \alpha(t), \\ g(t, \alpha(t), \alpha'(t)) + \frac{x-\alpha(t)}{1+|x-\alpha(t)|}, & \text{if } x \leq \alpha(t) - \epsilon, \end{cases}$$

where $\epsilon > 0$ is a small fixed number. Note that $g^*_{\alpha\beta}$ is continuous and bounded on $I \times \mathbb{R}^2$ and hence satisfies the Nagumo condition.

Consider the modified BVP

$$\begin{aligned} x''(t) + x(t) &= g^*_{\alpha\beta}(t, x(t), x'(t)),\ t \in I, \\ x(a) &= x(c),\ x(b) = x(d). \end{aligned} \tag{3.5}$$

As before, we note that any solution of the BVP (3.5) such that

$$\alpha(t) \leq x(t) \leq \beta(t) \text{ and } \left|x'(t)\right| \leq C, t \in I,$$

is a solution of the BVP (3.4).

Using the definition of $g^*_{\alpha\beta}$ and that of upper and lower solutions, we obtain

$$\begin{aligned} g^*_{\alpha\beta}(t,\alpha(t),\alpha'(t)) &= g(t,\alpha(t),\alpha'(t)) \\ &= f(t,\alpha(t),\alpha'(t)) + \alpha(t) \leq \alpha''(t) + \alpha(t), \ t \in I \end{aligned}$$

and

$$\begin{aligned} g^*_{\alpha\beta}(t,\beta(t),\beta'(t)) &= g(t,\beta(t),\beta'(t)) \\ &= f(t,\beta(t),\beta'(t)) + \beta(t) \geq \beta''(t) + \beta(t), \ t \in I, \end{aligned}$$

which imply that α, β are lower and upper solutions of (3.5). Hence by Theorem (11), any solution of the modified BVP (3.5) satisfies the following inequalities

$$\alpha(t) \leq x(t) \leq \beta(t), \ \left|x'(t)\right| \leq C, \ t \in I$$

and hence is solution of (3.4). Thus, it is sufficient to show that the modified BVP (3.5) has at least three solutions x_i such that

$$\alpha(t) \leq x_i(t) \leq \beta(t), \ t \in I, \ i = 1, 2, 3.$$

Define an integral operator $T : C^1(I) \to C^1(I)$ by

$$\widetilde{T}_{\alpha_\beta} x(t) = \int_a^b G(t,s) g^*_{\alpha\beta}(s, x(s), x'(s)) ds. \tag{3.6}$$

Differentiating (3.5) with respect to t , we have

$$(\widetilde{T}_{\alpha_\beta} x(t))' = \int_a^b G_t(t,s) g^*_{\alpha\beta}(s, x(s), x'(s)) ds. \tag{3.7}$$

Since $g^*_{\alpha\beta}$ is continuous and bounded on $I \times \mathbb{R}^2$, there exists $M_1 > 0$ such that

$$\left|g^*_{\alpha\beta}(t, x(t), x'(t))\right| \leq M_1 \text{ on } I \times \mathbb{R}^2.$$

Also $G(t, s)$ is continuous and bounded on $I \times I$, hence there exists $M_2 > 0$ such that

$$|G(t, s)| \leq M_2, \text{ on } I \times I.$$

The derivative of $G(t, s)$, that is, $G_t(t, s)$ is discontinuous at $t = s$, but have a finite jump-discontinuity, hence there also exists $M_3 > 0$ such that

$$|G_t(t, s)| \leq M_3 \text{ on } I \times I.$$

Consequently, from (3.6) and (3.7), it follows that

$$\left|\widetilde{T}_{\alpha_\beta} x(t)\right| \leq \int_a^b G(t, s) g^*_{\alpha\beta}(s, x(s), x'(s)) ds \leq M_2 M_1 (b - a) \tag{3.8}$$

and

$$\left|(\widetilde{T}_{\alpha_\beta} x(t))'\right| \leq \int_a^b G_t(t, s) g^*_{\alpha\beta}(s, x(s), x'(s)) ds \leq M_3 M_1 (b - a), \tag{3.9}$$

which imply that the sequence $\left\{\widetilde{T}_{\alpha_\beta} x(t)\right\}$ is uniformly bounded and equicontinuous. Hence by Arzela-Ascolai theorem, $\widetilde{T}_{\alpha_\beta}$ is compact.

Choose,

$$M > \max\left\{M_2 M_1 (b - a),\ M_3 M_1 (b - a)\right\}.$$

Then, by (3.8) and (3.9), we get

$$\left|\widetilde{T}_{\alpha_\beta} x(t)\right| < M \text{ and } \left|(\widetilde{T}_{\alpha_\beta} x(t))'\right| < M,\ \forall\ x \in C^1(I).$$

Let

$$\Omega = \left\{x \in C^1(I) : \|x\| < M,\ \ \left\|x'\right\| < M\right\}.$$

Then Ω is a bounded open subset of $C^1(I)$.

Let $x, y \in \Omega$, then $\|x\| < M$, and $\|y\| < M$. Also $\|x'\| < M$, $\|y'\| < M$. Since

$$\begin{aligned} \|\alpha x + (1-\alpha)\, y\| &\leq \|\alpha x\| + (1-\alpha)\,\|y\| \\ &= \alpha\,\|x\| + (1-\alpha)\,\|y\| \\ &= \alpha M + (1-\alpha)\, M = M \end{aligned}$$

and

$$\begin{aligned} \left\|\alpha x' + (1-\alpha)\, y'\right\| &\leq \left\|\alpha x'\right\| + (1-\alpha)\left\|y'\right\| \\ &= \alpha\left\|x'\right\| + (1-\alpha)\left\|y'\right\| \\ &= \alpha M + (1-\alpha)\, M = M. \end{aligned}$$

Hence

$$\alpha x + (1-\alpha)\, y \in \Omega,$$

which implies that Ω is convex.

Now, we show that $\widetilde{T}_{\alpha_\beta}(\Omega) \subseteq \Omega$.

Let $x \in \Omega$, then $x \in C^1(I)$ and $\|x\| < M$, $\|x'\| < M$. Since

$$\widetilde{T}_{\alpha_\beta}\left(x\left(t\right)\right) = \int_a^b G(t,s) g^*_{\alpha\beta}(s, x(s), x'(s)) ds,$$

$$\widetilde{T}_{\alpha_\beta}\left(x\left(t\right)\right)' = \int_a^b G_t(t,s) g^*_{\alpha\beta}(s, x(s), x'(s)) ds.$$

Therefore

$$\begin{aligned} \left\|\widetilde{T}_{\alpha_\beta} x\right\| &\leq \int_a^b \left\|G(t,s) g^*_{\alpha\beta}(s, x(s), x'(s))\right\| ds \\ &\leq \int_a^b M_1 M_2 ds = M_1 M_2\,(b-a) < M \end{aligned}$$

and

$$\left\|\left(\widetilde{T}_{\alpha_\beta}x\right)'\right\| \leq \int_a^b \left\|G_t(t,s)g^*_{\alpha\beta}(s,x(s),x'(s))\right\| ds$$
$$\leq \int_a^b M_3M_1 ds = M_3M_1\left(b-a\right) < M,$$

which implies that $\widetilde{T}_{\alpha_\beta}x \in \Omega$. Hence $\widetilde{T}_{\alpha_\beta}(\Omega) \subseteq \Omega$.

Hence by Schauder's fixed point theorem, $\widetilde{T}_{\alpha_\beta}$ has a fixed point, that is, the BVP (3.4) has a solution in Ω.

Since $\widetilde{T}_{\alpha_\beta}(\Omega) \subseteq \Omega$ and for $x \in \partial\Omega$ and $\lambda \in [0,1]$, we have $\lambda\widetilde{T}_{\alpha_\beta}x \neq x$, this implies that

$$(1-\lambda\widetilde{T}_{\alpha_\beta})x \neq 0, \text{ for } \lambda[0,1].$$

Using the homotopy invariance property of degree, we get

$$\deg(I-\lambda\widetilde{T}_{\alpha_\beta},\Omega,0) = 1, \text{ for } \lambda[0,1]$$

and

$$\deg(I-\widetilde{T}_{\alpha_\beta},\Omega,0) = 1 \text{ for } \lambda[0,1].$$

Consider open bounded sets

$$\Omega_{\alpha_1} = \{x \in \Omega : \ x > \alpha_1 \text{ on } I\}$$

and

$$\Omega^{\beta_1} = \{x \in \Omega : \ x < \beta_1 \text{ on } I\}.$$

Then $\overline{\Omega}_{\alpha_1} \cap \overline{\Omega}^{\beta_1} = \varnothing$ and since $\alpha_1 \nleq \beta_1$ on I, therefore the set $\Omega \setminus \overline{\Omega_{\alpha_1} \cup \Omega^{\beta_1}} \neq \varnothing$. Moreover, there are no solutions of (3.5) on $\partial\Omega_{\alpha_1} \cup \partial\Omega^{\beta_1}$. Because, if x is a solution of (3.5) such that

$$x \in \partial\Omega_{\alpha_1} \cup \partial\Omega^{\beta_1},$$

then $x \in \partial\Omega_{\alpha_1}$ or $x \in \partial\Omega^{\beta_1}$, which implies that

$$x = \alpha_1 \text{ or } x = \beta_1,$$

an impossible result as α_1 and β_1 are strict lower and upper solutions.

Using the additivity property of degree, it follows

$$1 = \deg(I-\widetilde{T}_{\alpha_\beta}, \Omega, 0) = \deg(I-\widetilde{T}_{\alpha_\beta}, \Omega_{\alpha_1}, 0)+\deg(I-\widetilde{T}_{\alpha_\beta}, \Omega^{\beta_1}, 0)+\deg(I-\widetilde{T}_{\alpha_\beta}, \Omega\diagdown\overline{\Omega_{\alpha_1} \cup \Omega^{\beta_1}}, 0). \tag{3.10}$$

We show that

$$\deg(I - \widetilde{T}_{\alpha_\beta}, \Omega_{\alpha_1}, 0) = \deg(I - \widetilde{T}_{\alpha_\beta}, \Omega^{\beta_1}, 0) = 1.$$

First, we show that

$$\deg(I - \widetilde{T}_{\alpha_\beta}, \Omega_{\alpha_1}, 0) = 1.$$

Let $G_{\alpha_1\beta}$ be the modification of g with respect to α_1, β, that is

$$G_{\alpha_1\beta}(t,x,x') = \begin{cases} g(t,\beta(t),\beta'(t)) + x - \frac{x-\beta(t)}{1+|x-\beta(t)|}, & \text{if } x \geq \beta(t)+\epsilon, \\ g(t,\beta(t),q(x')) + [g(t,\beta(t),\beta'(t) - g(t,\beta(t),q(x')) + \frac{x-\beta(t)}{1+[x-\beta(t)]}] \\ \frac{x-\beta(t)}{\epsilon}, & \text{if } \beta(t) \leq x < \beta(t)+\epsilon, \\ g(t,x,q(x')), & \text{if } \alpha(t) \leq x \leq \beta(t), \\ g(t,\alpha_1(t),x') - [g(t,\alpha_1(t),\alpha_1'(t)) - g(t,\alpha_1(t),q(x')) + \frac{x-\alpha_1(t)}{1+|x-\alpha_1(t)|}] \\ \frac{x-\alpha_1(t)}{\epsilon}, & \text{if } \alpha_1(t)-\epsilon < x \leq \alpha_1(t), \\ g(t,\alpha_1(t),\alpha_1'(t)) + \frac{x-\alpha_1(t)}{1+|x-\alpha_1(t)|}, & \text{if } x \leq \alpha_1(t)-\epsilon. \end{cases}$$

Note that $G_{\alpha_1\beta} = g^*_{\alpha_1\beta}$.

Consider the BVP

$$\begin{aligned} x''(t) &= G_{\alpha_1\beta}(t,x(t),x'(t)),\ t \in I, \\ x(a) &= x(c),\ x(b) = x(d). \end{aligned} \tag{3.11}$$

This is equivalent to the integral equation

$$(I - \widetilde{T}_{\alpha_{1\beta}})x = 0,$$

where

$$\widetilde{T}_{\alpha_{1\beta}}x(t) = \int_a^b G(t,s)G_{\alpha_1\beta}(s, x(s), x'(s))ds,$$

is a compact operator.

As in the proof of Theorem (11), we can show that any solution x of the BVP (3.11) satisfies $x \geq \alpha_1$ on I and since α_1 is strict lower solution of (1.1), this implies that $x \neq \alpha_1$ on (a, b) and hence $x \in \Omega_{\alpha_1}$. It follows that the BVP (3.11) has no solution in $\Omega \diagdown \Omega_{\alpha_1}$, which implies that the degree

$$d(I - \widetilde{T}_{\alpha_{1\beta}}, \Omega_\alpha \diagdown \overline{\Omega_{\alpha_1}}, 0) = 0. \tag{3.12}$$

Moreover, since $\widetilde{T}_{\alpha_{1\beta}}(\overline{\Omega}) \subseteq \Omega$, hence

$$d(I - \widetilde{T}_{\alpha_{1\beta}}, \Omega, 0) = 1. \tag{3.13}$$

From(3.12) and (3.13), it follows that

$$d(I - \widetilde{T}_{\alpha_{1\beta}}, \Omega_{\alpha_1}, 0) = 1. \tag{3.14}$$

Since $\widetilde{T}_{\alpha_{1\beta}} = \widetilde{T}_{\alpha_\beta}$ on Ω_{α_1}, so by the property of degree

$$d(I - \widetilde{T}_{\alpha_\beta}, \Omega_{\alpha_1}, 0) = 1. \tag{3.15}$$

Similarly, we can show that

$$d(I - \widetilde{T}_{\alpha_\beta}, \Omega^{\beta_1}, 0) = 1.$$

Thus, from (3.10)

$$d(I - \widetilde{T}_{\alpha_\beta}, \Omega \diagdown \overline{\Omega_{\alpha_1} \cup \Omega^{\beta_1}}, 0) = -1.$$

Hence there are at least three solutions one in each of the sets Ω_{α_1}, Ω^{β_1} and $\Omega \diagdown \Omega_{\alpha_1} \cup \Omega^{\beta_1}$. Thus, we have proved that in the presence of two lower solutions and two upper solutions, the

BVP (1.1) has at least three solutions in some specific regions determined by the lower and upper solutions. ■

Chapter 4

Conclusion

In this dissertation, the existence of solutions of four-point BVPs of the type

$$\begin{aligned} x''(t) &= f(t,x,x'),\ t \in I = [a,b], \\ x(a) &= x(c),\ x(b) = x(d), \end{aligned}$$

is discussed, where $a, b, c, d \in \mathbb{R}$, are parameters such that $a < c \leq d < b$, and $f : I \times \mathbb{R}^2 \to \mathbb{R}$ is a continuous function. The method of upper and lower solutions is used to establish existence of solution of the BVP (1.1). The basic idea is to modify the given problem suitably and then employ Leray-Schauder theory or known existence results of the modified problem together with the theory of differential and integral inequalities to establish existence of solution of the given problem. Here f is modified in such a way that solutions of the modified problem lie in a region where f is unmodified and hence are solutions of the original nonlinear BVP. Such methods allow us to ensure the existence of at least one solution of the given problem in the region bounded by the well ordered upper and lower solutions.

Chapter 1 contains the basic definitions and notions. In Chapter 2, the existence of at least one solution of the four point BVP (1.1) is discussed in the presence of lower and upper solutions. It is assumed that the nonlinearity is bounded on $I \times \mathbb{R}^2$ and established the existence of at least one solution in the presence of upper and lower solutions. Then, the problem under much weaker assumptions is studied by replacing the boundedness assumption on f by the Nagumo type growth condition. Finally, the result is further generalized by allowing f to satisfy some sign conditions without demanding the Nagumo condition.

In Chapter 3, new results on the existence of at least three solutions of the BVP (1.1) are proved in the presence of two lower solutions and two upper solutions. Such types of results were studied for some other BVPs in [5] and [10] . We proved that more than one solutions of the BVP (1.1) exist in the presence of two lower solutions α, α_1 and two upper solutions β, β_1.

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