TRAVELING WAVE SOLUTIONS OF REACTION DIFFUSION EQUATIONS

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Dedicated to

MY BROTHER

SHAHEED MAHBOOB ALI KHAN

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ABSTRACT

Reaction-diffusion systems are mathematical models that describe how the concentration of one or more substances distributed in space changes under the influence of two processes namely chemical reactions and diffusion. In chemical reactions substances are converted into each other and diffusion causes the substances to spread out. The simplest system with no reaction part, is the heat equation. More complex systems can model different phenomena, like chemical reactions or ecological systems. For example, Fisher's linear model of advance of advantageous genes [1], Fitzhaugh's model of the propagation of voltage impulse through a nerve axon [2], etc are modeled by reaction-diffusion equations. Some of these systems show different behavior like traveling waves. The main focus of this dissertation is on the existence, uniqueness and approximation of traveling wave solutions of the reaction-diffusion system of the type

$$\begin{aligned} \frac{\partial S}{\partial t} &= \rho \frac{\partial^2 S}{\partial x^2} - \theta \frac{\partial S}{\partial x} - f(S)P, \\ \frac{\partial P}{\partial t} &= \frac{\partial^2 P}{\partial x^2} - \theta \frac{\partial P}{\partial x} + [f(S) - K]P, \end{aligned} \tag{0.0.1}$$

subject to the boundary conditions

$$\begin{aligned} &S(-\infty) = S^0,\ S(\infty) = S_0, \\ &P(-\infty) = 0,\ P(\infty) = 0. \end{aligned} \tag{0.0.2}$$

The system (0.0.1) has been used in references [3, 4] as a model for a single population microbial growth for a limiting nutrient in a flow reactor, where

$\theta\,(\geq 0)$ is the flow velocity, $S(x,t)$, $P(x,t)$ are the concentration of nutrient and microbial population in the reactor at position x and time t, $\rho > 0$ is the diffusion coefficient, $K > 0$ is the cell death rate and f is the nutrient uptake function (or birth rate) of bacteria cells.

This dissertation consists of three chapters. In **Chapter One,** some basic definitions and notations are presented. These consist of stable, unstable, center manifold, equilibrium points, reaction-diffusion equations, divergence theorem, Gronwall's inequality, traveling waves, Adomian's decomposition method and the homotopy perturbation method. **Chapter Two,** deals with existence of traveling wave solutions of the system (0.0.1) for any given positive constant ρ. Some properties of the solution are also discussed in this chapter. Uniqueness of traveling wave solution is proved under the monotone condition on f. In **Chapter Three,** Adomian's decomposition method, the homotopy perturbation method and the generalized approximation method are used to study approximation of traveling wave solutions of the system (0.0.1) on a finite interval. Existence results are also discussed by the method of upper and lower solutions. An example is included for comparison purposes. Graphs of solutions are also presented in the cases of the Adomian decomposition method and the homotopy perturbation method.

CONTENTS

1. **PRELIMINARIES**

1.1 Introduction

In this chapter, we recall some basic definitions and known results that are necessary for the existence, uniqueness and approximation of traveling wave solutions of the reaction-diffusion equations of the type

$$\begin{aligned} \frac{\partial S}{\partial t} &= \rho \frac{\partial^2 S}{\partial x^2} - \theta \frac{\partial S}{\partial x} - f(S)P, \\ \frac{\partial P}{\partial t} &= \frac{\partial^2 P}{\partial x^2} - \theta \frac{\partial P}{\partial x} + [f(S) - K]P. \end{aligned} \tag{1.1.1}$$

The system (1.1.1) is used as a mathematical model to study some problems in Biology. For example, the system (1.1.1) has been used in references [3, 4] as a model for a single population microbial growth for a limiting nutrient in a flow reactor, where $\theta\,(\geq 0)$ is the flow velocity, $S(x,t)$, $P(x,t)$ are the concentrations of nutrient and microbial population in the reactor at position x and time t, $\rho > 0$ is the diffusion coefficient, $K > 0$ is the cell death rate and f is the nutrient uptake function (or birth rate) of bacteria cells. For $f(S) = S$ and $\theta = 0$, the system (1.1.1) describes a simple diffusive epidemic model, in which $S(x,t)$ and $P(x,t)$ represent the densities of susceptible and infective population [5, 6].

Before discussing the system (1.1.1) in detail let us introduce some basic definitions and concepts.

1.2 Basic definitions and concepts

Definition 1.2.1. A system that describes the time evolution of variables known as states of the system is called a *dynamical system*.

For example, for the dynamical system representing a moving body the position $q(t)$ and velocity $\dot{q}(t)$ are state variables, where $\dot{q}$ is the time derivative of q. Dynamical systems are generally defined by differential equations, that describe the variations of different state variables. In this case

$$\dot{q} = f(q).$$

Definition 1.2.2. *Metabolism* is a set of chemical reactions that occur in living organisms in order to maintain life.

Metabolism allow organisms to grow, reproduce, maintain their structure and respond to their environment.

Definition 1.2.3. A substance taken from the environment that is used in an organism's metabolism is called a *nutrient*.

For example, carbohydrates, fats, proteins, vitamins, etc are nutrients.

Definition 1.2.4. *Diffusion* is a phenomenon by which matter, particle groups, population, etc., spread out according to individual's random motion.

Fick's law: The net movement of a diffusing substance is proportional to the concentration gradient. Mathematically,

$$\mathbf{J}(\mathbf{x}, t) = -D(\mathbf{x})\nabla_{\mathbf{x}} C(\mathbf{x}, t),$$

where $\mathbf{J}$ is the particle flux, $C(\mathbf{x}, t)$ is the concentration of the solute, $D(\mathbf{x})$ is diffusion coefficient.

Definition 1.2.5. An *equilibrium point* p for a system $\dot{\mathbf{X}} = f(\mathbf{X})$ is a point such that $f(p) = 0$.

Definition 1.2.6. A *trajectory* is a solution to a dynamical system passing through a specified initial condition.

Definition 1.2.7. An equilibrium point p for a system $\dot{\mathbf{X}} = f(\mathbf{X})$ is said to be *hyperbolic* if $Re(\lambda) \neq 0$ for all the eigenvalues λ of the linearized system (linearized about p). If $Re(\lambda) < 0$ for all eigenvalues λ of the linearized system, the point p is said be a *sink* and is stable because all trajectories approach it as t increases. If $Re(\lambda) > 0$, for all eigenvalues, it is said to be a *source* and all the trajectories move away from it as t increases. As such it is unstable. It is called a *saddle point* if it is neither a source nor a sink i.e, $Re(\lambda) < 0$ for at least one eigenvalue and $Re(\lambda) > 0$ for at least one eigenvalue.

Definition 1.2.8. A subset S of a dynamical system is called an *invariant set* if every trajectory that starts in S remains in S.

For example, a sink is an invariant set.

Definition 1.2.9. An invariant subset which is also a manifold (a separable, connected, Hausdorff space with a homeomorphism from each element of its open cover into $\mathbb{R}^n$) is called an *invariant sub-manifold.*

Definition 1.2.10. For a hyperbolic point p, a *stable manifold* W^s in the neighborhood u' of p is defined to be the set

$$W^s(p, u', f) = \{q \in u' : f^j(q) \in u' \text{ for } j > 0 \text{ and } d(f^j, p) \to 0 \text{ as } j \to \infty\}.$$

A *past history* of a point q is defined to be a sequence of points $\{q_{-j}\}_{j=0}^{\infty}$ such that $q_0 = q$ and $f(q_{-j-1}) = q_{-j}$ for $j \geq 0$.

Definition 1.2.11. The *unstable manifold* for a hyperbolic point p in a neighborhood $u'$of p is defined to be the following set

$$\begin{aligned} W^u(p, u', f) &= \{q \in u' : \text{there exists a past history } \{q_{-j}\}_{j=0}^{\infty} \subset u' \\ &\qquad \text{such that } d(q_{-j}, p) \to 0 \; as \; j \to \infty\}. \end{aligned}$$

Definition 1.2.12. An invariant manifold, which at a stationary point is tangential to the space spanned by the eigenvectors associated with imaginary eigenvalues, is called a *center manifold.*

Notice that for an $n \times n$ matrix associated with a linearized system of differential equations at a point p there are n eigenvalues. Stable manifolds correspond to eigenvalues with negative real parts, unstable manifolds correspond to eigenvalues with positive real parts and center manifolds correspond to eigenvalues with zero real parts.

For example, consider the system of differential equations [7],

$$\begin{aligned} \dot{x} &= \sigma(y - x) = f_1, \\ \dot{y} &= -y + rx - xz = f_2, \\ \dot{z} &= -bz + xy = f_3, \end{aligned} \tag{1.2.1}$$

where σ, r and b are positive constants. The point $P(0,0,0)$ is an equilibrium point of the system (1.2.1). The linearized system about the point $P(0,0,0)$ is given by

$$\begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{pmatrix} = \begin{pmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_2}{\partial y} & \frac{\partial f_1}{\partial z} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} & \frac{\partial f_2}{\partial z} \\ \frac{\partial f_3}{\partial x} & \frac{\partial f_3}{\partial y} & \frac{\partial f_3}{\partial z} \end{pmatrix}_{(0,0,0)} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \sigma(y - x) \\ rx - y \\ -bz \end{pmatrix} \tag{1.2.2}$$

and the matrix associated with this linearized system at $P(0,0,0)$ is

$$M = \begin{pmatrix} -\sigma & \sigma & 0 \\ r & -1 & 0 \\ 0 & 0 & -b \end{pmatrix}.$$

If $0 < r < 1$, then all the eigenvalues of M are negative, so the stable manifold of the linear system is the whole space and the unstable manifold is the empty set. If $r > 1$, then M has two negative eigenvalues and one positive eigenvalue, so the unstable manifold of the linear system is a line and the stable manifold is a plane. If $r = 1$, then M has one zero eigenvalue. Hence the linearized system (1.2.2) has a one dimensional center manifold.

Theorem 1.2.13. *(Gauss's divergence theorem) Let Ω be a bounded open subset of $\mathbb{R}^n$ with boundary $\partial\Omega$. Let $\boldsymbol{X}$=(X_1, ..., X_n) be a vector field in $\mathbb{R}^n$. Let $\boldsymbol{n}$ be the unit outward-pointing normal of $\partial\Omega$. Then,*

$$\int_\Omega div\boldsymbol{X}\,dV = \int_{\partial\Omega} \boldsymbol{X}.\boldsymbol{n}\,dA,$$

where dV is the element of volume in $\mathbb{R}^n$ and dA is the element of surface area on $\partial\Omega$.

1.3 Reaction-diffusion equation

The diffusion mechanism models the movement of many individuals in an environment. The individuals can be very small such as bacteria, molecules, cells, etc, or very large such as animals, plants, certain kind of events like epidemics, rumors, etc. The particles reside in a region, denoted by Ω. Assume that Ω is an open subset of $\mathbb{R}^n$. Let $P(\mathbf{x}, t)$ be a population density. Assume that $P(\mathbf{x}, t)$ is continuous and differentiable. The total population in any subregion $\mathcal{D}$ of Ω at time t is given by

$$\int_{\mathcal{D}} P(\mathbf{x}, t)dV. \tag{1.3.1}$$

The population density function $P(\mathbf{x}, t)$ changes as time evolves and as the location $\mathbf{x}$ varies. Population can change in two ways:

1. The individual particles can move around;
2. The number of individuals may change because of birth, death, hunting, etc.

Using Fick's law, it follows that

$$\mathbf{J}(\mathbf{x}, t) = -D(\mathbf{x})\nabla_{\mathbf{x}}P(\mathbf{x}, t), \tag{1.3.2}$$

where $\mathbf{J}$ is the flux of $P(\mathbf{x}, t)$ and $D(\mathbf{x})$ is the diffusion coefficient at $\mathbf{x}$. Assume that the rate of change of the density function $P(\mathbf{x}, t)$ due to the reasons stated in (2) is $f(t, \mathbf{x}, P)$. The rate of change of the total population in the region $\mathcal{D}$ is given by

$$\frac{\partial}{\partial t}\int_{\mathcal{D}} P(\mathbf{x}, t)dV. \tag{1.3.3}$$

The net growth of the population inside the region $\mathcal{D}$ is

$$\int_{\mathcal{D}} f(t, \mathbf{x}, P)dV \tag{1.3.4}$$

and the total out-flux is

$$\int_{\partial\mathcal{D}} \mathbf{J}(\mathbf{x}, t) \cdot \mathbf{n}\, dA, \tag{1.3.5}$$

where $\partial\mathcal{D}$ is the boundary of $\mathcal{D}$ and $\mathbf{n}$ is the outer normal direction at $\mathbf{x}$. According to the conservation law, the rate of change of the amount of material in a region is equal to the rate of flow across the boundary plus any amount of material that is created within the boundary. Mathematically

$$\frac{\partial}{\partial t}\int_{\mathcal{D}} P(\mathbf{x}, t)dV = -\int_{\partial\mathcal{D}} \mathbf{J}(\mathbf{x}, t).\mathbf{n}(\mathbf{x})dA + \int_{\mathcal{D}} f(t, \mathbf{x}, P)dV. \tag{1.3.6}$$

From the divergence theorem, it follows that

$$\int_{\partial\mathcal{D}} \mathbf{J}(\mathbf{x}, t).\mathbf{n}(\mathbf{x})dA = \int_{\mathcal{D}} div(\mathbf{J}(\mathbf{x}, t))dV. \tag{1.3.7}$$

Using (1.3.2), (1.3.6) and (1.3.7), it follows that

$$\int_{\mathcal{D}} [\frac{\partial P}{\partial t} - div(D(\mathbf{x})\nabla_{\mathbf{x}} P) - f(\mathbf{x}, t, P)]dV = 0. \tag{1.3.8}$$

Since the choice of the region $\mathcal{D}$ is arbitrary, hence

$$\frac{\partial P}{\partial t} = div(D(\mathbf{x})\nabla_{\mathbf{x}} P) + f(\mathbf{x}, t, P). \tag{1.3.9}$$

The equation (1.3.9) is known as the reaction-diffusion equation. In Eq.(1.3.9), the term $div(D(\mathbf{x})\nabla_{\mathbf{x}} P(\mathbf{x}, t))$ is the diffusion term which describes the movement of the individuals and $f(\mathbf{x}, t, P)$ is the reaction term which describes the birth-death or reaction occurring inside the habitat. The diffusion coefficient $D(\mathbf{x})$ also known as diffusivity, is a measure of how efficiently the particle disperse. When the region of the diffusion is approximately homogeneous, $D(\mathbf{x})$ is assumed to be constant. In such a case, (1.3.9) takes the form

$$\frac{\partial P}{\partial t} = D \triangle P + f(\mathbf{x}, t, P), \tag{1.3.10}$$

where $\triangle P = div(\nabla P)$ is the laplacian operator. When there is no reaction term, then (1.3.10) takes the form

$$\frac{\partial P}{\partial t} = D \triangle P, \tag{1.3.11}$$

which is also known as the heat equation.

If Ω is an unbounded domain, one require some initial conditions of the type

$$P(\mathbf{x}, 0) = P_0(\mathbf{x}), \quad \mathbf{x} \in \Omega.$$

If Ω is a bounded domain with boundary $\partial\Omega$, then some boundary conditions of the type

$$B(\mathbf{x}, t, u, \nabla u) = 0, \;\; \mathbf{x} \in \partial\Omega, \quad t > 0,$$

are required, where B is a boundary operator. For example:
1) Neumann or **no flux** boundary conditions:

$$\mathbf{n}.\nabla P = 0, \quad \mathbf{x} \in \partial\Omega, \quad t > 0, \tag{1.3.12}$$

where $\mathbf{n}$ is the outer normal to Ω at $\mathbf{x}\in \partial\Omega$. For equations like the one above this means that there can be no flux of particles either into or out of the domain Ω.
2) Dirichlet boundary conditions:

$$P = b(\mathbf{x}, t), \quad \mathbf{x} \in \partial\Omega, \quad t > 0, \tag{1.3.13}$$

where b is some prescribed function. These conditions are called homogeneous provided $b = 0$.
3) Robin or **mixed** boundary conditions:
Conditions of the type

$$\alpha(\mathbf{x}, t)P + \beta(\mathbf{x}, t)\mathbf{n}.\nabla P = b(\mathbf{x}, t), \quad \mathbf{x} \in \partial\Omega, \quad t > 0, \tag{1.3.14}$$

where $\mathbf{n}$ and b are as defined in **(1)** and **(2)** and α, β are given functions are known as Robin or mixed boundary conditions.

One might also impose some combination of **(1)**-**(3)** on separate parts of the boundary.

1.4 Traveling waves

The theory of traveling wave solutions of partial differential equations begins with the mathematical work by Kolmogorov, Petroski and Piskunov, see reference [8]. A traveling wave is any kind of wave which propagates with negligible change in its shape. Mathematically, if the function $u(x, t)$ is of the type

$$u(x, t) = u(x - ct) = u(z), \qquad z = x - ct, \tag{1.4.1}$$

then $u(x,t)$ is a traveling wave, which travels with a speed c. The variable z is also known as the wave variable. One of the important properties of nonlinear reaction diffusion systems is their ability to support traveling waves.

1.5 Existence of traveling wave solutions

A number of approaches are available in the literature to prove the existence of traveling wave solutions of reaction diffusion equations [8]. The most commonly used method is reduction of the system of equations of second order to a system of first order ordinary equations and then analyzing the trajectories of the reduced system. By substituting $u(x,t) = u(z)$, where $z = x - ct$ into the reaction diffusion equation

$$\frac{\partial u}{\partial t} = D\frac{\partial^2 u}{\partial x^2} + f(u). \tag{1.5.1}$$

The following ordinary differential equation is obtained

$$Du'' + cu' + f(u) = 0. \tag{1.5.2}$$

Equation (1.5.2) can be reduced to the system of equations

$$u' = p, \qquad Dp' = -cp - f(u). \tag{1.5.3}$$

Thus the problem of proving the existence of a traveling wave can be reduced to the study of trajectories of the system (1.5.3). Particularly, if the waves satisfy the boundary conditions

$$\lim_{z\to\pm\infty} u(z) = u_{\pm}, \tag{1.5.4}$$

then the system (1.5.2) has trajectories joining the stationary points $(u_+, 0)$ and $(u_-, 0)$ in the phase space (u, p). Hence the problem of proving the existence of a traveling wave solution reduces to the problem of proving the existence of the corresponding trajectories of the system (1.5.3). To prove the existence of a traveling wave solution, a trajectory is drawn from one of the stationary points $(u_+, 0)$ or $(u_-, 0)$ and the constant c is selected so that this trajectory reaches the other stationary point. This method was used for the first time by Kolmogorov to prove the existence of a traveling wave, see reference [8].

The following result will be used for our later work [9].

Lemma 1.5.1. *Let a and b be two positive numbers. Then there are positive numbers γ_1 and γ_2 such that for all $t \geq 0$,*

$$\gamma_1(1 - e^{-bt}) \leq 1 - e^{-at} \leq \gamma_2(1 - e^{-bt}).$$

Proof. If $a \leq b$, then the second inequality holds for $\gamma_2 = 1$. If $a > b$, choose $\gamma_2 > 0$ such that $\gamma_2 b > a$ and let $h(t) = \gamma_2(1 - e^{-bt}) - (1 - e^{-at})$, then

$$h(0) = 0, \quad \dot{h}(t) = \gamma_2 b e^{-bt} - a e^{-at} > 0, \quad t \geq 0.$$

Hence $h(t) > 0$ for $t \geq 0$. This implies $1 - e^{-at} \leq \gamma_2(1 - e^{-bt})$. Similarly $\gamma_1(1 - e^{-bt}) \leq 1 - e^{-at}$. □

Theorem 1.5.2. *(Gronwall's Inequality) Let $v(t)$ and $g(t)$ be continuous nonnegative functions on (a, b), $a < t_0 < b$, $C \geq 0$, and*

$$v(t) \leq C + \left|\int_{t_0}^{t} v(s)g(s)ds\right|, \quad a < t < b,$$

then

$$v(t) \leq C exp\left(\left|\int_{t_0}^{t} g(s)ds\right|\right).$$

1.6 Adomian's decomposition method

In the eighties G. Adomian [10] (1923-1996) proposed a method for solving nonlinear differential equations, which is based on a decomposition of the nonlinear operator as a series. Adomian's decomposition method has been successfully applied to wide class of problems.

To explain the basic ideas, consider a general nonlinear differential equation

$$\mathcal{F}u = g, \tag{1.6.1}$$

where $\mathcal{F}$ represent a general nonlinear differential operator involving both linear and nonlinear parts and g is a continuous function. Decompose $\mathcal{F}$ into two parts, linear and nonlinear N. The linear part is decomposed as $L + R$, where L is invertible and R is the reminder of the linear operator. Thus equation (1.6.1) can be written as

$$Lu + Ru + Nu = g, \tag{1.6.2}$$

where N represent nonlinear terms. Solving for Lu,

$$Lu = g - Ru - Nu. \tag{1.6.3}$$

Applying L^{-1}, it follows that

$$L^{-1}Lu = L^{-1}g - L^{-1}Ru - L^{-1}Nu. \tag{1.6.4}$$

If L is of order n, then L^{-1} is the n-fold integral. Thus $L^{-1}Lu = u + \phi$, where ϕ is the term emerging from the integration and one gets

$$u = L^{-1}g - \phi - L^{-1}Ru - L^{-1}Nu. \tag{1.6.5}$$

The solution u of the problem (1.6.1) is assumed to be in the form of a series of the type

$$u = \sum_{n=0}^{\infty} u_n \tag{1.6.6}$$

and the nonlinear part Nu is assumed to be an infinite series of polynomials of the type

$$Nu = \sum_{n=0}^{\infty} A_n, \tag{1.6.7}$$

where A_n are the so-called Adomian's polynomials. Substituting (1.6.6) and (1.6.7) into (1.6.5), it follows that

$$\sum_{n=0}^{\infty} u_n = L^{-1}g - \phi - L^{-1}R\sum_{n=0}^{\infty} u_n - L^{-1}\sum_{n=0}^{\infty} A_n. \tag{1.6.8}$$

Identifying u_0 as $L^{-1}g - \phi$, the components, $u_n(x,t),\ n = 0, 1, 2.......$, can be determined from the following relations

$$\begin{aligned}
u_0 &= L^{-1}g - \phi,\\
u_1 &= -L^{-1}Ru_0 - L^{-1}A_0,\\
u_2 &= -L^{-1}Ru_1 - L^{-1}A_1,\\
u_3 &= -L^{-1}Ru_2 - L^{-1}A_2,\\
&.\\
&.\\
&.\\
u_{n+1} &= -L^{-1}Ru_n - L^{-1}A_n.
\end{aligned}$$

The polynomials $A_0, A_1, A_2, A_3, ...$ can be computed as follows. Choose a parameter λ and set

$$u(\lambda) = \sum_{n=0}^{\infty} \lambda^n u_n, \tag{1.6.9}$$

then

$$N(u(\lambda)) = \sum_{n=0}^{\infty} \lambda^n A_n. \tag{1.6.10}$$

From (1.6.10), it follows that

$$\begin{aligned}
&N(u(\lambda)) = A_0 + \lambda A_1 + \lambda^2 A_2 + \lambda^3 A_3 + \lambda^4 A_4 + ... + \lambda^n A_n \\
&\frac{d}{d\lambda} N(u(\lambda)) = A_1 + 2!\lambda A_2 + 3\lambda^2 A_3 + 4\lambda^3 A_4 + ... + n\lambda^{n-1} A_n \\
&\frac{d^2}{d\lambda^2} N(u(\lambda)) = 2!A_2 + 3!\lambda A_3 + 4.3\lambda^2 A_4 + ... + n(n-1)\lambda^{n-2} A_n \\
&\frac{d^3}{d\lambda^3} N(u(\lambda)) = 3!A_3 + 4!\lambda A_4 + ... + n(n-1)(n-2)\lambda^{n-3} A_n \\
&\frac{d^4}{d\lambda^4} N(u(\lambda)) = 4!A_4 + ... + n(n-1)(n-2)(n-3)\lambda^{n-4} A_n \\
&\text{and so on.}
\end{aligned} \tag{1.6.11}$$

From (1.6.11), it follows that

$$A_n = \frac{[\frac{d^n}{d\lambda^n} N(u(\lambda))]_{\lambda=0}}{n!}, \quad n = 0, 1, 2, 3..... \tag{1.6.12}$$

As an example, consider the following two point boundary value problem [11]

$$u'' = f(x, u, u'), \quad a < x < b, \tag{1.6.13}$$

subject to the boundary conditions

$$u(a) = \alpha, \quad u(b) = \beta, \tag{1.6.14}$$

where f is continues on the set $\mathcal{D}$=$\{(x, u, u') \mid a \leq x \leq b, u, u' \in \mathbb{R}\}$.
The boundary value problem (1.6.13) can be written as

$$Lu = Nu + g, \tag{1.6.15}$$

where $L = \frac{d^2}{dx^2}$ is the second order operator and Nu is the nonlinear operator. Clearly L is invertible and

$$L^{-1} = \int_a^x \int_b^x (.)\, dx\, dx. \tag{1.6.16}$$

From (1.6.15), it follows

$$u = u(a) + (x-a)u'(b) + L^{-1}g + L^{-1}Nu. \quad (1.6.17)$$

Let

$$u = \sum_{n=0}^{\infty} u_n$$

and

$$Nu = \sum_{n=0}^{\infty} A_n,$$

then (1.6.17) implies

$$\sum_{n=0}^{\infty} u_n = u(a) + (x-a)u'(b) + L^{-1}g + L^{-1}\sum_{n=0}^{\infty} A_n. \quad (1.6.18)$$

Letting

$$u_0 = u(a) + (x-a)u'(b) + L^{-1}g,$$

then the following recursive scheme can be obtained

$$u_{n+1} = L^{-1}A_n, \quad n = 0, 1, 2, 3, 4..... \quad (1.6.19)$$

In order to determine all the components of u, the zeroth component u_0 has to be determined first. However, $u'(b)$ is not defined by the boundary condition as such the zeroth component u_0 of the solution u can not be determined directly. The best way to determine $u'(b)$ is proposed in reference [11], which is as under.

From (1.6.17), the value of u at $x = b$ is given by

$$u(b) = u(a) + (b-a)u'(b) + [L^{-1}g + L^{-1}\sum_{n=0}^{\infty} A_n]_{x=b}, \quad (1.6.20)$$

which implies

$$u'(b) = \frac{u(b)-u(a)}{b-a} - \frac{1}{b-a}[L^{-1}g + L^{-1}\sum_{n=0}^{\infty} A_n]_{x=b}. \quad (1.6.21)$$

Substituting (1.6.21) into (1.6.18), it follows that

$$\sum_{n=0}^{\infty} u_n = u(a)+q(x)[u(b)-u(a)]-q(x)[L^{-1}g+L^{-1}\sum_{n=0}^{\infty} A_n]_{x=b}+L^{-1}g+L^{-1}\sum_{n=0}^{\infty} A_n, \tag{1.6.22}$$

where

$$q(x) = \frac{x-a}{b-a}.$$

From (1.6.22) combined with boundary conditions (1.6.14), the following is proposed

$$\begin{aligned} u_0 &= \alpha + q(x)(\beta - \alpha) + L^{-1}g - q(x)[L^{-1}g]_{x=b}, \\ u_n &= L^{-1}A_n - q(x)[L^{-1}A_n]_{x=b}, \quad n > 0. \end{aligned} \tag{1.6.23}$$

1.7 *Homotopy perturbation method*

The homotopy perturbation method (HPM) [14, 15], proposed by Ji-Huan He in 1998, is a method for finding approximate solution of nonlinear differential and integral equations. The method, couples the perturbation method and homotopy theory.

To explain the basic ideas of the method, consider the following nonlinear differential equation,

$$\mathcal{A}(u) - f(r) = 0, \quad r \in \Omega, \tag{1.7.1}$$

with boundary conditions

$$\mathcal{B}(u, \frac{\partial u}{\partial n}) = 0, \tag{1.7.2}$$

where $\mathcal{A}$ is a general differential operator, $\mathcal{B}$ is an operator known as the boundary operator, $f(r)$ is a known analytic function. The operator $\mathcal{A}$ can be divided into two parts L and N, where L is linear part and N is nonlinear part. Therefore, equation (1.7.1) can be written as

$$L(u) + N(u) - f(r) = 0. \tag{1.7.3}$$

Defines the homotopy $v(r,p) : \Omega \times [0,1] \to \mathbb{R}$ which satisfies

$$\mathcal{H}(v,p) = (1-p)[L(v) - L(u_0)] + p\,[L(v) + N(v) - f(r)] = 0, \tag{1.7.4}$$

or equivalently,

$$\mathcal{H}(v,p) = L(v) - L(u_0) + p\,L(u_0) + p\,[N(v) - f(r)] = 0, \tag{1.7.5}$$

where $p \in [0,1]$ is an embedding parameter, u_0 is an initial approximation. From equation (1.7.5) it follows that

$$\mathcal{H}(v,0) = L(v) - L(u_0) = 0, \tag{1.7.6}$$

$$\mathcal{H}(v,1) = L(v) + N(v) - f(r) = \mathcal{A}(v) - f(r) = 0. \tag{1.7.7}$$

The changing process of p from zero to unity is just that of $v(r,p)$ from $u_0(r)$ to $u(r)$. In topology, this is called deformation and $L(v) - L(u_0)$, $\mathcal{A}(v) - f(r)$ are homotopic. The basic assumption is that the solution of equation (1.7.5) can be expressed as a power series in p, that is

$$v = v_0 + pv_1 + p^2 v_2 + p^3 v_3 + \ldots\ldots\ldots \tag{1.7.8}$$

The approximate solution of Eq.(1.7.1), can be obtained as

$$u = \lim_{p \to 1} v = v_0 + v_1 + v_2 + v_3 + \ldots\ldots \tag{1.7.9}$$

Consider the following PDE [16]

$$\nabla^2 u + (\frac{\partial u}{\partial y})^2 = 2y + x^4, \tag{1.7.10}$$

subject to the boundary conditions

$$u(0,y) = 0, \quad u(1,y) = y + a,$$
$$u(x,0) = ax, \quad u(x,1) = x(x+a), \tag{1.7.11}$$

where a is a constant. According to the homotopy perturbation method, construct the following homotopy:

$$\nabla^2 u - 2y = p\,[x^4 - (\frac{\partial u}{\partial y})^2]. \tag{1.7.12}$$

The homotopy parameter p changes from zero to unity. In case $p = 0$, equation (1.7.12) becomes $\nabla^2 u - 2y = 0$, which is easy to be solved and when it is one, equation (1.7.12) turns out to be the original equation (1.7.10). According to homotopy perturbation method the solution is written as

$$u = u_0 + pu_1 + p^2 u_2 + p^3 u_3 + p^4 u_4 + \ldots\ldots\ldots\ldots. \tag{1.7.13}$$

The approximate solution can be obtained by setting $p = 1$ in the equation (1.7.13), that is

$$u = u_0 + u_1 + u_2 + u_3 + u_4 + \ldots\ldots., \tag{1.7.14}$$

is the approximate solution of the problem (1.7.10). Substituting (1.7.13) into (1.7.12),

$$\nabla^2 (\sum_{n=0}^{\infty} p^n u_n) - 2y = p\,[x^4 - (\frac{\partial(\sum_{n=0}^{\infty} p^n u_n)}{\partial y})^2], \tag{1.7.15}$$

equating the terms with the identical powers of p, only the first two linear equations are written

$$\nabla^2 u_0 - 2y = 0, \tag{1.7.16}$$
$$\nabla^2 u_1 = x^4 - (\frac{\partial u_0}{\partial y})^2. \tag{1.7.17}$$

Solution of the equation (1.7.16) gives,

$$u_0 = x^2 y. \tag{1.7.18}$$

Substituting u_0 into the equation (1.7.17), a differential equation for u_1 is obtained as,

$$\nabla^2 u_1 = 0. \tag{1.7.19}$$

If the first order approximate solution is sought, $u = u_0 + u_1$, then the boundary conditions for u_1 are

$$\begin{aligned}
u_1(0, y) &= u(0, y) - u_0(0, y) = 0,\\
u_1(1, y) &= u(1, y) - u_0(1, y) = y + a - y = a,\\
u_1(x, 0) &= u(x, 0) - u_0(x, 0) = ax,\\
u_1(x, 1) &= u(x, 1) - u_0(x, 1) = x(x + a) - x^2 = ax.
\end{aligned} \tag{1.7.20}$$

Considering the boundary conditions (1.7.20), equation (1.7.19) can be solved to obtain

$$u_1 = ax.$$

The first order approximate solution is

$$u = u_0 + u_1 = x^2 y + ax.$$

2. EXISTENCE AND UNIQUENESS OF TRAVELING WAVE SOLUTIONS

2.1 Introduction

In this chapter, the existence and uniqueness of traveling wave solutions of the system of reaction-diffusion equations of the type,

$$\begin{aligned} \frac{\partial S}{\partial t} &= \rho \frac{\partial^2 S}{\partial x^2} - \theta \frac{\partial S}{\partial x} - f(S)P, \\ \frac{\partial P}{\partial t} &= \frac{\partial^2 P}{\partial x^2} - \theta \frac{\partial P}{\partial x} + [f(S) - K]P, \end{aligned} \tag{2.1.1}$$

is discussed. The system (2.1.1) is used as a mathematical model to study some problems in Biology. For example, the system (2.1.1) has been used in references [3, 4] as a model for a single population microbial growth for a limiting nutrient in a flow reactor, where $\theta\,(\geq 0)$ is the flow velocity, $S(x,t)$, $P(x,t)$ are the concentrations of nutrient and microbial population in the reactor at position x and time t, $\rho > 0$ is the diffusion coefficient, $K > 0$ is the cell death rate and f is the nutrient uptake function (or birth rate) of the bacteria cells. If $f(S) = S$ and $\theta = 0$, then the system (2.1.1) describes a simple diffusive epidemic model, in which $S(x,t)$ and $P(x,t)$ represent the densities of susceptible and infective population [5, 6].

Consider a tubular reactor through which liquid medium flows at constant velocity. Fresh nutrient at constant concentration enters at the inflow to the reactor while un-utilized nutrients exit at the outflow from the reactor. This reactor is known as a flow reactor. The natural environment most closely approximated by the flow reactor is the gut of an animal [3]. Let the flow reactor be long enough and assume that the amount S^0 of nutrient is input at a constant velocity θ at one end of the flow reactor, say at $x = -\infty$. If there is no bacteria population, then the concentration of nutrient keeps on constant and is washed out at the other end of the reactor. On the other hand, suppose that the nutrient uptake function (or birth rate of bacteria cell) $f(S)$ is increasing with respect to S and there exists a constant K such that $f(S^0) - K > 0$. If a small amount of bacteria be introduced, then the population increases when the growth rate $f(S) - K > 0$. The growth rate eventually becomes negative because of the reduction of the nutrient due to which the bacteria population declines. Hence one might expect that a hump-shaped bacteria population density $P(x,t)$ moves toward the other end of the reactor. That is to say, it is expected that there are constants c and S_0 with $f(S_0) < K$ and a non-negative traveling wave $(S(x-ct), P(x-ct))$ such that

$$\begin{aligned} &S(-\infty) = S^0, \quad S(\infty) = S_0, \\ &P(-\infty) = 0, \quad P(\infty) = 0. \end{aligned} \tag{2.1.2}$$

Assume that the function $f : [0,\infty) \to \mathbb{R}$ is Lipschitz continuous and satisfies the following conditions:

(A_1) $f(0) = 0$ and there is a unique $S_K > 0$ such that $f(S_K) = K$ and
$$0 < f(S) < K, \quad S \in (0, S_K), \quad K < f(S), \quad S \in (S_K, \infty).$$

(A_2) There is a positive number ξ^* such that $f(S)$ is increasing for $S \in [0,\xi^*]$ and
$$\lim_{\epsilon \to 0^+} \int_\epsilon^{\xi^*} \frac{1}{f(S)} dS = +\infty.$$

Introducing the new variable $z = x - ct$ and using

$$(S(x,t), P(x,t)) = (U(x-ct), V(x-ct)),$$

into the equations (2.1.1) and (2.1.2), it follows that

$$\begin{aligned} C\dot{U} &= \rho\ddot{U} - f(U)V, \\ C\dot{V} &= \ddot{V} + [f(U) - K]V \end{aligned} \tag{2.1.3}$$

and

$$\begin{aligned} &U(-\infty) = S^0, \quad U(\infty) = S_0, \\ &V(-\infty) = 0, \quad V(\infty) = 0, \end{aligned} \tag{2.1.4}$$

where $C = -c + \theta$ and (.) denote differentiation with respect to z. Choose $C > 0$.

By the change of variables

$$u(z) = U(-z), \quad v(z) = V(-z), \quad z \in \mathbb{R},$$

the system (2.1.3) takes the form

$$\begin{aligned} \rho\ddot{u} &= -C\dot{u} + f(u)v, \\ \ddot{v} &= -C\dot{v} + [K - f(u)]v \end{aligned} \tag{2.1.5}$$

and the boundary conditions (2.1.4) can be written as

$$u(-\infty) = u_0 < u_k < u^0 = u(\infty), \quad v(-\infty) = v(\infty) = 0, \tag{2.1.6}$$

where $u_K = S_K$.

Let

$$u_1 = u, \quad u_2 = \dot{u}, \quad v_1 = v, \quad v_2 = \dot{v}.$$

Then the system (2.1.5) can be written equivalently as a system of first-order equations

$$\begin{aligned} \dot{u_1} &= u_2, \\ \dot{u_2} &= \frac{1}{\rho}[f(u_1)v_1 - Cu_2], \\ \dot{v_1} &= v_2, \\ \dot{v_2} &= [K - f(u_1)]v_1 - Cv_2. \end{aligned} \tag{2.1.7}$$

The equilibrium points of the system (2.1.7) are

$$u_1 = u_0, \quad \dot{u} = v = \dot{v} = 0, \quad u_0 \in [0, \infty).$$

The linearized system of the system (2.1.7), which provides an approximation to the behavior of the original system near the equilibrium point, in the matrix form is

$$\begin{pmatrix} \dot{u_1} \\ \dot{u_2} \\ \dot{v_1} \\ \dot{v_2} \end{pmatrix} = \begin{pmatrix} \frac{\partial f_1}{\partial u_1} & \frac{\partial f_1}{\partial u_2} & \frac{\partial f_1}{\partial v_1} & \frac{\partial f_1}{\partial v_2} \\ \frac{\partial f_2}{\partial u_1} & \frac{\partial f_2}{\partial u_2} & \frac{\partial f_2}{\partial v_1} & \frac{\partial f_2}{\partial v_2} \\ \frac{\partial f_3}{\partial u_1} & \frac{\partial f_3}{\partial u_2} & \frac{\partial f_3}{\partial v_1} & \frac{\partial f_3}{\partial v_2} \\ \frac{\partial f_4}{\partial u_1} & \frac{\partial f_4}{\partial u_2} & \frac{\partial f_4}{\partial v_1} & \frac{\partial f_4}{\partial v_2} \end{pmatrix}_{(u_0,0,0,0)} \begin{pmatrix} u_1 \\ u_2 \\ v_1 \\ v_2 \end{pmatrix}, \qquad (2.1.8)$$

where $f_1 = u_2$, $f_2 = \frac{1}{\rho}[f(u_1)v_1 - Cu_2]$, $f_3 = v_2$ and $f_4 = [K - f(u_1)]v_1 - Cv_2$. The characteristic equation of the linear system (2.1.8) at the equilibrium point $(u_0, 0, 0, 0)$ is

$$det \begin{pmatrix} -\lambda & 1 & 0 & 0 \\ 0 & \frac{-C}{\rho} - \lambda & 0 & 0 \\ 0 & 0 & -\lambda & 1 \\ 0 & 0 & K - f(u_0) & -C - \lambda \end{pmatrix} = 0$$

or equivalently

$$\lambda(\frac{C}{\rho} + \lambda)(\lambda^2 + C\lambda + f(u_0) - K) = 0, \qquad (2.1.9)$$

whose roots known as eigenvalues of the system (2.1.8) are

$$\lambda_1 = \frac{-C + \sqrt{C^2 + 4[K - f(u_0)]}}{2},$$
$$\lambda_2 = \frac{-C - \sqrt{C^2 + 4[K - f(u_0)]}}{2},$$
$$\lambda_3 = \frac{-C}{\rho}, \quad \lambda_4 = 0.$$

If $u_0 < u_K$, then of these four eigenvalues, λ_1 is positive and λ_2, λ_3 are negative. Corresponding to the positive eigenvalue λ_1, there is a one dimensional unstable manifold with one branch positive. For $u_0 \in (0, u_k)$, assume that $(u(t, u_0), v(t, u_0))$ be a solution of the system (2.1.5) that stays in the positive part of the unstable manifold for all sufficiently negative t and satisfy

$$u(-\infty) = u_0, \qquad \dot{u}(-\infty) = v(-\infty) = \dot{v}(-\infty) = 0. \tag{2.1.10}$$

Also, assume that there exists a $T_M(u_0) \leq \infty$ such that the interval $(-\infty, T_M)$ is the maximum interval on which the solution $(u(t, u_0), v(t, u_0))$ exists. Now multiplying the first equation of the system (2.1.5) by $e^{\int \frac{C}{\rho} ds}$ and then integrating, the following expression is obtained

$$\dot{u}(t) = \dot{u}(t_0) e^{\frac{-C(t-t_0)}{\rho}} + \frac{1}{\rho} \int_{t_0}^{t} e^{\frac{-C(t-s)}{\rho}} f(u(s)) v(s) ds. \tag{2.1.11}$$

Integrating (2.1.11) from t_0 to t, it follows that

$$\begin{aligned} u(t) &= u(t_0) + \dot{u}(t_0) \int_{t_0}^{t} e^{\frac{-C(s-t_0)}{\rho}} ds \\ &+ \frac{1}{\rho} \int_{t_0}^{t} \int_{t_0}^{\tau} e^{\frac{-C(\tau-s)}{\rho}} f(u(s)) v(s) ds d\tau, \end{aligned} \tag{2.1.12}$$

which implies that

$$\begin{aligned} u(t) &= u(t_0) + \frac{\rho}{C} \dot{u}(t_0) [1 - e^{\frac{-C(t-t_0)}{\rho}}] \\ &+ \frac{1}{C} \int_{t_0}^{t} [1 - e^{\frac{-C(t-s)}{\rho}}] f(u(s)) v(s) ds. \end{aligned} \tag{2.1.13}$$

Similarly, from the second equation of the system (2.1.5), it follows that

$$\dot{v}(t) = \dot{v}(t_0) e^{-C(t-t_0)} + \int_{t_0}^{t} e^{-C(t-s)} [K - f(u(s))] v(s) ds. \tag{2.1.14}$$

Integrating (2.1.14) from t_0 to t,

$$\begin{aligned} v(t) &= v(t_0) + \frac{1}{C} \dot{v}(t_0) [1 - e^{-C(t-t_0)}] \\ &+ \frac{1}{C} \int_{t_0}^{t} [1 - e^{-C(t-s)}][K - f(u(s))] v(s) ds. \end{aligned} \tag{2.1.15}$$

At $t_0 = -\infty$, using the boundary conditions (2.1.10), it follows that

$$\dot{u}(t) = \frac{1}{\rho}\int_{-\infty}^{t} e^{\frac{-C(t-s)}{\rho}} f(u(s))v(s)ds, \tag{2.1.16}$$

$$u(t) = u_0 + \frac{1}{C}\int_{-\infty}^{t} [1 - e^{\frac{-C(t-s)}{\rho}}] f(u(s))v(s)ds, \tag{2.1.17}$$

$$\dot{v}(t) = \int_{-\infty}^{t} e^{-C(t-s)}[K - f(u(s))]v(s)ds \tag{2.1.18}$$

and

$$v(t) = \frac{1}{C}\int_{-\infty}^{t} [1 - e^{-C(t-s)}][K - f(u(s))]v(s)ds. \tag{2.1.19}$$

Since

$$e^{\frac{-C(t-s)}{\rho}} f(u(s))v(s) > 0, \qquad s \in (-\infty, t],$$

for some sufficiently negative t, hence from equation (2.1.16), $\dot{u}(t) > 0$ for some sufficiently negative t. In a similar way, from (2.1.18), it can be deduced that there exists a sufficiently negative t_1 such that $\dot{v}(t) > 0$ for all $t \in (-\infty, t_1]$. Hence $\dot{u}(t, u_0)$, $v(t, u_0)$ and $\dot{v}(t, u_0)$ are all positive for all sufficiently negative t.

Let

$$t_M = sup\{t \le T_M(u_0) : v(s, u_0) > 0, \quad s \in (-\infty, t]\}.$$

Since

$$e^{\frac{-C(t-s)}{\rho}} f(u(s))v(s),$$

is positive for all $s \in (-\infty, t_M)$, hence from (2.1.16), $\dot{u}(t) > 0$ for all $t \in (-\infty, t_M)$, that is, $u(t)$ is increasing on $(-\infty, t_M)$. Now define

$$u^+(u_0) = \lim_{t \to t_M} u(t, u_0).$$

We need the following result for later use [9].

Lemma 2.1.1. *For a given u_0, the solution $(u(t), v(t)) = (u(t, u_0), v(t, u_0))$ of the system* (2.1.5) *is bounded on $(-\infty, t_M)$.*

Proof. Firstly, it is required to show that $\lim_{t\to t_M} u(t, u_0) = u^+(u_0) < +\infty$. Suppose that $u^+(u_0) = \infty$. Then, by (2.1.17),

$$\lim_{t\to t_M} u(t) = \frac{1}{C} \lim_{t\to t_M} \int_{-\infty}^{t} [1 - e^{\frac{-C(t-s)}{\rho}}] f(u(s))v(s)ds = \infty. \tag{2.1.20}$$

Hence there exists a $t_1 < t_M$ such that $u(t) > u(t_1) > u_k$ for all $t \in (t_1, t_M)$. The equation (2.1.20) implies that

$$\lim_{t\to t_M} \int_{-\infty}^{t} [1 - e^{\frac{-C(t-s)}{\rho}}] f(u(s))v(s)ds = \infty. \tag{2.1.21}$$

From (2.1.19) the expression for $v(t)$ is

$$v(t) \;=\; \frac{1}{C} \int_{-\infty}^{t} [1 - e^{-C(t-s)}][K - f(u(s))]v(s)ds. \tag{2.1.22}$$

Since $u(t) > u(t_1) > u_k$ for all $t \in (t_1, t_M)$, by assumption (A_1),

$$\psi = sup\{\frac{K}{f(u(t))} : t \in [t_1, t_M)\} < 1. \tag{2.1.23}$$

Hence $\psi f(u(t)) \geq K$ for $t \in [t_1, t_M)$. Let $d = 1 - \psi$ then $d > 0$,

$$\begin{aligned} f(u(t)) - K &= \psi f(u(t)) - \psi f(u(t)) + f(u(t)) - K \\ &= (1-\psi) f(u(t)) + \psi f(u(t)) - K \\ &= d\, f(u(t)) + \psi f(u(t)) - K \\ &\geq df(u(t)), \;\; t \in [t_1, t_M). \end{aligned} \tag{2.1.24}$$

Now, by Lemma (1.5.1), there is a positive γ_1 such that

$$1 - e^{-C\tau} \geq \gamma_1 (1 - e^{\frac{-C\tau}{\rho}}),\; \tau \geq 0. \tag{2.1.25}$$

Hence, it follows from (2.1.23)-(2.1.25) that

$$\begin{aligned} v(t) &= \frac{-1}{C}\int_{-\infty}^{t}[1-e^{-C(t-s)}][f(u(s))-K]v(s)ds, \\ &\le \frac{-1}{C}\int_{-\infty}^{t}\gamma_1[1-e^{\frac{-C(t-s)}{\rho}}]df(u(s))v(s)ds, \\ &= \frac{-d\gamma_1}{C}\int_{-\infty}^{t}[1-e^{\frac{-C(t-s)}{\rho}}]f(u(s))v(s)ds \to -\infty \text{ as } t\to t_M, \end{aligned} \tag{2.1.26}$$

which contradicts the definition of t_M. Thus $u^+(u_0) < \infty$. Since $u(t)$ is increasing on $(-\infty, t_M)$ and $\lim_{t\to t_M} u(t) = u^+(u_0) < \infty$, hence, $u(t)$ is bounded on $(-\infty, t_M)$.

Now, it is required to show that v is bounded on $(-\infty, t_M)$. From (2.1.19) it follows that

$$\begin{aligned} v(t) &= \frac{1}{C}\int_{-\infty}^{t}[1-e^{-C(t-s)}][K-f(u(s))]v(s)ds \\ &< \frac{1}{C}\int_{-\infty}^{t}[1-e^{-C(t-s)}]Kv(s)ds. \end{aligned} \tag{2.1.27}$$

By Lemma (1.5.1), there exists a $\gamma_2 > 0$ such that

$$1-e^{-Ct} \le \gamma_2(1-e^{\frac{-Ct}{\rho}}),\ t\ge 0.$$

Moreover, since $u(t) > u_0 > 0$ for $t\in(-\infty, t_M)$. By assumption (A_1) there exists a $\xi > 0$ such that

$$\xi \le f(u(t)) \,\text{or}\, K \le \frac{K}{\xi}f(u(t)),\ t\in(-\infty, t_M).$$

Hence, from the above inequalities and (2.1.27), it follows that

$$\begin{aligned} v(t) &\le \frac{K\gamma_2}{\xi C}\int_{-\infty}^{t}[1-e^{\frac{-C(t-s)}{\rho}}]f(u(s))v(s)ds \\ &< \frac{K\gamma_2}{\xi}(u_0+\frac{1}{C}\int_{-\infty}^{t}[1-e^{\frac{-C(t-s)}{\rho}}]f(u(s))v(s)ds) \\ &= \frac{K\gamma_2}{\xi}u(t) \le \frac{K\gamma_2}{\xi}u^+(u_0) < \infty, \quad t\in(-\infty, t_M), \end{aligned} \tag{2.1.28}$$

which implies that $v(t)$ is bounded on $(-\infty, t_M)$. $\square$

Corollary 2.1.2. *Let $u_0 \in (0, u_K)$ and $(u(t, u_0), v(t, u_0))$ be a solution of the system* (2.1.5)*. Then, $\dot{u}(t, u_0)$ and $\dot{v}(t, u_0)$ are bounded on $(-\infty, t_M)$.*

Proof. Let $u(t) = u(t, u_0)$ and $v(t) = v(t, u_0)$. By Lemma (2.1.1), there exists $M_1 > 0$ such that

$$0 < v(t) < M_1, \qquad t \in (-\infty, t_M).$$

Since f is continuous and $u(t)$ is bounded, so there exists $M_2 > 0$ such that

$$f(u(t)) < M_2, \qquad t \in (-\infty, t_M).$$

Let

$$M = max\{M_1, M_2\},$$

then

$$0 < v(t) < M,\ f(u(t)) < M, \qquad t \in (-\infty, t_M).$$

Hence, using (2.1.16), It follows that

$$0 \leq \dot{u}(t) \leq \frac{M^2}{\rho} \int_{-\infty}^{t} e^{\frac{-C(t-s)}{\rho}} ds = \frac{M^2}{C}, \qquad t \in (-\infty, t_M).$$

Similarly, from (2.1.18), it follows that

$$|\dot{v}(t)| \leq \frac{(K+M)M}{C}, \qquad t \in (-\infty, t_M).$$

Hence, $\dot{u}(t)$ and $\dot{v}(t)$ are bounded on $(-\infty, t_M)$. $\square$

Corollary 2.1.3. *For each $u_0 \in (0, u_K)$, the following are true:*

(1) $u^+(u_0) > u_K$

(2) There is a $t_1 < t_M$ such that $\dot{v}(t, u_0) > 0$ for $t < t_1$ and $\dot{v}(t, u_0) < 0$ for $t \in (t_1, t_M)$.

Proof. Let $u(t) = u(t, u_0)$ and $v(t) = v(t, u_0)$. It is claimed that there exists a $t_1 < t_M$ such that $\dot{v}(t_1) = 0$. If this is not the case then $\dot{v}(t) > 0$ for all $t \in (t_1, t_M)$. The boundedness of $v(t)$ implies that $\lim_{t \to t_M} v(t) = v(t_M) < \infty$. So $t_M = \infty$, for otherwise the boundedness of $u(t), \dot{u}(t), v(t)$ and $\dot{v}(t)$ would imply that the solution $(u(t), v(t))$ can be extended to a right interval of t_M on which $v(t)$ is positive. This contradicts the definition of t_M. On the other hand, if $t_M = \infty$, then from monotone increasing property of $v(t)$, it follows that

$$u^+(u_0) = \lim_{t \to \infty}(u_0 + \frac{1}{C}\int_{-\infty}^{t_1}[1 - e^{\frac{-C(t-s)}{\rho}}]f(u(s))v(s)ds) = \infty.$$

This leads to a contradiction of Lemma (2.1.1). Now let $t_1 < t_M$ is a first time at which $\dot{v}(t) = 0$. Then from (2.1.18), it follows that

$$0 = \dot{v}(t_1) = \int_{-\infty}^{t_1} e^{-C(t_1 - s)}[K - f(u(s))]v(s)ds. \qquad (2.1.29)$$

It is claimed that there is a $t' < t_1$ such that $u(t') = u_K$. Because, if $u(s) < u_K$ for all $s \in (-\infty, t_1)$, then

$$[K - f(u(s))]v(s) > 0, \; s \in (-\infty, t_1).$$

which implies that

$$\dot{v}(t_1) = \int_{-\infty}^{t_1} e^{-C(t_1 - s)}[K - f(u(s))]v(s)ds > 0,$$

a contradiction to (2.1.29). Hence, there is a $t' < t_1 < t_M$ such that $u(t') = u_K$. This proves the inequality $u^+(u_0) > u_K$. Moreover, since $u(s)$ is monotone increasing, hence $f(u(s)) - K > 0$ for all $s > t_1 > t'$. Now from (2.1.14) with the use of equality $\dot{v}(t_1) = 0$, for $t \in (t_1, t_M)$,

$$\begin{aligned} \dot{v}(t) &= \dot{v}(t_1)e^{-C(t-t_1)} + \int_{t_1}^{t} e^{-C(t-s)}[K - f(u(s))]v(s)ds \\ &= \int_{t_1}^{t} e^{-C(t-s)}[K - f(u(s))]v(s)ds < 0. \end{aligned} \qquad (2.1.30)$$

Hence there is a $t_1 < t_M$ such that $\dot{v}(t, u_0) > 0$ for $t < t_1$ and $\dot{v}(t, u_0) < 0$ for $t \in (t_1, t_M)$. □

2.2 Existence of traveling wave solutions

In this section existence of a traveling wave solution of the system (2.1.1) is discussed. For each $u^0 > u_k$, define

$$f^M(u^0) = max\{f(u) : u \in [u_K, u^0]\}.$$

Lemma 2.2.1. *[9] If for each $u_0 \in (0, u_K)$, $u^+(u_0) \leq u^0$,*

$$4[f^M(u^0) - K] < C^2,$$

and $t_M = +\infty$, then $(u(t, u_0), v(t, u_0))$ is a traveling wave solution connecting $(u_0, 0)$ and $(u^0, 0)$ such that $u(t, u_0)$ is monotone increasing and $v(t, u_0) > 0$ for all $t \in \mathbb{R}$.

Proof. Let $(u(t), v(t)) = (u(t, u_0), v(t, u_0))$. As shown in the proof of Corollary (2.1.3), there exists a $t' < t_M$, such that $u(t') = u_K$. Without loss of generality assume that $u(0) = u_K$. By Corollary (2.1.3), $v(t)$ has a maximum value at $t = t_1 > 0$ and $\dot{v}(t) < 0$ for all $t \in (t_1, t_M)$. Let

$$v_1(t) = v(t), \ v_2(t) = \dot{v}(t).$$

Since

$$v_1(t) = v(t) > 0, \ v_2(t) = \dot{v}(t) < 0, \text{ when } t \in (t_1, t_M),$$

this implies that the solution $(v_1(t), v_2(t))$ enters the region $\{(v_1, v_2) : v_1 > 0, v_2 < 0\}$ in the plane (v_1, v_2) for $t \in (t_1, t_M)$. Since $t \geq t_1 > 0$ implies that $u(t) \geq u(t_1) > u(0) = u_K$, hence $f(u(t)) - K > 0$. Also $\dot{v}_2(t) = -[f(u(t)) - K]v_1 < 0$, at the positive v_1 axis of the plane (v_1, v_2) for $t \geq t_1$.

Hence $(v_1(t), v_2(t))$ can not leave this region along the positive v_1-axis. Select $\delta = C^2 - 4[f^M(u^0) - K]$ then $\delta > 0$. Now choose $0 < 4\epsilon < \delta$. Define

$$\beta = \frac{C - \sqrt{\delta - 4\epsilon}}{2}.$$

Then

$$(\beta - \frac{C}{2})^2 = \frac{\delta}{4} - \epsilon.$$

Expanding and re-arranging terms

$$C\beta = \beta^2 + f^M(u^0) + \epsilon - K. \tag{2.2.1}$$

Now consider the region Ω in the $4th$ quadrant of (v_1, v_2) plane bounded by the positive v_1-axis and the ray $L = \{v_2 = -\beta v_1 : v_1 > 0\}$. Let $(v_1, v_2) \in L$, then

$$\dot{v}_1 = v_2 < 0.$$

Using (2.2.1), it follows that

$$\begin{aligned}
\dot{v}_2 &= -[f(u(t)) - K]v_1 - Cv_2 \\
&= -[f(u(t)) - K]v_1 + C\beta v_1 \\
&= v_1\{C\beta - [f(u(t)) - K]\} \\
&= v_1[\beta^2 + f^M(u^0) + \epsilon - f(u(t))] > 0.
\end{aligned} \tag{2.2.2}$$

Moreover, from (2.2.2), it follows that

$$\begin{aligned}
|\tfrac{\dot{v}_2}{\dot{v}_1}| &= \frac{v_1\{\beta^2 + [f^M(u^0) + \epsilon - f(u(t))]\}}{|v_2|} \\
&= \frac{v_1\{\beta^2 + [f^M(u^0) - f(u(t)) + \epsilon]\}}{\beta v_1} \\
&\geq \frac{\beta^2 + \epsilon}{\beta} > \beta.
\end{aligned} \tag{2.2.3}$$

From the above inequalities it follows that the vector $(\dot{v}_1(t), \dot{v}_2(t))$ points to the interior of Ω. Hence the solution $(v_1(t), v_2(t))$ must stay in Ω for all $t \geq t_0$. Thus, $t_M = +\infty$ and $u(t) \to u^+(u_0) \leq u^0$ as $t \to +\infty$. □

Let $u^0 > u_K$ be fixed. Since $\frac{1}{K}(u^0 - u_0) \le \frac{u^0}{K}$ for all $u_0 \in (0, u_K)$ and from assumption (A_2),

$$\lim_{u_0 \to 0^+} \int_{u_0}^{\xi^*} \frac{1}{f(u)} du = +\infty. \tag{2.2.4}$$

Hence there exists a $u_0^* \in (0, \xi^*)$ such that

$$\int_{u_0}^{\xi^*} \frac{1}{f(u)} du > \frac{1}{K}(u^0 - u_0) \quad \text{for all } u_0 \in (0, u_0^*], \tag{2.2.5}$$

where $\xi^* \le u_K$.

Lemma 2.2.2. *[9] For each $u^0 > u_K$ such that $4[f^M(u^0) - K] < C^2$, let u_0^* be defined as in* (2.2.5)*. Then for all $u_0 \in (0, u_0^*]$ the relation $u^+(u_0) > u^0$ holds.*

Proof. In order to prove this result, it is shown that $u^+(u_0) \le u^0$ leads to a contradiction. Let $u_0 \in (0, u_0^*]$ and that $u^+(u_0) \le u^0$. Then by Lemma (2.2.1) $t_M = \infty$. Hence $\lim_{t\to\infty} u(t) = u^+(u_0)$, also $\lim_{t\to\infty} \dot{u}(t) = \lim_{t\to\infty} v(t) = \lim_{t\to\infty} \dot{v}(t) = 0$. By integrating the first equation of the system (2.1.5) from $-\infty$ to $+\infty$, it follows that

$$\rho[\dot{u}(+\infty) - \dot{u}(-\infty)] = -C[u(+\infty) - u(-\infty)] + \int_{-\infty}^{+\infty} f(u(t))v(t)dt,$$

which implies that

$$C[u^+(u_0) - u_0] = \int_{-\infty}^{+\infty} f(u(t))v(t)dt. \tag{2.2.6}$$

Similarly integrating the second equation of the system (2.1.5) from $-\infty$ to $+\infty$, it follows that

$$\int_{-\infty}^{+\infty} [K - f(u(t))]v(t)dt = 0. \tag{2.2.7}$$

From (2.2.6) and (2.2.7), it follows that

$$K\int_{-\infty}^{\infty} v(t)dt = \int_{-\infty}^{\infty} f(u(t))v(t)dt = C[u^{+}(u_0) - u_0]. \quad (2.2.8)$$

Select $\xi^* \leq u_K$, then by Part (1) of Corollary (2.1.3) there is a t^* such that $u(t^*) = \xi^*$. Since $u(t)$ is monotone increasing for all $t \in \mathbb{R}$, hence $f(u(t)) < K$ for all $t \in (-\infty, t^*)$. From equation (2.1.18), the expression for $\dot{v}(t)$ is

$$\dot{v}(t) = \int_{-\infty}^{t} e^{-C(t-s)}[K - f(u(s))]v(s)ds.$$

As $K - f(u(t)) > 0, \quad t \in (-\infty, t^*)$. Hence $\dot{v}(t) \geq 0, \;\; t \in (-\infty, t^*]$. That is, $v(t)$ is increasing on $(-\infty, t^*]$. Now by the equation (2.1.16)

$$\dot{u}(t) = \frac{1}{\rho}\int_{-\infty}^{t} e^{\frac{-C(t-s)}{\rho}} f(u(s))v(s)ds. \quad (2.2.9)$$

Since $u(s)$ is increasing and $u(s) \leq \xi^*$ for all $s \in (-\infty, t^*]$, then by the assumption $A(2)$ $f(u)$ is increasing for $u \in (0, \xi^*]$. It follows from (2.2.9) that $\dot{u}(t)$ is increasing for $t \in (-\infty, t^*]$, so that

$$\ddot{u}(t) \geq 0, \quad t \in (-\infty, t^*]. \quad (2.2.10)$$

Now dividing the first equation of the system (2.1.5) by $f(u(t))$, it follows that

$$\frac{\rho K \ddot{u}}{f(u(t))} = -CK\frac{\dot{u}}{f(u(t))} + Kv. \quad (2.2.11)$$

Integrating (2.2.11) from $-\infty$ to t^*

$$\int_{-\infty}^{t^*} \frac{\rho K \ddot{u}(t)}{f(u(t))}dt = -CK\int_{-\infty}^{t^*} \frac{\dot{u}(t)}{f(u(t))}dt + K\int_{-\infty}^{t^*} v(t)dt. \quad (2.2.12)$$

using equations (2.2.8), (2.2.10) and the equalities $u(-\infty) = u_0$, $u(t^*) = \xi^*$, in the equation (2.2.12), it follows that

$$\begin{aligned} 0 &\leq \int_{-\infty}^{t^*} \frac{\rho K \ddot{u}(t)}{f(u(t))}\, dt \\ &= -CK \int_{-\infty}^{t^*} \frac{\dot{u}(t)}{f(u(t))}\, dt + K \int_{-\infty}^{t^*} v(t) dt \\ &= -CK \int_{u_0}^{\xi^*} \frac{1}{f(u)}\, du + k \int_{-\infty}^{t^*} v(t) dt \\ &< -CK \int_{u_0}^{\xi^*} \frac{1}{f(u)}\, du + k \int_{-\infty}^{\infty} v(t) dt \\ &= -CK \int_{u_0}^{\xi^*} \frac{1}{f(u)}\, du + C[u^+(u_0) - u_0]. \end{aligned}$$

From the above inequality it follows that

$$\int_{u_0}^{\xi^*} \frac{1}{f(u)}\, du \leq \frac{1}{K}(u^+(u_0) - u_0) \leq \frac{1}{K}(u^0 - u_0). \tag{2.2.13}$$

Equation (2.2.13) contradicts (2.2.5) and therefore $u^+(u_0) > u^0$. □

The following Lemma provides an estimate for the bounds of $v(t)$ and $\dot{v}(t)$ on $t \in (-\infty, 0)$.

Lemma 2.2.3. *[9] There is a $\bar{u}_0 < u_K$ such that for each $u_0 \in [\bar{u}_0, u_K)$, if $(u(t), v(t)) = (u(t, u_0), v(t, u_0))$ is a solution of the system (2.1.5) with $u(0) = u_K$, then*

$$0 < v(t) \leq u_K - u_0,\ 0 < \dot{v}(t) \leq \frac{K(u_K - u_0)}{C},\ t \in (-\infty, 0].$$

Proof. The continuity of f implies that for a given $\epsilon > 0$, there exists $\delta > 0$ such that

$$|f(u) - K| < \epsilon \text{ whenever } |u - u_K| < \delta.$$

Consequently, there exists $\bar{u}_0 < u_K$ such that

$$K - f(u) \leq \frac{1}{\gamma_2} f(u),\ u \in [\bar{u}_0, u_K]. \tag{2.2.14}$$

Now, let $u_0 \in [\bar{u}_0, u_K)$ and $(u(t), v(t)) = (u(t, u_0), v(t, u_0))$ be a solution of the system (2.1.5) with $u(0) = u_K$. Since u is increasing and $u(-\infty) = u_0$, hence $u_0 \leq u(s) \leq u_K$ for $s \in (-\infty, 0]$. By Lemma (1.5.1), there exists $\gamma_2 > 0$ such that

$$1 - e^{Cs} \leq \gamma_2[1 - e^{\frac{C}{\rho}s}], \text{ for } s \leq 0. \tag{2.2.15}$$

Using the inequalities (2.2.15) and (2.2.14), the following inequality is obtained,

$$[1 - e^{Cs}][K - f(u(s))] \leq [1 - e^{\frac{C}{\rho}s}]f(u(s)), \ s \leq 0. \tag{2.2.16}$$

By formula (2.1.17), it follows that

$$u_K = u(0) = u_0 + \frac{1}{C}\int_{-\infty}^{0} [1 - e^{\frac{C}{\rho}s}]f(u(s))v(s)ds, \tag{2.2.17}$$

which implies that

$$\frac{1}{C}\int_{-\infty}^{0} [1 - e^{\frac{C}{\rho}s}]f(u(s))v(s)ds = u_K - u_0. \tag{2.2.18}$$

From (2.1.19),

$$v(0) = \frac{1}{C}\int_{-\infty}^{0} [1 - e^{Cs}][K - f(u(s))]v(s)ds.$$

Using (2.2.16) and (2.2.18), it follows that

$$\begin{aligned} v(0) &= \frac{1}{C}\int_{-\infty}^{0} [1 - e^{Cs}][K - f(u(s))]v(s)ds \\ &\leq \frac{1}{C}\int_{-\infty}^{0} [1 - e^{\frac{C}{\rho}s}]f(u(s))v(s)ds = u_K - u_0. \end{aligned} \tag{2.2.19}$$

From (2.1.18) $v(t)$ is increasing for $t \in (-\infty, 0]$. Hence,

$$0 < v(t) \leq v(0) \leq u_K - u_0, \ t \in (-\infty, 0]. \tag{2.2.20}$$

Moreover from (2.1.18) and (2.2.20),

$$\begin{aligned} 0 \leq \quad \dot{v}(t) \quad &= \int_{-\infty}^{t} e^{-C(t-s)}[K - f(u(s))]v(s) \\ &\leq K(u_K - u_0) \int_{-\infty}^{t} e^{-C(t-s)} ds \\ &= \frac{K(u_K - u_0)}{C}, \quad t \in (-\infty, 0]. \end{aligned}$$

□

Lemma 2.2.4. *[9] Let $u^0 > u_K$ be fixed. Then for each $\delta > 0$ with $\delta < u^0 - u_K$, there exists a $\sigma > 0$ such that if $(u(t), v(t))$ is a solution of the system* (2.1.5) *with*

$$|u(t_0) - u^0| < \sigma, \quad |\dot{u}(t_0)| < \sigma, \quad \sqrt{v^2(t) + [\dot{v}(t_0)]^2} < \sigma, \quad (2.2.21)$$

then $(u(t), v(t))$ exists for all $t \geq t_0$ and

$$|u^0 - u(t)| < \delta, \quad t \geq t_0, \quad lim_{t\to\infty} v(t) = 0.$$

Define the quadratic form

$$\begin{aligned} V_0(v_1, v_2) \quad &= C^2 v_1^2 + 2Cv_1 v_2 + 2v_2^2, \quad (v_1, v_2) \in \mathbb{R}^2, \\ &= (Cv_1 + v_2)^2 + v_2^2, \quad (v_1, v_2) \in \mathbb{R}^2. \end{aligned} \quad (2.2.22)$$

Hence V_0 is positive and there are positive constants $\eta_1 < \eta_2$ such that

$$\eta_1(v_1^2 + v_2^2) \leq V_0(v_1, v_2) \leq \eta_2(v_1^2 + v_2^2), \quad (v_1, v_2) \in \mathbb{R}^2. \quad (2.2.23)$$

Since $\lim_{u\to u_K} f(u) = K$, define $\bar{u}_0 < u_K$ such that

$$K - f(u) \leq \frac{1}{\gamma} f(u), \text{ where } \gamma > 0,$$

where $u \in [\bar{u}_0, u_K]$.

Lemma 2.2.5. *[9] Let* $\bar{u}_0$ *be defined as above, for each fixed* $u^0 > u_K$*, if* $u_0 \in [\bar{u}_0, u_K)$ *is sufficiently close to* u_K*, then the relation* $u^+(u_0) < u^0$ *holds.*

Proof. Since $f(u)$ is continues, so $f(u) \to K$ as $u \to u_K$, there is a $\bar{u}$ with $u_K < \bar{u} < u^0$ such that

$$f(u) - K \leq \frac{C^2}{2}, \quad u \in [u_K, \bar{u}]. \tag{2.2.24}$$

Consider the following two cases.

Case I. $u^+(u_0) \leq \bar{u}$.

Since $u^+(u_0) \leq \bar{u}$ and $\bar{u} < u^0$, hence $u^+(u_0) < u^0$.

Case II. $u^+(u_0) > \bar{u}$.

Without loss of generality suppose that $u(0) = u_K$. Then $\exists \ \ t_0 \in (0, t_M)$ such that

$$u(t_0) = \bar{u}, \quad u_K < u(t) \leq \bar{u}, \quad t \in (0, t_0).$$

So that $f(u(t)) - K > 0$ for $t \in (0, t_0]$. Let $v_1(t) = v(t)$, $\dot{v}_1(t) = v_2(t)$, and $V_0(t) = V_0(v_1(t), v_2(t))$ for $t \in (0, t_0]$. From the second equation of the system (2.1.5), it follows that

$$\dot{v}_2(t) = \ddot{v}_1(t) = \ddot{v}(t) = -C\dot{v}_1(t) - [f(u(t)) - K]v_1(t). \tag{2.2.25}$$

Now from equation (2.2.22) with the use of equation (2.2.25), it follows that

$$\begin{aligned} \dot{V}_0(t) &= 2C^2 v_1(t)\dot{v}_1(t) + 2C[v_1(t)\dot{v}_2(t) + v_2(t)\dot{v}_1(t)] + 4v_2(t)\dot{v}_2(t) \\ &= 2C^2 v_1(t)v_2(t) + 2C[v_1(t)\{-C\dot{v}_1(t) - [f(u(t)) - K]v_1(t)\} + v_2^2(t)] \\ &\quad +4v_2(t)[-C\dot{v}_1(t) - [f(u(t)) - K]v_1(t)]. \end{aligned} \tag{2.2.26}$$

Rearranging the equation (2.2.26), it follows that

$$\begin{aligned}\dot{V}_0(t) &= -2C[f(u(t)) - K]v_1^2 - 4[f(u(t)) - K]v_1v_2 - 2Cv_2^2 \\ &= -\frac{2}{C}[f(u(t)) - K](C^2v_1^2 + 2Cv_1v_2 + \frac{C^2}{f(u(t)) - K}v_2^2).\end{aligned} \tag{2.2.27}$$

Equation (2.2.24) implies that

$$\frac{C^2}{f(u(t)) - K} \geq 2, \quad t \in (0, t_0].$$

The above inequality, with the use of equation (2.2.27), implies that

$$\begin{aligned}\dot{V}(t) &\leq -\frac{2}{C}[f(u(t)) - K](C^2v_1^2 + 2Cv_1v_2 + 2v_2^2) \\ &= -\frac{2}{C}[f(u(t)) - K]V_0(t) < 0, \quad t \in (0, t_0].\end{aligned} \tag{2.2.28}$$

Hence $V(t)$ is monotone decreasing for $t \in (0, t_0]$. Using the monotone decreasing property of $V(t)$ and equation (2.2.23), it follows that

$$\eta_1(v^2(t) + [\dot{v}(t)]^2) < V_0(t) < V_0(0)) \leq \eta_2(v^2(0) + [\dot{v}(0)]^2), \quad t \in (0, t_0]. \tag{2.2.29}$$

Using Lemma (2.2.3), it follows that

$$\begin{aligned}(v^2(0) + [\dot{v}(0)]^2) &\leq (u_K - u_0)^2 + \frac{K^2(u_K - u_0)^2}{C^2} \\ &= (u_K - u_0)^2(1 + \frac{K^2}{C^2}) \\ &= (u_K - u_0)^2\frac{(C^2 + K^2)}{C^2}.\end{aligned} \tag{2.2.30}$$

Equations (2.2.29) and (2.2.30) implies that

$$(v^2(t) + [\dot{v}(t)]^2) < \frac{\eta_2(v^2(0) + [\dot{v}(0)]^2)}{\eta_1} \leq (u_K - u_0)^2\frac{\eta_2(C^2 + K^2)}{\eta_1 C^2}.$$

From the above equation it follows that

$$v(t) \leq \sqrt{(v^2(t) + [\dot{v}(t)]^2)} < (u_K - u_0)\sqrt{\frac{\eta_2(C^2+K^2)}{\eta_1 C^2}}, \ \ t \in (0, t_0]. \tag{2.2.31}$$

Since $\eta_2 > \eta_1$ and by Lemma (2.2.3) $v(t) \leq u_K - u_0$ for $t \in (-\infty, 0]$, hence from equation (2.2.31), it follows that

$$v(t) \leq u_K - u_0 \leq (u_K - u_0)\sqrt{\frac{\eta_2(C^2+K^2)}{\eta_1 C^2}}, \ \ t \in (-\infty, 0]. \tag{2.2.32}$$

It follows from (2.1.16) and (2.2.32) that

$$\begin{aligned} 0 &\leq \dot{u}(t_0) \\ &\leq \frac{f^M(\bar{u})}{\rho}\sqrt{\frac{\eta_2(C^2+K^2)}{\eta_1 C^2}}(u_K - u_0)\int_{-\infty}^{t_0} e^{\frac{-C(t_0-s)}{\rho}} ds \\ &= \frac{f^M(\bar{u})}{C}\sqrt{\frac{\eta_2(C^2+K^2)}{\eta_1 C^2}}(u_K - u_0), \end{aligned} \tag{2.2.33}$$

where $f^M(\bar{u}) = \max\{f(u) : u \in [u_K, \bar{u}]\}$. Suppose that $\delta > 0$ with $\delta < \min\{\bar{u} - u_K, u^0 - \bar{u}\}$, $\sigma > 0$ be the corresponding number defined in Lemma (2.2.4), such that

$$\begin{aligned} &\sqrt{\frac{\eta_2(K^2+C^2)}{\eta_1 C^2}}(u_K - u_0) < \sigma, \\ &\frac{f^M(\bar{u})}{C}\sqrt{\frac{\eta_2(K^2+c^2)}{\eta_1 C^2}}(u_K - u_0) < \sigma. \end{aligned} \tag{2.2.34}$$

From equations (2.2.31), (2.2.33), (2.2.34) and the fact $u(t_0) = \bar{u}$, it follows that

$$|u(t_0) - \bar{u}| < \sigma, \ \ |\dot{u}(t_0)| < \sigma, \ \ \sqrt{v^2(t_0) + [\dot{v}(t_0)]^2} < \sigma.$$

By Lemma (2.2.4), $u(t) \leq \bar{u} + \delta < u^0$ for all $t \geq t_0$ and hence $u^+(u_0) < u^0$. □

The following result [9] establishes the existence of a traveling wave solution of the system (2.1.1).

Theorem 2.2.6. *Suppose that $f : [0, \infty) \to \mathbb{R}$ is Lipschitz continuous and satisfy the assumptions (A_1) and (A_2). Then for each $S^0 > u_K$ and*

$$C \geq \sqrt{4[sup\{f(u) : u \in [u_K, S^0]\} - K]}, \tag{2.2.35}$$

there is a $S_0 \in (0, u_K)$ such that system (2.1.1) has a traveling wave solution $(S(x,t), P(x,t)) = (U(x-ct)), V(x-ct))$, connecting $(S^0, 0)$ and $(S_0, 0)$ satisfying boundary conditions (2.1.2). Moreover $U(z)$ is strictly monotone decreasing and there is a unique t_1 such that $V(z)$ is increasing on $(-\infty, t_1)$ and decreasing on (t_1, ∞).

Proof. Consider the system

$$\rho\ddot{u} = -C\dot{u} + f(u)v,$$
$$\ddot{v} = -C\dot{v} + [f(u) - K]v,$$

with boundary conditions

$$u(-\infty) = u_0 < u_k < u^0 = u(\infty), \qquad v(-\infty) = v(\infty) = 0.$$

Firstly, consider the strict inequality for the above system, that is,

$$4[f^M(u^0) - K] < C^2.$$

Let

$$\Omega = \{u_0 \in (0, u_K) : u^+(u_0) < u^0\}.$$

By Lemmas (2.2.2) and (2.2.5) the set Ω is nonempty and it has a positive lower bound. Let

$$\inf\{u_0 : u_0 \in \Omega\} = u_0^m > 0$$

and $u_0^m < u_K$. It is claimed that $u^+(u_0^m) = u^0$, if this is not true, then one of the following cases must occur

Case I. $u^+(u_0^m) < u^0$.

Let $W^u(u_0)$ be the unstable manifold of the equilibrium $(u_0, 0, 0, 0)$. Then, $W^u(u_0)$ is continuously depending on u_0. Moreover, a solution depends continuously on its initial value on any finite interval. If $\acute{u}_0 < u_0^m$ is very close to u_0^m, then with the use of continuity property and Lemma (2.2.5) it is concluded that $u^+(\acute{u}_0) < u^0$. This contradicts the definition of u_0^m. Hence $u^+(u_0^m) \nless u^0$.

Case II. $u^+(u_0^m) > u^0$.

By continuity, there is an $\epsilon > 0$ such that for each $u_0 \in (u_0^m + \epsilon, u_0^m]$, $u^+(u_0) > u_0$. Hence $u_0 \notin \Omega$ for all $u_0 \in [u_0^m, u_0^m + \epsilon)$. It follows that $inf\{u_0 : u_0 \in \Omega\} > u_0^m$, which again leads to a contradiction. It follows that $u^+(u_0^m) \ngtr u^0$. Since both the cases leads to contradictions so $u^+(u_0^m) = u^0$ and by Lemma (2.2.1), $t_M(u_0) = +\infty$.

Secondly, consider $4[f^M(u^0) - K] = C^2$. Choose a sequence $C_n > C$, $n = 1, 2, 3, 4....$ with $C_n \to C$ as $n \to \infty$. Since $4[f^M(u^0) - K] = C^2 < C_n^2$ for $n = 1, 2, 3, 4......$ By the above arguments, for each n there exists u_0^n such that $u^+(u_0^n) = u^0$ and $t_M(u_0^n) = +\infty$. Since there exists an $\epsilon > 0$ such that $0 < \epsilon \le u_0^n \le u_K - \epsilon$ for all n, the sequence $\{u_0^n\}$ has a subsequence that converges to some number $u_0 \in (0, u_K)$. So that $u^+(u_0) = u^0$. Hence by Lemma (2.2.1), $t_M(u_0) = +\infty$. □

Remark 2.2.7. If f is monotone increasing then the condition (2.2.35) becomes

$$C \ge \sqrt{4[f(u^0) - K]}.$$

2.3 Uniqueness of traveling wave solutions

In this section the uniqueness of a traveling wave solution of the system (2.1.1) is established. Suppose that (A_1), (A_2) holds and $f(u)$ also satisfies

(A_3) $f(u)$ in the system (2.1.1) is monotone increasing.

The second equation of the system (2.1.5) can be rewritten as

$$\begin{aligned}\ddot{v} &= -C\dot{v} - [f(u(t)) - K]v \\ &= -C\dot{v} - [f(u) - f(u^0) + f(u^0) - K]v \\ &= -C\dot{v} - [f(u^0) - K]v + [f(u^0) - f(u)]v \\ &= -C\dot{v} - [f(u^0) - K]v + g(t)v(t),\end{aligned}$$

or

$$\ddot{v} + C\dot{v} + [f(u^0) - K]v = \phi(t), \tag{2.3.1}$$

where $\phi(t) = g(t)v(t)$ and $g(t) = f(u^0) - f(u(t))$, $t \in \mathbb{R}$.

Using the increasing property of f, it follows that

$$g(t) = f(u^0) - f(u(t)) > 0,$$

as $u \leq u^0$. The characteristic equation of the equation (2.3.1) is

$$\lambda^2 + C\lambda + f(u^0) - K = 0,$$

whose roots are

$$\lambda_1 = \frac{-C + \sqrt{C^2 - 4[f(u^0) - K]}}{2},$$
$$\lambda_2 = \frac{-C - \sqrt{C^2 - 4[f(u^0) - K]}}{2}.$$

When $C^2 > 4[f(u^0) - K]$, then clearly $\lambda_1 > \lambda_2$. Applying the variation of constant formula to (2.3.1), it follows that

$$\begin{aligned} v(t) &= c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t} - e^{\lambda_1 t} \int_0^t \frac{e^{\lambda_2 s} g(s) v(s)}{(\lambda_2 - \lambda_1) e^{(\lambda_1 + \lambda_2)s}} ds \\ &\quad + e^{\lambda_2 t} \int_0^t \frac{e^{\lambda_1 s} g(s) v(s)}{(\lambda_2 - \lambda_1) e^{(\lambda_1 + \lambda_2)s}} ds \\ &= c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t} + \frac{1}{\lambda_1 - \lambda_2} [\int_0^t [e^{\lambda_1 (t-s)}] g(s) v(s) ds \\ &\quad - \int_0^t [e^{\lambda_2 (t-s)}] g(s) v(s) ds] \\ &= c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t} + \delta \int_0^t [e^{\lambda_1 (t-s)} - e^{\lambda_2 (t-s)}] g(s) v(s) ds, \end{aligned} \tag{2.3.2}$$

where

$$\delta = \frac{1}{\lambda_1 - \lambda_2} = \frac{1}{\sqrt{C^2 - 4[f(u^0) - K]}} > 0$$

and the constants c_1, c_2 satisfy the following equations

$$v(0) = c_1 + c_2, \quad \dot{v}(0) = c_1 \lambda_1 + c_2 \lambda_2.$$

Solving for c_1 and c_2, it follows that

$$\begin{aligned} c_1 &= \frac{1}{\lambda_1 - \lambda_2} [-\lambda_2 v(0) + \dot{v}(0)] = \delta[-\lambda_2 v(0) + \dot{v}(0)], \\ c_2 &= \frac{1}{\lambda_1 - \lambda_2} [\lambda_1 v(0) - \dot{v}(0)] = \delta[\lambda_1 v(0) - \dot{v}(0)]. \end{aligned} \tag{2.3.3}$$

Since $\lambda_2 - \lambda_1 < 0$ and $g(t)v(t) = (f(u^0) - f(u(t)))v(t) > 0$, hence from (2.3.2)

$$\begin{aligned}
v(t) &= c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t} + \delta \int_0^t [e^{\lambda_1 (t-s)} - e^{\lambda_2 (t-s)}] g(s) v(s) ds \\
e^{-\lambda_1 t} v(t) &= c_1 + c_2 e^{(\lambda_2 - \lambda_1)t} \\
&\quad + \delta [\int_0^t e^{-\lambda_1 s} v(s) g(s) ds - \int_0^t e^{\lambda_2 (t-s) - \lambda_1 t + \lambda_1 s} e^{-\lambda_1 s} v(s) g(s) ds] \\
&= c_1 + c_2 e^{(\lambda_2 - \lambda_1)t} \\
&\quad + \delta [\int_0^t e^{-\lambda_1 s} v(s) g(s) ds - \int_0^t e^{(\lambda_2 - \lambda_1)(t-s)} e^{-\lambda_1 s} v(s) g(s) ds] \\
&\le |c_1| + |c_2| + \delta \int_0^t e^{-\lambda_1 s} v(s) g(s) ds, \quad t \ge 0. \qquad (2.3.4)
\end{aligned}$$

Since $(u(t), v(t))$ converges to $(u^0, 0)$ as $t \to \infty$ and $v(t) \neq 0$, $(u(t), \dot{u}(t), v(t), \dot{v}(t))$ must converges to $(u^0, 0, 0, 0)$ along the stable manifold of the equilibrium $(u^0, 0, 0, 0)$. So that $u(t)$ converges to u^0 exponentially as $t \to \infty$, hence $g(t) = f(u^0) - f(u(t))$ converges to 0 exponentially as $t \to \infty$, also $g(t)$ is continuous and bounded on $[0, \infty)$, hence,

$$\int_0^\infty g(s) ds < \infty.$$

Applying the Gronwall's inequality to (2.3.4), the following is obtained

$$e^{-\lambda_1 t} v(t) \le (|c_1| + |c_2|) e^{\delta \int_0^t g(s) ds} \le (|c_1| + |c_2|) e^{\delta \int_0^\infty g(s) ds} < \infty, \ t \ge 0. \qquad (2.3.5)$$

Hence,

$$\lim_{t \to \infty} \int_t^\infty g(s) v(s) e^{-\lambda_1 s} ds = 0. \qquad (2.3.6)$$

Moreover, $g(t)v(t)e^{-\lambda_1 t} \to 0$ as $t \to \infty$ implies that

$$\lim_{t \to \infty} \int_0^t e^{(\lambda_2 - \lambda_1)(t-s)} g(s) v(s) e^{-\lambda_1 s} ds = 0. \qquad (2.3.7)$$

From (2.3.2), the expression for $v(t)$ is

$$\begin{aligned} v(t) &= c_1e^{\lambda_1 t} + c_2e^{\lambda_2 t} + \delta\int_0^t [e^{\lambda_1(t-s)} - e^{\lambda_2(t-s)}]g(s)v(s)ds \\ &= e^{\lambda_1 t}[c_1 + c_2e^{(\lambda_2-\lambda_1)}t + \delta\int_0^\infty g(s)v(s)e^{-\lambda_1 s}ds \\ &\quad -\delta\int_t^\infty g(s)v(s)e^{-\lambda_1 s}ds - \delta\int_0^t e^{\lambda_2(t-s)-\lambda_1 t}g(s)v(s)ds] \\ &= e^{\lambda_1 t}[\delta(-\lambda_2 v(0) + \dot{v}(0)) + \delta\int_0^\infty g(s)v(s)e^{-\lambda_1 s}ds + \delta(\lambda_1 v(0) - \dot{v}(0)) \\ &\quad e^{(\lambda_2-\lambda_1)t} - \delta\int_t^\infty g(s)v(s)e^{-\lambda_1 s}ds - \delta\int_0^t e^{(\lambda_2-\lambda_1)(t-s)}g(s)v(s)e^{-\lambda_1 s}ds]. \end{aligned} \tag{2.3.8}$$

The equation (2.3.8) can be rewritten as:

$$v(t) = e^{\lambda_1 t}[M + H(t)]. \tag{2.3.9}$$

with

$$M = \delta[-\lambda_2 v(0) + \dot{v}(0) + \int_0^\infty g(s)v(s)e^{-\lambda_1 s}ds] \tag{2.3.10}$$

and

$$H(t) = \delta(\lambda_1 v(0) - \dot{v}(0))e^{(\lambda_2-\lambda_1)t} - \delta\int_t^\infty g(s)v(s)e^{-\lambda_1 s}ds - \delta\int_0^t e^{(\lambda_2-\lambda_1)(t-s)}g(s)v(s)e^{-\lambda_1 s}ds.$$

From (2.3.6)-(2.3.7) and the expression for $H(t)$ one can easily see that

$$\lim_{t\to\infty} H(t) = 0.$$

Remark 2.3.1. When $C^2 = 4[f(u^0) - K]$, that is, $\lambda_1 = \lambda_2$, then the expression for $v(t)$ is

$$v(t) = e^{\lambda_1 t}([-\lambda_1 v(0) + \dot{v}(0)]t + v(0)) + \int_0^t e^{\lambda_1(t-s)}(t-s)g(s)v(s)ds. \tag{2.3.11}$$

Lemma 2.3.2. *[17] Suppose that* $(u_i(t), v_i(t))$, $i = 1, 2$ *are two traveling wave solutions of the system* (2.1.5) *connecting* $(\xi_i, 0)$ *and* $(u^0, 0)$, *where* $0 < \xi_1 < \xi_2 < u_K$. *If there is a* $t_1 \leq \infty$ *such that* $u_1(t) < u_2(t)$ *for* $t \in (-\infty, t_1)$ *and* $u_1(t_1) = u_2(t_1)$, *then there is a* $t_0 < t_1$ *such that* $v_1(t_0) = v_2(t_0)$ *and* $\dot{v}_1(t_0) > \dot{v}_2(t_0)$.

Proof. The characteristic equation for the equilibrium point $(\xi_i, 0)$ of the system (2.1.5) is

$$\lambda(\frac{C}{\rho} + \lambda)(\lambda^2 + C\lambda + f(\xi_i) - K) = 0, \tag{2.3.12}$$

roots of the above equation are

$$\begin{aligned} \lambda_{1,i} &= \frac{-C + \sqrt{C^2 + 4[K - f(\xi_i)]}}{2}, \\ \lambda_{2,i} &= \frac{-C - \sqrt{C^2 + 4[K - f(\xi_i)]}}{2}, \\ \lambda_{3,i} &= \frac{-C}{\rho}, \quad \lambda_{4,i} = 0, \quad i = 1, 2. \end{aligned}$$

If $u_0 < u_K$, then of these four eigenvalues, $\lambda_{1,i}$ is positive and $\lambda_{2,i}$, $\lambda_{3,i}$ are negative. Corresponding to the positive eigenvalue $\lambda_{1,i}$, there is a one dimensional unstable manifold. The positive traveling wave $(u_i(t), v_i(t))$ must converges to $(\xi_i, 0)$ as $t \to -\infty$, along the unstable manifold. Since f is monotone increasing and $\xi_1 < \xi_2$, so $f(\xi_1) < f(\xi_2)$ and one has $\lambda_{1,1} > \lambda_{1,2}$. Hence $v_1(t)$ goes to 0 faster than $v_2(t)$ as $t \to -\infty$. So there is a t' such that

$$v_1(t) < v_2(t), \quad t \in (-\infty, t']. \tag{2.3.13}$$

Next it is claimed that there is a $t_0' \in (-\infty, t_1)$ such that $v_1(t_0') > v_2(t_0')$. If this is not the case, then $v_1(t) \leq v_2(t)$ for all $t \in (-\infty, t_1)$. From the monotone increasing property of f it follows that

$$f(u_2(t))v_2(t) \geq f(u_1(t))v_1(t). \tag{2.3.14}$$

From (2.1.17)

$$u(t) = u_0 + \frac{1}{C}\int_{-\infty}^{t}[1 - e^{\frac{-C(t-s)}{\rho}}]f(u(s))v(s)ds. \qquad (2.3.15)$$

Using (2.3.14), it follows that

$$\begin{aligned} u_2(t_1) &= \xi_2 + \frac{1}{C}\int_{-\infty}^{t_1}[1 - e^{\frac{-C(t_1-s)}{\rho}}]f(u_2(s))v_2(s)ds \\ &> \xi_1 + \frac{1}{C}\int_{-\infty}^{t_1}[1 - e^{\frac{-C(t_1-s)}{\rho}}]f(u_1(s))v_1(s)ds \\ &= u_1(t_1). \end{aligned} \qquad (2.3.16)$$

Equation (2.3.16) is a contradiction to the assumption $u_1(t_1) = u_2(t_1)$. Hence there exists a $t_0' \leq t_1$ such that $v_1(t_0') > v_2(t_0')$. Now from (2.3.13) and the continuity it follows that there is a $t_0 < t_1$ such that

$$v_1(t) < v_2(t), \quad t \in (-\infty, t_0), \quad v_1(t_0) = v_2(t_0). \qquad (2.3.17)$$

Obviously $\dot{v}_1(t_0) \geq \dot{v}_2(t_0)$. Suppose that $\dot{v}_1(t_0) = \dot{v}_2(t_0)$. Then, from the inequality $f(u_1(t_0)) < f(u_2(t_0))$ and second equation of the system (2.1.5), it follows that

$$\begin{aligned} \ddot{v}_1(t_0) &= -C\dot{v}_1(t_0) + [K - f(u_1(t_0))]v_1(t_0) \\ &= -C\dot{v}_2(t_0) + [K - f(u_1(t_0))]v_2(t_0) \\ &> -C\dot{v}_2(t_0) + [K - f(u_2(t_0))]v_2(t_0) \\ &= \ddot{v}_2(t_0). \end{aligned} \qquad (2.3.18)$$

Hence $\ddot{v}_1(t_0) - \ddot{v}_2(t_0) > 0$. Thus the function $v_1(t) - v_2(t)$ has a local minimum zero at t_0. This is a contradiction to (2.3.17). Hence $\dot{v}_1(t_0) > \dot{v}_2(t_0)$. □

The following result will be used in our later work.

Lemma 2.3.3. *[17] Let* $(u_i(t), v_i(t))$, $i = 1, 2$ *be two positive solutions of the system* (2.1.5) *satisfying*

$$v_1(t_0) = v_2(t_0), \quad \dot{v}_1(t_0) > \dot{v}_2(t_0),$$

for some $t_0 \in \mathbb{R}$ *and if there exists* $t^* > t_0$ *such that*

$$u_1(t) < u_2(t) < u^0, \quad t \in [t_0, t^*),$$

then

$$v_1(t) > v_2(t), \ \ \forall \ \ t \in (t_0, t^*].$$

Proof. Without loss of generality suppose that $t_0 = 0$. Also let $g_i(t) = f(u^0) - f(u_i(t))$. Since $u_1(t) < u_2(t)$ for $t \in [0, t^*)$, so that $g_1(t) > g_2(t) > 0$ for $t \in [0, t^*)$. Since

$$v_1(0) = v_2(0), \qquad \dot{v}_1(0) > \dot{v}_2(0),$$

implies that $v_1(t) > v_2(t)$ for small $t > 0$. It is claimed that $v_1(t) > v_2(t)$ remains true for all $t \in (0, t^*]$. If this is not true, then there must be a $t_1 \in (0, t^*]$ such that $v_1(t_1) = v_2(t_1)$ and

$$v_1(t) > v_2(t), \qquad t \in [0, t_1).$$

Then clearly

$$g_1(t)v_1(t) - g_2(t)v_2(t) > 0, \qquad t \in [0, t_1).$$

By formulas (2.3.4) and (2.3.5), it follows that

$$\begin{aligned}
v_1(t_1) - v_2(t_1) &= \delta[-\lambda_2 v_1(0) + \dot{v}_1(0)]e^{\lambda_1 t_1} + \delta[\lambda_1 v_1(0) - \dot{v}_1(0)]e^{\lambda_2 t_1} \\
&\quad + \int_0^{t_1} [e^{\lambda_1(t_1 - s)} - e^{\lambda_2(t_1 - s)}]g_1(s)v_1(s)ds - \delta[-\lambda_2 v_2(0) \\
&\quad + \dot{v}_2(0)]e^{\lambda_1 t_1} - \delta[\lambda_1 v_2(0) - \dot{v}_2(0)]e^{\lambda_2 t_1} \\
&\quad - \delta \int_0^{t_1} [e^{\lambda_1(t_1 - s)} - e^{\lambda_2(t_1 - s)}]g_2(s)v_2(s)ds \\
&= \delta e^{\lambda_1 t_1}[\dot{v}_1(0) - \dot{v}_2(0)] - \delta e^{\lambda_2 t_1}[\dot{v}_1(0) - \dot{v}_2(0)] \\
&\quad + \delta \int_0^{t_1} [e^{\lambda_1(t_1 - s)} - e^{\lambda_2(t_1 - s)}][g_1(s)v_1(s) - g_2(s)v_2(s)]ds \\
&= \delta[\dot{v}_1(0) - \dot{v}_2(0)](e^{\lambda_1 t_1} - e^{\lambda_2 t_1}) \\
&\quad + \delta \int_0^{t_1} [e^{\lambda_1(t_1 - s)} - e^{\lambda_2(t_1 - s)}][g_1(s)v_1(s) - g_2(s)v_2(s)]ds.
\end{aligned} \tag{2.3.19}$$

Since

$$\dot{v_1}(0) - \dot{v_2}(0) > 0, \quad g_1(t)v_1(t) - g_2(t)v_2(t) > 0, \qquad t \in [0, t_1),$$

hence

$$\int_0^{t_1} [e^{\lambda_1(t_1-s)} - e^{\lambda_2(t_1-s)}][g_1(s)v_1(s) - g_2(s)v_2(s)]ds > 0.$$

It follows from the equation (2.3.19) that $v_1(t_1) - v_2(t_1) > 0$, which contradicts the assumption $v_1(t_1) = v_2(t_1)$. Hence $v_1(t) > v_2(t)$ is true for all $t \in (0, t^*]$. □

Remark 2.3.4. If $\lambda_1 = \lambda_2$, then using the equation (2.3.11) the following results are obtained

$$v(t) = e^{\lambda_1 t}[M^0 t + H^0(t)] = te^{\lambda_1 t}[M^0 t + \frac{1}{t}H^0(t)], \tag{2.3.20}$$

where

$$M^0 = [-\lambda_1 v(0) + \dot{v}(0) + \int_0^\infty e^{-\lambda_1 s} g(s)v(s)\, ds] \tag{2.3.21}$$

and

$$H^0(t) = v(0) - t\int_t^\infty e^{-\lambda_1 s} g(s)v(s)ds - \int_0^t se^{-\lambda_1 s} g(s)v(s)ds,$$

$\frac{1}{t}H^0(t) \to 0$ as $t \to \infty$.

Remark 2.3.5. The Lemma (2.3.3) remains true for the case $C = \sqrt{4[f(u^0) - K]}$.

Since $v(t) > 0$ for all $t \in \mathbb{R}$ and $v(-\infty) = v(\infty) = 0$, hence there exists a time $t_0 \in \mathbb{R}$ such that $\dot{v}(t_0) = 0$. Let $(\tilde{u}(t), \tilde{v}(t)) = (u(t + t_0), v(t + t_0))$ be a translation. Then $(\tilde{u}(t), \tilde{v}(t))$ is a traveling wave and

$$\tilde{u}(-\infty) = u(-\infty + t_0) = u(-\infty) = u_0,$$
$$\tilde{u}(\infty) = u(\infty + t_0) = u(\infty) = u^0.$$

Also,

$$\tilde{v}(-\infty) = v(-\infty + t_0) = v(-\infty) = 0,$$
$$\tilde{v}(\infty) = v(\infty + t_0) = v(\infty) = 0,$$

which implies that $(\tilde{u}(t), \tilde{v}(t))$ connects the same points as $(u(t), v(t))$ does. Further, from $\dot{\tilde{v}}(0) = \dot{v}(t_0) = 0$, (2.3.8) and (2.3.9), it follows that

$$\begin{aligned}\tilde{v}(t) &= e^{\lambda_1 t}[\delta(-\lambda_2 \tilde{v}(0)) + \delta \int_0^\infty g(s)\tilde{v}(s)e^{-\lambda_1 s}ds + \delta(\lambda_1 \tilde{v}(0))e^{(\lambda_2-\lambda_1)t} \\ &\quad -\delta \int_t^\infty g(s)\tilde{v}(s)e^{-\lambda_1 s}ds - \delta \int_t^\infty e^{(\lambda_2-\lambda_1)(t-s)}g(s)\tilde{v}(s)e^{-\lambda_1 s}ds \\ &= e^{\lambda_1 t}[\tilde{M} + \tilde{H}(t)],\end{aligned}$$

where

$$\tilde{M} = \delta[-\lambda_2 \tilde{v}(0) + \int_0^\infty [f(u^0) - f(\tilde{u}(s)]\tilde{v}(s)e^{-\lambda_1 s}ds]$$

and

$$\tilde{H}(t) = \delta(\lambda_1 \tilde{v}(0))e^{(\lambda_2-\lambda_1)t} - \delta \int_t^\infty g(s)\tilde{v}(s)e^{-\lambda_1 s}ds - \delta \int_0^t e^{(\lambda_2-\lambda_1)(t-s)}g(s)\tilde{v}(s)e^{-\lambda_1 s}ds.$$

From equations (2.3.6) and (2.3.7), it follows that $\tilde{H}(t) \to 0$ as $t \to \infty$.
Since $f(u^0) - f(\tilde{u}(s)) > 0$ for all $s \in (0, \infty)$ and $\tilde{v}(s) > 0$ for all $s \in (0, \infty)$, hence,

$$\int_0^\infty [f(u^0) - f(\tilde{u}(s)]\tilde{v}(s)e^{-\lambda_1 s}ds > 0,$$

which implies

$$\tilde{M} > 0.$$

Now,

$$v(t) = \tilde{v}(t - t_0) = e^{\lambda_1(t-t_0)}[\tilde{M} + \tilde{H}(t - t_0)] = e^{\lambda_1 t}[e^{-\lambda_1 t_0}\tilde{M} + e^{\lambda_1 t_0}\tilde{H}(t - t_0)].$$

Since $e^{\lambda_1 t_0}\tilde{H}(t - t_0) \to 0$ as $t \to \infty$, by comparing the above inequality and (2.3.9), it follows that $M = e^{-\lambda_1 t_0}\tilde{M} > 0$.

Remark 2.3.6. The above result remains true for the case when $C = \sqrt{4[f(u^0) - K]}$ that is, when $\lambda_1 = \lambda_2$.

Proposition 2.3.7. *[17] Suppose there are two traveling waves $(u_i(t), v_i(t)), i = 1, 2$ connecting $(\xi_i, 0)$ and $(u^0, 0)$, where $0 < \xi_1 < \xi_2 < u_K$. If $u_1(t) < u_2(t)$, $t \in \mathbb{R}$, then $M_1 > M_2$ where M_i is the number in (2.3.9) associated with $(u_i(t), v_i(t))$ for $i = 1, 2$.*

Proof. Let $(u_i(t), v_i(t))$, $i = 1, 2$ be the two traveling wave solutions connecting $(\xi_i, 0)$ and $(u^0, 0)$, where $0 < \xi_1 < \xi_2 < u_K$ and $u_1(t) < u_2(t)$, $t \in \mathbb{R}$. By Lemma (2.3.2), there exists a $t_0 \in \mathbb{R}$ such that

$$v_1(t_0) = v_2(t_0), \quad \dot{v}_1(t_0) > \dot{v}_2(t_0).$$

Without loss of generality suppose that $t_0 = 0$. Since $u_1(t) < u_2(t)$ for all $t \in \mathbb{R}$, by Lemma (2.3.3), it follows that $v_1(t) > v_2(t)$ for all $t > 0$. Since f is monotone increasing, thus the inequality $u_1(t) < u_2(t)$ for $t > 0$ implies that $f(u_1(t)) < f(u_2(t))$ for $t > 0$, hence

$$g_1(t) = [f(u^0) - f(u_1(t))] > [f(u^0) - f(u_2(t))] = g_2(t), \ t > 0.$$

The above equation together with $v_1(t) > v_2(t)$, for $t > 0$, implies that

$$g_1(t)v_1(t) > g_2(t)v_2(t), \quad t > 0. \tag{2.3.22}$$

It follows from (2.3.10) that

$$\begin{aligned} M_1 &= \delta[-\lambda_2 v_1(0) + \dot{v}_1(0) + \int_0^\infty g_1(s)v_1(s)e^{-\lambda_1 s}ds \\ &> \delta[-\lambda_2 v_2(0) + \dot{v}_2(0) + \int_0^\infty g_2(s)v_2(s)e^{-\lambda_1 s}ds = M_2. \end{aligned}$$

□

Remark 2.3.8. The Proposition (2.3.7) remains true when $C = \sqrt{4[f(u^0) - K]}$.

Lemma 2.3.9. *[17] Let $y_i : \mathbb{R} \to \mathbb{R}$, $i = 1, 2$, be monotone increasing, differentiable function and $y_1(-\infty) < y_2(-\infty)$. If there are $t_1 < t_2$ such that*

$$y_1(t_1) > y_2(t_1), \qquad y_1(t_2) = y_2(t_2),$$

then there exist numbers t_0 and $\varepsilon^* > 0$ such that

$$\begin{aligned} &y_1(t_0 - \varepsilon^*) = y_2(t_0), \qquad \dot{y}_1(t_0 - \varepsilon^*) = \dot{y}_2(t_0), \\ &y_1(t - \varepsilon^*) < y_2(t), \qquad t \in (-\infty, t_0). \end{aligned}$$

Proof. Define $y^\varepsilon : (-\infty, t_2] \to \mathbb{R}$ by

$$y^\varepsilon(t) = y_1(t - \varepsilon), \qquad t \in (-\infty, t_2],$$

where $\varepsilon > 0$. Since y_1, y_2 are continuous, hence $y_1(-\infty) < y_2(-\infty)$, implies that there is a $\tau < t_1$ such that

$$y_1(t) < y_2(t), \qquad t \in (-\infty, \tau]. \tag{2.3.23}$$

Further, from the monotone increasing property of y_1 and y_2, it follows that

$$y^\varepsilon(t) = y_1(t - \varepsilon) < y_1(t) < y_2(t), \qquad t \in (-\infty, \tau], \ \varepsilon \geq 0.$$

Since $y_1(t_1) > y_2(t_1)$ and y_1, y_2 are continuous, so select a small enough $\varepsilon > 0$ such that $y^\varepsilon(t_1) = y_1(t_1 - \varepsilon) > y_2(t_1)$. Next, if $t \in [\tau, t_2]$, then $t - t_2 + \tau \leq \tau$. Hence from (2.3.23), it follows that

$$y^{t_2 - \tau}(t) = y_1(t - t_2 + \tau) < y_2(t - t_2 + \tau), \qquad t \in [\tau, t_2].$$

From the above inequality

$$\inf\{\varepsilon : y^\varepsilon(t) < y_2(t),\, t \in [\tau, t_2]\} = \varepsilon^* > 0.$$

From the definition of ε^* and continuity, it follows that $y^{\varepsilon^*}(t) \leq y_2(t)$ for $t \in [\tau, t_2]$. Hence there exists $\bar{t} \in [\tau, t_2)$, such that $y^\varepsilon(\bar{t}) = y_2(\bar{t})$. Further, let

$t_0 = \min\{\bar{t} \in [\tau, t_2) : y^{\varepsilon^*}(t) = y_2(\bar{t})\}$, then $t_0 < t_2$. From the definition of t_0 and (2.3.23), it is concluded that

$$y_1(t - \varepsilon^*) = y^{\varepsilon^*}(t) < y_2(t), \quad t < t_0,$$
$$y_1(t_0 - \varepsilon^*) = y_2(t_0), \quad \dot{y}_1(t_0 - \varepsilon^*) = \dot{y}_2(t_0).$$

□

Lemma 2.3.10. *[17] Suppose there are two traveling waves* $(u_i(t), v_i(t)), i = 1, 2$ *connecting* $(\xi_i, 0)$ *and* $(u^0, 0)$*, where* $0 < \xi_1 < \xi_2 < u_K$*. If there is a* $t_1 \in \mathbb{R}$ *such that* $u_1(t_1) = u_2(t_1)$ *and* $u_1(t) < u_2(t)$ *for all* $t \in (-\infty, t_1)$*, then* $\dot{u}_1(t_1) > \dot{u}_2(t_1)$*.*

Proof. As $u_1(t_1) = u_2(t_1)$ and $u_1(t) < u_2(t)$ for all $t \in (-\infty, t_1)$, hence $\dot{u}_1(t_1) \not< \dot{u}_2(t_1)$. It is claimed that $\dot{u}_1(t_1) > \dot{u}_2(t_1)$, if this not the case then $\dot{u}_1(t_1) = \dot{u}_2(t_1)$. By Lemma (2.3.2), there exists a $t_0 < t_1$ such that $v_1(t_0) = v_2(t_0)$ and $\dot{v}_1(t_0) > \dot{v}_2(t_0)$. Hence, it follows from Lemma (2.3.3) that $v_1(t) > v_2(t)$ for $t \in (t_0, t_1]$. Set $\vartheta(t) = u_2(t) - u_1(t)$, then $\vartheta(t) > 0$, $t \in (-\infty, t_1)$ and

$$\vartheta(t_1) = u_1(t_1) - u_2(t_1) = 0,$$
$$\dot{\vartheta}(t_1) = \dot{u}_1(t_1) - \dot{u}_2(t_1) = 0.$$

Since $\vartheta(t_1)$ is not a local maximum of ϑ in $(t_1 - \epsilon, t_1 + \epsilon)$ for small $\epsilon > 0$, hence

$$0 \leq \ddot{\vartheta}(t_1) = \ddot{u}_2(t_1) - \ddot{u}_1(t_1). \tag{2.3.24}$$

Further, using the first equation of the system (2.1.5), it follows that

$$\begin{aligned} \rho\ddot{u}_2(t_1) &= -C\dot{u}_2(t_1) + f(u_2(t_1))v_2(t_1) = -C\dot{u}_1(t_1) + f(u_1(t_1))v_2(t_1) \\ &< -C\dot{u}_1(t_1) + f(u_1(t_1))v_1(t_1) = \rho\ddot{u}_1(t_1), \end{aligned}$$

which implies that

$$\ddot{\vartheta}(t_1) = \ddot{u}_2(t_1) - \ddot{u}_1(t_1) < 0,$$

which is a contradiction to (2.3.24). Hence $\dot{u}_1(t_1) > \dot{u}_2(t_1)$. □

Proposition 2.3.11. *[17] Suppose there are two points* $0 < \xi_1 < \xi_2 < u_K$ *and two positive traveling wave solutions* $(u_i(t), v_i(t)), i = 1, 2$ *such that* $(u_i(t), v_i(t))$ *connects* $(\xi_i, 0)$ *and* $(u^0, 0)$*. If there is a* $t_1 \in \mathbb{R}$ *such that* $u_1(t_1) = u_2(t_1)$*, then the following holds:*

(1) $\dot{u}_1(t_1) > \dot{u}_2(t_1)$.

(2) $u_1(t) < u_2(t), \quad t \in (-\infty, t_1)$ *and* $u_1(t) > u_2(t), \quad t \in (t_1, -\infty)$.

(3) There is at least a $t_2 > t_1$ *such that* $v_1(t_2) < v_2(t_2)$.

Proof. Consider the set

$$\mathcal{A} = \{t : u_1(t) = u_2(t)\}.$$

Since it is given that there is a $t_1 \in \mathbb{R}$ such that $u_1(t_1) = u_2(t_1)$, so the set $\mathcal{A}$ is nonempty and has lower bound. Let

$$\tau = \inf\{t : u_1(t) = u_2(t)\} \leq t_1.$$

From the definition of τ it follows that $u_1(\tau) = u_2(\tau)$ and $u_1(t) < u_2(t)$ for all $t \in (-\infty, \tau)$. By Lemma (2.3.10) $\dot{u}_1(\tau) > \dot{u}_2(\tau)$. It is claimed that $\tau = t_1$. If this is not the case then $\tau < t_1$, which implies that $\dot{u}_1(\tau) > \dot{u}_2(\tau)$, hence that there is a $\tilde{t} \in (\tau, t_1)$ such that $u_1(\tilde{t}) > u_2(\tilde{t})$. Since

$$u_1(-\infty) = \xi_1 < \xi_2 = u_2(-\infty)$$

and there exists a $\tilde{t} < t_1$ such that

$$u_1(\tilde{t}) > u_2(\tilde{t}) \text{ and } u_1(t_1) = u_2(t_1),$$

using Lemma (2.3.9), it follows that there is a t_0 and $\alpha^* > 0$ such that

$$u_1(t_0 - \alpha^*) = u_2(t_0),\ \dot{u}_1(t_0 - \alpha^*) = \dot{u}_2(t_0),\ u_1(t - \alpha^*) < u_2(t),\ t \in (-\infty, t_0).$$

Let $\tilde{u}_1(t) = u_1(t-\alpha^*)$ and $\tilde{v}_1(t) = v_1(t-\alpha^*)$, then

$$\begin{aligned}
\tilde{u}_1(-\infty) &= u_1(-\infty-\alpha^*) = u_1(-\infty) = u_0, \quad i = 1,2,\\
\tilde{u}_1(\infty) &= u_1(\infty-\alpha^*) = u_1(\infty) = u^0, \quad i = 1,2,\\
\tilde{v}_1(-\infty) &= v_1(-\infty-\alpha^*) = v_1(-\infty) = 0, \quad i = 1,2,\\
\tilde{v}_1(\infty) &= v_1(\infty-\alpha^*) = v_1(\infty) = 0, \quad i = 1,2.
\end{aligned}$$

Hence $(\tilde{u}_1(t), \tilde{v}_1(t))$ is also a traveling wave solution connecting $(\xi_1, 0)$ and $(u^0, 0)$. Moreover

$$\tilde{u}_1(t) = u_1(t-\alpha^*) < u_2(t), \quad t \in (-\infty, t_0)$$

and

$$\tilde{u}_1(t_0) = u_1(t_0-\alpha^*) = u_2(t_0), \ \dot{\tilde{u}}_1(t_0) = \dot{u}_2(t_0),$$

that is, $\tilde{u}_1(t) < u_2(t)$ for $t \in (-\infty, t_0)$ and $\tilde{u}_1(t_0) = u_2(t_0)$, $\dot{\tilde{u}}_1(t_0) = \dot{u}_2(t_0)$. This leads to a contradiction of the Lemma (2.3.10). Hence $\tau = t_1$. This implies that $u_1(t) < u_2(t)$ for all $t < t_1$ and $\dot{u}_1(t_1) > \dot{u}_2(t_1)$. From the above discussion, it follows that τ is unique. The uniqueness of τ implies that $u_1(t) \neq u_2(t)$ for all $t > t_1$. Hence $u_1(t) < u_2(t)$, $t \in (-\infty, t_1)$ and $\dot{u}_1(t_1) > \dot{u}_2(t_1)$, implies that $u_1(t) > u_2(t)$ for all $t > t_1$.

Next, assume that $v_1(t) \geq v_2(t)$ for all $t \in [t_1, \infty)$. Integrating the first equation of the system (2.1.5) from t_1 to ∞ and with the use of Part (2), it follows that

$$\begin{aligned}
\rho[\dot{u}_1(\infty) - \dot{u}_1(t_1)] &= -C[u_1(\infty) - u_1(t_1)] + \int_{t_1}^{\infty} f(u_1(s))v_1(s)ds\\
-\rho\dot{u}_1(t_1) &= -C[u_2(\infty) - u_2(t_1)] + \int_{t_1}^{\infty} f(u_1(s))v_1(s)ds\\
&> -C[u_2(\infty) - u_2(t_1)] + \int_{t_1}^{\infty} f(u_2(s))v_2(s)ds\\
&= -\rho\dot{u}_2(t_1),
\end{aligned}$$

which implies that $\dot{u}_1(t_1) < \dot{u}_2(t_1)$. This leads to a contradiction of Part (1). Hence there must be a $t_2 > t_1$ such that $v_1(t_2) < v_2(t_2)$. □

Theorem 2.3.12. *[17] Let $u^0 > u_K$ and $C > 0$ be given with*

$$C \geq \sqrt{4[f(u^0) - K]}.$$

Then there is a unique $u_0 < u_K$ such that the system (2.1.5) *has a positive traveling wave solution connecting* $(u_0, 0)$ *and* $(u^0, 0)$.

Proof. **Case I.** Let $C > \sqrt{4[f(u^0) - K]}$.

Suppose there are two traveling waves $(u_i(t), v_i(t))$ connecting $(\xi_i, 0)$ and $(u^0, 0)$, $i = 1, 2$, with $0 < \xi_1 < \xi_2 < u_K$. Since

$$u_i(-\infty) = \xi_i < u_K < u^0 = u_i(\infty),\ i = 1, 2.$$

In view of the Corollary (2.1.3), with out loss of generality suppose that $u_1(0) = u_2(0) = u_K$. By Proposition (2.3.11), $\dot{u}_1(0) > \dot{u}_2(0)$. Hence, it follows that $u_1(a) > u_2(a)$ for all small $a > 0$. The inequality $u_1(-\infty) = \xi_1 < \xi_2 = u_2(-\infty)$ implies that for each small $a > 0$, there is a t_a such that $u_1(t_a - a) = u_2(t_a)$. By Part (2) the of preposition (2.3.11), it follows that

$$u_1(t - a) < u_2(t) \text{ for } t \in (-\infty, t_a) \text{ and } u_1(t - a) > u_2(t) \text{ for } t \in (t_a, \infty).$$

Let

$$\Omega = \{a \in \mathbb{R} : \text{there is a}\, t_a \, \text{such that}\, u_1(t_a - a) = u_2(t_a)\},$$

then Ω is nonempty. If $a \in \Omega$, then there exists a t_a such that $u_1(t_a - a) = u_2(t_a)$. Since $u_1(t)$ is monotone increasing, hence,

$$u_1(t_a - a') > u_1(t_a - a) = u_2(t_a), \quad a' < a.$$

Thus there is a $t_{a'}$ such that $u_1(t_{a'} - a') = u_2(t_{a'})$, which implies that $a' \in \Omega$. That is, $(-\infty, a] \subset \Omega$ whenever $a \in \Omega$. Hence Ω is an interval. Let $a_1, a_2 \in j$ with $a_1 < a_2$. Then

$$\begin{aligned} u_1(t_{a_1} - a_1) &= u_2(t_{a_1}) \\ u_1(t_{a_2} - a_2) &= u_2(t_{a_2}). \end{aligned} \tag{2.3.25}$$

From Part (2) of the Proposition (2.3.11), it follows that

$$u_1(t - a_1) < u_2(t),\ t < t_{a_1}.$$

The monotone increasing nature of $u_1(t)$ implies that

$$u_1(t - a_2) < u_1(t - a_1) \leq u_2(t),\ t \leq t_{a_1}.$$

It is claimed that $t_{a_2} > t_{a_1}$, for if $t_{a_2} \leq t_{a_1}$, then $u_1(t_{a_2} - a_2) < u_2(t_{a_2})$, a contradiction to the equation (2.3.25). Hence, $t_{a_2} > t_{a_1}$, that is, t_a is monotone increasing for $a \in \Omega$. Further, suppose that

$$a^* = sup\,\{a : a \in \Omega\}.$$

Since t_a is monotone increasing for $a \in \Omega$, hence, $\lim_{a \to a^*} t_a = t_{a^*}$ is well defined. It is claimed that $t_{a^*} = \infty$. If $t_{a^*} < \infty$, then, from the equality $u_1(t_a - a) = u_2(t_a)$ as $a \to a^*$, it follows that

$$u_1(t_{a^*} - a^*) = u_2(t_{a^*}) > \xi_2 > \xi_1.$$

Since $u_2(-\infty) = \xi_2 > \xi_1 = u_1(-\infty)$. It therefore follows that $t_{a^*} - a^* > -\infty$ or $a^* < \infty$. Thus both $t_{a^*} - a^*$ and t_{a^*} are real numbers. Hence, by Part (1) of Proposition (2.3.11), it follows that $\dot{u}_1(t_{a^*} - a^*) > \dot{u}_2(t_{a^*})$. From this inequality it is easily concluded that for each small $\epsilon > 0$, there is a $t_{a^*+\epsilon}$ such that $u_1(t_{a^*+\epsilon} - (a^* + \epsilon)) = u_2(t_{a^*+\epsilon})$, which contradicts the definition of a^*. Hence $t_{a^*} = \infty$. Let $a^* < \infty$, since $t_a \to \infty$ as $a \to a^*$, for each $t \in \mathbb{R}$

there is an $a < a^*$ such that $t < t_a$. Thus $u_1(t-a^*) < u_1(t-a) < u_2(t)$ for $t < t_a$. Let $(\tilde{u}_1(t), \tilde{v}_1(t)) = (u_1(t-a^*), v_1(t-a^*))$. Then $(\tilde{u}_1(t), \tilde{v}_1(t))$ is a traveling wave solution that connects the same points as (u_1, v_1) does and

$$\tilde{u}_1(t) < u_2(t). \tag{2.3.26}$$

Using the equation (2.3.9), it follows that

$$\tilde{v}_1(t) = e^{\lambda_1 t}[\tilde{M}_1 + \tilde{H}_1(t)],$$

where $\tilde{M}_1$ is the number associated with $\tilde{v}_1$ in the equation (2.3.9) and $\tilde{H}_1(t) \to 0$ as $t \to \infty$ and

$$v_2(t) = e^{\lambda_1 t}[M_2 + H_2(t)],$$

where M_2 is the number associated with v_2 in the equation (2.3.9) and $H_2(t) \to 0$ as $t \to \infty$. From (2.3.26), $\tilde{u}_1(t) < u_2(t)$ and Proposition (2.3.7), it follows that $\tilde{M}_1 > M_2$. Thus there exists a small number $\delta > 0$ and sufficiently large T such that for any $\epsilon \in [0, \delta]$ and $t \geq T$,

$$e^{\lambda_1 \epsilon}\tilde{M}_1 + e^{\lambda_1 \epsilon} H_1(t+\epsilon) > M_2 + H_2(t), \tag{2.3.27}$$

which implies that, for any $\epsilon \in [0, \delta]$ and all $t \geq T$, with the use of (2.3.27)

$$\begin{aligned} \tilde{v}_1(t+\epsilon) &= e^{\lambda_1 t + \lambda_1 \epsilon}[\tilde{M}_1 + H_1(t+\epsilon)] \\ &= e^{\lambda_1 t}[e^{\lambda_1 \epsilon}\tilde{M}_1 + e^{\lambda_1 \epsilon} H_1(t+\epsilon)] \\ &> e^{\lambda_1 t}[M_2 + H_2(t)] = v_2(t). \end{aligned} \tag{2.3.28}$$

Since $t_a \to t_{a^*} = \infty$ as $a \to a^*$, hence there exists an $\epsilon \in (0, \delta]$ such that $t_{a^*-\epsilon} > T$. By definition

$$\tilde{u}_1(t_{a^*-\epsilon} + \epsilon) = u_1(t_{a^*-\epsilon} + \epsilon - a^*) = u_1(t_{a^*-\epsilon} - (a^* - \epsilon)) = u_2(t_{a^*-\epsilon}) \tag{2.3.29}$$

Since $(\tilde{u}_1(t+\epsilon), \tilde{v}_1(t+\epsilon))$ is also a traveling wave solution connecting $(\xi_1, 0)$ and $(u^0, 0)$, in the view of (2.3.29) and Part (3) of Proposition (2.3.11) there exists at least a $t^{'} > t_{a^*-\epsilon} > T$ such that $\tilde{v}_1(t^{'} + \epsilon) < v_2(t^{'})$. This leads to a contradiction of the equation (2.3.28). Hence $a^* \not< \infty$.

If $a^* = \infty$. Then since $t_a \to \infty$ as $a \to a^* = \infty$. Clearly

$$\lim_{a \to \infty} u_1(t_a - a) = \lim_{a \to \infty} u_2(t_a) = u_2(\infty) = u^0 = u_1(\infty).$$

The monotone increasing property of $u_1(t)$ in the view of the above equality implies that, $t_a - a \to \infty$ as $a \to \infty$. Now by (2.3.9) $v_1(t)$ and $v_2(t)$ are expressed as

$$v_i(t) = e^{\lambda_1 t}[M_i + H_i(t)],$$

with $H_i(t) \to 0$ as $t \to \infty$, and $M_i > 0$ for $i = 1, 2$. Since $\lambda_1 < 0$, a sufficiently large number $\eta^* > 0$ is chosen, such that

$$\frac{e^{-\lambda_1 \eta^*} M_1}{2} > 2M_2.$$

Since $H_i(t) \to 0$ as $t \to \infty$, there is a sufficiently large t^* such that for all $t \geq t^*$,

$$|H_1(t)| \leq \frac{M_1}{2},\ |H_2(t)| \leq M_2.$$

Hence for $a \geq \eta^*$ and all $t \geq a + t^*$ we have

$$\begin{aligned} v_1(t-a) &= e^{\lambda_1 t} e^{-\lambda_1 a}[M_1 + H_1(t-a)] \\ &= e^{\lambda_1 t}[e^{-\lambda_1 a} M_1 + e^{-\lambda_1 a} H_1(t-a)] \\ &\geq e^{\lambda_1 t} \frac{e^{-\lambda_1 a} M_1}{2} \geq e^{\lambda_1 t} 2M_2 \\ &\geq e^{\lambda_1 t}[M_2 + H_2(t)] = v_2(t). \end{aligned} \tag{2.3.30}$$

Since $t_a - a \to \infty$ as $a \to \infty$, there is a $\bar{a} > \eta^*$ such that $t_{\bar{a}} - \bar{a} > t^*$, that is $t_{\bar{a}} > t^* + \bar{a}$,. Let $\tilde{u}_1(t) = u_1(t - \bar{a})$ and $\tilde{v}_1(t) = v_1(t - \bar{a})$, then

$$\tilde{u}_1(t_{\bar{a}}) = u_1(t - \bar{a}) > u_2(t_{\bar{a}}),\ t > t_{\bar{a}}. \tag{2.3.31}$$

Since $t_{\bar{a}} - \bar{a} > t^*$, implies that $t_{\bar{a}} > \bar{a} + t^*$. So in the view of (2.3.30)

$$\tilde{v}_1(t) \geq v_1(t - \bar{a}) > v_2(t),\ t > t_{\bar{a}}. \tag{2.3.32}$$

Equation (2.3.31) and equation (2.3.32) contradicts Part (3) of the Proposition (2.3.11). Hence $a^* \neq \infty$. That is for each given $u^0 > u_K$ and $C > 0$ with

$$C \geq \sqrt{4[f(u^0) - K]},$$

there exists a unique $u_0 < u_K$ such that the system (2.1.5) has a positive traveling wave solution connecting $(u_0, 0)$ and $(u^0, 0)$.

Case II. When $C = \sqrt{4[f(u^0) - K]}$.

In this case the theorem can be proved in the same fashion by using the equation (2.3.20) instead of equation (2.3.9). □

3. APPROXIMATION OF TRAVELING WAVE SOLUTIONS

3.1 *Introduction*

In this chapter, we study traveling wave solutions of the system of reaction-diffusion equations of the type

$$\begin{aligned}\frac{\partial S}{\partial t} &= \rho\frac{\partial^2 S}{\partial x^2} - \theta\frac{\partial S}{\partial x} - f(S)P,\\ \frac{\partial P}{\partial t} &= \frac{\partial^2 P}{\partial x^2} - \theta\frac{\partial P}{\partial x} + [f(S) - K]P.\end{aligned} \tag{3.1.1}$$

The system (3.1.1) has many useful applications. It is used as a mathematical model to study some problems in Biology. For example, the system (3.1.1) has been studied in references [3, 4] as a Mathematical model for a single population microbial growth for a limiting nutrient in a flow reactor, where $\theta\,(\geq 0)$ represents the flow velocity, $S(x,t)$ and $P(x,t)$ are respectively the concentrations of nutrient and the microbial population density in the reactor at position x and time t, $\rho > 0$ is the diffusion coefficient, $K > 0$ is the cell death rate and f is the nutrient uptake function (or birth rate) of bacteria cells and is assumed to be increasing. If $f(S) = S$ and $\theta = 0$, the system (3.1.1) describes a simple diffusive epidemic model, in which $S(x,t)$ and $P(x,t)$ represent the densities of susceptible and infective population [5, 6]. For a long flow reactor the boundary conditions are

$$\begin{aligned}&S(-\infty) = S^0, \quad S(\infty) = S_0,\\ &P(-\infty) = 0, \quad P(\infty) = 0,\end{aligned} \tag{3.1.2}$$

where $S^0\,(> S_0)$ is the amount of nutrient that enters at one end (inflow) of the reactor with constant velocity θ, S_0 is the amount of un-utilized nutrients that exit at the other end (outflow) of the reactor and the constant K is such that $f(S^0) > K > f(S_0)$. Introducing the new variable $z = x - ct$, $c \in \mathbb{R}$ and using

$$(S(x,t), P(x,t)) = (U(x-ct), V(x-ct)),$$

into (3.1.1) and (3.1.2), we have the following coupled system of boundary value problems

$$\begin{aligned}
&C\dot{U} = \rho\ddot{U} - f(U)V,\\
&C\dot{V} = \ddot{V} + [f(U) - K]V\\
&U(-\infty) = S^0,\, U(\infty) = S_0,\\
&V(-\infty) = 0,\, V(\infty) = 0,
\end{aligned} \tag{3.1.3}$$

where $C = -c + \theta$ and (.) denote differentiation with respect to z. Solutions of the system (3.1.3) are the corresponding traveling waves of the reaction diffusion system (3.1.1),(3.1.2). The existence of a traveling wave reflects the important phenomenon of wave propagation and has been studied extensively by many authors, see for example, [5, 6, 9, 17, 18] etc.

This chapter deals with the analytical approximate solutions to the system (3.1.3). We present the Adomian decomposition method [10, 19] and numerical simulations are also presented to illustrate and confirm the theoretical results. We also present the homotopy perturbation method [14, 15] and the generalized approximation method, see reference [23] to study approximation of solutions. We also present the upper and lower solution method [20] for the system (3.1.1) to establish new existence results.

3.2 Adomian's decomposition method [10]

Since $V(z)$ is nonnegative for $z \in \mathbb{R}$, it follows from the boundary conditions that there exist z_1, $z_2 \in \mathbb{R}$ with $z_2 > z_1$ such that $V(z_1) = V(z_2) = \xi$ for some $\xi > 0$. The monotone decreasing property of U implies that there exist some α, β with $\alpha > \beta$ such that $U(z_1) = \alpha,\ U(z_2) = \beta$. Hence, we consider the following system of boundary value problems on a finite domain

$$\begin{aligned} &C\dot{U} = \rho\ddot{U} - f(U)V,\\ &C\dot{V} = \ddot{V} + [f(U) - K]V,\ z \in [z_1, z_2],\\ &U(z_1) = \alpha,\ U(z_2) = \beta,\\ &V(z_1) = \xi,\ V(z_2) = \xi. \end{aligned} \tag{3.2.1}$$

Define the operator L_{zz} by $L_{zz} = \frac{d^2}{dz^2}$. The system (3.2.1) can be written as

$$\begin{aligned} L_{zz}U &= \frac{C}{\rho}L_zU + \frac{1}{\rho}f(U)V, \quad z \in [z_1, z_2],\\ L_{zz}V &= CL_zV + [K - f(U)]V, \quad z \in [z_1, z_2]. \end{aligned} \tag{3.2.2}$$

The inverse operator $L_{zz}^{-1}(.)$ is defined by

$$L_{zz}^{-1}(.) = \int_{z_1}^{z}\int_{z_2}^{z'} (.)\, dz'\, dz.$$

Applying the inverse operator to both the sides of the system (3.2.2), we obtain

$$\begin{aligned} L_{zz}^{-1}L_{zz}U &= \frac{C}{\rho}L_{zz}^{-1}L_zU + \frac{1}{\rho}L_{zz}^{-1}f(U)V,\\ L_{zz}^{-1}L_{zz}V &= CL_{zz}^{-1}L_zV + L_{zz}^{-1}[K - f(U)]V, \end{aligned} \tag{3.2.3}$$

which imply that

$$\begin{aligned} U(z) &= U(z_1) + (z - z_1)U'(z_2) + \frac{C}{\rho}L_{zz}^{-1}L_zU + \frac{1}{\rho}L_{zz}^{-1}f(U)V,\\ V(z) &= V(z_1) + (z - z_1)V'(z_2) + CL_{zz}^{-1}L_zV + L_{zz}^{-1}[K - f(U)]V. \end{aligned} \tag{3.2.4}$$

Using the boundary conditions, we obtain

$$\begin{aligned} U(z) &= \alpha + (z - z_1)U'(z_2) + \frac{C}{\rho} L_{zz}^{-1} L_z U + \frac{1}{C} L_{zz}^{-1} f(U)V, \\ V(z) &= \xi + (z - z_1)V'(z_2) + C L_{zz}^{-1} L_z V + L_{zz}^{-1}\{K - f(U)\}V. \end{aligned} \tag{3.2.5}$$

At $z = z_2$, we have

$$\begin{aligned} \beta &= \alpha + (z_2 - z_1)U'(z_2) + [\frac{C}{\rho} L_{zz}^{-1} L_z U + \frac{1}{C} L_{zz}^{-1} f(U)V]_{z=z_2}, \\ 0 &= (z_2 - z_1)V'(z_2) + [C L_{zz}^{-1} L_z V + L_{zz}^{-1}\{K - f(U)\}V]_{z=z_2}. \end{aligned} \tag{3.2.6}$$

Solving the system (3.2.6) for $U'(z_2)$ and $V'(z_2)$, we obtain

$$\begin{aligned} U'(z_2) &= \frac{\beta - \alpha}{z_2 - z_1} - \frac{1}{z_2 - z_1}[\frac{C}{\rho} L_{zz}^{-1} L_z U + \frac{1}{\rho} L_{zz}^{-1} f(U)V]_{z=z_2}, \\ V'(z_2) &= -\frac{1}{z_2 - z_1}[C L_{zz}^{-1} L_z V + L_{zz}^{-1}\{K - f(U)\}V]_{z=z_2}. \end{aligned} \tag{3.2.7}$$

Substituting (3.2.7) into (3.2.4), we obtain

$$\begin{aligned} U(z) &= \alpha + (z - z_1)[\frac{\beta - \alpha}{z_2 - z_1} - \frac{1}{z_2 - z_1}[\frac{C}{\rho} L_{zz}^{-1} L_z U + \frac{1}{\rho} L_{zz}^{-1} f(U)V]_{z=z_2}] \\ &\quad + \frac{C}{\rho} L_{zz}^{-1} L_z U + \frac{1}{\rho} L_{zz}^{-1} f(U)V, \\ V(z) &= \xi - (z - z_1)[\frac{1}{z_2 - z_1}[C L_{zz}^{-1} L_z V + L_{zz}^{-1}\{K - f(U)\}V]_{z=z_2}] \\ &\quad + C L_{zz}^{-1} L_z V + L_{zz}^{-1}[K - f(U)]V. \end{aligned} \tag{3.2.8}$$

For simplicity, let us denote $\frac{z - z_1}{z_2 - z_1}$ by $q(z)$, then

$$\begin{aligned} U(z) &= \alpha + q(z)(\beta - \alpha) - q(z)[\frac{C}{\rho} L_{zz}^{-1} L_z U + \frac{1}{\rho} L_{zz}^{-1} f(U)V]_{z=z_2} \\ &\quad + \frac{C}{\rho} L_{zz}^{-1} L_z U + \frac{1}{\rho} L_{zz}^{-1} f(U)V, \\ V(z) &= \xi - q(z)[C L_{zz}^{-1} L_z V + L_{zz}^{-1}\{K - f(U)\}V]_{z=z_2} \\ &\quad + C L_{zz}^{-1} L_z V + L_{zz}^{-1}[K - f(U)]V. \end{aligned} \tag{3.2.9}$$

According to the Adomian decomposition method, the solution $(U(z), V(z))$ can be written as infinite series of the form

$$U(z) = \sum_{n=0}^{\infty} U_n(z),\ V(z) = \sum_{n=0}^{\infty} V_n(z)$$

and the nonlinear function $f(U)$ as

$$f(U) = \sum_{n=0}^{\infty} A_n,$$

where A_n are the so-called Adomian's polynomials and can be computed using the formula

$$A_n = \frac{1}{n!}\Big[\frac{d^n}{d\lambda^n} f(S(\lambda))\Big]_{\lambda=0}, \quad n = 0, 1, 2, 3.... \tag{3.2.10}$$

The system (3.2.9) takes the form

$$\begin{aligned}
\sum_{n=0}^{\infty} U_n(z) &= \alpha + q(z)(\beta-\alpha) - q(z)\big[\frac{C}{\rho}L_{zz}^{-1}L_z\sum_{n=0}^{\infty} U_n(z) \\
&\quad +\frac{1}{\rho}L_{zz}^{-1}\sum_{n=0}^{\infty} A_n \sum_{n=0}^{\infty} V_n(z)\big]_{z=z_2} + \frac{C}{\rho}L_{zz}^{-1}L_z\sum_{n=0}^{\infty} U_n(z) \\
&\quad +\beta L_{zz}^{-1}\sum_{n=0}^{\infty} A_n \sum_{n=0}^{\infty} V_n(z), \qquad (3.2.11)\\
\sum_{n=0}^{\infty} V_n(z) &= \xi - q(z)\big[CL_{zz}^{-1}L_z\sum_{n=0}^{\infty} V_n(z) + L_{zz}^{-1}(K-\sum_{n=0}^{\infty} A_n)\sum_{n=0}^{\infty} V_n(z)\big]_{z=z_2} \\
&\quad +CL_{zz}^{-1}L_z\sum_{n=0}^{\infty} V_n(z) + L_{zz}^{-1}K\sum_{n=0}^{\infty} V_n(z) \\
&\quad -L_{zz}^{-1}\sum_{n=0}^{\infty} A_n \sum_{n=0}^{\infty} V_n(z). \qquad (3.2.12)
\end{aligned}$$

The components $U_n(z)$, $V_n(z)$ for $n \geq 0$ are given by

$$U_0(z) = \alpha + q(z)(\beta - \alpha),$$
$$V_0(z) = \xi,$$

$$
\begin{aligned}
U_1(z) &= -q(z)\{\frac{C}{\rho}L_{zz}^{-1}L_zU_0+\frac{1}{\rho}L_{zz}^{-1}A_0V_0\}_{z=z_2}+\frac{C}{\rho}L_{zz}^{-1}L_zU_0+\frac{1}{\rho}L_{zz}^{-1}A_0V_0\\
V_1(z) &= q(z)[L_{zz}^{-1}A_0V_0-KL_{zz}^{-1}V_0-CL_{zz}^{-1}L_zV_0]_{z=z_2}+CL_{zz}^{-1}L_zV_0\\
&\quad +KL_{zz}^{-1}V_0-L_{zz}^{-1}A_0V_0,\\
U_2(z) &= -q(z)\{\frac{C}{\rho}L_{zz}^{-1}L_zU_1+\frac{1}{\rho}L_{zz}^{-1}(A_0V_1+A_1V_0)\}_{z=z_2}+\frac{C}{\rho}L_{zz}^{-1}L_zU_1\\
&\quad +\frac{1}{\rho}L_{zz}^{-1}(A_0V_1+A_1V_0),\\
V_2(z) &= q(z)[L_{zz}^{-1}(A_0V_1+A_1V_0)-KL_{zz}^{-1}V_1-CL_{zz}^{-1}L_zV_1]_{z=z_2}\\
&\quad +CL_{zz}^{-1}L_zV_1+KL_{zz}^{-1}V_1-L_{zz}^{-1}(A_0V_1+A_1V_0),
\end{aligned}
$$

and so on,

$$
\begin{aligned}
U_{n+1}(z) &= -q(z)\{\frac{C}{\rho}L_{zz}^{-1}L_zU_n+\frac{1}{\rho}L_{zz}^{-1}(A_nV_0+A_{n-1}V_1+.....+A_0V_n)\}_{z=z_2}\\
&\quad +\frac{C}{\rho}L_{zz}^{-1}L_zU_n+\frac{1}{\rho}L_{zz}^{-1}(A_nV_0+A_{n-1}V_1+.....+A_0V_n),\\
V_{n+1}(z) &= q(z)[L_{zz}^{-1}(A_nV_0+A_{n-1}V_1+.....+A_0V_n)-KL_{zz}^{-1}V_n-CL_{zz}^{-1}L_zV_n]_{z=z_2}\\
&\quad +CL_{zz}^{-1}L_zV_n+KL_{zz}^{-1}V_n-L_{zz}^{-1}(A_nV_0+A_{n-1}V_1+.....+A_0V_n).
\end{aligned}
$$

Define

$$\Psi_n(z)=\Sigma_{i=0}^{n-1}U_i(z),\ \Phi_n(z)=\Sigma_{i=0}^{n-1}V_i(z),\ n\geq 1,$$

then the solution of the system (3.2.1) is given by

$$(U(z),V(z))=(\lim_{n\to\infty}\Psi_n(z),\lim_{n\to\infty}\Phi_n(z)).$$

Example 3.2.1. (Diffusive epidemic model [5]): As an illustration, choose $f(U)=U$ and $z_1=0$, $z_2=1$, then the system (3.2.1) takes the form

$$
\begin{aligned}
&C\dot{U}=\rho\ddot{U}-UV,\\
&C\dot{V}=\ddot{V}+[U-K]V,\ z\in[0,1],\\
&U(0)=\alpha,\ U(1)=\beta,\\
&V(0)=\xi,\ V(1)=\xi.
\end{aligned}
\tag{3.2.13}
$$

The Adomian's polynomials are

$$A_0 = U_0,\ A_1 = U_1,\ A_2 = U_2,\ .... \tag{3.2.14}$$

The components $U_0(z)$, $V_0(z)$, $U_1(z)$, $V_1(z)$, $U_2(z)$, $V_2(z)$, ... of solution of the system (3.2.13) are:

$$\begin{aligned}
U_0(z) &= \alpha + (\beta - \alpha)z,\\
V_0(z) &= \xi,\\
U_1(z) &= \frac{z(1-z)}{6\rho}[3C(\alpha-\beta) + ((z-2)\alpha - (1+z)\beta)\xi],\\
V_1(z) &= \frac{z(1-z)\xi}{6}[(2-z)\alpha + (1+z)\beta - 3K],
\end{aligned}$$

$$\begin{aligned}
U_2(z) = \frac{1}{360\rho^2}&[z(z-1)[30C^2(2z-1)(\alpha-\beta) + 30C\xi(-2z\alpha + z^2(\alpha-\beta)+\beta]\\
&+ \xi[(-5-5z+15z^2-10z^3+2z^4)\alpha^2\rho + \alpha\{(8+8z-12z^2+3z^3)\xi\\
&+ \{3K(3+3z-7z^2+3Z^3) + (-7-7z+3z^2+8z^3-4z^4)\beta\}\rho\}\\
&+ \beta\{(7+7z-3z^2-3z^3)\xi + \{K(6+6z+6z^2-9z^3)\\
&+ (-3-3z-3z^2+2z^3+2z^4)\beta\}\rho\}\}]],
\end{aligned}$$

$$\begin{aligned}
V_2(z) = \frac{1}{360\rho}&[(-1+z)z\xi[7\beta\xi + 7z\beta\xi - 3z^3\beta\xi - 15K^2\rho - 15K^2z\rho + 15K^2z^2\rho\\
&+ (-5-5z+15z^2-10z^3+2z^4)\alpha^2\rho + 13K\beta\rho + 13Kz\beta\rho + 3Kz^2\beta\rho\\
&- 12Kz^3\beta\rho - 3\beta^2\rho - 3z\beta^2\rho - 3z^2\beta^2\rho + 2z^3\beta^2\rho + 2z^4\beta^2\rho + \alpha\{(8+8z\\
&- 12z^2+3z^3)\xi + \{K(17+17z-33z^2+12z^3) + (-7-7z+3z^2+8z^3\\
&- 4z^2)\beta\}\rho\} + 15C\{2K(-1+2z)\rho + \beta\{1+z+\rho-z\rho-z^2(1+\rho)\}\\
&+ \alpha\{-1+\rho+z^2(1+\rho)-z(1+3\rho)\}\}]].
\end{aligned}$$

Hence the three term solution of the system (3.2.13) is

$$U(z) = U_0(z) + U_1(z) + U_2(z),$$

$$V(z) = V_0(z) + V_1(z) + V_2(z).$$

3.3 Graphs of solutions

For the existence of a traveling wave solution, we must have a unique positive number U_K such that $f(U_K) = K$, $f(\alpha) > K$ and $f(\beta) < K$. Also $\rho > 0$, $K > 0$, $C > 0$ are arbitrary constants. Hence, select $K = 1$, $C = 2$, we must have $U_K = 1$, $S^0 > 1 > S_0$. Choose $S^0 = 2$, $S_0 = 0.5$, i.e $U(0) = 2$, $U(1) = 0.5$ and $V(0) = 2 = V(1)$. Result obtained for different values of $\rho = 0.3$, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9, 1 are shown graphically, see appendix.

3.4 The homotopy perturbation method [14, 15]

Writing the system of equations

$$\begin{aligned} \ddot{U} &= \frac{C}{\rho}\dot{U} + \frac{1}{\rho}f(U)V, \\ \ddot{V} &= C\dot{V} + [K - f(U)]V, \end{aligned} \tag{3.4.1}$$

as

$$\begin{aligned} \ddot{U} - \frac{C}{\rho}\dot{U} - \frac{1}{\rho}f(U)V = 0, \\ \ddot{V} - C\dot{V} + [f(U) - K]V = 0. \end{aligned} \tag{3.4.2}$$

Construct the following homotopy for system (3.4.2)

$$\begin{aligned} \ddot{U} &= \mu[\frac{C}{\rho}\dot{U} + \frac{1}{\rho}f(U)V], \\ \ddot{V} &= \mu[C\dot{V} + (K - f(U))V], \end{aligned} \tag{3.4.3}$$

where the parameter $\mu \in [0, 1]$. For $\mu = 0$, the system (3.4.3) reduces to the following simpler system

$$\begin{aligned} \ddot{U} &= 0, \\ \ddot{V} &= 0, \end{aligned}$$

whose solution is

$$U_0 = Az + B,$$
$$V_0 = Cz + D.$$

For $\mu = 1$, the system (3.4.3) reduces to the original system (3.4.2).
Using homotopy perturbation method, we write

$$\begin{aligned} U &= U_0 + \mu U_1 + \mu^2 U_2 + \mu^3 U_3 + ... \\ V &= V_0 + \mu V_1 + \mu^2 V_2 + \mu^3 V_3 + ... \end{aligned} \tag{3.4.4}$$

and letting $\mu \to 1$, we obtain the approximate solution of the system (3.4.2), that is,

$$\begin{aligned} U &= U_0 + U_1 + U_2 + U_3 + ... \\ V &= V_0 + V_1 + V_2 + V_3 + ... \end{aligned} \tag{3.4.5}$$

Substitute (3.4.4) into (3.4.3) and expanding $f(U)$ by Taylor's series, we obtain

$$\begin{aligned} &\ddot{U}_0 + \mu\ddot{U}_1 + \mu^2\ddot{U}_2 + ... - \mu\,[\frac{C}{\rho}(\dot{U}_0 + \mu\dot{U}_1 + \mu^2\dot{U}_2 + ...)] \\ &-\frac{\mu}{\rho}[f(U_0) + f'(U_0)(\mu U_1 + \mu^2 U_2 + ...) + \frac{f''(U_0)}{2!}(\mu U_1 + \mu^2 U_2 + ...)^2 + ...] \\ &[V_0 + \mu V_1 + \mu^2 V_2 + ...] = 0, \end{aligned} \tag{3.4.6}$$

$$\begin{aligned} &\ddot{V}_0 + \mu\ddot{V}_1 + \mu^2\ddot{V}_2 + ... - \mu\,[C(\dot{V}_0 + \mu\dot{V}_1 + \mu^2\dot{V}_2 + ...)] \\ &+[f(U_0) + f'(U_0)(\mu U_1 + \mu^2 U_2 + ...) + \frac{f''(U_0)}{2!}(\mu U_1 + \mu^2 U_2 + ...)^2 + ... - K] \\ &[V_0 + \mu V_1 + \mu^2 V_2 + ...] = 0. \end{aligned} \tag{3.4.7}$$

Collecting terms of the same powers of μ, we obtain

$$\mu^0 \ : \ \ddot{U}_0 = 0, \quad \ddot{V}_0 = 0, \tag{3.4.8}$$

$$\mu^1 \ : \ \ddot{U}_1 - \frac{1}{\rho} f(U_0)V_0 - \frac{C}{\rho}\dot{U}_0 = 0, \quad \ddot{V}_1 + f(U_0)V_0 - KV_0 - C\dot{V}_0 = 0, \tag{3.4.9}$$

$$\mu^2 \ : \ \ddot{U}_2 - \frac{1}{\rho} f(U_0)V_1 - \frac{C}{\rho}\dot{U}_1 - \frac{1}{\rho} f^{'}(U_0)V_0U_1 = 0, \quad \ddot{V}_2 + f(U_0)V_1 - KV_1 - C\dot{V}_1 + V_0U_1f^{'}(U_0) = 0, \tag{3.4.10}$$

$$\mu^3 \ : \ \ddot{U}_3 - \frac{1}{\rho} f(U_0)V_2 - \frac{C}{\rho}\dot{U}_2 - \frac{1}{\rho}V_1U_1 f^{'}(U_0) - \frac{1}{\rho}V_0U_2f^{'}(U_0) = 0, \quad \ddot{V}_3 - f(U_0)V_2 - KV_2 - C\dot{V}_2 + V_1U_1f^{'}(U_0) + V_0U_2f^{'}(U_0) = 0, \tag{3.4.11}$$

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Solving (3.4.8), we obtain

$$U_0 = c_1 + c_2 z, \quad V_0 = \eta_1 + \eta_2 z. \tag{3.4.12}$$

Using boundary conditions

$$U_0(z_1) = \alpha,\ U_0(z_2), = \beta$$

$$V_0(z_1) = \xi,\ V_0(z_2) = \xi,$$

we obtain

$$c_1 = \frac{z_1\beta - z_2\alpha}{z_1 - z_2}, \ \ c_2 = \frac{\alpha - \beta}{z_1 - z_2}, \ \ \eta_1 = \xi, \ \ \eta_2 = 0.$$

If the first order approximate solution is required, that is, $U = U_0 + U_1$ and $V = V_0 + V_1$, then we have

$$U_1 = U - U_0 \,\text{and}\, V_1 = V - V_0.$$

Hence,

$$U_1(z_1) = U(z_1) - U_0(z_1) = \alpha - \alpha = 0,$$

$$V_1(z_1) = V(z_1) - V_0(z_1) = \xi - \xi = 0.$$

Similarly,

$$U_1(z_2) = 0 \,\text{and}\, V_1(z_2) = 0.$$

Thus, U_1 and V_1 satisfy the homogenous boundary conditions, that is,

$$\begin{aligned} &U_1(z_1) = 0,\ U_1(z_2) = 0 \\ &V_1(z_1) = 0,\ V_1(z_2) = 0. \end{aligned} \tag{3.4.13}$$

Next, we solve the system (3.4.9) subject to the boundary conditions (3.4.13), to obtain the first order approximate solution $U = U_0 + U_1$ and $V = V_0 + V_1$ of the system (3.4.2). For the second order approximate solution, i.e

$$U = U_0 + U_1 + U_2,$$

$$V = V_0 + V_1 + V_2,$$

we solve the system (3.4.10) subject to the boundary conditions

$$U_2(z_1) = 0,\ U_2(z_2) = 0,\ V_2(z_1) = 0,\ \text{and}\, V_2(z_2) = 0. \tag{3.4.14}$$

Continuing in the same fashion, we can obtain the 3rd, 4th, etc order approximate solutions.

Example 3.4.1. Consider the following equation with $f(U) = U$

$$\ddot{U} - \frac{C}{\rho}\dot{U} - \frac{1}{\rho}UV = 0,$$
$$\ddot{V} - C\dot{V} + [U - K]V = 0, \tag{3.4.15}$$

with boundary conditions as

$$U(0) = \alpha,\ U(1) = \beta,$$

$$V(0) = \xi,\ V(1) = \xi.$$

According to the homotopy perturbation method putting

$$U = U_0 + \mu U_1 + \mu^2 U_2 + \mu^3 U_3 + ...,$$
$$V = V_0 + \mu V_1 + \mu^2 V_2 + \mu^3 V_3 + ..., \tag{3.4.16}$$

and collecting terms of the same powers of μ we obtain the following systems of differential equation with boundary conditions as

$$\begin{aligned}
\ddot{U}_0(z) &= 0,\quad \ddot{V}_0(z) = 0,\\
U_0(0) &= \alpha,\ U_0(1) = \beta,\quad (\alpha > \beta)\\
V_0(0) &= 1,\ V_0(1) = 1,
\end{aligned} \tag{3.4.17}$$

$$\begin{aligned}
\ddot{U}_1(z) &= 2U_0(z)V_0(z) + 4\dot{U}_0(z),\\
\ddot{V}_1(z) &= -U_0(z)V_0(z) + 2V_0(z) + 2\dot{V}_0(z),\\
U_1(0) &= 0,\ U_1(1) = 0,\quad V_1(0) = 0,\ V_1(1) = 0,
\end{aligned} \tag{3.4.18}$$

$$\begin{aligned}
\ddot{U}_2(z) &= 2U_0(z)V_1(z) + 4\dot{U}_1(z) + 2V_0(z)V_1(z),\\
\ddot{V}_2(z) &= -U_0(z)V_1(z) + 2V_1(z) + 2\dot{V}_1(z) - V_0(z)U_1(z),\\
U_2(0) &= 0,\ U_2(1) = 0,\quad V_2(0) = 0,\ V_2(1) = 0,
\end{aligned} \tag{3.4.19}$$

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Solving these equations with the corresponding boundary conditions, we get

$$
\begin{aligned}
U_0(z) &= \alpha - z\alpha + z\beta, \\
V_0(z) &= \xi, \\
U_1(z) &= \frac{(1-z)z(3c(\alpha-\beta)+((z-2)\alpha-(1+z)\beta)\xi}{6\rho}, \\
V_1(z) &= \frac{1}{6}(z-1)z(3K+(z-2)\alpha-(1+z)\beta)\xi, \\
U_2(z) &= -\frac{1}{360\rho^2}[(-1+z)z[30C^2(-1+2z)(\alpha-\beta)+30C(-2z\alpha+z^2)(\alpha-\beta) \\
&\quad +\beta]\xi+\xi\{(-5-5z+15z^2-10z^3+2z^4)\alpha^2\rho+\alpha\{(8+8z-12z^2+3z^3)\xi \\
&\quad +\{3K(3+3z-7z^2+3z^3)+(-7-7z+3z^2+8z^3-4z^4)\beta\}\rho\} \\
&\quad +\beta\{(7+7z-3z^2-3z^3)\xi+\{K(6+6z+6z^2-9z^3) \\
&\quad +(-3-3z-3z^2+2z^3+2z^4)\beta\}\rho\}\}], \\
V_2(z) &= \frac{1}{360\rho}[(-1+z)z\xi[7\beta\xi+7z\beta\xi-3z^2\beta\xi-3z^3\beta\xi-15K^2\rho+15K^2z\rho \\
&\quad +(-5-5z+15z^2-10z^3+2z^4)\alpha^2\rho+13K\beta\rho+13Kz\beta\rho \\
&\quad +3z^2K\beta\rho-12Kz^3\beta\rho-3\beta^2\rho-3z\beta^2\rho-3z^2\beta^2\rho+2z^3\beta^2\rho \\
&\quad +2z^4\beta^2\rho+\alpha\{(8+8z-12z^2+3z^3)\xi+\{K(17+17z-33z^2+12z^3) \\
&\quad +(-7-7z+3z^2+8z^3-4z^4)\beta\}\rho\}+15C\{2K(-1+2z)\rho+\beta\{1+z \\
&\quad +\rho-z\rho-z^2(1+\rho)\}+\alpha\{-1+\rho+z^2(1+\rho)-z(1+3\rho)\}\}]],
\end{aligned}
\tag{3.4.20}
$$

The second order approximate solution is obtained as

$$U(z) = U_0(z) + U_1(z) + U_2(z),$$

$$V(z) = V_0(z) + V_1(z) + V_2(z).$$

3.5 Graphs of solutions

Select $K = 1$, $C = 2$, we must have $U_K = 1$, $S^0 > 1 > S_0$. Choose $S^0 = 2$, $S_0 = 0.5$, i.e $U(0) = 2$, $U(1) = 0.5$ and $V(0) = 2 = V(1)$. For these values the homotopy perturbation method correspond to the Adomian decomposition method. The graphs for various values $\rho = 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9, 1$ are exactly the same as given in appendix.

3.6 The method of upper and lower solutions [20]

We write the system of equations

$$\begin{aligned} C\dot{U}(z) &= \rho\ddot{U}(z) - f(U)V, \quad z \in [z_1, z_2], \\ C\dot{V}(z) &= \ddot{V}(z) + [f(U) - K]V, \quad z \in [z_1, z_2], \end{aligned} \tag{3.6.1}$$

in more general form of the type

$$\begin{aligned} \ddot{U}(z) - \frac{C}{\rho}\dot{U}(z) &= g(U, V), \quad z \in [z_1, z_2], \\ \ddot{V}(z) - C\dot{V}(z) &= h(U, V), \quad z \in [z_1, z_2], \end{aligned} \tag{3.6.2}$$

where

$$g(U, V) = \frac{1}{\rho}f(U)V, \quad h(U, V) = [K - f(U)]V$$

The boundary conditions are

$$\begin{aligned} U(z_1) &= \alpha, \quad U(z_2) = \beta, \\ V(z_1) &= \xi, \quad V(z_2) = \xi. \end{aligned} \tag{3.6.3}$$

We write the systems (3.6.2), (3.6.3) as an equivalent system of the integral equations

$$\begin{aligned} U(z) &= \frac{1}{(e^{\frac{c}{\rho}z_2}-e^{\frac{c}{\rho}z_1})}(\alpha(e^{\frac{c}{\rho}z_2}-e^{\frac{c}{\rho}z})+\beta(e^{\frac{c}{\rho}z}-e^{\frac{c}{\rho}z_1})) \\ &\quad + \int_{z_1}^{z_2} K_1(z,s)\, g(U,V)dt \\ V(z) &= \xi + \int_{z_1}^{z_2} K_2(z,s)\, h(U,V)dt, \end{aligned} \tag{3.6.4}$$

where

$$K_1(z,s) = \frac{\rho}{c\, e^{\frac{c}{\rho}s}(e^{\frac{c}{\rho}z_2}-e^{\frac{c}{\rho}z_1})}\begin{cases}(e^{\frac{c}{\rho}s}-e^{\frac{c}{\rho}z_2})(e^{\frac{c}{\rho}z}-e^{\frac{c}{\rho}z_1}), & z \le s\\ (e^{\frac{c}{\rho}s}-e^{\frac{c}{\rho}z_1})(e^{\frac{c}{\rho}z}-e^{\frac{c}{\rho}z_2}), & s \le z.\end{cases}$$

$$K_2(z,s)\frac{1}{c\, e^{cs}(e^{cz_2}-e^{cz_1})}\begin{cases}(e^{cs}-e^{cz_2})(e^{cz}-e^{cz_1}), & z \le s\\ (e^{cs}-e^{cz_1})(e^{cz}-e^{cz_2}), & s \le z.\end{cases}$$

Recall the definition of upper and lower solutions for a system.

Definition 3.6.1. A pair of functions $(\underline{U},\underline{V})$ is called a lower solution of the system (3.6.2), if it satisfies the following inequalities

$$\begin{aligned} &\ddot{\underline{U}} - \frac{C}{\rho}\dot{\underline{U}} \ge g(\underline{U},\underline{V}) = \frac{1}{\rho}f(\underline{U})\underline{V}, \\ &\ddot{\underline{V}} - C\dot{\underline{V}} \ge h(\underline{U},\underline{V}) = [K - f(\underline{U})]\underline{V}, \\ &(\underline{U}(z_1),\underline{V}(z_1)) \le (\alpha,\xi),\ (\underline{U}(z_2),\underline{V}(z_2)) \le (\beta,\xi). \end{aligned} \tag{3.6.5}$$

A pair of functions $(\bar{U},\bar{V})$ is called an upper solution of the system (3.6.2), if it satisfies the following inequalities

$$\begin{aligned} &\ddot{\bar{U}} - \frac{C}{\rho}\dot{\bar{U}} \le g(\bar{U},\bar{V}) = \frac{1}{\rho}f(\bar{U})\bar{V}, \\ &\ddot{\bar{V}} - C\dot{\bar{V}} \le h(\bar{U},\bar{V}) = [K - f(\bar{U})]\bar{V}, \\ &(\bar{U}(z_1),\bar{V}(z_1)) \ge (\alpha,\xi),\ (\bar{U}(z_2),\bar{V}(z_2)) \ge (\beta,\xi). \end{aligned} \tag{3.6.6}$$

Let us choose $\underline{U}$, $\underline{V}$, $\bar{U}$, $\bar{V} \in C^1([z_1, z_2], \mathbb{R})$ such that $\underline{U} \leq \bar{U}$ and $\underline{V} \leq \bar{V}$. Define the modification of g and h as follows:

$$\hat{g}(U,V) = \begin{cases} g(\bar{U},\bar{V}), & \text{if } g(U,V) \not\leq g(\bar{U},\bar{V}) \\ g(U,V), & \text{if } g(\underline{U},\underline{V}) \leq g(U,V) \leq g(\bar{U},\bar{V}) \\ g(\underline{U},\underline{V}), & \text{if } g(\underline{U},\underline{V}) \not\leq g(U,V), \end{cases}$$

$$\hat{h}(U,V) = \begin{cases} h(\bar{U},\bar{V}), & \text{if } h(U,V) \not\leq h(\bar{U},\bar{V}) \\ h(U,V), & \text{if } h(\underline{U},\underline{V}) \leq h(U,V) \leq h(\bar{U},\bar{V}) \\ h(\underline{U},\underline{V}), & \text{if } h(\underline{U},\underline{V}) \not\leq h(U,V). \end{cases}$$

Clearly, the modified functions are $\hat{g}$ and $\hat{h}$ continuous and bounded.

Consider the following modified system of BVPs

$$\begin{aligned} \ddot{U}(z) - \frac{C}{\rho}\dot{U}(z) &= \hat{g}(U,V), \quad z \in [z_1, z_2], \\ \ddot{V}(z) - C\dot{V}(z) &= \hat{h}(U,V), \quad z \in [z_1, z_2], \end{aligned} \tag{3.6.7}$$

subject to the boundary conditions (3.6.3). Note that if the modified system (3.6.7) has a solution (U, V) such that

$$(\underline{U},\underline{V}) \leq (U,V) \leq (\bar{U},\bar{V}), \quad \text{on } [z_1, z_2], \tag{3.6.8}$$

then the solution (U, V) is a solution of the original system (3.6.2). On the other hand, the system (3.6.7) can be equivalently written as

$$\begin{aligned} U(z) &= \frac{1}{(e^{\frac{c}{\rho}z_2} - e^{\frac{c}{\rho}z_1})}(\alpha(e^{\frac{c}{\rho}z_2} - e^{\frac{c}{\rho}z}) + \beta(e^{\frac{c}{\rho}z} - e^{\frac{c}{\rho}z_1})) \\ &\quad + \int_{z_1}^{z_2} K_1(z,s)\,\hat{g}(U,V)dt \quad z \in [z_1, z_2] \\ V(z) &= \xi + \int_{z_1}^{z_2} K_2(z,s)\,\hat{h}(U,V)dt, \quad z \in [z_1, z_2]. \end{aligned} \tag{3.6.9}$$

By a solution of the modified system (3.6.7), we mean a solution of the system of integral equations (3.6.9). Since $\hat{g}$ and $\hat{h}$ are continuous and bounded, by Schauder's fixed point theorem, the system of integral equation (3.6.9) has a solution, which implies that the modified system (3.6.7) has a solution.

Theorem 3.6.2. *Every solution* (U, V) *of the modified system* (3.6.7) *satisfies* (3.6.8) *and hence is a solution of the original system* (3.6.2)*.*

Proof. From the boundary condition (3.6.3) and the definition of lower and upper solutions, we have

$$(\underline{U}(z_1), \underline{V}(z_1)) \leq (\alpha, \xi) = (U(z_1), V(z_1)) \leq (\bar{U}(z_1), \bar{V}(z_1)),$$
$$(\underline{U}(z_2), \underline{V}(z_2)) \leq (\beta, \xi) = (U(z_2), V(z_2)) \leq (\bar{U}(z_2), \bar{V}(z_2)).$$

Hence, the relation (3.6.8) hold for $z = z_1,\ z_2$.

Assume that $(U, V) \leq (\bar{U}, \bar{V})$ is not true on (z_1, z_2).

Case I. $U \not\leq \bar{U}$ on (z_1, z_2).

Then either $U \not\leq \bar{U}$ and $V > \bar{V}$ on (z_1, z_2) or $U \not\leq \bar{U}$ and $V < \bar{V}$ on (z_1, z_2). If $U \not\leq \bar{U}$ and $V > \bar{V}$ on (z_1, z_2). Then, there exists some $z_0 \in (z_1, z_2)$ such that the function $U - \bar{U}$ has a positive maximum at z_0. Consequently,

$$U(z_0) > \bar{U}(z_0),\ \dot{U}(z_0) = \dot{\bar{U}}(z_0), \text{ and } \ddot{U}(z_0) - \ddot{\bar{U}}(z_0) \leq 0.$$

However, using the definition of $\hat{g}$ and that of upper solution, we obtain

$$\begin{aligned} \ddot{U}(z_0) - \ddot{\bar{U}}(z_0) &= \frac{C}{\rho}\dot{U}(z_0) + \hat{g}(U, V) - \ddot{\bar{U}}(t_0) \\ &= \hat{g}(U, V) - (\ddot{\bar{U}}(z_0) - \frac{C}{\rho}\dot{\bar{U}}(z_0)) \\ &\geq g(\bar{U}, \bar{V}) - g(\bar{U}, \bar{V}) = 0, \end{aligned} \tag{3.6.10}$$

which is a contradiction to the choice $\ddot{U}(z_0) - \ddot{\bar{U}}(z_0) \leq 0$. If $U \not\leq \bar{U}$ and $V < \bar{V}$ on (z_1, z_2). Then there exists at least $z_0 \in (z_1, z_2)$ such that

$$V(z_0) > \bar{V}(z_0), \quad \dot{V}(z_0) = \dot{\bar{V}}(z_0) \,\text{and}\, \ddot{V}(z_0) - \ddot{\bar{V}}(z_0) \leq 0.$$

But,

$$\begin{aligned} \ddot{V}(z_0) - \ddot{\bar{V}}(z_0) \quad &= C\dot{V}(z_0) + \hat{h}(U, V) - \ddot{\bar{U}}(z_0) \\ &= \hat{h}(U, V) - ((\ddot{\bar{V}}(z_0) - C\dot{\bar{V}}(z_0)) \\ &\geq h(\bar{U}, \bar{V}) - h(\bar{U}, \bar{V}) = 0, \end{aligned} \tag{3.6.11}$$

which is again a contradiction. Hence $U \leq \bar{U}$ on $[z_1, z_2]$.

Case II. $V \not\leq \bar{V}$ on (z_1, z_2).

Then either $V \not\leq \bar{V}$ and $U > \bar{U}$ or $V \not\leq \bar{V}$ and $U < \bar{U}$. We can prove in the same fashion as above that both these possibilities lead to contradictions. Hence $V \leq \bar{V}$, on $[z_1, z_2]$. That is $(U, V) \leq (\bar{U}, \bar{V})$ on $[z_1, z_2]$.

Now, we show that $(\underline{U}, \underline{V}) \leq (U, V)$. Assume that $(\underline{U}, \underline{V}) \leq (U, V)$ is not true on (z_1, z_2). Again consider

Case. I. $\underline{U} \not\leq U$ on (z_1, z_2).

Then either $\underline{U} \not\leq U$ and $\underline{V} > V$ or $\underline{U} \not\leq U$ and $\underline{V} < V$. If $\underline{U} \not\leq U$ and $\underline{V} > V$. Then, there exists at least $y_0 \in (z_1, z_2)$ such that

$$\underline{U}(y_0) > U(y_0),\, \dot{\underline{U}}(y_0) = \dot{U}(y_0),\, \text{and}\, \ddot{\underline{U}}(y_0) - \ddot{U}(y_0) \leq 0.$$

Using the definition of $\hat{g}$ and that of lower solution, we obtain

$$\begin{aligned} \ddot{\underline{U}}(y_0) - \ddot{U}(y_0) &= \ddot{\underline{U}}(y_0) - \frac{C}{\rho}\dot{U}(y_0) - \hat{g}(U, V) \\ &= \ddot{\underline{U}}(y_0) - \frac{C}{\rho}\dot{\underline{U}}(y_0) - g(\underline{U}, \underline{V}) \\ &\geq g(\underline{U}, \underline{V}) - g(\underline{U}, \underline{V}) = 0, \end{aligned} \tag{3.6.12}$$

which leads to a contradiction. If $\underline{U} \not\leq U$ and $\underline{V} < V$, then by the same arguments, as above there exists a point $y_0 \in (z_1, z_2)$ such that the function $\underline{V} - V$ has a positive maximum at y_0. But

$$\begin{aligned} \ddot{\underline{V}}(y_0) - \ddot{V}(y_0) &= \ddot{\underline{V}}(y_0) - C\dot{V}(y_0) - \hat{h}(U, V) \\ &= \ddot{\underline{V}}(z_0) - C\dot{\underline{V}}(z_0) - h(\underline{U}, \underline{V}) \\ &\geq h(\underline{U}, \underline{V}) - h(\underline{U}, \underline{V}) = 0, \end{aligned} \tag{3.6.13}$$

which again leads to a contradiction. Hence $\underline{U} \leq U$ on $[z_1, z_2]$.

Case II. $\underline{V} \not\leq V$ on (z_1, z_2).

We can prove in the same fashion as above that this also leads to a contradiction. Hence every solution (U, V) of (3.6.7) satisfies (3.6.8) and hence is a solution of the system of equations (3.6.2). □

3.7 *Generalized approximation method*

In this section, we show that there exists a bounded monotone sequence of solutions of linear system that converges to a solution of the original system (3.6.2).

Assume that the quadratic forms

$$H(g) = (U - Y)^2 \frac{\partial^2 g}{\partial U^2} + 2(U - Y)(V - Z)\frac{\partial^2 g}{\partial U \partial V} + (V - Z)^2 \frac{\partial^2 g}{\partial V^2},$$

$$H(h) = (U-Y)^2\frac{\partial^2 h}{\partial U^2} + 2(U-Y)(V-Z)\frac{\partial^2 h}{\partial U \partial V} + (V-Z)^2\frac{\partial^2 h}{\partial V^2},$$

are non positive, that is, $H(g) \leq 0$, $H(h) \leq 0$. Hence,

$$\begin{aligned} g(U,V) &\leq g(Y,Z) + \frac{\partial g}{\partial U}(Y,Z)(U-Y) + \frac{\partial g}{\partial V}(Y,Z)(V-Z), \\ &\leq g(Y,Z) + m_1(U-Y) + m_2(V-Z), \quad U \geq Y, V \geq Z, \end{aligned} \tag{3.7.1}$$

where

$$m_1 = max\{\frac{1}{\rho}\frac{\partial f(U)}{\partial U}V : U \in (\underline{U}, \bar{U}),\, V \in (\underline{V}, \bar{V})\}$$

and

$$m_2 = max\{\frac{1}{\rho}f(U) : U \in (\underline{U}, \bar{U}),\, V \in (\underline{V}, \bar{V})\}.$$

Similarly

$$\begin{aligned} h(U,V) &\leq h(Y,Z) + \frac{\partial h}{\partial U}(Y,Z)(U-Y) + \frac{\partial h}{\partial V}(Y,Z)(V-Z) \\ &\leq h(Y,Z) + m_3(U-Y) + m_4(V-Z), \text{ for } U \geq Y, V \geq Z, \end{aligned} \tag{3.7.2}$$

where

$$\begin{aligned} m_3 &= max\{-\frac{\partial f(U)}{\partial U}V : U \in (\underline{U}, \bar{U})\ , V \in (\underline{V}, \bar{V})\}, \\ m_4 &= max\{[K - f(U)] : U \in (\underline{U}, \bar{U})\ , V \in (\underline{V}, \bar{V})\}. \end{aligned}$$

Define

$$\begin{aligned} G(U,V;Y,Z) &= g(Y,Z) + m_1(U-Y) + m_2(V-Z), \\ H(U,V;Y,Z) &= h(Y,Z) + m_3(U-Y) + m_4(V-Z). \end{aligned} \tag{3.7.3}$$

From (3.7.1), (3.7.2) and (3.7.3), it follows that

$$\begin{cases} g(U,V) \leq G(U,V;Y,Z), & \text{for } U \geq Y,\ V \geq Z, \\ g(U,V) = G(U,V;U,V), \end{cases}$$

$$\begin{cases} h(U,V) \leq H(U,V;Y,Z), & \text{for } U \geq Y,\ V \geq Z, \\ h(U,V) = H(U,V;U,V). \end{cases} \tag{3.7.4}$$

Consider the following system of linear BVPs,

$$\begin{aligned}\ddot{U}(z)-\frac{C}{\rho}\dot{U}(z)=G(U,V,\underline{U},\underline{V}) &= g(\underline{U},\underline{V})+m_1(U-\underline{U})\\ &\quad +m_2(V-\underline{V}),\ z\in[z_1,z_2],\\ \ddot{V}(z)-C\dot{V}(z)=H(U,V,\underline{U},\underline{V}) &= h(\underline{U},\underline{V})+m_3(U-\underline{U})\\ &\quad +m_4(V-\underline{V}),\ z\in[z_1,z_2], \quad (3.7.5)\end{aligned}$$

subject to the boundary conditions

$$\begin{aligned}U(z_1)=\alpha,\quad U(z_2)=\beta,\\ V(z_1)=\xi,\quad V(z_2)=\xi.\end{aligned}$$

Using (3.7.4) and the definition of lower solution, we obtain

$$\begin{aligned}G(\underline{U},\underline{V},\underline{U},\underline{V})=g(\underline{U},\underline{V})\leq\ddot{\underline{U}}-\frac{C}{\rho}\dot{\underline{U}},\ \text{for } [z_1,z_2],\\ H(\underline{U},\underline{V},\underline{U},\underline{V})=h(\underline{U},\underline{V})\leq\ddot{\underline{V}}-C\dot{\underline{V}},\ \text{for } [z_1,z_2],\end{aligned} \quad (3.7.6)$$

which implies that$(\underline{U},\underline{V})$ is a lower solution of the system (3.7.5).

Similarly, using (3.7.4) and the definition of upper solution, we obtain

$$\begin{aligned}G(\bar{U},\bar{V},\underline{U},\underline{V})\geq g(\bar{U},\bar{V})\geq\ddot{\bar{U}}-\frac{C}{\rho}\dot{\bar{U}},\ \text{for } [z_1,z_2],\\ H(\bar{U},\bar{P},\underline{U},\underline{V})\geq h(\bar{U},\bar{V})\geq\ddot{\bar{V}}-\dot{\bar{V}},\ \text{for } [z_1,z_2],\end{aligned} \quad (3.7.7)$$

which implies that $(\bar{U},\bar{V})$ is upper solution of the system of (3.7.5). Hence by Theorem (3.6.2) the system (3.7.5) has a solution (U_1,V_1) such that

$$(\underline{U},\underline{V})\leq(U_1,V_1)\leq(\bar{U},\bar{V}),\ \text{on } [z_1,z_2].$$

In view of (3.7.4) and the fact that (U_1,V_1) is solution of the system (3.7.5), we obtain

$$\begin{aligned}\ddot{U}_1(z)-\frac{C}{\rho}\dot{U}_1(z)=G(U_1,V_1;\underline{U},\underline{V})\geq g(U_1,V_1),\ \text{on } [z_1,z_2],\\ \ddot{V}_1(z)-C\dot{V}_1(z)=H(U_1,V_1;\underline{U},\underline{V})\geq h(U_1,V_1),\ \text{on } [z_1,z_2],\end{aligned} \quad (3.7.8)$$

which implies that (U_1, V_1) is lower solution of the system (3.6.2).

Now, consider the following linear system

$$\begin{aligned}
&\ddot{U}(z) - \frac{C}{\rho}\dot{U}(z) = G(U, V; U_1, V_1), \;\; z \in [z_1, z_2],\\
&\ddot{V}(z) - C\dot{V}(z) = H(U, V; U_1, V_1), \;\; z \in [z_1, z_2],\\
&U(z_1) = \alpha, \quad U(z_2) = \beta,\\
&V(z_1) = \xi, \quad V(z_2) = \xi. \qquad (3.7.9)
\end{aligned}$$

Using (3.7.8) and the fact that (U_1, V_1) is a solution of (3.7.5), we obtain

$$\begin{aligned}
&\ddot{U}_1(z) - \frac{C}{\rho}\dot{U}_1(z) \geq g(U_1, V_1) = G(U_1, V_1; U_1, V_1) \;\; z \in [z_1, z_2],\\
&\ddot{V}_1(z) - CV_1(z) \geq h(U_1, V_1) = H(U_1, V_1; U_1, V_1) \;\; z \in [z_1, z_2] \qquad (3.7.10)
\end{aligned}$$

and

$$\begin{aligned}
&G(\bar{U}, \bar{V}; U_1, V_1) \geq g(\bar{U}, \bar{V}) \geq \ddot{\bar{U}} - \frac{C}{\rho}\dot{\bar{U}} \;\; \text{on}\, [z_1, z_2],\\
&H(\bar{U}, \bar{P}; U_1, V_1) \geq h(\bar{U}, \bar{V}) \geq \ddot{\bar{V}} - C\dot{\bar{V}} \;\; \text{on}\, [z_1, z_2]. \qquad (3.7.11)
\end{aligned}$$

which implies that (U_1, V_1) and $(\bar{U}, \bar{V})$ are lower and upper solutions of the system (3.7.9). Hence by theorem (3.6.2), there exists a solution (U_2, V_2) of the system (3.7.9) such that

$$(U_1, V_1) \leq (U_2, V_2) \leq (\bar{U}, \bar{V}) \text{ on}\, [z_1, z_2].$$

From (3.7.4) and the fact that (U_2, V_2) is a solution of (3.7.9), we obtain

$$\begin{aligned}
&\ddot{U}_2(z) - \frac{C}{\rho}\dot{U}_2(z) = G(U_2, V_2; U_1, V_1) \geq g(U_2, V_2),\\
&\ddot{V}_2(z) - C\dot{V}_2(z) = H(U_2, V_2; U_1, V_1) \geq h(U_2, V_2).
\end{aligned} \qquad (3.7.12)$$

Hence, (U_2, V_2) is lower solution of the system (3.6.2). Continuing in the same passion, we obtain a monotone sequence $\{(U_n, V_n)\}$ of solutions of linear system such that

$$(\underline{U}, \underline{V}) \le (U_1, V_1) \le (U_2, V_2) \le (U_3, V_3) \le .... \le (\bar{U}, \bar{V}) \text{ on } [z_1, z_2],$$

where (U_n, V_n) is a solution of the system

$$\begin{aligned} \ddot{U}(z) - \frac{C}{\rho}\dot{U} &= G(U, V; U_{n-1}, V_{n-1}), \\ \ddot{V} - C\dot{V} &= H(U, V; U_{n-1}, V_{n-1}) \end{aligned}$$

and is given by

$$\begin{aligned} U_n(z) &= \frac{1}{(e^{\frac{c}{\rho}z_2} - e^{\frac{c}{\rho}z_1})}(\alpha(e^{\frac{c}{\rho}z_2} - e^{\frac{c}{\rho}z}) + \beta(e^{\frac{c}{\rho}z} - e^{\frac{c}{\rho}z_1})) \\ &\quad + \int_{z_1}^{z_2} K_1(z,s)\, G(U_n, V_n; U_{n-1}, V_{n-1})dt, \\ P_n(z) &= \xi + \int_{z_1}^{z_2} K_2(z,s)\, H(U_n, V_n; U_{n-1}, V_{n-1})dt, \end{aligned} \tag{3.7.13}$$

where

$$K_1(z,s) = \frac{\rho}{c\, e^{\frac{c}{\rho}s}(e^{\frac{c}{\rho}z_2} - e^{\frac{c}{\rho}z_1})} \begin{cases} (e^{\frac{c}{\rho}s} - e^{\frac{c}{\rho}z_2})(e^{\frac{c}{\rho}z} - e^{\frac{c}{\rho}z_1}), & z \le s \\ (e^{\frac{c}{\rho}s} - e^{\frac{c}{\rho}z_1})(e^{\frac{c}{\rho}z} - e^{\frac{c}{\rho}z_2}), & s \le z. \end{cases}$$

$$K_2(z,s)\frac{1}{c\, e^{cs}(e^{cz_2} - e^{cz_1})} \begin{cases} (e^{cs} - e^{cz_2})(e^{cz} - e^{cz_1}), & z \le s \\ (e^{cs} - e^{cz_1})(e^{cz} - e^{cz_2}), & s \le z. \end{cases}$$

The sequence of functions $\{(U_n, V_n)\}$ is uniformly bounded and equi-continuous. Hence by Arzela Ascoli theorem the sequence $\{(U_n, V_n)\}$ has a convergent subsequence $\{(U_{n_j}, V_{n_j})\}$, which converges to $(U, V) \in C^1[z_1, z_2] \times C^1[z_1, z_2]$.

Passing to the limit as $n \to \infty$ (3.7.13) gives

$$\begin{aligned} U(z) &= \frac{1}{(e^{\frac{c}{\rho}z_2} - e^{\frac{c}{\rho}z_1})}(\alpha(e^{\frac{c}{\rho}z_2} - e^{\frac{c}{\rho}z}) + \beta(e^{\frac{c}{\rho}z} - e^{\frac{c}{\rho}z_1})) \\ &\quad + \int_{z_1}^{z_2} K_1(z,s)\, g(U,V)dt, \\ V(z) &= \xi + \int_{z_1}^{z_2} K_2(z,s)\, h(U,V)dt, \end{aligned} \tag{3.7.14}$$

which is a solution of the system (3.6.2).

Remark 3.7.1. The condition $H(g) \leq 0$, $H(h) \leq 0$ does not generally hold for every g and h. If for some problems, g, h produce $H(g) > 0$, $H(h) > 0$, then one can select an auxiliary function ϕ (say) with $H(\phi) < 0$ such that $H(g + \phi) \leq 0$, $H(h + \phi) \leq 0$, see for example [23].

3.8 Conclusion

In this dissertation we address the existence and uniqueness of traveling wave solutions of the system of reaction diffusion equations (1.1.1). We further obtain some approximate traveling wave solutions. Using the method of upper and lower solutions we establish results for the existence of solutions on a finite interval. Moreover, we are able to eliminate the usual condition $C \geq \sqrt{4[sup\{f(u) : u \in [u_K, S^0]\} - K]}$ imposed on traveling wave solutions of the system (1.1.1) [9, 17].

We approximate traveling wave solutions on a finite interval by using three different methods: Adomian's decomposition method; the homotopy perturbation method and the generalized approximation method. Neither the upper and lower solutions method nor the three methods mentioned above, were studied previously for the traveling wave solutions of the system

(1.1.1). Graphs of solutions are also presented in the cases of the Adomian decomposition method and the homotopy perturbation method.

There are many approximation techniques available for approximating solutions of non-linear differential equations. Among these approximation techniques the homotopy perturbation method and the Adomian decomposition method have received much attention recently. In reference [21], the author says that Adomian's decomposition method and the homotopy perturbation method are theoretically equivalent in solving non-linear problems. Further in reference [22] it is claimed that He's homotopy perturbation corresponds to Adomian's decomposition method for certain classes of nonlinear problems, namely problems involving exponential nonlinearities. The claim is confirmed by Bratu's type problems [22]. In this dissertation we use Adomian's decomposition method and the homotopy perturbation method for approximating the solutions. Both the methods lead to the same second order approximate solutions. In reference [23], R. A. Khan studied a one-dimensional steady state heat transfer problem in a slab made of a material with temperature dependent conductivity. He proved that the results obtained by the generalized approximation method for the heat transfer problem are much more accurate and consistent than the results obtained by the homotopy perturbation method. The solution obtained by the homotopy perturbation method may not converge to the solution of the problem in some cases. A deeper understanding of these three approximation methods is required. It is expected that the generalized approximation method will produce better results than Adomian's decomposition method and the homotopy perturbation method.

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APPENDIX I

GRAPHICAL RESULTS AND COMPUTER PROGRAMMING FOR ADOMIAN'S DECOMPOSITION METHOD

In this appendix graphical results for the traveling wave solutions of the reaction-diffusion system (1.1.1), obtained by Adomian's decomposition method with the help of Mathematica (5.2) are shown.

$$U_0[z] = \alpha + (\beta - \alpha)\, z$$

$$\alpha + z\,(-\alpha + \beta)$$

$$V_0[z] = \xi + (\xi - \xi)\, z$$

$$\xi$$

$$U_1[z] = -z\left(\frac{c}{\rho}\left(\int_0^z\int_1^z D[U_0[z], z]\, dz\, dz \;/.\; z \to 1\right) + \frac{1}{\rho}\left(\int_0^z\int_1^z U_0[z]\, V_0[z]\, dz\, dz \;/.\; z \to 1\right)\right) + \frac{c}{\rho}\int_0^z\int_1^z D[U_0[z], z]\, dz\, dz + \frac{1}{\rho}\int_0^z\int_1^z U_0[z]\, V_0[z]\, dz\, dz$$

$$-z\left(\frac{c\,(\alpha - \beta)}{2\rho} + \frac{(-\alpha - 2\beta)\,\xi}{6\rho}\right) - \frac{c\,(-2 + z)\, z\,(\alpha - \beta)}{2\rho} + \frac{z\,(-(3 - 3z + z^2)\,\alpha + (-3 + z^2)\,\beta)\,\xi}{6\rho}$$

`Simplify[%]`

$$-\frac{(-1 + z)\, z\,(3c\,(\alpha - \beta) + ((-2 + z)\,\alpha - (1 + z)\,\beta)\,\xi)}{6\rho}$$

$$V_1[z] = z\left(-c\left(\int_0^z\int_1^z D[V_0[z], z]\, dz\, dz \;/.\; z \to 1\right) - K\left(\int_0^z\int_1^z V_0[z]\, dz\, dz \;/.\; z \to 1\right) + \left(\int_0^z\int_1^z U_0[z]\, V_0[z]\, dz\, dz \;/.\; z \to 1\right)\right) + c\int_0^z\int_1^z D[V_0[z], z]\, dz\, dz + K\int_0^z\int_1^z V_0[z]\, dz\, dz - \int_0^z\int_1^z U_0[z]\, V_0[z]\, dz\, dz$$

$$\frac{1}{2} K (-2 + z) z \xi - \frac{1}{6} z \left(-(3 - 3 z + z^2) \alpha + (-3 + z^2) \beta\right) \xi + z \left(\frac{K \xi}{2} + \frac{1}{6} (-\alpha - 2 \beta) \xi\right)$$

Simplify[%]

$$\frac{1}{6} (-1 + z) z (3 K + (-2 + z) \alpha - (1 + z) \beta) \xi$$

$$\mathbf{U_2[z]} = -z \left(\frac{c}{\rho} \left(\int_0^z \int_1^z D[U_1[z], z]\, dz\, dz \;/.\; z \to 1\right) + \frac{1}{\rho} \left(\int_0^z \int_1^z (U_0[z]\, V_1[z] + U_1[z]\, V_0[z])\, dz\, dz \;/.\; z \to 1\right)\right) + \frac{c}{\rho} \int_0^z \int_1^z D[U_1[z], z]\, dz\, dz + \frac{1}{\rho} \int_0^z \int_1^z (U_0[z]\, V_1[z] + U_1[z]\, V_0[z])\, dz\, dz$$

$$-\frac{c z^2 \left(2 c (-3 + 2 z) (\alpha - \beta) + \left((-2 + z)^2 \alpha - (-2 + z^2) \beta\right) \xi\right)}{24 \rho^2} + \frac{1}{360 \rho^2} \Big(z \xi \big(-15 c (2 + (-2 + z) z^2) (\alpha - \beta) + 15 \alpha \xi + 15 \beta \xi + 15 K \alpha \rho - 8 \alpha^2 \rho - 2 z^5 (\alpha - \beta)^2 \rho + 15 K \beta \rho - 14 \alpha \beta \rho - 8 \beta^2 \rho + 3 z^4 (\alpha - \beta) (-\xi - 3 K \rho + 4 \alpha \rho) + 10 z^2 (-(2 \alpha + \beta) \xi + \alpha (-3 K + 2 \alpha + \beta) \rho) - 5 z^3 (-3 \alpha \xi - (6 K \alpha - 5 \alpha^2 - 3 K \beta + \alpha \beta + \beta^2) \rho)\big)\Big) - z \left(-\frac{c (-2 c (\alpha - \beta) + (\alpha + \beta) \xi)}{24 \rho^2} + \frac{1}{360 \rho^2} \Big(\xi \big(-15 c (\alpha - \beta) + 15 \alpha \xi + 15 \beta \xi + 15 K \alpha \rho - 8 \alpha^2 \rho - 2 (\alpha - \beta)^2 \rho + 15 K \beta \rho - 14 \alpha \beta \rho - 8 \beta^2 \rho + 3 (\alpha - \beta) (-\xi - 3 K \rho + 4 \alpha \rho) + 10 (-(2 \alpha + \beta) \xi + \alpha (-3 K + 2 \alpha + \beta) \rho) - 5 (-3 \alpha \xi - (6 K \alpha - 5 \alpha^2 - 3 K \beta + \alpha \beta + \beta^2) \rho)\big)\Big)\right)$$

Simplify[%]

$$-\frac{1}{360\rho^2}\left((-1+z)z\left(30c^2(-1+2z)(\alpha-\beta)+30c\left(-2z\alpha+z^2(\alpha-\beta)+\beta\right)\xi+\xi\left((-5-5z+15z^2-10z^3+2z^4)\alpha^2\rho+\alpha\left((8+8z-12z^2+3z^3)\xi+(3K(3+3z-7z^2+3z^3)+(-7-7z+3z^2+8z^3-4z^4)\beta)\rho\right)+\beta\left((7+7z-3z^2-3z^3)\xi+(K(6+6z+6z^2-9z^3)+(-3-3z-3z^2+2z^3+2z^4)\beta)\rho\right)\right)\right)\right)$$

V2[z] =

$$z\left(-c\left(\int_0^z\int_1^z D[V_1[z],z]\,dz\,dz\ /.\ z\to 1\right)-K\left(\int_0^z\int_1^z V_1[z]\,dz\,dz\ /.\ z\to 1\right)+\left(\int_0^z\int_1^z (U_0[z]\,V_1[z]+U_1[z]\,V_0[z])\,dz\,dz\ /.\ z\to 1\right)\right)+c\int_0^z\int_1^z D[V_1[z],z]\,dz\,dz+K\int_0^z\int_1^z V_1[z]\,dz\,dz-\int_0^z\int_1^z (U_0[z]\,V_1[z]+U_1[z]\,V_0[z])\,dz\,dz$$

$$\frac{1}{24}cz^2\left(K(-6+4z)+(-2+z)^2\alpha-(-2+z^2)\beta\right)\xi+\frac{1}{360}Kz\left(15K\left(2+(-2+z)z^2\right)-15(\alpha+\beta)+z^2\left((20+3(-5+z)z)\alpha+(10-3z^2)\beta\right)\right)\xi-\frac{1}{360\rho}\left(z\xi\left(-15c\left(2+(-2+z)z^2\right)(\alpha-\beta)+15\alpha\xi+15\beta\xi+15K\alpha\rho-8\alpha^2\rho-2z^5(\alpha-\beta)^2\rho+15K\beta\rho-14\alpha\beta\rho-8\beta^2\rho+3z^4(\alpha-\beta)(-\xi-3K\rho+4\alpha\rho)+10z^2\left(-(2\alpha+\beta)\xi+\alpha(-3K+2\alpha+\beta)\rho\right)-5z^3\left(-3\alpha\xi-(6K\alpha-5\alpha^2-3K\beta+\alpha\beta+\beta^2)\rho\right)\right)\right)+$$

$$z\left(-\frac{1}{24}c(-2K+\alpha+\beta)\xi-\frac{1}{360}K(15K+8\alpha+7\beta-15(\alpha+\beta))\xi+\frac{1}{360\rho}\right.$$
$$(\xi(-15c(\alpha-\beta)+15\alpha\xi+15\beta\xi+15K\alpha\rho-8\alpha^2\rho-2(\alpha-\beta)^2\rho+$$
$$15K\beta\rho-14\alpha\beta\rho-8\beta^2\rho+3(\alpha-\beta)(-\xi-3K\rho+4\alpha\rho)+$$
$$10(-(2\alpha+\beta)\xi+\alpha(-3K+2\alpha+\beta)\rho)-$$
$$\left.5(-3\alpha\xi-(6K\alpha-5\alpha^2-3K\beta+\alpha\beta+\beta^2)\rho)))\right)$$

```
Simplify[%]
```

$$\frac{1}{360\rho}$$
$$((-1+z)z\xi(7\beta\xi+7z\beta\xi-3z^2\beta\xi-3z^3\beta\xi-15K^2\rho-15K^2z\rho+$$
$$15K^2z^2\rho+(-5-5z+15z^2-10z^3+2z^4)\alpha^2\rho+13K\beta\rho+13Kz\beta\rho+$$
$$3Kz^2\beta\rho-12Kz^3\beta\rho-3\beta^2\rho-3z\beta^2\rho-3z^2\beta^2\rho+2z^3\beta^2\rho+$$
$$2z^4\beta^2\rho+\alpha((8+8z-12z^2+3z^3)\xi+(K(17+17z-33z^2+12z^3)+$$
$$(-7-7z+3z^2+8z^3-4z^4)\beta)\rho)+$$
$$15c(2K(-1+2z)\rho+\beta(1+z+\rho-z\rho-z^2(1+\rho))+$$
$$\alpha(-1+\rho+z^2(1+\rho)-z(1+3\rho)))))$$

U[z_] = U0[z] + U1[z] + U2[z]

$$\alpha+z(-\alpha+\beta)-z\left(\frac{c(\alpha-\beta)}{2\rho}+\frac{(-\alpha-2\beta)\xi}{6\rho}\right)-$$
$$\frac{cz^2(2c(-3+2z)(\alpha-\beta)+((-2+z)^2\alpha-(-2+z^2)\beta)\xi)}{24\rho^2}-$$
$$\frac{c(-2+z)z(\alpha-\beta)}{2\rho}+\frac{z(-(3-3z+z^2)\alpha+(-3+z^2)\beta)\xi}{6\rho}+\frac{1}{360\rho^2}$$
$$(z\xi(-15c(2+(-2+z)z^2)(\alpha-\beta)+15\alpha\xi+15\beta\xi+15K\alpha\rho-$$
$$8\alpha^2\rho-2z^5(\alpha-\beta)^2\rho+15K\beta\rho-14\alpha\beta\rho-8\beta^2\rho+3z^4(\alpha-\beta)$$
$$(-\xi-3K\rho+4\alpha\rho)+10z^2(-(2\alpha+\beta)\xi+\alpha(-3K+2\alpha+\beta)\rho)-$$

$$5 z^3 (-3 \alpha \xi - (6 K \alpha - 5 \alpha^2 - 3 K \beta + \alpha \beta + \beta^2) \rho))) -$$

$$z \left(-\frac{c (-2 c (\alpha - \beta) + (\alpha + \beta) \xi)}{24 \rho^2} + \frac{1}{360 \rho^2} (\xi (-15 c (\alpha - \beta) + 15 \alpha \xi + 15 \beta \xi + 15 K \alpha \rho - 8 \alpha^2 \rho - 2 (\alpha - \beta)^2 \rho + 15 K \beta \rho - 14 \alpha \beta \rho - 8 \beta^2 \rho + 3 (\alpha - \beta) (-\xi - 3 K \rho + 4 \alpha \rho) + 10 (-(2 \alpha + \beta) \xi + \alpha (-3 K + 2 \alpha + \beta) \rho) - 5 (-3 \alpha \xi - (6 K \alpha - 5 \alpha^2 - 3 K \beta + \alpha \beta + \beta^2) \rho))) \right)$$

Simplify[%]

$$\frac{1}{360 \rho^2} (-30 c^2 z (1 - 3 z + 2 z^2) (\alpha - \beta) + 180 z^2 \alpha \xi \rho - 2 z^6 (\alpha - \beta)^2 \xi \rho + 360 \alpha \rho^2 + 3 z^5 (\alpha - \beta) \xi (-\xi - 3 K \rho + 4 \alpha \rho) - 30 c (-1 + z) z (-2 z \alpha \xi + z^2 (\alpha - \beta) \xi + \beta \xi + 6 \alpha \rho - 6 \beta \rho) - 10 z^3 \xi (\beta (\xi - 6 \rho) - 2 \alpha^2 \rho + \alpha (2 \xi + (6 + 3 K - \beta) \rho)) + 5 z^4 \xi (-5 \alpha^2 \rho + \beta (-3 K + \beta) \rho + \alpha (3 \xi + 6 K \rho + \beta \rho)) + z (-5 \alpha^2 \xi \rho + \alpha (8 \xi^2 + (-120 + 9 K - 7 \beta) \xi \rho - 360 \rho^2) + \beta (7 \xi^2 + 3 (-20 + 2 K - \beta) \xi \rho + 360 \rho^2)))$$

N[U[0]]

α

N[U[1]]

β

V[z_] = V$_0$[z] + V$_1$[z] + V$_2$[z]

$$\xi + \frac{1}{2} K (-2 + z) z \xi - \frac{1}{6} z (-(3 - 3 z + z^2) \alpha + (-3 + z^2) \beta) \xi + \frac{1}{24} c z^2 (K (-6 + 4 z) + (-2 + z)^2 \alpha - (-2 + z^2) \beta) \xi + \frac{1}{360} K z (15 K (2 + (-2 + z) z^2) - 15 (\alpha + \beta) +$$

$$z^2\,((20+3\,(-5+z)\,z)\,\alpha+(10-3\,z^2)\,\beta))\,\xi+$$

$$z\left(\frac{K\,\xi}{2}+\frac{1}{6}\,(-\alpha-2\,\beta)\,\xi\right)-\frac{1}{360\,\rho}$$

$$(z\,\xi\,(-15\,c\,(2+(-2+z)\,z^2)\,(\alpha-\beta)+15\,\alpha\,\xi+15\,\beta\,\xi+15\,K\,\alpha\,\rho-8\,\alpha^2\,\rho-$$
$$2\,z^5\,(\alpha-\beta)^2\,\rho+15\,K\,\beta\,\rho-14\,\alpha\,\beta\,\rho-8\,\beta^2\,\rho+3\,z^4\,(\alpha-\beta)$$
$$(-\xi-3\,K\,\rho+4\,\alpha\,\rho)+10\,z^2\,(-(2\,\alpha+\beta)\,\xi+\alpha\,(-3\,K+2\,\alpha+\beta)\,\rho)-$$
$$5\,z^3\,(-3\,\alpha\,\xi-(6\,K\,\alpha-5\,\alpha^2-3\,K\,\beta+\alpha\,\beta+\beta^2)\,\rho)))+$$

$$z\left(-\frac{1}{24}\,c\,(-2\,K+\alpha+\beta)\,\xi-\frac{1}{360}\,K\,(15\,K+8\,\alpha+7\,\beta-15\,(\alpha+\beta))\,\xi+\frac{1}{360\,\rho}\right.$$
$$(\xi\,(-15\,c\,(\alpha-\beta)+15\,\alpha\,\xi+15\,\beta\,\xi+15\,K\,\alpha\,\rho-8\,\alpha^2\,\rho-2\,(\alpha-\beta)^2\,\rho+$$
$$15\,K\,\beta\,\rho-14\,\alpha\,\beta\,\rho-8\,\beta^2\,\rho+3\,(\alpha-\beta)\,(-\xi-3\,K\,\rho+4\,\alpha\,\rho)+$$
$$10\,(-(2\,\alpha+\beta)\,\xi+\alpha\,(-3\,K+2\,\alpha+\beta)\,\rho)-$$
$$\left.5\,(-3\,\alpha\,\xi-(6\,K\,\alpha-5\,\alpha^2-3\,K\,\beta+\alpha\,\beta+\beta^2)\,\rho)))\right)$$

Simplify[%]

$$\frac{1}{360\,\rho}\,(\xi\,(360\,\rho+180\,z^2\,(K-\alpha)\,\rho+$$
$$2\,z^6\,(\alpha-\beta)^2\,\rho-3\,z^5\,(\alpha-\beta)\,(-\xi-4\,K\,\rho+4\,\alpha\,\rho)+$$
$$10\,z^3\,(-3\,K^2\,\rho-2\,\alpha^2\,\rho+\beta\,(\xi+(-6+K)\,\rho)+\alpha\,(2\,\xi+(6+5\,K-\beta)\,\rho))+$$
$$5\,z^4\,(5\,\alpha^2\,\rho+(3\,K^2+3\,K\,\beta-\beta^2)\,\rho-\alpha\,(3\,\xi+(9\,K+\beta)\,\rho))+$$
$$z\,(15\,(-12+K)\,K\,\rho+5\,\alpha^2\,\rho+3\,\beta^2\,\rho+$$
$$\beta\,(-7\,\xi+60\,\rho-13\,K\,\rho)+\alpha\,(-8\,\xi+(120-17\,K+7\,\beta)\,\rho))+$$
$$15\,c\,(-1+z)\,z\,(2\,K\,(-1+2\,z)\,\rho+\beta\,(1+z+\rho-z\,\rho-z^2\,(1+\rho))+$$
$$\alpha\,(-1+\rho+z^2\,(1+\rho)-z\,(1+3\,\rho)))))$$

```
N[V[0]]
```
ξ
```
N[V[1]]
```
ξ
```
α=2
β=0.5
ξ=2
K=1
c=2
```

2
0.5
2
1

U1[z_] = U_0[z] + U_1[z] + U_2[z]

$$2-1.5z-\frac{z^2\left(6.\left(-3+2z\right)+2\left(2\left(-2+z\right)^2-0.5\left(-2+z^2\right)\right)\right)}{12\rho^2}-\frac{0.5z}{\rho}-\frac{1.5\left(-2+z\right)z}{\rho}+\frac{z\left(0.5\left(-3+z^2\right)-2\left(3-3z+z^2\right)\right)}{3\rho}+\frac{1}{180\rho^2}\left(z\left(75.-45.\left(2+\left(-2+z\right)z^2\right)-10.5\rho-4.5z^5\rho+10z^2\left(-9.+3.\rho\right)+4.5z^4\left(-2+5\rho\right)-5z^3\left(-12+8.25\rho\right)\right)\right)-z\left(\frac{0.0833333}{\rho^2}+\frac{30.-15.\rho+10\left(-9.+3.\rho\right)+4.5\left(-2+5\rho\right)-5\left(-12+8.25\rho\right)}{180\rho^2}\right)$$

V1[z_] = V_0[z] + V_1[z] + V_2[z]

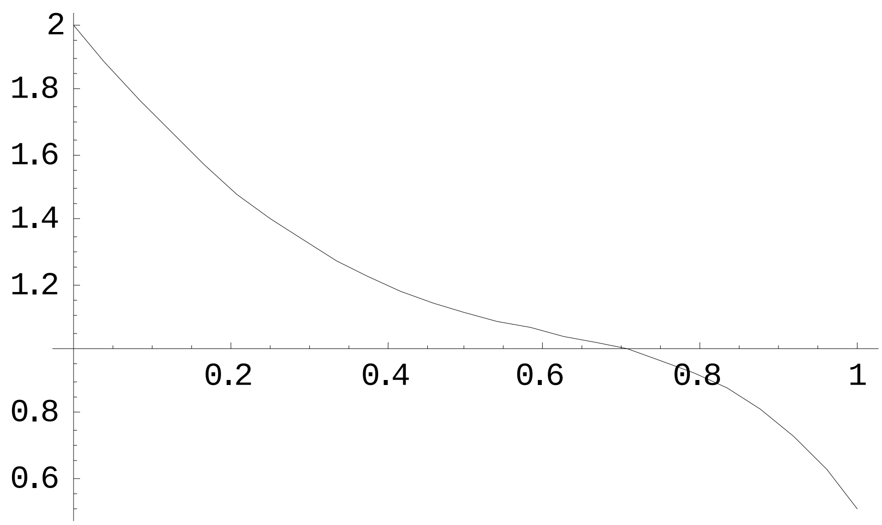
$$2+0.z+\left(-2+z\right)z+\frac{1}{6}z^2\left(-6+2\left(-2+z\right)^2+4z-0.5\left(-2+z^2\right)\right)-\frac{1}{3}z\left(0.5\left(-3+z^2\right)-2\left(3-3z+z^2\right)\right)+\frac{1}{180}z\left(-37.5+15\left(2+\left(-2+z\right)z^2\right)+z^2\left(2\left(20+3\left(-5+z\right)z\right)+0.5\left(10-3z^2\right)\right)\right)-\frac{1}{180\rho}\left(z\left(75.-45.\left(2+\left(-2+z\right)z^2\right)-10.5\rho-4.5z^5\rho+10z^2\left(-9.+3.\rho\right)+4.5z^4\left(-2+5\rho\right)-5z^3\left(-12+8.25\rho\right)\right)\right)+z\left(-0.0666667+\frac{1}{180\rho}\left(30.-15.\rho+10\left(-9.+3.\rho\right)+4.5\left(-2+5\rho\right)-5\left(-12+8.25\rho\right)\right)\right)$$

ρ=0.3

0.3

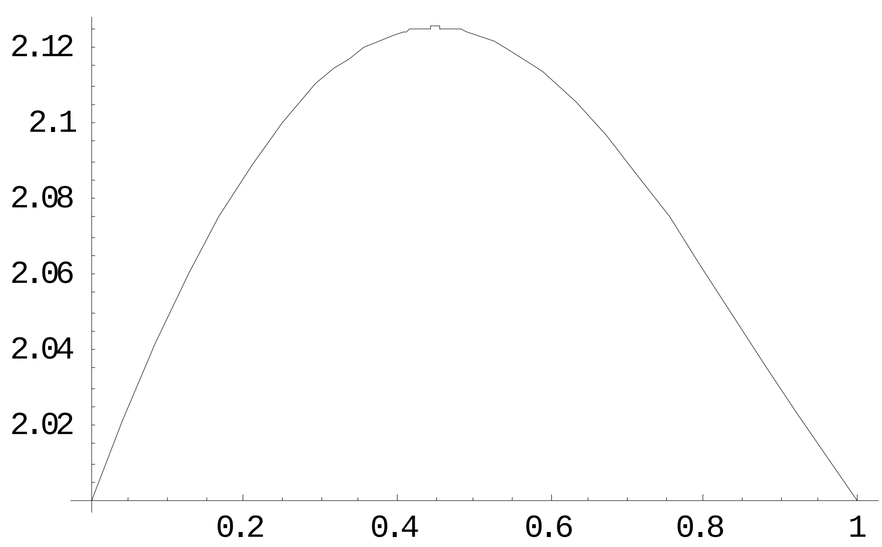
```
Table[U1[z],{z,0,1,0.1}]
{2,1.72687,1.49891,1.32203,1.19535,1.11097,1.05353,0.999832
,0.918273,0.768347,0.5}
Table[V1[z],{z,0,1,0.1}]
{2,2.04879,2.08627,2.11147,2.12392,2.12374,2.11182,2.08994,
2.06098,2.02911,2.}

Plot[U1[z],{z,0,1}]
```



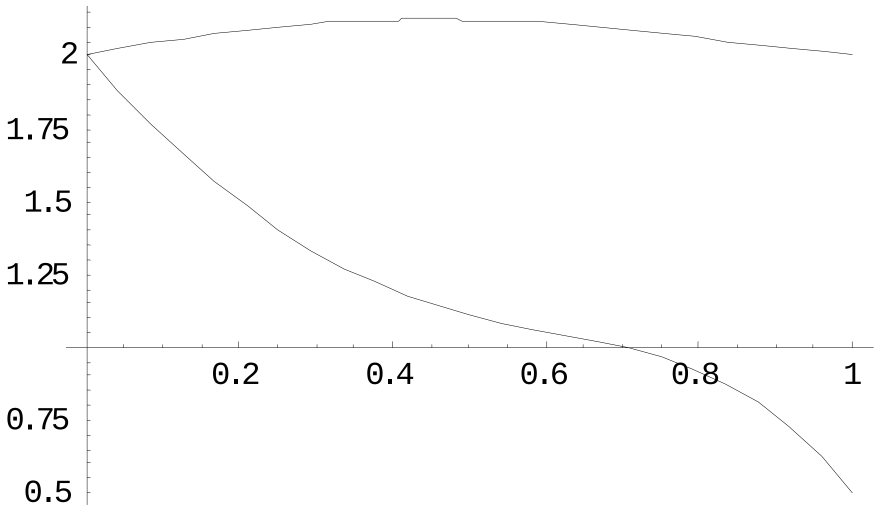
-Graphics-

```
Plot[V1[z],{z,0,1}]
```



-Graphics-

```
Plot[{U1[z],V1[z]},{z,0,1}]
```



-Graphics-

```
ρ=0.4
```

0.4

U2[z_] = U₀[z] + U₁[z] + U₂[z]

$$2-2.90625\,z-3.75\,(-2+z)\,z+0.833333\,z\,(0.5\,(-3+z^2)-2\,(3-3z+z^2))+0.0347222\,z\,(70.8-78.\,z^2+43.5\,z^3+0.\,z^4-1.8\,z^5-45.\,(2+(-2+z)\,z^2))-0.520833\,z^2\,(6.\,(-3+2z)+2\,(2\,(-2+z)^2-0.5\,(-2+z^2)))$$

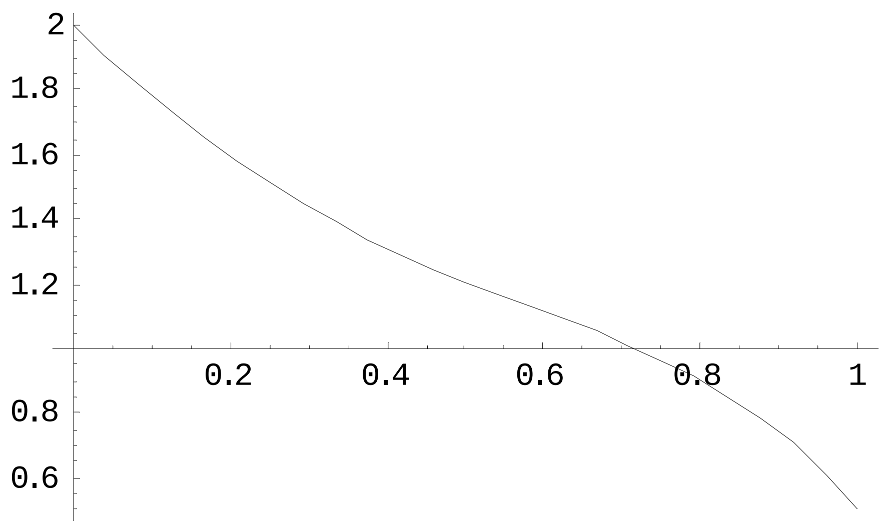
V2[z_] = V₀[z] + V₁[z] + V₂[z]

$$2-0.2125\,z+(-2+z)\,z+\frac{1}{6}z^2\,(-6+2\,(-2+z)^2+4z-0.5\,(-2+z^2))-\frac{1}{3}z\,(0.5\,(-3+z^2)-2\,(3-3z+z^2))-0.0138889\,z\,(70.8-78.\,z^2+43.5\,z^3+0.\,z^4-1.8\,z^5-45.\,(2+(-2+z)\,z^2))+\frac{1}{180}z\,(-37.5+15\,(2+(-2+z)\,z^2)+z^2\,(2\,(20+3\,(-5+z)\,z)+0.5\,(10-3z^2)))$$

```
Table[U2[z],{z,0,1,0.1}]
```

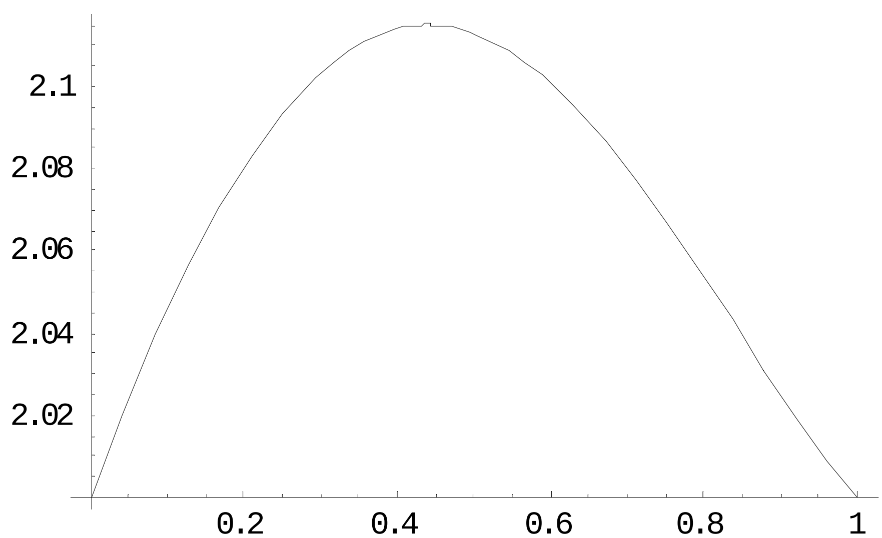
{2,1.7813,1.59283,1.43625,1.30924,1.2054,1.11408,1.02019,

```
0.903949,0.740582,0.5}
Table[V2[z],{z,0,1,0.1}]
{2,2.04602,2.08081,2.1036,2.11416,2.11289,2.10091,2.08016,
2.05355,2.02507,2.}
Plot[U2[z],{z,0,1}]
```



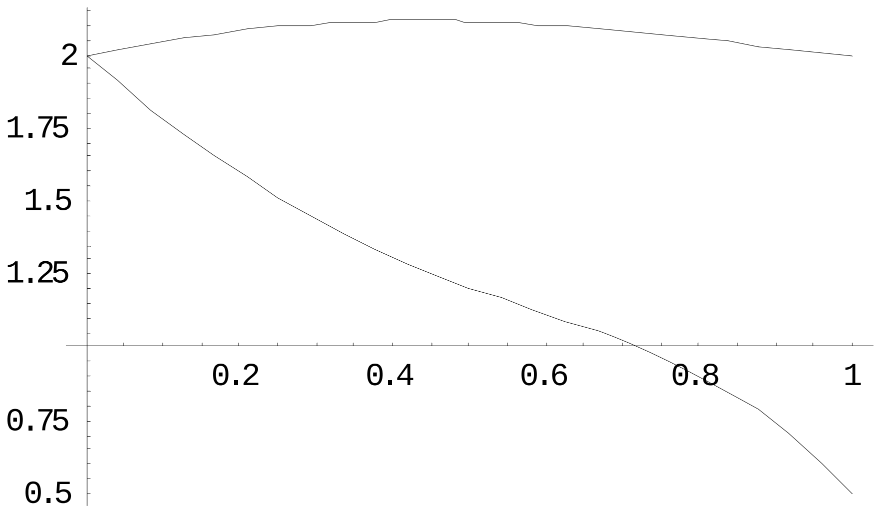
-Graphics-

```
Plot[V2[z],{z,0,1}]
```



-Graphics-

```
Plot[{U2[z],V2[z]},{z,0,1}]
```



-Graphics-

ρ=0.5

0.5

U3[z_] = U_0[z] + U_1[z] + U_2[z]

$$2-2.59167\,z-3.\,(-2+z)\,z+0.666667\,z\,(0.5\,(-3+z^2)-2\,(3-3\,z+z^2))+0.0222222\,z\,(69.75-75.\,z^2+39.375\,z^3+2.25\,z^4-2.25\,z^5-45.\,(2+(-2+z)\,z^2))-0.333333\,z^2\,(6.\,(-3+2\,z)+2\,(2\,(-2+z)^2-0.5\,(-2+z^2)))$$

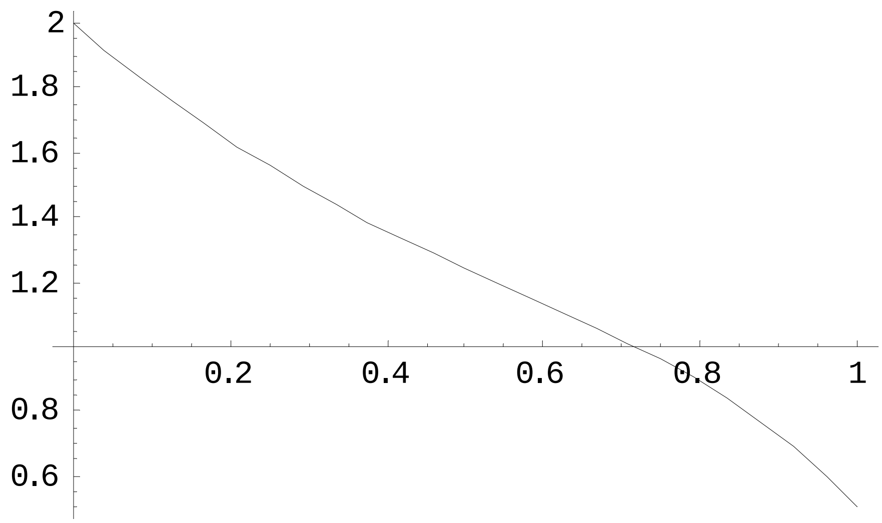
V3[z_] = V_0[z] + V_1[z] + V_2[z]

$$2-0.1875\,z+(-2+z)\,z+\frac{1}{6}\,z^2\,(-6+2\,(-2+z)^2+4\,z-0.5\,(-2+z^2))-\frac{1}{3}\,z\,(0.5\,(-3+z^2)-2\,(3-3\,z+z^2))-0.0111111\,z\,(69.75-75.\,z^2+39.375\,z^3+2.25\,z^4-2.25\,z^5-45.\,(2+(-2+z)\,z^2))+\frac{1}{180}\,z\,(-37.5+15\,(2+(-2+z)\,z^2)+z^2\,(2\,(20+3\,(-5+z)\,z)+0.5\,(10-3\,z^2)))$$

```
Table[U3[z],{z,0,1,0.1}]
{2,1.80639,1.63521,1.48647,1.35751,1.24297,1.13476,1.02191,
```

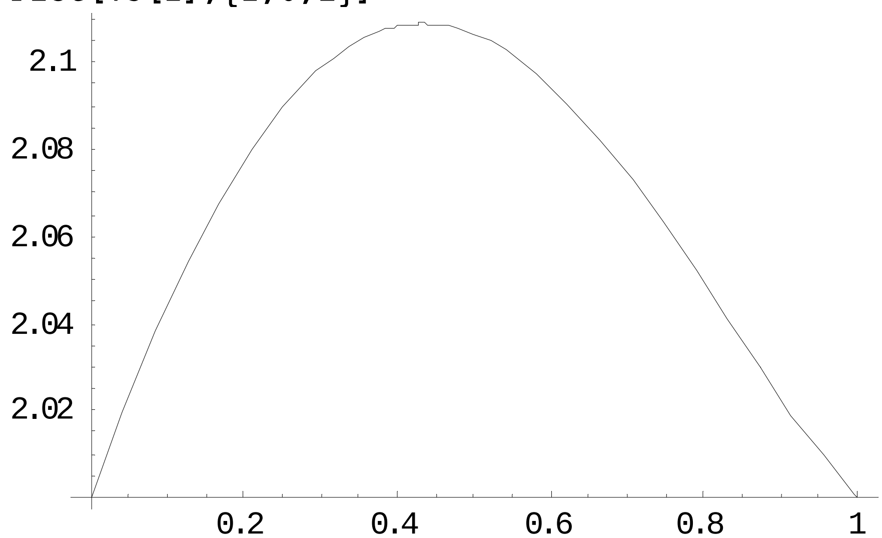
```
0.890477,0.72334,0.5}
Table[V3[z],{z,0,1,0.1}]
{2,2.04436,2.07753,2.09887,2.1083,2.10638,2.09437,2.0743,
2.04909,2.02264,2.}

Plot[U3[z],{z,0,1}]
```



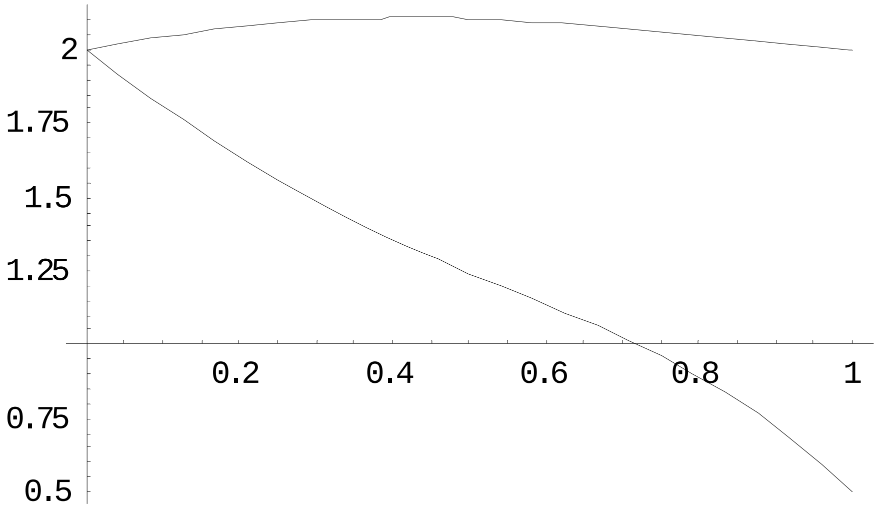

```
-Graphics-

Plot[V3[z],{z,0,1}]
```




```
-Graphics-
```

```
Plot[{U3[z],V3[z]},{z,0,1}]
```



-Graphics-

```
ρ=0.6
```

0.6

U4[z_] = U_0[z] + U_1[z] + U_2[z]

$$2-2.3912\,z-2.5\,(-2+z)\,z+0.555556\,z\,(0.5\,(-3+z^2)-2\,(3-3\,z+z^2))+0.0154321\,z\,(68.7-72.\,z^2+35.25\,z^3+4.5\,z^4-2.7\,z^5-45.\,(2+(-2+z)\,z^2))-0.231481\,z^2\,(6.\,(-3+2\,z)+2\,(2\,(-2+z)^2-0.5\,(-2+z^2)))$$

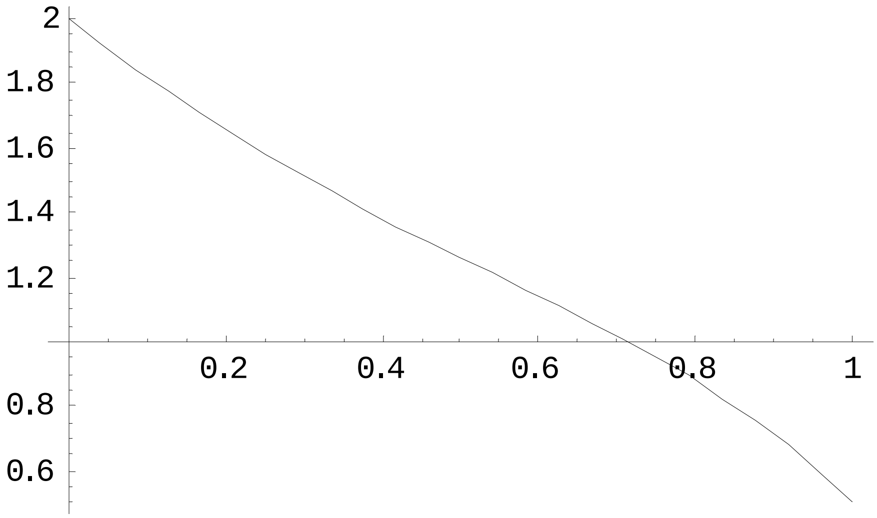
V4[z_] = V_0[z] + V_1[z] + V_2[z]

$$2-0.170833\,z+(-2+z)\,z+\frac{1}{6}\,z^2\,(-6+2\,(-2+z)^2+4\,z-0.5\,(-2+z^2))-\frac{1}{3}\,z\,(0.5\,(-3+z^2)-2\,(3-3\,z+z^2))-0.00925926\,z\,(68.7-72.\,z^2+35.25\,z^3+4.5\,z^4-2.7\,z^5-45.\,(2+(-2+z)\,z^2))+\frac{1}{180}\,z\,(-37.5+15\,(2+(-2+z)\,z^2)+z^2\,(2\,(20+3\,(-5+z)\,z)+0.5\,(10-3\,z^2)))$$

```
Table[U4[z],{z,0,1,0.1}]
```

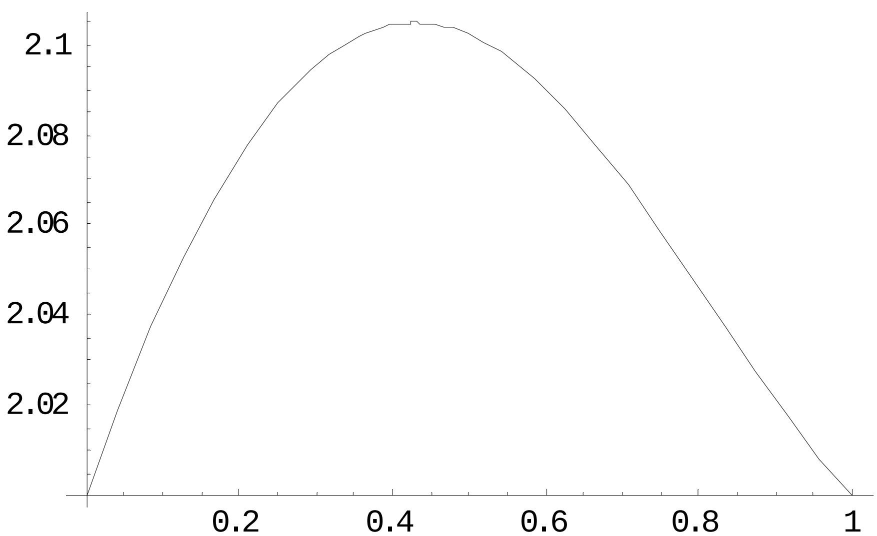
{2,1.81996,1.65765,1.51232,1.38132,1.26005,1.14201,1.01868,
0.879463,0.711602,0.5}

```
Table[V4[z],{z,0,1,0.1}]
{2,2.04325,2.07535,2.09573,2.1044,2.10204,2.09,2.07039,
2.04611,2.02103,2.}
Plot[U4[z],{z,0,1}]
```



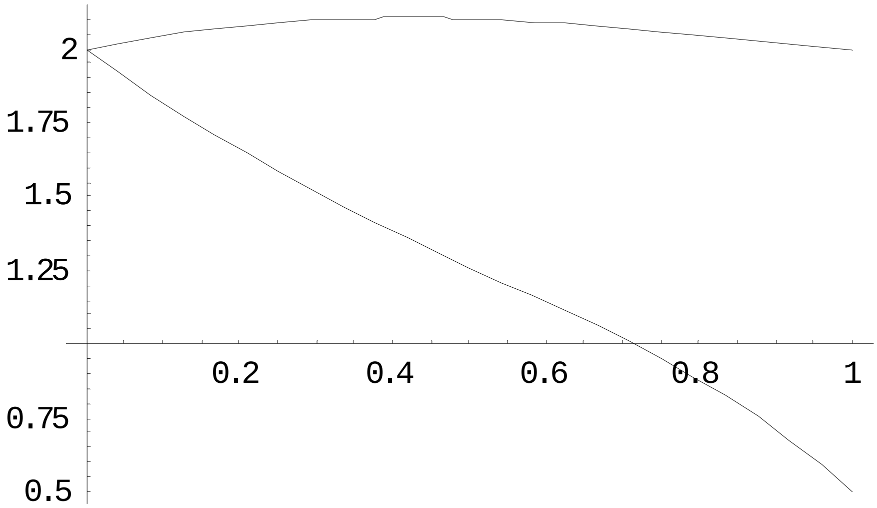
-Graphics-

```
Plot[V4[z],{z,0,1}]
```



-Graphics-

```
Plot[{U4[z],V4[z]},{z,0,1}]
```



-Graphics-

```
ρ=0.7
```

0.7

U5[z_] = U_0[z] + U_1[z] + U_2[z]

$$2 - 2.25255\,z - 2.14286\,(-2+z)\,z + 0.47619\,z\,(0.5\,(-3+z^2) - 2\,(3-3\,z+z^2)) + 0.0113379\,z\,(67.65 - 69.\,z^2 + 31.125\,z^3 + 6.75\,z^4 - 3.15\,z^5 - 45.\,(2+(-2+z)\,z^2)) - 0.170068\,z^2\,(6.\,(-3+2\,z) + 2\,(2\,(-2+z)^2 - 0.5\,(-2+z^2)))$$

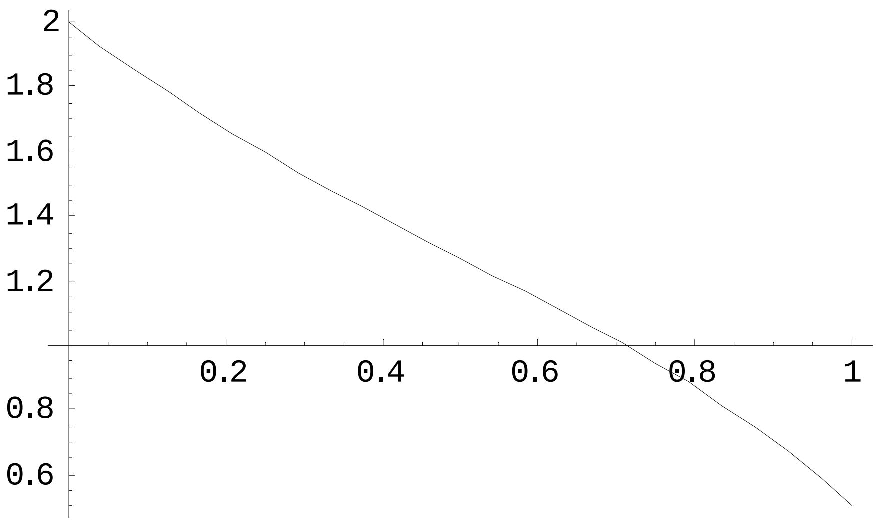
V5[z_] = V_0[z] + V_1[z] + V_2[z]

$$2 - 0.158929\,z + (-2+z)\,z + \frac{1}{6}\,z^2\,(-6 + 2\,(-2+z)^2 + 4\,z - 0.5\,(-2+z^2)) - \frac{1}{3}\,z\,(0.5\,(-3+z^2) - 2\,(3-3\,z+z^2)) - 0.00793651\,z\,(67.65 - 69.\,z^2 + 31.125\,z^3 + 6.75\,z^4 - 3.15\,z^5 - 45.\,(2+(-2+z)\,z^2)) + \frac{1}{180}\,z\,(-37.5 + 15\,(2+(-2+z)\,z^2) + z^2\,(2\,(20+3\,(-5+z)\,z) + 0.5\,(10-3\,z^2)))$$

```
Table[U5[z],{z,0,1,0.1}]
```

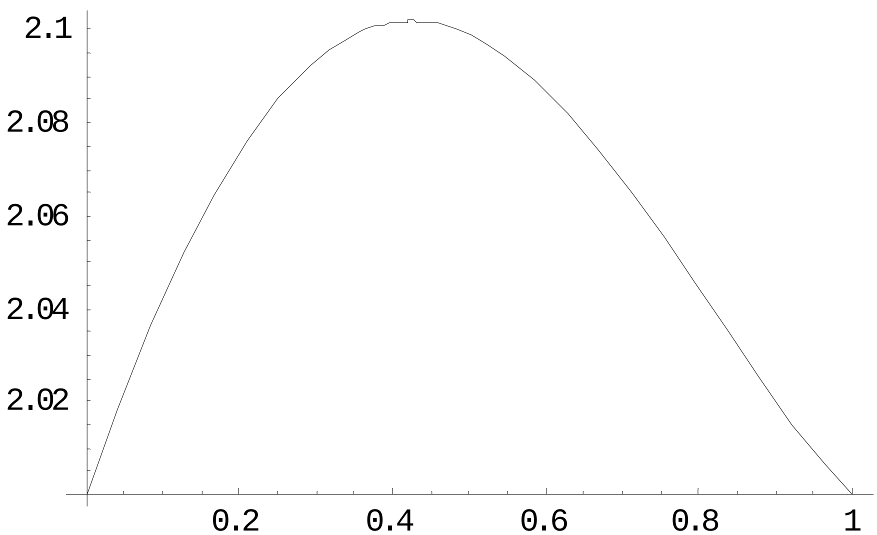
{2,1.82811,1.67082,1.52705,1.39423,1.26836,1.144,1.01422,
0.8706,0.7031,0.5}

```
Table[V5[z],{z,0,1,0.1}]
{2,2.04246,2.07379,2.09348,2.10161,2.09894,2.08689,2.0676,
2.04399,2.01987,2.}
Plot[U5[z],{z,0,1}]
```



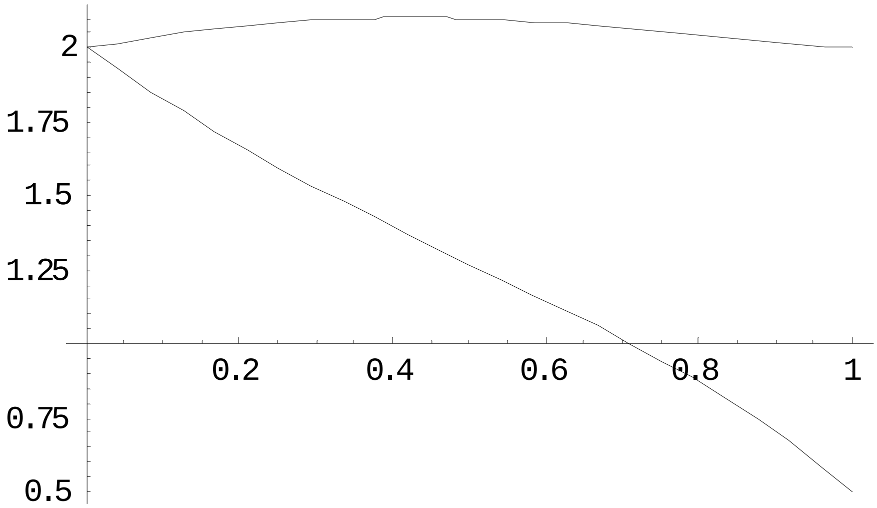

```
-Graphics-

Plot[V5[z],{z,0,1}]
```




```
-Graphics-
```

```
Plot[{U5[z],V5[z]},{z,0,1}]
```



-Graphics-

```
ρ=0.8
```

0.8

U6[z_] = U_0[z] + U_1[z] + U_2[z]

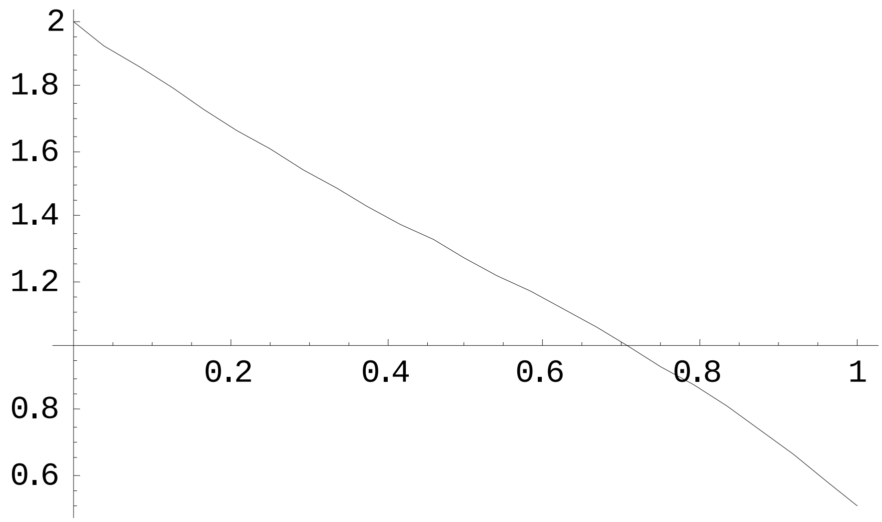
$$2-2.15104\,z-1.875\,(-2+z)\,z+0.416667\,z\,(0.5\,(-3+z^2)-2\,(3-3\,z+z^2))+0.00868056\,z\,(66.6-66.\,z^2+27.\,z^3+9.\,z^4-3.6\,z^5-45.\,(2+(-2+z)\,z^2))-0.130208\,z^2\,(6.\,(-3+2\,z)+2\,(2\,(-2+z)^2-0.5\,(-2+z^2)))$$

V6[z_] = V_0[z] + V_1[z] + V_2[z]

$$2-0.15\,z+(-2+z)\,z+\frac{1}{6}\,z^2\,(-6+2\,(-2+z)^2+4\,z-0.5\,(-2+z^2))-\frac{1}{3}\,z\,(0.5\,(-3+z^2)-2\,(3-3\,z+z^2))-0.00694444\,z\,(66.6-66.\,z^2+27.\,z^3+9.\,z^4-3.6\,z^5-45.\,(2+(-2+z)\,z^2))+\frac{1}{180}\,z\,(-37.5+15\,(2+(-2+z)\,z^2)+z^2\,(2\,(20+3\,(-5+z)\,z)+0.5\,(10-3\,z^2)))$$

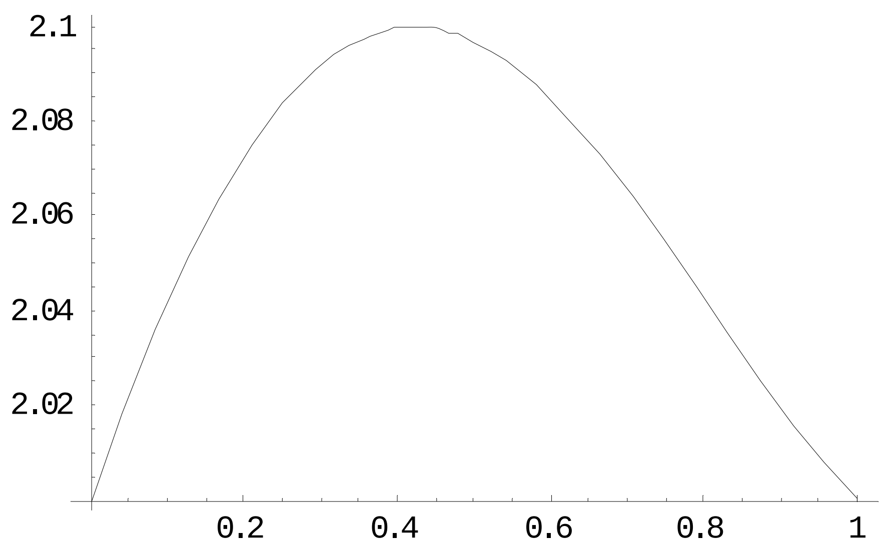
```
Table[U6[z],{z,0,1,0.1}]
{2,1.83338,1.67915,1.53605,1.40167,1.27246,1.14374,1.00971,
0.863408,0.696657,0.5}
Table[V6[z],{z,0,1,0.1}]
{2,2.04186,2.07262,2.09179,2.09952,2.09661,2.08455,2.0655,
2.0424,2.01901,2.}
```

```
Plot[U6[z],{z,0,1}]
```



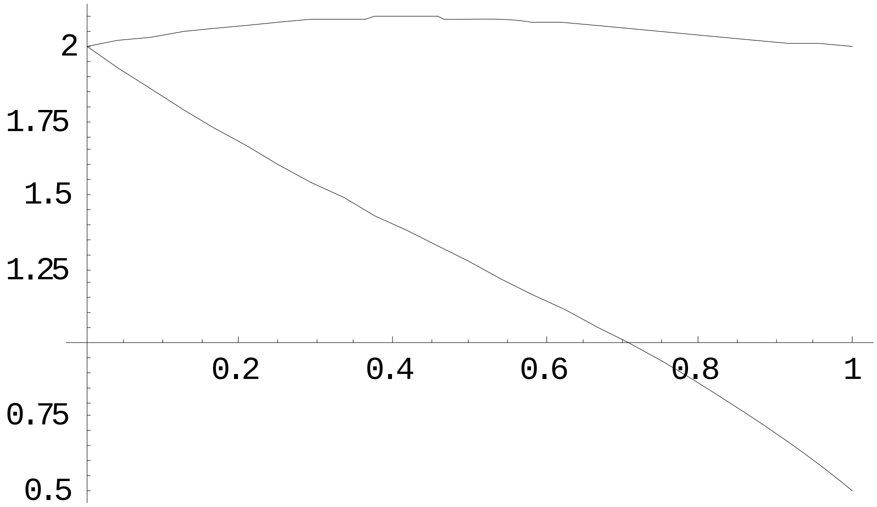
-Graphics-

```
Plot[V6[z],{z,0,1}]
```



-Graphics-

```
Plot[{U6[z],V6[z]},{z,0,1}]
```



-Graphics-

```
ρ=0.9
0.9
```

U7[z_] = U_0[z] + U_1[z] + U_2[z]

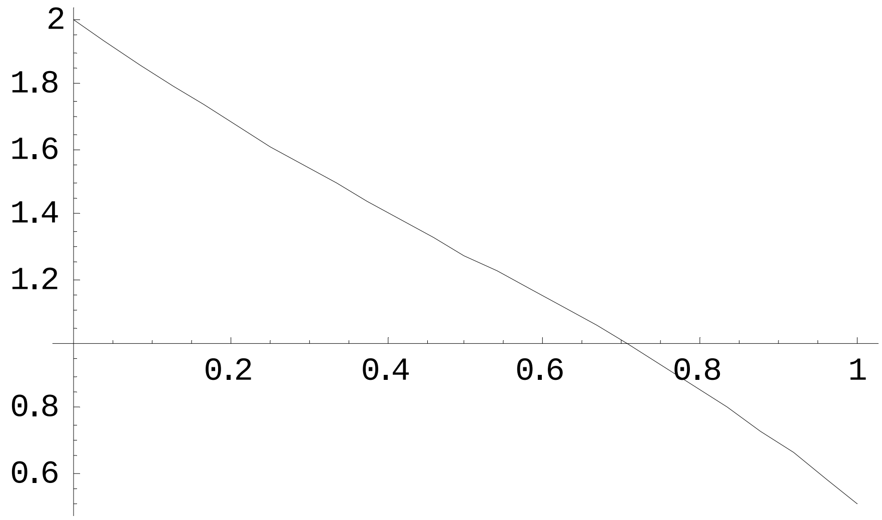
$$2-2.07356\,z-1.66667\,(-2+z)\,z+0.37037\,z\,(0.5\,(-3+z^2)-2\,(3-3\,z+z^2))+0.00685871\,z\,(65.55-63.\,z^2+22.875\,z^3+11.25\,z^4-4.05\,z^5-45.\,(2+(-2+z)\,z^2))-0.102881\,z^2\,(6.\,(-3+2\,z)+2\,(2\,(-2+z)^2-0.5\,(-2+z^2)))$$

V7[z_] = V_0[z] + V_1[z] + V_2[z]

$$2-0.143056\,z+(-2+z)\,z+\frac{1}{6}\,z^2\,(-6+2\,(-2+z)^2+4\,z-0.5\,(-2+z^2))-\frac{1}{3}\,z\,(0.5\,(-3+z^2)-2\,(3-3\,z+z^2))-0.00617284\,z\,(65.55-63.\,z^2+22.875\,z^3+11.25\,z^4-4.05\,z^5-45.\,(2+(-2+z)\,z^2))+\frac{1}{180}\,z\,(-37.5+15\,(2+(-2+z)\,z^2)+z^2\,(2\,(20+3\,(-5+z)\,z)+0.5\,(10-3\,z^2)))$$

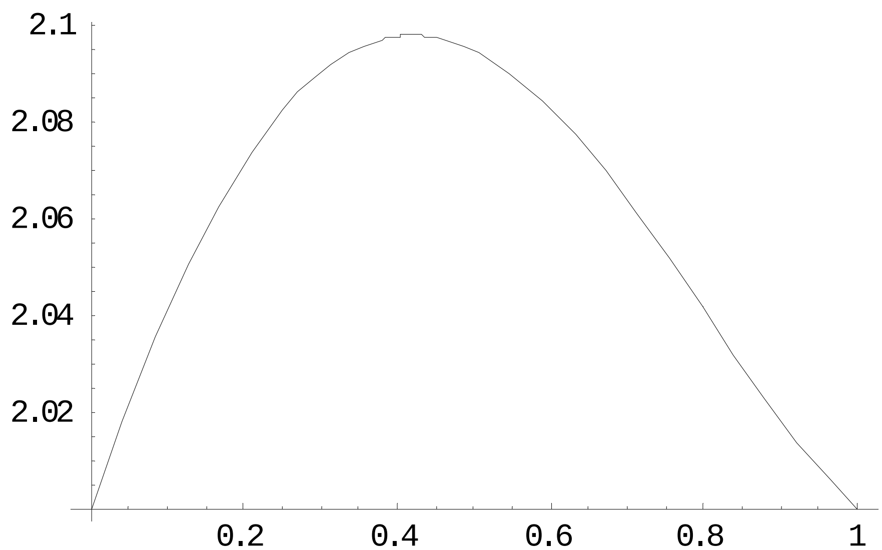
```
Table[U7[z],{z,0,1,0.1}]
{2,1.83698,1.6847,1.54184,1.40613,1.27439,1.14251,1.00551,
0.857492,0.691608,0.5}
Table[V7[z],{z,0,1,0.1}]
{2,2.0414,2.07171,2.09048,2.09789,2.09481,2.08273,2.06387,
2.04116,2.01833,2.}
```

Plot[U7[z],{z,0,1}]



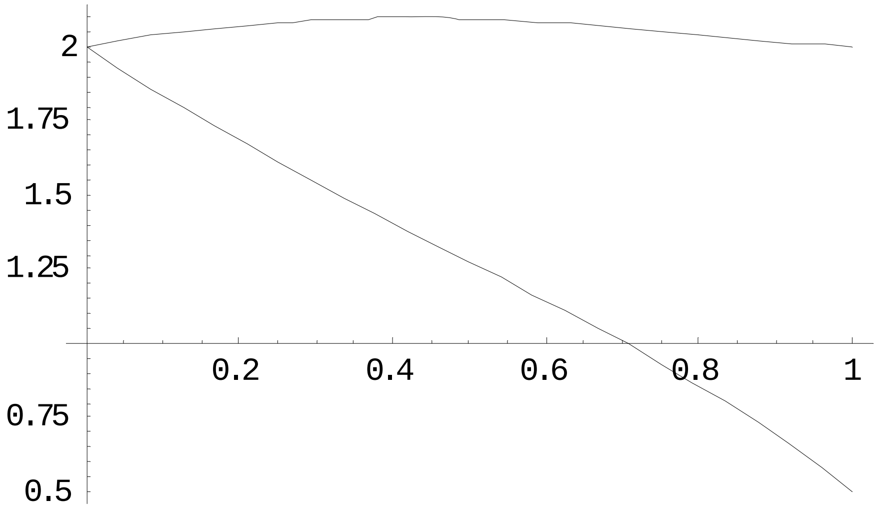
-Graphics-

Plot[V7[z],{z,0,1}]



-Graphics-

```
Plot[{U7[z],V7[z]},{z,0,1}]
```



-Graphics-

ρ=1

1

U8[z_] = U_0[z] + U_1[z] + U_2[z]

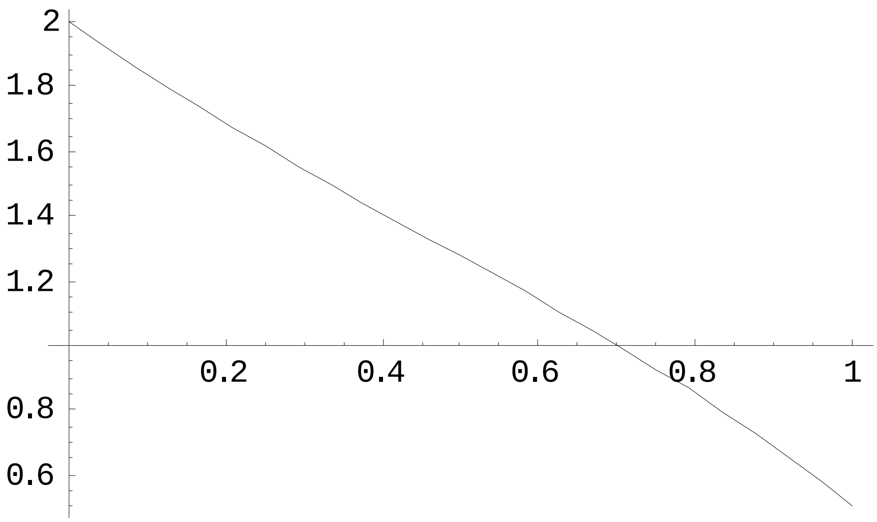
$$2-2.0125\,z-1.5\,(-2+z)\,z+\frac{1}{3}\,z\,(0.5\,(-3+z^2)-2\,(3-3\,z+z^2))+\frac{1}{180}\,z\,(64.5-60.\,z^2+18.75\,z^3+13.5\,z^4-4.5\,z^5-45.\,(2+(-2+z)\,z^2))-\frac{1}{12}\,z^2\,(6.\,(-3+2\,z)+2\,(2\,(-2+z)^2-0.5\,(-2+z^2)))$$

V8[z_] = V_0[z] + V_1[z] + V_2[z]

$$2-0.1375\,z+(-2+z)\,z+\frac{1}{6}\,z^2\,(-6+2\,(-2+z)^2+4\,z-0.5\,(-2+z^2))-\frac{1}{3}\,z\,(0.5\,(-3+z^2)-2\,(3-3\,z+z^2))-\frac{1}{180}\,z\,(64.5-60.\,z^2+18.75\,z^3+13.5\,z^4-4.5\,z^5-45.\,(2+(-2+z)\,z^2))+\frac{1}{180}\,z\,(-37.5+15\,(2+(-2+z)\,z^2)+z^2\,(2\,(20+3\,(-5+z)\,z)+0.5\,(10-3\,z^2)))$$

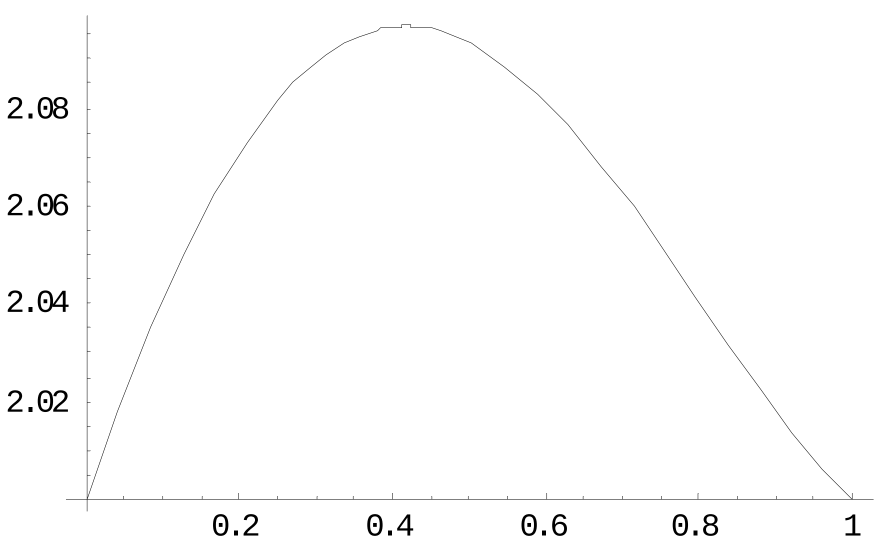
```
Table[U8[z],{z,0,1,0.1}]
{2,1.83954,1.68856,1.54571,1.40887,1.27513,1.14087,1.00171,
0.852556,0.687544,0.5}
```

```
Table[V8[z],{z,0,1,0.1}]
{2,2.04103,2.07099,2.08943,2.09659,2.09336,2.08128,2.06257,
2.04017,2.01779,2.}
Plot[U8[z],{z,0,1}]
```



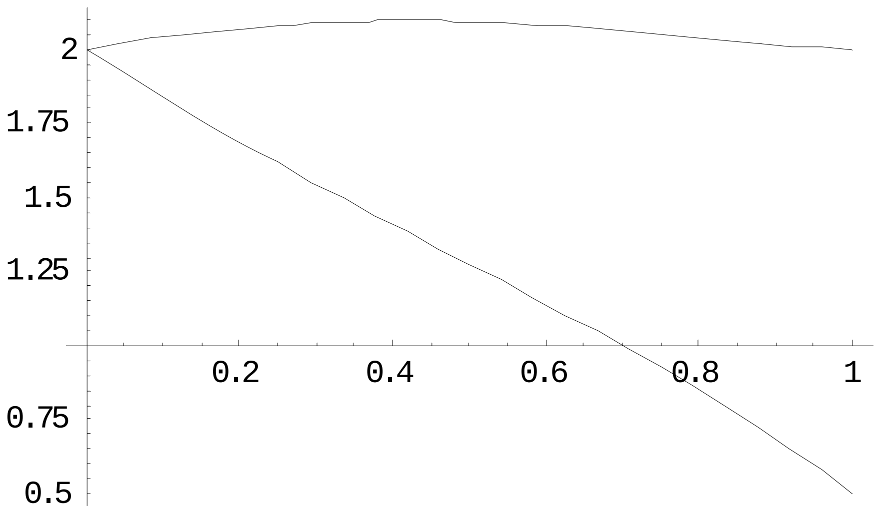
-Graphics-

```
Plot[V8[z],{z,0,1}]
```



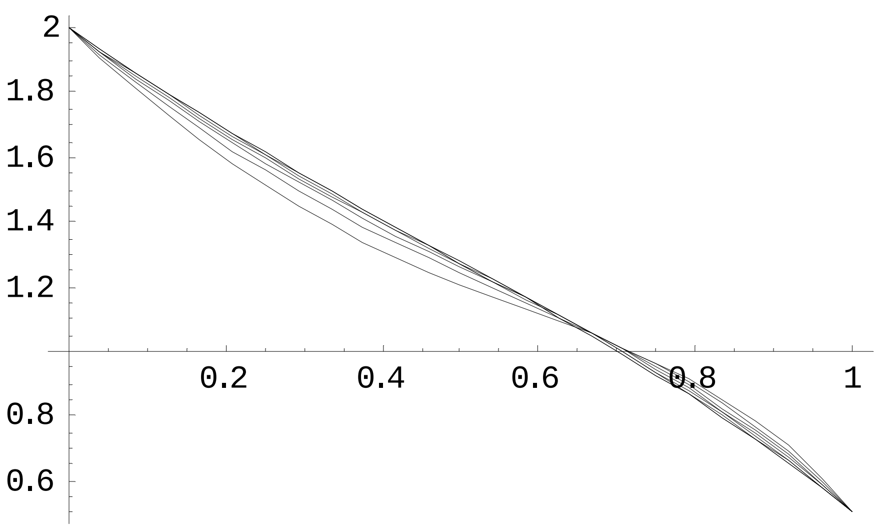
-Graphics-

```
Plot[{U8[z],V8[z]},{z,0,1}]
```



-Graphics-

```
Plot[{U1[z],U2[z],U3[z],U4[z],U5[z],U6[z],U7[z],U8[z]},{z,0
,1}]
```



-Graphics-