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Chapter 1

Introduction

No matter what the branch of science, classification of objects has always helped study and understand them better and in a more convenient manner. This work is about one of the advances in the research program of classification of Lie algebras over the field of complex or real numbers. For a detailed review on classification of Lie algebras we refer to [12] .

Lie algebras can be obtained from Lie groups and vice versa. Lie groups were defined by Sophus Lie as a step in a program to provide a systematic procedure of solving differential equations. Lie groups are continuous groups and can be looked at as smooth manifolds. Lie groups will be discussed in a little more detail later. They have a profound impact on many areas of mathematics, physics and other mathematical sciences [10] . Some of these are briefly given below.

1- Particle Physics. Lie groups were used to classify elementary particles as well as their interactions. The classification required the knowledge of irreducible representation of Lie groups. For example, neutrons and protons have masses very close to one another, but with the difference that the proton is charged whereas the neutron is neutral. Thus from the point of view of strong interactions (neglecting the electromagnetic interaction), they were assumed to form a 2-dimensional representation of the SU(2) group. In the 60's, SU(3) was discovered as a symmetry of strong interactions and the 8-dimensional representation was used to group together the baryons $n, p, \sum^{+}, \sum^{0}, \sum^{-}, \equiv^{-}, \equiv^{0}, \Lambda$. Similarly 8 mesons $\pi^{+}, \pi^{0}, \pi^{-}, k^{+}, k^{0}, \bar{k}^{+}, \bar{k}^{0}, \eta$ were put together in an 8-dimensional irreducible representation.

2- Relativistic theories. A special relativistic theory is required to be invariant under the group of Lorentz transformations. However, the Lorentz group is only a subgroup of the whole symmetry group. Additional symmetries which are not part of the Lorentz group, are shifts (or translations) in space and time. Adjoining these transformations to the Lorentz group yields a Lie group called the Poincare group [14] . The relativistic theories are invariant under the Poincare group also.

3- Differential equations. Specific Lie groups may appear as consequences of specific dynamics. For any physical system with dynamics described by a system of ordinary or partial differential equations, the system of equations can be invariant under some local Lie group of local point transformations, taking solutions into solutions. This symmetry group G and its Lie algebra L can be determined in an algorithmic manner [10] . The Lie algebra L is obtained as an algebra of vector fields in some basis, depending on the way in which the algorithm is applied [7] .

The reason we study **Lie algebras** is that the Lie theory of continuous groups reduces the "local" problems using Lie groups to corresponding problems on Lie algebras and thus it becomes easier to solve them using linear algebra. Lie algebras can be obtained from Lie groups by expanding near the identity and Lie groups can be obtained back from the Lie algebras by exponentiation. There is a unique Lie algebra for a Lie group but the converse may not be true.

This work mainly concerns itself with the classification of solvable Lie algebras which helps understand the structure of the Lie algebras in general. The reason being that any Lie algebra is isomorphic to a semidirect sum of a **semisimple Lie algebra** and a **solvable one** (a theorem due to Levi [8] , which will be mentioned briefly later). Since semisimple Lie algebras have already been classified [7], it is only needed to classify the solvable Lie algebras. It is also known that every solvable Lie algebra has a uniquely defined nilradical, which is a maximal nilpotent ideal, therefore we can consider the classification of **nilpotent Lie algebras** and then find their solvable extensions. Classification of solvable Lie algebras of dim$\leq$ 6 and classification of nilpotent Lie algebras of dim $6, 7, 8$ and some results for dim 9 are known [16] , but this classification is far from complete.[1]

[1] The mathematical terms used here will be defined later in section 1.1

Classification of nilpotent Lie algebras is a step forward in the classification of solvable Lie algebras and consequently the classification of all finite dimensional Lie algebras. But the problem is that infinitely many different series of nilpotent Lie algebras exist and these have not been classified. For n$\geqslant$ 7 the number of non-equivalent or non-isomorphic nilpotent Lie algebras becomes infinite [16] . Therefore we consider 'specific' nilradicals and construct all solvable Lie algebras with this nilradical. These solvable Lie algebras are extensions of this nilradical.

The nilpotent Lie algebra N, which will be considered, is particular in the sense that it is of dimension n and has degree of nilpotency $n-1$ and its Lie brackets are all known and are given (in an appropriate basis $(e_1, ...e_n)$) by

$$\begin{aligned}
[e_i, e_j] &= 0 \; ; 1 \leq i, j \leq n-2 \\
[e_1, e_{n-1}] &= [e_2, e_{n-1}] = 0, \\
[e_k, e_{n-1}] &= e_{k-2} \; ; 3 \leq k \leq n-2, \\
[e_1, e_n] &= 0, \\
[e_k, e_n] &= e_{k-1}; 2 \leq k \leq n-1.
\end{aligned}$$

All solvable extensions are then constructed by using derivations obtained by considering automorphisms close to the identity.

Casimir operators, a particular kind of invariants of Lie algebras, have also been discussed since they play an important role in the representation theory of Lie algebras and Lie groups. Also, in physics Casimir operators represent important quantities such as magnitude of angular momentum, elementary particle mass and spin, Lagrangians of various physical systems etc.

At the end we find the extension of a nilpotent Lie algebra of degree of nilpotency $n-1$ where n is the dimension of the Lie algebra and for which not all the Lie brackets are known. This gives results which are more generalized. The Casimir invariants of this more general case have also been discussed.

After this brief introduction and mathematical background we present the calculation of all possible solvable extensions of a particular nilpotent Lie algebra in Chapter 2. In Chapter 3, the Casimir invariants of the nilpotent Lie algebra and its solvable extension are obtained. Chapter 4 is an extension of the work presented in Chapters 2 and 3. Section A.1 of the appendix is an explanation of how an automorphism and derivation are related. In section A.2 it is shown that the derived algebra of a solvable Lie algebra is contained in its nilradical.[2]

[2] As mentioned earlier, the terms used here will be defined in section 1.1

1.1 Mathematical background

In this section some basic definitions along with examples and mathematical background has been presented which provides a base for the later work done in this dissertation.

1.1.1 Basic definitions

Lie Algebra

Let F be a field. A Lie algebra over F is an F-vector space L, together with a bilinear map, the Lie bracket

$$L \times L \to L \qquad (x, y) \longrightarrow [x, y] .$$

satisfying

(i) $[x, x] = 0,$ for all $x \in L$.

which is equivalent to saying $[x, y] = -[y, x]$, for all $x, y \in L$.

(ii) $[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0$, for all $x, y, z \in L$.

(Jacobi identity).

Example let V be a finite dimensional vector space over F. The set $gl(V)$ of all linear maps from V to V is a vector space over F. It becomes a Lie algebra if we define the Lie brackets $[,]$ by

$$[x, y] = x \circ y - y \circ x \quad , x, y \in gl(V) .$$

where $\circ$ denotes composition of maps [4].

Lie subalgebra

If L is a Lie algebra and K a subset of L, such that

$[x, y] \in K$ for all $x, y \in K$.Then K is a Lie subalgebra of L. Lie subalgebras are Lie algebras on their own right.

Example Let $gl(n, F)$ be the vector space of all $n \times n$ matrices over F (a field) with the Lie brackets defined by

$$[x, y] = xy - yx,$$

where xy is the usual product of the matrices x and y.

Let $sl(n,F)$ be the subspace of $gl(n,F)$ consisting of all matrices of trace 0. For arbitrary square matrices x and y, the matrix $xy-yx$ has trace 0, so $[x,y]=xy-yx$ defines a Lie algebra structure on $sl(n,F)$, properties (i) and (ii) are inherited from $gl(n,F)$. This algebra $sl(n,F)$ is a subalgebra of $gl(n,F)$ [4] .

Ideal

An Ideal of a Lie algebra L is a subspace M of L such that

$$[x,y]\in M, \qquad \text{for all } x\in L, y\in M.$$

Example Let $b(n,F)$ be the upper triangular matrices in $gl(n,F)$. This is a Lie algebra with the same Lie brackets as $gl(n,F)$. Let $\mathbf{n}(n,F)$ be the strictly upper triangular matrices in $b(n,F)$.This is an ideal in $b(n,F)$. It has the same Lie brackets as $gl(n,F)$. We see that for arbitrary $x\in b(n,F)$ and $y\in \mathbf{n}(n,F); xy\in \mathbf{n}(n,F)$. Thus $[x,y]\in \mathbf{n}(n,F)$ for all $x\in b(n,F)$ and $y\in \mathbf{n}(n,F)$ [4] .

Homomorphism and Isomorphism

Let L be a Lie algebras over a field F. A mapping $\Phi : L \longrightarrow L_1$, L_1 another Lie algebra, is a homomorphism if Φ is a linear map and

$$\Phi([x,y]) = [\Phi(x), \Phi(y)] \quad \text{for all } x,y\in L.$$

Φ is an isomorphism if Φ is also bijective [4] .

Adjoint homomorphism

Let L is a Lie algebra . We define the adjoint homomorphism by

$ad : L \longrightarrow gl(L)$ where $gl(V)$ is the set of all linear maps from V to V and

$$(ad(x))(y) = [x,y] \quad \text{for } x,y\in L.$$

It follows from the bilinearity of the Lie brackets that the map $ad(x)$ is linear for each $x\in L$ [4] .

Derivations

Let A be an algebra over a field F. A derivation of A is an F-linear map $D : A \longrightarrow A$, such that

$$D(ab) = aD(b) + D(a)b \quad \text{for all } a,b\in A.$$

Example Let $A = C^{\infty}(\mathbb{R})$, the usual derivative, $Df = f'$,is a derivation D of A since by the product rule

$$D(fg) = (fg)' = f'g + fg' = (Df)g + f(Dg) \quad [4] .$$

Structure constants

Let L is a Lie algebra over a field F with basis $(e_1, ..., e_n)$, then the bracket $[,]$ is completely determined by the products $[e_i, e_j] \quad 1 \le i, j \le n$.

We define scalars $c_{ij}^k \in F$ such that

$$[e_i, e_j] = \sum_{k=1}^{n} c_{ij}^k e_k.$$

The c_{ij}^k are called the structure constants of L with respect to this basis [4] .

Derived series of a Lie algebra

Let L be a Lie algebra. Define

$L^{(0)} = L,$

$L^{(1)} = L'$, where L' is the derived algebra $[L, L]$.

and $L^{(k)} = \left[L^{(k-1)}, L^{(k-1)}\right]$, for $k \geqq 2$.

Then

$L = L^{(0)} \supseteq L^{(1)} \supseteq L^{(2)} \supseteq ...$

is the derived series of L [4] .

Solvable Lie algebra

A Lie algebra L is said to be solvable if for some $m \ge 1 (m \in \mathbb{N})$ we have $L^{(m)} = 0$. [4] .

Radical of a Lie algebra

The largest solvable ideal of a Lie algebra is called the radical 'rad(L)' of L [4] .

Semisimple Lie algebra

A non-zero Lie algebra L is called semisimple if it has no non-zero solvable ideals [4] .

Simple Lie algebra

A Lie algebra L is called simple if it is not abelian and has no ideals other than 0 and L [4].

Lower central series

Consider the Lie algebra L. Define

$$\begin{aligned} L^1 &= L'. \\ L^k &= \left[L^{k-1}, L\right]; \quad k \geq 2. \end{aligned}$$

Then

$$L \supseteq L^1 \supseteq L^2 \supseteq ...$$

is called the lower central series. The reason for the name central comes from the fact that $\frac{L^k}{L^{k+1}}$ is contained in the centre of $\frac{L}{L^{k+1}}$ [4].

Nilpotent Lie algebra

A Lie algebra L is said to be nilpotent if for some $m \geqslant 1 \, (m \in N)$ we have $L^m = 0$. Then $(m-1)$ is the degree of nilpotency if m is the least such positive integer which gives $L^m = 0$ [4].

Centralizer

The centralizer L_K of a given subalgebra $K \subset L$ in L is the set of all elements in L commuting with all elements in K. i.e.

$L_K = \{x \in L \mid [x, y] = 0, \text{for all } y \in K\}$ [16].

Automorphism

An automorphism of a given Lie algebra L is a bijective linear map

$$\Phi : L \longrightarrow L,$$

such that any pair x, y of elements of L

$\Phi([x, y]) = [\Phi(x), \Phi(y)]$.

All automorphisms of L form a Lie group $Aut(L)$. Its Lie algebra is then the algebra of derivations of L [16].

Inner and outer derivations

Let D be a derivation. If an element z exists, such that $D = ad\left(z\right)$ i.e. $D\left(x\right) = \left(ad\left(z\right)\right)\left(x\right) = \left[z, x\right]$ for all $x \in G$ then the derivation is called an inner derivation. Any other derivation is called an outer derivation [16].

The relation of automorphism with nilpotent derivations is given in appendix A.1.

Nilradical of a Lie algebra

The maximal nilpotent ideal of a Lie algebra is called its nilradical [16].

1.1.2 Lie groups and Lie algebras

An r parameter Lie group G is a group which also carries the structure of an r-dimensional smooth manifold in such a way that both the group operations

$$m : G \times G \to G \qquad m\left(g, h\right) = g.h,\ g, h \in G$$

and the inversion

$$i : G \to G \qquad i\left(g\right) = g^{-1},\ g \in G$$

are smooth maps between manifolds [10].

An r-parameter Lie group of transformation is given by

$$\bar{x}' = f(\bar{x}, \bar{a}),$$

or

$$x_i' = f_i(x_1, ...x_n; a_1, ..., a_r), \quad 1 \leq i \leq n,$$

where $\bar{a} = \left(a_1, ..., a_r\right)$,and $\bar{x} = \left(x_1, ...x_n\right)$ and the parameters $a_1, ..., a_r$ are considered as independent.

If

$$x_i'' = f_i(x_1', ...x_n'; b_1, ..., b_r),$$

then because of the group structure, we can find parameter set c such that

$$x_i'' = f_i(x_1, ..., x_n; c_1, ..., c_r),$$

where $c = (c_1, ...c_r)$; $c_k = \varphi_k(a_1, ...a_r; b_1, ..., b_r)$, $1 \leq k \leq r$, and the functions f_i and φ_k are analytic. Also because of the group structure, we can find a parameter set $\bar{a}'$ [6] such that

$$f(\bar{x}'; \bar{a}') = f(f(\bar{x}; \bar{a}); \bar{a}') = \bar{x}.$$

Then

$$\bar{x} = f(\bar{x}; \bar{d}),$$

where, $\bar{d} = (d_1, ..., d_k)$ and

$$\bar{d}_k = \psi_k\left(a_1, ..., a_r; a'_1, ..., a'_r\right), \quad 1 \leq k \leq r,$$

This gives the identity mapping.

Infinitesimal generators of the algebra of the group

Consider the r-parameter Lie group of transformations

$$x_i = f_i(x_1, ..., x_n; a_1, ..., a_r); \qquad 1 \leq i \leq n. \tag{1.1}$$

or

$$\bar{x}' = f(\bar{x}; \bar{a}),$$

in an abbreviated notation, where f_i are analytic functions of the parameter set $\bar{a}$ for each $i, (1 \leq i \leq n)$.

Consider the transformation

$$x'_i + dx'_i = f_i(x'_1, ..., x'_n; \delta a_1, ..., \delta a_r), \tag{1.2}$$

where we are taking the identity map to be $\bar{x}' = f(\bar{x}'; 0)$.

Expanding the right hand side of (1.2) using Taylor series expansion and using $x'_i = f_i(x_1, ...x_n; a_1, ..., a_r)$, we get

$$\begin{aligned} dx'_i &= \sum_{k=1}^{r} \left[\frac{\partial f_i(x'_i, ..., x'_n; a_1, ..., a_r)}{\partial a_k}\right]_{a=0} \delta a_k \\ &= \sum_{k=1}^{r} f_{ik}(x')\delta a_k. \end{aligned} \tag{1.3}$$

where

$$f_{ik}(x') = \left[\frac{\partial f_i(x'_i,...,x'_n;a_1,...,a_r)}{\partial a_k}\right]_{a=0}.$$

Again writing

$$a_l + da_l = \varphi_l(a_1,...a_r;\delta a_1,...,\delta a_r), \tag{1.4}$$

and using Taylor series expansion, we obtain

$$\begin{aligned} da_l &= \sum_{m=1}^{r}\left[\frac{\partial\varphi_l(a_1,...,a_r;b_1,...,b_r)}{\partial b_m}\right]_{b=0}\delta a_m \\ &= \sum_{m=1}^{r}\Theta_{lm}\delta a_m, \end{aligned} \tag{1.5}$$

where $\Theta_{lm} = \Theta_{lm}\left(a_1,...,a_n\right)$, and at $a = 0$, $\Theta_{lm}\left(0\right) = \delta_{lm}$.

Inverting the system of linear equations in equation (1.5), we obtain

$$\delta a_k = \sum \Psi_{kl}\left(a\right) da_l, \quad 1 \leq k \leq r. \tag{1.6}$$

Obviously the $r \times r$ matrices Ψ and Θ satisfy $\Psi\Theta = 1$. Also using $\Theta_{lm}\left(0\right) = \delta_{lm}$, we get $\Psi_{kl}\left(0\right) = \delta_{kl}$.

Substituting the expression for δa_k from equation (1.6) in equation (1.3), we get

$$dx'_i = \sum_{k,l=1}^{r} f_{ik}\left(x'\right)\Psi_{kl}\left(a\right) da_l, \quad 1 \leq i \leq n. \tag{1.7}$$

In other words

$$\frac{\partial x'_i}{\partial a_l} = \sum_{k=1}^{r} f_{ik}\left(x'\right)\Psi_{kl}\left(a\right), \tag{1.8}$$

and the free indices i and l have the ranges $1 \leq i \leq n$ and $1 \leq l \leq r$ respectively. Here $\bar{x}'$ are functions of the parameter sets $\bar{a}$ and $\bar{x}$ are initial values of $\bar{x}'$ for $\bar{a} = 0$.

If we examine the change of a function $F(x)$ under the infinitesimal transformation given in equation (1.2) we find that

$$\begin{aligned} dF &= \sum_{i=1}^{n} \frac{\partial F}{\partial x_i} dx_i \\ &= \sum_{i=1}^{n} \frac{\partial F}{\partial x_i} \sum_{l=1}^{r} f_{il}(x)\, \delta a_l \\ &= \sum_{l=1}^{r} \delta a_l \left(\sum_{i=1}^{n} f_{il}(x) \frac{\partial}{\partial x_i} \right) F \\ &= \sum_{l=1}^{r} \delta a_l X_l F. \end{aligned} \tag{1.9}$$

The operators

$$X_l = \sum_{i=1}^{n} f_{il}(x) \frac{\partial}{\partial x_i}; \quad 1 \leq l \leq r, \tag{1.10}$$

are called the infinitesimal generators of the group and are exactly in number as the number of parameters [6] .

One parameter subgroup

Let G be a Lie Group. A one parameter subgroup of G is a group homomorphism $\Phi : \mathbb{R} \to G$ that is also a differentiable map at the same time. We view $\mathbb{R}$ additively and G multiplicatively, so that $\Phi(r+s){=}\Phi(r)\Phi(s)$.

Structure constants

The commutators of the infinitesimal generators are linearly expressible in terms of the infinitesimal generators. The transformations f form a group with r essential parameters. This implies that f_{ik} are linearly independent. Using x instead of x' in equation (1.8)

$$\frac{\partial x_i}{\partial a_\lambda} = \sum_{k=1}^{r} f_{ik}(x)\, \Psi_{k\lambda}(a)\,; 1 \leq i \leq n \quad ; 1 \leq \lambda \leq r. \tag{1.11}$$

Now a one parameter continuous group (one parameter group having continuous group operations) is equivalent to an abelian group of transformations. For a one parameter Lie group one can always introduce a canonical parameter 't' such that

$$\begin{aligned} g(t_1)\, g(t_2) &= g(t_1+t_2)\,, \\ [g(t)]^{-1} &= g(-t)\,. \end{aligned}$$

(Here $g(x)$ are elements of the group).

If the above transformation equations are to be obtained from equation (1.11), with arbitrary initial conditions, we must have

$$\frac{\partial^2 x_i}{\partial a_\lambda \partial a_\mu} = \frac{\partial^2 x_i}{\partial a_\mu \partial a_\lambda}, \tag{1.12}$$

or

$$\sum_{k=1}^{r} \frac{\partial}{\partial a_\lambda}\left[f_{ik}\Psi_{k\mu}\right] - \frac{\partial}{\partial a_\mu}\left[f_{ik}\Psi_{k\lambda}\right] = 0.$$

$$\sum_{k=1}^{r} \left\{ f_{ik}\left[\frac{\partial \Psi_{k\mu}}{\partial a_\lambda} - \frac{\partial \Psi_{k\lambda}}{\partial a_\mu}\right] + \Psi_{k\mu}\frac{\partial f_{ik}}{\partial a\lambda} - \Psi_{k\lambda}\frac{\partial f_{ik}}{\partial a\mu} \right\} = 0. \tag{1.13}$$

Using equation (1.11), we obtain

$$\frac{\partial f_{ik}}{\partial a_\lambda} = \sum_{j=1}^{n} \frac{\partial f_{ik}}{\partial x_j}\frac{\partial x_j}{\partial a_\lambda} = \sum_{j=1}^{n}\sum_{\nu=1}^{r} \frac{\partial f_{ik}}{\partial x_j} f_{j\nu}\Psi_{\nu\lambda}. \tag{1.14}$$

Substituting in equation (1.13), we get

$$\sum_{k=1}^{r} f_{ik}\left[\frac{\partial \Psi_{k\mu}}{\partial a_\lambda} - \frac{\partial \Psi_{k\lambda}}{\partial a_\mu}\right] + \sum_{j=1}^{n}\left[f_{jv}\frac{\partial f_{ik}}{\partial x_j} - f_{jk}\frac{\partial f_{iv}}{\partial x_j}\right]\Psi_{k\mu}\Psi_{\rho\lambda} = 0. \tag{1.15}$$

But $\Psi_{k\mu}\Theta_{\mu\lambda} = \delta_{k\lambda}$, thus we have

$$\begin{aligned} \sum_{j=1}^{n}\left\{ f_{j\sigma}\frac{\partial f_{i\tau}}{\partial x_j} - f_{j\tau}\frac{\partial f_{i\sigma}}{\partial x_j} \right\} &= \sum_{\lambda=1}^{r}\sum_{\mu=1}^{r}\sum_{k=1}^{r}\left[\frac{\partial \Psi_{k\mu}}{\partial a_\lambda} - \frac{\partial \Psi_{k\lambda}}{\partial a_\mu}\right]\Theta_{\mu\tau}\Theta_{\lambda\sigma} f_{ik} \\ &= \sum_{k=1}^{r} c^k_{\tau\sigma}(a)\, f_{ik}(x)\,, \end{aligned} \tag{1.16}$$

where

$$c^k_{\tau\sigma}(a) = \sum_{\lambda=1}^{r}\sum_{\mu=1}^{r}\left[\frac{\partial \Psi_{k\mu}}{\partial a_\lambda} - \frac{\partial \Psi_{k\lambda}}{\partial a_\mu}\right]\Theta_{\mu\tau}\Theta_{\lambda\sigma}. \tag{1.17}$$

Differentiating equation (1.16) with respect to a_ρ and using the operator $\frac{\partial x_k}{\partial a_\rho}\frac{\partial}{\partial x_k}$ we get

$$\sum_{k=1}^{r} \frac{\partial c_{\tau\sigma}^{k}}{\partial a_{\rho}} f_{ik} = 0. \tag{1.18}$$

Since $f_{ik}(x)$ are linearly independent, we conclude that the $c_{\tau\sigma}^{k}$ are independent of a, i.e. are constants. These are called the structure constants of the Lie Group (or algebra). From (1.16) and (1.17), we obtain

$$\sum_{j=1}^{n} \left\{ f_{j\sigma} \frac{\partial f_{i\tau}}{\partial x_j} - f_{j\tau} \frac{\partial f_{i\sigma}}{\partial x_j} \right\} = \sum_{k=1}^{r} c_{\tau\sigma}^{k} f_{ik}, \tag{1.19}$$

$$\frac{\partial \Psi_{k\mu}}{\partial a_{\lambda}} - \frac{\partial \Psi_{k\lambda}}{\partial a_{\mu}} = \sum_{\sigma=1}^{r} \sum_{\tau=1}^{r} c_{\tau\sigma}^{k} \Psi_{\tau\mu} \Psi_{\sigma\lambda}. \tag{1.20}$$

The infinitessimal generators

$$X_{\rho} = \sum_{i=1}^{n} f_{i\rho} \frac{\partial}{\partial x_i},$$

in view of (1.16) yield

$$\begin{aligned}
[X_{\rho}, X_{\sigma}] &= X_{\rho} X_{\sigma} - X_{\sigma} X_{\rho} \\
&= \sum_{i=1}^{n} \sum_{j=1}^{n} \left\{ f_{i\rho} \frac{\partial}{\partial x_i} f_{j\sigma} \frac{\partial}{\partial x_j} - f_{j\sigma} \frac{\partial}{\partial x_j} f_{i\rho} \frac{\partial}{\partial x_i} \right\} \\
&= \sum_{i=1}^{n} \sum_{j=1}^{n} \left\{ \left[f_{i\rho} \frac{\partial f_{j\sigma}}{\partial x_i} - f_{i\sigma} \frac{\partial f_{j\rho}}{\partial x_i} \right] \frac{\partial}{\partial x_j} \right\} \\
&= \sum_{k=1}^{r} \sum_{j=1}^{n} c_{\rho\sigma}^{k} f_{jk} \frac{\partial}{\partial x_j} \\
&= \sum_{k=1}^{r} c_{\rho\sigma}^{k} X_k.
\end{aligned} \tag{1.21}$$

This shows that all commutators are linearly expressible in terms of the infinitesimal generators, where $c_{\rho\sigma}^{k}$ are structure constants of the Lie group. We see that

$$c_{\rho\sigma}^{k} = -c_{\sigma\rho}^{k}. \tag{1.22}$$

Also substituting equation (1.21) into the Jacobi identity

$$[[X_{\rho}, X_{\sigma}], X_{\tau}] + [[X_{\sigma}, X_{\tau}], X_{\rho}] + [[X_{\tau}, X_{\rho}], X_{\sigma}] = 0,$$

we obtain

$$c^{\mu}_{\rho\sigma}c^{\upsilon}_{\mu\tau} + c^{\mu}_{\sigma\tau}c^{\upsilon}_{\mu\rho} + c^{\mu}_{\tau\rho}c^{\upsilon}_{\mu\sigma} = 0. \tag{1.23}$$

The above procedure shows how, starting from a group of transformations we arrive at equations (1.22) and (1.23) involving the structure constants.

A remarkable work by Lie is the proof that starting from the structure constants of the algebra the above procedure can be reversed. If we have structure constants satisfying equations (1.22) and (1.23) we can find f's and Ψ's satisfying equations (1.19) and (1.20) and thus we can find functions which are integrals of equation (1.11) and form a group [6] . This can be done by exponentiation, the details can be found in 'Theory of Lie groups' by Claude Chevalley [3] .

Lie algebras [6]

Using

$$X_\rho = \sum_{i=1}^{n} f_{i\rho}\frac{\partial}{\partial x_i},$$

we can write equation (1.11) in the form

$$\frac{\partial x_i}{\partial a_\lambda} = \Psi_{k\lambda}(a)\, X_k(x_i)\,; \quad \Psi_{k\lambda}(0) = \delta_{k\lambda}. \tag{1.24}$$

Any transformation of the group can be reached by letting the parameters a_λ vary along a line

$$a_\lambda = s_\lambda \tau, \quad 1 \leq \lambda \leq r, \tag{1.25}$$

where s_λ is a real vector. Clearly for $\tau = 0$ we get the identity transformation. Different values of τ give different transformation operators $S(\tau)$

$$x_i(\tau) = S(\tau)\, x_i(0)\,, \qquad S(0) = 1; \quad 1 \leq i \leq n. \tag{1.26}$$

Using equation (1.26) in equation (1.24) and $\frac{da_\lambda}{d\tau} = s_\lambda$ from equation (1.25), we get

$$\frac{dx_i}{d\tau} = \frac{dx_i}{da_\lambda}\frac{da_\lambda}{d\tau} = s_\lambda \Psi_{k\lambda}(s_\rho\tau)\, X_k(x_i)\,. \tag{1.27}$$

But

$$x_i(\tau) = S(\tau)\, x_i(0)\,.$$

Thus we get

$$\frac{dS(\tau)}{d\tau} x_i(0) = s_\lambda \Psi_{k\lambda}(s_\rho \tau)\, X_k S(\tau)\, x_i(0)\,, \tag{1.28}$$

i.e.

$$\frac{dS(\tau)}{d\tau} = s_\lambda \Psi_{k\lambda}(s_\rho \tau)\, X_k S(\tau)\,. \tag{1.29}$$

At $\tau = 0$,

$$\frac{dS}{d\tau} = s_k X_k, \tag{1.30}$$

where $\Psi_{k\lambda}(0) = \delta_{k\lambda}$ and $S(0) = 0$.

The Taylor expansion for $S(\tau)$ is then given by

$$S(\tau) = 1 + \tau s_k X_k + ... \tag{1.31 a}$$

Substituting value of $S(\tau)$ in equation (1.26), we get

$$\begin{aligned} x_i(\tau) &= S(\tau)\, x_i(0) \qquad (1.32)\\ &= (1 + \tau s_k X_k + ...) x_i(0)\,. \end{aligned}$$

Now

(i) Equations (1.29) and (1.31) show that $S(\tau)$ depends on the infinitesimal operator and $s_k X_k$ is determined by $S(\tau)$.

(ii) Consider another vector t_λ giving $t_\lambda X_\lambda$. Let $T(\tau)$ be the corresponding transformation operator satisfying equations (1.28) to (1.32) and

$$T(\tau) = 1 + \tau t_k X_k + ... \tag{1.31 b}$$

The product $S(\tau)\, T(\tau)$ gives

$$(1 + \tau s_k X_k + ...)(1 + \tau t_k X_k + ...) = 1 + \tau (s_k + t_k) X_k + ... \tag{1.33}$$

The **product** $S(\tau)\, T(\tau)$ corresponds to the sum $(s_k + t_k) X_k$.

(iii) The **commutator** of $S(\tau)$ and $T(\tau)$ is the transformation operator

$$S^{-1}(\tau)\,T^{-1}(\tau)\,S(\tau)\,T(\tau)\,,$$

which, using equations (1.31a) and (1.31b), becomes

$$\begin{aligned} & (1-\tau s_k X_k+...)\,(1-t_\lambda X_\lambda+...)\,(1+\tau s_k X_k+...)\,(1+\tau t_\lambda X_\lambda+..) \\ = \; & 1+\tau^2\,(s_k X_k t_\lambda X_\lambda - t_\lambda X_\lambda s_k X_k)+... \\ = \; & 1+\tau^2\,[s_k X_k, t_\lambda X_\lambda]+... \end{aligned} \tag{1.34}$$

i.e. the commutator of $S(\tau)$ and $T(\tau)$ corresponds to the commutator of the infinitesimal operators corresponding to $S(\tau)$ and $T(\tau)$. Also we see that if $S(\tau)$ and $T(\tau)$ commute we get the identity operator for the commutators and the corresponding infinitesimal operators satisfy

$$[X_k, X_\lambda]=0.$$

(iv) If the given group is **abelian** all of its elements commute and we get

$$[X_k, X_\lambda]=0, \quad 1\le k,\lambda\le r.$$

(v) Let H be a p-parameter proper subgroup of G $(p<r)$. Then we can select p infinitesimal operators corresponding to elements of H. The remaining $(r-p)$ infinitesimal operators correspond to elements in $G\setminus H$. Since H is a **subgroup**, the commutators of the infinitesimal operators $X_1,...,X_p$ of H must be expressible in terms of $X_1,...X_p$ alone so that

$$c^\rho_{k\lambda}=0 \qquad\qquad 1\le k,\lambda\le p \quad ; \; p+1\le\rho\le r. \tag{1.35}$$

(vi) Let H be an **invariant subgroup** of G. For $S\in H$ and $T\in G$, $T^{-1}ST\in H$. But then $S^{-1}T^{-1}ST$ is also in H. Thus we see from equation (1.34) that the commutators $[s_k X_k, t_\lambda X_\lambda]$ must be expressible as a linear combination of the infinitesimal operators of H alone. In other words

$$c^\rho_{k\lambda}=0 \qquad\qquad 1\le k\le p \;\; ; p+1\le\rho\le r. \tag{1.36}$$

(vii) If G is a **direct product** of H and G/H , then

$$c^\rho_{k\lambda}=0\ , \tag{1.37}$$

for $\quad 1 \leq k \leq p;\ p+1 \leq \rho \leq r$ and for $p+1 \leq k \leq r;\ 1 \leq \rho \leq p$.

(viii) If G is **simple** i.e. it has no proper invariant subgroups, then equation (1.36) cannot be satisfied for any choice of basis X_ρ. If G is **semisimple** then equation

$$c^{\rho}_{k\lambda} = 0 \qquad\qquad 1 \leq k, \lambda, \rho \leq p,$$

and equation (1.36) cannot be satisfied for any choice of basis X_ρ.

(ix) Thus for an r-parameter transformation group, there are associated with it r linearly independent infinitesimal operators X_ρ. Their linear combinations can be formed to give an r-dimensional vector space. That is, an r-parameter Lie group has associated with it an r-dimensional real vector space of quantities $\sum_\rho a_\rho X_\rho$ which is closed under multiplication defined by

$$[X_\rho, X_\sigma] = c^k_{\rho\sigma} X_k,$$

in terms of the structure constants which satisfy

$$c^k_{\rho\sigma} = -c^k_{\sigma\rho}.$$

Also the Jacobi Identity

$$\left[[X_\rho, X_\sigma], X_\tau\right] + \left[[X_\sigma, X_\tau], X_\rho\right] + \left[[X_\tau, X_\rho], X_\sigma\right] = 0,$$

is satisfied which gives

$$c^\mu_{\rho\sigma} c^\upsilon_{\mu\tau} + c^\mu_{\sigma\tau} c^\upsilon_{\mu\rho} + c^\mu_{\tau\rho} c^\upsilon_{\mu\sigma} = 0.$$

This is the Lie algebra of the Lie group.

An example

Consider the orthogonal group in two dimensions given by the parameter $0 \leq \theta < 2\pi$ and

$$\begin{aligned} x' &= x\cos\theta - y\sin\theta, \\ y' &= y\cos\theta + x\sin\theta, \end{aligned}$$

or

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}.$$

The matrix $A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ satisfies

$$AA^T = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I.$$

Any 2x2 orthogonal transformation is given parametrically by

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}, \quad 0 \leq \theta \leq 2\pi.$$

Now we find the corresponding infinitesimal generator. Identity corresponds to $\theta = 0$. Near the identity

$$cos\,(0+\delta\theta) = \cos\delta\theta = 1 - \frac{(\delta\theta)^2}{2!} + \frac{(\delta\theta)^4}{4!} - ...,$$

$$sin\,(0+\delta\theta) = \sin(\delta\theta) = \delta\theta - \frac{(\delta\theta)^3}{3!} + \frac{(\delta\theta)^5}{5!} - ...,$$

we have

$$\begin{aligned} x' &= x\left(1 - \frac{(\delta\theta)^2}{2!} + ...\right) - y\left(\delta\theta - \frac{(\delta\theta)^3}{3!} + ...\right) \\ &= x - y\delta\theta, \end{aligned}$$

neglecting 2nd and higher powers of $\delta\theta$ since $\delta\theta$ is small. Similarly

$$\begin{aligned} y' &= y\left(1 - \frac{(\delta\theta)^2}{2!} + ...\right) + x\left(\delta\theta - \frac{(\delta\theta)^3}{3!} + ...\right) \\ &= y + x\delta\theta, \end{aligned}$$

again neglecting 2nd and higher powers of $\delta\theta$, i.e.

$$\begin{aligned} x + dx &= x - y\delta\theta, \\ y + dy &= y + x\delta\theta, \end{aligned}$$

or

$$dx = -y\delta\theta,$$

and

$$dy = x\delta\theta.$$

We have only one parameter θ so we get only one infinitesimal generator

$$X_1 = \sum_{i=1}^{2} u_{i1}(x, y) \frac{\partial}{\partial x_i}.$$

Here

$$u_{11} = -y \qquad , u_{21} = x,$$

or

$$X_1 = -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}.$$

We write X_1 as X_θ . Thus

$$X_\theta = -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}.$$

This is the infinitesimal generator corresponding to rotation about the z-axis.

Now we exponentiate to see if we get back our orthogonal transformation. Consider $e^{X\theta}$ where $X = x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}$.We first calculate

$$\begin{aligned} x' &= e^{X\theta} x \\ &= \left(I + X\theta + \frac{X^2}{2!}\theta^2 + ... \right) x. \end{aligned} \tag{1.38}$$

But

$$X(x) = \left(-y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y} \right)(x) = -y,$$

and

$$X^2(x) = X(X(x)) = \left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x} \right)(-y) = -x.$$

Similarly

$$X^3(x) = y,$$

and so on. Putting these values in equation (1.38), we get

$$
\begin{aligned}
x' &= e^{X\theta}(x) = x + X(x)\theta + \frac{X^2(x)}{2!}\theta^2 + \frac{X^3(x)}{3!}\theta^3 + ... \qquad (1.39) \\
&= x - y\theta - x\frac{\theta^2}{2!} + y\frac{\theta^3}{3!} + ... \\
&= x\left(1 - \frac{\theta^2}{2!} + ...\right) - y\left(\theta - \frac{\theta^3}{3!} + ...\right) \\
&= x\cos\theta - y\sin\theta.
\end{aligned}
$$

Next we calculate

$$
y' = e^{X\theta}(y) = (I + X\theta + \frac{X^2}{2!}\theta^2 + \frac{X^3}{3!}\theta^3 + ...)(y). \qquad (1.40)
$$

Now

$$
X(y) = \left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right)(y) = x,
$$

$$
X^2(y) = -y,
$$

$$
X^3(y) = -x,
$$

and so on. Putting in equation (1.40), we get

$$
\begin{aligned}
y' &= e^{X\theta}(y) \qquad (1.41) \\
&= y + X(y)\theta + \frac{X^2(y)}{2!}\theta^2 + \frac{X^3(y)}{3!}\theta^3 + ... \\
&= y + x\theta - y\frac{\theta^2}{2!} - x\frac{\theta^3}{3!} + ... \\
&= y\left(1 - \frac{\theta^2}{2!} + ...\right) + x\left(\theta - \frac{\theta^3}{3!} + ...\right) \\
&= y\cos\theta + x\sin\theta.
\end{aligned}
$$

Equations (1.39) and (1.41) show that we get back to our group by exponentiating the infinitesimal generator where θ is the parameter.

Comparison of group and algebra

Group	**Algebra**
$x' = x\cos\theta - y\sin\theta,$	$\left\{X_\theta = x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right\}.$
$y' = y\cos\theta + x\sin\theta.$	
Group is abelian	$[X_\theta, X_\theta] = 0.$
$\bar{x}_1' = f(\bar{x}, \theta_1),$	
$\bar{x}_2' = f(\bar{x}_1', \theta_2),$	
$\bar{x}_1' + \bar{x}_2' = \bar{x}_2' + \bar{x}_1' = \bar{x}_3',$	
$\bar{x}_3' = f(\bar{x}, \theta_1 + \theta_2).$	

Now the work that was done before the classification of solvable Lie algebra was started can be summarized as follows:

1.1.3 Levi's theorem

Any finite dimensional Lie algebra L is isomorphic to a semidirect sum of a semisimple Lie algebra and a solvable one i.e. if L is a finite dimensional Lie algebra of characteristic 0 with radical S then there exist a semisimple Lie subalgebra B of L such that

$$L = B \oplus S.$$

(The proof of the theorem, being very lengthy, has not been given here. It is available in Jacobson's *Lie Algebras* [8]).

1.1.4 Dynkin diagrams [5]

Dynkin diagrams have a central role in the classification of semisimple Lie algebras. In this context we give a few basic definitions before presenting the role of Dynkin diagrams in the classification.

Cartan subalgebra:

A subalgebra K of a Lie algebra L is called a Cartan subalgebra if: **(1)** K is nilpotent and **(2)** K is its own normalizer in L.

Root:

A root of an algebra L relative to the Cartan subalgebra B is a nonzero linear functional α on B such that there exists a nonzero element X of L with

$$[H, X] = \alpha(H) X,$$

for all H in B. The symbol R denotes the set of all real roots.

Root system:

Any collection of roots in a finite dimensional real inner-product space having the following properties is called a *root system.*

1. The roots span E (E is the real inner-product space such that the roots form a finite set of non-zero elements of E)

2. If α is a root, then -α is also a root and the only multiples of α that are roots are α and -α.

3. If α is a root, let w_α denote the linear transformation of E given by

$$w_\alpha.\beta = \beta - 2\frac{\langle \alpha, \beta \rangle}{\langle \alpha, \alpha \rangle}\alpha.$$

Then, for all roots α and $\beta, w_\alpha.\beta$ is also a root.

4. If α and β are roots, then the quantity

$$2\frac{\langle \alpha, \beta \rangle}{\langle \alpha, \alpha \rangle},$$

is an integer.

Dynkin diagrams:

Let $\Delta = \{\alpha_1, \alpha_2, ..., \alpha_r\}$ be a basis for a root system R. Then the Dynkin diagram for R is a graph having vertices $v_1, v_2, ..., v_r$.The construction of the graph is such that we place either no edge, one edge, two edges or three edges between the vertices as follows:

Let i and j be two distinct indices. If the corresponding roots α_i and α_j are orthogonal then we put no edge between v_i and v_j. If α_i and α_j are not orthogonal, we put one two or three edges between v_i and v_j. The ratio of the lengths of the roots α_i and α_j is $1.\sqrt{2}, \sqrt{3}$. If α_i and α_j are of same length then one edge, two edges if longer root is $\sqrt{2}$ times longer than the shorter and three edges if longer is $\sqrt{3}$ longer.

Also if α_i and α_j are not orthogonal and not of the same length then we decorate the edges between v_i and v_j with an arrow pointing from the vertex associated with the longer towards the vertex associated with the shorter root.

The only possible length ratios are 1 , $\sqrt{2}$,$\sqrt{3}$ and these three cases correspond to angles of 120° ,135° , 150° respectively.

Dynkin diagrams are said to be equivalent if there exists a 1-1, onto map of the vertices of one to the vertices of the other which preserves the number of edges and the direction of the arrows. Since any two bases for the same root system can be mapped into one another by the action of the Weyl group, the equivalent class of Dynkin diagrams is independent of the choice of the base.

Classifying all the connected diagrams that can arise as Dynkin diagrams of root systems amounts to the classification of irreducible root system since a root system is irreducible if and only if its Dynkin diagram is connected. Further, two root systems with equivalent Dynkin diagrams are equivalent.

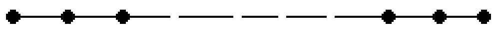
The analysis of the root systems of classical Lie algebras allows us to read off the Dynkin diagrams for the classical Lie algebras $sl\,(n;\mathbb{C})\,, so\,(n;\mathbb{C})$ and $sp\,(n;\mathbb{C})$

A_n : The root system A_n is the root system of $sl\,(n+1;\mathbb{C})$, which has rank n.

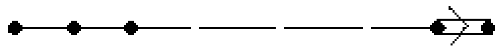
B_n : The root system B_n is the root system of $so\,(2n+1;\mathbb{C})$, which has rank n.

C_n : The root system C_n is the root system of $sp\,(n;\mathbb{C})$, which has rank n.

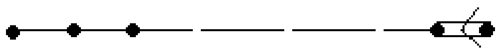
D_n : The root system D_n is the root system of $so\,(2n;\mathbb{C})$, which has rank n.



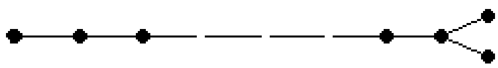
The Dynkin diagram for An



The Dynkin diagram for Bn

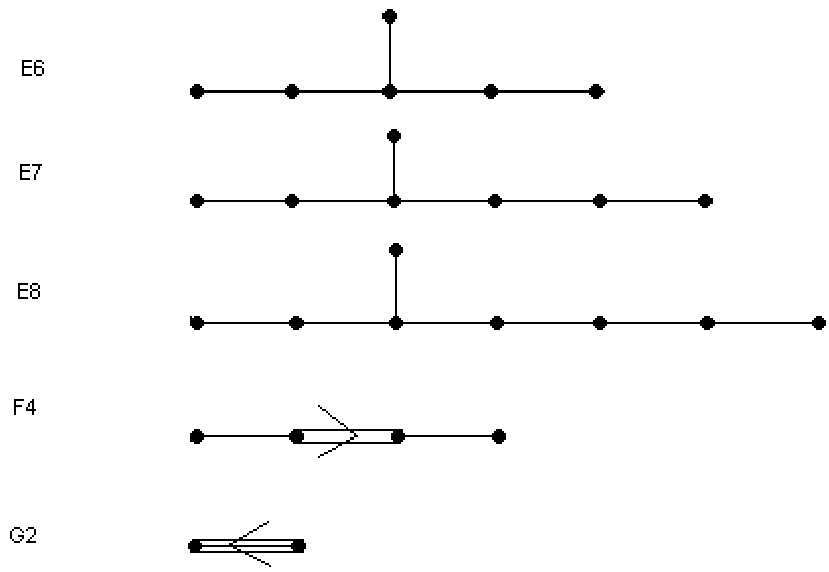


The Dynkin diagram for Cn



The Dynkin diagram for Dn

In addition to root systems associated to the classical Lie algebras, there are five exceptional irreducible root systems, denoted G_2, F_4, E_6, E_7 and E_8.



so that every irreducible root system is isomorphic to precisely one root system from the following list:

1.The classical root systems $A_n, n \geq 1$.

2.The classical root systems $B_n, n \geq 2$.

3.The classical root systems $C_n, n \geq 3$.

4.The classical root systems $D_n, n \geq 4$.

5.The exceptional root systems G_2, F_4, E_6, E_7 and $E_{8.}$

The classification of semisimple Lie algebras is equivalent to the classification of root systems, since

1.If R_1 and R_2 are the root systems for two different Cartan subalgebras of the same complex semisimple Lie algebra, then R_1 and R_2 are isomorphic.

2.A semisimple Lie algebra is simple if and only if its root system is irreducible.

3.If two complex semisimple Lie algebras have isomorphic root systems, then the semisimple Lie algebras are isomorphic.

4.Every root system arises as the root system of some complex semisimple Lie algebra.

Hence we have that every complex simple Lie algebra is isomorphic to precisely one algebra from the following list:

1. $sl\,(n+1;\mathbb{C})\,, n \geq 1$.

2. $so\,(2n+1;\mathbb{C})\,, n \geq 2$.

3. $sp\,(n;\mathbb{C})\,, n \geq 3$.

4. $so\,(2n;\mathbb{C})\,, n \geq 4$.

5. The exceptional Lie algebras G_2, F_4, E_6, E_7 and E_8.

A semisimple Lie algebra is determined up to isomorphism by specifying which simple summands occur and how many times each one occurs.

This completes the classification of all semisimple Lie algebras.

1.1.5 An important theorem

STATEMENT: Every solvable Lie algebra has a uniquely defined Nilradical [8].

PROOF:

Proof involves the following four steps:

(I). In this context, we first prove

$$[...\,[[U_1, U_2]\,, U_3]\,, ..., U_k] \subseteq B^h \quad ; h \leq k; h, k \in \mathbb{N}$$

where B is an ideal in L, $B^0 = L$ and h of the U_i are equal to B and the remaining $(k-h)$ U_i are equal to L.

For $k = 1$

$[U_1] \subseteq B$ is true since either **(i)** $h = 0$, then $U_1 = L$ implies $[U_1] \subseteq B^0 = L$. or **(ii)** $h = 1$,then $U_1 = B$ implies $[U_1] \subseteq B$.

Assume that the statement is true for $k = n$ i.e.

$[...[[U_1, U_2], U_3], ..., U_n] \subseteq B^h$.

Consider

$[...[[[U_1, U_2], U_3], ...U_n], U_{n+1}]$.

There are two cases; either **(i)** $U_{n+1} = L$, or **(ii)** $U_{n+1} = B$.

Case **(i)** If $U_{n+1} = L$

Let $[...[[U_1, U_2], U_3], ..., U_n] = M \subseteq B^h$.

Then $[M, U_{n+1}] \subseteq B^h$.

Since if B is an ideal, $[B, L] \subseteq B$ and Lie product of ideals is an ideal.

Case **(ii)** If $U_{n+1} = B$

$[M, B] \subseteq B^h$,

which shows that if the statement if true for $k = n$ then it is also true for $k = n + 1$

Hence the result is established by induction.

(II). Sum of nilpotent ideals is nilpotent

Consider two ideals $B_1$and B_2 and the ideal $B_1 + B_2$. Then $(B_1 + B_2)^m$ is contained in a sum of terms $[...[[U_1, U_2], U_3], ..., U_m]$ where $U_i = B_1 or B_2$

We prove below by induction that any such term contains $\left[\frac{m}{2}\right]$ B_1's or $\left[\frac{m}{2}\right]$ B_2's, where $\left[\frac{m}{2}\right]$ is the integral value of $\frac{m}{2}$.

Obviously $(B_1 + B_2)^1 = [B_1] + [B_2]$, here $\left[\frac{1}{2}\right] = 0$

Assume that the above statement is true for $k = n$

$(B_1 + B_2)^n \subseteq sum$ of terms $[...[[U_1, U_2], U_3], ..., Un]$ where each term contains either $\left[\frac{n}{2}\right]$ $B_1's$ or $\left[\frac{n}{2}\right]$ $B_2's$.

Then for $k = n+1$; $(B_1 + B_2)^{n+1} = [(B_1 + B_2)^n, B_1 + B_2] = [(B_1 + B_2)^n, B_1] + [(B_1 + B_2)^n, B_2]$.

i.e. $(B_1 + B_2)^{n+1} \subseteq$ sum of terms with $\left[\frac{n}{2}\right] + 1$ B_1's or $\left[\frac{n}{2}\right] + 1$ B_2's

$\subseteq$ sum of terms with $\left[\frac{n+1}{2}\right]$ B_1's or $\left[\frac{n+1}{2}\right]$ B_2's

Hence the result is established by induction.

From this and the previous result (I) we see that the terms $[...[[U_1, U_2], U_3], ..., U_n]$ (where $U_i = B_1 or B_2$) , containing $\left[\frac{m}{2}\right]$ B_1's or $\left[\frac{m}{2}\right]$ B_2's, is contained in $B_1^{\left[\frac{m}{2}\right]}$ or $B_2^{\left[\frac{m}{2}\right]}$ consequently

$(B_1 + B_2)^m \subseteq B_1^{\left[\frac{m}{2}\right]} + B_2^{\left[\frac{m}{2}\right]}$

Let $B_1^n = 0$, $B_2^l = 0$

Set m such that $\left[\frac{m}{2}\right] = l.c.m\,(l, n)$

Then $(B_1 + B_2)^m \subseteq 0 + 0$

$0 \subseteq (B_1 + B_2)^m \subseteq 0$

implies $(B_1 + B_2)^m = 0$ which further implies that $B_1 + B_2$ is nilpotent.

As a corollary, sum of finite number of nilpotent ideals is nilpotent.

(III). Main result: Every solvable Lie algebra has a uniquely defined nilradical.

Let L be a finite dimensional solvable Lie algebra and let R be a nilpotent ideal of maximal dimensionality. If B is another nilpotent ideal then $R + B$ is also nilpotent but since R is of maximal dimensionality

$R + B = R$ thus $B \subseteq R$.

We call R the Nilradical of L.

(IV). The nilradical obtained in **(III)** above is unique.

Let M be another nilpotent ideal of maximal dimensionality. Then

$M + R = M$.

But

$M + R = R$, because R is maximal ideal which implies that $M = R$.

Thus the nilradical is unique. This is the main result which will be used in the sequel.

1.1.6 Conclusion

Levis theorem tells us that every finite dimensional Lie algebra is isomorphic to a semidirect sum of a semisimple Lie algebra and a solvable one. Since the semisimple Lie algebras have already been classified, so the classification of all finite dimensional Lie algebras can be completed if we are able to classify solvable Lie algebras and this is what is being attempted these days.

Chapter 2

Solvable extension of a nilpotent Lie algebra

In this chapter, we find all solvable extensions of a nilpotent Lie algebra using the sets of nonequivalent nil-independent outer derivations as in [16].

Since we are to construct all possible solvable extensions of a nilpotent Lie algebra, we must see what information we can get about the extension from the given nilpotent Lie algebra.

Let N be a nilpotent Lie algebra, with (in some basis $(e_1, ..., e_n)$ of N) the Lie brackets.

$$[e_j, e_k] = H^l_{jk} e_l. \tag{2.1}$$

In the above and within the sequel, summation over repeated indices is assumed. To extend N we have to add further elements $f_1, ..., f_r$ to the basis which can together form basis for its solvable extension S.

Now the derived algebra of any finite dimensional solvable Lie algebra of characteristic zero is nilpotent. Thus it is contained in the nilradical. (Here we will consider N as the unique maximal nilpotent ideal of S i.e. nilradical of S) i.e.

$$[S, S] \subseteq N,$$

(see appendix A.2)

The above statement shows that the Lie brackets on S will be of the form

$$\begin{aligned} [f_a, e_j] &= (A_a)^k_j e_k, 1 \le a \le r, 1 \le j \le n, \\ [f_a, f_b] &= \gamma^j_{ab} e_j, 1 \le a, b \le r, \\ [e_j, e_k] &= H^l_{jk} e_l; 1 \le j, k \le n. \end{aligned} \tag{2.2}$$

To know the solvable algebra completely we must find out the matrices A_a and the structure constants γ_{ab}^{j}. Since S is to be a Lie algebra, the matrix elements of the matrices A_a must satisfy certain linear relations following from the Jacobi identities between the basis elements (f_a, e_j, e_k). Using equations (2.1) and (2.2)

$$[f_a, [e_j, e_k]] + [e_j, [e_k, f_a]] + [e_k, [f_a, e_j]] = 0,$$

we have

$$[f_a, [e_j, e_k]] + \left[e_j, -\left(A_a\right)_k^l e_l\right] + \left[e_k, -\left(A_a\right)_j^l e_l\right] = 0,$$

or

$$\left[f_a, H_{jk}^l e_l\right] - \left(A_a\right)_k^l [e_j, e_l] - \left(A_a\right)_j^l [e_k, e_l] = 0,$$

or

$$H_{jk}^l \left(A_a\right)_l^m e_m - \left(A_a\right)_k^l H_{jl}^m e_m + \left(A_a\right)_j^l H_{kl}^m e_m = 0.$$

Similarly the Jacobi identities between (f_a, f_b, e_j) will provide linear expression for the structure constants γ_{ab}^{j} in terms of the matrix elements of the commutators of the matrices A_a and A_b. Using equations(2.1) and (2.2) and

$$[f_a, [f_b, e_j]] + [f_b, [e_j, f_a]] + [e_j, [f_a, f_b]] = 0,$$

we have

$$\left[f_a, \left(A_b\right)_j^k e_k\right] - \left[f_b, \left(A_a\right)_j^k e_k\right] + \left[e_j, \gamma_{ab}^k e_k\right] = 0,$$

or

$$\left(A_b\right)_j^k [f_a, e_k] - \left(A_a\right)_j^k [f_b, e_k] + \gamma_{ab}^k [e_j, e_k] = 0,$$

or

$$\left(A_b\right)_j^k \left(A_a\right)_k^l e_l - \left(A_a\right)_j^k \left(A_b\right)_k^l e_l + \gamma_{ab}^k H_{jk}^l e_l = 0,$$

or

$$\gamma_{ab}^k H_{jk}^l e_l = \left(A_a\right)_j^k \left(A_b\right)_k^l e_l - \left(A_b\right)_j^k \left(A_a\right)_k^l e_l.$$

The set of matrices A_i $(1 \leq i \leq n)$ are **linearly nil-independent** i.e. no nontrivial linear combination of the matrices A_i is a nilpotent matrix. Otherwise it will contradict the maximality of N. Here we see how this statement is true:

Since the restriction $ad\mid_N (f_a)$ of the adjoint representation $ad \mid (f_a)$ on N satisfies $ad\mid_N (f_a) = A_a$, any non-trivial linear combination of the matrices A_i is of the form

$$\alpha \left(ad\mid_N (f_a)\right) + \beta \left(ad\mid_N (f_b)\right) + ... + \gamma \left(ad\mid_N (f_c)\right),$$

where $\alpha, \beta, ..., \gamma$ are constants and not all are zero.

Now the above could not be a nilpotent matrix as is shown below. Consider $M = \{e_1, e_2, ..., e_n, f_a, f_b\}$ such that $M \supseteq N$ with Lie brackets given by

$$\begin{aligned} [e_j, e_k] &= H^l_{jk} e_l; 1 \leq j, k \leq n, \\ [f_a, e_j] &= (A_a)^k_j e_k, 1 \leq j \leq n, \\ [f_a, f_b] &= \gamma^j_{ab} e_j. \end{aligned}$$

The ideal M can be nilpotent if at some stage $M^k = 0$ for some positive integer k. Consider

$$[f_a, [f_a, e_j]] = (A_a)^2 (e_j),$$

where $(A_a)^2$ is an $n \times n$ matrix and (e_j) is an $n \times 1$ matrix. Also

$$[\alpha f_a + \beta f_b, e_j] = (\alpha A_a + \beta A_b)(e_j),$$

is a linear combination of matrices A_a and A_b.

From here we see that if the matrices of the form $\alpha A_a + \beta A_b$ (where not both α, β are 0) are nilpotent then M will be nilpotent and since M is an ideal and $M \supseteq N$ it contradicts the maximality of N. Therefore the set of matrices A_i; $1 \leq i \leq n$ are linearly nil-independent.

Sets of outer nil-independent derivations of *N:* The representation $ad(f_a)$ is an adjoint representation of S. We restrict it to N i.e. $ad\mid_N (f_a)$. It follows from the Jacobi identities that $ad\mid_N (f_a)$ is a derivation of N

$$ad\mid_N (f_a)([x, y]) = [f_a, [x, y]],$$

and

$$[f_a, [x, y]] + [x, [y, f_a]] + [y, [f_a, x]] = 0,$$

by Jacobi identity. Thus

$$\begin{aligned} [f_a, [x, y]] &= -[x, [y, f_a]] - [y, [f_a, x]] \\ &= [x, [f_a, y]] + [[f_a, x], y], \end{aligned}$$

or

$$ad\mid_N (f_a)([x, y]) = [x, ad\mid_N (f_a)(y)] + [ad\mid_N (f_a)(x), y].$$

This proves that $ad\mid_N (f_a)$ is a derivation of N. It is an outer derivation since $f_a \notin N$.

We know that finding all sets of matrices A_a (to know the algebra of the extension of N) satisfying Jacobi identities is equivalent to finding all sets of outer nil-independent derivations of N. Here

$$D^1 = ad|_N(f_1), ..., D^r = ad|_N(f_r),$$

Again $[D^a, D^b]$ must be inner derivation of N.

$$[D^a, D^b] = D^a D^b - D^b D^a,$$

and for $ad(x) \in \mathbb{N}$,

$$\begin{aligned}
[D^a, D^b](x) &= (D^a D^b - D^b D^a)(x); \text{ for all } x \in n \\
&= D^a D^b(x) - D^b D^a(x) \\
&= D^a [f_b, x] - D^b [f_a, x] \\
&= [f_a, [f_b, x]] - [f_b, [f_a, x]] \\
&= [f_a, [f_b, x]] + [f_b, [x, f_a]] \\
&= -[x, [f_a, f_b]]; \text{by Jacobi identity} \\
&= [[f_a, f_b], x], \\
[D^a, D^b]([x, y]) &= [[f_a, f_b], [x, y]] \\
&= -[x, [y, [f_a, f_b]]] - [y, [[f_a, f_b], x]]; \text{by Jacobi identity} \\
&= [x, [[f_a, f_b], y]] + [[[f_a, f_b], x], y].
\end{aligned}$$

Hence $[D^a, D^b]$ is a derivation in N which is an inner derivation since $[D^a, D^b] \in N$.

2.1 A particular nilpotent algebra N

Let us define a Lie algebra by the Lie brackets

$$\begin{aligned}
[e_i, e_j] &= 0; 1 \le i, j \le n-2, \\
[e_1, e_{n-1}] &= [e_2, e_{n-1}] = 0, \\
[e_k, e_{n-1}] &= e_{k-2}; 3 \le k \le n-2, \\
[e_1, e_n] &= 0, \\
[e_k, e_n] &= e_{k-1}; 2 \le k \le n-1.
\end{aligned} \tag{2.3}$$

We get the following information from the above Lie brackets:

(i) This Lie algebra is nilpotent and has **degree of nilpotency** $n-1$ i.e. $N^{n-1} \neq 0$ and $N^n = 0$. To prove it we observe that the only non zero commutations are $[e_k, e_{n-1}] = e_{k-2}$ and $[e_k, e_n] = e_{k-1}$, with $[e_k, e_{n-1}] = e_{k-2}$, $[e_{k-2}, e_{n-1}] = e_{k-4}$ ... This series will continue if we keep on taking commutation relations of the resultant with e_{n-1} i.e. $[e_{k-4}, e_{n-1}] = e_{k-6}$ and so on. But this will terminate earlier than N^{n-1} because it reduces the indices of e_i by 2; $1 \leq i \leq n-1$. With $[e_k, e_n] = e_{k-1}$, the series will continue if we take commutators repeatedly with e_n and the farthest we can go is with e_{n-1}. Consider

$$\begin{aligned} [e_{n-1}, e_n] &= e_{n-2} \in \ N^2, \\ [e_{n-2}, e_n] &= e_{n-3} \in \ N^3, \\ &\quad . \\ &\quad . \\ &\quad . \\ \left[e_{n-(n-2)}, e_n\right] &= e_1 \in \ N^{n-1} \neq 0, \\ \left[e_{n-(n-1)}, e_n\right] &= [e_1, e_n] \in \ N^n = 0. \end{aligned}$$

This shows that N has degree of nilpotency $n-1$.

(ii) The nilpotent Lie algebra N has a **uniquely defined maximal abelian ideal ă** of dimension $(n-2)$ equal to its derived algebra; Consider N^2, N has basis $(e_1, ..., e_n)$ and from the Lie brackets (2.3) we see that N^2 has elements as linear combinations of $(e_1, ..., e_{n-2})$. Thus N^2 has basis $(e_1, ..., e_{n-2})$. Since $(e_1, ..., e_{n-2})$ are given N^2 is uniquely defined and has dimension $n-2$. Also N^2 is abelian since $[e_i, e_j] = 0$ for $1 \leq i, j \leq n-2$. This $N^2 = [N, N]$ is the derived algebra of N.

(iii) Derived series

For an arbitrary Lie algebra L

$$L = L^{(0)} \supseteq L^{(1)} \supseteq .... \supseteq L^{(k)} \supseteq ...$$

where

$$L^{(0)} = L, \ L^{(k)} = \left[L^{(k-1)}, L^{(k-1)}\right] \text{ for } k \geq 1.$$

We investigate

$$N = N^{(0)} \supseteq N^{(1)} \supseteq ... \supseteq N^{(k)} \supseteq ...,$$

where

$$N = span\{e_1, ..., e_n\}.$$

Now

$$N^{(1)} = [N, N] = span\{e_1, ..., e_{n-2}\} \text{ by equation(2.3)},$$

and

$$N^{(2)} = \left[N^{(1)}, N^{(1)}\right] = 0 \text{ by equation (2.3)},$$

since

$$[e_i, e_j] = 0; 1 \leq i, j \leq n-2,$$

i.e. the derived series for N terminates since

$$N = N^{(0)} \supseteq N^{(1)} \supseteq N^{(2)} = 0.$$

This shows that N is a solvable Lie algebra.

(iv) Lower central series

For an arbitrary Lie algebra L we define the series

$$L = L^1 \supseteq L^2 \supseteq .... \supseteq L^k \supseteq ...,$$

where

$$L^1 = L \qquad L^k = \left[L^{k-1}, L\right] \text{ for } k \geq 2.$$

We investigate

$$N = N^1 \supseteq N^2 \supseteq ... \supseteq N^k \supseteq ...$$

Here

$$N^1 = N,$$

and

$$N^2 = [N, N] = span\{e_1, ..., e_{n-2}\},$$

and

$$N^3 = \left[N^2, N\right].$$

Now

$$\begin{aligned}
[e_k, e_j] &= 0;\ 1 \leq i, j \leq n-2, \\
[e_k, e_{n-1}] &= 0 \text{ for } k = 1, 2, \\
[e_k, e_{n-1}] &= e_{k-2} \text{ for } 3 \leq k \leq n-2, \\
[e_1, e_n] &= 0, \\
[e_k, e_n] &= e_{k-1} \text{ for } 2 \leq k \leq n-1.
\end{aligned}$$

Thus

$$N^3 = span\{e_1, ..., e_{n-3}\}.$$

Similarly

$$\begin{aligned} N^4 &= span\{e_1, ..., e_{n-4}\}, \\ &. \\ &. \\ &. \\ N^{n-2} &= span\{e_1, e_2\}, \\ N^{n-1} &= span\{e_1\}. \end{aligned}$$

The lower central series for N is

$$N = N^1 \supseteq N^2 \supseteq ... \supseteq N^{n-1} \supseteq N^n = 0.$$

This shows that N is nilpotent of degree of nilpotency $(n-1)$.

(v) We see that for the derived series and the lower central series of this algebra the dimension is DS and CS respectively where these are notations for ordered list of integers denoting the dimensions of subalgebras in the derived and lower central series respectively i.e.

$$DS = [n, n-2, 0],$$

since

$$\begin{aligned} \dim N^{(0)} &= n, \\ \dim N^{(1)} &= n-2, \\ \dim N^{(2)} &= 0. \end{aligned}$$

Also for the central series

$$CS = [n, n-2, n-3, ..., 1, 0],$$

since

$$\begin{aligned} \dim N^1 &= n, \\ \dim N^2 &= n-2, \\ \dim N^3 &= n-3, \\ &\ . \\ &\ . \\ &\ . \\ \dim N^{n-1} &= 1, \\ \dim N^n &= 0. \end{aligned}$$

(vi) The **maximal abelian ideal $\breve{\mathfrak{a}}$** coincides with the derived algebra $N^{(1)} = N^2$ i.e. $\breve{\mathfrak{a}} = span\{e_1, ..., e_{n-2}\}$ because of equation (2.3).

(vii) There exists a **flag of ideals** which is invariant under any automorphism

$$N \supset C_{N^{n-2}} \supset N^2 \supseteq N^3 ... \supseteq N^{n-1}.$$

We calculated the ideals $N^2, N^3...$. Let us investigate $C_{N^{n-2}}$ which is the centralizer of N^{n-2}. Now we know that $N^{n-2} = span\{e_1, e_2\}$. From equation (2.3) the elements which commute with e_1 and e_2 are $e_3, ..., e_{n-2}, e_{n-1}$. Also e_1 commutes with e_2, thus the centralizer of N^{n-2} is $span\{e_1, ..., e_{n-1}\}$ and we see that $N \supset C_{N^{n-2}} \supset N^2$ where each element in the flag has codimension one in the previous one, as is evident from the structure of the flag.

(viii) For the Lie brackets given in equation (2.3), any automorphism will be represented by a **triangular matrix.** From Lie brackets given in equation (2.3) the only non zero commutators are

$$[e_k, e_{n-1}] = e_{k-2}; \quad 3 \leq k \leq n-2, \tag{2.4}$$

and

$$[e_k, e_n] = e_{k-1}; \quad 2 \leq k \leq n-1. \tag{2.5}$$

Equation (2.4) shows that the 3rd row has non-zero element only in the 1st column (or 3rd column,1st row) 4th column in 2nd row (or 2nd column in 4th row) ... Thus it gives a triangular matrix.

(ix) An automorphism in N.

The whole algebra N is generated via multiple commutators of the elements e_{n-1}, e_n

$$\begin{aligned} e_{n-2} &= [e_{n-1}, e_n], \\ e_{n-3} &= [[e_{n-1}, e_n], e_n], \\ e_{n-4} &= [e_{n-3}, e_n], \\ &\cdot \\ &\cdot \\ &\cdot \\ e_1 &= [e_2, e_n]. \end{aligned}$$

Thus from the definition of automorphism

$$\Phi [e_i, e_j] = [\Phi (e_i), \Phi (e_j)],$$

the knowledge of

$$\Phi (e_{n-1}) = \sum_{k=1}^{n-1} \phi_k e_k; \quad \Phi (e_n) = \sum_{k=1}^{n} \psi_k e_k,$$

amounts to the full knowledge of Φ. The automorphism is chosen to be triangular because the basis of N follow the flag of ideals

$$N \supset C_{N^{n-2}} \supset N^2 \supseteq N^3 ... \supseteq N^{n-1}.$$

(x) Choices of $\phi_k; 1 \leq k \leq n-1$ and $\psi_k; 1 \leq k \leq n$ which are consistent with the definition of automorphism.

Due to the definition of automorphism

$$\begin{aligned} [\Phi (e_j), \Phi (e_k)] &= 0; \quad 1 \leq j, k \leq n-2, \\ [\Phi (e_1), \Phi (e_{n-1})] &= [\Phi (e_2), \Phi (e_{n-1})] = 0, \\ [\Phi (e_1), \Phi (e_n)] &= 0, \end{aligned}$$

and

$$[\Phi (e_k), \Phi (e_n)] = \Phi (e_{k-1}); \quad 2 \leq k \leq n-1. \tag{2.6}$$

The above relation is due to the definition of e_{k-1} and

$$[\Phi (e_k), \Phi (e_{n-1})] = \Phi (e_{k-2}); \quad 3 \leq k \leq n-2.$$

Now

$$\begin{aligned}
[\Phi(e_{n-1}), \Phi(e_k)] &= -[\Phi(e_k), \Phi(e_{n-1})] \\
&= -\Phi(e_{k-2}) \\
&= -\Phi[e_{k-1}, e_n] \\
&= -[\Phi(e_{k-1}), \Phi(e_n)] \\
&= [\Phi(e_n), \Phi(e_{k-1})] \\
&= [\Phi(e_n), \Phi[e_k, e_n]] \\
&= -[\Phi(e_n), [\Phi(e_n), \Phi(e_k)]].
\end{aligned}$$

Since $\Phi(e_k)$ is common on the two of the sides, we conclude

$$ad(\Phi(e_{n-1})) = -ad(\Phi(e_n))^2. \tag{2.7}$$

We can write $\Phi(e_k)$ in terms of e_k since Φ is an automorphism. Since automorphism is bijective, for every e_k we can find an $a \in N$ such that $e_k = \Phi(a)$ so that equation (2.7) is also true for e_k i.e. $ad(\Phi(e_{n-1})) = -ad(\Phi(e_n))^2$ also for e_k.

Now to find $ad\mid_{N^2} \Phi(e_{n-1})$, the restriction of $ad \mid \Phi(e_{n-1})$ on N^2. Since $\Phi(e_{n-1}) = \sum_{j=1}^{n-1} \phi_j e_j$, we have

$$\begin{aligned}
[e_k, \Phi(e_{n-1})] &= \left[e_k, \sum_{j=1}^{n-1} \phi_j e_j\right]; 3 \leq k \leq n-2 \\
&= [e_k, \phi_1 e_1] + [e_k, \phi_2 e_2] + ... + [e_k, \phi_{n-1} e_{n-1}] \\
&= \phi_1 [e_k, e_1] + \phi_2 [e_k, e_2] + ... + \phi_{n-1} [e_k, e_{n-1}] \\
&= \phi_{n-1} e_{k-2}, \\
[\Phi(e_{n-1}), e_k] &= -\phi_{n-1} e_{k-2},
\end{aligned}$$

which gives

$$ad\mid_{N^2}(\Phi(e_{n-1})) = \begin{bmatrix} 0 & 0 & -\phi_{n-1} & 0 & 0 & \cdot\ \cdot\ \cdot & 0 \\ 0 & 0 & 0 & -\phi_{n-1} & 0 & & 0 \\ 0 & 0 & 0 & 0 & -\phi_{n-1} & & 0 \\ \cdot & & \cdot & & & \cdot & \\ \cdot & & & \cdot & & \cdot & \\ \cdot & & & & \cdot & \cdot & \\ 0 & 0 & 0 & 0 & 0 & \cdot\ \cdot\ \cdot & -\phi_{n-1} \\ 0 & 0 & 0 & 0 & 0 & & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdot\ \cdot\ \cdot & 0 \end{bmatrix}.$$

Similarly for $ad\mid_{N^2}(\Phi(e_n))$, we have

$$\begin{aligned} [\Phi(e_n), e_k] &= -\left[e_k, \sum_{j=1}^{n} \psi_j e_j\right]; \quad 3 \le k \le n-1 \\ &= -\left\{[e_k, \psi_1 e_1] + [e_k, \psi_2 e_2] + ... + [e_k, \psi_{n-1} e_{n-1}] + [e_k, \psi_n e_n]\right\} \\ &= -\left\{0 + 0 + ... + \psi_{n-1} e_{k-2} + \psi_n e_{k-1}\right\} \\ &= -\psi_{n-1} e_{k-2} - \psi_n e_{k-1}. \end{aligned}$$

Thus

$$ad\mid_{N^2}\Phi(e_n) = \begin{bmatrix} 0 & -\psi_n & -\psi_{n-1} & 0 & 0 & \cdot\ \cdot & \cdot & 0 \\ 0 & 0 & -\psi_n & -\psi_{n-1} & 0 & \cdot\ \cdot & \cdot & 0 \\ 0 & 0 & 0 & -\psi_n & -\psi_{n-1} & \cdot\ \cdot & \cdot & 0 \\ \cdot & & \cdot & & \cdot & \cdot & & \\ \cdot & & & \cdot & & \cdot\ \cdot & & \\ \cdot & & & & \cdot & \cdot & \cdot & \\ 0 & 0 & 0 & 0 & 0 & & -\psi_n & -\psi_{n-1} \\ 0 & 0 & 0 & 0 & 0 & & 0 & -\psi_n \\ 0 & 0 & 0 & 0 & 0 & & 0 & 0 \end{bmatrix}.$$

Reduction of parameters in the automorphism.

Initially we had $\phi_k, 1 \le k \le n-1; \psi_k 1 \le k \le n$ i.e. $2n-1$ parameters in all. But since $ad(\Phi(e_{n-1})) = -ad(\Phi(e_n))^2$ we have $\phi_{n-1} = (\psi_n)^2, \psi_{n-1} = 0$ so that we are left with $\phi_1, ..., \phi_{n-2}, \psi_n, \psi_1, ..., \psi_{n-2}$ i.e. $n-2+1+n-2 = 2n-3$ parameters.

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(xi) Computation of $\Phi(e_k)$.

From $e_{n-2} = [e_{n-1}, e_n]$, we get

$$\Phi(e_{n-2}) = \Phi[e_{n-1}, e_n].$$

Using $\phi_{n-1} = (\psi_n)^2, \psi_{n-1} = 0$, we have

$$\begin{aligned}
\Phi(e_{n-2}) &= \left[\sum_{j=1}^{n-2} \phi_j e_j + (\psi_n)^2 e_{n-1}, \sum_{k=1}^{n-2} \psi_k e_k + \psi_n e_n\right] \quad , \text{(Since } \phi_{n-1} = (\psi_n)^2, \psi_{n-1} = 0) \\
&= \left[\sum_{j=1}^{n-2} \phi_j e_j, \sum_{k=1}^{n-2} \psi_k e_k\right] + \left[\sum_{j=1}^{n-2} \phi_j e_j, \psi_n e_n\right] \\
&\quad + \left[(\psi_n)^2 e_{n-1}, \sum_{k=1}^{n-2} \psi_k e_k\right] + \left[(\psi_n)^2 e_{n-1}, \psi_n e_n\right] \\
&= 0 + \left\{\phi_2 \psi_n e_1 + \phi_3 \psi_n e_2 + ... + \phi_{n-2} \psi_n e_{n-3}\right\} \\
&\quad - \left\{\psi_3 (\psi_n)^2 e_1 + \psi_4 (\psi_n)^2 e_2 + ... + \psi_{n-2} (\psi_n)^2 e_{n-4}\right\} + (\psi_n)^2 \psi_n e_{n-2}
\end{aligned}$$

$$\begin{aligned}
\Phi(e_{n-2}) &= \left(\phi_2 \psi_n - \psi_3 (\psi_n)^2\right) e_1 + \left(\phi_3 \psi_n - \psi_4 (\psi_n)^2\right) e_2 + \left(\phi_4 \psi_n - \psi_5 (\psi_n)^2\right) e_3 ... \\
&\quad + \left(\phi_{n-3} \psi_n - \psi_{n-2} (\psi_n)^2\right) e_{n-4} + \left(\phi_{n-2} \psi_n\right) e_{n-3} + (\psi_n)^3 e_{n-2}. \qquad (2.8)
\end{aligned}$$

Similarly

$$\begin{aligned}
\Phi(e_{n-3}) &= \Phi[[e_{n-1}, e_n], e_n] \\
&= [\Phi[e_{n-1}, e_n], \Phi(e_n)] \\
&= \left[\begin{array}{c} \left\{\begin{array}{c} \left(\phi_2 \psi_n - \psi_3 (\psi_n)^2\right) e_1 + \left(\phi_3 \psi_n - \psi_4 (\psi_n)^2\right) e_2 + \left(\phi_4 \psi_n - \psi_5 (\psi_n)^2\right) e_3 \\ +... + \left(\phi_{n-3} \psi_n - \psi_{n-2} (\psi_n)^2\right) e_{n-4} + \left(\phi_{n-2} \psi_n\right) e_{n-3} + (\psi_n)^3 e_{n-2}. \end{array}\right\}, \\ \sum_{k=1}^{n-2} \psi_k e_k + \psi_n e_n \end{array}\right] \\
&= \left(\phi_3 \psi_n - \psi_4 (\psi_n)^2\right) \psi_n e_1 + \left(\phi_4 \psi_n - \psi_5 (\psi_n)^2\right) \psi_n e_2 + ... \\
&\quad + \left(\phi_{n-3} \psi_n - \psi_{n-2} (\psi_n)^2\right) \psi_n e_{n-5} + \left(\phi_{n-2} \psi_n\right) \psi_n e_{n-4} + (\psi_n)^4 e_{n-3}.
\end{aligned}$$

Generalizing we get

$$\Phi(e_k) = \sum_{j=1}^{k-1} \left(\phi_{n-k+j-1} \psi_n - \psi_{n-k+j} (\psi_n)^2\right) (\psi_n)^{n-k-2} e_j + (\psi_n)^{n-k+1} e_k; k = 1, ..., n-2. \qquad (2.9)$$

Also

$$\begin{aligned}\Phi\left(e_{n-1}\right) &= \sum_{j=1}^{n-2}\phi_j e_j + \left(\psi_n\right)^2 e_{n-1},\\ \Phi\left(e_n\right) &= \sum_{j=1}^{n-2}\psi_j e_j + \psi_n e_n.\end{aligned}$$

(xii) The derivations of N by considering automorphisms infinitesimally close to the identity.

The equation (2.9) is

$$\Phi\left(e_k\right) = \sum_{j=1}^{k-1}\left\{\left(\phi_{n-k+j-1}\psi_n - \psi_{n-k+j}\left(\psi_n\right)^2\right)\left(\psi_n\right)^{n-k-2}\right\} e_j + \left(\psi_n\right)^{n-k+1} e_k; k = 1, ..., n-2.$$

Near the identity the coefficients $\phi_{n-k+j-1} = 0, \psi_{n-k+j} = 0$ and $\psi_n = 1$. Thus

$$\begin{aligned}\Phi\left(e_k\right) &= e_k + D\left(e_k\right),\\ &= \sum_{j=1}^{k-1}\left\{\left(\delta\phi_{n-k+j-1}\left(1+\delta\psi_n\right) - \delta\psi_{n-k+j}\left(1+\delta\psi_n\right)^2\right)\left(1+\delta\psi_n\right)^{n-k-2}\right\} e_j\\ &\quad + \left(1+\delta\psi_n\right)^{n-k+1} e_k\\ &= \sum_{j=1}^{k-1}\left\{\left[\begin{array}{c}\delta\phi_{n-k+j-1} + \left(\delta\phi_{n-k+j-1}\right)\delta\psi_n -\\ \delta\psi_{n-k+j} - \left(\delta\psi_{n-k+j}\right)\delta\psi_n^2 - 2\left(\delta\psi_{n-k+j}\right)\delta\psi_n\end{array}\right]\left(1+(n-k-2)\,\delta\psi_n + ...\right)\right\} e_j\\ &\quad + \left(1 + (n-k+1)\,\delta\psi_n + ...\right) e_k\end{aligned}$$

where $\left(\delta\psi_n\right)^2$ and higher powers of $\delta\psi_n$ are neglected. The products of $\delta\phi_{n-k+j-1}, \delta\psi_{n-k+j}$ and $\delta\psi_n$ are also neglected. Then we have

$$\begin{aligned}e_k + D\left(e_k\right) &= \sum_{j=1}^{k-1}\left\{\left[\delta\phi_{n-k+j-1} - \delta\psi_{n-k+j}\right]\left(1+(n-k-2)\,\delta\psi_n\right)\right\} e_j\\ &\quad + e_k + (n-k+1)\,\delta\psi_n e_k,\end{aligned}$$

or

$$D\left(e_k\right) = \sum_{j=1}^{k-1}\left\{\left[\delta\phi_{n-k+j-1} - \delta\psi_{n-k+j}\right]\right\} e_j + (n-k+1)\,\delta\psi_n e_k,$$

or, renaming $\delta\psi_n = q_n$ and $\delta\phi_{n-k+j-1} - \delta\psi_{n-k+j} = o_j$, we found

$$D\left(e_k\right) = \sum_{j=1}^{k-1}\left(o_j\right) e_j + (n-k+1)\, q_n e_k.$$

Similarly for $\Phi(e_{n-1})$ and $\Phi(e_n)$, near the identity we have $\phi_j = 0, \psi_j = 0$ and $\psi_n = 1$. Thus

$$\begin{aligned}\Phi(e_{n-1}) &= e_{n-1} + D(e_{n-1}) \\ &= \sum_{j=1}^{n-2} \delta\phi_j e_j + (1+\delta\psi_n)^2 e_{n-1} \\ &= \sum_{j=1}^{n-2} \delta\phi_j e_j + \left\{(1 + 2\delta\psi_n + (\delta\psi_n)^2\right\} e_{n-1},\end{aligned}$$

or neglecting $(\delta\psi_n)^2$

$$D(e_{n-1}) = \sum_{j=1}^{n-1} \delta\phi_j e_j + 2\delta\psi_n e_{n-1}.$$

Renaming $\delta\phi_j = p_j$ and $\delta\psi_n = q_n$, we have

$$D(e_{n-1}) = \sum_{j=1}^{n-1} p_j e_j + 2q_n e_{n-1}.$$

For $\Phi(e_n)$

$$\begin{aligned}\Phi(e_n) &= e_n + D(e_n) \\ &= \sum_{j=1}^{n-2} \delta\psi_j e_j + (1+\delta\psi_n) e_n.\end{aligned}$$

Renaming $\delta\psi_j = q_j$ and $\delta\psi_n = q_n$, we arrive at

$$D(e_n) = \sum_{j=1}^{n-2} q_j e_j + q_n e_n.$$

Thus we have

$$D(e_k) = \sum_{j=1}^{k-1} (o_j) e_j + (n-k+1) q_n e_k, \quad \text{for } \ 1 \leq k \leq n-2. \tag{2.10 a}$$

Also

$$D(e_{n-1}) = \sum_{j=1}^{n-1} p_j e_j + 2q_n e_{n-1}, \tag{2.10 b}$$

and

$$D\left(e_n\right)=\sum_{j=1}^{n-2} q_j e_j+q_n e_n. \qquad (2.10\text{ c})$$

Here $o_j = p_{n-k+j-1} - q_{n-k+j}$ for $1 \leq j \leq k-1$. The only independent parameters are p_k, q_k, q_n with $1 \leq k \leq n-2$ and $n \in \mathbb{N}$ fixed (n is the dimension of N). Hence the algebra of derivations is $(2n-3)$ dimensional since an arbitrary derivation depends on $(2n-3)$ parameters p_k, q_k, q_n; $1 \leq k \leq n-2$.

2.2 All possible solvable extensions of N

For finding all solvable Lie algebras with nilradical N we need to find all non-equivalent nil-independent sets $\left\{D^1, ..., D^f\right\}$ of derivations of N. Since for example

$$\begin{aligned} D^1 &= ad\left(f_1\right), \\ D^1\left(e_k\right) &= \left[f_1, e_k\right], \\ D^1\left(f_a\right) &= \left[f_1, f_a\right]. \end{aligned}$$

If we know D^1 we have the required information about the additional base element. We need to find nil-independent outer derivations of N (since N is maximal nilpotent ideal).

Examining equations (2.10) we immediately see that there can be atmost one nil-independent derivation such that $q_n \neq 0$. If there would be more of them say D and $\tilde{D}$, then by taking a linear combination $\tilde{q}_n D - q_n \tilde{D}$ we obtain a nilpotent operator (namely one represented by a strictly upper triangular matrix). Since the set of derivations is independent of such linear combinations, we will contradict the maximality of N since then we obtain a set $M = (e_1, .., e_n, e_m), e_m \neq e_k, 1 \leq k \leq n$ which is nilpotent and $M \supset N$.

Therefore any solvable but not nilpotent Lie algebra with the nilradical N must be $n+1$ dimensional.

(i) By proper choice of multiple of D and adding suitable inner derivations we can transform D into the form

$$\begin{aligned} D\left(e_{k}\right) &= \sum_{j=1}^{k-1} o_{j} e_{j}+(n-k+1) e_{k} \text{ for } 1 \leq k \leq n-2, \\ D\left(e_{n-1}\right) &= \sum_{j=1}^{n-3} p_{j} e_{j}+2 e_{n-1}, \\ D\left(e_{n}\right) &= e_{n}. \end{aligned} \tag{2.11}$$

There are $n-1$ nontrivial inner derivations $ad\left(e_k\right)$; $2 \leq k \leq n$ (since $ad\left(e_1\right)=0$) and one choice of scaling so we are able to remove $n-1+1=n$ parameters in a non nilpotent outer derivation in equations (2.10). Using scaling we put $q_n=1$, then

$$\begin{aligned} D\left(e_{n-1}\right) \mid_{new} &= D\left(e_{n-1}\right)-p_{n-2} ad\left(e_{n}\right)\left(e_{n-1}\right) ; \; q_{n}=1 \\ &= \sum_{j=1}^{n-3} p_{j} e_{j}+2 e_{n-1}. \end{aligned}$$

Further

$$D\left(e_{k}\right) |_{new}= D\left(e_{k}\right)-\sum_{i=2}^{n-1} q_{n-k+j-1} ad\left(e_{i}\right)\left(e_{n}\right) ; \; q_{n}=1,$$

which gives

$$\begin{aligned} D\left(e_{n}\right) \mid_{new} &= D\left(e_{n}\right)-\sum_{i=2}^{n-1} q_{i-1} ad\left(e_{i}\right)\left(e_{n}\right) ; \; q_{n}=1 \\ &= e_{n}. \end{aligned}$$

There are still $n-3$ parameters p_j $1 \leq j \leq n-3$ remaining.

(ii) Next we perform a change of basis in N such that the Lie brackets in equation (2.3) are preserved. This can be done by using conjugate of the derivation D by a suitable automorphism Φ i.e.

$$D \rightarrow \tilde{D}=\Phi^{-1} \circ D \circ \Phi.$$

which will help diagonalize the action of D if possible and hence simplify it.

For our convenience, we perform this process in $n-3$ steps to eliminate $n-3$ parameters p_k, setting one parameter p_k equal to zero in each step. Thus our Φ here will be

$$\Phi = \Phi_{n-3} \circ \Phi_{n-4} \circ ... \circ \Phi_1.$$

We construct the automorphism Φ_k. Let us assume that for given $k \leq n-3$ we have $p_j = 0$ for all $k < j \leq n-2$.

From the form in equation (2.11) we construct an automorphism Φ_k :

$$\Phi_k\left(e_{n-1}\right) = \omega_k e_k + e_{n-1}, \Phi_k\left(e_n\right) = e_n,$$

where ω_k is to be determined. We have

$$\begin{aligned} D\left(\Phi_k\left(e_{n-1}\right)\right) &= D\left(e_{n-1}\right) + \omega_k D\left(e_k\right) \\ &= 2e_{n-1} + p_k e_k + (n-k+1)\omega_k e_k + \sum_{j=1}^{k-1} \omega_k(o_j)e_j \\ &= 2\left(e_{n-1} + \frac{1}{2}\left(p_k + (n-k+1)\omega_k\right)e_k\right) + \sum_{j=1}^{k-1} \omega_k(o_j)e_j. \end{aligned}$$

Thus we find that

$$D\left(\Phi_k\left(e_{n-1}\right)\right) = 2\Phi_k\left(e_{n-1}\right) + \sum_{j=1}^{k-1}(...)e_j,$$

precisely when $\omega_k = \frac{p_k}{n-k+1}$. By this choice of ω_k we can set $p_k = 0$ and proceed to the elimination of p_{k-1}. We are ultimately able to eliminate all p_k's using suitably chosen automorphisms Φ_k. Thus we calculated that:

up to addition of inner derivations, conjugation by automotphisms and rescaling, there exist just one nil-independent set of outer derivations, consisting of a unique element D

$$D\left(e_k\right) = (n-k+1)e_k, \ 1 \leq k \leq n.$$

2.3 Conclusion

The one nil-independent outer derivation corresponds to an additional base element other than the base elements of N. This additional element added to the basis of N gives the basis of the solvable extension S of N.

Chapter 3

Generalized Casimir invariants

The invariants of Lie algebras are one of their defining features. In particular, the polynomial invariants of a Lie algebra exhaust its set of Casimir operators, i.e. the center of its universal enveloping algebra. Since the structure of invariants strongly depends on the structure of the algebra and the classification of all (finite-dimensional) Lie algebras is an inherently difficult problem (actually unsolvable), it seems to be impossible to elaborate a complete theory for generalized Casimir operators in the general case [2]. In this chapter we calculate the Casimir invariant of N and its solvable extension S using a more convenient method as in [16] .

3.1 Definitions

Coadjoint representation

Let L be the Lie algebra of the Lie group G and $L^{\star}$ the dual space of L. Then the map $ad^{\star} : G \to gl\left(L^{\star}\right)$ defined for any $g \in G$ by the relation $\left\langle \left(ad^{\star}\left(g\right)\right) f, a\right\rangle = \left\langle f, ad\left(g^{-1}\right) a\right\rangle$ for all $f \in L^{\star}$ and $a \in L$ is called the coadjoint representation of the Lie group G.

Here $ad : G \to gl\left(L\right)$ is the usual adjoint representation of G in L and the image $ad\left(G\right)$ of G under ad is the inner automorphism group $Int\left(L\right)$ of the Lie algebra L. The image of G under $ad^{\star}$ is a subgroup of $gl\left(L^{\star}\right)$ and is denoted by $ad^{\star}\left(G\right)$ [1] .

Invariant

A function $I \in C^{\infty}\left(L^{\star}\right)$ is called an invariant of $ad^{\star}(G)$ if

$$I\left(\left(ad^{\star}(g)\right)f\right) = I(f) \text{ for all } g \in G, f \in L^{\star} \qquad [1].$$

Casimir invariants

Casimir invariants are the elements in the centre of the enveloping algebra of a Lie algebras L [16].

Generalized Casimir invariants

Casimir operators are in one-to-one correspondence with the polynomial invariants of the coadjoint representation of L. 'Non-polynomial invariants are called generalized Casimir invariants'.

For algebraic Lie algebras (Lie algebras that are isomorphic to the Lie algebra of an affine algebraic group), it is possible to choose a basis for all invariants of the coadjoint representation consisting entirely of polynomials (examples of algebraic Lie algebras are semisimple, perfect, nilpotent Lie algebras etc.) [16].

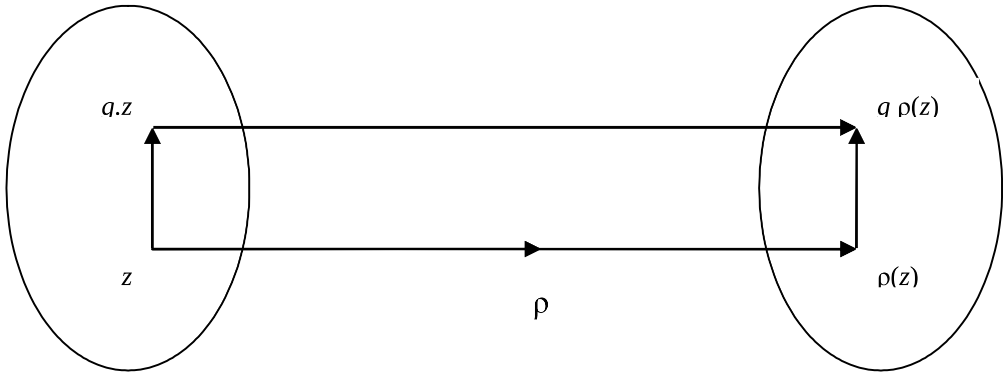
Flow

The non-zero Lie algebra elements $0 \neq \hat{v} \in L$ are in one-to-one correspondence with the connected one parameter (or one-dimensional subgroup) of G, a Lie group, identified as its Flow $\exp(t\hat{v})\,e$ through the identity e [11].

Moving Frames

A moving frame is a smooth G-equivariant map $\rho : M \to G$.i.e. $\rho(g.z) = g\ \rho(z)$, $g \in G, z \in M$. Here G is a group and M is the manifold on which it acts.

G is acting on M so $g.z$ is an element of M. Now ρ is a map from M to G so $\rho(g.z)$ is an element of G. Also $g\rho(z)$ is an element of G since it is the product of two elements of G. Thus the equation $\rho(g.z) = g\ \rho(z)$ is consistent as shown in the diagram.



3.2 Importance

Casimir invariants are of great use in Physics .They represent such important quantities as angular momentum elementary particle masses and spin and Hamiltonian of various physical systems etc [16] .

3.3 Methods of computation of invariants

There are two different methods which exist for the construction of invariants of group action and in particular of the coadjoint representation of a Lie group G.

3.3.1 The first method uses the differential operators

Consider a first order differential operator

$$\hat{X}_k = x_a c^a_{kb} \frac{\partial}{\partial x_b}. \tag{3.1}$$

This serves as a basis for the coadjoint representation of the Lie algebra L [16] . Here c^k_{ij} are the structure constants of the Lie algebra L in the basis

$(x_1, x_2, ..., x_l)$ where l is the dimension of L. In equation (3.1), x_a are commuting independent variables which are the coordinates in the basis of the space $L^\star$ dual to the basis $(x_1, x_2, ..., x_l)$.

The generalized Casimir invariants are solutions of the set of partial differential equations

$$\hat{X}_k I\,(x_1, ..., x_l) = 0\ ; 1 \leq k \leq l. \tag{3.2}$$

The method of characteristics may be used to solve the system of partial differential equations given in equation (3.2).

The number n_I of functionally independent invariants comes out to be equal to the number of functionally independent solutions of the system (3.2), [1].

i.e.

$$n_I = l - r, \tag{3.3}$$

where r is the generic rank of the antisymmetric matrix

$$C = \begin{bmatrix} 0 & \sum_{b=1}^{l} c_{12}^{b} x_b & \cdot\ \cdot\ \cdot & \cdot & \sum_{b=1}^{l} c_{1l}^{b} x_b \\ -\sum_{b=1}^{l} c_{12}^{b} x_b & 0 & & & \sum_{b=1}^{l} c_{2l}^{b} x_b \\ \cdot & & \cdot & & \cdot \\ \cdot & & \cdot & & \cdot \\ \cdot & & \cdot & & \cdot \\ -\sum_{b=1}^{l} c_{1,l-1}^{b} x_b & \cdot & \cdot\ \cdot\ \cdot & 0 & \sum_{b=1}^{l} c_{l-1,l}^{b} x_b \\ -\sum_{b=1}^{l} c_{1l}^{b} x_b & \cdot & \cdot\ \cdot\ \cdot & -\sum_{b=1}^{l} c_{l-1,l}^{b} x_b & 0 \end{bmatrix}.$$

The matrix C being antisymmetric has an even rank and thus n_I has the same parity as l.

3.3.2 The second method uses Moving Frames

We will use the recent formulation of this method as given in [10] by P Olver , to calculate the Casimir invariants of the coadjoint action of the nilpotent Lie algebra N and the solvable Lie algebra S. The method can be divided into the following steps:

(i) This step involves the integration of the coadjoint action of the Lie algebra L on its dual $L^\star$ as given by the vector fields in equation (3.1) to the local action of the group G.

We choose a convenient parameterization of G in terms of its one parameter subgroups;

$$g(\bar{\alpha}) = \exp(\alpha_N x_N) \ldots \exp(\alpha_2 x_2) \exp(\alpha_1 x_1) \in G, \tag{3.4}$$
$$\bar{\alpha} = (\alpha_1, \ldots, \alpha_l).$$

We then compose the flows $\Psi_{X_k}^{\alpha_k}$ of the vector fields $\hat{X}_k$ given in equation (3.1),

$$\frac{d\Psi_{X_k}^{\alpha_k}(p)}{d\alpha_k} = \hat{X}_k\left(\Psi_{\hat{X}_k}^{\alpha_k}(p)\right), p \in L^{\star}, \tag{3.5}$$

then

$$\Psi(g(\bar{\alpha})) = \Psi_{\hat{X}_N}^{\alpha_N} \circ \ldots \circ \Psi_{\hat{X}_2}^{\alpha_2} \circ \Psi_{\hat{X}_1}^{\alpha_1}. \tag{3.6}$$

Let $p \in L^{\star}$ with coordinates $x_k = x_k(p), \bar{x} = (x_1, \ldots, x_l)$ then the coordinates of the transformed point $\Psi(g(\bar{\alpha}))p$ are denoted by $\tilde{x}_k$ i.e.

$$\tilde{x}_k = \Psi_k(\bar{\alpha})(\bar{x}) = x_k(\Psi(g(\bar{\alpha}))p), \tag{3.7}$$

In the above $\tilde{x}_k$ is a function of both the group parameters $\bar{\alpha}$ and the coordinates $\bar{x}$ of the original point p.

(ii) In this step we choose a section $\sum$ through the orbits of the action Ψ

A single point on each of the orbits of the action of the group G is chosen in a smooth way. For this we find a subset of r coordinates say on which the group G acts transitively (group action is transitive if X is non-empty and for any x, y in X there exists $g \in G$ such that $g.x = y$). G acts transitively at least locally in an open neighborhood of chosen values $\left(x_{\pi(i)}^0\right)_{i=1}^r$ here $\pi : \{1, \ldots, r\} \to \{1, \ldots, l\}$ is an injection, and r is the rank of C.

Points whose coordinates satisfy

$$x_{\pi(i)} = x_{\pi(i)}^0, \tag{3.8}$$

form our desired section $\sum$ intersecting each generic orbit once.

(iii) Now we construct the invariants .

For $p \in L^{\star}$ given we find group elements transforming p into $\tilde{p} \in \sum$ by the action of Ψ. Here

$$\Psi_{\hat{X}_k}^{\alpha_k}(p) = \exp(\alpha_k x_k).$$

Then we express as many of their parameters as possible in terms of the original coordinates $\bar{x}$ and substitute their value in equation (3.7) so that we get $\tilde{x}_k$ as function of $\bar{x}$ only. Out of these $\left(\tilde{x}_{\pi(i)}\right)_{i=1}^{r}$ have the prescribed fixed values. The remaining $l-r$ functions $\tilde{x}_k$ are by construction invariant under the coadjoint action of G. These define the required invariants of the coadjoint representation.

We shall see that in our case, a lesser number of parameters appear. It is then not necessary to evaluate all the functions $\tilde{x}_k$ and a suitable choice of the basis in L can substantially simplify the whole procedure. In this case the other parameters can be ignored and are then specified by the remaining equations

$$\tilde{x}_i = x_i^0, l-r+r_0 \leq i \leq l.$$

But these do not enter into the expression for $\tilde{x}_k, 1 \leq k \leq l-r+r_0$ which define the invariants.

We shall see that our method may naturally lead to non-polynomial invariants but it is usually quite easy to construct polynomials out of them.

An Example

Here we use the method of moving frames as applied by Boyko, Patera and Popovych in [1] .

Consider the non-zero commutation relations:
$[e_1, e_4] = ae_1, \quad [e_2, e_4] = be_2 - e_3, \quad [e_3, e_4] = e_2 + be_3, a \geq 0, b \in \mathbb{R}.$
For this algebra

$$C = \begin{bmatrix} 0 & 0 & 0 & ae_1 \\ 0 & 0 & 0 & be_2 - e_3 \\ 0 & 0 & 0 & e_2 + be_3 \\ -ae_1 & -(be_2 - e_3) & -(e_2 + be_3) & 0 \end{bmatrix},$$

$$C \cong \begin{bmatrix} 0 & 0 & 0 & ae_1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -ae_1 & -(be_2 - e_3) & -(e_2 + be_3) & 0 \end{bmatrix}; R_2 - \frac{(be_2 - e_3)R_1}{ae_1}, \; R_3 - \frac{(e_2 + be_3)R_1}{ae_1}$$

According to equation (3.3) $n_I = 2$ i.e. the algebra has two functionally independent invariants.

The matrices of the adjoint representation $\hat{a}d_{e_i}$ of the basis elements e_1, e_2, e_3 and e_4 correspondingly have the form

$$\begin{bmatrix} 0 & 0 & 0 & a \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & b \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & b \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} -a & 0 & 0 & 0 \\ 0 & -b & -1 & 0 \\ 0 & 1 & b & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

The product of their exponentiation is the matrix of the inner automorphisms

$$\Pi_{i=1}^{4} \exp\left(-\theta_i \hat{a}d_{e_i}\right) = A\left(\theta\right) = \begin{bmatrix} e^{a\theta_4} & 0 & 0 & -a\theta_1 \\ 0 & e^{b\theta_4}\cos\theta_4 & e^{b\theta_4}\sin\theta_4 & -b\theta_2 - \theta_3 \\ 0 & -e^{b\theta_4}\sin\theta_4 & e^{b\theta_4}\cos\theta_4 & \theta_2 - b\theta_3 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Now from equation (3.7)

$$\tilde{x}_k = \sum_{l=1}^{4} \left(A\left(\theta\right)\right)_{kl} x_l.$$

Thus

$$\begin{aligned} \tilde{x}_1 &= x_1 e^{a\theta_4}, \\ \tilde{x}_2 &= e^{b\theta_4}\left(x_2\cos\theta_4 - x_3\sin\theta_4\right), \\ \tilde{x}_3 &= e^{b\theta_4}\left(x_2\sin\theta_4 + x_3\cos\theta_4\right), \\ \tilde{x}_4 &= -ax_1\theta_1 - x_2\left(b\theta_2 + \theta_3\right) + x_3\left(\theta_2 - b\theta_3\right). \end{aligned}$$

Combining the first three equations we get

$$\begin{aligned} \frac{\tilde{x}_1}{x_1} &= e^{a\theta_4}, \\ \left(\tilde{x}_2\right)^2 + \left(\tilde{x}_3\right)^2 &= e^{2b\theta_4}\left(\left(x_2\right)^2 + \left(x_3\right)^2\right), \\ \frac{\tilde{x}_3}{\tilde{x}_2} &= \tan\left(\arctan\frac{x_3}{x_2} + \theta_4\right). \end{aligned}$$

Thus two θ-free relations that we get are

$$\begin{aligned} \frac{\left(\tilde{x}_1\right)^b}{\left(\left(\tilde{x}_2\right)^2 + \left(\tilde{x}_3\right)^2\right)^a} &= \frac{\left(x_1\right)^b}{\left(\left(x_2\right)^2 + \left(x_3\right)^2\right)^a}, \\ \left(\tilde{x}_2\right)^2 + \left(\tilde{x}_3\right)^2 \exp\left(-2\arctan\frac{\tilde{x}_3}{\tilde{x}_2}\right) &= \left(x_2\right)^2 + \left(x_3\right)^2 \exp\left(-2\arctan\frac{x_3}{x_2}\right). \end{aligned}$$

Consequently we arrive at our final results i.e. the two invariants are [1]

$$\frac{(e_1)^b}{\left((e_2)^2+(e_3)^2\right)^a} and\, (e_2)^2+(e_3)^2\exp\left(-2\arctan\frac{e_3}{e_2}\right).$$

Invariants of Lie algebras with various additional structural restrictions have also been found. Some of these are the solvable Lie algebras with the nilradicals isomorphic to the Heisenberg algebras [13], with Abelian nilradical [9], with nilradicals containing Abelian ideals of codimension 1 [15], solvable triangular algebras [17], etc.

3.4 Casimir invariants of the Lie algebra *N*

We use the method of moving frames to calculate the Casimir invariants. Its main advantage is that it is purely algebraic. Unlike the conventional method, it eliminates the need of solving system of differential equations, replacing them by algebraic ones.

Using equation (3.1), the differential operators corresponding to the basis elements of N are given by

$$\begin{aligned} \hat{E}_1 &= 0, \hat{E}_2 = e_1\frac{\partial}{\partial e_n}, \hat{E}_k = e_{k-2}\frac{\partial}{\partial e_{n-1}} + e_{k-1}\frac{\partial}{\partial e_n}, \\ \hat{E}_{n-1} &= -\sum_{k=3}^{n-2} e_{k-2}\frac{\partial}{\partial e_k} + e_{n-2}\frac{\partial}{\partial e_n}, \hat{E}_n = -\sum_{k=2}^{n-1} e_{k-1}\frac{\partial}{\partial e_k}. \end{aligned} \tag{3.9}$$

The form of $\hat{E}_k, 1 \leq k \leq n$ implies that the invariants do not depend on e_{n-1}, e_n. Using equation (3.3) we find that the nilpotent Lie algebra N has $(n-4)$ functionally independent invariants. For N, we find the rank of the matrix C:

$$C=\begin{bmatrix}
0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & & 0 & 0 & e_1\\
0 & 0 & 0 & 0 & 0 & & 0 & e_1 & e_2\\
0 & 0 & 0 & 0 & 0 & & 0 & e_2 & e_3\\
. & . & & & & . & . & . & .\\
. & & . & & & . & . & . & .\\
. & & .. & & & . & . & . & .\\
0 & 0 & 0 & 0 & 0 & \cdots & 0 & e_{n-5} & e_{n-4}\\
0 & 0 & 0 & 0 & 0 & \cdots & 0 & e_{n-4} & e_{n-3}\\
0 & 0 & -e_1 & -e_2 & . & \cdots & -e_{n-4} & 0 & e_{n-2}\\
0 & -e_1 & -e_2 & -e_3 & . & \cdots & -e_{n-3} & -e_{n-2} & 0
\end{bmatrix}.$$

To find the rank we do the following steps: First $R_i-\frac{e_{i-1}}{e_1}R_2$ and then $R_i-\frac{e_{i-2}}{e_1}R_3$ for the i rows. We get

$$C\cong\begin{bmatrix}
0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & & 0 & 0 & e_1\\
0 & 0 & 0 & 0 & 0 & & 0 & e_1 & 0\\
0 & 0 & 0 & 0 & 0 & & & . & .\\
. & & & & . & & & . & .\\
. & & & & & . & & . & .\\
. & & & & & . & & 0 & 0\\
0 & 0 & 0 & 0 & & . & & 0 & 0\\
0 & 0 & 0 & 0 & & & . & 0 & 0\\
0 & 0 & -e_1 & -e_2 & . & \cdots & -e_{n-4} & 0 & e_{n-2}\\
0 & -e_1 & -e_2 & -e_3 & . & \cdots & -e_{n-3} & -e_{n-2} & 0
\end{bmatrix}.$$

i.e. rank of the above matrix is 4.

We have thus found that the nilpotent Lie algebra N has $n-4$ functionally independent invariants. It is rather complicated to directly solve the remaining two differential equations defining the invariants which are

$$\hat{E}_{n-1}I\left(e_1,e_2,..,e_{n-2}\right)=0,\ \ \hat{E}_nI\left(e_1,e_2,..,e_{n-2}\right)=0.$$

Therefore we will use the method of moving frames. We construct the flows of the vector fields $\hat{E}_{n-1},\hat{E}_n$ acting on the space spanned by $e_1,e_2,..,e_{n-2}$ only.

From [16] flows satisfy

$$\frac{d\Psi_{\hat{X}_k}^{\alpha_k}(p)}{d\alpha_k} = \hat{X}_k\left(\Psi_{\hat{X}_k}^{\alpha_k}(p)\right). \tag{3.10}$$

For $\hat{E}_{n-1}$

$$\begin{aligned} \hat{X}_k &= \hat{E}_{n-1}, \alpha_k = -\alpha_{n-1}, \\ \Psi_{\hat{E}_{n-1}}^{\alpha_{n-1}} &= \exp\left(-\alpha_{n-1}e_{n-1}\right), \end{aligned}$$

$$\begin{aligned} e_k\left(\Psi_{\hat{E}_{n-1}}^{\alpha_{n-1}}(p)\right) &= \left[e_k, \left(\begin{array}{c} 1-\frac{\alpha_{n-1}}{1!}e_{n-1}+\frac{(\alpha_{n-1})^2}{2!}(e_{n-1})^2 \\ -\frac{(\alpha_{n-1})^3}{3!}(e_{n-1})^3+... \end{array}\right)\right] \\ &= e_k - \frac{\alpha_{n-1}}{1!}e_{k-2} + \frac{(\alpha_{n-1})^2}{2!}e_{k-4} - \frac{(\alpha_{n-1})^3}{3!}e_{k-6}+... \end{aligned}$$

Now to check if it satisfies equation (3.10), we calculate

$$\begin{aligned} L.H.S &= \frac{d}{d\alpha_{n-1}}\left\{e_k - \frac{\alpha_{n-1}}{1!}e_{k-2} + \frac{(\alpha_{n-1})^2}{2!}e_{k-4} - \frac{(\alpha_{n-1})^3}{3!}e_{k-6}+...\right\} \\ &= 0 - \frac{e_{k-2}}{1!} + \frac{\alpha_{n-1}}{1!}e_{k-4} - \frac{(\alpha_{n-1})^2}{2!}e_{k-6}+... \end{aligned} \tag{3.11}$$

$$\begin{aligned} R.H.S &= \left[\begin{array}{c} \left\{-\sum_{k=3}^{n-2} e_{k-2}\frac{\partial}{\partial e_k} + e_{n-2}\frac{\partial}{\partial e_n}\right\}, (e_k - \frac{\alpha_{n-1}}{1!}e_{k-2} \\ +\frac{(\alpha_{n-1})^2}{2!}e_{k-4} - \frac{(\alpha_{n-1})^3}{3!}e_{k-6}+...) \end{array}\right] \\ &= -e_1 \\ &= -e_2 \\ &= -\left(e_3 - \frac{\alpha_{n-1}}{1!}e_1\right) \\ &= -\left(e_4 - \frac{\alpha_{n-1}}{1!}e_2\right) \\ &= -\left(e_5 - \frac{\alpha_{n-1}}{1!}e_3 + \frac{(\alpha_{n-1})^2}{2!}e_1\right) \\ & \quad . \\ & \quad . \\ & \quad . \\ &= -\frac{e_{k-2}}{1!} + \frac{\alpha_{n-1}}{1!}e_{k-4} - \frac{(\alpha_{n-1})^2}{2!}e_{k-6}+... \end{aligned} \tag{3.12}$$

From equation (3.11) and (3.12) we see that equation (3.10) is satisfied. Similarly for $\hat{E}_n$

$$\begin{aligned}
\left(\Psi^{\alpha_n}_{\hat{E}_n}(p)\right) &= \left(1-\frac{\alpha_n}{1!}e_n+\frac{(\alpha_n)^2}{2!}(e_n)^2-\frac{(\alpha_n)^3}{3!}(e_n)^3+...\right)\\
e_k\left(\exp\left(-\alpha_n e_n\right)\right) &= e_k-\frac{\alpha_n}{1!}\left[e_k,e_n\right]+\frac{(\alpha_n)^2}{2!}\left[\left[e_k,e_n\right],e_n\right]-...\\
&= e_k-\frac{\alpha_n}{1!}e_{k-1}+\frac{(\alpha_n)^2}{2!}e_{k-2}-\frac{(\alpha_n)^3}{3!}e_{k-3}+... \qquad (3.14)
\end{aligned}$$

Now to see if it satisfies equation (3.12), we calculate

$$\begin{aligned}
L.H.S &= \frac{d}{d\alpha_n}\left\{e_k-\frac{\alpha_n}{1!}e_{k-1}+\frac{(\alpha_n)^2}{2!}e_{k-2}-\frac{(\alpha_n)^3}{3!}e_{k-3}+...\right\} \qquad (3.15)\\
&= 0-e_{k-1}+\alpha_n e_{k-2}-\frac{(\alpha_n)^2}{2!}e_{k-3}+...
\end{aligned}$$

$$\begin{aligned}
R.H.S &= \left[\left(-\sum_{k=2}^{n-1}e_{k-1}\frac{\partial}{\partial e_k}\right),\left\{\begin{array}{c} e_k-\frac{\alpha_n}{1!}e_{k-1}+\frac{(\alpha_n)^2}{2!}e_{k-2}\\ -\frac{(\alpha_n)^3}{3!}e_{k-3}+...\end{array}\right\}\right] \qquad (3.16)\\
&= -e_1\\
&= (e_2-\alpha_n e_1)\\
&= -\left(e_3-\alpha_n e_2+\frac{(\alpha_n)^2}{2!}e_1\right)\\
&\;.\\
&\;.\\
&\;.\\
&= -e_{k-1}+\alpha_n e_{k-2}-\frac{(\alpha_n)^2}{2!}e_{k-3}+...
\end{aligned}$$

From equations (3.15) and (3.16) we see that (3.10) is satisfied. Next we have

$$\begin{aligned}
e_k\left(\Psi^{\alpha_{n-1}}_{\hat{E}_{n-1}}(p)\right) &= \sum_{j=0}^{\left[\frac{k-1}{2}\right]}\frac{(-1)^j}{j!}(\alpha_{n-1})^j e_{k-2j}(p),\, 1\le k\le n-2\\
e_k\left(\Psi^{\alpha_n}_{\hat{E}_n}(p)\right) &= \sum_{j=0}^{k-1}\frac{(-1)^j}{j!}(\alpha_n)^j e_{k-j}(p),\, 1\le k\le n-2
\end{aligned}$$

Combining these two expressions

$$\tilde{e}_k=\sum_{l=0}^{k-1}\sum_{m=0}^{\left[\frac{k-l-1}{2}\right]}\frac{(-1)^{l+m}}{l!m!}(\alpha_n)^l(\alpha_{n-1})^m e_{k-l-2m},\, 1\le k\le n-2, \qquad (3.17)$$

where

$$\begin{aligned}
\tilde{e}_k &= \left(e_k\left(\Psi^{\alpha_n}_{\hat{E}_n}(p)\right)\right)\left(\Psi^{\alpha_{n-1}}_{\hat{E}_{n-1}}\right)\\
&= \left(\sum_{l=0}^{k-1}\frac{(-1)^l}{l!}(\alpha_n)^l e_{k-l}\right)\left(\Psi^{\alpha_{n-1}}_{\hat{E}_{n-1}}\right)\\
&= \sum_{l=0}^{k-1}\sum_{m=0}^{\left[\frac{(k-l)-1}{2}\right]}\frac{(-1)^{l+m}}{l!m!}(\alpha_n)^l(\alpha_{n-1})^m e_{(k-l)-2m}, 1\le k\le n-2.
\end{aligned}$$

These $(n-2)$ functions involve only the group parameters α_{n-1}, α_n. These can be easily determined as follows. We choose them to satisfy

$$\begin{aligned}
0 &= \tilde{e}_2 = e_2-\alpha_n e_1, \qquad 0=\tilde{e}_3 = e_3-\alpha_n e_2+\frac{(\alpha_n)^2}{2!}e_1-\alpha_{n-1}e_1 \quad (3.18)\\
\alpha_n &= \frac{e_2}{e_1}, \qquad \alpha_{n-1}=\frac{\left(e_1e_3-\frac{1}{2}(e_2)^2\right)}{(e_1)^2} \quad \text{using value of } \alpha_n.
\end{aligned}$$

Putting these values in equation (3.17),

$$\tilde{e}_k=\sum_{l=0}^{k-1}\sum_{m=0}^{\left[\frac{k-l-1}{2}\right]}\frac{(-1)^{l+m}}{l!m!}\left(\frac{e_2}{e_1}\right)^l\left[\frac{\left(e_1e_3-\frac{1}{2}(e_2)^2\right)}{(e_1)^2}\right]^m e_{k-l-2m}, 1\le k\le n-2.$$

In order to get polynomial expression we multiply by e_1^{k-2} since in the above we have $\left(\frac{1}{e_1}\right)^{l+2m}e_1$ with the maximum negative $e_1^{-(l+k-l-1-1)}=e_1^{-(k-2)}$. Therefore we multiply by $e_1^{(k-2)}$ to find

$$e_1^{(k-2)}\tilde{e}_k=\sum_{l=0}^{k-1}\sum_{m=0}^{\left[\frac{k-l-1}{2}\right]}e_1^{(k-2)}\frac{(-1)^{l+m}}{l!m!}\left(\frac{e_2}{e_1}\right)^l\left[\frac{\left(e_1e_3-\frac{1}{2}(e_2)^2\right)}{(e_1)^2}\right]^m e_{k-l-2m}, 1\le k\le n-2.$$

Now we find the invariants. Here we take

$$\xi_0=\tilde{e}_1=\sum_{l=0}^{0}\sum_{m=0}^{0}\frac{(-1)^{l+m}}{l!m!}(\alpha_n)^l(\alpha_{n-1})^m e_{(k-l)-2m}, k=1,$$

i.e.

$$\xi_0=e_1.$$

We had chosen

$$\tilde{e}_2=0=\tilde{e}_3.$$

Now relabeling $k = j + 3$

$$\xi_j = e_1^{(j+1)}\tilde{e}_{j+3} = e_1^{(j+1)} \sum_{l=0}^{j+2} \sum_{m=0}^{\left[\frac{j-1}{2}+1\right]} \frac{(-1)^{l+m}}{l!m!} (\alpha_n)^l (\alpha_{n-1})^m e_{(j+3-l)-2m}, 1 \leq j \leq n-5 \quad ,$$

$$\text{where } \alpha_n = \frac{e_2}{e_1}, \qquad \alpha_{n-1} = \frac{\left(e_1 e_3 - \frac{1}{2}(e_2)^2\right)}{(e_1)^2}.$$

or

$$\begin{aligned}
\xi_j &= \sum_{l=0}^{j+2} \sum_{m=0}^{\left[\frac{j-1}{2}+1\right]} \frac{(-1)^{l+m}}{l!m!} e_1^{(j+1)} \left(\frac{e_2}{e_1}\right)^l \left[\frac{\left(e_1 e_3 - \frac{1}{2}(e_2)^2\right)}{(e_1)^2}\right]^m e_{(j+3-l)-2m} \\
&= \sum_{l=0}^{j+2} \sum_{m=0}^{\left[\frac{j-1}{2}+1\right]} \frac{(-1)^{l+m}}{l!m!} e_1^{j+1-l-2m} e_2^l \left(e_1 e_3 - \frac{1}{2}(e_2)^2\right)^m e_{(j+3-l)-2m}. \quad (3.19)
\end{aligned}$$

where $1 \leq j \leq n-5$. In equation (3.19) we have the $(n-5)$ Casimir Invariants of N.

3.5 The generalized Casimir invariants of the Lie algebra S

For this $(n+1)$dimensional solvable Lie algebra S, the operators $\hat{E}_i$ representing elements on the nilradical N will each contain an additional term involving a derivative with respect to f_1 i.e.

$$\begin{aligned}
\hat{E}_1 &= (n-k+1)e_1\frac{\partial}{\partial f_1}, \hat{E}_2 = e_1\frac{\partial}{\partial e_n} + (n-k+1)e_2\frac{\partial}{\partial f_1}, \\
\hat{E}_k &= e_{k-2}\frac{\partial}{\partial e_{n-1}} + e_{k-1}\frac{\partial}{\partial e_n} + (n-k+1)e_k\frac{\partial}{\partial f_1}, \\
\hat{E}_{n-1} &= -\sum_{k=3}^{n-2} e_{k-2}\frac{\partial}{\partial e_k} + e_{n-2}\frac{\partial}{\partial e_n} + (n-k+1)e_{n-1}\frac{\partial}{\partial f_1}, \\
\hat{E}_n &= -\sum_{k=2}^{n-1} e_{k-1}\frac{\partial}{\partial e_k} + (n-k+1)e_n\frac{\partial}{\partial f_1},
\end{aligned}$$

and there is one additional operator which is

$$\hat{F}_1 = -\sum_{k=1}^{n}(n-k+1)e_k\frac{\partial}{\partial e_k}.$$

Since $\hat{E}_1 = (n-k+1)e_1\frac{\partial}{\partial f_1}$ and any invariant must satisfy$\left(\hat{E}_1(I)=0\right)$ we see that the invariants cannot depend on f_1. Define

$$\hat{F}_{1T} = -\sum_{k=1}^{n-1}(n-k+1)e_k\frac{\partial}{\partial e_k};\quad \text{the truncated differential operator acting only on}\,(e_1,...,e_{n-2})\,.$$

We now employ the method of moving frames. Finding the flow of the vector fields $\hat{F}_{1T}$

$$\begin{aligned}
e_k\left(\Psi^{\alpha_{n+1}}_{\hat{F}_{1T}}(p)\right) &= \exp\left(-\alpha_{n+1}f_1\right)\\
&= e_k\left(1-\frac{\alpha_{n+1}}{1!}f_1+\frac{(\alpha_{n+1})^2}{2!}(f_1)^2-\frac{(\alpha_{n+1})^3}{3!}(f_1)^3+...\right)\\
&= e_k-\frac{\alpha_{n+1}}{1!}(n-k+1)e_k+\frac{(\alpha_{n+1})^2}{2!}(n-k+1)^2e_k-...\\
&= \exp\left(-(n-k+1)\alpha_{n+1}\right)e_k\left(p\right).
\end{aligned}$$

The full action of the group S on the space with coordinates $(e_1,...,e_{n-2})$ gives

$$\tilde{e}_k = \exp\left(-(n-k+1)\alpha_{n+1}\right)\sum_{l=0}^{k-1}\sum_{m=0}^{\left[\frac{k-l-1}{2}\right]}\frac{(-1)^{l+m}}{l!m!}(\alpha_n)^l(\alpha_{n-1})^m e_{k-l-2m},\, 1\leq k\leq n-2, \tag{3.20}$$

where

$$\tilde{e}_k = \left[\left(e_k\left(\Psi^{\alpha_n}_{\hat{E}_n}(p)\right)\right)\left(\Psi^{\alpha_{n-1}}_{\hat{E}_{n-1}}\right)\right]\left(\Psi^{\alpha_{n+1}}_{\hat{F}_{1T}}(p)\right).$$

We choose in the truncated space $\{(1,0,0,e_4,...,e_{n-2})\}$. Thus in addition to $\tilde{e}_2=0=\tilde{e}_3$ we have $\tilde{e}_1=1$. Then

$$1=\tilde{e}_1=\exp\left(-n\alpha_{n+1}\right)e_1 \text{ putting } k=1 \text{ in equation (3.20)},$$

and

$$\begin{aligned}
\exp\left(-n\alpha_{n+1}\right) &= \frac{1}{e_1},\\
\exp\left(-\alpha_{n+1}\right) &= \left(\frac{1}{e_1}\right)^{\frac{1}{n}}.
\end{aligned}$$

Substituting the above and using equation (3.19) in equation (3.20) we find invariants which can be expressed in the form

$$\begin{aligned}\tilde{e}_k &= \frac{\xi_{k-3}}{e_1^{\frac{jn+2n-j-2}{n}}}, 4 \leq k \leq n-2,\\ \exp\left(-(n-k+1)\alpha_{n+1}\right) &= \exp\left(-\alpha_{n+1}\right)^{(n-k+1)} = \left(\frac{1}{e_1}\right)^{\frac{n-k+1}{n}}.\end{aligned}$$

Putting $k = j+3; n-k+1 = n-j-2$

$$\exp\left(-(n-k+1)\alpha_{n+1}\right) = \left(\frac{1}{e_1}\right)^{\frac{n-j-2}{n}}.$$

Now from equation (3.19)

$$\begin{aligned}\xi_j &= \xi_{k-3} = e_1^{(j+1)} \sum_{l=0}^{j+2} \sum_{m=0}^{\left[\frac{j-l}{2}+1\right]} \frac{(-1)^{l+m}}{l!m!} \left(\alpha_n\right)^l \left(\alpha_{n-1}\right)^m e_{(j+3-l)-2m},\\ \xi_j &= e_1^{(j+1)} \tilde{e}_{j+3} = e_1^{(j+1)} \tilde{e}_k.\end{aligned}$$

Therefore we multiply and divide equation (3.20) by $e_1^{(j+1)}$ to use the value of the ξ_j given above i.e.

$$\begin{aligned}\tilde{e}_k &= \frac{e_1^{(j+1)}}{e_1^{(j+1)}} \left(\frac{1}{e_1}\right)^{\frac{n-j-2}{n}} \tilde{e}_k \mid_{previuos},\\ \tilde{e}_k &= \left(\frac{1}{e_1}\right)^{\frac{n-j-2}{n}+j+1} e_1^{(j+1)} \tilde{e}_k \mid_{previous}\\ &= \frac{\xi_{k-3}}{(e_1)^{\frac{jn+2n-j-2}{n}}}\\ &= \frac{\xi_j}{(e_1)^{\frac{jn+2n-j-2}{n}}}, \quad j = k-3,\end{aligned}$$

or

$$(\tilde{e}_k)^n = \frac{\left(\xi_j\right)^n}{(e_1)^{jn+2n-j-2}}, \quad j = k-3.$$

Since $\xi_0 = e_1$ we have

$$\chi_j = \frac{\left(\xi_j\right)^n}{\xi_0^{(n-1)(j+2)}}, 1 \leq j \leq n-5.$$

3.5.1 Conclusion

We have arrived at the result:

The $(n+1)$ dimensional solvable Lie algebra S (with Lie brackets given in equation (2.3)) has $(n-5)$ functionally independent invariants. They can be chosen in the form

$$\chi_j = \frac{\left(\xi_j\right)^n}{\xi_0^{(n-1)(j+2)}}, 1 \leq j \leq n-5.$$

These are rational in ξ_j and consequently in e_j.

Chapter 4

Extension

We discuss the case of the nilradical N of dimension n and degree of nilpotency $(n-1)$ with Lie brackets in an appropriate basis $(e_1, ..., e_n)$ given by

$$\begin{aligned}
[e_i, e_j] &= 0 \text{ for all } 1 \leq i, j \leq n-2, \qquad (4.1)\\
[e_1, e_{n-1}] &= 0 = [e_2, e_{n-1}],\\
[e_k, e_{n-1}] &= e_{k-2}, 3 \leq k \leq n-2,\\
[e_1, e_n] &= 0,\\
[e_k, e_n] &= e_{k-1}, 2 \leq k \leq n-1.
\end{aligned}$$

We observe that:

(i) Since $e_{k-1} = [e_k, e_n]$, all elements of N can be defined in terms of repeated commutations of e_{n-1}, e_n. Hence the knowledge of $\Phi(e_{n-1})$ and $\Phi(e_n)$ suffices to know all $\Phi(e_k)$ for $1 \leq k \leq n$, for the automorphism Φ .

(ii) The difference of 1 i.e. $[e_k, e_n] = e_{k-1}$ and the difference of 2 i.e. $[e_k, e_{n-1}] = e_{k-2}$ enables us to define a relation in adjoint representation which reduces the unknown parameters in the automorphism.

Hence the loss of information about any of the above commutation relations will not lead anywhere.

The case of a particular nilradical with Lie brackets

$$\begin{aligned}
[e_i, e_j] &= 0;\ 1 \leq i, j \leq n-1\\
[e_1, e_n] &= 0\\
[e_k, e_n] &= e_{k-1};\ 2 \leq k \leq n-1
\end{aligned}$$

is discussed in [15] .

4.1 A more general case.

Consider the nilpotent Lie algebra with Lie brackets in an appropriate basis given by:

$$\begin{aligned} [e_i, e_j] &= 0 \text{ for all } 1 \leq i, j \leq n-2, \\ [e_1, e_{n-1}] &= 0 = [e_2, e_{n-1}], \\ [e_k, e_{n-1}] &= e_{k-2}, 3 \leq k \leq n-2, \\ [e_1, e_n] &= 0, \\ [e_m, e_n] &= \text{unknown}, \\ [e_k, e_n] &= e_{k-1}, 2 \leq k \leq n-1, k \neq m, \end{aligned} \tag{4.2}$$

for **some** m: $2 \leq m \leq n-2$.

4.1.1 Solvable extension.

(i) Here we consider $[e_m, e_n]$ is unknown. But since we have a nilpotent Lie algebra N of degree of nilpotency $(n-1)$ we can take

$$[e_m, e_n] = \sum_{a=1}^{n-1} \zeta_{mn}^a e_a, a = 1, ..., n-1; a \neq m.$$

We will use ζ_a for ζ_{mn}^a. Here $a \neq m$ because if we include the term with $a = m$, then there will not exist any $k \in \mathbb{N}$ such that $[[e_m, e_n], ..., e_n] = 0$ where commutations with e_n are taken k times.

(ii) Let $\Phi(e_{n-1}) = \sum_{k=1}^{n-1} \phi_k e_k$ and $\Phi(e_n) = \sum_{k=1}^{n} \psi_k e_k$.

Now we find $ad|_{N^2} \Phi(e_{n-1})$. Indeed

$$\begin{aligned} [e_k, \Phi(e_{n-1})] &= \left[e_k, \sum_{j=1}^{n-1} \phi_j e_j\right]; \quad 3 \leq k \leq n-2; \; k \neq m \\ &= [e_k, \phi_1 e_1] + [e_k, \phi_2 e_2] + ... + [e_k, \phi_{n-1} e_{n-1}] \\ &= \phi_1 [e_k, e_1] + \phi_2 [e_k, e_2] + ... + \phi_{n-1} [e_k, e_{n-1}] \\ &= \phi_{n-1} e_{k-2}, \end{aligned}$$

or

$$[\Phi(e_{n-1}), e_k] = -\phi_{n-1} e_{k-2},$$

which gives

$$ad\,|_{N^2}\left(\Phi\left(e_{n-1}\right)\right)=\begin{bmatrix} 0 & 0 & -\phi_{n-1} & 0 & 0 & . & . & . & 0 \\ 0 & 0 & 0 & -\phi_{n-1} & 0 & & & & 0 \\ 0 & 0 & 0 & 0 & -\phi_{n-1} & & & & 0 \\ . & & . & & & . & & & \\ . & & & . & & & . & & \\ . & & & & . & & & . & \\ 0 & 0 & 0 & 0 & 0 & . & . & . & -\phi_{n-1} \\ 0 & 0 & 0 & 0 & 0 & & & & 0 \\ 0 & 0 & 0 & 0 & 0 & . & . & . & 0 \end{bmatrix}. \quad (4.3)$$

Similarly to determine $ad\,|_{N^2}\left(\Phi\left(e_n\right)\right)$, we have

$$\begin{aligned} \left[\Phi\left(e_n\right), e_k\right] &= -\left[e_k, \sum_{k=1}^{n}\psi_k e_k\right]; \quad 3\leq k\leq n-1; \quad k\neq m \\ &= -\left\{\left[e_k,\psi_1 e_1\right]+\left[e_k,\psi_2 e_2\right]+...+\left[e_k,\psi_{n-1}e_{n-1}\right]+\left[e_k,\psi_n e_n\right]\right\} \\ &= -\left\{0+0+...+\Psi_{n-1}e_{k-2}+\Psi_n e_{k-1}\right\} \\ &= -\Psi_{n-1}e_{k-2}-\Psi_n e_{k-1}. \end{aligned}$$

Thus

$$ad\,|_{N^2}\,\Phi\left(e_n\right)=\begin{bmatrix} 0 & -\psi_n & -\psi_{n-1} & 0 & 0 & . & . & . & 0 \\ 0 & 0 & -\psi_n & -\psi_{n-1} & 0 & . & . & . & 0 \\ 0 & 0 & 0 & -\psi_n & -\psi_{n-1} & . & . & . & 0 \\ . & & . & & . & . & & & \\ . & & & . & & . & . & & \\ . & & & & . & & . & . & \\ 0 & 0 & 0 & 0 & 0 & & & -\psi_n & -\psi_{n-1} \\ 0 & 0 & 0 & 0 & 0 & & & 0 & -\psi_n \\ 0 & 0 & 0 & 0 & 0 & & & 0 & 0 \end{bmatrix}. \quad (4.4)$$

In the above we have $1\leq k\leq n-2, k\neq m$. Using equation (2.7) in this case with $k\neq m$ we get the same results i.e. $\phi_{n-1}=\left(\psi_n\right)^2, \psi_{n-1}=0$.

(iii) We calculate $\Phi\left(e_k\right)$ using the relation

$$e_{n-2}=\left[e_{n-1},e_n\right],$$

which implies

$$\Phi\left(e_{n-2}\right)=\Phi\left[e_{n-1},e_{n}\right]=\left[\Phi\left(e_{n-1}\right),\Phi\left(e_{n}\right)\right].$$

After calculation we get the generalized form for $\Phi\left(e_{k}\right)$ where $m\leq k\leq n-2$, given by

$$\Phi\left(e_{k}\right)=\sum_{j=1}^{k-1}\left(\phi_{n-k+j-1}\psi_{n}-\psi_{n-k+j}\left(\psi_{n}\right)^{2}\right)\left(\psi_{n}\right)^{n-k-2}e_{j}+\left(\psi_{n}\right)^{n-k+1}e_{k}.$$

(iv) Now we do not know $\Phi\left(e_{m-1}\right)$

Here

$$\begin{aligned}\left[e_{m},e_{n}\right] &= \sum_{a=1}^{n-1}\zeta_{a}e_{a} \qquad ;a\neq m,\\ \left[\Phi\left(e_{m}\right),\Phi\left(e_{n}\right)\right] &= \sum_{a=1}^{n-1}\zeta_{a}\Phi\left(e_{a}\right),\end{aligned}$$

which gives

$$\left(\begin{array}{c}\zeta_{1}\Phi\left(e_{1}\right)+\zeta_{2}\Phi\left(e_{2}\right)+...+\zeta_{m-1}\Phi\left(e_{m-1}\right)\\ +\underline{\zeta_{m+1}\Phi\left(e_{m+1}\right)+...+\zeta_{n-1}\Phi\left(e_{n-1}\right)}\end{array}\right)=\underline{\left[\Phi\left(e_{m}\right),\Phi\left(e_{n}\right)\right]}. \tag{4.5}$$

Here the underlined are the parts which are known. If we are able to find $\Phi\left(e_{m-1}\right)$ we can find all $\Phi\left(e_{m-2}\right),...,\Phi\left(e_{1}\right)$ since

$$\begin{aligned}\Phi\left(e_{m-2}\right) &= \left[\Phi\left(e_{m-1}\right),\Phi\left(e_{n}\right)\right],\\ &\;\;.\\ &\;\;.\\ &\;\;.\\ \Phi\left(e_{1}\right) &= \left[\Phi\left(e_{2}\right),\Phi\left(e_{n}\right)\right].\end{aligned}$$

Now

$$\begin{aligned}\Phi\left(e_{m-1}\right) &= \left[\Phi\left(e_{m+1}\right),\Phi\left(e_{n-1}\right)\right]\\ &= \left[\begin{array}{c}\sum_{j=1}^{m}\left(\phi_{n-m+j-2}\psi_{n}-\psi_{n-m+j-1}\left(\psi_{n}\right)^{2}\right)\left(\psi_{n}\right)^{n-m-3}e_{j}+\left(\psi_{n}\right)^{n-m}e_{m+1},\sum_{j=1}^{n-2}\phi_{j}e_{j}\\ +\left(\psi_{n}\right)^{2}e_{n-1}\end{array}\right.\\ &= \left[\sum_{j=1}^{m}\left(\phi_{n-m+j-2}\psi_{n}-\psi_{n-m+j-1}\left(\psi_{n}\right)^{2}\right)\left(\psi_{n}\right)^{n-m-3}e_{j}+\left(\psi_{n}\right)^{n-m}e_{m+1},\left(\psi_{n}\right)^{2}e_{n-1}\right]\\ &= \sum_{j=3}^{m}\left(\phi_{n-m+j-2}\psi_{n}-\psi_{n-m+j-1}\left(\psi_{n}\right)^{2}\right)\left(\psi_{n}\right)^{n-m-1}e_{j}+\left(\psi_{n}\right)^{n-m+2}e_{m-1}.\end{aligned}$$

We take

$$\begin{aligned}\left(\phi_{n-m+i}\psi_n-\psi_{n-m+i+1}\left(\psi_n\right)^2\right)\left(\psi_n\right)^{n-m-1} &= \alpha_i, \quad 1\leq i\leq m-2,\\ \left(\psi_n\right)^{n-m+2} &= \alpha_{m-1}.\end{aligned}$$

Defining

$$\Phi\left(e_{m-1}\right)=\sum_i \alpha_i e_i, \quad 1\leq i\leq m-1 \quad \text{(automorphism)},$$

we have

$$\begin{aligned}
\Phi\left(e_{m-2}\right) &= \left[\sum_{i=1}^{m-1}\alpha_i e_i, \sum_{j=1}^{n-2}\psi_j e_j+\psi_n e_n\right];\\
&= \left[\sum_{i=1}^{m-1}\alpha_i e_i, \sum_{j=1}^{n-2}\psi_j e_j\right]+\left[\sum_{i=1}^{m-1}\alpha_i e_i, \psi_n e_n\right]\\
&= 0+\left[\alpha_1 e_1,\psi_n e_n\right]+\left[\alpha_2 e_2,\psi_n e_n\right]+...+\left[\alpha_{m-1}e_{m-1},\psi_n e_n\right]\\
&= 0+\alpha_2\psi_n e_1+\alpha_3\psi_n e_2+\alpha_4\psi_n e_3+...+\alpha_{m-1}\psi_n e_{m-2},\\
\Phi\left(e_{m-3}\right) &= \alpha_3\left(\psi_n\right)^2 e_1+\alpha_4\left(\psi_n\right)^2 e_2+\alpha_5\left(\psi_n\right)^2 e_3+...+\alpha_{m-1}\left(\psi_n\right)^2 e_{m-3},\\
\Phi\left(e_{m-4}\right) &= \alpha_4\left(\psi_n\right)^3 e_1+...+\alpha_{m-1}\left(\psi_n\right)^3 e_{m-4},\\
\Phi\left(e_{m-5}\right) &= \alpha_5\left(\psi_n\right)^4 e_1+...+\alpha_{m-1}\left(\psi_n\right)^4 e_{m-5},\\
& .\\
& .\\
& .\\
\Phi\left(e_2\right) &= \alpha_{m-2}\left(\psi_n\right)^{m-3} e_1+\alpha_{m-1}\left(\psi_n\right)^{m-3} e_2,\\
\Phi\left(e_1\right) &= \alpha_{m-1}\left(\psi_n\right)^{m-2} e_1.
\end{aligned}$$

Putting these values in equation (4.5)

$$\zeta_1\alpha_{m-1}(\psi_n)^{m-2}e_1+\zeta_2\left\{\alpha_{m-2}(\psi_n)^{m-3}e_1+\alpha_{m-1}(\psi_n)^{m-3}e_2\right\}$$
$$+\zeta_3\left\{\alpha_{m-3}(\psi_n)^{m-4}e_1+\alpha_{m-2}(\psi_n)^{m-4}e_2+\alpha_{m-1}(\psi_n)^{m-4}e_3\right\}+...$$
$$+\zeta_{m-4}\left\{\alpha_4(\psi_n)^3e_1+...+\alpha_{m-1}(\psi_n)^3e_{m-4}\right\}$$
$$+\zeta_{m-3}\left\{\alpha_3(\psi_n)^2e_1+\alpha_4(\psi_n)^2e_2+\alpha_5(\psi_n)^2e_3+...+\alpha_{m-1}(\psi_n)^2e_{m-3}\right\}$$
$$+\zeta_{m-2}\left\{\alpha_2\psi_ne_1+\alpha_3\psi_ne_2+\alpha_4\psi_ne_3+...+\alpha_{m-1}\psi_ne_{m-2}\right\}+\zeta_{m-1}\left\{\sum_{i=1}^{m-1}\alpha_ie_i\right\}$$
$$+\zeta_{m+1}\left\{\sum_{j=1}^{m}(...)e_j+(\psi_n)^{n-m}e_{m+1}\right\}+\zeta_{m+2}\left\{\sum_{j=1}^{m+1}(...)e_j+(\psi_n)^{n-m-1}e_{m+2}\right\}$$
$$+...+\zeta_{n-1}\left\{\sum_{j=1}^{n-2}(\phi_j)e_j+(\psi_n)^2e_{n-1}\right\}$$

$$\begin{aligned}
&= \left[\sum_{j=1}^{m-1}(...)e_j+(\psi_n)^{n-m+1}e_m,\sum_{j=1}^{n-2}\psi_je_j+\psi_ne_n\right]\\
&= \left[\sum_{j=1}^{m-1}(...)e_j,\psi_ne_n\right]+\left[(\psi_n)^{n-m+1}e_m,\psi_ne_n\right]\\
&= \psi_n\sum_{j=1}^{m-1}(...)e_{j-1}+(\psi_n)^{n-m+2}\zeta_ae_a; a=1,...n-1; a\neq m.
\end{aligned}$$

Here (...) is used for $\left(\phi_{n-k+j-1}\psi_n-\psi_{n-k+j}(\psi_n)^2\right)(\psi_n)^{n-k-2}$. Here k is the value in the summation from $j=1$ to $k-1$.

Now we equate the coefficients of the basis elements in e_i on both sides which gives the following $n-2$ equations;

For e_1

$$\begin{aligned}
&\left(\zeta_1\alpha_{m-1}(\psi_n)^{m-2}+\zeta_2\alpha_{m-2}(\psi_n)^{m-3}+\zeta_3\alpha_{m-3}(\psi_n)^{m-4}+...\right.\\
&\left.+\zeta_{m-4}\alpha_4(\psi_n)^3+\zeta_{m-3}\alpha_3(\psi_n)^2+\zeta_{m-2}\alpha_2\psi_n+\zeta_{m-1}\alpha_1\right)\\
&+\left\{\begin{array}{c}\zeta_{m+1}\left(\phi_{n-m-1}\psi_n-\psi_{n-m}\psi_n^2\right)(\psi_n)^{n-m-3}+\zeta_{m+2}\left(\phi_{n-m-2}\psi_n-\psi_{n-m-1}\psi_n^2\right)(\psi_n)^{n-m-4}\\ +\zeta_{m+3}\left(\phi_{n-m-3}\psi_n-\psi_{n-m-2}(\psi_n)^2\right)(\psi_n)^{n-m-5}+...+\zeta_{n-1}\phi_1\end{array}\right\}\\
= \;&\psi_n\left(\phi_{n-m+1}\psi_n-\psi_{n-m+2}(\psi_n)^2\right)(\psi_n)^{n-m-2}+(\psi_n)^{n-m+2}\zeta_1.
\end{aligned}$$

For e_2

$$\begin{aligned}
& \left(\zeta_2\alpha_{m-1}\left(\psi_n\right)^{m-3}+\zeta_3\alpha_{m-2}\left(\psi_n\right)^{m-4}+\zeta_4\alpha_{m-3}\left(\psi_n\right)^{m-5}+\ldots\right.\\
& \left.+\zeta_{m-4}\alpha_5\left(\psi_n\right)^3+\zeta_{m-3}\alpha_4\left(\psi_n\right)^2+\zeta_{m-2}\alpha_3\psi_n+\zeta_{m-1}\alpha_2\right)\\
& +\left\{\begin{array}{c}\zeta_{m+1}\left(\phi_{n-m}\psi_n-\psi_{n-m+1}\left(\psi_n\right)^2\right)\left(\psi_n\right)^{n-m-3}+\zeta_{m+2}\left(\phi_{n-m-1}\psi_n-\psi_{n-m}\psi_n^2\right)\left(\psi_n\right)^{n-m-4}\\
+\zeta_{m+3}\left(\phi_{n-m-2}\psi_n-\psi_{n-m-1}\left(\psi_n\right)^2\right)\left(\psi_n\right)^{n-m-5}+\ldots+\zeta_{n-1}\phi_2\end{array}\right\}\\
= & \ \psi_n\left(\phi_{n-m+2}\psi_n-\psi_{n-m+3}\left(\psi_n\right)^2\right)\left(\psi_n\right)^{n-m-2}+\left(\psi_n\right)^{n-m+2}\zeta_2.
\end{aligned}$$

For e_3

$$\begin{aligned}
& \left(\zeta_3\alpha_{m-1}\left(\psi_n\right)^{m-4}+\zeta_4\alpha_{m-2}\left(\psi_n\right)^{m-5}+\zeta_5\alpha_{m-3}\left(\psi_n\right)^{m-6}+\ldots\right.\\
& \left.+\zeta_{m-4}\alpha_6\left(\psi_n\right)^3+\zeta_{m-3}\alpha_5\left(\psi_n\right)^2+\zeta_{m-2}\alpha_4\psi_n+\zeta^{m-1}\alpha_3\right)\\
& +\left\{\begin{array}{c}\zeta_{m+1}\left(\phi_{n-m+1}\psi_n-\psi_{n-m+2}\psi_n^2\right)\left(\psi_n\right)^{n-m-3}+\zeta_{m+2}\left(\phi_{n-m}\psi_n-\psi_{n-m+1}\psi_n^2\right)\left(\psi_n\right)^{n-m-4}\\
+\zeta_{m+3}\left(\phi_{n-m-1}\psi_n-\psi_{n-m}\left(\psi_n\right)^2\right)\left(\psi_n\right)^{n-m-5}+\ldots+\zeta_{n-1}\phi_3\end{array}\right\}\\
= & \ \psi_n\left(\phi_{n-m+3}\psi_n-\psi_{n-m+4}\left(\psi_n\right)^2\right)\left(\psi_n\right)^{n-m-2}+\left(\psi_n\right)^{n-m+2}\zeta_3.
\end{aligned}$$

For e_4

$$\begin{aligned}
& \left(\zeta_4\alpha_{m-1}\left(\psi_n\right)^{m-5}+\zeta_5\alpha_{m-2}\left(\psi_n\right)^{m-6}+\zeta_6\alpha_{m-3}\left(\psi_n\right)^{m-7}+\ldots\right.\\
& \left.+\zeta_{m-4}\alpha_7\left(\psi_n\right)^3+\zeta_{m-3}\alpha_6\left(\psi_n\right)^2+\zeta_{m-2}\alpha_5\psi_n+\zeta_{m-1}\alpha_4\right)\\
& +\left\{\begin{array}{c}\zeta_{m+1}\left(\phi_{n-m+2}\psi_n-\psi_{n-m+3}\psi_n^2\right)\left(\psi_n\right)^{n-m-3}+\zeta_{m+2}\left(\phi_{n-m+1}\psi_n-\psi_{n-m+2}\psi_n^2\right)\left(\psi_n\right)^{n-m-4}\\
+\zeta_{m+3}\left(\phi_{n-m}\psi_n-\psi_{n-m+1}\left(\psi_n\right)^2\right)\left(\psi_n\right)^{n-m-5}+\ldots+\zeta_{n-1}\phi_4\end{array}\right.\\
= & \ \psi_n\left(\phi_{n-m+4}\psi_n-\psi_{n-m+5}\left(\psi_n\right)^2\right)\left(\psi_n\right)^{n-m-2}+\left(\psi_n\right)^{n-m+2}\zeta_4,
\end{aligned}$$

and so on, we get for e_{m-2}

$$\begin{aligned}
& \left(\zeta_{m-2}\alpha_{m-1}\psi_n+\zeta_{m-1}\alpha_{m-2}\right)+\\
& \left\{\begin{array}{c}\zeta_{m+1}\left(\phi_{n-4}\psi_n-\psi_{n-3}\left(\psi_n\right)^2\right)\left(\psi_n\right)^{n-m-3}+\zeta_{m+2}\left(\phi_{n-5}\psi_n-\psi_{n-4}\left(\psi_n\right)^2\right)\left(\psi_n\right)^{n-m-4}\\
+\ldots+\zeta_{n-1}\phi_{m-2}\end{array}\right\}\\
= & \ \psi_n\left(\phi_{n-2}\psi_n-\psi_{n-1}\left(\psi_n\right)^2\right)\left(\psi_n\right)^{n-m-2}+\left(\psi_n\right)^{n-m+2}\zeta_{m-2}.
\end{aligned}$$

For e_{m-1}

$$\begin{aligned}
& \zeta_{m-1}\alpha_{m-1}+\left\{\begin{array}{c}\zeta_{m+1}\left(\phi_{n-3}\psi_n-\psi_{n-2}\psi_n^2\right)\left(\psi_n\right)^{n-m-3}+\zeta_{m+2}\left(\phi_{n-4}\psi_n-\psi_{n-3}\psi_n^2\right)\left(\psi_n\right)^{n-m-4}\\
+\ldots+\zeta_{n-1}\phi_{m-1}\end{array}\right.\\
= & \ \zeta_{m-1}\left(\psi_n\right)^{n-m+2}.
\end{aligned}$$

For e_{m+1}

$$\begin{aligned} &\left\{ \begin{array}{l} \zeta_{m+1}(\psi_n)^{n-m-3} + \zeta_{m+2}\left(\phi_{n-2}\psi_n - \psi_{n-1}(\psi_n)^2\right)(\psi_n)^{n-m-4} \\ +\zeta_{m+3}\left(\phi_{n-3}\psi_n - \psi_{n-2}(\psi_n)^2\right)(\psi_n)^{n-m-5} + ... + \zeta_{n-1}\phi_{m+1} \end{array} \right\} \\ = \ & \zeta_{m+1}(\psi_n)^{n-m+2}. \end{aligned}$$

For e_{m+2}

$$\begin{aligned} &\left\{ \begin{array}{l} \zeta_{m+2}(\psi_n)^{n-m-1} + \zeta_{m+3}\left(\phi_{n-2}\psi_n - \psi_{n-1}(\psi_n)^2\right)(\psi_n)^{n-m-5} \\ +\zeta_{m+4}\left(\phi_{n-3}\psi_n - \psi_{n-2}(\psi_n)^2\right)(\psi_n)^{n-m-6} + ... + \zeta_{n-1}\phi_{m+2} \end{array} \right\} \\ = \ & \zeta_{m+2}(\psi_n)^{n-m+2}, \end{aligned}$$

and so on, we get for e_{n-2}

$$\zeta_{n-2}(\psi_n)^4 + \zeta_{n-1}\phi_{n-2} = \zeta_{n-2}(\psi_n)^{n-m+2}.$$

For e_{n-1}

$$\begin{aligned} \zeta_{n-1}(\psi_n)^2 &= \zeta_{n-1}(\psi_n)^{n-m+2}, \\ \zeta_{n-1}\left((\psi_n)^2 - (\psi_n)^{n-m+2}\right) &= 0, \\ \zeta_{n-1}(\psi_n)^2\left(1 - (\psi_n)^{n-m}\right) &= 0. \end{aligned}$$

Now here

$$\text{either } \zeta_{n-1} = 0 \ \text{ or } \ (\psi_n)^2 = 0 \text{ or } \left(1 - (\psi_n)^{n-m}\right) = 0, \tag{4.6}$$

We cannot take $\psi_n = 0$. For if we take $\psi_n = 0$ then in the particular case discussed in Chapter 2 with $\zeta_{m-1} = 1, \zeta_i = 0$; for all other i, we get

$$\begin{aligned} D(e_k) &= 0, \\ D(e_{n-1}) &= \sum_1^{n-2} c_j e_j, \\ D(e_n) &= \sum_1^{n-2} d_j e_j, \end{aligned}$$

which is not an outer nil-independent derivation. *This shows that we must have $\psi_n \neq 0$ to obtain the extension.*

We check $\left(1 - (\psi_n)^{n-m}\right) = 0$, this gives $m = n$ as $\psi_n \neq 0$ i.e. $[e_n, e_n]$ is the unknown quantity. But we know from the properties of the Lie algebra that it is equal to zero so this is not actually unknown. Now we are left with only one choice i.e. $\zeta_{n-1} = 0$. If we substitute this value in the equation for e_{n-2} we find $\zeta_{n-2} = 0$ for the same reason. If we go on in a similar fashion we arrive at

$$\zeta_{n-1} = \zeta_{n-2} = ... = \zeta_{m+1} = 0,$$

This also shows that this automorphism is also represented by a triangular matrix.

Now we have, from the equation for e_{m-1},

$$\begin{aligned} \zeta_{m-1}\alpha_{m-1} - \zeta_{m-1}(\psi_n)^{n-m+2} &= 0, \\ \zeta_{m-1}\left(\alpha_{m-1} - (\psi_n)^{n-m+2}\right) &= 0. \end{aligned}$$

If we take $\zeta_{m-1} \neq 0$ then $\left(\alpha_{m-1} - (\psi_n)^{n-m+3}\right) = 0$ which gives

$$\alpha_{m-1} = (\psi_n)^{n-m+2}.$$

From equation for e_{m-2} we have

$$\alpha_{m-2} = \frac{\psi_n\left(\phi_{n-2}\psi_n - \psi_{n-1}(\psi_n)^2 + \zeta_{m-2}\right)(\psi_n)^{n-m-2} - \zeta_{m-2}\alpha_{m-1}\psi_n}{\zeta_{m-1}}.$$

The equation for e_{m-3} gives

$$\alpha_{m-3} = \frac{\psi_n\left(\phi_{n-3}\psi_n - \psi_{n-2}(\psi_n)^2 + \zeta_{m-3}\right)(\psi_n)^{n-m-2} - \left(\zeta_{m-2}\alpha_{m-2}\psi_n + \zeta_{m-3}\alpha_{m-1}(\psi_n)^2\right)}{\zeta_{m-1}}.$$

Generalizing we find

$$\alpha_i = \frac{\left(\phi_{n-m+i}\psi_n - \psi_{n-m+i+1}\psi_n^2 + \zeta_i\right)(\psi_n)^{n-m-1} - \begin{pmatrix} \zeta_i\alpha_{m-1}(\psi_n)^{m-i-1} + \zeta_{i+1}\alpha_{m-2}(\psi_n)^{m-2-i} \\ +... + \zeta_{m-2}\alpha_{i+1}\psi_n \end{pmatrix}}{\zeta_{m-1}}.$$

If we take $m = k+1$ or $m-1 = k$ we find for $1 \leq i \leq m-2$.

$$\alpha_i = \frac{\left(\phi_{n-k+i-1}\psi_n - \psi_{n-k+i}(\psi_n)^2 + \zeta_i\right)(\psi_n)^{n-k-2} - \begin{pmatrix} \zeta_i\alpha_k(\psi_n)^{k-i} + \zeta_{i+1}\alpha_{k-1}(\psi_n)^{k-1-i} \\ +... + \zeta_{k-1}\alpha_{i+1}\psi_n \end{pmatrix}}{\zeta_k}.$$

Next we calculate the value for $\Phi(e_k)$

$$\begin{aligned}
\Phi\left(e_{m-1}\right) &= \sum_{i=1}^{m-2} \alpha_i e_i + \left(\psi_n\right)^{n-m+2} e_{m-1}, \\
\Phi\left(e_{m-2}\right) &= \left[\Phi\left(e_{m-1}\right), \Phi\left(e_n\right)\right] \\
&= \left[\alpha_i e_i + \left(\psi_n\right)^{n-m+2} e_{m-1}, \sum_{j=1}^{n-2} \psi_j e_j + \psi_n e_n\right] \\
&= \left[\alpha_i e_i, \psi_n e_n\right] + \left[\left(\psi_n\right)^{n-m+2} e_{m-1}, \psi_n e_n\right] \\
&= \psi_n \left(\alpha_i e_{i-1} + \left(\psi_n\right)^{n-m+2} e_{m-2}\right), \\
\Phi\left(e_{m-2}\right) &= \sum_{i=1}^{m-2} \psi_n \alpha_i e_{i-1} + \left(\psi_n\right)^{n-m+3} e_{m-2} \\
&= \sum_{i=2}^{m-2} \psi_n \alpha_i e_{i-1} + \left(\psi_n\right)^{n-m+3} e_{m-2} \\
&= \sum_{i=1}^{m-3} \psi_n \alpha_{i+1} e_i + \left(\psi_n\right)^{n-m+3} e_{m-2}.
\end{aligned}$$

Now

$$\psi_n \alpha_{i+1} = \frac{\begin{array}{c}\left(\phi_{n-m+i+1}\psi_n - \psi_{n-m+i+2}\left(\psi_n\right)^2 + \zeta_{i+1}\right)\left(\psi_n\right)^{n-m} \\ -\left(\zeta_{i+1}\alpha_{m-1}\left(\psi_n\right)^{m-i-1} + \zeta_{i+2}\alpha_{m-2}\left(\psi_n\right)^{m-2-i} + \ldots + \zeta_{m-3}\alpha_{i+2}\psi_n\right)\end{array}}{\zeta_{m-1}},$$

and

$$\begin{aligned}
\Phi\left(e_k\right) &= \sum_{j=1}^{k-1} \left(\frac{\left(\phi_{n-k+j-1}\psi_n - \psi_{n-k+j}\left(\psi_n\right)^2 + \zeta_j\right)\left(\psi_n\right)^{n-k-2}}{\zeta_{m-1}}\right) e_j + \left(\psi_n\right)^{n-k+1} e_k \\
&\quad + \sum_{j=1}^{k} \left(\frac{\zeta_j \alpha_k \left(\psi_n\right)^{k-j} + \alpha_{k-1}\zeta_{j+1}\left(\psi_n\right)^{k-1-j} + \ldots + \zeta_{k-1}\alpha_{j+1}\psi_n}{\zeta_{m-1}}\right) e_j,
\end{aligned}$$

which is a generalized case since we get back the case in equation (2.9) if we put $\zeta_{m-1} = 1$ and $\zeta_j = 0$ for all $1 \leq j \leq k-1$.

So we find

$$\begin{aligned}
\Phi(e_k) &= \sum_{j=1}^{k-1}\left(\phi_{n-k+j-1}\psi_n - \psi_{n-k+j}(\psi_n)^2\right)(\psi_n)^{n-k-2}e_j + (\psi_n)^{n-k+1}e_k; \quad 1 \le k \le m-1 \\
\Phi(e_k) &= \sum_{j=1}^{k-1}\left(\frac{\left(\phi_{n-k+j-1}\psi_n - \psi_{n-k+j}(\psi_n)^2 + \zeta_j\right)(\psi_n)^{n-k-2}}{\zeta_{m-1}}\right)e_j + (\psi_n)^{n-k+1}e_k \\
&\quad + \sum_{j=1}^{k}\left(\frac{\zeta_j\alpha_k(\psi_n)^{k-j} + \alpha_{k-1}\zeta_{j+1}(\psi_n)^{k-1-j} + \dots + \zeta_{k-1}\alpha_{j+1}\psi_n}{\zeta_{m-1}}\right)e_j, \quad m \le k \le n \\
\Phi(e_{n-1}) &= \sum_{j=1}^{n-2}\phi_j e_j + (\psi_n)^2 e_{n-1} \\
\Phi(e_n) &= \sum_{j=1}^{n-2}\psi_j e_j + \psi_n e_n.
\end{aligned}$$

(v) Now we consider the automorphism infinitesimally close to the identity. Near the identity, we take

$$\begin{aligned}
\psi_n &= 1 + \delta\psi_n \\
\phi_{n-k+j-1} &= 0 + \delta\phi_{n-k+j-1} \\
\psi_{n-k+j} &= 0 + \delta\psi_{n-k+j} \\
\alpha_i &= 0 + \delta\alpha_i \quad , \ j+1 \le i \le k.
\end{aligned}$$

Then we have

$$\begin{aligned}
\Phi(e_k) &= e_k + D(e_k) \\
&= \sum_{j=1}^{k-1}\left(\frac{\begin{pmatrix}\delta\phi_{n-k+j-1}(1+\delta\psi_n) \\ -\delta\psi_{n-k+j}(1+\delta\psi_n)^2 + \zeta_j\end{pmatrix}(1+\delta\psi_n)^{n-k-2}}{\zeta_k}\right)e_j + (1+\delta\psi_n)^{n-k+1}e_k \\
&\quad + \sum_{j=1}^{k}\left(\frac{\begin{matrix}\zeta_j\delta\alpha_k(1+\delta\psi_n)^{k-j} + \delta\alpha_{k-1}\zeta_{j+1}(1+\delta\psi_n)^{k-1-j} + \dots \\ +\zeta_{k-1}\delta\alpha_{j+1}(1+\delta\psi_n)\end{matrix}}{\zeta_k}\right)e_j.
\end{aligned}$$

Neglecting $(\delta\psi_n)^2$ and higher powers and all products of $\delta\psi_n, \delta\phi_{n-k+j-1}, \delta\psi_{n-k+j}$ and $\delta\alpha_i$, $(j+1 \le i \le k)$, we find

$$D(e_k) = \sum_{j=1}^{k-1} \frac{o_j}{\zeta_k} + (n-k+1) q_n e_k + \sum_{j=1}^{k} \left(\frac{\zeta_j \delta\alpha_k + \delta\alpha_{k-1}\zeta_{j+1} + ... + \zeta_{k-1}\delta\alpha_{j+1}}{\zeta_k} \right) e_j$$

Here the values of o_j and q_n are the same as in equation (2.10).

Now

$$\delta\alpha_i = \left(\delta\phi_{n-m+i}(1+\delta\psi_n) - \delta\psi_{n-m+i+1}(1+\delta\psi_n)^2\right)(1+\delta\Psi_n)^{n-m+3}$$

neglecting δ^2 and higher powers, we get

$$\begin{aligned} \delta\alpha_i &= \left(\delta\phi_{n-m+i} - \delta\psi_{n-m+i+1}\right)[1+(n-m-1)\delta\Psi_n] \\ \delta\alpha_i &= \left(\delta\phi_{n-m+i} - \delta\psi_{n-m+i+1}\right), \\ \delta\alpha_{m-2} &= 1+(n-m+2) q_n. \end{aligned}$$

After renaming $\delta\alpha_i = h_i$; $j+1 \le i \le k$ and $h_{m-2}=\delta\alpha_{m-2}$, we get

$$D(e_k) = \sum_{j=1}^{k-1} \frac{o_j}{\zeta_k} + (n-k+1) q_n e_k + \sum_{j=1}^{k} \left(\frac{h_k\zeta_j + h_{k-1}\zeta_{j+1} + ... + \zeta_{k-1}h_{j+1}}{\zeta_k} \right) e_j.$$

So that we have

$$\begin{aligned} D(e_k) &= \sum_{j=1}^{k-1} \frac{o_j}{\zeta_k} + \sum_{j=1}^{k} \left(\frac{h_k\zeta_j + h_{k-1}\zeta_{j+1} + ... + \zeta_{k-1}h_{j+1}}{\zeta_k} \right) e_j \quad (4.7) \\ &\quad + (n-k+1) q_n e_k, \\ D(e_{n-1}) &= \sum_{j=1}^{n-1} p_j e_j + 2q_n e_{n-1}, \\ D(e_n) &= \sum_{j=1}^{n-2} q_j e_j + q_n e_n. \end{aligned}$$

Here o_j, p_j, q_j and q_n are the same as in equation (2.10). Also $h_i = (p_{n-m+i} - q_{n-m+i+1})$; $j+1 \le i \le k$ and $h_{m-2} = 1+(n-m+2) q_n$. The only independent parameters are p_k, q_k and q_n where $1 \le k \le n-2$ and $n \in \mathbb{N}$ fixed. Hence the algebra of derivations is $2n-3$dimensional in this case also.

From equation (4.7) we observe that we have obtained an outer derivation which is equivalent to that in equation (2.10) since there is only a difference of an addition of a linear multiple of inner derivation. Because of this fact the extension in this case is the same as that obtained in section 2.2

4.1.2 Casimir invariants.

Since $[e_m, e_n] = \sum_a \zeta_a e_a, a = 1, ..., n-1; a \neq m$ and $[e_m, e_{n-1}] = e_{m-2}$ therefore $\hat{E}_m = e_{m-2}\frac{\partial}{\partial e_{n-1}} + e_a\zeta_a\frac{\partial}{\partial e_n}$ and $\hat{E}_n = -\sum_{\substack{k=2 \\ k\neq m}}^{n-1} e_{k-1}\frac{\partial}{\partial e_k} + \zeta_a e_a\frac{\partial}{\partial e_m}$ is the only information that we get different from the case discussed in equation (3.10). But this does not affect the results since we still get the same two differential equations defining the invariants i.e.

$$\hat{E}_{n-1}I\left(e_1, e_2, .., e_{n-2}\right) = 0, \hat{E}_n I\left(e_1, e_2, .., e_{n-2}\right) = 0.$$

We also calculate the rank of the matrix C to find the number of functionally independent invariants.

To calculate the rank consider

$$\begin{bmatrix}
0 & 0 & 0 & 0 & 0 & . & . & . & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & & & & 0 & 0 & e_1 \\
0 & 0 & 0 & 0 & 0 & & & & 0 & e_1 & e_2 \\
0 & 0 & 0 & 0 & 0 & & & & 0 & e_2 & e_3 \\
. & . & & & & . & & & . & . & . \\
. & & . & & & & . & & . & . & \sum_{\substack{a=1 \\ a\neq m}}^{n-1}\zeta_a e_a \\
. & & & .. & & & & . & . & . & . \\
0 & 0 & 0 & 0 & 0 & . & . & . & 0 & e_{n-5} & e_{n-4} \\
0 & 0 & 0 & 0 & 0 & . & . & . & 0 & e_{n-4} & e_{n-3} \\
0 & 0 & -e_1 & -e_2 & . & . & . & . & -e_{n-4} & 0 & e_{n-2} \\
0 & -e_1 & -e_2 & -e_3 & . & -\sum_{\substack{a=1 \\ a\neq m}}^{n-1}\zeta_a e_a & . & . & -e_{n-3} & -e_{n-2} & 0
\end{bmatrix}.$$

To find the rank we do the following steps: First $R_i - \frac{e_{i-1}}{e_1}R_2$ and then $R_i - \frac{e_{i-2}}{e_1}R_3$ for the i rows but for $i = m$ we apply $R_m - \frac{\left(\sum_{\substack{a=1 \\ a\neq m}}^{n-1}\zeta_a e_a\right)}{e_1}R_2$ and then $R_m - \frac{e_{m-2}}{e_1}R_3$.We get

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & & 0 & 0 & e_1 \\ 0 & 0 & 0 & 0 & 0 & & 0 & e_1 & 0 \\ 0 & 0 & 0 & 0 & 0 & & & . & . \\ . & & & & . & & & . & . \\ . & & & & & . & & . & . \\ . & & & & & & . & 0 & 0 \\ 0 & 0 & 0 & 0 & & & . & 0 & 0 \\ 0 & 0 & 0 & 0 & & & . & 0 & 0 \\ 0 & 0 & -e_1 & -e_2 & \cdots & & -e_{n-4} & 0 & e_{n-2} \\ 0 & -e_1 & -e_2 & -e_3 & \cdots & & -e_{n-3} & -e_{n-2} & 0 \end{bmatrix}.$$

So the rank is still 4.

Since the commutation of e_{n-1} with all the basis vectors is the same as in equation (2.3) the flow is the same as in equation(3.11)

Now we find the flow for $\hat{E}_n$.There are three cases;

(i) For $k < m$ we get the result of equation (3.14).

(ii) For $k = m$ we have

$$\begin{aligned} e_m\left(\exp\left(-\alpha_n e_n\right)\right) &= e_m - \frac{\alpha_n}{1!}\left[e_m, e_n\right] + \frac{(\alpha_n)^2}{2!}\left[\left[e_m, e_n\right], e_n\right] - ... \\ &= e_m - \frac{\alpha_n}{1!}\zeta_a e_a + \frac{(\alpha_n)^2}{2!}\zeta_a\left[e_a, e_n\right] - ... \\ &= e_m - \frac{\alpha_n}{1!}\zeta_a e_a + \frac{(\alpha_n)^2}{2!}\left\{\zeta_2 e_1 + \zeta_3 e_2 + \zeta_4 e_3 + ... + \zeta_{n-1}e_{n-2}\right\} \\ &\quad - \frac{(\alpha_n)^3}{3!}\left\{\zeta_3 e_1 + \zeta_4 e_2 + \zeta_5 e_3 + ... + \zeta_{n-1}e_{n-3}\right\} + ... + \frac{(\alpha_n)^{n-1}}{n-1!}\zeta_{n-1}e_1 \\ &= e_m + \sum_{i=1}^{n-1}\frac{(-1)^i(\alpha_n)^i}{i!}\sum_{\substack{j=i \\ j\neq m}}^{n-1}\zeta_j e_{j-i+1}. \end{aligned}$$

Here we see that if we put $j = m - 1$ and $\zeta_j = 1$ for $j = m - 1$ and $\zeta_j = 0$ for all $j \neq m - 1$ we get back equation (4.14) for $k = m$.

(iii) For $k > m$.

Let $l \in \mathbb{N}$, $1 \leq l \leq n - m$ and $k = m + l$. We give different values to l and then find a general answer.

For $l = 1$ i.e. $k = m + 1$

$$\begin{aligned}
e_{m+1}(\exp(-\alpha_n e_n)) &= e_{m+1} - \frac{\alpha_n}{1!}[e_{m+1}, e_n] + \frac{(\alpha_n)^2}{2!}[[e_{m+1}, e_n], e_n] - \ldots \\
&= e_{m+1} - \frac{\alpha_n}{1!}e_m + \frac{(\alpha_n)^2}{2!}[e_m, e_n] + \frac{(\alpha_n)^3}{3!}[[e_m, e_n], e_n] - \ldots \\
&= e_{m+1} - \frac{\alpha_n}{1!}e_m + \sum_{i=1}^{n-1}\frac{(-1)^{i+1}(\alpha_n)^{i+1}}{(i+1)!}\sum_{\substack{j=i \\ j\neq m}}^{n-1}\zeta_j e_{j-i+1}.
\end{aligned}$$

For $k = m + 2$

$$\begin{aligned}
e_{m+2}(\exp(-\alpha_n e_n)) &= e_{m+2} - \frac{\alpha_n}{1!}[e_{m+2}, e_n] + \frac{(\alpha_n)^2}{2!}[[e_{m+2}, e_n], e_n] - \ldots \\
&= e_{m+2} - \frac{\alpha_n}{1!}e_{m+1} + \frac{(\alpha_n)^2}{2!}[e_{m+1}, e_n] + \frac{(\alpha_n)^3}{3!}[[e_{m+1}, e_n], e_n] - \ldots \\
&= e_{m+2} - \frac{\alpha_n}{1!}e_{m+1} + \frac{(\alpha_n)^2}{2!}e_m + \frac{(\alpha_n)^3}{3!}[e_m, e_n] - \ldots \\
&= e_{m+2} - \frac{\alpha_n}{1!}e_{m+1} + \frac{(\alpha_n)^2}{2!}e_m + \sum_{i=1}^{n-1}\frac{(-1)^{i+2}(\alpha_n)^{i+2}}{(i+2)!}\sum_{j=i}^{n-1}\zeta_j e_{j-i+1}.
\end{aligned}$$

Generalizing, for $k = m + l$ we get

$$\begin{aligned}
e_{m+l}(\exp(-\alpha_n e_n)) &= e_{m+l} - \frac{\alpha_n}{1!}e_{m+l-1} + \frac{(\alpha_n)^2}{2!}e_{m+l-2} + \frac{(-1)^l(\alpha_n)^l}{l!}e_m \cdots \\
&\quad + \sum_{i=1}^{n-1}\frac{(-1)^{i+l}(\alpha_n)^{i+l}}{(i+l)!}\sum_{j=i}^{n-1}\zeta_j e_{j-i+1} \\
&= \sum_{r=0}^{l}\frac{(-1)^r(\alpha_n)^r}{r!}e_{m+l-r} + \sum_{i=1}^{n-1}\frac{(-1)^{i+l}(\alpha_n)^{i+l}}{(i+l)!}\sum_{\substack{j=i \\ j\neq m}}^{n-1}\zeta_j e_{j-i+1}.
\end{aligned}$$

Now we combine the expression for the flow of $\hat{E}_{n-1}$ and $\hat{E}_n$ to get the value for $\tilde{e}_k$. Here there are also three cases depending on the value of k. (Values of α_{n-1} and α_n remain the same as in equation(3.18)).

(i) For $k < m$ we get the same value as in equation (3.19).

(ii) For $k = m$,

$$\begin{aligned}
\tilde{e}_m &= \left(e_m + \sum_{i=1}^{n-1}\frac{(-1)^i(\alpha_n)^i}{i!}\sum_{j=i}^{n-1}\zeta_j e_{j-i+1}\right)\left(\Psi_{\hat{E}_{n-1}}^{\alpha_{n-1}}\right) \\
&= \sum_{j=0}^{\left[\frac{m-1}{2}\right]}\frac{(-1)^j(\alpha_{n-1})^j}{j!}e_{m-2j} + \sum_{i=1}^{n-1}\frac{(-1)^i(\alpha_n)^i}{i!}\sum_{\substack{j=i \\ j\neq m}}^{n-1}\zeta_j\sum_{r=0}^{\left[\frac{(j-i+1)-1}{2}\right]}\frac{(-1)^r(\alpha_{n-1})^r}{r!}e_{(j-i+1)-2r}.
\end{aligned}$$

To get polynomial invariant,

$$\xi_m = (e_1)^{n-2}\,\tilde{e}_m. \tag{4.8}$$

(iii) For $k > m$

$$\begin{aligned}
\tilde{e}_{m+l} &= \left(\sum_{r=0}^{l}\frac{(-1)^r(\alpha_n)^r}{r!}e_{m+l-r} + \sum_{i=1}^{n-1}\frac{(-1)^{i+l}(\alpha_n)^{i+l}}{(i+l)!}\sum_{\substack{j=i\\ j\neq m}}^{n-1}\zeta_j e_{j-i+1}\right)\left(\Psi_{\hat{E}_{n-1}}^{\alpha_{n-1}}\right)\\
&= \sum_{r=0}^{l}\sum_{j=0}^{\left[\frac{(m+l-r)-1}{2}\right]}\frac{(-1)^{r+j}(\alpha_n)^r(\alpha_{n-1})^j}{r!j!}e_{(m+l-r)-2j}\\
&\quad+\sum_{i=1}^{n-1}\frac{(-1)^{i+l}(\alpha_n)^{i+l}}{(i+l)!}\sum_{\substack{j=i\\ j\neq m}}^{n-1}\zeta_j\sum_{r=0}^{\left[\frac{(j-i+1)-1}{2}\right]}\frac{(-1)^r(\alpha_{n-1})^r}{r!}e_{(j-i+1)-2r}.
\end{aligned}$$

To get polynomial invariants,

$$\xi_{m+l} = (e_1)^{n+l-2}\,\tilde{e}_{m+l}. \tag{4.9}$$

These are the Casimir invariants.

For the **solvable extension** we have the same extra operator $\hat{F}_{1T} = -\sum_{k=1}^{n-1}(n-k+1)e_k\frac{\partial}{\partial e_k}$ as in section 3.5. Following the same pattern we obtain the generalized Casimir invariants of the solvable extension.

We employ the method of moving frames, finding the flow of the vector fields $\hat{F}_{1T}$

$$\begin{aligned}
e_k\left(\Psi_{\hat{F}_{1T}}^{\alpha_{n+1}}(p)\right) &= \exp\left(-\alpha_{n+1}f_1\right)\\
&= e_k\left(1-\frac{\alpha_{n+1}}{1!}f_1+\frac{(\alpha_{n+1})^2}{2!}(f_1)^2-\frac{(\alpha_{n+1})^3}{3!}(f_1)^3+...\right)\\
&= e_k-\frac{\alpha_{n+1}}{1!}(n-k+1)e_k+\frac{(\alpha_{n+1})^2}{2!}(n-k+1)^2e_k-...\\
&= \exp\left(-(n-k+1)\alpha_{n+1}\right)e_k(p).
\end{aligned}$$

The full action of the group on the space with coordinates $(e_1,...,e_{n-2})$ gives

$$\tilde{e}_k = \left(\exp\left(-(n-k+1)\alpha_{n+1}\right)\right)\left[\left(e_k\left(\Psi_{\hat{E}_n}^{\alpha_n}(p)\right)\right)\left(\Psi_{\hat{E}_{n-1}}^{\alpha_{n-1}}\right)\right]. \tag{4.10}$$

We choose in the truncated space $\{(1,0,0,e_4,...,e_{n-2})\}$ so that in addition to $\tilde{e}_2 = 0 = \tilde{e}_3$ we have $\tilde{e}_1 = 1$.

Thus

$$1 = \tilde{e}_1 = \exp\left(-n\alpha_{n+1}\right) e_1 \text{ putting } k = 1 \text{ in equation (3.20),}$$

and

$$\begin{aligned} \exp\left(-n\alpha_{n+1}\right) &= \frac{1}{e_1}, \\ \exp\left(-\alpha_{n+1}\right) &= \left(\frac{1}{e_1}\right)^{\frac{1}{n}}. \end{aligned}$$

Substituting the above and using equation (3.19),(4.8),(4.9) in equation (4.10) we find invariants which can be expressed in the form

$$\begin{aligned} \tilde{e}_k &= \frac{\xi_{k-3}}{(e_1)^{\frac{jn+2n-j-2}{n}}}; \quad 4 \le k \le m-1, \\ \tilde{e}_m &= \frac{\xi_m}{(e_1)^{\frac{n-m+1}{n}(n-2)}}, \\ \tilde{e}_{m+l} &= \frac{\xi_{m+l}}{(e_1)^{\frac{n-m-l+1}{n}(n+l-2)}}; \quad 1 \le l \le (n-2)-m, \end{aligned}$$

and rationalizing,

$$\begin{aligned} \chi_k &= \left(\tilde{e}_k\right)^n; \quad 4 \le k \le m-1, \\ \chi_k &= \left(\tilde{e}_k\right)^n; \qquad k = m, \\ \chi_{m+l} &= \left(\tilde{e}_{m+l}\right)^n; \quad l \in \mathbb{N}; \quad 1 \le l \le (n-2)-m, \end{aligned}$$

are the $(n-5)$ generalized Casimir invariants of the solvable extension.

4.2 Significance of our work

(i) We started with the limited knowledge in the sense that one of the brackets was unknown and tried to see if this lack of knowledge has any consequences i.e. can we still move forward and if yes then what are the results?

(ii) Here we had $[e_m, e_n]$ =unknown for **some** $m : 2 \le m \le n-2$.

(iii) Since we wanted N to be nilpotent we took $[e_m, e_n] = \zeta_a e_a, a = 1, ..., n-1; a \ne m$.

(iv) After calculations, we found out the formula for the automorphism $\Phi\left(e_k\right)$ to be

$$\begin{aligned}\Phi\left(e_k\right) &= \sum_{j=1}^{k-1}\left(\frac{\left(\phi_{n-k+j-1}\psi_n - \psi_{n-k+j}\left(\psi_n\right)^2 + \zeta_j\right)\left(\psi_n\right)^{n-k-2}}{\zeta_{m-1}}\right) e_j + \left(\psi_n\right)^{n-k+1} e_k \\ &\quad + \sum_{j=1}^{k}\left(\frac{\zeta_j\alpha_k\left(\psi_n\right)^{k-j} + \alpha_{k-1}\zeta_{j+1}\left(\psi_n\right)^{k-1-j} + \ldots + \zeta_{k-1}\alpha_{j+1}\psi_n}{\zeta_{m-1}}\right) e_j; \quad m \leq k \leq n\end{aligned}$$

(v) We see that the change is only in the coefficients of $e_j, 1 \leq j \leq k-1$ and not in the coefficients of e_k and also if we take $\zeta_{m-1} = 1$ and all other $\zeta' s$ equal to zero, we get back the result of equation (2.9) as a particular case.

(vi) Now if we find derivations using this automorphism close to identity, we see that, since in this more general case we have a change in the automorphism only in the coefficients of $e_j, 1 \leq j \leq k-1$ and not in the coefficients of e_k we obtain derivation similar to the one in the paper with a change only in the coefficients of $e_j, 1 \leq j \leq k-1$ for $D\left(e_k\right)$

(vii) Also adding an inner derivation gives a set of derivations which gives an isomorphic Lie algebra. This more general case shows that $\left[e_m, e_n\right]$ being unknown only changes the desired derivation set to give isomorphic Lie algebra. That is, following the procedure of expanding N to solvable Lie algebra we arrive at a solvable Lie algebra of order $n+1$ and $D\left(f\right)\left(e_k\right) = \left(n-k+1\right)e_k$ i.e. $\left[f, e_k\right] = \left(n-k+1\right)e_k$; diagonal action.

(viii) So we see that the lack of knowledge about one of the Lie brackets $\left[e_m, e_n\right]; 1 \leq m \leq n-2$, gives the same results up to isomorphism in the expansion of a nilpotent Lie algebra of order n and degree of nilpotency $n-1$. But the results are more generalized.

(ix) We also calculated the Casimir invariants in this case and again obtained more generalized results.

Appendix A

Two proofs used in the text

A.1 Connection between automorphism and derivation D when $D^n = 0$ for some $n \in \mathbb{N}$.

Let D be a nilpotent derivation of an arbitrary non-associative algebra L, such that $D^n = 0$. Consider the mapping

$$E = \exp D = 1 + D + \frac{D^2}{2!} + ... + \frac{D^{n-1}}{(n-1)!}.$$

This can also be written as

$$\begin{aligned} E &= 1 + Z, \\ Z &= D + \frac{D^2}{2!} + ... + \frac{D^{n-1}}{(n-1)!}. \end{aligned}$$

From $D^n = 0$, $Z^n = 0$ follows. Hence $E = 1 + Z$ has the inverse

$$1 - Z + Z^2 + ... + (-1)^{n-1} Z^{n-1}.$$

Thus E is one to one and onto L. Then

$$\begin{aligned} (xE)(yE) &= \left(\sum_{i=0}^{n-1} \frac{xD^i}{i!} \right) \left(\sum_{j=0}^{n-1} \frac{yD^j}{j!} \right) \\ &= \sum_{m=0}^{2n-2} \left(\sum_{i=0}^{m} \left(\frac{xD^i}{i!} \right) \left(\frac{yD^{m-i}}{(m-i)!} \right) \right) \\ &= \sum_{m=0}^{2n-2} (xy) \frac{D^m}{m!} \\ &= (xy) E. \end{aligned}$$

Hence E is an automorphism of L [8] .

Also it can be proved that an automorphism is related to a derivation by expanding the automorphism near the identity.

Let Φ be an automorphism of a Lie group G. Then

$$\Phi(xy) = \Phi(x)\Phi(y); \quad x, y \in G.$$

Expanding Φ near identity

$$\Phi = I + \epsilon D.$$

We get

$$(I + \epsilon D)(xy) = (I + \epsilon D)(x)(I + \epsilon D)(y),$$

by the definition of automorphism. Hence

$$I(xy) + \epsilon D(xy) = \{I(x) + \epsilon D(x)\}\{I(y) + \epsilon D(y)\},$$

or

$$(xy) + \epsilon D(xy) = \{x + \epsilon D(x)\}\{y + \epsilon D(y)\},$$

or

$$(xy) + \epsilon D(xy) = (x)(y) + x\epsilon D(y) + \epsilon D(x)y + \epsilon^2 D(x)D(y).$$

Neglecting ϵ^2 as being very small compared to ϵ, we get

$$(xy) + \epsilon D(xy) = (x)(y) + x\epsilon D(y) + \epsilon D(x)y.$$

$$D(xy) = xD(y) + D(x)y.$$

Thus D is a derivation.

Also we can prove the relation in a Lie algebra L as follows;

Let Φ be an automorphism of a Lie algebra i.e.

$$\Phi([x, y]) = [\Phi(x), \Phi(y)]; \quad x, y \in L.$$

Expanding Φ near identity

$$\Phi = I + \epsilon D.$$

We get

$$(I + \epsilon D)([x, y]) = [(I + \epsilon D)(x), (I + \epsilon D)(y)],$$

by the definition of automorphism. Hence

$$I([x, y]) + \epsilon D([x, y]) = [I(x) + \epsilon D(x), I(y) + \epsilon D(y)],$$

or

$$[x, y] + \epsilon D([x, y]) = [x + \epsilon D(x), (y) + \epsilon D(y)],$$

or

$$[x, y] + \epsilon D([x, y]) = [x, y] + \epsilon [x, D(y)] + \epsilon [D(x), y] + \epsilon^2 [D(x), D(y)].$$

Neglecting ϵ^2 as being very small, we get

$$[x, y] + \epsilon D([x, y]) = [x, y] + \epsilon [x, D(y)] + \epsilon [D(x), y],$$

$$D([x, y]) = [x, D(y)] + [D(x), y],$$

which is a derivation by definition.

A.2 The derived algebra of any finite dimensional solvable Lie algebra of characteristic zero is nilpotent and hence contained in the nilradical.

For a Lie algebra L of linear transformations in a finite dimensional vector space over a field of characteristic 0 with semisimple enveloping associative algebra $L^{\star}, L = L_1 \oplus C$, where C is the centre of L and L_1 is an ideal of L which is semisimple (as a Lie algebra) [8]. If R_1 is the radical of L and R_2 the radical of $L^{\star}$, then $L \cap R_2$, is the totality of nilpotent elements of R_1 and $[R_1, L] \subseteq R_2$ .Hence $[adL, adR_1]$ is contained in the radical of $\left(adL^{\star}\right)$. Also if we take $L = R_1$ we find that, $[adL, adL]$ is contained in the radical of $\left(adL^{\star}\right)$. This implies that there exists an integer n such that for any transformation of the form $[ada_i, adb_i]\,; a_i, b_i \in L$ we have

$$[ada_1, adb_1]\,[ada_2, adb_2] \ldots [ada_n, adb_n] = 0.$$

Hence

$$ad\,[a_1, b_1]\, ad\,[a_2, b_2] \ldots ad\,[a_n, b_n] = 0.$$

Thus for any $x \in L$,

$$[\ldots [x, [a_1, b_1]], [a_2, b_2], \ldots, [a_n, b_n]] = 0$$

This implies that $([L, L])^{n+1} = 0$, so $[L, L]$ is nilpotent. Since this is an ideal, $[L, L] \subseteq R$, the nilradical of L [8].