Linear Invariants of a Cartesian Tensor under *SO(4)*

By

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بسم الله الرحمن الرحيم

"If we value independence, if we are disturbed by the growing conformity of knowledge, of values, of attitudes, which our present system induces, then we may wish to set up conditions of learning which make for uniqueness, for self-direction,and for self-initiated learning".

(Carl Rogers)

Dedicated This Humble Task

To My

Loving Parents & Family.

Without their knowledge, wisdom, and guidance, I would not have the goals I have to strive and be the best to reach my dreams!

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Abstract

Tensors are of great importance in Mathematics, Physics and Engineering. They provide a natural and concise mathematical framework for formulating and solving problems in areas such as elasticity, fluid mechanics, and general relativity. In this dissertation we discuss a general theory of finding independent linear invariants of a Cartesian tensor of arbitrary rank under SO(4) . A linear form is defined in terms of elements of a tensor. Group theoretic methods produce formulas which precisely determine the number of linear invariants for an arbitrary tensor of rank r. Explicit results are obtained for simple cases.

Contents

Chapter 1

Introduction

1.1 Background

This dissertation entitled "Linear Invariants of a cartesian tensor under $SO(4)$" is a blend of mathematical flavors. As its title indicates, the dissertation revolves around three interconnected and particularly fertile themes, each arising in a wide variety of mathematical disciplines.

The most basic of the themes to be discussed is the concept of tensors. It is very important to note that physical laws are independent of any particular coordinate system. Consequently, equations describing physical laws, when referred to a particular coordinate system, must transform in a definite manner under transformation of coordinate systems. This leads to the concept of a tensor, that is, a quantity that does not depend on the choice of coordinate system.

Tensors were first conceived by Bernhard Riemann and Elwin Bruno Christoffel. Later they were developed by Tullio Levi-Civita and Gregorio Ricci-Curbastro, in order to formulate the intrinsic differential geometry of a manifold in the form of the Riemann curvature tensor. Because they express a relationship between vectors, tensors themselves are independent of a particular choice of coordinate system.

Tensors in the general sense relate to arbitrary coordinate transformations, as required in general relativity, whereas Cartesian tensors relate to rotations between orthogonal axes, as in elasticity and other branches of classical physics. This makes tensor analysis an important tool in theoretical physics, continuum mechanics and many other areas of science and engineering.

In many areas of physics, there are certain systems which are homogeneous on a macroscopic scale. In dealing with the behaviour of such systems, it is necessary to make use of isotropic tensors. Isotropic tensors are tensors whose components referred to any Cartesian frame are invariant under rotation of the frame axes. Such tensors play an important role in the theory of many physical processes which take place in gases and liquids.

The concept of an invariant is useful in many areas of mathematics and physics e.g. elasticity, relativity etc. By definition, an invariant is a quantity which is unaffected by the change of variables. An invariant of a tensor is a scalar associated with that tensor. It does not vary under co-ordinate changes.

In geometrical and physical applications, group theory is closely associated with symmetry tranformations. In classical physics, the interest lies in the effect of symmetry transformations on the solutions to partial differential or integral equations of "mathematical physics". These solutions usually form a linear vector space. With the advent of quantum mechanics, this connection becomes more explicit, as linear vector space is adopted as the formal mathemaical framework for the underlying theory. The interest in group theory, therefore, centers on the realization of group transformations as linear transformations on vector spaces of classical and quantum physics.

The symmetry groups that arise most often in the applications to geometry and differential equations are Lie groups of transformations acting on a finite-dimensional manifold. Often, an r-dimensional Lie group is referred to as an r parameter group, the "group parameters" referring to a choice of local coordinates on the group manifold. The simplest example of an r parameter Lie group is the abelian Lie group $\mathbb{R}^r$. The group operation is given by vector addition. The identity element is the zero vector, and the inverse of a vector x is the vector $-x$.

In elasticity, the elasticity tensor or the stiffness tensor c_{ijkl} plays an important role. By definition, a medium is said to be elastic if it returns to its initial state after the external forces are removed. The generalized Hooke's law is defined as

$$\sigma_{ij} = c_{ijkl}\epsilon_{kl},$$

where σ_{ij} is the stress tensor, ϵ_{kl} is the strain tensor and c_{ijkl} is the fourth rank elasticity tensor which is the coefficient of linearity.

It is well known [1,2] that, under arbitrary orthogonal rotations of the coordinate axes, the stiffness tensor c_{ijkl} possesses only two linear invariants, namely

$$\begin{aligned} A_1 &= c_{iijj} = c_{11} + c_{22} + c_{33} + 2\left(c_{12} + c_{23} + c_{13}\right), \\ A_2 &= c_{ijij} = c_{11} + c_{22} + c_{33} + 2\left(c_{44} + c_{55} + c_{66}\right). \end{aligned}$$

Here, the familiar Voigt's two-index notation has been used. Index pairs ij used for detailed notation are contracted to a single index I with summation range $I = 1....6$. This is done according to the following convention

$$11 \to 1,\ 22 \to 2,\ 33 \to 3,\ 23 \to 4,\ 13 \to 5,\ 12 \to 6.$$

Then the component c_{11} represents c_{1111}, c_{12} represents c_{1122} and c_{55} denotes c_{1313} etc.

In 1987, Ting [3] considered quadratic invariants of the stiffness tensor. He reported two invariants of second order with respect to an arbitrary orthogonal transformation and fifteen invariants when the transformation is confined to rotation about a fixed axis. In 2002, F. Ahmad [4] extended the work done by Ting [3]. He re-examined the invariants of the elasticity tensor. His analysis enlarged these numbers to four and seventeen, respectively. Further he demonstrated that the seven quadratic invariants are independent but still it was an open question whether the list of invariants is complete or not. In 2007, A. N. Norris [5] showed that the seven invariants identified by Ahmad form a complete basis. He also proved that the corresponding set under $SO(2)$ consists of 35 quadratic invariants. In 2009, F. Ahmad and M. A. Rashid [6] considered the problem of finding the number of independent linear invariants of an arbitray tensor. This problem was related to the work of Ting [3], Ahmad [4] and Norris [5]. They applied methods of group theory to derive a general formula for finding the number of independent linear invariants for a tensor of arbitrary rank r in three dimensions. In addition to this, they developed a method for finding linear invariants explicitly. Recently Ahmad and Rashid [7] have described a general theory for finding independent invariants of an arbitrary tensor. Group theoretic methods have been applied to find the formulas which exactly determine the number of independent linear invariants for an arbitrary tensor.

1.2 Objective of the dissertation

In this dissertation, we consider the problem of finding the number of linearly independent invariants of a Cartesian tensor under $SO(4)$. The main tool to be used in obtaining the number of independent linear invariants is the density function. In case of $SO(4)$ we do not have a direct expression for the density function. But it is well known that

$$SO(4) \cong SO(3) \otimes SO(3).$$

Due to this isomorphism, we can use the density function of $SO(3)$ in order to calculate the number of independent linear invariants of an arbitrary tensor of rank r. In this dissertation, we have calculated explicitly the independent linear invariants of tensors of rank r, with $2 \leq r \leq 8$. Also, by applying the methods of group theory, we have derived a general formula for finding the number of independent linear invariants of an arbitrary tensor of rank r.

1.3 Scheme of work

The dissertation has been organized in the following manner:

In Chapter 1 we have provided a brief introduction of the three interconnected themes that have been used throughout the sequel and some previous work done in this field. This Chapter presents the manner in which the Chapters are being organized in the dissertation and also includes a brief overview of the contents of each chapter.

Chapter 2 deals with the preliminary notions along with examples, which provide us necessary background for the later work. In this Chapter, we also give the terminology which will be used throughout this dissertation.

In Chapter 3, we have provided a review of the work done by F. Ahmad and M. A. Rashid [6,7] for finding the number of independent linear invariants of a Cartesian tensor of an arbitrary rank r, under $SO(2)$ and $SO(3)$. Explicit expressions obtained for simpler cases and formulas for finding the number of independent invariants in the two cases are also part of this Chapter. This dissertation is an extension of their work to the group $SO(4)$.

In Chapter 4, we have described a general theory for finding the independent linear invariants of a tensor of an arbitrary rank r under $SO(4)$. Linearly independent invariants of a tensor of rank r with $2 \leq r \leq 8$, have been determined explicitly. We have derived a general formula for finding the number of independent linear invariants of a tensor of arbitrary rank r. Also, we have verified that the number of invariants calculated by finding the dimension of the space of isotropic tensors agrees with the number produced by the general formula. The direct approach has the advantage that it not only gives the number of independent invariants but, at the same time, it produces the corresponding invariants as well. However, for large r, it is difficult to find the invariants, or merely their number, by using this approach. The formula becomes handy in such a situation.

Chapter 2

Mathematical preliminaries

In this Chapter, we recall some fundamental concepts and relevant results used throughout this dissertation. We also provide examples to make the definitions easily understood. Necessary notations and the terminology used in the sequel are also introduced. The definitions presented in the Chapter have been divided into three sections. We begin our exposition with a brief review of the basic concepts related to group theory. Afterwards, definitions relating to tensor analysis have been introduced. Famous Wallis cosine formula has been included in the end of this chapter.

2.1 Basic concepts in group theory

2.1.1 Group representation

In the mathematical field of representation theory, group representations describe abstract groups in terms of linear transformations of vector spaces; in particular, they can be used to represent group elements as matrices so that the group operation can be represented by matrix multiplication. Representations of groups are important because they allow many group-theoretic problems to be reduced to problems in linear algebra, which is well-understood. They are also important in physics because, for example, they describe how the symmetry group of a physical system affects the solutions of equations describing the system.

A representation of a group is a (continuous) mapping that sends each element of the group into a continuous linear operator that acts on some vector space, and which preserves the group operation. If we map an arbitrary group G homomorphically on a group of operators $D(G)$ acting in the vector space V, we say that the operator group $D(G)$ is a representation of the group G in the representation space V. If the dimensionality of V is n, we say that the representation is of degree n (or is an n-dimensional representation). The operator corresponding to the element R of G is denoted by $D(R)$. If R, S and E are elements of the group G, then

$$D(RS) = D(R)D(S), \tag{1}$$

$$D(R^{-1}) = [D(R)]^{-1}, \tag{2}$$

$$D(E) = I. \tag{3}$$

As an example, let us consider the general linear group $GL(n, \mathbb{R})$. The simplest possible representation is the trivial one-dimensional representation that assigns to each matrix $A \in GL(n, \mathbb{R})$ the real number 1. Slightly more interesting is the one-dimensional determinantal representation $D(A) = \det A$. Let us now consider the complex number $u = e^{2\pi i/3}$ which has the property $u^3 = 1$. The cyclic group $C_3 = \{1, u, u^2\}$ has a representation D on $\mathbb{C}^2$ given by

$$D(1) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \; D(u) = \begin{bmatrix} 1 & 0 \\ 0 & u \end{bmatrix}, \; D(u^2) = \begin{bmatrix} 1 & 0 \\ 0 & u^2 \end{bmatrix}.$$

If we choose a basis in the n-dimensional space V, the linear operators of the representation can be described by their matrix representatives. We then obtain a homomorphic mapping of the group G on a group of $n \times n$ matrices $D(G)$, i.e., a matrix representation of the group G. From equations (1), (2) and (3), we see that all the matrices are nonsingular, and that

$$D_{ij}(E) = \delta_{ij} = \begin{Bmatrix} 1 & \text{for } i = j \\ 0 & \text{for } i \neq j \end{Bmatrix}, \qquad i,\ j = 1, ..., \ n;$$
$$D_{ij}(RS) = (D(R)D(S))_{ij} = \sum_k D_{ik}(R)D_{kj}(S) = D_{ik}(R)D_{kj}(S),$$

where in the last expression, summation over the index k is implied.

Representations have applications to many branches of mathematics, physics and chemistry. The name of the theory (representation theory) depends on the group G and on the vector space V. Different approaches are required depending on whether G is a finite group, an infinite discrete group, or an infinite continuous (Lie) group. Another important ingredient is the field of scalars for V. The vector space V can be infinite dimensional such as a Hilbert space. Also, special kinds of representations may require that a vector space structure is preserved. For instance, a unitary representation is a group homomorphism $\phi : G \longrightarrow D(V)$ into the group of unitary transformations which preserve a Hermitian inner product on V.

2.1.2 Group character

If we change the basis in the n-dimensional space V, the matrices $D(R)$ are replaced by their transforms by some matrix T. The matrices

$$D'(R) = TD(R)T^{-1},$$

also provide a representation of the group G, which is equivalent to the representation $D(R)$. If we take the sum of the diagonal elements of the matrix, or trace of a matrix

$D(R)$, we obtain

$$\begin{aligned}
\sum_i \left(TD(R)T^{-1}\right)_{ii} &= \sum_{ijk} T_{ij} D_{jk}(R) T^{-1}_{ki} \\
&= \sum_{ijk} \left(T^{-1}_{ki}\right) T_{ij} D_{jk}(R) \\
&= \sum_{jk} \delta_{kj} D_{jk}(R) \\
&= \sum_{k} D_{kk}(R).
\end{aligned}$$

When we are dealing with group representations, this trace is called the *character* of R in the representation D and is denoted by $\chi(R)$. Thus

$$\chi(R) = \sum_i D_{ii}(R).$$

We see that equivalent representations have the same set of characters. In essence, group characters can be thought of as the matrix traces of a special set of matrices (a so-called irreducible representation) used to represent group elements and whose multiplication corresponds to the multiplication table of the group. Charaters are invariant on conjugacy classes. All members of the same conjugacy class in the same representation have the same character. Therefore we can say that character of a representation is a *class function*. Also the charaters of irreducible representations are orthogonal.

2.1.3 Scalar product

In order to bring the theory of representations into closer contact with physics, we define a metric in the n-dimensional space V. For this purpose we associate with each pair of vectors x, y in V a complex number (x, y). The complex number (x, y) is called the scalar product of x and y and is required to satisfy the following conditions:

$$\begin{aligned}
&(x, y) = (y, x)^*\,, \\
&(x, \alpha y) = \alpha\,(x, y)\,, \\
&(x_1 + x_2, y) = (x_1, y) + (x_2, y)\,, \\
&(x, x) \geq 0, \\
&\text{and} \quad (x, x) = 0 \text{ only if } x = 0.
\end{aligned}$$

2.1.4 Unitary representation

Before defining unitary representation we define a unitary operator and a unitary matrix.

The operator O is called *unitary operator* if

$$(Ox, Oy) = (x, y)\,, \qquad \text{for all } x, y$$

where (x, y) is the scalar product.

In an orthonormal system the matrix representative of O is represented by a *unitary matrix*

$$O^\dagger O = OO^\dagger = I\,.$$

Now if the operators of a representation of the group G are unitary operators or if the matrices of the representation are unitary matrices, then the representation is called a *unitary representation*.

2.1.5 Irreducible representation

Irreducible representations are the building blocks of all other representations. Mathematically, they are useful because one can often reduce an idea or a calculation involving representations to an easier one involving only irreducible representations. Physically, irreducible representations correspond to fundamental physical entities.

An irreducible representation of a group is a group representation that has no nontrivial invariant subspace. For example, the orthogonal group $O(n)$ has an irreducible representation on $\mathbb{R}^n$. Any representation of a finite or semisimple Lie group breaks up into a direct sum of irreducible representations. But in general, this is not the case.

2.1.6 Group orthogonality relation

Let the order of a group be h, and the dimension of the i^{th} representation (the order of each constituent matrix) be l_i (a positive integer). Let any operator be denoted by R, and let the m^{th} row and n^{th} column of the matrix corresponding to a matrix R in the i^{th} irreducible representation be $D_i\left(R\right)_{mn}$, then

$$\sum_R D_i\left(R\right)_{mn} D_j\left(R\right)^*_{m'n'} = \frac{h}{\sqrt{l_i l_j}}\delta_{ij}\delta_{mm'}\delta_{nn'}.$$

The orthogonality relation for the characters of the unitary representation is given by

$$\sum_R \chi^{(\mu)}\left(R\right)\chi^{(\nu)*}\left(R\right) = h\delta_{\mu\nu}.$$

2.1.7 Continuous groups

Consider a set of elements R which depend on a number of real continuous parameters, $R(a) = R(a_1, a_2, ..., a_r)$. These elements are said to form a continuous group if they fulfill the requirements of a group and there is some notion of 'nearness' or 'continuity' imposed on the elements of the group in the sense that a small change in one of the factors of a product produces a correspondingly small change in their product.

If the group elements depend on r parameters, then this is called a r-parameter continuous group. Our interest in physical applications centers around transformations on n-dimensional spaces. Examples include Minkowski spaces, where the variables are space-time coordinates. In this case, these are mappings of the space onto itself and have the general form

$$x_i' = \varphi_i\left(x_1, ..., x_n; a_1, ..., a_r\right), \qquad i = 1, ..., n.$$

If the functions φ_i are analytic, then this defines an r-parameter Lie group of transformaions.

2.1.8 Lie groups

The symmetry groups that arise most often in the applications to geometry and differential equations are Lie groups of transformations acting on a finite-dimensional manifold. The general mathematical theory of continuous groups is usually called the theory of Lie groups. Roughly speaking, a Lie group is an infinite group whose elements can be parametrized smoothly and analytically.

Lie group is a differentiable manifold obeying the group properties which satisfies the additional condition that the group operations are differentiable. We are mainly interested in groups of linear transformations. An *r-parameter Lie group* of *transformations* is a group of transformations

$$x_i' = \varphi_i\left(x_1, ..., x_n; a_1, ..., a_r\right), \qquad i = 1, ..., n.$$

or, symbolically,

$$x' = \varphi\left(x; a\right),$$

for which the functions φ_i are analytic functions of the parameter a. The r real parameters a_j are assumed to be essential.

Example:

Consider the one-dimensional transformations

$$x' = ax, \tag{1}$$

where a is a nonzero real number. This transformation corresponds to stretching the real line by a factor a. The product of two such operatios, $x'' = ax'$ and $x' = bx$ is

$$x'' = ax' = abx.$$

By writing $x'' = cx$, we have

$$c = ab, \tag{2}$$

so the multiplication of two transformations is described by an analytic function that yields another transformation of the form in (1). This operation is clearly associative, as well as abelian, since the product transformation corresponds to the multiplication of real numbers. By setting $c = 1$ in (2), so that $x'' = x$, the inverse of (1) is seen to correspond to the transformation with $a' = a^{-1}$, which explains the requirement that $a \neq 0$. Finally, the identity is determined from $x' = x$, which clearly corresponds to the transformation with $a = 1$. Hence, the transformations defined in (1) form a one parameter abelian Lie group.

In general, a Lie group may have a more complicated group structure, such as the orthogonal group $O(n)$ (i.e., the set of all $n \times n$ orthogonal matrices), or the general linear group $GL(n, \mathbb{R})$ (i.e., the set of all $n \times n$ invertible matrices). The Lorentz group is also a Lie group.

2.1.9 Orthogonal group $O(n)$

Many transformations in physical applications are required to preserve length in the appropriate space. If the space is ordinary Euclidean n-dimensional space, then the restriction that lengths be preserved means that

$$x_1'^2 + x_2'^2 + ... + x_n'^2 = x_1^2 + x_2^2 + ... + x_n^2,$$

i.e., we restrict the transformations of the general linear group to those which leave $\sum_{i=1}^{n} x_i^2$ invariant. Thus we impose $n + \frac{n(n-1)}{2}$ conditions on the n^2 parameters, which leaves us with $\frac{n(n-1)}{2}$ essential parameters. The corresponding group, which is subgroup of the general linear group $GL(n, F)$ is called orthogonal group. Thus, the orthogonal group of degree n over a field F (written as $O(n, F)$) is the set of all $n \times n$ orthogonal matrices with entries from F, with the group operation of matrix multiplication, given by

$$O(n, F) = \left\{Q \in GL(n, F) \mid Q^T Q = QQ^T = I\right\},$$

where Q^T is the transpose of Q. Elements of $O(n, F)$ are called orthogonal transformations. Every orthogonal matrix has determinant either 1 or -1. The orthogonal $n \times n$ matrices with determinant 1 form a normal subgroup of $O(n, \mathbb{R})$ known as the special orthogonal group $SO(n, \mathbb{R})$. Over the field $\mathbb{R}$ of real numbers, the orthogonal group $O(n, \mathbb{R})$ and the special orthogonal group $SO(n, \mathbb{R})$ are often simply denoted by $O(n)$ and $SO(n)$. Geometrically, elements of $O(n)$, are either rotations or combinations of rotations and reflections.

2.1.10 Special orthogonal group $SO(n)$

Let V be a n-dimensional real inner product space. The group of orthogonal operators on V with positive determinant (i.e., the group of "rotations" on V) is called the special orthogonal group, denoted $SO(n)$. The special orthogonal group $SO(n)$ is a subgroup of the orthogonal group $O(n)$. $SO(n)$ is isomorphic to the group of rotations of $\mathbb{R}^n$ that keep the origin fixed. Geometrically, elements of $SO(n)$, are pure rotations.

$SO(2)$ is isomorphic (as a Lie group) to the circle S^1 (circle group). This isomorphism sends the complex number $exp(\varphi i) = cos(\varphi) + isin(\varphi)$ to the orthogonal matrix

$$\begin{bmatrix} cos\varphi & -sin\varphi \\ sin\varphi & cos\varphi \end{bmatrix},$$

The group $SO(3)$, understood as the set of rotations of three-dimensional space, is of major importance in science and engineering.

2.1.11 Special orthogonal group in four dimensions $SO(4)$

The group $SO(4)$ is the four-dimensional rotation group; i.e, the group of rotations about a fixed point in a four-dimensional Euclidean space. This is the group whose elements are 4×4 matrices S such that $S^T S = I$, where I is the 4×4 unit matrix, with the condition $\det S = +1$. The Euclidean (length)^2 $x_1^2 + x_2^2 + x_3^2 + x_4^2$ is left invariant under $SO(4)$ transformations. The group $SO(4)$ is a noncommutative six-parameter Lie group. The six infinitesimal operators can be chosen as

$$\begin{aligned} A_1 &= x_3\frac{\partial}{\partial x_2} - x_2\frac{\partial}{\partial x_3},\ A_2 = x_1\frac{\partial}{\partial x_3} - x_3\frac{\partial}{\partial x_1},\ A_3 = x_2\frac{\partial}{\partial x_1} - x_1\frac{\partial}{\partial x_2}; \\ B_1 &= x_1\frac{\partial}{\partial x_4} - x_4\frac{\partial}{\partial x_1},\ B_2 = x_2\frac{\partial}{\partial x_4} - x_4\frac{\partial}{\partial x_2},\ B_3 = x_3\frac{\partial}{\partial x_4} - x_4\frac{\partial}{\partial x_3}. \end{aligned}$$

Evaluating the commutators, we obtain the relations

$$\begin{aligned} [A_i, A_j] &= \epsilon_{ijk}A_k, \\ [A_i, B_j] &= \epsilon_{ijk}B_k, \\ [B_i, B_j] &= \epsilon_{ijk}A_k, \end{aligned}$$

which describe the structure of the corresponding Lie algebra. This algebra may be simplified by introducing the linear combinations

$$M_i = \frac{A_i + B_i}{2}, \quad N_i = \frac{A_i - B_i}{2},$$

which satisfy

$$\begin{aligned} [M_i, M_j] &= \epsilon_{ijk} M_k, \\ [N_i, N_j] &= \epsilon_{ijk} N_k, \\ [M_i, N_j] &= 0. \end{aligned}$$

We see that, in this form, the six generators have seperated into two sets of three, each set obeying the algebra of $SO(3)$ (or of $SU(2)$) and commuting with the other set. Consequently $SO(4)$ is isomorphic to the direct product of $SO(3) \otimes SO(3)$.

2.1.12 Conjugacy classes

In mathematics, especially group theory, the elements of any group may be partitioned into conjugacy classes. Suppose G is a group. A conjugacy class in G is a nonempty subset C of G such that the following two conditions hold

1. Given any $x, y \in C$, there exists $g \in G$ such that $gxg^{-1} = y$.
2. If $x \in C$ and $g \in G$ then $gxg^{-1} \in C$.

In other words, it is closed under the action of the group on itself by conjugation, and the action is transitive when restricted to the conjugacy class. Any group is a disjoint union of conjugacy classes. The identity element forms a class by itself. In an abelian group every conjugacy class is a set containing one element (singleton set). If two elements a and b of G belong to the same conjugacy class (i.e., if they are conjugate), then they have the same order. For a finite group, the size of any conjugacy class divides the order of the group. In general, the size of C is not greater (as a cardinal) than the size of G.

Example:

Let us consider the symmetric group on three letters i.e.,

$$S_3 = \{e, (12), (13), (23), (123), (132)\}.$$

This is the smallest non-Abelian group. This group has three conjugacy classes, the class of the identity element (size 1), the class of the transpositions (size 3) and the class of the 3-cycles (size 2). We may write these distinct classes explicitly as

1. e;
2. $(12), (13), (23)$;
3. $(123), (132)$.

2.1.13 Invariant

An invariant is a quantity which remains unchanged under certain classes of transformations. Invariants are extremely useful for classifying mathematical objects because they usually reflect intrinsic properties of the object of study. The most fundamental example of invariance is expressed in our ability to count. For a finite collection of objects of any kind, there appears to be a number to which we invariably arrive regardless of how we count the objects in the set. The quantity–cardinal number–is associated with the set and is invariant under the process of counting.

Another simple example of an invariant is the distance between two points on a number line which is not changed by adding the same quantity to both numbers. On the other hand multiplication of the numbers by the same quantity does not have this property. Thus, distance is not invariant under multiplication.

2.1.14 Invariant integration

The symmetry transformations that are based on continuous quantities, occur in many physical applications. For example, the Hamiltonian of a system with spherical symmetry (e.g., atoms) is invariant under all three-dimensional rotations. To address the consequences of this invariance within the framework of group theory, we need to examine the theory of the representations for continuous groups. A fundamental tool in the study of continuous groups is invariant integration. In order to define characters and derive the orthogonality relations for continuous groups, we use invariant integration. For the generalization of the orthogonality relations from finite groups to infinite or continuous groups, it is necessary to replace the summation by an integration over the group.

In case of finite groups, in the derivation of orthogonality relation we need that if $f(R)$ is a function defined on the group manifold, then

$$\sum_R f(R) = \sum_R f(SR), \tag{1}$$

where S is any element of the group. Now if we want to extend the orthogonality relation to the continuous groups, then we need to establish some analog of the above equation. The important point in the derivation of equation (1) is that equal weights are attached to all elements R of the finite group. For Lie groups, we must have some replacement of the statement that the weight attached to an element A of the group is equal to the weight attached to the element BA which is obtained by left translation from A. We associate a volume measure $d\tau_A$ with a set of elements H in the neighborhood of A such that

$$d\tau_A = d\tau_{BA},$$

where $d\tau_{BA}$ is the volume measure of the set of elements BH which is obtained by making a left translation of the set H with the element B. In order to make the measure of the group elements H and BH the same, we choose a density function $\rho(a)$ such that

$$d\tau_H = \rho(a)\, d\tau_a = \rho(c)\, d\tau_c = d\tau_{BH}.$$

Here $c_k = \varphi_k(a; b)$, where the functions φ_k are analytic.

In order to find the density function for group integration, we arbitrarily fix the value of $\rho(0)$ in the neighborhood of identity. The set in the neighborhood of identity is carried by the left translation with B into a region of the parameter space in the neighborhood of b, so that

$$\begin{aligned} b_k &= \varphi_k(0; b), \\ db_k &= \sum_{l=1}^{r} \left[\frac{\partial \varphi_k(a; b)}{\partial a_l} \right]_{a=0} da_l. \end{aligned}$$

Thus the relation of the volume elements da and db in the parameter space is given by

$$db = \begin{vmatrix} \frac{\partial \varphi_1(a;b)}{\partial a_1}|_{a=0} & \cdots & \frac{\partial \varphi_r(a;b)}{\partial a_1}|_{a=0} \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \frac{\partial \varphi_1(a;b)}{\partial a_r}|_{a=0} & \cdots & \frac{\partial \varphi_r(a;b)}{\partial a_r}|_{a=0} \end{vmatrix} da = J(b) da.$$

If we let

$$\rho(b) = \frac{\rho(0)}{J(b)},$$

then we have

$$\rho(b)\, db = \rho(0)\, da.$$

Thus the density function $\rho(b)$ is determined for all b by making a left translation with the element B. For integrating a function $f(a)$ over the group, we must form $\int da f(a)$. To make the theory of invariant integration clearer, let us consider an example.

Example:

Let us consider a linear group $GL(2)$ of linear transformation in two dimensions, given by:

$$\begin{aligned} x' &= a_1 x + a_2 y \ , \\ y' &= a_3 x + a_4 y \ , \end{aligned} \qquad \begin{vmatrix} a_1 & a_2 \\ a_3 & a_4 \end{vmatrix} \neq 0.$$

This group is isomorphic to the group of 2×2 invertible matrices, with matrix multiplication as the law of combination. Here

$$\begin{aligned} \text{Identity element:} \quad & A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = 1, \\ \text{Inverse element:} \quad & A \ = A^{-1}, \\ \text{Product element:} \quad & C = BA. \end{aligned}$$

Now

$$\begin{aligned} x'' &= b_1 x' + b_2 y' \\ &= b_1 (a_1 x + a_2 y) + b_2 (a_3 x + a_4 y) \\ &= (b_1 a_1 + b_2 a_3) x + (b_1 a_2 + b_2 a_4) y \\ &= c_1 x + c_2 y, \\ \text{where} \quad c_1 &= \varphi_1 (a; b) = b_1 a_1 + b_2 a_3, \\ \text{and} \quad c_2 &= \varphi_2 (a; b) = b_1 a_2 + b_2 a_4. \end{aligned}$$

Similarly

$$\begin{aligned}
y'' &= b_3 x' + b_4 y' \\
&= b_3 (a_1 x + a_2 y) + b_4 (a_3 x + a_4 y) \\
&= (b_3 a_1 + b_4 a_3) x + (b_3 a_2 + b_4 a_4) y \\
&= c_3 x + c_4 y, \\
\text{where} \quad c_3 &= \varphi_3 (a; b) = b_3 a_1 + b_4 a_3, \\
\text{and} \quad c_4 &= \varphi_4 (a; b) = b_3 a_2 + b_4 a_4.
\end{aligned}$$

Now

$$\frac{\partial \varphi_1 (a; b)}{\partial a_1}|_{a=0} = \frac{\partial (b_1 a_1 + b_2 a_3)}{\partial a_1}|_{a=0} = b_1$$

Similarly,

$$\frac{\partial \varphi_1 (a; b)}{\partial a_2}|_{a=0} = 0, \ \frac{\partial \varphi_1 (a; b)}{\partial a_3}|_{a=0} = b_2 \text{ and } \frac{\partial \varphi_1 (a; b)}{\partial a_4}|_{a=0} = 0$$

Note that in the above, by $a = 0$, we mean that

$$a_1 = 1, a_2 = 0, a_3 = 0, a_4 = 1.$$

On similar lines as above, we find that

$$\begin{aligned}
\frac{\partial \varphi_2 (a; b)}{\partial a_1}|_{a=0} &= \frac{\partial (b_1 a_2 + b_2 a_4)}{\partial a_1}|_{a=0} = 0, \\
\frac{\partial \varphi_2 (a; b)}{\partial a_2}|_{a=0} &= b_1, \ \frac{\partial \varphi_2 (a; b)}{\partial a_3}|_{a=0} = 0 \text{ and } \frac{\partial \varphi_2 (a; b)}{\partial a_4}|_{a=0} = b_2. \\
\frac{\partial \varphi_3 (a; b)}{\partial a_1}|_{a=0} &= \frac{\partial (b_3 a_1 + b_4 a_3)}{\partial a_1}|_{a=0} = b_3, \\
\frac{\partial \varphi_3 (a; b)}{\partial a_2}|_{a=0} &= 0, \ \frac{\partial \varphi_3 (a; b)}{\partial a_3}|_{a=0} = b_4 \text{ and } \frac{\partial \varphi_3 (a; b)}{\partial a_4}|_{a=0} = 0. \\
\frac{\partial \varphi_4 (a; b)}{\partial a_1}|_{a=0} &= \frac{\partial (b_3 a_2 + b_4 a_4)}{\partial a_1}|_{a=0} = 0, \\
\frac{\partial \varphi_4 (a; b)}{\partial a_2}|_{a=0} &= b_3, \ \frac{\partial \varphi_4 (a; b)}{\partial a_3}|_{a=0} = 0 \text{ and } \frac{\partial \varphi_4 (a; b)}{\partial a_4}|_{a=0} = b_4.
\end{aligned}$$

The Jacobian of the transformation is

$$J(b) = \begin{vmatrix} b_1 & 0 & b_3 & 0 \\ 0 & b_1 & 0 & b_3 \\ b_2 & 0 & b_4 & 0 \\ 0 & b_2 & 0 & b_4 \end{vmatrix} = (b_1 b_4 - b_2 b_3)^2 = [\det(B)]^2 ,$$

where

$$B = \begin{bmatrix} b_1 & b_2 \\ b_3 & b_4 \end{bmatrix} ,$$

so that the density function $\rho(b)$ is

$$\rho(b) = \frac{1}{(b_1 b_4 - b_2 b_3)^2} = \frac{1}{[\det(B)]^2}.$$

To integrate over the group we form $\int da_1 da_2 da_3 da_4 \frac{1}{(b_1 b_4 - b_2 b_3)^2} f(a_1, a_2, a_3, a_4)$.

2.2 Basic concepts in tensor analysis

2.2.1 Tensor

A tensor is a physical quantity which is independent of co-ordinate system changes. The simplest tensor is a scalar, a zeroth-rank tensor. A scalar is represented by a single component that is invariant under coordinate transformation. A first rank tensor is a vector. The representation of a second order tensor in a co-ordinate system is a square matrix. The representation matrices of a second rank tensor are square matrices similar to each other.

An n^{th}-rank tensor in m-dimensional space is a mathematical object that has n indices and m^n components and obeys certain transformation rules. Each index of a tensor ranges over the dimension of the space. However, the dimension of the space is largely irrelevant in most tensor equations (with the notable exception of the contracted Kronecker delta). Tensors are basically generalizations of scalars (which have no indices), vectors (which have exactly one index), and matrices (which have exactly two indices) to an arbitrary number of indices.

The tensor transformation rules are linear in the components and are homogeneous (there is no additive constant term). A tensor itself is (as for a vector) an invariant entity, independent of the coordinate system used: it is the components that transform when the coordinates are changed.

2.2.2 Invariant of a tensor

An invariant of a tensor is a scalar associated with that tensor. It does not vary under co-ordinate changes. For example, the magnitude of a vector is an invariant of that vector. For second order tensors, there is a well-developed theory of invariants. The coefficients of the characteristic polynomial of a second order tensor are invariants of that tensor.

2.2.3 Cartesian tensor

A Cartesian tensor is a tensor in three-dimensional Euclidean space. Cartesian tensors are widely used in various branches of continuum mechanics, such as fluid mechanics and elasticity. In classical continuum mechanics, the space of interest is usually three-dimensional Euclidean space. Cartesian tensors behave as tensors under orthogonal linear transformations, representing rotations and translations of axes. Unlike general tensors, there is no distinction between covariant and contravariant indices for Cartesian tensors. However, tensors in non-Euclidean spaces (e.g., Lorentzian spaces) do require this distinction.

A Cartesian tensor T of rank r is an array of components denoted by $T_{ijk...m}$ with r indices $ijk...m$. In three-dimensional space T has 3^r components. The defining property of a Cartesian tensor is the following law:

From coordinate system S to S' by a rotation, the components of a tensor transform according to

$$T'_{ijk...m} = a_{is}a_{jt}a_{ku}....a_{mv}T_{stu...v},$$

where $a_{ij} = cos$(angle between the x_i-axis and the transformed x_j-axis). As special cases, a scalar is a zero-th rank tensor $T' = T$. A vector is a first rank tensor which is transformed according to $T'_i = a_{ij}T_j$. A second rank tensor is transformed according to $T'_{ij} = a_{is}a_{jt}T_{st}$.

2.2.4 Isotropic tensor

Something is isotropic at a particular point if it looks the same in all directions when one stands at that point. On the largest scale, the universe is thought to be isotropic at every point. An isotropic tensor is a tensor which has the same components in all rotated coordinate systems. All rank-0 tensors (scalars) are isotropic, but no rank-1 tensor (vectors) is isotropic. The unique rank-2 isotropic tensor is the Kronecker delta, and the unique rank-3 isotropic tensor is the permutation tensor (Goldstein 1980, p. 172).

2.2.5 Permutation symbol

The Levi-Civita symbol (Weinberg 1972, p. 38; Arfken 1985, p. 132), also called the permutation symbol (Goldstein 1980, p. 172), antisymmetric symbol, or alternating symbol, is a three-index mathematical symbol used in particular in tensor calculus. It is named after the Italian mathematician and physicist Tullio Levi-Civita.

In three dimensions, the Levi-Civita symbol is defined as follows

$$\epsilon_{ijk} = \begin{cases} +1 & \text{for } (i,j,k) \in \{(1,2,3)\,,(2,3,1)\,,(3,1,2)\}\,, \\ -1 & \text{for } (i,j,k) \in \{(1,3,2)\,,(3,2,1)\,,(2,1,3)\}\,, \\ 0 & \text{for } i=j \text{ or } j=k \text{ or } k=i, \end{cases}$$

i.e., ϵ_{ijk} is 1 if (i, j, k) is an even permutation of (1,2,3), -1 if it is an odd permutation, and 0 if any index is repeated.

The Levi-Civita symbol can be generalized to higher dimensions

$$\epsilon_{ijkl.....} = \begin{cases} +1 & \text{if } (i,j,k,l,.....) \text{ is an even permutation of } (1,2,3,4,....)\,, \\ -1 & \text{if } (i,j,k,l,.....) \text{ is an odd permutation of } (1,2,3,4,....)\,, \\ 0 & \text{if any two labels are the same.} \end{cases}$$

Thus, it is the sign of the permutation in the case of a permutation, and zero otherwise.

The permutation symbol satisfies the following identities:

$$\begin{aligned} \delta_{ij}\,\epsilon_{ijk} &= 0, \\ \epsilon_{ijk}\,\epsilon_{pqk} &= \delta_{ip}\,\delta_{jq} - \delta_{iq}\,\delta_{jp}, \\ \epsilon_{ipq}\,\epsilon_{jpq} &= 2\delta_{ij}, \\ \epsilon_{ijk}\,\epsilon_{ijk} &= 6, \end{aligned}$$

where δ_{ij} is the Kronecker delta (Arfken 1985, p. 136).

2.2.6 Permutation tensor

The tensor whose components in an orthonormal basis are given by the Levi-Civita symbol (a tensor of covariant rank n) is sometimes called the permutation tensor. The permutation tensor, also called the Levi-Civita tensor or isotropic tensor of rank

3 (Goldstein 1980, p. 172), is actually a pseudotensor because under an orthogonal transformation of Jacobian determinant -1 (i.e., a rotation composed with a reflection), it acquires a minus sign. Because the Levi-Civita symbol is a pseudotensor, the result of taking a cross product of two vectors is a pseudovector, not a vector.

The permutation tensor of rank four is important in general relativity, and has components defined as

$$\epsilon^{\alpha\beta\gamma\delta} = \begin{cases} +1 & \text{if } \alpha\beta\gamma\delta \text{ is an even permutation of } 0123, \\ -1 & \text{if } \alpha\beta\gamma\delta \text{ is an odd permutation of } 0123, \\ 0 & \text{otherwise.} \end{cases}$$

(Weinberg 1972, p. 38).

2.3 Wallis cosine formula

Wallis cosine formula is given by

$$\begin{aligned} \int_0^{\pi/2} \cos^n x dx &= \frac{\sqrt{\pi}\Gamma\left(\frac{1}{2}(n+1)\right)}{n\Gamma\left(\frac{1}{2}n\right)} \\ &= \frac{(n-1)!!}{n!!}\begin{cases} \pi/2 & \text{for } n = 2, 4, ... \\ 1 & \text{for } n = 3, 5, ... \end{cases} \end{aligned}$$

where $\Gamma(n)$ is a gamma function and $n!!$ is a double factorial.

The double factorial of a positive integer is a generalization of the usual factorial and is defined by

$$n!! \equiv \begin{cases} n.(n-2).....5.3.1 & n > 0 \ \text{odd}, \\ n.(n-2).....6.4.2 & n > 0 \ \text{even}, \\ 1 & n = -1, 0 \end{cases}$$

Note that $(-1)!! = 0!! = 1$, by definition (Arfken 1985, p. 547).

The double factorial can be expressed in terms of the gamma function by

$$\begin{aligned} (2n-1)!! &= \frac{1}{\sqrt{\pi}} 2^n \Gamma\left(n + \frac{1}{2}\right), \\ \text{and} \qquad (2n)!! &= 2^n . n! = 2^n \Gamma\left(n+1\right). \end{aligned}$$

(Arfken 1985, p. 548).

Chapter 3

Linear invariants of a Cartesian tensor under $SO(2)$ and $SO(3)$

Consider group of special orthogonal coordinate transformations. Let us first examine the proper rotations in two dimensions. This group is called the special orthogonal group in two dimensions and is denoted by $SO(2)$. The parametrization of this group that we use in this dissertation is given by

$$R(\varphi) = \begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix},$$

where φ, the single parameter in this Lie group, is the rotation angle of the transformation. The matrix representing an orthogonal coordinate transformation, in three dimensions, through an angle φ about z-axis passing through the origin can be written as

$$R(\varphi) = \begin{pmatrix} \cos\varphi & -\sin\varphi & 0 \\ \sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

Now consider a Cartesian tensor, $T_{i_1 i_2}$, of rank two. In two dimensions, the indices take values 1 and 2 only and the tensor has $2^2 = 4$ components but, in case of three dimensions the indices take values 1, 2 and 3, and the number of components increases to $3^2 = 9$. We denote dimension by letter d and rank by the letter r. Let us now consider invariants associated with a tensor $T_{i_1 i_2}$, of rank 2. When $d = 2$, i.e., in two dimensions, it is obvious that $T_{11} + T_{22}$ is an invariant of the tensor $T_{i_1 i_2}$. However it is not so obvious that $T_{12} - T_{21}$ is also an invariant and that this set of invariants is complete in the sense that any other linear invariant must be a linear combination of the two. But we will see later that $T_{12} - T_{21}$ is also a linear invariant of the tensor $T_{i_1 i_2}$ and that this set of invariants is complete. On the other hand, if $d = 3$ and $r = 2$, $T_{i_1 i_2}$ possesses only one invariant, $T_{11} + T_{22} + T_{33}$.

Here we have just discussed the case of a second rank tensor. In the papers written by F. Ahmad and M. A. Rashid [6,7], results pertaining to a tensor of arbitrary rank in two and three dimensions have been presented. These results have been applied to tensors of different ranks and linearly independent invariants have been found explicitly. By using the methods of group theory, formulas for finding the number of linear invariants of a tensor of an arbitrary rank have also been derived.

In the beginning of this Chapter we have discussed the independent linear invariants of an arbitrary tensor of rank two for $d = 2$ and $d = 3$. Now we provide the independent linear invariants of a tensor of arbitrary rank under $SO(3)$, and some invariants under $SO(2)$.

3.1 Linear invariants under $SO(2)$

The linear invariants of a tensor of rank r, with $2 \leq r \leq 4$. have been mentioned below.

1. $d = 2$, $r = 2$

In this case any isotropic tensor must be a linear combination of the tensors $\delta_{i_1i_2}$ and $\epsilon_{i_1i_2}$. Thus there are two linear invariants of the tensor of rank 2, $\delta_{i_1i_2}T_{i_1i_2}$ and $\epsilon_{i_1i_2}T_{i_1i_2}$ for $1 \leq i_1, i_2 \leq 2$.

2. $d = 2$, $r = 3$

In this case no isotropic tensor can be constructed in terms of products of $\delta_{i_1i_2}$. Also in two dimensions $\epsilon_{i_1i_2i_3} = 0$. Thus for a tensor of rank 3, no linear invariant exists. It is obvious that this result is true for any tensor of odd rank.

3. $d = 2$, $r = 4$

Here the possible members of the basis for the space of isotropic tensors are

$$\delta_{i_1i_2}\delta_{i_3i_4}, \delta_{i_1i_3}\delta_{i_2i_4}, \delta_{i_1i_4}\delta_{i_2i_3}, \delta_{i_1i_2}\epsilon_{i_3i_4}, \delta_{i_1i_3}\epsilon_{i_2i_4}, \delta_{i_1i_4}\epsilon_{i_2i_3},$$
$$\delta_{i_2i_3}\epsilon_{i_1i_4}, \delta_{i_2i_4}\epsilon_{i_1i_3}, \delta_{i_3i_4}\epsilon_{i_1i_2}, \epsilon_{i_1i_2}\epsilon_{i_3i_4}, \epsilon_{i_1i_3}\epsilon_{i_2i_4}, \epsilon_{i_1i_4}\epsilon_{i_2i_3}.$$

However calculations show that out of these, only six are linearly independent and remaining six depend upon these independent ones. A ser of independent linear isotropic tensors is

$$\delta_{i_1i_2}\delta_{i_3i_4}, \delta_{i_1i_3}\delta_{i_2i_4}, \delta_{i_1i_4}\delta_{i_2i_3}, \delta_{i_2i_3}\epsilon_{i_1i_4}, \delta_{i_2i_4}\epsilon_{i_1i_3}, \delta_{i_3i_4}\epsilon_{i_1i_2},$$

which results in the linear invariants T_{iijj}, T_{ijij}, T_{ijji}, $\epsilon_{i_1i_2}T_{i_1i_2ii}$, $\epsilon_{i_1i_3}T_{i_1ii_3i}$, $\epsilon_{i_1i_4}T_{i_1iii_4}$, *where, as in the sequel, summation over repeated indices is implied.*

3.2 Linear invariants under $SO(3)$

Theorem 1 (given below) has been applied to find the linear invariants of a tensor of rank r, with $2 \leq r \leq 5$.

1. $d = 3, \quad r = 2$

Here, $\delta_{i_1 i_2}$ is the only isotropic tensor. Thus T_{ii} is the only linear invariant of the tensor $T_{i_1 i_2}$.

2. $d = 3, \quad r = 3$

In three dimensions, for a tensor of rank 3, the situation is different from the one in two dimensions. Here $\epsilon_{i_1 i_2 i_3} \neq 0$, rather this is the only member of the basis for the space of isotropic tensors. Thus there is only one linear invariant namely, $\epsilon_{i_1 i_2 i_3} T_{i_1 i_2 i_3}$.

3. $d = 3, \quad r = 4$

For $d = 3$ and $r = 4$, the possible candidates for the membership of basis for the space of isotropic tensors are

$$\delta_{i_1 i_2} \delta_{i_3 i_4}, \delta_{i_1 i_3} \delta_{i_2 i_4}, \delta_{i_1 i_4} \delta_{i_2 i_3}.$$

For different choices of i_1, i_2 and i_3 we see that all of the above are linearly independent. Thus for a tensor of rank 4 there are three independent linear invariants namely T_{iijj}, T_{ijij} and T_{ijji}.

4. $d = 3, \quad r = 5$

Here we have

$$\frac{5!}{2!.3!} = 10 \text{ isotropic tensors of type } \delta_{i_1 i_2} \epsilon_{i_3 i_4 i_5}.$$

Thus the possible elements of the basis in this case are

$$\delta_{i_1 i_2} \epsilon_{i_3 i_4 i_5}, \delta_{i_1 i_3} \epsilon_{i_2 i_4 i_5}, \delta_{i_1 i_4} \epsilon_{i_2 i_3 i_5}, \delta_{i_1 i_5} \epsilon_{i_2 i_3 i_4}, \delta_{i_2 i_3} \epsilon_{i_1 i_4 i_5},$$
$$\delta_{i_2 i_4} \epsilon_{i_1 i_3 i_5}, \delta_{i_2 i_5} \epsilon_{i_1 i_3 i_4}, \delta_{i_3 i_4} \epsilon_{i_1 i_2 i_5}, \delta_{i_3 i_5} \epsilon_{i_1 i_2 i_4}, \delta_{i_4 i_5} \epsilon_{i_1 i_2 i_3},$$

for $1 \leq i_1, i_2, i_3, i_4, i_5 \leq 3$.

Calculations reveal that among these 10 isotropic tensors only six are linearly independent. For example, a basis for the independent linear isotropic tensors in this case consists of

$$\delta_{i_2 i_3}\epsilon_{i_1 i_4 i_5}, \delta_{i_2 i_4}\epsilon_{i_1 i_3 i_5}, \delta_{i_2 i_5}\epsilon_{i_1 i_3 i_4}, \delta_{i_3 i_4}\epsilon_{i_1 i_2 i_5}, \delta_{i_3 i_5}\epsilon_{i_1 i_2 i_4}, \delta_{i_4 i_5}\epsilon_{i_1 i_2 i_3},$$

which leads to the independent linear invariants as $\epsilon_{i_1 i_4 i_5} T_{i_1 i i i_4 i_5}$, $\epsilon_{i_1 i_3 i_5} T_{i_1 i i_3 i i_5}$ etc.

Obviously this is not the only linearly independent set. There can be many different choices. Another linearly independent set for the isotropic tensors is

$$\delta_{i_1 i_2}\epsilon_{i_3 i_4 i_5}, \delta_{i_1 i_3}\epsilon_{i_2 i_4 i_5}, \delta_{i_1 i_4}\epsilon_{i_2 i_3 i_5}, \delta_{i_2 i_3}\epsilon_{i_1 i_4 i_5}, \delta_{i_2 i_4}\epsilon_{i_1 i_3 i_5}, \delta_{i_3 i_4}\epsilon_{i_1 i_2 i_5}.$$

3.3 Main results

Now we present without proofs some main results used in these two papers [6,7] and the formulas which have been derived for finding the independent linear invariants of a tensor of an arbitrary rank in two and three dimensions respectively.

Let $T = T_{i_1, \ldots\ldots, i_n}$ be a Cartesian tensor of rank r in three dimensions. The main results of the reviewd work are embodied in the following theorems.

Theorem 1:

The number of independent linear invariants of T under $SO(3)$ is the same as the dimension of the space of the space of the isotropic tensors of rank r possessing the same symmetries, if any, as T.

Define $v_1$and v_2, respectively, as the number of 1's and 2's among the subscripts $_{i_1, \ldots\ldots\ldots, i_n}$. For example, in T_{122312}, v_1= 2 and v_2= 3. Now the result concerning linear invariants under $SO(2)$ can be stated.

Theorem 2:

The number of independent linear invariants of T under the group of rotations about a fixed axis, say x_3-axis, is the same as the number of components $T_{i_1,\ldots\ldots,i_n}$ with v_1=v_2.

Theorem 1 in [6] (stated above) was given in the context of arbitrary r and $d = 3$. However the proof holds for arbitrary d and in [7] it has been restated as

Theorem 3:

The number of independent linear invariants of a tensor of rank r equals the dimension of the space of the isotropic tensors of the same rank.

Theorem 4:

The number of invariants under $SO(2)$ is the same as the number of ways S_n, in which n elements consisting of 1s, 2s and 3s can be arranged with the stipulation that the number of 1s and 2s are equal. It is easy to see that

$$S_n = \sum_{r=0}^{k} \frac{(2k)!}{(2r)!\,(k-r)!\,(k-r)!}, \qquad \text{if} \quad n = 2k,$$

$$\text{and} \qquad S_n = \sum_{r=0}^{k} \frac{(2k+1)!}{(2r+1)!\,(k-r)!\,(k-r)!}, \qquad \text{if} \quad n = 2k+1.$$

Theorem 5:

The number of invariants $S_3(r)$ in three dimensions is found to be

$$S_3\,(r) = \sum_{i=0}^{k} \frac{(2k)!}{(2r)!\,(k-i)!\,(k-i)!}, \qquad \text{if} \quad r = 2k,$$

$$\text{and} \qquad S_3\,(r) = \sum_{i=0}^{k} \frac{(2k+1)!}{(2i+1)!\,(k-i)!\,(k-i)!}, \qquad \text{if} \quad r = 2k+1.$$

Obviously the set of invariants in two dimensions is a subset of the above invariants consisting of those members which are independent of the third dimension.

Theorem 6:

The number $I_2(r)$ of independent linear invariants of a tensor of rank r in two dimensions is zero if r is an odd integer and if r is an even integer then this number is given by

$$I_2(r) = \binom{r}{\frac{r}{2}}.$$

Theorem 7:

The number $I_3(r)$ of independent linear invariants for a tensor of arbitrary rank r in three dimensions is given by

$$I_3(r) = \sum_{i=0}^{[\frac{r+1}{2}]} \frac{r!(r+1-3i)}{i!i!(r+1-2i)!}.$$

Chapter 4

Linear invariants of a Cartesian tensor under $SO(4)$

In this Chapter, we find independent linear invariants of a tensor of rank r, with $2 \leq r \leq 8$ under $SO(4)$. But for higher ranks, it is not possible to find invariants by using the direct approach. So, by applying methods of group theory we find a general formula for finding the number of independent linear invariants.

General results for three dimensions were presented by F. Ahmad and M. A. Rashid [6,7]. Theorem 1 in [6] was given in context of arbitrary r and $d = 3$. However the proof holds for arbitrary d and it has been restated by Ahmad and Rashid [7] as follows:

Theorem 1: *The number of independent linear invariants of a tensor of rank r equals the dimension of the space of the isotropic tensors of the same rank.*

4.1 Case 1: $d = 4$, $r = 2$:

For $d = 4$ and $r = 2$, we see that $\delta_{i_1 i_2}$ is the only isotropic tensor. Thus T_{ii} is the only linear invariant of the tensor $T_{i_1 i_2}$, that is we have 1^2=1 linearly independent invariants of a tensor of rank 2.

4.2 Case 2: $d = 4$, $r = 3$:

No isotropic tensor of rank 3 can be constructed in terms of products of $\delta_{i_1 i_2}$ and $\epsilon_{i_1 i_2 i_3 i_4}$, since both $\delta_{i_1 i_2}$ and $\epsilon_{i_1 i_2 i_3 i_4}$ are even rank tensors. It is obvious that this result must hold for any tensor of odd rank. Thus we conclude that for $d = 4$, we only find the linear invariants of tensors of even rank because number of linear invariants is zero for any tensor of odd rank.

4.3 Case 3: $d = 4$, $r = 4$:

Now let $d = 4$ and $r = 4$. Here number of possible candidates for the membership of a basis for the space of isotropic tensors of rank 4 is four. These candidates are the following

$$\delta_{i_1 i_2}\delta_{i_3 i_4}\ ,\ \delta_{i_1 i_3}\delta_{i_2 i_4},\ \delta_{i_1 i_4}\delta_{i_2 i_3},\ \epsilon_{i_1 i_2 i_3 i_4},$$

where $1 \leq i_1, i_2, i_3, i_4 \leq 4$.

Let

$$a_1\delta_{i_1 i_2}\delta_{i_3 i_4} + a_2\delta_{i_1 i_3}\delta_{i_2 i_4} + a_3\delta_{i_1 i_4}\delta_{i_2 i_3} + a_4\epsilon_{i_1 i_2 i_3 i_4} = 0. \qquad \text{(A)}$$

where a_1, a_2, a_3, a_4 are scalars, not all zero.

Now we determine that out of the above four possible candidates, how many are linearly independent. For this purpose, we assign different values to i_1, i_2, i_3 and i_4 in equation (A) and proceed as follow:

Let $i_1 = i_2 = 1, i_3 = i_4 = 2$, then

$a_1\delta_{11}\,\delta_{22} + a_2\delta_{12}\,\delta_{12} + a_3\delta_{12}\,\delta_{12} + a_4\epsilon_{1122} = 0,$

which implies that $a_1 = 0$.

Let $i_1 = i_3 = 1,\ i_2 = i_4 = 2$, then

$a_1\delta_{12}\,\delta_{12} + a_2\delta_{11}\,\delta_{22} + a_3\delta_{12}\,\delta_{12} + a_4\epsilon_{1212} = 0,$

which implies that $a_2 = 0$.

Similarly for $i_1 = i_4 = 1,\ i_2 = i_3 = 2$

$a_1\delta_{12}\,\delta_{12} + a_2\delta_{12}\,\delta_{12} + a_3\delta_{11}\,\delta_{22} + a_4\epsilon_{1221} = 0,$

we arrive at $a_3 = 0$.

Finally taking $i_1 = 1, i_2 = 2,\ i_3 = 3, i_4 = 4$

$a_1\delta_{12}\,\delta_{34} + a_2\delta_{13}\,\delta_{24} + a_3\delta_{14}\,\delta_{23} + a_4\epsilon_{1234} = 0,$

we arrive at $a_4 = 0$.

As all the scalars come out to be zero, so all the four isotropic tensors are linearly independent and form basis for the space of isotropic tensors of rank 4. Thus for $d = 4$ and $r = 4$, there are $2^2 = 4$ linearly independent isotropic tensors namely

$$\delta_{i_1 i_2}\delta_{i_3 i_4}\ , \delta_{i_1 i_3}\delta_{i_2 i_4}, \delta_{i_1 i_4}\delta_{i_2 i_3}, \epsilon_{i_1 i_2 i_3 i_4},$$

which results in the independent linear invariants

$$T_{iijj},\ T_{ijij},\ T_{ijji},\ \epsilon_{i_1 i_2 i_3 i_4} T_{i_1 i_2 i_3 i_4}.$$

4.4 Case 4: $d = 4$, $r = 6$:

Now when $d = 4$ and $r = 6$, we have

$$\frac{6!}{2!.2!.2!}.\frac{1}{3!} = 15 \text{ isotropic tensors of type } \delta_{i_1 i_2}\delta_{i_3 i_4}\delta_{i_5 i_6},$$

and

$$\frac{6!}{2!.4!} = 15 \text{ isotropic tensors of type } \delta_{i_1 i_2}\epsilon_{i_3 i_4 i_5 i_6}.$$

Thus the number of possible candidates for the membership of a basis for the space of isotropic tensors of rank 6 is 30. These possible candidates are:

$$\delta_{i_1 i_2}\delta_{i_3 i_4}\delta_{i_5 i_6}, \delta_{i_1 i_2}\delta_{i_3 i_5}\delta_{i_4 i_6}, \delta_{i_1 i_2}\delta_{i_3 i_6}\delta_{i_4 i_5}, \delta_{i_1 i_3}\delta_{i_2 i_4}\delta_{i_5 i_6}, \delta_{i_1 i_3}\delta_{i_2 i_5}\delta_{i_4 i_6},$$
$$\delta_{i_1 i_3}\delta_{i_2 i_6}\delta_{i_4 i_5}, \delta_{i_1 i_4}\delta_{i_2 i_3}\delta_{i_5 i_6}, \delta_{i_1 i_4}\delta_{i_2 i_5}\delta_{i_3 i_6}, \delta_{i_1 i_4}\delta_{i_2 i_6}\delta_{i_3 i_5}, \delta_{i_1 i_5}\delta_{i_2 i_3}\delta_{i_4 i_6},$$
$$\delta_{i_1 i_5}\delta_{i_2 i_4}\delta_{i_3 i_6}, \delta_{i_1 i_5}\delta_{i_2 i_6}\delta_{i_3 i_4}, \delta_{i_1 i_6}\delta_{i_2 i_3}\delta_{i_4 i_5}, \delta_{i_1 i_6}\delta_{i_2 i_4}\delta_{i_3 i_5}, \delta_{i_1 i_6}\delta_{i_2 i_5}\delta_{i_3 i_4},$$
$$\delta_{i_1 i_2}\epsilon_{i_3 i_4 i_5 i_6}, \delta_{i_1 i_3}\epsilon_{i_2 i_4 i_5 i_6}, \delta_{i_1 i_4}\epsilon_{i_2 i_3 i_5 i_6}, \delta_{i_1 i_5}\epsilon_{i_2 i_3 i_4 i_6}, \delta_{i_1 i_6}\epsilon_{i_2 i_3 i_4 i_5},$$
$$\delta_{i_2 i_3}\epsilon_{i_1 i_4 i_5 i_6}, \delta_{i_2 i_4}\epsilon_{i_1 i_3 i_5 i_6}, \delta_{i_2 i_5}\epsilon_{i_1 i_3 i_4 i_6}, \delta_{i_2 i_6}\epsilon_{i_1 i_3 i_4 i_5}, \delta_{i_3 i_4}\epsilon_{i_1 i_2 i_5 i_6},$$
$$\delta_{i_3 i_5}\epsilon_{i_1 i_2 i_4 i_6}, \delta_{i_3 i_6}\epsilon_{i_1 i_2 i_4 i_5}, \delta_{i_4 i_5}\epsilon_{i_1 i_2 i_3 i_6}, \delta_{i_4 i_6}\epsilon_{i_1 i_2 i_3 i_5}, \delta_{i_5 i_6}\epsilon_{i_1 i_2 i_3 i_4},$$

where $1 \leq i_1, i_2, i_3, i_4, i_5, i_6 \leq 4$.

Now in order to find the independent linear invariants, we let

$$a_1\delta_{i_1 i_2}\delta_{i_3 i_4}\delta_{i_5 i_6} + a_2\delta_{i_1 i_2}\delta_{i_3 i_5}\delta_{i_4 i_6} + a_3\delta_{i_1 i_2}\delta_{i_3 i_6}\delta_{i_4 i_5} + a_4\delta_{i_1 i_3}\delta_{i_2 i_4}\delta_{i_5 i_6} +$$
$$a_5\delta_{i_1 i_3}\delta_{i_2 i_5}\delta_{i_4 i_6} + a_6\delta_{i_1 i_3}\delta_{i_2 i_6}\delta_{i_4 i_5} + a_7\delta_{i_1 i_4}\delta_{i_2 i_3}\delta_{i_5 i_6} + a_8\delta_{i_1 i_4}\delta_{i_2 i_5}\delta_{i_3 i_6} +$$
$$a_9\delta_{i_1 i_4}\delta_{i_2 i_6}\delta_{i_3 i_5} + a_{10}\delta_{i_1 i_5}\delta_{i_2 i_3}\delta_{i_4 i_6} + a_{11}\delta_{i_1 i_5}\delta_{i_2 i_4}\delta_{i_3 i_6} + a_{12}\delta_{i_1 i_5}\delta_{i_2 i_6}\delta_{i_3 i_4} +$$
$$a_{13}\delta_{i_1 i_6}\delta_{i_2 i_3}\delta_{i_4 i_5} + a_{14}\delta_{i_1 i_6}\delta_{i_2 i_4}\delta_{i_3 i_5} + a_{15}\delta_{i_1 i_6}\delta_{i_2 i_5}\delta_{i_3 i_4} + a_{16}\delta_{i_1 i_2}\epsilon_{i_3 i_4 i_5 i_6} +$$
$$a_{17}\delta_{i_1 i_3}\epsilon_{i_2 i_4 i_5 i_6} + a_{18}\delta_{i_1 i_4}\epsilon_{i_2 i_3 i_5 i_6} + a_{19}\delta_{i_1 i_5}\epsilon_{i_2 i_3 i_4 i_6} + a_{20}\delta_{i_1 i_6}\epsilon_{i_2 i_3 i_4 i_5} +$$
$$a_{21}\delta_{i_2 i_3}\epsilon_{i_1 i_4 i_5 i_6} + a_{22}\delta_{i_2 i_4}\epsilon_{i_1 i_3 i_5 i_6} + a_{23}\delta_{i_2 i_5}\epsilon_{i_1 i_3 i_4 i_6} + a_{24}\delta_{i_2 i_6}\epsilon_{i_1 i_3 i_4 i_5} +$$
$$a_{25}\delta_{i_3 i_4}\epsilon_{i_1 i_2 i_5 i_6} + a_{26}\delta_{i_3 i_5}\epsilon_{i_1 i_2 i_4 i_6} + a_{27}\delta_{i_3 i_6}\epsilon_{i_1 i_2 i_4 i_5} + a_{28}\delta_{i_4 i_5}\epsilon_{i_1 i_2 i_3 i_6} +$$
$$a_{29}\delta_{i_4 i_6}\epsilon_{i_1 i_2 i_3 i_5} + a_{30}\delta_{i_5 i_6}\epsilon_{i_1 i_2 i_3 i_4} = 0, \quad \text{(A)}$$

where a_n , $1 \leq n \leq 30$, not all zero, are the members of the field F.

In order to find the linearly independent invariants in this case we assign different values to i_1, i_2, i_3, i_4, i_5 and $i_6$in equation (A) and proceed as follow,

Let $i_1 = i_2 = 1,\ \ i_3 = i_4 = 2,\ \ i_5 = i_6 = 3$, then

$$a_1\delta_{11}\delta_{22}\delta_{33} + a_2\delta_{11}\delta_{23}\delta_{23} + a_3\delta_{11}\delta_{23}\delta_{23,} + a_4\delta_{12}\delta_{12}\delta_{33} + a_5\delta_{12}\delta_{13}\delta_{23} +$$
$$a_6\delta_{12}\delta_{13}\delta_{23} + a_7\delta_{12}\delta_{12}\delta_{33} + a_8\delta_{12}\delta_{13}\delta_{23} + a_9\delta_{12}\delta_{13}\delta_{23} + a_{10}\delta_{13}\delta_{12}\delta_{23} +$$
$$a_{11}\delta_{13}\delta_{12}\delta_{23} + a_{12}\delta_{13}\delta_{13}\delta_{22} + a_{13}\delta_{13}\delta_{12}\delta_{23} + a_{14}\delta_{13}\delta_{12}\delta_{23} + a_{15}\delta_{13}\delta_{13}\delta_{22} +$$
$$a_{16}\delta_{11}\epsilon_{2233} + a_{17}\delta_{12}\epsilon_{1233} + a_{18}\delta_{12}\epsilon_{1233} + a_{19}\delta_{13}\epsilon_{1223} + a_{20}\delta_{13}\epsilon_{1223} + a_{21}\delta_{12}\epsilon_{1233} +$$
$$a_{22}\delta_{12}\epsilon_{1233} + a_{23}\delta_{13}\epsilon_{1223} + a_{24}\delta_{13}\epsilon_{1223} + a_{25}\delta_{22}\epsilon_{1133} + a_{26}\delta_{23}\epsilon_{1123} + a_{27}\delta_{23}\epsilon_{1123} +$$
$$a_{28}\delta_{23}\epsilon_{1123} + a_{29}\delta_{23}\epsilon_{1123} + a_{30}\delta_{33}\epsilon_{1122} = 0,$$

which implies that $a_1 = 0$.

Now let $i_1 = i_2 = 1,\ \ i_3 = i_5 = 2,\ \ i_4 = i_6 = 3$, then

$$a_1\delta_{11}\delta_{23}\delta_{23} + a_2\delta_{11}\delta_{22}\delta_{33} + a_3\delta_{11}\delta_{23}\delta_{23} + a_4\delta_{12}\delta_{13}\delta_{23} + a_5\delta_{12}\delta_{12}\delta_{33} +$$
$$a_6\delta_{12}\delta_{13}\delta_{23} + a_7\delta_{13}\delta_{12}\delta_{23} + a_8\delta_{13}\delta_{12}\delta_{23} + a_9\delta_{13}\delta_{13}\delta_{22} + a_{10}\delta_{12}\delta_{12}\delta_{33} +$$
$$a_{11}\delta_{12}\delta_{23}\delta_{23} + a_{12}\delta_{12}\delta_{13}\delta_{23} + a_{13}\delta_{13}\delta_{12}\delta_{23} + a_{14}\delta_{13}\delta_{13}\delta_{22} + a_{15}\delta_{13}\delta_{12}\delta_{23} +$$
$$a_{16}\delta_{11}\epsilon_{2323} + a_{17}\delta_{12}\epsilon_{1323} + a_{18}\delta_{13}\epsilon_{1223} + a_{19}\delta_{12}\epsilon_{1233} + a_{20}\delta_{13}\epsilon_{1232} + a_{21}\delta_{12}\epsilon_{1323} +$$
$$a_{22}\delta_{13}\epsilon_{1223} + a_{23}\delta_{12}\epsilon_{1233} + a_{24}\delta_{13}\epsilon_{1232} + a_{25}\delta_{23}\epsilon_{1123} + a_{26}\delta_{22}\epsilon_{1133} + a_{27}\delta_{23}\epsilon_{1132} +$$
$$a_{28}\delta_{23}\epsilon_{1123} + a_{29}\delta_{33}\epsilon_{1122} + a_{30}\delta_{23}\epsilon_{1123} = 0,$$

which implies that $a_2 = 0$

Proceeding in the same way as above we find that

$$a_3 = a_4 = ... = a_{15} = 0.$$

Hence

$$a_1 = a_2 = a_3 = ... = a_{15} = 0.$$

Thus we have found that out of 30, first 15 isotropic tensors given above are linearly independent. These 15 members are part of the basis for the isotropic tensor of rank 6.

As $a_1 = a_2 = a_3 = ... = a_{15} = 0$, so equation (A) reduces to

$$a_{16}\delta_{i_1 i_2}\epsilon_{i_3 i_4 i_5 i_6} + a_{17}\delta_{i_1 i_3}\epsilon_{i_2 i_4 i_5 i_6} + a_{18}\delta_{i_1 i_4}\epsilon_{i_2 i_3 i_5 i_6} + a_{19}\delta_{i_1 i_5}\epsilon_{i_2 i_3 i_4 i_6} + a_{20}\delta_{i_1 i_6}\epsilon_{i_2 i_3 i_4 i_5} + a_{21}\delta_{i_2 i_3}\epsilon_{i_1 i_4 i_5 i_6} + a_{22}\delta_{i_2 i_4}\epsilon_{i_1 i_3 i_5 i_6} + a_{23}\delta_{i_2 i_5}\epsilon_{i_1 i_3 i_4 i_6} + a_{24}\delta_{i_2 i_6}\epsilon_{i_1 i_3 i_4 i_5} + a_{25}\delta_{i_3 i_4}\epsilon_{i_1 i_2 i_5 i_6} + a_{26}\delta_{i_3 i_5}\epsilon_{i_1 i_2 i_4 i_6} + a_{27}\delta_{i_3 i_6}\epsilon_{i_1 i_2 i_4 i_5} + a_{28}\delta_{i_4 i_5}\epsilon_{i_1 i_2 i_3 i_6} + a_{29}\delta_{i_4 i_6}\epsilon_{i_1 i_2 i_3 i_5} + a_{30}\delta_{i_5 i_6}\epsilon_{i_1 i_2 i_3 i_4} = 0. \quad \text{(B)}$$

Now we find that out of the remaining 15 isotropic tensors, how many are linearly independent. For this purpose we assign different values to i_1, i_2, i_3, i_4, i_5 and i_6 in equation (B). We do this in a systematic manner. The symbol

$$(12)\begin{pmatrix} 3 \\ 4 \\ 5 \\ 6 \end{pmatrix}$$

is a collection of the next four equations.

For (12) (3), we take $i_1 = i_2 = i_3 = 1$ and the remaining $i's$ are given values 2, 3, 4 where the indices appear in ascending order. We mention only the resulting equations.

(12) (3)

$i_3 = 1, i_4 = 2, i_5 = 3, i_6 = 4,$

$i_1 = i_2 = 1,$

$a_{16}\delta_{11}\epsilon_{1234} + a_{17}\delta_{11}\epsilon_{1234} + a_{21}\delta_{11}\epsilon_{1234} = 0$

$$\Rightarrow a_{16} + a_{17} + a_{21} = 0. \quad (1)$$

The same pattern is followed in the sequel upto equation (20)

(12) (4)

$$i_2 = 1, i_3 = 2, i_5 = 3, i_6 = 4,$$
$$i_1 = i_4 = 1,$$
$$a_{16}\delta_{11}\epsilon_{2134} + a_{18}\delta_{11}\epsilon_{1234} + a_{22}\delta_{11}\epsilon_{1234} = 0$$
$$\Rightarrow -a_{16} + a_{18} + a_{22} = 0. \tag{2}$$

(12) (5)

$$i_2 = 1, i_3 = 2, i_4 = 3, i_6 = 4,$$
$$i_1 = i_5 = 1,$$
$$a_{16}\delta_{11}\epsilon_{2314} + a_{19}\delta_{11}\epsilon_{1234} + a_{23}\delta_{11}\epsilon_{1234} = 0$$
$$\Rightarrow a_{16} + a_{19} + a_{23} = 0. \tag{3}$$

(12) (6)

$$i_2 = 1, i_3 = 2, i_4 = 3, i_5 = 4,$$
$$i_1 = i_6 = 1,$$
$$a_{16}\delta_{11}\epsilon_{2341} + a_{20}\delta_{11}\epsilon_{1234} + a_{24}\delta_{11}\epsilon_{1234} = 0$$
$$\Rightarrow -a_{16} + a_{20} + a_{24} = 0. \tag{4}$$

$$(13)\begin{pmatrix}4\\5\\6\end{pmatrix}$$

(13) (4)

$$i_1 = 1, i_2 = 2, i_5 = 3, i_6 = 4,$$
$$i_3 = i_4 = 1,$$
$$a_{17}\delta_{11}\epsilon_{2134} + a_{18}\delta_{11}\epsilon_{2134} + a_{25}\delta_{11}\epsilon_{1234} = 0$$
$$\Rightarrow -a_{17} - a_{18} + a_{25} = 0. \quad (5)$$

(13) (5)

$$i_1 = 1, i_2 = 2, i_4 = 3, i_6 = 4,$$
$$i_3 = i_5 = 1,$$
$$a_{17}\delta_{11}\epsilon_{2314} + a_{19}\delta_{11}\epsilon_{2134} + a_{26}\delta_{11}\epsilon_{1234} = 0$$
$$\Rightarrow a_{17} - a_{19} + a_{26} = 0. \quad (6)$$

(13) (6)

$$i_1 = 1, i_2 = 2, i_4 = 3, i_5 = 4,$$
$$i_3 = i_6 = 1,$$
$$a_{17}\delta_{11}\epsilon_{2341} + a_{20}\delta_{11}\epsilon_{2134} + a_{27}\delta_{11}\epsilon_{1234} = 0$$
$$\Rightarrow -a_{17} - a_{20} + a_{27} = 0. \quad (7)$$

(14) $\begin{pmatrix} 5 \\ 6 \end{pmatrix}$

(14) (5)

$$i_1 = 1, i_2 = 2, \ i_3 = 3, i_6 = 4,$$
$$i_4 = i_5 = 1,$$
$$a_{18}\delta_{11}\epsilon_{2314} + a_{19}\delta_{11}\epsilon_{2314} + a_{28}\delta_{11}\epsilon_{1234} = 0$$
$$\Rightarrow a_{18} + a_{19} + a_{28} = 0. \quad (8)$$

$(14)\,(6)$

$$i_1 = 1, i_2 = 2,\ i_3 = 3, i_5 = 4,$$
$$i_4 = i_6 = 1,$$
$$a_{18}\delta_{11}\epsilon_{2341} + a_{20}\delta_{11}\epsilon_{2314} + a_{29}\delta_{11}\epsilon_{1234} = 0$$
$$\Rightarrow -a_{18} + a_{20} + a_{29} = 0. \tag{9}$$

$$\Big(15\Big)\Big(6\Big)$$

$(15)\,(6)$

$$i_1 = 1, i_2 = 2,\ i_3 = 3, i_4 = 4,$$
$$i_5 = i_6 = 1,$$
$$a_{19}\delta_{11}\epsilon_{2341} + a_{20}\delta_{11}\epsilon_{2341} + a_{30}\delta_{11}\epsilon_{1234} = 0$$
$$\Rightarrow -a_{19} - a_{20} + a_{30} = 0. \tag{10}$$

$$(23)\begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix}$$

$(23)\,(4)$

$$i_1 = 1, i_4 = 2, i_5 = 3, i_6 = 4,$$
$$i_2 = i_3 = 2,$$
$$a_{21}\delta_{22}\epsilon_{1234} + a_{22}\delta_{22}\epsilon_{1234} + a_{25}\delta_{22}\epsilon_{1234} = 0$$
$$\Rightarrow a_{21} + a_{22} + a_{25} = 0. \tag{11}$$

(23) (5)

$$i_1 = 1, i_3 = 2, i_4 = 3, i_6 = 4,$$
$$i_2 = i_5 = 2,$$
$$a_{21}\delta_{22}\epsilon_{1324} + a_{23}\delta_{22}\epsilon_{1234} + a_{26}\delta_{22}\epsilon_{1234} = 0$$
$$\Rightarrow -a_{21} + a_{23} + a_{26} = 0. \tag{12}$$

(23) (6)

$$i_1 = 1, i_3 = 2, i_4 = 3, i_5 = 4,$$
$$i_2 = i_6 = 2,$$
$$a_{21}\delta_{22}\epsilon_{1342} + a_{24}\delta_{22}\epsilon_{1234} + a_{27}\delta_{22}\epsilon_{1234} = 0$$
$$\Rightarrow a_{21} + a_{24} + a_{27} = 0. \tag{13}$$

$$(24)\begin{pmatrix}5\\6\end{pmatrix}$$

(24) (5)

$$i_1 = 1, i_2 = 2, i_3 = 3, i_6 = 4,$$
$$i_4 = i_5 = 1,$$
$$a_{22}\delta_{22}\epsilon_{1324} + a_{23}\delta_{22}\epsilon_{1324} + a_{28}\delta_{22}\epsilon_{1234} = 0$$
$$\Rightarrow -a_{22} - a_{23} + a_{28} = 0. \tag{14}$$

(24) (6)

$$i_1 = 1, i_2 = 2, i_3 = 3, i_5 = 4,$$
$$i_4 = i_6 = 1,$$
$$a_{22}\delta_{22}\epsilon_{1342} + a_{24}\delta_{22}\epsilon_{1324} + a_{29}\delta_{22}\epsilon_{1234} = 0$$
$$\Rightarrow a_{22} - a_{24} + a_{29} = 0. \tag{15}$$

$\begin{pmatrix}25\end{pmatrix}\begin{pmatrix}6\end{pmatrix}$

(25) (6)

$$\begin{aligned}
&i_1 = 1, i_2 = 2, i_3 = 3, i_4 = 4,\\
&i_5 = i_6 = 1,\\
&a_{23}\delta_{22}\epsilon_{1342} + a_{24}\delta_{22}\epsilon_{1342} + a_{30}\delta_{22}\epsilon_{1234} = 0\\
&\Rightarrow a_{23} + a_{24} + a_{30} = 0. \qquad (16)
\end{aligned}$$

$(34)\begin{pmatrix}5\\6\end{pmatrix}$

(34) (5)

$$\begin{aligned}
&i_1 = 1, i_2 = 2, i_5 = 3, i_6 = 4,\\
&i_3 = i_4 = 3,\\
&a_{25}\delta_{33}\epsilon_{1234} + a_{26}\delta_{33}\epsilon_{1234} + a_{28}\delta_{33}\epsilon_{1234} = 0\\
&\Rightarrow a_{25} + a_{26} + a_{28} = 0. \qquad (17)
\end{aligned}$$

(34) (6)

$$\begin{aligned}
&i_1 = 1, i_2 = 2, i_3 = 3, i_5 = 4,\\
&i_4 = i_6 = 3,\\
&a_{25}\delta_{33}\epsilon_{1243} + a_{27}\delta_{33}\epsilon_{1234} + a_{29}\delta_{33}\epsilon_{1234} = 0\\
&\Rightarrow -a_{25} + a_{27} + a_{29} = 0. \qquad (18)
\end{aligned}$$

$\begin{pmatrix}35\end{pmatrix}\begin{pmatrix}6\end{pmatrix}$

(35) (6)

$$i_1 = 1, i_2 = 2, i_3 = 3, i_4 = 4,$$
$$i_5 = i_6 = 3,$$
$$a_{26}\delta_{33}\epsilon_{1243} + a_{27}\delta_{33}\epsilon_{1243} + a_{30}\delta_{33}\epsilon_{1234} = 0$$
$$\Rightarrow -a_{26} - a_{27} + a_{30} = 0. \tag{19}$$

$\binom{45}{}\binom{6}{}$

(45) (6)

$$i_1 = 1, i_2 = 2, i_3 = 3, i_4 = 4,$$
$$i_5 = i_6 = 4,$$
$$a_{28}\delta_{44}\epsilon_{1234} + a_{29}\delta_{44}\epsilon_{1234} + a_{30}\delta_{44}\epsilon_{1234} = 0$$
$$\Rightarrow a_{28} + a_{29} + a_{30} = 0. \tag{20}$$

In order to find the remaining linear invariants for rank 6, we consider these 20 equations and determine that among these 20 equations how many are independent. Starting from equation (1) our observation is that equations (1), (2), (3) and (4) are independent because none of these can be written as linear combination of one another. Let us move further.

From equations (5) and (11) we have

$$-a_{17} - a_{18} + a_{25} = 0, \tag{5}$$
$$a_{21} + a_{22} + a_{25} = 0. \tag{11}$$

By eliminating a_{25} from the above equations we arrive at

$$a_{17} + a_{18} + a_{21} + a_{22} = 0.$$

(1) + (2) gives us

$$a_{17} + a_{18} + a_{21} + a_{22} = 0.$$

This equation is identical to the one which we obtained by combining equations (5) and (11). This shows that we need only one of the two equations for a_{25}.

From equations (6) and (12) we have

$$a_{17} - a_{19} + a_{26} = 0, \tag{6}$$

$$-a_{21} + a_{23} + a_{26} = 0. \tag{12}$$

By combining above two equations we get

$$a_{17} - a_{19} + a_{21} - a_{23} = 0.$$

(1) $-$ (3) gives us

$$a_{17} - a_{19} + a_{21} - a_{23} = 0.$$

This equation is identical to the one which we obtained by combining equations (6) and (12). Thus equations for a_{26} does not yield a new equation. So we consider any one of the two equations for a_{26}.

From equations (7) and (13) we have

$$-a_{17} - a_{20} + a_{27} = 0, \tag{7}$$

$$a_{21} + a_{24} + a_{27} = 0. \tag{13}$$

By eliminating a_{27} from the above equations we arrive at

$$a_{17} + a_{20} + a_{21} + a_{24} = 0.$$

(1) + (4) gives us

$$a_{17} + a_{20} + a_{21} + a_{24} = 0.$$

This equation is the same as above equation which we obtained by combining equations (7) and (13). Thus equations for a_{27} does not give us a new equation. So, we keep any one of the two equations for a_{27}.

From equations (8) and (14) we have

$$a_{18} + a_{19} + a_{28} = 0, \tag{8}$$

$$-a_{22} - a_{23} + a_{28} = 0. \tag{14}$$

By combining above two equations we get

$$a_{18} + a_{19} + a_{22} + a_{23} = 0.$$

(2) + (3) gives us

$$a_{18} + a_{19} + a_{22} + a_{23} = 0.$$

This equation is identical to the one which we obtained by combining equations (8) and (14). Thus equations for a_{28} does not provide us a new equation. So we consider any one of the two equations for a_{28}.

From equations (9) and (15) we have

$$-a_{18} + a_{20} + a_{29} = 0, \tag{9}$$

$$a_{22} - a_{24} + a_{29} = 0. \tag{15}$$

By combining above two equations we get

$$a_{18} - a_{20} + a_{22} - a_{24} = 0.$$

(2) − (4) gives us

$$a_{18} - a_{20} + a_{22} - a_{24} = 0.$$

This equation is the same as above equation which we obtained by combining equations (9) and (15). Thus equations for a_{29} does not yield a new equation. So, we keep any one of the two equations for a_{29}.

From equations (10) and (16) we have

$$-a_{19} - a_{20} + a_{30} = 0, \tag{10}$$

$$a_{23} + a_{24} + a_{30} = 0. \tag{16}$$

By eliminating a_{30} from the above equations we arrive at

$$a_{19} + a_{20} + a_{23} + a_{24} = 0.$$

(3) + (4) gives us

$$a_{19} + a_{20} + a_{23} + a_{24} = 0.$$

This equation is identical to the equation which we obtained by combining equations (10) and (16). Thus equations for a_{30} does not provide us a new equation. So, we keep any one of the two equations for a_{30}.

From eq (17) we have

$$a_{25} + a_{26} + a_{28} = 0,$$

using equations (5) and (6) in above equation we get

$$a_{18} + a_{19} + a_{28} = 0.$$

This is not a new equation. This is identical to equation (8)

From equation (18) we have

$$-a_{25} + a_{27} + a_{29} = 0.$$

Using equations (5) and (7) in above equation we get

$$-a_{18} + a_{20} + a_{29} = 0.$$

This is not a new equation. This is identical to equation (9)

From equation (19) we have

$$-a_{26} - a_{27} + a_{30} = 0,$$

using equations (6) and (7) in above equation we get

$$-a_{19} - a_{20} + a_{30} = 0.$$

This is not a new equation. This is identical to equation (10)

From equation (20) we have

$$a_{28} + a_{29} + a_{30} = 0,$$

using equations (8) and (9) in above equation we get

$$-a_{19} - a_{20} + a_{30} = 0.$$

This is not a new equation. This is identical to equation (10)

Observation:

Out of above 20 equations we found that first 10 equations are linearly independent and remaining 10 equations depend upon these first 10 equations. For the sake of convienience, let us rewrite these linearly independent equations. These linearly independent equations are

$$a_{21} = -\ a_{16} - a_{17},$$

$$a_{22} = +\ a_{16} - a_{18},$$

$$a_{23} = -\ a_{16} - a_{19},$$

$$a_{24} = +\ a_{16} - a_{20},$$

$$a_{25} = +\ a_{17} + a_{18},$$

$$a_{26} = -\ a_{17} + a_{19},$$

$$a_{27} = +\ a_{17} + a_{20},$$

$$a_{28} = -\ a_{18} - a_{19},$$
$$a_{29} = +\ a_{18} - a_{20},$$
$$a_{30} = +\ a_{19} - a_{20}.$$

Now $a_{21}, a_{22}, \ldots\ldots.., a_{30}$ will be zero if $a_{16} = a_{17} = \ldots.. = a_{20} = 0$. Thus we conclude that for $d = 4$ and $r = 6$, we have $5^2 = 25$ independent linear invariants. These linear invariants are constructed from the following linearly independent isotropic tensors of rank 6

$$\delta_{i_1i_2}\delta_{i_3i_4}\delta_{i_5i_6}, \delta_{i_1i_2}\delta_{i_3i_5}\delta_{i_4i_6}, \delta_{i_1i_2}\delta_{i_3i_6}\delta_{i_4i_5}, \delta_{i_1i_3}\delta_{i_2i_4}\delta_{i_5i_6}, \delta_{i_1i_3}\delta_{i_2i_5}\delta_{i_4i_6},$$
$$\delta_{i_1i_3}\delta_{i_2i_6}\delta_{i_4i_5}, \delta_{i_1i_4}\delta_{i_2i_3}\delta_{i_5i_6}, \delta_{i_1i_4}\delta_{i_2i_5}\delta_{i_3i_6}, \delta_{i_1i_4}\delta_{i_2i_6}\delta_{i_3i_5}, \delta_{i_1i_5}\delta_{i_2i_3}\delta_{i_4i_6},$$
$$\delta_{i_1i_5}\delta_{i_2i_4}\delta_{i_3i_6}, \delta_{i_1i_5}\delta_{i_2i_6}\delta_{i_3i_4}, \delta_{i_1i_6}\delta_{i_2i_3}\delta_{i_4i_5}, \delta_{i_1i_6}\delta_{i_2i_4}\delta_{i_3i_5}, \delta_{i_1i_6}\delta_{i_2i_5}\delta_{i_3i_4},$$
$$\delta_{i_2i_3}\epsilon_{i_1i_4i_5i_6}, \delta_{i_2i_4}\epsilon_{i_1i_3i_5i_6}, \delta_{i_2i_5}\epsilon_{i_1i_3i_4i_6}, \delta_{i_2i_6}\epsilon_{i_1i_3i_4i_5}, \delta_{i_3i_4}\epsilon_{i_1i_2i_5i_6},$$
$$\delta_{i_3i_5}\epsilon_{i_1i_2i_4i_6}, \delta_{i_3i_6}\epsilon_{i_1i_2i_4i_5}, \delta_{i_4i_5}\epsilon_{i_1i_2i_3i_6}, \delta_{i_4i_6}\epsilon_{i_1i_2i_3i_5}, \delta_{i_5i_6}\epsilon_{i_1i_2i_3i_4}.$$

These appear in equation (A) on p. 28, together with the coefficients $a_1\ , a_2\ , \ldots.., a_{15}\ ,$ $a_{21}\ , a_{22}\ , \ldots.., a_{30}$. In this case a set of independent linear invariants of an arbitrary tensor is given by $T_{iijjkk}, T_{iijkjk}, T_{iikkjj}, T_{ijijkk}, T_{ijikjk}, \epsilon_{i_1i_4i_5i_6}T_{i_1iii_4i_5i_6}, \epsilon_{i_1i_3i_5i_6}T_{i_1ii_3ii_5i_6},$ $\epsilon_{i_1i_3i_4i_6}T_{i_1ii_3i_4ii_6}$ etc.

4.5 Case 5: $d = 4$, $r = 8$:

Now when $d =\ 4$ and $r = 8$, we have

$$\frac{8!}{2!.2!.2!.2!}.\frac{1}{4!} \;=\; 105 \text{ isotropic tensors of type } \delta_{i_1i_2}\delta_{i_3i_4}\delta_{i_5i_6}\delta_{i_7i_8},$$
$$\frac{8!}{2!.2!.4!}.\frac{1}{2!} \;=\; 210 \text{ isotropic tensors of type } \delta_{i_1i_2}\delta_{i_3i_4}\epsilon_{i_5i_6i_7i_8},$$

and

$$\frac{8!}{4!.4!}.\frac{1}{2!} \;=\; 35 \text{ isotropic tensors of type } \epsilon_{i_1i_2i_3i_4}\epsilon_{i_5i_6i_7i_8}.$$

Thus, the number of possible candidates for the membership of a basis for the space of isotropic tensors of rank 8 is 350. Here we list a few of these possible candidates, however, in equation (A), we list them all in detail.

$$\delta_{i_1i_2}\delta_{i_3i_4}\delta_{i_5i_6}\delta_{i_7i_8}, \delta_{i_1i_2}\delta_{i_3i_4}\delta_{i_5i_7}\delta_{i_6i_8}, \delta_{i_1i_2}\delta_{i_3i_4}\delta_{i_5i_8}\delta_{i_6i_7}, ..., \delta_{i_1i_2}\delta_{i_3i_8}\delta_{i_4i_7}\delta_{i_5i_6},$$

$$\delta_{i_1i_3}\delta_{i_2i_4}\delta_{i_5i_6}\delta_{i_7i_8}, \delta_{i_1i_3}\delta_{i_2i_4}\delta_{i_5i_7}\delta_{i_6i_8}, \delta_{i_1i_3}\delta_{i_2i_4}\delta_{i_5i_8}\delta_{i_6i_7}, ..., \delta_{i_1i_3}\delta_{i_2i_8}\delta_{i_4i_7}\delta_{i_5i_6},$$

$$\delta_{i_1i_4}\delta_{i_2i_3}\delta_{i_5i_6}\delta_{i_7i_8}, \delta_{i_1i_4}\delta_{i_2i_3}\delta_{i_5i_7}\delta_{i_6i_8}, \delta_{i_1i_4}\delta_{i_2i_3}\delta_{i_5i_8}\delta_{i_6i_7}, ..., \delta_{i_1i_4}\delta_{i_2i_8}\delta_{i_3i_7}\delta_{i_5i_6},$$

$$\delta_{i_1i_5}\delta_{i_2i_3}\delta_{i_4i_6}\delta_{i_7i_8}, \delta_{i_1i_5}\delta_{i_2i_3}\delta_{i_4i_7}\delta_{i_6i_8}, \delta_{i_1i_5}\delta_{i_2i_3}\delta_{i_4i_8}\delta_{i_6i_7}, ..., \delta_{i_1i_5}\delta_{i_2i_8}\delta_{i_3i_7}\delta_{i_4i_6},$$

$$\delta_{i_1i_6}\delta_{i_2i_3}\delta_{i_4i_5}\delta_{i_7i_8}, \delta_{i_1i_6}\delta_{i_2i_3}\delta_{i_4i_7}\delta_{i_5i_8}, \delta_{i_1i_6}\delta_{i_2i_3}\delta_{i_4i_8}\delta_{i_5i_7}, ..., \delta_{i_1i_6}\delta_{i_2i_8}\delta_{i_3i_7}\delta_{i_4i_5},$$

$$\delta_{i_1i_7}\delta_{i_2i_3}\delta_{i_4i_5}\delta_{i_6i_8}, \delta_{i_1i_7}\delta_{i_2i_3}\delta_{i_4i_6}\delta_{i_5i_8}, \delta_{i_1i_7}\delta_{i_2i_3}\delta_{i_4i_8}\delta_{i_5i_6}, ..., \delta_{i_1i_7}\delta_{i_2i_8}\delta_{i_3i_6}\delta_{i_4i_5},$$

$$\delta_{i_1i_8}\delta_{i_2i_3}\delta_{i_4i_5}\delta_{i_6i_7}, \delta_{i_1i_8}\delta_{i_2i_3}\delta_{i_4i_6}\delta_{i_5i_7}, \delta_{i_1i_8}\delta_{i_2i_3}\delta_{i_4i_7}\delta_{i_5i_6}, ..., \delta_{i_1i_8}\delta_{i_2i_7}\delta_{i_3i_6}\delta_{i_4i_5},$$

$$\delta_{i_1i_2}\delta_{i_3i_4}\epsilon_{i_5i_6i_7i_8}, \delta_{i_1i_2}\delta_{i_3i_5}\epsilon_{i_4i_6i_7i_8}, \delta_{i_1i_2}\delta_{i_3i_6}\epsilon_{i_4i_5i_7i_8}, ..., \delta_{i_1i_2}\delta_{i_7i_8}\epsilon_{i_3i_4i_5i_6},$$

$$\delta_{i_1i_3}\delta_{i_2i_4}\epsilon_{i_5i_6i_7i_8}, \delta_{i_1i_3}\delta_{i_2i_5}\epsilon_{i_4i_6i_7i_8}, \delta_{i_1i_3}\delta_{i_2i_6}\epsilon_{i_4i_5i_7i_8}, ..., \delta_{i_1i_3}\delta_{i_7i_8}\epsilon_{i_2i_4i_5i_6},$$

$$\delta_{i_1i_4}\delta_{i_2i_3}\epsilon_{i_5i_6i_7i_8}, \delta_{i_1i_4}\delta_{i_2i_5}\epsilon_{i_3i_6i_7i_8}, \delta_{i_1i_4}\delta_{i_2i_6}\epsilon_{i_3i_5i_7i_8}, ..., \delta_{i_1i_4}\delta_{i_7i_8}\epsilon_{i_2i_3i_5i_6},$$

$$\delta_{i_1i_5}\delta_{i_2i_3}\epsilon_{i_4i_6i_7i_8}, \delta_{i_1i_5}\delta_{i_2i_4}\epsilon_{i_3i_6i_7i_8}, \delta_{i_1i_5}\delta_{i_2i_6}\epsilon_{i_3i_4i_7i_8}, ..., \delta_{i_1i_5}\delta_{i_7i_8}\epsilon_{i_2i_3i_4i_6},$$

$$\delta_{i_1i_6}\delta_{i_2i_3}\epsilon_{i_4i_5i_7i_8}, \delta_{i_1i_6}\delta_{i_2i_4}\epsilon_{i_3i_5i_7i_8}, \delta_{i_1i_6}\delta_{i_2i_5}\epsilon_{i_3i_4i_7i_8}, ..., \delta_{i_1i_6}\delta_{i_7i_8}\epsilon_{i_2i_3i_4i_5},$$

$$\delta_{i_1i_7}\delta_{i_2i_3}\epsilon_{i_4i_5i_6i_8}, \delta_{i_1i_7}\delta_{i_2i_4}\epsilon_{i_3i_5i_6i_8}, \delta_{i_1i_7}\delta_{i_2i_5}\epsilon_{i_3i_4i_6i_8}, ..., \delta_{i_1i_7}\delta_{i_6i_8}\epsilon_{i_2i_3i_4i_5},$$

$$\delta_{i_1i_8}\delta_{i_2i_3}\epsilon_{i_4i_5i_6i_7}, \delta_{i_1i_8}\delta_{i_2i_4}\epsilon_{i_3i_5i_6i_7}, \delta_{i_1i_8}\delta_{i_2i_5}\epsilon_{i_3i_4i_6i_7}, ..., \delta_{i_1i_8}\delta_{i_6i_7}\epsilon_{i_2i_3i_4i_5},$$

$$\delta_{i_2i_3}\delta_{i_4i_5}\epsilon_{i_1i_6i_7i_8}, \delta_{i_2i_3}\delta_{i_4i_6}\epsilon_{i_1i_5i_7i_8}, \delta_{i_2i_3}\delta_{i_4i_7}\epsilon_{i_1i_5i_6i_8}, ..., \delta_{i_2i_3}\delta_{i_7i_8}\epsilon_{i_1i_4i_5i_6},$$

$$\delta_{i_2i_4}\delta_{i_3i_5}\epsilon_{i_1i_6i_7i_8}, \delta_{i_2i_4}\delta_{i_3i_6}\epsilon_{i_1i_5i_7i_8}, \delta_{i_2i_4}\delta_{i_3i_7}\epsilon_{i_1i_5i_6i_8}, ..., \delta_{i_2i_4}\delta_{i_7i_8}\epsilon_{i_1i_3i_5i_6},$$

$$\delta_{i_2i_5}\delta_{i_3i_4}\epsilon_{i_1i_6i_7i_8}, \delta_{i_2i_5}\delta_{i_3i_6}\epsilon_{i_1i_4i_7i_8}, \delta_{i_2i_5}\delta_{i_3i_7}\epsilon_{i_1i_4i_6i_8}, ..., \delta_{i_2i_5}\delta_{i_7i_8}\epsilon_{i_1i_3i_4i_6},$$

$$\delta_{i_2i_6}\delta_{i_3i_4}\epsilon_{i_1i_5i_7i_8}, \delta_{i_2i_6}\delta_{i_3i_5}\epsilon_{i_1i_4i_7i_8}, \delta_{i_2i_6}\delta_{i_3i_7}\epsilon_{i_1i_4i_5i_8}, ..., \delta_{i_2i_6}\delta_{i_7i_8}\epsilon_{i_1i_3i_4i_5},$$

$$\delta_{i_2i_7}\delta_{i_3i_4}\epsilon_{i_1i_5i_6i_8}, \delta_{i_2i_7}\delta_{i_3i_5}\epsilon_{i_1i_4i_6i_8}, \delta_{i_2i_7}\delta_{i_3i_6}\epsilon_{i_1i_4i_5i_8}, ..., \delta_{i_2i_7}\delta_{i_6i_8}\epsilon_{i_1i_3i_4i_5},$$

$$\delta_{i_2i_8}\delta_{i_3i_4}\epsilon_{i_1i_5i_6i_7}, \delta_{i_2i_8}\delta_{i_3i_5}\epsilon_{i_1i_4i_6i_7}, \delta_{i_2i_8}\delta_{i_3i_6}\epsilon_{i_1i_4i_5i_7}, ..., \delta_{i_2i_8}\delta_{i_6i_7}\epsilon_{i_1i_3i_4i_5},$$

$$\delta_{i_3i_4}\delta_{i_5i_6}\epsilon_{i_1i_2i_7i_8}, \delta_{i_3i_4}\delta_{i_5i_7}\epsilon_{i_1i_2i_6i_8}, \delta_{i_3i_4}\delta_{i_5i_8}\epsilon_{i_1i_2i_6i_7}, ..., \delta_{i_3i_4}\delta_{i_7i_8}\epsilon_{i_1i_2i_5i_6},$$

$$\delta_{i_3i_5}\delta_{i_4i_6}\epsilon_{i_1i_2i_7i_8}, \delta_{i_3i_5}\delta_{i_4i_7}\epsilon_{i_1i_2i_6i_8}, \delta_{i_3i_5}\delta_{i_4i_8}\epsilon_{i_1i_2i_6i_7}, ..., \delta_{i_3i_5}\delta_{i_7i_8}\epsilon_{i_1i_2i_4i_6},$$

$$\delta_{i_3i_6}\delta_{i_4i_5}\epsilon_{i_1i_2i_7i_8}, \delta_{i_3i_6}\delta_{i_4i_7}\epsilon_{i_1i_2i_5i_8}, \delta_{i_3i_6}\delta_{i_4i_8}\epsilon_{i_1i_2i_5i_7}, \ldots\ldots\ldots\ldots, \delta_{i_3i_6}\delta_{i_7i_8}\epsilon_{i_1i_2i_4i_5},$$
$$\delta_{i_3i_7}\delta_{i_4i_5}\epsilon_{i_1i_2i_6i_8}, \delta_{i_3i_7}\delta_{i_4i_6}\epsilon_{i_1i_2i_5i_8}, \delta_{i_3i_7}\delta_{i_4i_8}\epsilon_{i_1i_2i_5i_6}, \ldots\ldots\ldots\ldots, \delta_{i_3i_7}\delta_{i_6i_8}\epsilon_{i_1i_2i_4i_5},$$
$$\delta_{i_3i_8}\delta_{i_4i_5}\epsilon_{i_1i_2i_6i_7}, \delta_{i_3i_8}\delta_{i_4i_6}\epsilon_{i_1i_2i_5i_7}, \delta_{i_3i_8}\delta_{i_4i_7}\epsilon_{i_1i_2i_5i_6}, \ldots\ldots\ldots\ldots, \delta_{i_3i_8}\delta_{i_6i_7}\epsilon_{i_1i_2i_4i_5},$$
$$\delta_{i_4i_5}\delta_{i_6i_7}\epsilon_{i_1i_2i_3i_8}, \delta_{i_4i_5}\delta_{i_6i_8}\epsilon_{i_1i_2i_3i_7}, \delta_{i_4i_5}\delta_{i_7i_8}\epsilon_{i_1i_2i_3i_6}, \ldots\ldots\ldots\ldots, \delta_{i_5i_8}\delta_{i_6i_7}\epsilon_{i_1i_2i_3i_4},$$
$$\epsilon_{i_1i_2i_3i_4}\epsilon_{i_5i_6i_7i_8}, \epsilon_{i_1i_2i_3i_5}\epsilon_{i_4i_6i_7i_8}, \epsilon_{i_1i_2i_3i_6}\epsilon_{i_4i_5i_7i_8}, \ldots\ldots\ldots\ldots, \epsilon_{i_1i_6i_7i_8}\epsilon_{i_2i_3i_4i_5},$$

where $1 \leq i_1, i_2, i_3, i_4, i_5, i_6, i_7, i_8 \leq 4$.

In order to find independent linear invariants of a tensor of rank 8, under $SO(4)$, we let

$$a_1\delta_{i_1i_2}\delta_{i_3i_4}\delta_{i_5i_6}\delta_{i_7i_8} + a_2\delta_{i_1i_2}\delta_{i_3i_4}\delta_{i_5i_7}\delta_{i_6i_8} + a_3\delta_{i_1i_2}\delta_{i_3i_4}\delta_{i_5i_8}\delta_{i_6i_7} +$$
$$a_4\delta_{i_1i_2}\delta_{i_3i_5}\delta_{i_4i_6}\delta_{i_7i_8} + a_5\delta_{i_1i_2}\delta_{i_3i_5}\delta_{i_4i_7}\delta_{i_6i_8} + a_6\delta_{i_1i_2}\delta_{i_3i_5}\delta_{i_4i_8}\delta_{i_6i_7} +$$
$$a_7\delta_{i_1i_2}\delta_{i_3i_6}\delta_{i_4i_5}\delta_{i_7i_8} + a_8\delta_{i_1i_2}\delta_{i_3i_6}\delta_{i_4i_7}\delta_{i_5i_8} + a_9\delta_{i_1i_2}\delta_{i_3i_6}\delta_{i_4i_8}\delta_{i_5i_7} +$$
$$a_{10}\delta_{i_1i_2}\delta_{i_3i_7}\delta_{i_4i_5}\delta_{i_6i_8} + a_{11}\delta_{i_1i_2}\delta_{i_3i_7}\delta_{i_4i_6}\delta_{i_5i_8} + a_{12}\delta_{i_1i_2}\delta_{i_3i_7}\delta_{i_4i_8}\delta_{i_5i_6} +$$
$$a_{13}\delta_{i_1i_2}\delta_{i_3i_8}\delta_{i_4i_5}\delta_{i_6i_7} + a_{14}\delta_{i_1i_2}\delta_{i_3i_8}\delta_{i_4i_6}\delta_{i_5i_7} + a_{15}\delta_{i_1i_2}\delta_{i_3i_8}\delta_{i_4i_7}\delta_{i_5i_6} +$$
$$a_{16}\delta_{i_1i_3}\delta_{i_2i_4}\delta_{i_5i_6}\delta_{i_7i_8} + a_{17}\delta_{i_1i_3}\delta_{i_2i_4}\delta_{i_5i_7}\delta_{i_6i_8} + a_{18}\delta_{i_1i_3}\delta_{i_2i_4}\delta_{i_5i_8}\delta_{i_6i_7} +$$
$$a_{19}\delta_{i_1i_3}\delta_{i_2i_5}\delta_{i_4i_6}\delta_{i_7i_8} + a_{20}\delta_{i_1i_3}\delta_{i_2i_5}\delta_{i_4i_7}\delta_{i_6i_8} + a_{21}\delta_{i_1i_3}\delta_{i_2i_5}\delta_{i_4i_8}\delta_{i_6i_7} +$$
$$a_{22}\delta_{i_1i_3}\delta_{i_2i_6}\delta_{i_4i_5}\delta_{i_7i_8} + a_{23}\delta_{i_1i_3}\delta_{i_2i_6}\delta_{i_4i_7}\delta_{i_5i_8} + a_{24}\delta_{i_1i_3}\delta_{i_2i_6}\delta_{i_4i_8}\delta_{i_5i_7} +$$
$$a_{25}\delta_{i_1i_3}\delta_{i_2i_7}\delta_{i_4i_5}\delta_{i_6i_8} + a_{26}\delta_{i_1i_3}\delta_{i_2i_7}\delta_{i_4i_6}\delta_{i_5i_8} + a_{27}\delta_{i_1i_3}\delta_{i_2i_7}\delta_{i_4i_8}\delta_{i_5i_6} +$$
$$a_{28}\delta_{i_1i_3}\delta_{i_2i_8}\delta_{i_4i_5}\delta_{i_6i_7} + a_{29}\delta_{i_1i_3}\delta_{i_2i_8}\delta_{i_4i_6}\delta_{i_5i_7} + a_{30}\delta_{i_1i_3}\delta_{i_2i_8}\delta_{i_4i_7}\delta_{i_5i_6} +$$
$$a_{31}\delta_{i_1i_4}\delta_{i_2i_3}\delta_{i_5i_6}\delta_{i_7i_8} + a_{32}\delta_{i_1i_4}\delta_{i_2i_3}\delta_{i_5i_7}\delta_{i_6i_8} + a_{33}\delta_{i_1i_4}\delta_{i_2i_3}\delta_{i_5i_8}\delta_{i_6i_7} +$$
$$a_{34}\delta_{i_1i_4}\delta_{i_2i_5}\delta_{i_3i_6}\delta_{i_7i_8} + a_{35}\delta_{i_1i_4}\delta_{i_2i_5}\delta_{i_3i_7}\delta_{i_6i_8} + a_{36}\delta_{i_1i_4}\delta_{i_2i_5}\delta_{i_3i_8}\delta_{i_6i_7} +$$
$$a_{37}\delta_{i_1i_4}\delta_{i_2i_6}\delta_{i_3i_5}\delta_{i_7i_8} + a_{38}\delta_{i_1i_4}\delta_{i_2i_6}\delta_{i_3i_7}\delta_{i_5i_8} + a_{39}\delta_{i_1i_4}\delta_{i_2i_6}\delta_{i_3i_8}\delta_{i_5i_7} +$$
$$a_{40}\delta_{i_1i_4}\delta_{i_2i_7}\delta_{i_3i_5}\delta_{i_6i_8} + a_{41}\delta_{i_1i_4}\delta_{i_2i_7}\delta_{i_3i_6}\delta_{i_5i_8} + a_{42}\delta_{i_1i_4}\delta_{i_2i_7}\delta_{i_3i_8}\delta_{i_5i_6} +$$
$$a_{43}\delta_{i_1i_4}\delta_{i_2i_8}\delta_{i_3i_5}\delta_{i_6i_7} + a_{44}\delta_{i_1i_4}\delta_{i_2i_8}\delta_{i_3i_6}\delta_{i_5i_7} + a_{45}\delta_{i_1i_4}\delta_{i_2i_8}\delta_{i_3i_7}\delta_{i_5i_6} +$$
$$a_{46}\delta_{i_1i_5}\delta_{i_2i_3}\delta_{i_4i_6}\delta_{i_7i_8} + a_{47}\delta_{i_1i_5}\delta_{i_2i_3}\delta_{i_4i_7}\delta_{i_6i_8} + a_{48}\delta_{i_1i_5}\delta_{i_2i_3}\delta_{i_4i_8}\delta_{i_6i_7} +$$

$$a_{49}\delta_{i_1i_5}\delta_{i_2i_4}\delta_{i_3i_6}\delta_{i_7i_8} + a_{50}\delta_{i_1i_5}\delta_{i_2i_4}\delta_{i_3i_7}\delta_{i_6i_8} + a_{51}\delta_{i_1i_5}\delta_{i_2i_4}\delta_{i_3i_8}\delta_{i_6i_7} +$$
$$a_{52}\delta_{i_1i_5}\delta_{i_2i_6}\delta_{i_3i_4}\delta_{i_7i_8} + a_{53}\delta_{i_1i_5}\delta_{i_2i_6}\delta_{i_3i_7}\delta_{i_4i_8} + a_{54}\delta_{i_1i_5}\delta_{i_2i_6}\delta_{i_3i_8}\delta_{i_4i_7} +$$
$$a_{55}\delta_{i_1i_5}\delta_{i_2i_7}\delta_{i_3i_4}\delta_{i_6i_8} + a_{56}\delta_{i_1i_5}\delta_{i_2i_7}\delta_{i_3i_6}\delta_{i_4i_8} + a_{57}\delta_{i_1i_5}\delta_{i_2i_7}\delta_{i_3i_8}\delta_{i_4i_6} +$$
$$a_{58}\delta_{i_1i_5}\delta_{i_2i_8}\delta_{i_3i_4}\delta_{i_6i_7} + a_{59}\delta_{i_1i_5}\delta_{i_2i_8}\delta_{i_3i_6}\delta_{i_4i_7} + a_{60}\delta_{i_1i_5}\delta_{i_2i_8}\delta_{i_3i_7}\delta_{i_4i_6} +$$
$$a_{61}\delta_{i_1i_6}\delta_{i_2i_3}\delta_{i_4i_5}\delta_{i_7i_8} + a_{62}\delta_{i_1i_6}\delta_{i_2i_3}\delta_{i_4i_7}\delta_{i_5i_8} + a_{63}\delta_{i_1i_6}\delta_{i_2i_3}\delta_{i_4i_8}\delta_{i_5i_7} +$$
$$a_{64}\delta_{i_1i_6}\delta_{i_2i_4}\delta_{i_3i_5}\delta_{i_7i_8} + a_{65}\delta_{i_1i_6}\delta_{i_2i_4}\delta_{i_3i_7}\delta_{i_5i_8} + a_{66}\delta_{i_1i_6}\delta_{i_2i_4}\delta_{i_3i_8}\delta_{i_5i_7} +$$
$$a_{67}\delta_{i_1i_6}\delta_{i_2i_5}\delta_{i_3i_4}\delta_{i_7i_8} + a_{68}\delta_{i_1i_6}\delta_{i_2i_5}\delta_{i_3i_7}\delta_{i_4i_8} + a_{69}\delta_{i_1i_6}\delta_{i_2i_5}\delta_{i_3i_8}\delta_{i_4i_7} +$$
$$a_{70}\delta_{i_1i_6}\delta_{i_2i_7}\delta_{i_3i_4}\delta_{i_5i_8} + a_{71}\delta_{i_1i_6}\delta_{i_2i_7}\delta_{i_3i_5}\delta_{i_4i_8} + a_{72}\delta_{i_1i_6}\delta_{i_2i_7}\delta_{i_3i_8}\delta_{i_4i_5} +$$
$$a_{73}\delta_{i_1i_6}\delta_{i_2i_8}\delta_{i_3i_4}\delta_{i_5i_7} + a_{74}\delta_{i_1i_6}\delta_{i_2i_8}\delta_{i_3i_5}\delta_{i_4i_7} + a_{75}\delta_{i_1i_6}\delta_{i_2i_8}\delta_{i_3i_7}\delta_{i_4i_5} +$$
$$a_{76}\delta_{i_1i_7}\delta_{i_2i_3}\delta_{i_4i_5}\delta_{i_6i_8} + a_{77}\delta_{i_1i_7}\delta_{i_2i_3}\delta_{i_4i_6}\delta_{i_5i_8} + a_{78}\delta_{i_1i_7}\delta_{i_2i_3}\delta_{i_4i_8}\delta_{i_5i_6} +$$
$$a_{79}\delta_{i_1i_7}\delta_{i_2i_4}\delta_{i_3i_5}\delta_{i_6i_8} + a_{80}\delta_{i_1i_7}\delta_{i_2i_4}\delta_{i_3i_6}\delta_{i_5i_8} + a_{81}\delta_{i_1i_7}\delta_{i_2i_4}\delta_{i_3i_8}\delta_{i_5i_6} +$$
$$a_{82}\delta_{i_1i_7}\delta_{i_2i_5}\delta_{i_3i_4}\delta_{i_6i_8} + a_{83}\delta_{i_1i_7}\delta_{i_2i_5}\delta_{i_3i_6}\delta_{i_4i_8} + a_{84}\delta_{i_1i_7}\delta_{i_2i_5}\delta_{i_3i_8}\delta_{i_4i_6} +$$
$$a_{85}\delta_{i_1i_7}\delta_{i_2i_6}\delta_{i_3i_4}\delta_{i_5i_8} + a_{86}\delta_{i_1i_7}\delta_{i_2i_6}\delta_{i_3i_5}\delta_{i_4i_8} + a_{87}\delta_{i_1i_7}\delta_{i_2i_6}\delta_{i_3i_8}\delta_{i_4i_5} +$$
$$a_{88}\delta_{i_1i_7}\delta_{i_2i_8}\delta_{i_3i_4}\delta_{i_5i_6} + a_{89}\delta_{i_1i_7}\delta_{i_2i_8}\delta_{i_3i_5}\delta_{i_4i_6} + a_{90}\delta_{i_1i_7}\delta_{i_2i_8}\delta_{i_3i_6}\delta_{i_4i_5} +$$
$$a_{91}\delta_{i_1i_8}\delta_{i_2i_3}\delta_{i_4i_5}\delta_{i_6i_7} + a_{92}\delta_{i_1i_8}\delta_{i_2i_3}\delta_{i_4i_6}\delta_{i_5i_7} + a_{93}\delta_{i_1i_8}\delta_{i_2i_3}\delta_{i_4i_7}\delta_{i_5i_6} +$$
$$a_{94}\delta_{i_1i_8}\delta_{i_2i_4}\delta_{i_3i_5}\delta_{i_6i_7} + a_{95}\delta_{i_1i_8}\delta_{i_2i_4}\delta_{i_3i_6}\delta_{i_5i_7} + a_{96}\delta_{i_1i_8}\delta_{i_2i_4}\delta_{i_3i_7}\delta_{i_5i_6} +$$
$$a_{97}\delta_{i_1i_8}\delta_{i_2i_5}\delta_{i_3i_4}\delta_{i_6i_7} + a_{98}\delta_{i_1i_8}\delta_{i_2i_5}\delta_{i_3i_6}\delta_{i_4i_7} + a_{99}\delta_{i_1i_8}\delta_{i_2i_5}\delta_{i_3i_7}\delta_{i_4i_6} +$$
$$a_{100}\delta_{i_1i_8}\delta_{i_2i_6}\delta_{i_3i_4}\delta_{i_5i_7} + a_{101}\delta_{i_1i_8}\delta_{i_2i_6}\delta_{i_3i_5}\delta_{i_4i_7} + a_{102}\delta_{i_1i_8}\delta_{i_2i_6}\delta_{i_3i_7}\delta_{i_4i_5} +$$
$$a_{103}\delta_{i_1i_8}\delta_{i_2i_7}\delta_{i_3i_4}\delta_{i_5i_6} + a_{104}\delta_{i_1i_8}\delta_{i_2i_7}\delta_{i_3i_5}\delta_{i_4i_6} + a_{105}\delta_{i_1i_8}\delta_{i_2i_7}\delta_{i_3i_6}\delta_{i_4i_5} +$$
$$a_{106}\delta_{i_1i_2}\delta_{i_3i_4}\epsilon_{i_5i_6i_7i_8} + a_{107}\delta_{i_1i_2}\delta_{i_3i_5}\epsilon_{i_4i_6i_7i_8} + a_{108}\delta_{i_1i_2}\delta_{i_3i_6}\epsilon_{i_4i_5i_7i_8} +$$
$$a_{109}\delta_{i_1i_2}\delta_{i_3i_7}\epsilon_{i_4i_5i_6i_8} + a_{110}\delta_{i_1i_2}\delta_{i_3i_8}\epsilon_{i_4i_5i_6i_7} + a_{111}\delta_{i_1i_2}\delta_{i_4i_5}\epsilon_{i_3i_6i_7i_8} +$$
$$a_{112}\delta_{i_1i_2}\delta_{i_4i_6}\epsilon_{i_3i_5i_7i_8} + a_{113}\delta_{i_1i_2}\delta_{i_4i_7}\epsilon_{i_3i_5i_6i_8} + a_{114}\delta_{i_1i_2}\delta_{i_4i_8}\epsilon_{i_3i_5i_6i_7} +$$
$$a_{115}\delta_{i_1i_2}\delta_{i_5i_6}\epsilon_{i_3i_4i_7i_8} + a_{116}\delta_{i_1i_2}\delta_{i_5i_7}\epsilon_{i_3i_4i_6i_8} + a_{117}\delta_{i_1i_2}\delta_{i_5i_8}\epsilon_{i_3i_4i_6i_7} +$$
$$a_{118}\delta_{i_1i_2}\delta_{i_6i_7}\epsilon_{i_3i_4i_5i_8} + a_{119}\delta_{i_1i_2}\delta_{i_6i_8}\epsilon_{i_3i_4i_5i_7} + a_{120}\delta_{i_1i_2}\delta_{i_7i_8}\epsilon_{i_3i_4i_5i_6} +$$
$$a_{121}\delta_{i_1i_3}\delta_{i_2i_4}\epsilon_{i_5i_6i_7i_8} + a_{122}\delta_{i_1i_3}\delta_{i_2i_5}\epsilon_{i_4i_6i_7i_8} + a_{123}\delta_{i_1i_3}\delta_{i_2i_6}\epsilon_{i_4i_5i_7i_8} +$$

$$a_{124}\delta_{i_1i_3}\delta_{i_2i_7}\epsilon_{i_4i_5i_6i_8} + a_{125}\delta_{i_1i_3}\delta_{i_2i_8}\epsilon_{i_4i_5i_6i_7} + a_{126}\delta_{i_1i_3}\delta_{i_4i_5}\epsilon_{i_2i_6i_7i_8} +$$
$$a_{127}\delta_{i_1i_3}\delta_{i_4i_6}\epsilon_{i_2i_5i_7i_8} + a_{128}\delta_{i_1i_3}\delta_{i_4i_7}\epsilon_{i_2i_5i_6i_8} + a_{129}\delta_{i_1i_3}\delta_{i_4i_8}\epsilon_{i_2i_5i_6i_7} +$$
$$a_{130}\delta_{i_1i_3}\delta_{i_5i_6}\epsilon_{i_2i_4i_7i_8} + a_{131}\delta_{i_1i_3}\delta_{i_5i_7}\epsilon_{i_2i_4i_6i_8} + a_{132}\delta_{i_1i_3}\delta_{i_5i_8}\epsilon_{i_2i_4i_6i_7} +$$
$$a_{133}\delta_{i_1i_3}\delta_{i_6i_7}\epsilon_{i_2i_4i_5i_8} + a_{134}\delta_{i_1i_3}\delta_{i_6i_8}\epsilon_{i_2i_4i_5i_7} + a_{135}\delta_{i_1i_3}\delta_{i_7i_8}\epsilon_{i_2i_4i_5i_6} +$$
$$a_{136}\delta_{i_1i_4}\delta_{i_2i_3}\epsilon_{i_5i_6i_7i_8} + a_{137}\delta_{i_1i_4}\delta_{i_2i_5}\epsilon_{i_3i_6i_7i_8} + a_{138}\delta_{i_1i_4}\delta_{i_2i_6}\epsilon_{i_3i_5i_7i_8} +$$
$$a_{139}\delta_{i_1i_4}\delta_{i_2i_7}\epsilon_{i_3i_5i_6i_8} + a_{140}\delta_{i_1i_4}\delta_{i_2i_8}\epsilon_{i_3i_5i_6i_7} + a_{141}\delta_{i_1i_4}\delta_{i_3i_5}\epsilon_{i_2i_6i_7i_8} +$$
$$a_{142}\delta_{i_1i_4}\delta_{i_3i_6}\epsilon_{i_2i_5i_7i_8} + a_{143}\delta_{i_1i_4}\delta_{i_3i_7}\epsilon_{i_2i_5i_6i_8} + a_{144}\delta_{i_1i_4}\delta_{i_3i_8}\epsilon_{i_2i_5i_6i_7} +$$
$$a_{145}\delta_{i_1i_4}\delta_{i_5i_6}\epsilon_{i_2i_3i_7i_8} + a_{146}\delta_{i_1i_4}\delta_{i_5i_7}\epsilon_{i_2i_3i_6i_8} + a_{147}\delta_{i_1i_4}\delta_{i_5i_8}\epsilon_{i_2i_3i_6i_7} +$$
$$a_{148}\delta_{i_1i_4}\delta_{i_6i_7}\epsilon_{i_2i_3i_5i_8} + a_{149}\delta_{i_1i_4}\delta_{i_6i_8}\epsilon_{i_2i_3i_5i_7} + a_{150}\delta_{i_1i_4}\delta_{i_7i_8}\epsilon_{i_2i_3i_5i_6} +$$
$$a_{151}\delta_{i_1i_5}\delta_{i_2i_3}\epsilon_{i_4i_6i_7i_8} + a_{152}\delta_{i_1i_5}\delta_{i_2i_4}\epsilon_{i_3i_6i_7i_8} + a_{153}\delta_{i_1i_5}\delta_{i_2i_6}\epsilon_{i_3i_4i_7i_8} +$$
$$a_{154}\delta_{i_1i_5}\delta_{i_2i_7}\epsilon_{i_3i_4i_6i_8} + a_{155}\delta_{i_1i_5}\delta_{i_2i_8}\epsilon_{i_3i_4i_6i_7} + a_{156}\delta_{i_1i_5}\delta_{i_3i_4}\epsilon_{i_2i_6i_7i_8} +$$
$$a_{157}\delta_{i_1i_5}\delta_{i_3i_6}\epsilon_{i_2i_4i_7i_8} + a_{158}\delta_{i_1i_5}\delta_{i_3i_7}\epsilon_{i_2i_4i_6i_8} + a_{159}\delta_{i_1i_5}\delta_{i_3i_8}\epsilon_{i_2i_4i_6i_7} +$$
$$a_{160}\delta_{i_1i_5}\delta_{i_4i_6}\epsilon_{i_2i_3i_7i_8} + a_{161}\delta_{i_1i_5}\delta_{i_4i_7}\epsilon_{i_2i_3i_6i_8} + a_{162}\delta_{i_1i_5}\delta_{i_4i_8}\epsilon_{i_2i_3i_6i_7} +$$
$$a_{163}\delta_{i_1i_5}\delta_{i_6i_7}\epsilon_{i_2i_3i_4i_8} + a_{164}\delta_{i_1i_5}\delta_{i_6i_8}\epsilon_{i_2i_3i_4i_7} + a_{165}\delta_{i_1i_5}\delta_{i_7i_8}\epsilon_{i_2i_3i_4i_6} +$$
$$a_{166}\delta_{i_1i_6}\delta_{i_2i_3}\epsilon_{i_4i_5i_7i_8} + a_{167}\delta_{i_1i_6}\delta_{i_2i_4}\epsilon_{i_3i_5i_7i_8} + a_{168}\delta_{i_1i_6}\delta_{i_2i_5}\epsilon_{i_3i_4i_7i_8} +$$
$$a_{169}\delta_{i_1i_6}\delta_{i_2i_7}\epsilon_{i_3i_4i_5i_8} + a_{170}\delta_{i_1i_6}\delta_{i_2i_8}\epsilon_{i_3i_4i_5i_7} + a_{171}\delta_{i_1i_6}\delta_{i_3i_4}\epsilon_{i_2i_5i_7i_8} +$$
$$a_{172}\delta_{i_1i_6}\delta_{i_3i_5}\epsilon_{i_2i_4i_7i_8} + a_{173}\delta_{i_1i_6}\delta_{i_3i_7}\epsilon_{i_2i_4i_5i_8} + a_{174}\delta_{i_1i_6}\delta_{i_3i_8}\epsilon_{i_2i_4i_5i_7} +$$
$$a_{175}\delta_{i_1i_6}\delta_{i_4i_5}\epsilon_{i_2i_3i_7i_8} + a_{176}\delta_{i_1i_6}\delta_{i_4i_7}\epsilon_{i_2i_3i_5i_8} + a_{177}\delta_{i_1i_6}\delta_{i_4i_8}\epsilon_{i_2i_3i_5i_7} +$$
$$a_{178}\delta_{i_1i_6}\delta_{i_5i_7}\epsilon_{i_2i_3i_4i_8} + a_{179}\delta_{i_1i_6}\delta_{i_5i_8}\epsilon_{i_2i_3i_4i_7} + a_{180}\delta_{i_1i_6}\delta_{i_7i_8}\epsilon_{i_2i_3i_4i_5} +$$
$$a_{181}\delta_{i_1i_7}\delta_{i_2i_3}\epsilon_{i_4i_5i_6i_8} + a_{182}\delta_{i_1i_7}\delta_{i_2i_4}\epsilon_{i_3i_5i_6i_8} + a_{183}\delta_{i_1i_7}\delta_{i_2i_5}\epsilon_{i_3i_4i_6i_8} +$$
$$a_{184}\delta_{i_1i_7}\delta_{i_2i_6}\epsilon_{i_3i_4i_5i_8} + a_{185}\delta_{i_1i_7}\delta_{i_2i_8}\epsilon_{i_3i_4i_5i_6} + a_{186}\delta_{i_1i_7}\delta_{i_3i_4}\epsilon_{i_2i_5i_6i_8} +$$
$$a_{187}\delta_{i_1i_7}\delta_{i_3i_5}\epsilon_{i_2i_4i_6i_8} + a_{188}\delta_{i_1i_7}\delta_{i_3i_6}\epsilon_{i_2i_4i_5i_8} + a_{189}\delta_{i_1i_7}\delta_{i_3i_8}\epsilon_{i_2i_4i_5i_6} +$$
$$a_{190}\delta_{i_1i_7}\delta_{i_4i_5}\epsilon_{i_2i_3i_6i_8} + a_{191}\delta_{i_1i_7}\delta_{i_4i_6}\epsilon_{i_2i_3i_5i_8} + a_{192}\delta_{i_1i_7}\delta_{i_4i_8}\epsilon_{i_2i_3i_5i_6} +$$
$$a_{193}\delta_{i_1i_7}\delta_{i_5i_6}\epsilon_{i_2i_3i_4i_8} + a_{194}\delta_{i_1i_7}\delta_{i_5i_8}\epsilon_{i_2i_3i_4i_6} + a_{195}\delta_{i_1i_7}\delta_{i_6i_8}\epsilon_{i_2i_3i_4i_5} +$$
$$a_{196}\delta_{i_1i_8}\delta_{i_2i_3}\epsilon_{i_4i_5i_6i_7} + a_{197}\delta_{i_1i_8}\delta_{i_2i_4}\epsilon_{i_3i_5i_6i_7} + a_{198}\delta_{i_1i_8}\delta_{i_2i_5}\epsilon_{i_3i_4i_6i_7} +$$

$$a_{199}\delta_{i_1i_8}\delta_{i_2i_6}\epsilon_{i_3i_4i_5i_7} + a_{200}\delta_{i_1i_8}\delta_{i_2i_7}\epsilon_{i_3i_4i_5i_6} + a_{201}\delta_{i_1i_8}\delta_{i_3i_4}\epsilon_{i_2i_5i_6i_7} +$$

$$a_{202}\delta_{i_1i_8}\delta_{i_3i_5}\epsilon_{i_2i_4i_6i_7} + a_{203}\delta_{i_1i_8}\delta_{i_3i_6}\epsilon_{i_2i_4i_5i_7} + a_{204}\delta_{i_1i_8}\delta_{i_3i_7}\epsilon_{i_2i_4i_5i_6} +$$

$$a_{205}\delta_{i_1i_8}\delta_{i_4i_5}\epsilon_{i_2i_3i_6i_7} + a_{206}\delta_{i_1i_8}\delta_{i_4i_6}\epsilon_{i_2i_3i_5i_7} + a_{207}\delta_{i_1i_8}\delta_{i_4i_7}\epsilon_{i_2i_3i_5i_6} +$$

$$a_{208}\delta_{i_1i_8}\delta_{i_5i_6}\epsilon_{i_2i_3i_4i_7} + a_{209}\delta_{i_1i_8}\delta_{i_5i_7}\epsilon_{i_2i_3i_4i_6} + a_{210}\delta_{i_1i_8}\delta_{i_6i_7}\epsilon_{i_2i_3i_4i_5} +$$

$$a_{211}\delta_{i_2i_3}\delta_{i_4i_5}\epsilon_{i_1i_6i_7i_8} + a_{212}\delta_{i_2i_3}\delta_{i_4i_6}\epsilon_{i_1i_5i_7i_8} + a_{213}\delta_{i_2i_3}\delta_{i_4i_7}\epsilon_{i_1i_5i_6i_8} +$$

$$a_{214}\delta_{i_2i_3}\delta_{i_4i_8}\epsilon_{i_1i_5i_6i_7} + a_{215}\delta_{i_2i_3}\delta_{i_5i_6}\epsilon_{i_1i_4i_7i_8} + a_{216}\delta_{i_2i_3}\delta_{i_5i_7}\epsilon_{i_1i_4i_6i_8} +$$

$$a_{217}\delta_{i_2i_3}\delta_{i_5i_8}\epsilon_{i_1i_4i_6i_7} + a_{218}\delta_{i_2i_3}\delta_{i_6i_7}\epsilon_{i_1i_4i_5i_8} + a_{219}\delta_{i_2i_3}\delta_{i_6i_8}\epsilon_{i_1i_4i_5i_7} +$$

$$a_{220}\delta_{i_2i_3}\delta_{i_7i_8}\epsilon_{i_1i_4i_5i_6} + a_{221}\delta_{i_2i_4}\delta_{i_3i_5}\epsilon_{i_1i_6i_7i_8} + a_{222}\delta_{i_2i_4}\delta_{i_3i_6}\epsilon_{i_1i_5i_7i_8} +$$

$$a_{223}\delta_{i_2i_4}\delta_{i_3i_7}\epsilon_{i_1i_5i_6i_8} + a_{224}\delta_{i_2i_4}\delta_{i_3i_8}\epsilon_{i_1i_5i_6i_7} + a_{225}\delta_{i_2i_4}\delta_{i_5i_6}\epsilon_{i_1i_3i_7i_8} +$$

$$a_{226}\delta_{i_2i_4}\delta_{i_5i_7}\epsilon_{i_1i_3i_6i_8} + a_{227}\delta_{i_2i_4}\delta_{i_5i_8}\epsilon_{i_1i_3i_6i_7} + a_{228}\delta_{i_2i_4}\delta_{i_6i_7}\epsilon_{i_1i_3i_5i_8} +$$

$$a_{229}\delta_{i_2i_4}\delta_{i_6i_8}\epsilon_{i_1i_3i_5i_7} + a_{230}\delta_{i_2i_4}\delta_{i_7i_8}\epsilon_{i_1i_3i_5i_6} + a_{231}\delta_{i_2i_5}\delta_{i_3i_4}\epsilon_{i_1i_6i_7i_8} +$$

$$a_{232}\delta_{i_2i_5}\delta_{i_3i_6}\epsilon_{i_1i_4i_7i_8} + a_{233}\delta_{i_2i_5}\delta_{i_3i_7}\epsilon_{i_1i_4i_6i_8} + a_{234}\delta_{i_2i_5}\delta_{i_3i_8}\epsilon_{i_1i_4i_6i_7} +$$

$$a_{235}\delta_{i_2i_5}\delta_{i_4i_6}\epsilon_{i_1i_3i_7i_8} + a_{236}\delta_{i_2i_5}\delta_{i_4i_7}\epsilon_{i_1i_3i_6i_8} + a_{237}\delta_{i_2i_5}\delta_{i_4i_8}\epsilon_{i_1i_3i_6i_7} +$$

$$a_{238}\delta_{i_2i_5}\delta_{i_6i_7}\epsilon_{i_1i_3i_4i_8} + a_{239}\delta_{i_2i_5}\delta_{i_6i_8}\epsilon_{i_1i_3i_4i_7} + a_{240}\delta_{i_2i_5}\delta_{i_7i_8}\epsilon_{i_1i_3i_4i_6} +$$

$$a_{241}\delta_{i_2i_6}\delta_{i_3i_4}\epsilon_{i_1i_5i_7i_8} + a_{242}\delta_{i_2i_6}\delta_{i_3i_5}\epsilon_{i_1i_4i_7i_8} + a_{243}\delta_{i_2i_6}\delta_{i_3i_7}\epsilon_{i_1i_4i_5i_8} +$$

$$a_{244}\delta_{i_2i_6}\delta_{i_3i_8}\epsilon_{i_1i_4i_5i_7} + a_{245}\delta_{i_2i_6}\delta_{i_4i_5}\epsilon_{i_1i_3i_7i_8} + a_{246}\delta_{i_2i_6}\delta_{i_4i_7}\epsilon_{i_1i_3i_5i_8} +$$

$$a_{247}\delta_{i_2i_6}\delta_{i_4i_8}\epsilon_{i_1i_3i_5i_7} + a_{248}\delta_{i_2i_6}\delta_{i_5i_7}\epsilon_{i_1i_3i_4i_8} + a_{249}\delta_{i_2i_6}\delta_{i_5i_8}\epsilon_{i_1i_3i_4i_7} +$$

$$a_{250}\delta_{i_2i_6}\delta_{i_7i_8}\epsilon_{i_1i_3i_4i_5} + a_{251}\delta_{i_2i_7}\delta_{i_3i_4}\epsilon_{i_1i_5i_6i_8} + a_{252}\delta_{i_2i_7}\delta_{i_3i_5}\epsilon_{i_1i_4i_6i_8} +$$

$$a_{253}\delta_{i_2i_7}\delta_{i_3i_6}\epsilon_{i_1i_4i_5i_8} + a_{254}\delta_{i_2i_7}\delta_{i_3i_8}\epsilon_{i_1i_4i_5i_6} + a_{255}\delta_{i_2i_7}\delta_{i_4i_5}\epsilon_{i_1i_3i_6i_8} +$$

$$a_{256}\delta_{i_2i_7}\delta_{i_4i_6}\epsilon_{i_1i_3i_5i_8} + a_{257}\delta_{i_2i_7}\delta_{i_4i_8}\epsilon_{i_1i_3i_5i_6} + a_{258}\delta_{i_2i_7}\delta_{i_5i_6}\epsilon_{i_1i_3i_4i_8} +$$

$$a_{259}\delta_{i_2i_7}\delta_{i_5i_8}\epsilon_{i_1i_3i_4i_6} + a_{260}\delta_{i_2i_7}\delta_{i_6i_8}\epsilon_{i_1i_3i_4i_5} + a_{261}\delta_{i_2i_8}\delta_{i_3i_4}\epsilon_{i_1i_5i_6i_7} +$$

$$a_{262}\delta_{i_2i_8}\delta_{i_3i_5}\epsilon_{i_1i_4i_6i_7} + a_{263}\delta_{i_2i_8}\delta_{i_3i_6}\epsilon_{i_1i_4i_5i_7} + a_{264}\delta_{i_2i_8}\delta_{i_3i_7}\epsilon_{i_1i_4i_5i_6} +$$

$$a_{265}\delta_{i_2i_8}\delta_{i_4i_5}\epsilon_{i_1i_3i_6i_7} + a_{266}\delta_{i_2i_8}\delta_{i_4i_6}\epsilon_{i_1i_3i_5i_7} + a_{267}\delta_{i_2i_8}\delta_{i_4i_7}\epsilon_{i_1i_3i_5i_6} +$$

$$a_{268}\delta_{i_2i_8}\delta_{i_5i_6}\epsilon_{i_1i_3i_4i_7} + a_{269}\delta_{i_2i_8}\delta_{i_5i_7}\epsilon_{i_1i_3i_4i_6} + a_{270}\delta_{i_2i_8}\delta_{i_6i_7}\epsilon_{i_1i_3i_4i_5} +$$

$$a_{271}\delta_{i_3i_4}\delta_{i_5i_6}\epsilon_{i_1i_2i_7i_8} + a_{272}\delta_{i_3i_4}\delta_{i_5i_7}\epsilon_{i_1i_2i_6i_8} + a_{273}\delta_{i_3i_4}\delta_{i_5i_8}\epsilon_{i_1i_2i_6i_7} +$$

$$
\begin{aligned}
&a_{274}\delta_{i_3i_4}\delta_{i_6i_7}\epsilon_{i_1i_2i_5i_8}+a_{275}\delta_{i_3i_4}\delta_{i_6i_8}\epsilon_{i_1i_2i_5i_7}+a_{276}\delta_{i_3i_4}\delta_{i_7i_8}\epsilon_{i_1i_2i_5i_6}+\\
&a_{277}\delta_{i_3i_5}\delta_{i_4i_6}\epsilon_{i_1i_2i_7i_8}+a_{278}\delta_{i_3i_5}\delta_{i_4i_7}\epsilon_{i_1i_2i_6i_8}+a_{279}\delta_{i_3i_5}\delta_{i_4i_8}\epsilon_{i_1i_2i_6i_7}+\\
&a_{280}\delta_{i_3i_5}\delta_{i_6i_7}\epsilon_{i_1i_2i_4i_8}+a_{281}\delta_{i_3i_5}\delta_{i_6i_8}\epsilon_{i_1i_2i_4i_7}+a_{282}\delta_{i_3i_5}\delta_{i_7i_8}\epsilon_{i_1i_2i_4i_6}+\\
&a_{283}\delta_{i_3i_6}\delta_{i_4i_5}\epsilon_{i_1i_2i_7i_8}+a_{284}\delta_{i_3i_6}\delta_{i_4i_7}\epsilon_{i_1i_2i_5i_8}+a_{285}\delta_{i_3i_6}\delta_{i_4i_8}\epsilon_{i_1i_2i_5i_7}+\\
&a_{286}\delta_{i_3i_6}\delta_{i_5i_7}\epsilon_{i_1i_2i_4i_8}+a_{287}\delta_{i_3i_6}\delta_{i_5i_8}\epsilon_{i_1i_2i_4i_7}+a_{288}\delta_{i_3i_6}\delta_{i_7i_8}\epsilon_{i_1i_2i_4i_5}+\\
&a_{289}\delta_{i_3i_7}\delta_{i_4i_5}\epsilon_{i_1i_2i_6i_8}+a_{290}\delta_{i_3i_7}\delta_{i_4i_6}\epsilon_{i_1i_2i_5i_8}+a_{291}\delta_{i_3i_7}\delta_{i_4i_8}\epsilon_{i_1i_2i_5i_6}+\\
&a_{292}\delta_{i_3i_7}\delta_{i_5i_6}\epsilon_{i_1i_2i_4i_8}+a_{293}\delta_{i_3i_7}\delta_{i_5i_8}\epsilon_{i_1i_2i_4i_6}+a_{294}\delta_{i_3i_7}\delta_{i_6i_8}\epsilon_{i_1i_2i_4i_5}+\\
&a_{295}\delta_{i_3i_8}\delta_{i_4i_5}\epsilon_{i_1i_2i_6i_7}+a_{296}\delta_{i_3i_8}\delta_{i_4i_6}\epsilon_{i_1i_2i_5i_7}+a_{297}\delta_{i_3i_8}\delta_{i_4i_7}\epsilon_{i_1i_2i_5i_6}+\\
&a_{298}\delta_{i_3i_8}\delta_{i_5i_6}\epsilon_{i_1i_2i_4i_7}+a_{299}\delta_{i_3i_8}\delta_{i_5i_7}\epsilon_{i_1i_2i_4i_6}+a_{300}\delta_{i_3i_8}\delta_{i_6i_7}\epsilon_{i_1i_2i_4i_5}+\\
&a_{301}\delta_{i_4i_5}\delta_{i_6i_7}\epsilon_{i_1i_2i_3i_8}+a_{302}\delta_{i_4i_5}\delta_{i_6i_8}\epsilon_{i_1i_2i_3i_7}+a_{303}\delta_{i_4i_5}\delta_{i_7i_8}\epsilon_{i_1i_2i_3i_6}+\\
&a_{304}\delta_{i_4i_6}\delta_{i_5i_7}\epsilon_{i_1i_2i_3i_8}+a_{305}\delta_{i_4i_6}\delta_{i_5i_8}\epsilon_{i_1i_2i_3i_7}+a_{306}\delta_{i_4i_6}\delta_{i_7i_8}\epsilon_{i_1i_2i_3i_5}+\\
&a_{307}\delta_{i_4i_7}\delta_{i_5i_6}\epsilon_{i_1i_2i_3i_8}+a_{308}\delta_{i_4i_7}\delta_{i_5i_8}\epsilon_{i_1i_2i_3i_6}+a_{309}\delta_{i_4i_7}\delta_{i_6i_8}\epsilon_{i_1i_2i_3i_5}+\\
&a_{310}\delta_{i_4i_8}\delta_{i_5i_6}\epsilon_{i_1i_2i_3i_7}+a_{311}\delta_{i_4i_8}\delta_{i_5i_7}\epsilon_{i_1i_2i_3i_6}+a_{312}\delta_{i_4i_8}\delta_{i_6i_7}\epsilon_{i_1i_2i_3i_5}+\\
&a_{313}\delta_{i_5i_6}\delta_{i_7i_8}\epsilon_{i_1i_2i_3i_4}+a_{314}\delta_{i_5i_7}\delta_{i_6i_8}\epsilon_{i_1i_2i_3i_4}+a_{315}\delta_{i_5i_8}\delta_{i_6i_7}\epsilon_{i_1i_2i_3i_4}+\\
&a_{316}\epsilon_{i_1i_2i_3i_4}\epsilon_{i_5i_6i_7i_8}+a_{317}\epsilon_{i_1i_2i_3i_5}\epsilon_{i_4i_6i_7i_8}+a_{318}\epsilon_{i_1i_2i_3i_6}\epsilon_{i_4i_5i_7i_8}+\\
&a_{319}\epsilon_{i_1i_2i_3i_7}\epsilon_{i_4i_5i_6i_8}+a_{320}\epsilon_{i_1i_2i_3i_8}\epsilon_{i_4i_5i_6i_7}+a_{321}\epsilon_{i_1i_2i_4i_5}\epsilon_{i_3i_6i_7i_8}+\\
&a_{322}\epsilon_{i_1i_2i_4i_6}\epsilon_{i_3i_5i_7i_8}+a_{323}\epsilon_{i_1i_2i_4i_7}\epsilon_{i_3i_5i_6i_8}+a_{324}\epsilon_{i_1i_2i_4i_8}\epsilon_{i_3i_5i_6i_7}+\\
&a_{325}\epsilon_{i_1i_2i_5i_6}\epsilon_{i_3i_4i_7i_8}+a_{326}\epsilon_{i_1i_2i_5i_7}\epsilon_{i_3i_4i_6i_8}+a_{327}\epsilon_{i_1i_2i_5i_8}\epsilon_{i_3i_4i_6i_7}+\\
&a_{328}\epsilon_{i_1i_2i_6i_7}\epsilon_{i_3i_4i_5i_8}+a_{329}\epsilon_{i_1i_2i_6i_8}\epsilon_{i_3i_4i_5i_7}+a_{330}\epsilon_{i_1i_2i_7i_8}\epsilon_{i_3i_4i_5i_6}+\\
&a_{331}\epsilon_{i_1i_3i_4i_5}\epsilon_{i_2i_6i_7i_8}+a_{332}\epsilon_{i_1i_3i_4i_6}\epsilon_{i_2i_5i_7i_8}+a_{333}\epsilon_{i_1i_3i_4i_7}\epsilon_{i_2i_5i_6i_8}+\\
&a_{334}\epsilon_{i_1i_3i_4i_8}\epsilon_{i_2i_5i_6i_7}+a_{335}\epsilon_{i_1i_3i_5i_6}\epsilon_{i_2i_4i_7i_8}+a_{336}\epsilon_{i_1i_3i_5i_7}\epsilon_{i_2i_4i_6i_8}+\\
&a_{337}\epsilon_{i_1i_3i_5i_8}\epsilon_{i_2i_4i_6i_7}+a_{338}\epsilon_{i_1i_3i_6i_7}\epsilon_{i_2i_4i_5i_8}+a_{339}\epsilon_{i_1i_3i_6i_8}\epsilon_{i_2i_4i_5i_7}+\\
&a_{340}\epsilon_{i_1i_3i_7i_8}\epsilon_{i_2i_4i_5i_6}+a_{341}\epsilon_{i_1i_4i_5i_6}\epsilon_{i_2i_3i_7i_8}+a_{342}\epsilon_{i_1i_4i_5i_7}\epsilon_{i_2i_3i_6i_8}+\\
&a_{343}\epsilon_{i_1i_4i_5i_8}\epsilon_{i_2i_3i_6i_7}+a_{344}\epsilon_{i_1i_4i_6i_7}\epsilon_{i_2i_3i_5i_8}+a_{345}\epsilon_{i_1i_4i_6i_8}\epsilon_{i_2i_3i_5i_7}\\
&+a_{346}\epsilon_{i_1i_4i_7i_8}\epsilon_{i_2i_3i_5i_6}+a_{347}\epsilon_{i_1i_5i_6i_7}\epsilon_{i_2i_3i_4i_8}+a_{348}\epsilon_{i_1i_5i_6i_8}\epsilon_{i_2i_3i_4i_7}+\\
&a_{349}\epsilon_{i_1i_5i_7i_8}\epsilon_{i_2i_3i_4i_6}+a_{350}\epsilon_{i_1i_6i_7i_8}\epsilon_{i_2i_3i_4i_5}=0, \qquad \text{(A)}
\end{aligned}
$$

where a_n, $1 \leq n \leq 350$, not all zero, belong to the field F.

In order to find the independent linear invariants we assign different values to $i_1, i_2, i_3, i_4, i_5, i_6, i_7$ and i_8 where $1 \leq i_1, i_2, ..., i_8 \leq 4$, in equation (A) and proceed further as

Let $\quad i_1 = i_2 = 1,\ i_3 = i_4 = 2,\ i_5 = i_6 = 3,\ i_7 = i_8 = 4$, then

$$a_1 \delta_{11} \delta_{22} \delta_{33} \delta_{44} = 0,$$

which implies $a_1 = 0$.

Let $\quad i_1 = i_2 = 1,\ i_3 = i_4 = 2,\ i_5 = i_7 = 3,\ i_6 = i_8 = 4$, then

$$a_2 \delta_{11} \delta_{22} \delta_{33} \delta_{44} = 0,$$

which implies $a_2 = 0$.

Similarly, by adopting the same process as above, we have

$$a_3 = a_4 = ... = a_{105} = 0.$$

Thus we have

$$a_1 = a_2 = a_3 = ... = a_{105} = 0.$$

Now we check that out of remaining 245 possible candidates of the basis, how many are linearly independent. As mentioned earlier, in case 4, this will be done in a systematic manner. The symbol

$$(123) \begin{pmatrix} 456 \\ 457 \\ 458 \\ 467 \\ 468 \\ 478 \end{pmatrix}$$

is a collection of the next six equations

For (123) (456) , we take $i_1 = i_2 = i_3 = 1$, $i_4 = i_5 = i_6 = 2$ and the remaining $i's$ are given values 3, 4 where the indices appear in ascending order.

(123) (456) $\quad i_3 = 1,\ i_6 = 2,\ i_7 = 3,\ i_8 = 4,$

$$i_1 = i_2 = 1,\ \ i_4 = i_5 = 2,$$

$$a_{111}\delta_{11}\delta_{22}\epsilon_{1234} + a_{112}\delta_{11}\delta_{22}\epsilon_{1234} + a_{115}\delta_{11}\delta_{22}\epsilon_{1234} +$$
$$a_{126}\delta_{11}\delta_{22}\epsilon_{1234} + a_{127}\delta_{11}\delta_{22}\epsilon_{1234} + a_{130}\delta_{11}\delta_{22}\epsilon_{1234} +$$
$$a_{211}\delta_{11}\delta_{22}\epsilon_{1234} + a_{212}\delta_{11}\delta_{22}\epsilon_{1234} + a_{215}\delta_{11}\delta_{22}\epsilon_{1234} = 0$$
$$\Rightarrow a_{111} + a_{112} + a_{115} + a_{126} + a_{127} + a_{130} + a_{211} + a_{212} + a_{215} = 0. \tag{1}$$

The same pattern is followed in the sequel upto equation (126)

(123) (457) $\quad i_3 = 1,\ i_5 = 2,\ i_6 = 3,\ i_8 = 4,$

$$i_1 = i_2 = 1,\ \ i_4 = i_7 = 2,$$

$$a_{111}\delta_{11}\delta_{22}\epsilon_{1324} + a_{113}\delta_{11}\delta_{22}\epsilon_{1234} + a_{116}\delta_{11}\delta_{22}\epsilon_{1234} +$$
$$a_{126}\delta_{11}\delta_{22}\epsilon_{1324} + a_{128}\delta_{11}\delta_{22}\epsilon_{1234} + a_{131}\delta_{11}\delta_{22}\epsilon_{1234} +$$
$$a_{211}\delta_{11}\delta_{22}\epsilon_{1324} + a_{213}\delta_{11}\delta_{22}\epsilon_{1234} + a_{216}\delta_{11}\delta_{22}\epsilon_{1234} = 0$$
$$\Rightarrow -a_{111} + a_{113} + a_{116} - a_{126} + a_{128} + a_{131} - a_{211} + a_{213} + a_{216} = 0. \tag{2}$$

(123) (458) $\quad i_3 = 1,\ i_5 = 2,\ i_6 = 3,\ i_7 = 4,$

$$i_1 = i_2 = 1,\ \ i_4 = i_8 = 2,$$

$$a_{111}\delta_{11}\delta_{22}\epsilon_{1342} + a_{114}\delta_{11}\delta_{22}\epsilon_{1234} + a_{117}\delta_{11}\delta_{22}\epsilon_{1234} +$$
$$a_{126}\delta_{11}\delta_{22}\epsilon_{1342} + a_{129}\delta_{11}\delta_{22}\epsilon_{1234} + a_{132}\delta_{11}\delta_{22}\epsilon_{1234} +$$
$$a_{211}\delta_{11}\delta_{22}\epsilon_{1342} + a_{214}\delta_{11}\delta_{22}\epsilon_{1234} + a_{217}\delta_{11}\delta_{22}\epsilon_{1234} = 0$$
$$\Rightarrow a_{111} + a_{114} + a_{117} + a_{126} + a_{129} + a_{132} + a_{211} + a_{214} + a_{217} = 0. \tag{3}$$

(123) (467) $\quad i_3 = 1,\ i_4 = 2,\ i_5 = 3,\ i_8 = 4,$

$$i_1 = i_2 = 1, \ \ i_6 = i_7 = 2,$$

$$a_{112}\delta_{11}\delta_{22}\epsilon_{1324} + a_{113}\delta_{11}\delta_{22}\epsilon_{1324} + a_{118}\delta_{11}\delta_{22}\epsilon_{1234} +$$
$$a_{127}\delta_{11}\delta_{22}\epsilon_{1324} + a_{128}\delta_{11}\delta_{22}\epsilon_{1324} + a_{133}\delta_{11}\delta_{22}\epsilon_{1234} +$$
$$a_{212}\delta_{11}\delta_{22}\epsilon_{1324} + a_{213}\delta_{11}\delta_{22}\epsilon_{1324} + a_{218}\delta_{11}\delta_{22}\epsilon_{1234} = 0$$
$$\Rightarrow -a_{112} - a_{113} + a_{118} - a_{127} - a_{128} + a_{133} - a_{212} - a_{213} + a_{218} = 0. \tag{4}$$

(123) (468) $\quad i_3 = 1,\ i_4 = 2,\ i_5 = 3,\ i_7 = 4,$

$$i_1 = i_2 = 1, \ \ i_6 = i_8 = 2,$$

$$a_{112}\delta_{11}\delta_{22}\epsilon_{1342} + a_{114}\delta_{11}\delta_{22}\epsilon_{1324} + a_{119}\delta_{11}\delta_{22}\epsilon_{1234} +$$
$$a_{127}\delta_{11}\delta_{22}\epsilon_{1342} + a_{129}\delta_{11}\delta_{22}\epsilon_{1324} + a_{134}\delta_{11}\delta_{22}\epsilon_{1234} +$$
$$a_{212}\delta_{11}\delta_{22}\epsilon_{1342} + a_{214}\delta_{11}\delta_{22}\epsilon_{1324} + a_{219}\delta_{11}\delta_{22}\epsilon_{1234} = 0$$
$$\Rightarrow a_{112} - a_{114} + a_{119} + a_{127} - a_{129} + a_{134} + a_{212} - a_{214} + a_{219} = 0. \tag{5}$$

(123) (478) $\quad i_3 = 1,\ i_4 = 2,\ i_5 = 3,\ i_6 = 4,$

$$i_1 = i_2 = 1, \ \ i_7 = i_8 = 2,$$

$$a_{113}\delta_{11}\delta_{22}\epsilon_{1342} + a_{114}\delta_{11}\delta_{22}\epsilon_{1342} + a_{120}\delta_{11}\delta_{22}\epsilon_{1234} +$$
$$a_{128}\delta_{11}\delta_{22}\epsilon_{1342} + a_{129}\delta_{11}\delta_{22}\epsilon_{1342} + a_{135}\delta_{11}\delta_{22}\epsilon_{1234} +$$
$$a_{213}\delta_{11}\delta_{22}\epsilon_{1342} + a_{214}\delta_{11}\delta_{22}\epsilon_{1342} + a_{220}\delta_{11}\delta_{22}\epsilon_{1234} = 0$$
$$\Rightarrow a_{113} + a_{114} + a_{120} + a_{128} + a_{129} + a_{135} + a_{213} + a_{214} + a_{220} = 0. \tag{6}$$

$$(124)\begin{pmatrix} 356 \\ 357 \\ 358 \\ 367 \\ 368 \\ 378 \end{pmatrix}$$

(124) (356) $$i_4 = 1,\, i_6 = 2,\, i_7 = 3,\, i_8 = 4,$$

$$i_1 = i_2 = 1, \;\; i_3 = i_5 = 2,$$

$$\begin{aligned}
&a_{107}\delta_{11}\delta_{22}\epsilon_{1234} + a_{108}\delta_{11}\delta_{22}\epsilon_{1234} + a_{115}\delta_{11}\delta_{22}\epsilon_{2134} + \\
&a_{141}\delta_{11}\delta_{22}\epsilon_{1234} + a_{142}\delta_{11}\delta_{22}\epsilon_{1234} + a_{145}\delta_{11}\delta_{22}\epsilon_{1234} + \\
&a_{221}\delta_{11}\delta_{22}\epsilon_{1234} + a_{222}\delta_{11}\delta_{22}\epsilon_{1234} + a_{225}\delta_{11}\delta_{22}\epsilon_{1234} = 0 \\
&\Rightarrow a_{107} + a_{108} - a_{115} + a_{141} + a_{142} + a_{145} + a_{221} + a_{222} + a_{225} = 0. \qquad (7)
\end{aligned}$$

(123) (457) $$i_4 = 1,\, i_5 = 2,\, i_6 = 3,\, i_8 = 4,$$

$$i_1 = i_2 = 1, \;\; i_3 = i_7 = 2,$$

$$\begin{aligned}
&a_{107}\delta_{11}\delta_{22}\epsilon_{1324} + a_{109}\delta_{11}\delta_{22}\epsilon_{1234} + a_{116}\delta_{11}\delta_{22}\epsilon_{2134} + \\
&a_{141}\delta_{11}\delta_{22}\epsilon_{1324} + a_{143}\delta_{11}\delta_{22}\epsilon_{1234} + a_{146}\delta_{11}\delta_{22}\epsilon_{1234} + \\
&a_{221}\delta_{11}\delta_{22}\epsilon_{1324} + a_{223}\delta_{11}\delta_{22}\epsilon_{1234} + a_{226}\delta_{11}\delta_{22}\epsilon_{1234} = 0 \\
&\Rightarrow -a_{107} + a_{109} - a_{116} - a_{141} + a_{143} + a_{146} - a_{221} + a_{223} + a_{226} = 0. \qquad (8)
\end{aligned}$$

(124) (358) $$i_4 = 1,\, i_5 = 2,\, i_6 = 3,\, i_7 = 4,$$

$$i_1 = i_2 = 1, \;\; i_3 = i_8 = 2,$$

$$\begin{aligned}
&a_{107}\delta_{11}\delta_{22}\epsilon_{1342} + a_{110}\delta_{11}\delta_{22}\epsilon_{1234} + a_{117}\delta_{11}\delta_{22}\epsilon_{2134} + \\
&a_{141}\delta_{11}\delta_{22}\epsilon_{1342} + a_{144}\delta_{11}\delta_{22}\epsilon_{1234} + a_{147}\delta_{11}\delta_{22}\epsilon_{1234} + \\
&a_{221}\delta_{11}\delta_{22}\epsilon_{1342} + a_{224}\delta_{11}\delta_{22}\epsilon_{1234} + a_{227}\delta_{11}\delta_{22}\epsilon_{1234} = 0 \\
&\Rightarrow a_{107} + a_{110} - a_{117} + a_{141} + a_{144} + a_{147} + a_{221} + a_{224} + a_{227} = 0. \qquad (9)
\end{aligned}$$

(124) (367) $$i_2 = 1,\, i_3 = 2,\, i_5 = 3,\, i_8 = 4,$$

$$i_1 = i_4 = 1, \;\; i_6 = i_7 = 2,$$

$$a_{108}\delta_{11}\delta_{22}\epsilon_{1324} + a_{109}\delta_{11}\delta_{22}\epsilon_{1324} + a_{118}\delta_{11}\delta_{22}\epsilon_{2134} +$$
$$a_{142}\delta_{11}\delta_{22}\epsilon_{1324} + a_{143}\delta_{11}\delta_{22}\epsilon_{1324} + a_{148}\delta_{11}\delta_{22}\epsilon_{1234} +$$
$$a_{222}\delta_{11}\delta_{22}\epsilon_{1324} + a_{223}\delta_{11}\delta_{22}\epsilon_{1324} + a_{228}\delta_{11}\delta_{22}\epsilon_{1234} = 0$$
$$\Rightarrow -a_{108} - a_{109} - a_{118} - a_{142} - a_{143} + a_{148} - a_{222} - a_{223} + a_{228} = 0. \tag{10}$$

(124) (368)

$$i_2 = 1, \; i_3 = 2, \; i_5 = 3, \; i_7 = 4,$$
$$i_1 = i_4 = 1, \;\; i_6 = i_8 = 2,$$

$$a_{108}\delta_{11}\delta_{22}\epsilon_{1342} + a_{110}\delta_{11}\delta_{22}\epsilon_{1324} + a_{119}\delta_{11}\delta_{22}\epsilon_{2134} +$$
$$a_{142}\delta_{11}\delta_{22}\epsilon_{1342} + a_{144}\delta_{11}\delta_{22}\epsilon_{1324} + a_{149}\delta_{11}\delta_{22}\epsilon_{1234} +$$
$$a_{222}\delta_{11}\delta_{22}\epsilon_{1342} + a_{224}\delta_{11}\delta_{22}\epsilon_{1324} + a_{229}\delta_{11}\delta_{22}\epsilon_{1234} = 0$$
$$\Rightarrow a_{108} - a_{110} - a_{119} + a_{142} - a_{144} + a_{149} + a_{222} - a_{224} + a_{229} = 0. \tag{11}$$

(124) (378)

$$i_2 = 1, i_3 = 2, \; i_5 = 3, \; i_6 = 4,$$
$$i_1 = i_4 = 1, \;\; i_7 = i_8 = 2,$$

$$a_{109}\delta_{11}\delta_{22}\epsilon_{1342} + a_{110}\delta_{11}\delta_{22}\epsilon_{1342} + a_{120}\delta_{11}\delta_{22}\epsilon_{2134} +$$
$$a_{143}\delta_{11}\delta_{22}\epsilon_{1342} + a_{144}\delta_{11}\delta_{22}\epsilon_{1342} + a_{150}\delta_{11}\delta_{22}\epsilon_{1234} +$$
$$a_{223}\delta_{11}\delta_{22}\epsilon_{1342} + a_{224}\delta_{11}\delta_{22}\epsilon_{1342} + a_{230}\delta_{11}\delta_{22}\epsilon_{1234} = 0$$
$$\Rightarrow a_{109} + a_{110} - a_{120} + a_{143} + a_{144} + a_{150} + a_{223} + a_{224} + a_{230} = 0. \tag{12}$$

$$(125)\begin{pmatrix} 346 \\ 347 \\ 348 \\ 367 \\ 368 \\ 378 \end{pmatrix}$$

(125) (346) $$i_5 = 1,\ i_6 = 2,\ i_7 = 3,\ i_8 = 4,$$

$$i_1 = i_2 = 1,\ \ i_3 = i_4 = 2,$$

$$\begin{aligned}
&a_{106}\delta_{11}\delta_{22}\epsilon_{1234} + a_{108}\delta_{11}\delta_{22}\epsilon_{2134} + a_{112}\delta_{11}\delta_{22}\epsilon_{2134} + \\
&a_{156}\delta_{11}\delta_{22}\epsilon_{1234} + a_{157}\delta_{11}\delta_{22}\epsilon_{1234} + a_{160}\delta_{11}\delta_{22}\epsilon_{1234} + \\
&a_{231}\delta_{11}\delta_{22}\epsilon_{1234} + a_{232}\delta_{11}\delta_{22}\epsilon_{1234} + a_{235}\delta_{11}\delta_{22}\epsilon_{1234} = 0 \\
&\Rightarrow a_{106} - a_{108} - a_{112} + a_{156} + a_{157} + a_{160} + a_{231} + a_{232} + a_{235} = 0. \qquad (13)
\end{aligned}$$

(125) (347) $$i_2 = 1,\ i_4 = 2,\ i_6 = 3,\ i_8 = 4,$$

$$i_1 = i_5 = 1,\ \ i_3 = i_7 = 2,$$

$$\begin{aligned}
&a_{106}\delta_{11}\delta_{22}\epsilon_{1324} + a_{109}\delta_{11}\delta_{22}\epsilon_{2134} + a_{113}\delta_{11}\delta_{22}\epsilon_{2134} + \\
&a_{156}\delta_{11}\delta_{22}\epsilon_{1324} + a_{158}\delta_{11}\delta_{22}\epsilon_{1234} + a_{161}\delta_{11}\delta_{22}\epsilon_{1234} + \\
&a_{231}\delta_{11}\delta_{22}\epsilon_{1324} + a_{233}\delta_{11}\delta_{22}\epsilon_{1234} + a_{236}\delta_{11}\delta_{22}\epsilon_{1234} = 0 \\
&\Rightarrow -a_{106} - a_{109} - a_{113} - a_{156} + a_{158} + a_{161} - a_{231} + a_{233} + a_{236} = 0. \qquad (14)
\end{aligned}$$

(125) (348) $$i_2 = 1,\ i_4 = 2,\ i_6 = 3,\ i_7 = 4,$$

$$i_1 = i_5 = 1,\ \ i_3 = i_8 = 2,$$

$$\begin{aligned}
&a_{106}\delta_{11}\delta_{22}\epsilon_{1342} + a_{110}\delta_{11}\delta_{22}\epsilon_{2134} + a_{114}\delta_{11}\delta_{22}\epsilon_{2134} + \\
&a_{156}\delta_{11}\delta_{22}\epsilon_{1342} + a_{159}\delta_{11}\delta_{22}\epsilon_{1234} + a_{162}\delta_{11}\delta_{22}\epsilon_{1234} + \\
&a_{231}\delta_{11}\delta_{22}\epsilon_{1342} + a_{234}\delta_{11}\delta_{22}\epsilon_{1234} + a_{237}\delta_{11}\delta_{22}\epsilon_{1234} = 0 \\
&\Rightarrow a_{106} - a_{110} - a_{114} + a_{156} + a_{159} + a_{162} + a_{231} + a_{234} + a_{237} = 0. \qquad (15)
\end{aligned}$$

(125) (367) $$i_2 = 1, i_3 = 2,\ i_4 = 3,\ i_8 = 4,$$

$$i_1 = i_5 = 1, \;\; i_6 = i_7 = 2,$$

$$a_{108}\delta_{11}\delta_{22}\epsilon_{3124} + a_{109}\delta_{11}\delta_{22}\epsilon_{3124} + a_{118}\delta_{11}\delta_{22}\epsilon_{2314} +$$
$$a_{157}\delta_{11}\delta_{22}\epsilon_{1324} + a_{158}\delta_{11}\delta_{22}\epsilon_{1324} + a_{163}\delta_{11}\delta_{22}\epsilon_{1234} +$$
$$a_{232}\delta_{11}\delta_{22}\epsilon_{1324} + a_{233}\delta_{11}\delta_{22}\epsilon_{1324} + a_{238}\delta_{11}\delta_{22}\epsilon_{1234} = 0$$
$$\Rightarrow a_{108} + a_{109} + a_{118} - a_{157} - a_{158} + a_{163} - a_{232} - a_{233} + a_{238} = 0. \quad (16)$$

(125) (368)

$$i_2 = 1,\, i_3 = 2,\, i_4 = 3,\, i_7 = 4,$$
$$i_1 = i_5 = 1, \;\; i_6 = i_8 = 2,$$

$$a_{108}\delta_{11}\delta_{22}\epsilon_{3142} + a_{110}\delta_{11}\delta_{22}\epsilon_{3124} + a_{119}\delta_{11}\delta_{22}\epsilon_{2314} +$$
$$a_{157}\delta_{11}\delta_{22}\epsilon_{1342} + a_{159}\delta_{11}\delta_{22}\epsilon_{1324} + a_{164}\delta_{11}\delta_{22}\epsilon_{1234} +$$
$$a_{232}\delta_{11}\delta_{22}\epsilon_{1342} + a_{234}\delta_{11}\delta_{22}\epsilon_{1324} + a_{239}\delta_{11}\delta_{22}\epsilon_{1234} = 0$$
$$\Rightarrow -a_{108} + a_{110} + a_{119} + a_{157} - a_{159} + a_{164} + a_{232} - a_{234} + a_{239} = 0. \quad (17)$$

(125) (378)

$$i_2 = 1, i_3 = 2,\, i_4 = 3,\, i_6 = 4,$$
$$i_1 = i_5 = 1, \;\; i_7 = i_8 = 2,$$

$$a_{109}\delta_{11}\delta_{22}\epsilon_{3142} + a_{110}\delta_{11}\delta_{22}\epsilon_{3142} + a_{120}\delta_{11}\delta_{22}\epsilon_{2314} +$$
$$a_{158}\delta_{11}\delta_{22}\epsilon_{1342} + a_{159}\delta_{11}\delta_{22}\epsilon_{1342} + a_{165}\delta_{11}\delta_{22}\epsilon_{1234} +$$
$$a_{233}\delta_{11}\delta_{22}\epsilon_{1342} + a_{234}\delta_{11}\delta_{22}\epsilon_{1342} + a_{240}\delta_{11}\delta_{22}\epsilon_{1234} = 0$$
$$\Rightarrow -a_{109} - a_{110} + a_{120} + a_{158} + a_{159} + a_{165} + a_{233} + a_{234} + a_{240} = 0. \quad (18)$$

$$(126)\begin{pmatrix} 345 \\ 347 \\ 348 \\ 357 \\ 358 \\ 378 \end{pmatrix}$$

(126) (345) $$i_2 = 1,\, i_5 = 2,\, i_7 = 3,\, i_8 = 4,$$

$$i_1 = i_6 = 1, \;\; i_3 = i_4 = 2,$$

$$\begin{aligned}
&a_{106}\delta_{11}\delta_{22}\epsilon_{2134} + a_{107}\delta_{11}\delta_{22}\epsilon_{2134} + a_{111}\delta_{11}\delta_{22}\epsilon_{2134} + \\
&a_{171}\delta_{11}\delta_{22}\epsilon_{1234} + a_{172}\delta_{11}\delta_{22}\epsilon_{1234} + a_{175}\delta_{11}\delta_{22}\epsilon_{1234} + \\
&a_{241}\delta_{11}\delta_{22}\epsilon_{1234} + a_{242}\delta_{11}\delta_{22}\epsilon_{1234} + a_{245}\delta_{11}\delta_{22}\epsilon_{1234} = 0 \\
&\Rightarrow -a_{106} - a_{107} - a_{111} + a_{171} + a_{172} + a_{175} + a_{241} + a_{242} + a_{245} = 0. \qquad (19)
\end{aligned}$$

(126) (347) $$i_2 = 1,\, i_4 = 2,\, i_5 = 3,\, i_8 = 4,$$

$$i_1 = i_6 = 1, \;\; i_3 = i_7 = 2,$$

$$\begin{aligned}
&a_{106}\delta_{11}\delta_{22}\epsilon_{3124} + a_{109}\delta_{11}\delta_{22}\epsilon_{2314} + a_{113}\delta_{11}\delta_{22}\epsilon_{2314} + \\
&a_{171}\delta_{11}\delta_{22}\epsilon_{1324} + a_{173}\delta_{11}\delta_{22}\epsilon_{1234} + a_{176}\delta_{11}\delta_{22}\epsilon_{1234} + \\
&a_{241}\delta_{11}\delta_{22}\epsilon_{1324} + a_{243}\delta_{11}\delta_{22}\epsilon_{1234} + a_{246}\delta_{11}\delta_{22}\epsilon_{1234} = 0 \\
&\Rightarrow a_{106} + a_{109} + a_{113} - a_{171} + a_{173} + a_{176} - a_{241} + a_{243} + a_{246} = 0. \qquad (20)
\end{aligned}$$

(126) (348) $$i_2 = 1,\, i_4 = 2,\, i_5 = 3,\, i_7 = 4,$$

$$i_1 = i_6 = 1, \;\; i_3 = i_8 = 2,$$

$$\begin{aligned}
&a_{106}\delta_{11}\delta_{22}\epsilon_{3142} + a_{110}\delta_{11}\delta_{22}\epsilon_{2314} + a_{114}\delta_{11}\delta_{22}\epsilon_{2314} + \\
&a_{171}\delta_{11}\delta_{22}\epsilon_{1342} + a_{174}\delta_{11}\delta_{22}\epsilon_{1234} + a_{177}\delta_{11}\delta_{22}\epsilon_{1234} + \\
&a_{241}\delta_{11}\delta_{22}\epsilon_{1342} + a_{244}\delta_{11}\delta_{22}\epsilon_{1234} + a_{247}\delta_{11}\delta_{22}\epsilon_{1234} = 0 \\
&\Rightarrow -a_{106} + a_{110} + a_{114} + a_{171} + a_{174} + a_{177} + a_{241} + a_{244} + a_{247} = 0. \qquad (21)
\end{aligned}$$

(126) (357) $$i_2 = 1, i_3 = 2,\, i_4 = 3,\, i_8 = 4,$$

$$i_1 = i_6 = 1, \;\; i_5 = i_7 = 2,$$

$$a_{107}\delta_{11}\delta_{22}\epsilon_{3124} + a_{109}\delta_{11}\delta_{22}\epsilon_{3214} + a_{116}\delta_{11}\delta_{22}\epsilon_{2314} +$$
$$a_{172}\delta_{11}\delta_{22}\epsilon_{1324} + a_{173}\delta_{11}\delta_{22}\epsilon_{1324} + a_{178}\delta_{11}\delta_{22}\epsilon_{1234} +$$
$$a_{242}\delta_{11}\delta_{22}\epsilon_{1324} + a_{243}\delta_{11}\delta_{22}\epsilon_{1324} + a_{248}\delta_{11}\delta_{22}\epsilon_{1234} = 0$$
$$\Rightarrow a_{107} - a_{109} + a_{116} - a_{172} - a_{173} + a_{178} - a_{242} - a_{243} + a_{248} = 0. \tag{22}$$

(126) (358) $$i_2 = 1, \, i_3 = 2, \, i_4 = 3, \, i_7 = 4,$$

$$i_1 = i_6 = 1, \;\; i_5 = i_8 = 2,$$

$$a_{107}\delta_{11}\delta_{22}\epsilon_{3142} + a_{110}\delta_{11}\delta_{22}\epsilon_{3214} + a_{117}\delta_{11}\delta_{22}\epsilon_{2314} +$$
$$a_{172}\delta_{11}\delta_{22}\epsilon_{1342} + a_{174}\delta_{11}\delta_{22}\epsilon_{1324} + a_{179}\delta_{11}\delta_{22}\epsilon_{1234} +$$
$$a_{242}\delta_{11}\delta_{22}\epsilon_{1342} + a_{244}\delta_{11}\delta_{22}\epsilon_{1324} + a_{249}\delta_{11}\delta_{22}\epsilon_{1234} = 0$$
$$\Rightarrow -a_{107} - a_{110} + a_{117} + a_{172} - a_{174} + a_{179} + a_{242} - a_{244} + a_{249} = 0. \tag{23}$$

(126) (378) $$i_2 = 1, i_3 = 2, \, i_4 = 3, \, i_5 = 4,$$

$$i_1 = i_6 = 1, \;\; i_7 = i_8 = 2,$$

$$a_{109}\delta_{11}\delta_{22}\epsilon_{3412} + a_{110}\delta_{11}\delta_{22}\epsilon_{3412} + a_{120}\delta_{11}\delta_{22}\epsilon_{2341} +$$
$$a_{173}\delta_{11}\delta_{22}\epsilon_{1342} + a_{174}\delta_{11}\delta_{22}\epsilon_{1342} + a_{180}\delta_{11}\delta_{22}\epsilon_{1234} +$$
$$a_{243}\delta_{11}\delta_{22}\epsilon_{1342} + a_{244}\delta_{11}\delta_{22}\epsilon_{1342} + a_{250}\delta_{11}\delta_{22}\epsilon_{1234} = 0$$
$$\Rightarrow a_{109} + a_{110} - a_{120} + a_{173} + a_{174} + a_{180} + a_{243} + a_{244} + a_{250} = 0. \tag{24}$$

$$(127)\begin{pmatrix} 345 \\ 346 \\ 348 \\ 356 \\ 358 \\ 368 \end{pmatrix}$$

(127) (345) $$i_2 = 1,\ i_5 = 2,\ i_6 = 3,\ i_8 = 4,$$

$$i_1 = i_7 = 1,\ \ i_3 = i_4 = 2,$$

$$a_{106}\delta_{11}\delta_{22}\epsilon_{2314} + a_{107}\delta_{11}\delta_{22}\epsilon_{2314} + a_{111}\delta_{11}\delta_{22}\epsilon_{2314} +$$
$$a_{186}\delta_{11}\delta_{22}\epsilon_{1234} + a_{187}\delta_{11}\delta_{22}\epsilon_{1234} + a_{190}\delta_{11}\delta_{22}\epsilon_{1234} +$$
$$a_{251}\delta_{11}\delta_{22}\epsilon_{1234} + a_{252}\delta_{11}\delta_{22}\epsilon_{1234} + a_{255}\delta_{11}\delta_{22}\epsilon_{1234} = 0$$
$$\Rightarrow a_{106} + a_{107} + a_{111} + a_{186} + a_{187} + a_{190} + a_{251} + a_{252} + a_{255} = 0. \tag{25}$$

(127) (346) $$i_2 = 1,\ i_4 = 2,\ i_5 = 3,\ i_8 = 4,$$

$$i_1 = i_7 = 1,\ \ i_3 = i_6 = 2,$$

$$a_{106}\delta_{11}\delta_{22}\epsilon_{3214} + a_{108}\delta_{11}\delta_{22}\epsilon_{2314} + a_{112}\delta_{11}\delta_{22}\epsilon_{2314} +$$
$$a_{186}\delta_{11}\delta_{22}\epsilon_{1324} + a_{188}\delta_{11}\delta_{22}\epsilon_{1234} + a_{191}\delta_{11}\delta_{22}\epsilon_{1234} +$$
$$a_{251}\delta_{11}\delta_{22}\epsilon_{1324} + a_{253}\delta_{11}\delta_{22}\epsilon_{1234} + a_{256}\delta_{11}\delta_{22}\epsilon_{1234} = 0$$
$$\Rightarrow -a_{106} + a_{108} + a_{112} - a_{186} + a_{188} + a_{191} - a_{251} + a_{253} + a_{256} = 0. \tag{26}$$

(127) (348) $$i_2 = 1,\ i_4 = 2,\ i_5 = 3,\ i_6 = 4,$$

$$i_1 = i_7 = 1,\ \ i_3 = i_8 = 2,$$

$$a_{106}\delta_{11}\delta_{22}\epsilon_{3412} + a_{110}\delta_{11}\delta_{22}\epsilon_{2341} + a_{114}\delta_{11}\delta_{22}\epsilon_{2341} +$$
$$a_{186}\delta_{11}\delta_{22}\epsilon_{1342} + a_{189}\delta_{11}\delta_{22}\epsilon_{1234} + a_{192}\delta_{11}\delta_{22}\epsilon_{1234} +$$
$$a_{251}\delta_{11}\delta_{22}\epsilon_{1342} + a_{254}\delta_{11}\delta_{22}\epsilon_{1234} + a_{257}\delta_{11}\delta_{22}\epsilon_{1234} = 0$$
$$\Rightarrow a_{106} - a_{110} - a_{114} + a_{186} + a_{189} + a_{192} + a_{251} + a_{254} + a_{257} = 0. \tag{27}$$

(127) (356) $$i_2 = 1, i_3 = 2,\ i_4 = 3,\ i_8 = 4,$$

$$i_1 = i_7 = 1, \;\; i_5 = i_6 = 2,$$

$$a_{107}\delta_{11}\delta_{22}\epsilon_{3214} + a_{108}\delta_{11}\delta_{22}\epsilon_{3214} + a_{115}\delta_{11}\delta_{22}\epsilon_{2314} +$$
$$a_{187}\delta_{11}\delta_{22}\epsilon_{1324} + a_{188}\delta_{11}\delta_{22}\epsilon_{1324} + a_{193}\delta_{11}\delta_{22}\epsilon_{1234} +$$
$$a_{252}\delta_{11}\delta_{22}\epsilon_{1324} + a_{253}\delta_{11}\delta_{22}\epsilon_{1324} + a_{258}\delta_{11}\delta_{22}\epsilon_{1234} = 0$$
$$\Rightarrow -a_{107} - a_{108} + a_{115} - a_{187} - a_{188} + a_{193} - a_{252} - a_{253} + a_{258} = 0. \quad (28)$$

(127) (358)

$$i_2 = 1, \; i_3 = 2, \; i_4 = 3, \; i_6 = 4,$$
$$i_1 = i_7 = 1, \;\; i_5 = i_8 = 2,$$

$$a_{107}\delta_{11}\delta_{22}\epsilon_{3412} + a_{110}\delta_{11}\delta_{22}\epsilon_{3241} + a_{117}\delta_{11}\delta_{22}\epsilon_{2341} +$$
$$a_{187}\delta_{11}\delta_{22}\epsilon_{1342} + a_{189}\delta_{11}\delta_{22}\epsilon_{1324} + a_{194}\delta_{11}\delta_{22}\epsilon_{1234} +$$
$$a_{252}\delta_{11}\delta_{22}\epsilon_{1342} + a_{254}\delta_{11}\delta_{22}\epsilon_{1324} + a_{259}\delta_{11}\delta_{22}\epsilon_{1234} = 0$$
$$\Rightarrow a_{107} + a_{110} - a_{117} + a_{187} - a_{189} + a_{194} + a_{252} - a_{254} + a_{259} = 0. \quad (29)$$

(127) (368)

$$i_2 = 1, i_3 = 2, \; i_4 = 3, \; i_5 = 4,$$
$$i_1 = i_7 = 1, \;\; i_6 = i_8 = 2,$$

$$a_{108}\delta_{11}\delta_{22}\epsilon_{3412} + a_{110}\delta_{11}\delta_{22}\epsilon_{3421} + a_{119}\delta_{11}\delta_{22}\epsilon_{2341} +$$
$$a_{188}\delta_{11}\delta_{22}\epsilon_{1342} + a_{189}\delta_{11}\delta_{22}\epsilon_{1342} + a_{195}\delta_{11}\delta_{22}\epsilon_{1234} +$$
$$a_{253}\delta_{11}\delta_{22}\epsilon_{1342} + a_{254}\delta_{11}\delta_{22}\epsilon_{1342} + a_{260}\delta_{11}\delta_{22}\epsilon_{1234} = 0$$
$$\Rightarrow a_{108} - a_{110} - a_{119} + a_{188} + a_{189} + a_{195} + a_{253} + a_{254} + a_{260} = 0. \quad (30)$$

$$(128)\begin{pmatrix} 345 \\ 346 \\ 347 \\ 356 \\ 357 \\ 367 \end{pmatrix}$$

(128) (345) $$i_2 = 1,\ i_5 = 2,\ i_6 = 3,\ i_7 = 4,$$

$$i_1 = i_8 = 1,\ \ i_3 = i_4 = 2,$$

$$\begin{aligned}
&a_{106}\delta_{11}\delta_{22}\epsilon_{2341} + a_{107}\delta_{11}\delta_{22}\epsilon_{2341} + a_{111}\delta_{11}\delta_{22}\epsilon_{2341} + \\
&a_{201}\delta_{11}\delta_{22}\epsilon_{1234} + a_{202}\delta_{11}\delta_{22}\epsilon_{1234} + a_{205}\delta_{11}\delta_{22}\epsilon_{1234} + \\
&a_{261}\delta_{11}\delta_{22}\epsilon_{1234} + a_{262}\delta_{11}\delta_{22}\epsilon_{1234} + a_{265}\delta_{11}\delta_{22}\epsilon_{1234} = 0 \\
&\Rightarrow -a_{106} - a_{107} - a_{111} + a_{201} + a_{202} + a_{205} + a_{261} + a_{262} + a_{265} = 0. \qquad (31)
\end{aligned}$$

(128) (346) $$i_2 = 1,\ i_4 = 2,\ i_5 = 3,\ i_7 = 4,$$

$$i_1 = i_8 = 1,\ \ i_3 = i_6 = 2,$$

$$\begin{aligned}
&a_{106}\delta_{11}\delta_{22}\epsilon_{3241} + a_{108}\delta_{11}\delta_{22}\epsilon_{2341} + a_{112}\delta_{11}\delta_{22}\epsilon_{2341} + \\
&a_{201}\delta_{11}\delta_{22}\epsilon_{1324} + a_{203}\delta_{11}\delta_{22}\epsilon_{1234} + a_{206}\delta_{11}\delta_{22}\epsilon_{1234} + \\
&a_{261}\delta_{11}\delta_{22}\epsilon_{1324} + a_{263}\delta_{11}\delta_{22}\epsilon_{1234} + a_{266}\delta_{11}\delta_{22}\epsilon_{1234} = 0 \\
&\Rightarrow a_{106} - a_{108} - a_{112} - a_{201} + a_{203} + a_{206} - a_{261} + a_{263} + a_{266} = 0. \qquad (32)
\end{aligned}$$

(128) (347) $$i_2 = 1,\ i_4 = 2,\ i_5 = 3,\ i_6 = 4,$$

$$i_1 = i_8 = 1,\ \ i_3 = i_7 = 2,$$

$$\begin{aligned}
&a_{106}\delta_{11}\delta_{22}\epsilon_{3421} + a_{109}\delta_{11}\delta_{22}\epsilon_{2341} + a_{113}\delta_{11}\delta_{22}\epsilon_{2341} + \\
&a_{201}\delta_{11}\delta_{22}\epsilon_{1342} + a_{204}\delta_{11}\delta_{22}\epsilon_{1234} + a_{207}\delta_{11}\delta_{22}\epsilon_{1234} + \\
&a_{261}\delta_{11}\delta_{22}\epsilon_{1342} + a_{264}\delta_{11}\delta_{22}\epsilon_{1234} + a_{267}\delta_{11}\delta_{22}\epsilon_{1234} = 0 \\
&\Rightarrow -a_{106} - a_{109} - a_{113} + a_{201} + a_{204} + a_{207} + a_{261} + a_{264} + a_{267} = 0. \qquad (33)
\end{aligned}$$

(128) (356) $$i_2 = 1, i_3 = 2,\ i_4 = 3,\ i_7 = 4,$$

$$i_1 = i_8 = 1, \;\; i_5 = i_6 = 2,$$

$$a_{107}\delta_{11}\delta_{22}\epsilon_{3241} + a_{108}\delta_{11}\delta_{22}\epsilon_{3241} + a_{115}\delta_{11}\delta_{22}\epsilon_{2341} +$$
$$a_{202}\delta_{11}\delta_{22}\epsilon_{1324} + a_{203}\delta_{11}\delta_{22}\epsilon_{1324} + a_{208}\delta_{11}\delta_{22}\epsilon_{1234} +$$
$$a_{262}\delta_{11}\delta_{22}\epsilon_{1324} + a_{263}\delta_{11}\delta_{22}\epsilon_{1324} + a_{268}\delta_{11}\delta_{22}\epsilon_{1234} = 0$$
$$\Rightarrow a_{107} + a_{108} - a_{115} - a_{202} - a_{203} + a_{208} - a_{262} - a_{263} + a_{268} = 0. \tag{34}$$

(128) (357) $$i_2 = 1,\, i_3 = 2,\, i_4 = 3,\, i_6 = 4,$$

$$i_1 = i_8 = 1, \;\; i_5 = i_7 = 2,$$

$$a_{107}\delta_{11}\delta_{22}\epsilon_{3421} + a_{109}\delta_{11}\delta_{22}\epsilon_{3241} + a_{116}\delta_{11}\delta_{22}\epsilon_{2341} +$$
$$a_{202}\delta_{11}\delta_{22}\epsilon_{1342} + a_{204}\delta_{11}\delta_{22}\epsilon_{1324} + a_{209}\delta_{11}\delta_{22}\epsilon_{1234} +$$
$$a_{262}\delta_{11}\delta_{22}\epsilon_{1342} + a_{264}\delta_{11}\delta_{22}\epsilon_{1324} + a_{269}\delta_{11}\delta_{22}\epsilon_{1234} = 0$$
$$\Rightarrow -a_{107} + a_{109} - a_{116} + a_{202} - a_{204} + a_{209} + a_{262} - a_{264} + a_{269} = 0. \tag{35}$$

(128) (367) $$i_2 = 1,\, i_3 = 2,\, i_4 = 3,\, i_5 = 4,$$

$$i_1 = i_8 = 1, \;\; i_6 = i_7 = 2,$$

$$a_{108}\delta_{11}\delta_{22}\epsilon_{3421} + a_{109}\delta_{11}\delta_{22}\epsilon_{3421} + a_{118}\delta_{11}\delta_{22}\epsilon_{2341} +$$
$$a_{203}\delta_{11}\delta_{22}\epsilon_{1342} + a_{204}\delta_{11}\delta_{22}\epsilon_{1342} + a_{210}\delta_{11}\delta_{22}\epsilon_{1234} +$$
$$a_{263}\delta_{11}\delta_{22}\epsilon_{1342} + a_{264}\delta_{11}\delta_{22}\epsilon_{1342} + a_{270}\delta_{11}\delta_{22}\epsilon_{1234} = 0$$
$$\Rightarrow -a_{108} - a_{109} - a_{118} + a_{203} + a_{204} + a_{210} + a_{263} + a_{264} + a_{270} = 0. \tag{36}$$

$$(134)\begin{pmatrix} 256 \\ 257 \\ 258 \\ 267 \\ 268 \\ 278 \end{pmatrix}$$

(134) (256) $$i_4 = 1,\ i_6 = 2,\ i_7 = 3,\ i_8 = 4,$$

$$i_1 = i_3 = 1,\ \ i_2 = i_5 = 2,$$

$$a_{122}\delta_{11}\delta_{22}\epsilon_{1234} + a_{123}\delta_{11}\delta_{22}\epsilon_{1234} + a_{130}\delta_{11}\delta_{22}\epsilon_{2134} +$$
$$a_{137}\delta_{11}\delta_{22}\epsilon_{1234} + a_{138}\delta_{11}\delta_{22}\epsilon_{1234} + a_{145}\delta_{11}\delta_{22}\epsilon_{2134} +$$
$$a_{231}\delta_{22}\delta_{11}\epsilon_{1234} + a_{241}\delta_{22}\delta_{11}\epsilon_{1234} + a_{271}\delta_{11}\delta_{22}\epsilon_{1234} = 0$$
$$\Rightarrow a_{122} + a_{123} - a_{130} + a_{137} + a_{138} - a_{145} + a_{231} + a_{241} + a_{271} = 0. \tag{37}$$

(134) (257) $$i_4 = 1,\ i_5 = 2,\ i_6 = 3,\ i_8 = 4,$$

$$i_1 = i_3 = 1,\ \ i_2 = i_7 = 2,$$

$$a_{122}\delta_{11}\delta_{22}\epsilon_{1324} + a_{124}\delta_{11}\delta_{22}\epsilon_{1234} + a_{131}\delta_{11}\delta_{22}\epsilon_{2134} +$$
$$a_{137}\delta_{11}\delta_{22}\epsilon_{1324} + a_{139}\delta_{11}\delta_{22}\epsilon_{1234} + a_{146}\delta_{11}\delta_{22}\epsilon_{2134} +$$
$$a_{231}\delta_{22}\delta_{11}\epsilon_{1324} + a_{251}\delta_{22}\delta_{11}\epsilon_{1234} + a_{272}\delta_{11}\delta_{22}\epsilon_{1234} = 0$$
$$\Rightarrow -a_{122} + a_{124} - a_{131} - a_{137} + a_{139} - a_{146} - a_{231} + a_{251} + a_{272} = 0. \tag{38}$$

(134) (258) $$i_4 = 1,\ i_5 = 2,\ i_6 = 3,\ i_7 = 4,$$

$$i_1 = i_3 = 1,\ \ i_2 = i_8 = 2,$$

$$a_{122}\delta_{11}\delta_{22}\epsilon_{1342} + a_{125}\delta_{11}\delta_{22}\epsilon_{1234} + a_{132}\delta_{11}\delta_{22}\epsilon_{2134} +$$
$$a_{137}\delta_{11}\delta_{22}\epsilon_{1342} + a_{140}\delta_{11}\delta_{22}\epsilon_{1234} + a_{147}\delta_{11}\delta_{22}\epsilon_{2134} +$$
$$a_{231}\delta_{22}\delta_{11}\epsilon_{1342} + a_{261}\delta_{22}\delta_{11}\epsilon_{1234} + a_{273}\delta_{11}\delta_{22}\epsilon_{1234} = 0$$
$$\Rightarrow a_{122} + a_{125} - a_{132} + a_{137} + a_{140} - a_{147} + a_{231} + a_{261} + a_{273} = 0. \tag{39}$$

(134) (267) $$i_1 = 1,\ i_2 = 2,\ i_5 = 3,\ i_8 = 4,$$

$$i_3 = i_4 = 1, \;\; i_6 = i_7 = 2,$$

$$a_{123}\delta_{11}\delta_{22}\epsilon_{1324} + a_{124}\delta_{11}\delta_{22}\epsilon_{1324} + a_{133}\delta_{11}\delta_{22}\epsilon_{2134} +$$
$$a_{138}\delta_{11}\delta_{22}\epsilon_{1324} + a_{139}\delta_{11}\delta_{22}\epsilon_{1324} + a_{148}\delta_{11}\delta_{22}\epsilon_{2134} +$$
$$a_{241}\delta_{22}\delta_{11}\epsilon_{1324} + a_{251}\delta_{22}\delta_{11}\epsilon_{1324} + a_{274}\delta_{11}\delta_{22}\epsilon_{1234} = 0$$
$$\Rightarrow -a_{123} - a_{124} - a_{133} - a_{138} - a_{139} - a_{148} - a_{241} - a_{251} + a_{274} = 0. \tag{40}$$

(134) (268)

$$i_1 = 1, \, i_2 = 2, \, i_5 = 3, \, i_7 = 4,$$
$$i_3 = i_4 = 1, \;\; i_6 = i_8 = 2,$$

$$a_{123}\delta_{11}\delta_{22}\epsilon_{1342} + a_{125}\delta_{11}\delta_{22}\epsilon_{1324} + a_{134}\delta_{11}\delta_{22}\epsilon_{2134} +$$
$$a_{138}\delta_{11}\delta_{22}\epsilon_{1342} + a_{140}\delta_{11}\delta_{22}\epsilon_{1324} + a_{149}\delta_{11}\delta_{22}\epsilon_{2134} +$$
$$a_{241}\delta_{22}\delta_{11}\epsilon_{1342} + a_{261}\delta_{22}\delta_{11}\epsilon_{1324} + a_{275}\delta_{11}\delta_{22}\epsilon_{1234} = 0$$
$$\Rightarrow a_{123} - a_{125} - a_{134} + a_{138} - a_{140} - a_{149} + a_{241} - a_{261} + a_{275} = 0. \tag{41}$$

(134) (278)

$$i_1 = 1, \, i_2 = 2, \, i_5 = 3, \, i_6 = 4,$$
$$i_3 = i_4 = 1, \;\; i_7 = i_8 = 2,$$

$$a_{124}\delta_{11}\delta_{22}\epsilon_{1342} + a_{125}\delta_{11}\delta_{22}\epsilon_{1342} + a_{135}\delta_{11}\delta_{22}\epsilon_{2134} +$$
$$a_{139}\delta_{11}\delta_{22}\epsilon_{1342} + a_{140}\delta_{11}\delta_{22}\epsilon_{1342} + a_{150}\delta_{11}\delta_{22}\epsilon_{2134} +$$
$$a_{251}\delta_{22}\delta_{11}\epsilon_{1342} + a_{261}\delta_{22}\delta_{11}\epsilon_{1342} + a_{276}\delta_{22}\delta_{11}\epsilon_{1234} = 0$$
$$\Rightarrow a_{124} + a_{125} - a_{135} + a_{139} + a_{140} - a_{150} + a_{251} + a_{261} + a_{276} = 0. \tag{42}$$

$$(135)\begin{pmatrix} 246 \\ 247 \\ 248 \\ 267 \\ 268 \\ 278 \end{pmatrix}$$

(135) (246) $$i_5 = 1,\ i_6 = 2,\ i_7 = 3,\ i_8 = 4,$$

$$i_1 = i_3 = 1, \ \ i_2 = i_4 = 2,$$

$$a_{121}\delta_{11}\delta_{22}\epsilon_{1234} + a_{123}\delta_{11}\delta_{22}\epsilon_{2134} + a_{127}\delta_{11}\delta_{22}\epsilon_{2134} +$$
$$a_{152}\delta_{11}\delta_{22}\epsilon_{1234} + a_{153}\delta_{11}\delta_{22}\epsilon_{1234} + a_{160}\delta_{11}\delta_{22}\epsilon_{2134} +$$
$$a_{221}\delta_{22}\delta_{11}\epsilon_{1234} + a_{242}\delta_{22}\delta_{11}\epsilon_{1234} + a_{277}\delta_{11}\delta_{22}\epsilon_{1234} = 0$$
$$\Rightarrow a_{121} - a_{123} - a_{127} + a_{152} + a_{153} - a_{160} + a_{221} + a_{242} + a_{277} = 0. \tag{43}$$

(135) (247) $$i_3 = 1,\ i_4 = 2,\ i_6 = 3,\ i_8 = 4,$$

$$i_1 = i_5 = 1, \ \ i_2 = i_7 = 2,$$

$$a_{121}\delta_{11}\delta_{22}\epsilon_{1324} + a_{124}\delta_{11}\delta_{22}\epsilon_{2134} + a_{128}\delta_{11}\delta_{22}\epsilon_{2134} +$$
$$a_{152}\delta_{11}\delta_{22}\epsilon_{1324} + a_{154}\delta_{11}\delta_{22}\epsilon_{1234} + a_{161}\delta_{11}\delta_{22}\epsilon_{2134} +$$
$$a_{221}\delta_{22}\delta_{11}\epsilon_{1324} + a_{252}\delta_{22}\delta_{11}\epsilon_{1234} + a_{278}\delta_{11}\delta_{22}\epsilon_{1234} = 0$$
$$\Rightarrow -a_{121} - a_{124} - a_{128} - a_{152} + a_{154} - a_{161} - a_{221} + a_{252} + a_{278} = 0. \tag{44}$$

(135) (248) $$i_3 = 1,\ i_4 = 2,\ i_6 = 3,\ i_7 = 4,$$

$$i_1 = i_5 = 1, \ \ i_2 = i_8 = 2,$$

$$a_{121}\delta_{11}\delta_{22}\epsilon_{1342} + a_{125}\delta_{11}\delta_{22}\epsilon_{2134} + a_{129}\delta_{11}\delta_{22}\epsilon_{2134} +$$
$$a_{152}\delta_{11}\delta_{22}\epsilon_{1342} + a_{155}\delta_{11}\delta_{22}\epsilon_{1234} + a_{162}\delta_{11}\delta_{22}\epsilon_{2134} +$$
$$a_{221}\delta_{22}\delta_{11}\epsilon_{1342} + a_{262}\delta_{22}\delta_{11}\epsilon_{1234} + a_{279}\delta_{11}\delta_{22}\epsilon_{1234} = 0$$
$$\Rightarrow a_{121} - a_{125} - a_{129} + a_{152} + a_{155} - a_{162} + a_{221} + a_{262} + a_{279} = 0. \tag{45}$$

(135) (267) $$i_1 = 1, i_2 = 2,\ i_4 = 3,\ i_8 = 4,$$

$$i_3 = i_5 = 1, \;\; i_6 = i_7 = 2,$$

$$a_{123}\delta_{11}\delta_{22}\epsilon_{3124} + a_{124}\delta_{11}\delta_{22}\epsilon_{3124} + a_{133}\delta_{11}\delta_{22}\epsilon_{2314} +$$
$$a_{153}\delta_{11}\delta_{22}\epsilon_{1324} + a_{154}\delta_{11}\delta_{22}\epsilon_{1324} + a_{163}\delta_{11}\delta_{22}\epsilon_{2134} +$$
$$a_{242}\delta_{22}\delta_{11}\epsilon_{1324} + a_{252}\delta_{22}\delta_{11}\epsilon_{1324} + a_{280}\delta_{11}\delta_{22}\epsilon_{1234} = 0$$
$$\Rightarrow a_{123} + a_{124} + a_{133} - a_{153} - a_{154} - a_{163} - a_{242} - a_{252} + a_{280} = 0. \tag{46}$$

(135) (268)

$$i_1 = 1, \, i_2 = 2, \, i_4 = 3, \, i_7 = 4,$$
$$i_3 = i_5 = 1, \;\; i_6 = i_8 = 2,$$

$$a_{123}\delta_{11}\delta_{22}\epsilon_{3142} + a_{125}\delta_{11}\delta_{22}\epsilon_{3124} + a_{134}\delta_{11}\delta_{22}\epsilon_{2314} +$$
$$a_{153}\delta_{11}\delta_{22}\epsilon_{1342} + a_{155}\delta_{11}\delta_{22}\epsilon_{1324} + a_{164}\delta_{11}\delta_{22}\epsilon_{2134} +$$
$$a_{242}\delta_{22}\delta_{11}\epsilon_{1342} + a_{262}\delta_{22}\delta_{11}\epsilon_{1324} + a_{281}\delta_{11}\delta_{22}\epsilon_{1234} = 0$$
$$\Rightarrow -a_{123} + a_{125} + a_{134} + a_{153} - a_{155} - a_{164} + a_{242} - a_{262} + a_{281} = 0. \tag{47}$$

(135) (278)

$$i_1 = 1, i_2 = 2, \, i_4 = 3, \, i_6 = 4,$$
$$i_3 = i_5 = 1, \;\; i_7 = i_8 = 2,$$

$$a_{124}\delta_{11}\delta_{22}\epsilon_{3142} + a_{125}\delta_{11}\delta_{22}\epsilon_{3142} + a_{135}\delta_{11}\delta_{22}\epsilon_{2314} +$$
$$a_{154}\delta_{11}\delta_{22}\epsilon_{1342} + a_{155}\delta_{11}\delta_{22}\epsilon_{1342} + a_{165}\delta_{11}\delta_{22}\epsilon_{2134} +$$
$$a_{252}\delta_{22}\delta_{11}\epsilon_{1342} + a_{262}\delta_{22}\delta_{11}\epsilon_{1342} + a_{282}\delta_{11}\delta_{22}\epsilon_{1234} = 0$$
$$\Rightarrow -a_{124} - a_{125} + a_{135} + a_{154} + a_{155} - a_{165} + a_{252} + a_{262} + a_{282} = 0. \tag{48}$$

$$(136)\begin{pmatrix} 245 \\ 247 \\ 248 \\ 257 \\ 258 \\ 278 \end{pmatrix}$$

(136) (245) $$i_3 = 1,\ i_5 = 2,\ i_7 = 3,\ i_8 = 4,$$

$$i_1 = i_6 = 1,\ \ i_2 = i_4 = 2,$$

$$\begin{aligned}&a_{121}\delta_{11}\delta_{22}\epsilon_{2134} + a_{122}\delta_{11}\delta_{22}\epsilon_{2134} + a_{126}\delta_{11}\delta_{22}\epsilon_{2134} +\\ &a_{167}\delta_{11}\delta_{22}\epsilon_{1234} + a_{168}\delta_{11}\delta_{22}\epsilon_{1234} + a_{175}\delta_{11}\delta_{22}\epsilon_{2134} +\\ &a_{222}\delta_{22}\delta_{11}\epsilon_{1234} + a_{232}\delta_{22}\delta_{11}\epsilon_{1234} + a_{283}\delta_{11}\delta_{22}\epsilon_{1234} = 0\\ &\Rightarrow -a_{121} - a_{122} - a_{126} + a_{167} + a_{168} - a_{175} + a_{222} + a_{232} + a_{283} = 0. \end{aligned} \tag{49}$$

(136) (247) $$i_3 = 1,\ i_4 = 2,\ i_5 = 3,\ i_8 = 4,$$

$$i_1 = i_6 = 1,\ \ i_2 = i_7 = 2,$$

$$\begin{aligned}&a_{121}\delta_{11}\delta_{22}\epsilon_{3124} + a_{124}\delta_{11}\delta_{22}\epsilon_{2314} + a_{128}\delta_{11}\delta_{22}\epsilon_{2314} +\\ &a_{167}\delta_{11}\delta_{22}\epsilon_{1324} + a_{169}\delta_{11}\delta_{22}\epsilon_{1234} + a_{176}\delta_{11}\delta_{22}\epsilon_{2134} +\\ &a_{222}\delta_{22}\delta_{11}\epsilon_{1324} + a_{253}\delta_{22}\delta_{11}\epsilon_{1234} + a_{284}\delta_{11}\delta_{22}\epsilon_{1234} = 0\\ &\Rightarrow a_{121} + a_{124} + a_{128} - a_{167} + a_{169} - a_{176} - a_{222} + a_{253} + a_{284} = 0. \end{aligned} \tag{50}$$

(136) (248) $$i_3 = 1,\ i_4 = 2,\ i_5 = 3,\ i_7 = 4,$$

$$i_1 = i_6 = 1,\ \ i_2 = i_8 = 2,$$

$$\begin{aligned}&a_{121}\delta_{11}\delta_{22}\epsilon_{3142} + a_{125}\delta_{11}\delta_{22}\epsilon_{2314} + a_{129}\delta_{11}\delta_{22}\epsilon_{2314} +\\ &a_{167}\delta_{11}\delta_{22}\epsilon_{1342} + a_{170}\delta_{11}\delta_{22}\epsilon_{1234} + a_{177}\delta_{11}\delta_{22}\epsilon_{2134} +\\ &a_{222}\delta_{22}\delta_{11}\epsilon_{1342} + a_{263}\delta_{22}\delta_{11}\epsilon_{1234} + a_{285}\delta_{11}\delta_{22}\epsilon_{1234} = 0\\ &\Rightarrow -a_{121} + a_{125} + a_{129} + a_{167} + a_{170} - a_{177} + a_{222} + a_{263} + a_{285} = 0. \end{aligned} \tag{51}$$

(136) (257) $$i_1 = 1, i_2 = 2,\ i_4 = 3,\ i_8 = 4,$$

$$i_3 = i_6 = 1, \;\; i_5 = i_7 = 2,$$

$$a_{122}\delta_{11}\delta_{22}\epsilon_{3124} + a_{124}\delta_{11}\delta_{22}\epsilon_{3214} + a_{131}\delta_{11}\delta_{22}\epsilon_{2314} +$$
$$a_{168}\delta_{11}\delta_{22}\epsilon_{1324} + a_{169}\delta_{11}\delta_{22}\epsilon_{1324} + a_{178}\delta_{11}\delta_{22}\epsilon_{2134} +$$
$$a_{232}\delta_{22}\delta_{11}\epsilon_{1324} + a_{253}\delta_{22}\delta_{11}\epsilon_{1324} + a_{286}\delta_{11}\delta_{22}\epsilon_{1234} = 0$$
$$\Rightarrow a_{122} - a_{124} + a_{131} - a_{168} - a_{169} - a_{178} - a_{232} - a_{253} + a_{286} = 0. \quad (52)$$

(136) (258)

$$i_1 = 1,\, i_2 = 2,\, i_4 = 3,\, i_7 = 4,$$
$$i_3 = i_6 = 1, \;\; i_5 = i_8 = 2,$$

$$a_{122}\delta_{11}\delta_{22}\epsilon_{3142} + a_{125}\delta_{11}\delta_{22}\epsilon_{3214} + a_{132}\delta_{11}\delta_{22}\epsilon_{2314} +$$
$$a_{168}\delta_{11}\delta_{22}\epsilon_{1342} + a_{170}\delta_{11}\delta_{22}\epsilon_{1324} + a_{179}\delta_{11}\delta_{22}\epsilon_{2134} +$$
$$a_{232}\delta_{22}\delta_{11}\epsilon_{1342} + a_{263}\delta_{22}\delta_{11}\epsilon_{1324} + a_{287}\delta_{11}\delta_{22}\epsilon_{1234} = 0$$
$$\Rightarrow -a_{122} - a_{125} + a_{132} + a_{168} - a_{170} - a_{179} + a_{232} - a_{263} + a_{287} = 0. \quad (53)$$

(136) (278)

$$i_1 = 1, i_2 = 2,\, i_4 = 3,\, i_5 = 4,$$
$$i_3 = i_6 = 1, \;\; i_7 = i_8 = 2,$$

$$a_{124}\delta_{11}\delta_{22}\epsilon_{3412} + a_{125}\delta_{11}\delta_{22}\epsilon_{3412} + a_{135}\delta_{11}\delta_{22}\epsilon_{2341} +$$
$$a_{169}\delta_{11}\delta_{22}\epsilon_{1342} + a_{170}\delta_{11}\delta_{22}\epsilon_{1342} + a_{180}\delta_{11}\delta_{22}\epsilon_{2134} +$$
$$a_{253}\delta_{22}\delta_{11}\epsilon_{1342} + a_{263}\delta_{22}\delta_{11}\epsilon_{1342} + a_{288}\delta_{11}\delta_{22}\epsilon_{1234} = 0$$
$$\Rightarrow a_{124} + a_{125} - a_{135} + a_{169} + a_{170} - a_{180} + a_{253} + a_{263} + a_{288} = 0. \quad (54)$$

$$(137)\begin{pmatrix} 245 \\ 246 \\ 248 \\ 256 \\ 258 \\ 268 \end{pmatrix}$$

(137) (245) $$i_3 = 1,\, i_5 = 2,\, i_6 = 3,\, i_8 = 4,$$

$$i_1 = i_7 = 1, \;\; i_2 = i_4 = 2,$$

$$a_{121}\delta_{11}\delta_{22}\epsilon_{2314} + a_{122}\delta_{11}\delta_{22}\epsilon_{2314} + a_{126}\delta_{11}\delta_{22}\epsilon_{2314} +$$
$$a_{182}\delta_{11}\delta_{22}\epsilon_{1234} + a_{183}\delta_{11}\delta_{22}\epsilon_{1234} + a_{190}\delta_{11}\delta_{22}\epsilon_{2134} +$$
$$a_{223}\delta_{22}\delta_{11}\epsilon_{1234} + a_{233}\delta_{22}\delta_{11}\epsilon_{1234} + a_{289}\delta_{11}\delta_{22}\epsilon_{1234} = 0$$
$$\Rightarrow a_{121} + a_{122} + a_{126} + a_{182} + a_{183} - a_{190} + a_{223} + a_{233} + a_{289} = 0. \quad (55)$$

(137) (246) $$i_3 = 1,\, i_4 = 2,\, i_5 = 3,\, i_8 = 4,$$

$$i_1 = i_7 = 1, \;\; i_2 = i_6 = 2,$$

$$a_{121}\delta_{11}\delta_{22}\epsilon_{3214} + a_{123}\delta_{11}\delta_{22}\epsilon_{2314} + a_{127}\delta_{11}\delta_{22}\epsilon_{2314} +$$
$$a_{182}\delta_{11}\delta_{22}\epsilon_{1324} + a_{184}\delta_{11}\delta_{22}\epsilon_{1234} + a_{191}\delta_{11}\delta_{22}\epsilon_{2134} +$$
$$a_{223}\delta_{22}\delta_{11}\epsilon_{1324} + a_{243}\delta_{22}\delta_{11}\epsilon_{1234} + a_{290}\delta_{11}\delta_{22}\epsilon_{1234} = 0$$
$$\Rightarrow -a_{121} + a_{123} + a_{127} - a_{182} + a_{184} - a_{191} - a_{223} + a_{243} + a_{290} = 0. \quad (56)$$

(137) (248) $$i_3 = 1,\, i_4 = 2,\, i_5 = 3,\, i_6 = 4,$$

$$i_1 = i_7 = 1, \;\; i_2 = i_8 = 2,$$

$$a_{121}\delta_{11}\delta_{22}\epsilon_{3412} + a_{125}\delta_{11}\delta_{22}\epsilon_{2341} + a_{129}\delta_{11}\delta_{22}\epsilon_{2341} +$$
$$a_{182}\delta_{11}\delta_{22}\epsilon_{1342} + a_{185}\delta_{11}\delta_{22}\epsilon_{1234} + a_{192}\delta_{11}\delta_{22}\epsilon_{2134} +$$
$$a_{223}\delta_{22}\delta_{11}\epsilon_{1342} + a_{264}\delta_{22}\delta_{11}\epsilon_{1234} + a_{291}\delta_{11}\delta_{22}\epsilon_{1234} = 0$$
$$\Rightarrow a_{121} - a_{125} - a_{129} + a_{182} + a_{185} - a_{192} + a_{223} + a_{264} + a_{291} = 0. \quad (57)$$

(137) (256) $$i_1 = 1, i_2 = 2,\, i_4 = 3,\, i_8 = 4,$$

$$i_3 = i_7 = 1, \;\; i_5 = i_6 = 2,$$

$$a_{122}\delta_{11}\delta_{22}\epsilon_{3214} + a_{123}\delta_{11}\delta_{22}\epsilon_{3214} + a_{130}\delta_{11}\delta_{22}\epsilon_{2314} +$$
$$a_{183}\delta_{11}\delta_{22}\epsilon_{1324} + a_{184}\delta_{11}\delta_{22}\epsilon_{1324} + a_{193}\delta_{11}\delta_{22}\epsilon_{2134} +$$
$$a_{233}\delta_{22}\delta_{11}\epsilon_{1324} + a_{243}\delta_{22}\delta_{11}\epsilon_{1324} + a_{292}\delta_{11}\delta_{22}\epsilon_{1234} = 0$$
$$\Rightarrow -a_{122} - a_{123} + a_{130} - a_{183} - a_{184} - a_{193} - a_{233} - a_{243} + a_{292} = 0. \tag{58}$$

(137) (258) $$i_1 = 1,\, i_2 = 2,\, i_4 = 3,\, i_6 = 4,$$

$$i_3 = i_7 = 1, \;\; i_5 = i_8 = 2,$$

$$a_{122}\delta_{11}\delta_{22}\epsilon_{3412} + a_{125}\delta_{11}\delta_{22}\epsilon_{3241} + a_{132}\delta_{11}\delta_{22}\epsilon_{2341} +$$
$$a_{183}\delta_{11}\delta_{22}\epsilon_{1342} + a_{185}\delta_{11}\delta_{22}\epsilon_{1324} + a_{194}\delta_{11}\delta_{22}\epsilon_{2134} +$$
$$a_{233}\delta_{22}\delta_{11}\epsilon_{1342} + a_{264}\delta_{22}\delta_{11}\epsilon_{1324} + a_{293}\delta_{11}\delta_{22}\epsilon_{1234} = 0$$
$$\Rightarrow a_{122} + a_{125} - a_{132} + a_{183} - a_{185} - a_{194} + a_{233} - a_{264} + a_{293} = 0. \tag{59}$$

(137) (268) $$i_1 = 1, i_2 = 2,\, i_4 = 3,\, i_5 = 4,$$

$$i_3 = i_7 = 1, \;\; i_6 = i_8 = 2,$$

$$a_{123}\delta_{11}\delta_{22}\epsilon_{3412} + a_{125}\delta_{11}\delta_{22}\epsilon_{3421} + a_{134}\delta_{11}\delta_{22}\epsilon_{2341} +$$
$$a_{184}\delta_{22}\delta_{11}\epsilon_{1342} + a_{185}\delta_{11}\delta_{22}\epsilon_{1342} + a_{195}\delta_{11}\delta_{22}\epsilon_{2134} +$$
$$a_{243}\delta_{11}\delta_{22}\epsilon_{1342} + a_{264}\delta_{22}\delta_{11}\epsilon_{1342} + a_{294}\delta_{11}\delta_{22}\epsilon_{1234} = 0$$
$$\Rightarrow a_{123} - a_{125} - a_{134} + a_{184} + a_{185} - a_{195} + a_{243} + a_{264} + a_{294} = 0. \tag{60}$$

$$(138)\begin{pmatrix} 245 \\ 246 \\ 247 \\ 256 \\ 257 \\ 267 \end{pmatrix}$$

(138) (245) $$i_3 = 1,\ i_5 = 2,\ i_6 = 3,\ i_7 = 4,$$

$$i_1 = i_8 = 1,\ \ i_2 = i_4 = 2,$$

$$\begin{aligned}
&a_{121}\delta_{11}\delta_{22}\epsilon_{2341} + a_{122}\delta_{11}\delta_{22}\epsilon_{2341} + a_{126}\delta_{11}\delta_{22}\epsilon_{2341} + \\
&a_{197}\delta_{11}\delta_{22}\epsilon_{1234} + a_{198}\delta_{11}\delta_{22}\epsilon_{1234} + a_{205}\delta_{11}\delta_{22}\epsilon_{2134} + \\
&a_{224}\delta_{22}\delta_{11}\epsilon_{1234} + a_{234}\delta_{22}\delta_{11}\epsilon_{1234} + a_{295}\delta_{11}\delta_{22}\epsilon_{1234} = 0 \\
&\Rightarrow -a_{121} - a_{122} - a_{126} + a_{197} + a_{198} - a_{205} + a_{224} + a_{234} + a_{295} = 0.
\end{aligned} \tag{61}$$

(138) (246) $$i_3 = 1,\ i_4 = 2,\ i_5 = 3,\ i_7 = 4,$$

$$i_1 = i_8 = 1,\ \ i_2 = i_6 = 2,$$

$$\begin{aligned}
&a_{121}\delta_{11}\delta_{22}\epsilon_{3241} + a_{123}\delta_{11}\delta_{22}\epsilon_{2341} + a_{127}\delta_{11}\delta_{22}\epsilon_{2341} + \\
&a_{197}\delta_{11}\delta_{22}\epsilon_{1324} + a_{199}\delta_{11}\delta_{22}\epsilon_{1234} + a_{206}\delta_{11}\delta_{22}\epsilon_{2134} + \\
&a_{224}\delta_{22}\delta_{11}\epsilon_{1324} + a_{244}\delta_{22}\delta_{11}\epsilon_{1234} + a_{296}\delta_{11}\delta_{22}\epsilon_{1234} = 0 \\
&\Rightarrow a_{121} - a_{123} - a_{127} - a_{197} + a_{199} - a_{206} - a_{224} + a_{244} + a_{296} = 0.
\end{aligned} \tag{62}$$

(138) (247) $$i_3 = 1,\ i_4 = 2,\ i_5 = 3,\ i_6 = 4,$$

$$i_1 = i_8 = 1,\ \ i_2 = i_7 = 2,$$

$$\begin{aligned}
&a_{121}\delta_{11}\delta_{22}\epsilon_{3421} + a_{124}\delta_{11}\delta_{22}\epsilon_{2341} + a_{128}\delta_{11}\delta_{22}\epsilon_{2341} + \\
&a_{197}\delta_{11}\delta_{22}\epsilon_{1342} + a_{200}\delta_{11}\delta_{22}\epsilon_{1234} + a_{207}\delta_{11}\delta_{22}\epsilon_{2134} + \\
&a_{224}\delta_{22}\delta_{11}\epsilon_{1342} + a_{254}\delta_{22}\delta_{11}\epsilon_{1234} + a_{297}\delta_{11}\delta_{22}\epsilon_{1234} = 0 \\
&\Rightarrow -a_{121} - a_{124} - a_{128} + a_{197} + a_{200} - a_{207} + a_{224} + a_{254} + a_{297} = 0.
\end{aligned} \tag{63}$$

(138) (256) $$i_1 = 1, i_2 = 2,\ i_4 = 3,\ i_7 = 4,$$

$$i_3 = i_8 = 1, \;\; i_5 = i_6 = 2,$$

$$\begin{aligned} & a_{122}\delta_{11}\delta_{22}\epsilon_{3241} + a_{123}\delta_{11}\delta_{22}\epsilon_{3241} + a_{130}\delta_{11}\delta_{22}\epsilon_{2341} + \\ & a_{198}\delta_{11}\delta_{22}\epsilon_{1324} + a_{199}\delta_{11}\delta_{22}\epsilon_{1324} + a_{208}\delta_{11}\delta_{22}\epsilon_{2134} + \\ & a_{234}\delta_{22}\delta_{11}\epsilon_{1324} + a_{244}\delta_{22}\delta_{11}\epsilon_{1324} + a_{298}\delta_{11}\delta_{22}\epsilon_{1234} = 0 \\ & \Rightarrow a_{122} + a_{123} - a_{130} - a_{198} - a_{199} - a_{208} - a_{234} - a_{244} + a_{298} = 0. \end{aligned} \tag{64}$$

(138) (257) $$i_1 = 1, \; i_2 = 2, \; i_4 = 3, \; i_6 = 4,$$

$$i_3 = i_8 = 1, \;\; i_5 = i_7 = 2,$$

$$\begin{aligned} & a_{122}\delta_{11}\delta_{22}\epsilon_{3421} + a_{124}\delta_{11}\delta_{22}\epsilon_{3241} + a_{131}\delta_{11}\delta_{22}\epsilon_{2341} + \\ & a_{198}\delta_{11}\delta_{22}\epsilon_{1342} + a_{200}\delta_{11}\delta_{22}\epsilon_{1324} + a_{209}\delta_{11}\delta_{22}\epsilon_{2134} + \\ & a_{234}\delta_{22}\delta_{11}\epsilon_{1342} + a_{254}\delta_{22}\delta_{11}\epsilon_{1324} + a_{299}\delta_{11}\delta_{22}\epsilon_{1234} = 0 \\ & \Rightarrow -a_{122} + a_{124} - a_{131} + a_{198} - a_{200} - a_{209} + a_{234} - a_{254} + a_{299} = 0. \end{aligned} \tag{65}$$

(138) (267) $$i_1 = 1, i_2 = 2, \; i_4 = 3, \; i_5 = 4,$$

$$i_3 = i_8 = 1, \;\; i_6 = i_7 = 2,$$

$$\begin{aligned} & a_{123}\delta_{11}\delta_{22}\epsilon_{3421} + a_{124}\delta_{11}\delta_{22}\epsilon_{3421} + a_{133}\delta_{11}\delta_{22}\epsilon_{2341} + \\ & a_{199}\delta_{11}\delta_{22}\epsilon_{1342} + a_{200}\delta_{11}\delta_{22}\epsilon_{1342} + a_{210}\delta_{11}\delta_{22}\epsilon_{2134} + \\ & a_{244}\delta_{22}\delta_{11}\epsilon_{1342} + a_{254}\delta_{22}\delta_{11}\epsilon_{1342} + a_{300}\delta_{11}\delta_{22}\epsilon_{1234} = 0 \\ & \Rightarrow -a_{123} - a_{124} - a_{133} + a_{199} + a_{200} - a_{210} + a_{244} + a_{254} + a_{300} = 0. \end{aligned} \tag{66}$$

$$(145)\begin{pmatrix} 267 \\ 268 \\ 278 \end{pmatrix}$$

(145) (267) $$i_1 = 1, i_2 = 2, \; i_3 = 3, \; i_8 = 4,$$

$$i_4 = i_5 = 1,\;\; i_6 = i_7 = 2,$$

$$\begin{aligned}
&a_{138}\delta_{11}\delta_{22}\epsilon_{3124} + a_{139}\delta_{11}\delta_{22}\epsilon_{3124} + a_{148}\delta_{11}\delta_{22}\epsilon_{2314} + \\
&a_{153}\delta_{11}\delta_{22}\epsilon_{3124} + a_{154}\delta_{11}\delta_{22}\epsilon_{3124} + a_{163}\delta_{11}\delta_{22}\epsilon_{2314} + \\
&a_{245}\delta_{22}\delta_{11}\epsilon_{1324} + a_{255}\delta_{22}\delta_{11}\epsilon_{1324} + a_{301}\delta_{11}\delta_{22}\epsilon_{1234} = 0 \\
&\Rightarrow a_{138} + a_{139} + a_{148} + a_{153} + a_{154} + a_{163} - a_{245} - a_{255} + a_{301} = 0.
\end{aligned} \tag{67}$$

(145) (268) $\quad i_1 = 1,\, i_2 = 2,\, i_3 = 3,\, i_7 = 4,$

$$i_4 = i_5 = 1,\;\; i_6 = i_8 = 2,$$

$$\begin{aligned}
&a_{138}\delta_{11}\delta_{22}\epsilon_{3142} + a_{140}\delta_{11}\delta_{22}\epsilon_{3124} + a_{149}\delta_{11}\delta_{22}\epsilon_{2314} + \\
&a_{153}\delta_{11}\delta_{22}\epsilon_{3142} + a_{155}\delta_{11}\delta_{22}\epsilon_{3124} + a_{164}\delta_{11}\delta_{22}\epsilon_{2314} + \\
&a_{245}\delta_{22}\delta_{11}\epsilon_{1342} + a_{265}\delta_{22}\delta_{11}\epsilon_{1324} + a_{302}\delta_{11}\delta_{22}\epsilon_{1234} = 0 \\
&\Rightarrow -a_{138} + a_{140} + a_{149} - a_{153} + a_{155} + a_{164} + a_{245} - a_{265} + a_{302} = 0.
\end{aligned} \tag{68}$$

(145) (278) $\quad i_1 = 1, i_2 = 2,\, i_3 = 3,\, i_6 = 4,$

$$i_4 = i_5 = 1,\;\; i_7 = i_8 = 2,$$

$$\begin{aligned}
&a_{139}\delta_{11}\delta_{22}\epsilon_{3142} + a_{140}\delta_{11}\delta_{22}\epsilon_{3142} + a_{150}\delta_{11}\delta_{22}\epsilon_{2314} + \\
&a_{154}\delta_{11}\delta_{22}\epsilon_{3142} + a_{155}\delta_{11}\delta_{22}\epsilon_{3142} + a_{165}\delta_{11}\delta_{22}\epsilon_{2314} + \\
&a_{255}\delta_{22}\delta_{11}\epsilon_{1342} + a_{265}\delta_{22}\delta_{11}\epsilon_{1342} + a_{303}\delta_{11}\delta_{22}\epsilon_{1234} = 0 \\
&\Rightarrow -a_{139} - a_{140} + a_{150} - a_{154} - a_{155} + a_{165} + a_{255} + a_{265} + a_{303} = 0.
\end{aligned} \tag{69}$$

(146) $\begin{pmatrix} 257 \\ 258 \\ 278 \end{pmatrix}$

(146) (257) $\quad i_1 = 1, i_2 = 2,\, i_3 = 3,\, i_8 = 4,$

$$i_4 = i_6 = 1,\;\; i_5 = i_7 = 2,$$

$$\begin{aligned}
&a_{137}\delta_{11}\delta_{22}\epsilon_{3124} + a_{139}\delta_{11}\delta_{22}\epsilon_{3214} + a_{146}\delta_{11}\delta_{22}\epsilon_{2314} + \\
&a_{168}\delta_{11}\delta_{22}\epsilon_{3124} + a_{169}\delta_{11}\delta_{22}\epsilon_{3124} + a_{178}\delta_{11}\delta_{22}\epsilon_{2314} + \\
&a_{235}\delta_{22}\delta_{11}\epsilon_{1324} + a_{256}\delta_{22}\delta_{11}\epsilon_{1324} + a_{304}\delta_{11}\delta_{22}\epsilon_{1234} = 0 \\
&\Rightarrow a_{137} - a_{139} + a_{146} + a_{168} + a_{169} + a_{178} - a_{235} - a_{256} + a_{304} = 0.
\end{aligned} \tag{70}$$

(146) (258) $\quad i_1 = 1,\, i_2 = 2,\, i_3 = 3,\, i_7 = 4,$

$$i_4 = i_6 = 1,\;\; i_5 = i_8 = 2,$$

$$\begin{aligned}
&a_{137}\delta_{11}\delta_{22}\epsilon_{3142} + a_{140}\delta_{11}\delta_{22}\epsilon_{3214} + a_{147}\delta_{11}\delta_{22}\epsilon_{2314} + \\
&a_{168}\delta_{11}\delta_{22}\epsilon_{3142} + a_{170}\delta_{11}\delta_{22}\epsilon_{3124} + a_{179}\delta_{11}\delta_{22}\epsilon_{2314} + \\
&a_{235}\delta_{22}\delta_{11}\epsilon_{1342} + a_{266}\delta_{22}\delta_{11}\epsilon_{1324} + a_{305}\delta_{11}\delta_{22}\epsilon_{1234} = 0 \\
&\Rightarrow -a_{137} - a_{140} + a_{147} - a_{168} + a_{170} + a_{179} + a_{235} - a_{266} + a_{305} = 0.
\end{aligned} \tag{71}$$

(146) (278) $\quad i_1 = 1, i_2 = 2,\, i_4 = 3,\, i_5 = 4,$

$$i_4 = i_6 = 1,\;\; i_7 = i_8 = 2,$$

$$\begin{aligned}
&a_{139}\delta_{11}\delta_{22}\epsilon_{3412} + a_{140}\delta_{11}\delta_{22}\epsilon_{3412} + a_{150}\delta_{11}\delta_{22}\epsilon_{2341} + \\
&a_{169}\delta_{11}\delta_{22}\epsilon_{3142} + a_{170}\delta_{11}\delta_{22}\epsilon_{3142} + a_{180}\delta_{11}\delta_{22}\epsilon_{2314} + \\
&a_{256}\delta_{22}\delta_{11}\epsilon_{1342} + a_{266}\delta_{22}\delta_{11}\epsilon_{1342} + a_{306}\delta_{11}\delta_{22}\epsilon_{1234} = 0 \\
&\Rightarrow a_{139} + a_{140} - a_{150} - a_{169} - a_{170} + a_{180} + a_{256} + a_{266} + a_{306} = 0.
\end{aligned} \tag{72}$$

$$(147)\begin{pmatrix} 256 \\ 258 \\ 268 \end{pmatrix}$$

(147) (256) $\quad i_1 = 1, i_2 = 2,\, i_3 = 3,\, i_8 = 4,$

$$i_4 = i_7 = 1, \;\; i_5 = i_6 = 2,$$

$$a_{137}\delta_{11}\delta_{22}\epsilon_{3214} + a_{138}\delta_{11}\delta_{22}\epsilon_{3214} + a_{145}\delta_{11}\delta_{22}\epsilon_{2314} +$$
$$a_{183}\delta_{11}\delta_{22}\epsilon_{3124} + a_{184}\delta_{11}\delta_{22}\epsilon_{3124} + a_{193}\delta_{11}\delta_{22}\epsilon_{2314} +$$
$$a_{236}\delta_{22}\delta_{11}\epsilon_{1324} + a_{246}\delta_{22}\delta_{11}\epsilon_{1324} + a_{307}\delta_{11}\delta_{22}\epsilon_{1234} = 0$$
$$\Rightarrow -a_{137} - a_{138} + a_{145} + a_{183} + a_{184} + a_{193} - a_{236} - a_{246} + a_{307} = 0. \quad (73)$$

(147) (258) $$i_1 = 1, \, i_2 = 2, \, i_3 = 3, \, i_6 = 4,$$

$$i_4 = i_7 = 1, \;\; i_5 = i_8 = 2,$$

$$a_{137}\delta_{11}\delta_{22}\epsilon_{3412} + a_{140}\delta_{11}\delta_{22}\epsilon_{3241} + a_{147}\delta_{11}\delta_{22}\epsilon_{2341} +$$
$$a_{183}\delta_{11}\delta_{22}\epsilon_{3142} + a_{185}\delta_{11}\delta_{22}\epsilon_{3124} + a_{194}\delta_{11}\delta_{22}\epsilon_{2314} +$$
$$a_{236}\delta_{22}\delta_{11}\epsilon_{1342} + a_{267}\delta_{22}\delta_{11}\epsilon_{1324} + a_{308}\delta_{11}\delta_{22}\epsilon_{1234} = 0$$
$$\Rightarrow a_{137} + a_{140} - a_{147} - a_{183} + a_{185} + a_{194} + a_{236} - a_{267} + a_{308} = 0. \quad (74)$$

(147) (268) $$i_1 = 1, i_2 = 2, \, i_4 = 3, \, i_5 = 4,$$

$$i_4 = i_7 = 1, \;\; i_6 = i_8 = 2,$$

$$a_{138}\delta_{11}\delta_{22}\epsilon_{3412} + a_{140}\delta_{11}\delta_{22}\epsilon_{3421} + a_{149}\delta_{11}\delta_{22}\epsilon_{2341} +$$
$$a_{184}\delta_{11}\delta_{22}\epsilon_{3142} + a_{185}\delta_{11}\delta_{22}\epsilon_{3142} + a_{195}\delta_{11}\delta_{22}\epsilon_{2314} +$$
$$a_{246}\delta_{22}\delta_{11}\epsilon_{1342} + a_{267}\delta_{22}\delta_{11}\epsilon_{1342} + a_{309}\delta_{11}\delta_{22}\epsilon_{1234} = 0$$
$$\Rightarrow a_{138} - a_{140} - a_{149} - a_{184} - a_{185} + a_{195} + a_{246} + a_{267} + a_{309} = 0. \quad (75)$$

$$(148) \begin{pmatrix} 256 \\ 257 \\ 267 \end{pmatrix}$$

(148) (256) $$i_1 = 1, i_2 = 2, \, i_3 = 3, \, i_7 = 4,$$

$$i_4 = i_8 = 1, \;\; i_5 = i_6 = 2,$$

$$\begin{aligned}
&a_{137}\delta_{11}\delta_{22}\epsilon_{3241} + a_{138}\delta_{11}\delta_{22}\epsilon_{3241} + a_{145}\delta_{11}\delta_{22}\epsilon_{2341} + \\
&a_{198}\delta_{11}\delta_{22}\epsilon_{3124} + a_{199}\delta_{11}\delta_{22}\epsilon_{3124} + a_{208}\delta_{11}\delta_{22}\epsilon_{2314} + \\
&a_{237}\delta_{22}\delta_{11}\epsilon_{1324} + a_{247}\delta_{22}\delta_{11}\epsilon_{1324} + a_{310}\delta_{11}\delta_{22}\epsilon_{1234} = 0 \\
&\Rightarrow a_{137} + a_{138} - a_{145} + a_{198} + a_{199} + a_{208} - a_{237} - a_{247} + a_{310} = 0. \qquad (76)
\end{aligned}$$

(148) (257) $$i_1 = 1,\, i_2 = 2,\, i_3 = 3,\, i_6 = 4,$$

$$i_4 = i_8 = 1, \;\; i_5 = i_7 = 2,$$

$$\begin{aligned}
&a_{137}\delta_{11}\delta_{22}\epsilon_{3421} + a_{139}\delta_{11}\delta_{22}\epsilon_{3241} + a_{146}\delta_{11}\delta_{22}\epsilon_{2341} + \\
&a_{198}\delta_{11}\delta_{22}\epsilon_{3142} + a_{200}\delta_{11}\delta_{22}\epsilon_{3124} + a_{209}\delta_{11}\delta_{22}\epsilon_{2314} + \\
&a_{237}\delta_{22}\delta_{11}\epsilon_{1342} + a_{257}\delta_{22}\delta_{11}\epsilon_{1324} + a_{311}\delta_{11}\delta_{22}\epsilon_{1234} = 0 \\
&\Rightarrow -a_{137} + a_{139} - a_{146} - a_{198} + a_{200} + a_{209} + a_{237} - a_{257} + a_{311} = 0. \qquad (77)
\end{aligned}$$

(148) (267) $$i_1 = 1, i_2 = 2,\, i_3 = 3,\, i_5 = 4,$$

$$i_4 = i_8 = 1, \;\; i_6 = i_7 = 2,$$

$$\begin{aligned}
&a_{138}\delta_{11}\delta_{22}\epsilon_{3421} + a_{139}\delta_{11}\delta_{22}\epsilon_{3421} + a_{148}\delta_{11}\delta_{22}\epsilon_{2341} + \\
&a_{199}\delta_{11}\delta_{22}\epsilon_{3142} + a_{200}\delta_{11}\delta_{22}\epsilon_{3142} + a_{210}\delta_{11}\delta_{22}\epsilon_{2314} + \\
&a_{247}\delta_{22}\delta_{11}\epsilon_{1342} + a_{257}\delta_{22}\delta_{11}\epsilon_{1342} + a_{312}\delta_{11}\delta_{22}\epsilon_{1234} = 0 \\
&\Rightarrow -a_{138} - a_{139} - a_{148} - a_{199} - a_{200} + a_{210} + a_{247} + a_{257} + a_{312} = 0. \qquad (78)
\end{aligned}$$

(156) (278) $$i_1 = 1, i_2 = 2,\, i_3 = 3,\, i_4 = 4,$$

$$i_5 = i_6 = 1, \;\; i_7 = i_8 = 2,$$

$$\begin{aligned}
&a_{154}\delta_{11}\delta_{22}\epsilon_{3412} + a_{155}\delta_{11}\delta_{22}\epsilon_{3412} + a_{165}\delta_{11}\delta_{22}\epsilon_{2341} + \\
&a_{169}\delta_{11}\delta_{22}\epsilon_{3412} + a_{170}\delta_{11}\delta_{22}\epsilon_{3412} + a_{180}\delta_{11}\delta_{22}\epsilon_{2341} + \\
&a_{258}\delta_{22}\delta_{11}\epsilon_{1342} + a_{268}\delta_{22}\delta_{11}\epsilon_{1342} + a_{313}\delta_{11}\delta_{22}\epsilon_{1234} = 0 \\
&\Rightarrow a_{154} + a_{155} - a_{165} + a_{169} + a_{170} - a_{180} + a_{258} + a_{268} + a_{313} = 0.
\end{aligned} \tag{79}$$

(157) (268) $$i_1 = 1, i_2 = 2, \; i_3 = 3, \; i_4 = 4,$$

$$i_5 = i_7 = 1, \;\; i_6 = i_8 = 2,$$

$$\begin{aligned}
&a_{153}\delta_{11}\delta_{22}\epsilon_{3412} + a_{155}\delta_{11}\delta_{22}\epsilon_{3421} + a_{164}\delta_{11}\delta_{22}\epsilon_{2341} + \\
&a_{184}\delta_{11}\delta_{22}\epsilon_{3412} + a_{185}\delta_{11}\delta_{22}\epsilon_{3412} + a_{195}\delta_{11}\delta_{22}\epsilon_{2341} + \\
&a_{248}\delta_{22}\delta_{11}\epsilon_{1342} + a_{269}\delta_{22}\delta_{11}\epsilon_{1342} + a_{314}\delta_{11}\delta_{22}\epsilon_{1234} = 0 \\
&\Rightarrow a_{153} - a_{155} - a_{164} + a_{184} + a_{185} - a_{195} + a_{248} + a_{269} + a_{314} = 0.
\end{aligned} \tag{80}$$

(158) (267) $$i_1 = 1, i_2 = 2, \; i_3 = 3, \; i_4 = 4,$$

$$i_5 = i_8 = 1, \;\; i_6 = i_7 = 2,$$

$$\begin{aligned}
&a_{153}\delta_{11}\delta_{22}\epsilon_{3421} + a_{154}\delta_{11}\delta_{22}\epsilon_{3421} + a_{163}\delta_{11}\delta_{22}\epsilon_{2341} + \\
&a_{199}\delta_{11}\delta_{22}\epsilon_{3412} + a_{200}\delta_{11}\delta_{22}\epsilon_{3412} + a_{210}\delta_{11}\delta_{22}\epsilon_{2341} + \\
&a_{249}\delta_{22}\delta_{11}\epsilon_{1342} + a_{259}\delta_{22}\delta_{11}\epsilon_{1342} + a_{315}\delta_{11}\delta_{22}\epsilon_{1234} = 0 \\
&\Rightarrow -a_{153} - a_{154} - a_{163} + a_{199} + a_{200} - a_{210} + a_{249} + a_{259} + a_{315} = 0.
\end{aligned} \tag{81}$$

(167) (258) $$i_1 = 1, i_2 = 2, \; i_3 = 3, \; i_4 = 4,$$

$$i_6 = i_7 = 1, \;\; i_5 = i_8 = 2,$$

$$\begin{aligned}
&a_{168}\delta_{11}\delta_{22}\epsilon_{3412} + a_{170}\delta_{11}\delta_{22}\epsilon_{3421} + a_{179}\delta_{11}\delta_{22}\epsilon_{2341} + \\
&a_{183}\delta_{11}\delta_{22}\epsilon_{3412} + a_{185}\delta_{11}\delta_{22}\epsilon_{3421} + a_{194}\delta_{11}\delta_{22}\epsilon_{2341} + \\
&a_{238}\delta_{22}\delta_{11}\epsilon_{1342} + a_{270}\delta_{22}\delta_{11}\epsilon_{1342} + a_{315}\delta_{22}\delta_{11}\epsilon_{1234} = 0 \\
&\Rightarrow a_{168} - a_{170} - a_{179} + a_{183} - a_{185} - a_{194} + a_{238} + a_{270} + a_{315} = 0.
\end{aligned} \tag{82}$$

(168) (257) $$i_1 = 1, i_2 = 2, \; i_3 = 3, \; i_4 = 4,$$

$$i_6 = i_8 = 1, \;\; i_5 = i_7 = 2,$$

$$\begin{aligned}
&a_{168}\delta_{11}\delta_{22}\epsilon_{3421} + a_{169}\delta_{11}\delta_{22}\epsilon_{3421} + a_{178}\delta_{11}\delta_{22}\epsilon_{2341} + \\
&a_{198}\delta_{11}\delta_{22}\epsilon_{3412} + a_{200}\delta_{11}\delta_{22}\epsilon_{3421} + a_{209}\delta_{11}\delta_{22}\epsilon_{2341} + \\
&a_{239}\delta_{22}\delta_{11}\epsilon_{1342} + a_{260}\delta_{22}\delta_{11}\epsilon_{1342} + a_{314}\delta_{22}\delta_{11}\epsilon_{1234} = 0 \\
&\Rightarrow -a_{168} - a_{169} - a_{178} + a_{198} - a_{200} - a_{209} + a_{239} + a_{260} + a_{314} = 0.
\end{aligned} \tag{83}$$

(178) (256) $$i_1 = 1, i_2 = 2, \; i_3 = 3, \; i_4 = 4,$$

$$i_7 = i_8 = 1, \;\; i_5 = i_6 = 2,$$

$$\begin{aligned}
&a_{183}\delta_{11}\delta_{22}\epsilon_{3421} + a_{184}\delta_{11}\delta_{22}\epsilon_{3421} + a_{193}\delta_{11}\delta_{22}\epsilon_{2341} + \\
&a_{198}\delta_{11}\delta_{22}\epsilon_{3421} + a_{199}\delta_{11}\delta_{22}\epsilon_{3421} + a_{208}\delta_{11}\delta_{22}\epsilon_{2341} + \\
&a_{240}\delta_{22}\delta_{11}\epsilon_{1342} + a_{250}\delta_{22}\delta_{11}\epsilon_{1342} + a_{313}\delta_{22}\delta_{11}\epsilon_{1234} = 0 \\
&\Rightarrow -a_{183} - a_{184} - a_{193} - a_{198} - a_{199} - a_{208} + a_{240} + a_{250} + a_{313} = 0.
\end{aligned} \tag{84}$$

(167) (248) $$i_1 = 1, i_2 = 2, \; i_3 = 3, \; i_5 = 4,$$

$$i_6 = i_7 = 1, \;\; i_4 = i_8 = 2,$$

$$\begin{aligned}
&a_{167}\delta_{11}\delta_{22}\epsilon_{3412} + a_{170}\delta_{11}\delta_{22}\epsilon_{3241} + a_{177}\delta_{11}\delta_{22}\epsilon_{2341} + \\
&a_{182}\delta_{11}\delta_{22}\epsilon_{3412} + a_{185}\delta_{11}\delta_{22}\epsilon_{3241} + a_{192}\delta_{11}\delta_{22}\epsilon_{2341} + \\
&a_{228}\delta_{22}\delta_{11}\epsilon_{1342} + a_{270}\delta_{22}\delta_{11}\epsilon_{1324} + a_{312}\delta_{22}\delta_{11}\epsilon_{1234} = 0 \\
&\Rightarrow a_{167} + a_{170} - a_{177} + a_{182} + a_{185} - a_{192} + a_{228} - a_{270} + a_{312} = 0. \qquad (85)
\end{aligned}$$

(157) (248) $$i_1 = 1, i_2 = 2, \, i_3 = 3, \, i_6 = 4,$$

$$i_5 = i_7 = 1, \;\; i_4 = i_8 = 2,$$

$$\begin{aligned}
&a_{152}\delta_{11}\delta_{22}\epsilon_{3412} + a_{155}\delta_{11}\delta_{22}\epsilon_{3241} + a_{162}\delta_{11}\delta_{22}\epsilon_{2341} + \\
&a_{182}\delta_{11}\delta_{22}\epsilon_{3142} + a_{185}\delta_{11}\delta_{22}\epsilon_{3214} + a_{192}\delta_{11}\delta_{22}\epsilon_{2314} + \\
&a_{226}\delta_{22}\delta_{11}\epsilon_{1342} + a_{269}\delta_{22}\delta_{11}\epsilon_{1324} + a_{311}\delta_{22}\delta_{11}\epsilon_{1234} = 0 \\
&\Rightarrow a_{152} + a_{155} - a_{162} - a_{182} - a_{185} + a_{192} + a_{226} - a_{269} + a_{311} = 0. \qquad (86)
\end{aligned}$$

(156) (248) $$i_1 = 1, i_2 = 2, \, i_3 = 3, \, i_7 = 4,$$

$$i_5 = i_6 = 1, \;\; i_4 = i_8 = 2,$$

$$\begin{aligned}
&a_{152}\delta_{11}\delta_{22}\epsilon_{3142} + a_{155}\delta_{11}\delta_{22}\epsilon_{3214} + a_{162}\delta_{11}\delta_{22}\epsilon_{2314} + \\
&a_{167}\delta_{11}\delta_{22}\epsilon_{3142} + a_{170}\delta_{11}\delta_{22}\epsilon_{3214} + a_{177}\delta_{11}\delta_{22}\epsilon_{2314} + \\
&a_{225}\delta_{22}\delta_{11}\epsilon_{1342} + a_{268}\delta_{22}\delta_{11}\epsilon_{1324} + a_{310}\delta_{22}\delta_{11}\epsilon_{1234} = 0 \\
&\Rightarrow -a_{152} - a_{155} + a_{162} - a_{167} - a_{170} + a_{177} + a_{225} - a_{268} + a_{310} = 0. \qquad (87)
\end{aligned}$$

(168) (247) $$i_1 = 1, i_2 = 2, \, i_3 = 3, \, i_5 = 4,$$

$$i_6 = i_8 = 1, \;\; i_4 = i_7 = 2,$$

$$\begin{aligned}
&a_{167}\delta_{11}\delta_{22}\epsilon_{3421} + a_{169}\delta_{11}\delta_{22}\epsilon_{3241} + a_{176}\delta_{11}\delta_{22}\epsilon_{2341} + \\
&a_{197}\delta_{11}\delta_{22}\epsilon_{3412} + a_{200}\delta_{11}\delta_{22}\epsilon_{3241} + a_{207}\delta_{11}\delta_{22}\epsilon_{2341} + \\
&a_{229}\delta_{22}\delta_{11}\epsilon_{1342} + a_{260}\delta_{22}\delta_{11}\epsilon_{1324} + a_{309}\delta_{22}\delta_{11}\epsilon_{1234} = 0 \\
&\Rightarrow -a_{167} + a_{169} - a_{176} + a_{197} + a_{200} - a_{207} + a_{229} - a_{260} + a_{309} = 0.
\end{aligned} \tag{88}$$

(158) (247)
$$i_1 = 1, i_2 = 2, \; i_3 = 3, \; i_6 = 4,$$

$$i_5 = i_8 = 1, \;\; i_4 = i_7 = 2,$$

$$\begin{aligned}
&a_{152}\delta_{11}\delta_{22}\epsilon_{3421} + a_{154}\delta_{11}\delta_{22}\epsilon_{3241} + a_{161}\delta_{11}\delta_{22}\epsilon_{2341} + \\
&a_{197}\delta_{11}\delta_{22}\epsilon_{3142} + a_{200}\delta_{11}\delta_{22}\epsilon_{3214} + a_{207}\delta_{11}\delta_{22}\epsilon_{2314} + \\
&a_{227}\delta_{22}\delta_{11}\epsilon_{1342} + a_{259}\delta_{22}\delta_{11}\epsilon_{1324} + a_{308}\delta_{22}\delta_{11}\epsilon_{1234} = 0 \\
&\Rightarrow -a_{152} + a_{154} - a_{161} - a_{197} - a_{200} + a_{207} + a_{227} - a_{259} + a_{308} = 0.
\end{aligned} \tag{89}$$

(156) (247)
$$i_1 = 1, i_2 = 2, \; i_3 = 3, \; i_8 = 4,$$

$$i_5 = i_6 = 1, \;\; i_4 = i_7 = 2,$$

$$\begin{aligned}
&a_{152}\delta_{11}\delta_{22}\epsilon_{3124} + a_{154}\delta_{11}\delta_{22}\epsilon_{3214} + a_{161}\delta_{11}\delta_{22}\epsilon_{2314} + \\
&a_{167}\delta_{11}\delta_{22}\epsilon_{3124} + a_{169}\delta_{11}\delta_{22}\epsilon_{3214} + a_{176}\delta_{11}\delta_{22}\epsilon_{2314} + \\
&a_{225}\delta_{22}\delta_{11}\epsilon_{1324} + a_{258}\delta_{22}\delta_{11}\epsilon_{1324} + a_{307}\delta_{22}\delta_{11}\epsilon_{1234} = 0 \\
&\Rightarrow a_{152} - a_{154} + a_{161} + a_{167} - a_{169} + a_{176} - a_{225} - a_{258} + a_{307} = 0.
\end{aligned} \tag{90}$$

(178) (246)
$$i_1 = 1, i_2 = 2, \; i_3 = 3, \; i_5 = 4,$$

$$i_7 = i_8 = 1, \;\; i_4 = i_6 = 2,$$

$$\begin{aligned}
&a_{182}\delta_{11}\delta_{22}\epsilon_{3421} + a_{184}\delta_{11}\delta_{22}\epsilon_{3241} + a_{191}\delta_{11}\delta_{22}\epsilon_{2341} + \\
&a_{197}\delta_{11}\delta_{22}\epsilon_{3421} + a_{199}\delta_{11}\delta_{22}\epsilon_{3241} + a_{206}\delta_{11}\delta_{22}\epsilon_{2341} + \\
&a_{230}\delta_{22}\delta_{11}\epsilon_{1342} + a_{250}\delta_{22}\delta_{11}\epsilon_{1324} + a_{306}\delta_{22}\delta_{11}\epsilon_{1234} = 0 \\
&\Rightarrow -a_{182} + a_{184} - a_{191} - a_{197} + a_{199} - a_{206} + a_{230} - a_{250} + a_{306} = 0.
\end{aligned} \tag{91}$$

(158) (246) $$i_1 = 1, i_2 = 2, \; i_3 = 3, \; i_7 = 4,$$

$$i_5 = i_8 = 1, \;\; i_4 = i_6 = 2,$$

$$\begin{aligned}
&a_{152}\delta_{11}\delta_{22}\epsilon_{3241} + a_{153}\delta_{11}\delta_{22}\epsilon_{3241} + a_{160}\delta_{11}\delta_{22}\epsilon_{2341} + \\
&a_{197}\delta_{11}\delta_{22}\epsilon_{3124} + a_{199}\delta_{11}\delta_{22}\epsilon_{3214} + a_{206}\delta_{11}\delta_{22}\epsilon_{2314} + \\
&a_{227}\delta_{22}\delta_{11}\epsilon_{1324} + a_{249}\delta_{22}\delta_{11}\epsilon_{1324} + a_{305}\delta_{22}\delta_{11}\epsilon_{1234} = 0 \\
&\Rightarrow a_{152} + a_{153} - a_{160} + a_{197} - a_{199} + a_{206} - a_{227} - a_{249} + a_{305} = 0.
\end{aligned} \tag{92}$$

(157) (246) $$i_1 = 1, i_2 = 2, \; i_3 = 3, \; i_8 = 4,$$

$$i_5 = i_7 = 1, \;\; i_4 = i_6 = 2,$$

$$\begin{aligned}
&a_{152}\delta_{11}\delta_{22}\epsilon_{3214} + a_{153}\delta_{11}\delta_{22}\epsilon_{3214} + a_{160}\delta_{11}\delta_{22}\epsilon_{2314} + \\
&a_{182}\delta_{11}\delta_{22}\epsilon_{3124} + a_{184}\delta_{11}\delta_{22}\epsilon_{3214} + a_{191}\delta_{11}\delta_{22}\epsilon_{2314} + \\
&a_{226}\delta_{22}\delta_{11}\epsilon_{1324} + a_{248}\delta_{22}\delta_{11}\epsilon_{1324} + a_{304}\delta_{22}\delta_{11}\epsilon_{1234} = 0 \\
&\Rightarrow -a_{152} - a_{153} + a_{160} + a_{182} - a_{184} + a_{191} - a_{226} - a_{248} + a_{304} = 0.
\end{aligned} \tag{93}$$

(178) (245) $$i_1 = 1, i_2 = 2, \; i_3 = 3, \; i_6 = 4,$$

$$i_7 = i_8 = 1, \;\; i_4 = i_5 = 2,$$

$$\begin{aligned}
&a_{182}\delta_{11}\delta_{22}\epsilon_{3241} + a_{183}\delta_{11}\delta_{22}\epsilon_{3241} + a_{190}\delta_{11}\delta_{22}\epsilon_{2341} + \\
&a_{197}\delta_{11}\delta_{22}\epsilon_{3241} + a_{198}\delta_{11}\delta_{22}\epsilon_{3241} + a_{205}\delta_{11}\delta_{22}\epsilon_{2341} + \\
&a_{230}\delta_{22}\delta_{11}\epsilon_{1324} + a_{240}\delta_{22}\delta_{11}\epsilon_{1324} + a_{303}\delta_{22}\delta_{11}\epsilon_{1234} = 0 \\
&\Rightarrow a_{182} + a_{183} - a_{190} + a_{197} + a_{198} - a_{205} - a_{230} - a_{240} + a_{303} = 0. \qquad (94)
\end{aligned}$$

(168) (245) $$i_1 = 1, i_2 = 2, \; i_3 = 3, \; i_7 = 4,$$

$$i_6 = i_8 = 1, \;\; i_4 = i_5 = 2,$$

$$\begin{aligned}
&a_{167}\delta_{11}\delta_{22}\epsilon_{3241} + a_{168}\delta_{11}\delta_{22}\epsilon_{3241} + a_{175}\delta_{11}\delta_{22}\epsilon_{2341} + \\
&a_{197}\delta_{11}\delta_{22}\epsilon_{3214} + a_{198}\delta_{11}\delta_{22}\epsilon_{3214} + a_{205}\delta_{11}\delta_{22}\epsilon_{2314} + \\
&a_{229}\delta_{22}\delta_{11}\epsilon_{1324} + a_{239}\delta_{22}\delta_{11}\epsilon_{1324} + a_{302}\delta_{22}\delta_{11}\epsilon_{1234} = 0 \\
&\Rightarrow a_{167} + a_{168} - a_{175} - a_{197} - a_{198} + a_{205} - a_{229} - a_{239} + a_{302} = 0. \qquad (95)
\end{aligned}$$

(167) (245) $$i_1 = 1, i_2 = 2, \; i_3 = 3, \; i_8 = 4,$$

$$i_6 = i_7 = 1, \;\; i_4 = i_5 = 2,$$

$$\begin{aligned}
&a_{167}\delta_{11}\delta_{22}\epsilon_{3214} + a_{168}\delta_{11}\delta_{22}\epsilon_{3214} + a_{175}\delta_{11}\delta_{22}\epsilon_{2314} + \\
&a_{182}\delta_{11}\delta_{22}\epsilon_{3214} + a_{183}\delta_{11}\delta_{22}\epsilon_{3214} + a_{190}\delta_{11}\delta_{22}\epsilon_{2314} + \\
&a_{228}\delta_{22}\delta_{11}\epsilon_{1324} + a_{238}\delta_{22}\delta_{11}\epsilon_{1324} + a_{301}\delta_{22}\delta_{11}\epsilon_{1234} = 0 \\
&\Rightarrow -a_{167} - a_{168} + a_{175} - a_{182} - a_{183} + a_{190} - a_{228} - a_{238} + a_{301} = 0. \qquad (96)
\end{aligned}$$

(167) (238) $$i_1 = 1, i_2 = 2, \; i_4 = 3, \; i_5 = 4,$$

$$i_6 = i_7 = 1, \;\; i_3 = i_8 = 2,$$

$$a_{166}\delta_{11}\delta_{22}\epsilon_{3412} + a_{170}\delta_{11}\delta_{22}\epsilon_{2341} + a_{174}\delta_{11}\delta_{22}\epsilon_{2341} +$$
$$a_{181}\delta_{11}\delta_{22}\epsilon_{3412} + a_{185}\delta_{11}\delta_{22}\epsilon_{2341} + a_{189}\delta_{11}\delta_{22}\epsilon_{2341} +$$
$$a_{218}\delta_{22}\delta_{11}\epsilon_{1342} + a_{270}\delta_{22}\delta_{11}\epsilon_{1234} + a_{300}\delta_{22}\delta_{11}\epsilon_{1234} = 0$$
$$\Rightarrow a_{166} - a_{170} - a_{174} + a_{181} - a_{185} - a_{189} + a_{218} + a_{270} + a_{300} = 0. \tag{97}$$

(157) (238) $$i_1 = 1, i_2 = 2, \; i_4 = 3, \; i_6 = 4,$$

$$i_5 = i_7 = 1, \;\; i_3 = i_8 = 2,$$

$$a_{151}\delta_{11}\delta_{22}\epsilon_{3412} + a_{155}\delta_{11}\delta_{22}\epsilon_{2341} + a_{159}\delta_{11}\delta_{22}\epsilon_{2341} +$$
$$a_{181}\delta_{11}\delta_{22}\epsilon_{3142} + a_{185}\delta_{11}\delta_{22}\epsilon_{2314} + a_{189}\delta_{11}\delta_{22}\epsilon_{2314} +$$
$$a_{216}\delta_{22}\delta_{11}\epsilon_{1342} + a_{269}\delta_{22}\delta_{11}\epsilon_{1234} + a_{299}\delta_{22}\delta_{11}\epsilon_{1234} = 0$$
$$\Rightarrow a_{151} - a_{155} - a_{159} - a_{181} + a_{185} + a_{189} + a_{216} + a_{269} + a_{299} = 0. \tag{98}$$

(156) (238) $$i_1 = 1, i_2 = 2, \; i_4 = 3, \; i_7 = 4,$$

$$i_5 = i_6 = 1, \;\; i_3 = i_8 = 2,$$

$$a_{151}\delta_{11}\delta_{22}\epsilon_{3142} + a_{155}\delta_{11}\delta_{22}\epsilon_{2314} + a_{159}\delta_{11}\delta_{22}\epsilon_{2314} +$$
$$a_{166}\delta_{11}\delta_{22}\epsilon_{3142} + a_{170}\delta_{11}\delta_{22}\epsilon_{2314} + a_{174}\delta_{11}\delta_{22}\epsilon_{2314} +$$
$$a_{215}\delta_{22}\delta_{11}\epsilon_{1342} + a_{268}\delta_{22}\delta_{11}\epsilon_{1234} + a_{298}\delta_{22}\delta_{11}\epsilon_{1234} = 0$$
$$\Rightarrow -a_{151} + a_{155} + a_{159} - a_{166} + a_{170} + a_{174} + a_{215} + a_{268} + a_{298} = 0. \tag{99}$$

(147) (238) $$i_1 = 1, i_2 = 2, \; i_5 = 3, \; i_6 = 4,$$

$$i_4 = i_7 = 1, \;\; i_3 = i_8 = 2,$$

$$\begin{aligned}
&a_{136}\delta_{11}\delta_{22}\epsilon_{3412} + a_{140}\delta_{11}\delta_{22}\epsilon_{2341} + a_{144}\delta_{11}\delta_{22}\epsilon_{2341} + \\
&a_{181}\delta_{11}\delta_{22}\epsilon_{1342} + a_{185}\delta_{11}\delta_{22}\epsilon_{2134} + a_{189}\delta_{11}\delta_{22}\epsilon_{2134} + \\
&a_{213}\delta_{22}\delta_{11}\epsilon_{1342} + a_{267}\delta_{22}\delta_{11}\epsilon_{1234} + a_{297}\delta_{22}\delta_{11}\epsilon_{1234} = 0 \\
&\Rightarrow a_{136} - a_{140} - a_{144} + a_{181} - a_{185} - a_{189} + a_{213} + a_{267} + a_{297} = 0.
\end{aligned} \tag{100}$$

(146) (238) $$i_1 = 1, i_2 = 2, \, i_5 = 3, \, i_7 = 4,$$

$$i_4 = i_6 = 1, \;\; i_3 = i_8 = 2,$$

$$\begin{aligned}
&a_{136}\delta_{11}\delta_{22}\epsilon_{3142} + a_{140}\delta_{11}\delta_{22}\epsilon_{2314} + a_{144}\delta_{11}\delta_{22}\epsilon_{2314} + \\
&a_{166}\delta_{11}\delta_{22}\epsilon_{1342} + a_{170}\delta_{11}\delta_{22}\epsilon_{2134} + a_{174}\delta_{11}\delta_{22}\epsilon_{2134} + \\
&a_{212}\delta_{22}\delta_{11}\epsilon_{1342} + a_{266}\delta_{22}\delta_{11}\epsilon_{1234} + a_{296}\delta_{22}\delta_{11}\epsilon_{1234} = 0 \\
&\Rightarrow -a_{136} + a_{140} + a_{144} + a_{166} - a_{170} - a_{174} + a_{212} + a_{266} + a_{296} = 0.
\end{aligned} \tag{101}$$

(145) (238) $$i_1 = 1, i_2 = 2, \, i_6 = 3, \, i_7 = 4,$$

$$i_4 = i_5 = 1, \;\; i_3 = i_8 = 2,$$

$$\begin{aligned}
&a_{136}\delta_{11}\delta_{22}\epsilon_{1342} + a_{140}\delta_{11}\delta_{22}\epsilon_{2134} + a_{144}\delta_{11}\delta_{22}\epsilon_{2134} + \\
&a_{151}\delta_{11}\delta_{22}\epsilon_{1342} + a_{155}\delta_{11}\delta_{22}\epsilon_{2134} + a_{159}\delta_{11}\delta_{22}\epsilon_{2134} + \\
&a_{211}\delta_{22}\delta_{11}\epsilon_{1342} + a_{265}\delta_{22}\delta_{11}\epsilon_{1234} + a_{295}\delta_{22}\delta_{11}\epsilon_{1234} = 0 \\
&\Rightarrow a_{136} - a_{140} - a_{144} + a_{151} - a_{155} - a_{159} + a_{211} + a_{265} + a_{295} = 0
\end{aligned} \tag{102}$$

(168) (237) $$i_1 = 1, i_2 = 2, \, i_4 = 3, \, i_5 = 4,$$

$$i_6 = i_8 = 1, \;\; i_3 = i_7 = 2,$$

$$\begin{aligned}
&a_{166}\delta_{11}\delta_{22}\epsilon_{3421} + a_{169}\delta_{11}\delta_{22}\epsilon_{2341} + a_{173}\delta_{11}\delta_{22}\epsilon_{2341} + \\
&a_{196}\delta_{11}\delta_{22}\epsilon_{3412} + a_{200}\delta_{11}\delta_{22}\epsilon_{2341} + a_{204}\delta_{11}\delta_{22}\epsilon_{2341} + \\
&a_{219}\delta_{22}\delta_{11}\epsilon_{1342} + a_{260}\delta_{22}\delta_{11}\epsilon_{1234} + a_{294}\delta_{22}\delta_{11}\epsilon_{1234} = 0 \\
&\Rightarrow -a_{166} - a_{169} - a_{173} + a_{196} - a_{200} - a_{204} + a_{219} + a_{260} + a_{294} = 0
\end{aligned} \tag{103}$$

(158) (237)

$$i_1 = 1, i_2 = 2, \, i_4 = 3, \, i_6 = 4,$$

$$i_5 = i_8 = 1, \;\; i_3 = i_7 = 2,$$

$$\begin{aligned}
&a_{151}\delta_{11}\delta_{22}\epsilon_{3421} + a_{154}\delta_{11}\delta_{22}\epsilon_{2341} + a_{158}\delta_{11}\delta_{22}\epsilon_{2341} + \\
&a_{196}\delta_{11}\delta_{22}\epsilon_{3142} + a_{200}\delta_{11}\delta_{22}\epsilon_{2314} + a_{204}\delta_{11}\delta_{22}\epsilon_{2314} + \\
&a_{217}\delta_{22}\delta_{11}\epsilon_{1342} + a_{259}\delta_{22}\delta_{11}\epsilon_{1234} + a_{293}\delta_{22}\delta_{11}\epsilon_{1234} = 0 \\
&\Rightarrow -a_{151} - a_{154} - a_{158} - a_{196} + a_{200} + a_{204} + a_{217} + a_{259} + a_{293} = 0.
\end{aligned} \tag{104}$$

(156) (237)

$$i_1 = 1, i_2 = 2, \, i_4 = 3, \, i_8 = 4,$$

$$i_5 = i_6 = 1, \;\; i_3 = i_7 = 2,$$

$$\begin{aligned}
&a_{151}\delta_{11}\delta_{22}\epsilon_{3124} + a_{154}\delta_{11}\delta_{22}\epsilon_{2314} + a_{158}\delta_{11}\delta_{22}\epsilon_{2314} + \\
&a_{166}\delta_{11}\delta_{22}\epsilon_{3124} + a_{169}\delta_{11}\delta_{22}\epsilon_{2314} + a_{173}\delta_{11}\delta_{22}\epsilon_{2314} + \\
&a_{215}\delta_{22}\delta_{11}\epsilon_{1324} + a_{258}\delta_{22}\delta_{11}\epsilon_{1234} + a_{292}\delta_{22}\delta_{11}\epsilon_{1234} = 0 \\
&\Rightarrow a_{151} + a_{154} + a_{158} + a_{166} + a_{169} + a_{173} - a_{215} + a_{258} + a_{292} = 0.
\end{aligned} \tag{105}$$

(148) (237)

$$i_1 = 1, i_2 = 2, \, i_5 = 3, \, i_6 = 4,$$

$$i_4 = i_8 = 1, \;\; i_3 = i_7 = 2,$$

$$a_{136}\delta_{11}\delta_{22}\epsilon_{3421} + a_{139}\delta_{11}\delta_{22}\epsilon_{2341} + a_{143}\delta_{11}\delta_{22}\epsilon_{2341} +$$
$$a_{196}\delta_{11}\delta_{22}\epsilon_{1342} + a_{200}\delta_{11}\delta_{22}\epsilon_{2134} + a_{204}\delta_{11}\delta_{22}\epsilon_{2134} +$$
$$a_{214}\delta_{22}\delta_{11}\epsilon_{1342} + a_{257}\delta_{22}\delta_{11}\epsilon_{1234} + a_{291}\delta_{22}\delta_{11}\epsilon_{1234} = 0$$
$$\Rightarrow -a_{136} - a_{139} - a_{143} + a_{196} - a_{200} - a_{204} + a_{214} + a_{257} + a_{291} = 0. \tag{106}$$

(146) (237) $$i_1 = 1, i_2 = 2, \; i_5 = 3, \; i_8 = 4,$$

$$i_4 = i_6 = 1, \;\; i_3 = i_7 = 2,$$

$$a_{136}\delta_{11}\delta_{22}\epsilon_{3124} + a_{139}\delta_{11}\delta_{22}\epsilon_{2314} + a_{143}\delta_{11}\delta_{22}\epsilon_{2314} +$$
$$a_{166}\delta_{11}\delta_{22}\epsilon_{1324} + a_{169}\delta_{11}\delta_{22}\epsilon_{2134} + a_{173}\delta_{11}\delta_{22}\epsilon_{2134} +$$
$$a_{212}\delta_{22}\delta_{11}\epsilon_{1324} + a_{256}\delta_{22}\delta_{11}\epsilon_{1234} + a_{290}\delta_{22}\delta_{11}\epsilon_{1234} = 0$$
$$\Rightarrow a_{136} + a_{139} + a_{143} - a_{166} - a_{169} - a_{173} - a_{212} + a_{256} + a_{290} = 0. \tag{107}$$

(145) (237) $$i_1 = 1, i_2 = 2, \; i_6 = 3, \; i_8 = 4,$$

$$i_4 = i_5 = 1, \;\; i_3 = i_7 = 2,$$

$$a_{136}\delta_{11}\delta_{22}\epsilon_{1324} + a_{139}\delta_{11}\delta_{22}\epsilon_{2134} + a_{143}\delta_{11}\delta_{22}\epsilon_{2134} +$$
$$a_{151}\delta_{11}\delta_{22}\epsilon_{1324} + a_{154}\delta_{11}\delta_{22}\epsilon_{2134} + a_{158}\delta_{11}\delta_{22}\epsilon_{2134} +$$
$$a_{211}\delta_{22}\delta_{11}\epsilon_{1324} + a_{255}\delta_{22}\delta_{11}\epsilon_{1234} + a_{289}\delta_{22}\delta_{11}\epsilon_{1234} = 0$$
$$\Rightarrow -a_{136} - a_{139} - a_{143} - a_{151} - a_{154} - a_{158} - a_{211} + a_{255} + a_{289} = 0. \tag{108}$$

(178) (236) $$i_1 = 1, i_2 = 2, \; i_4 = 3, \; i_5 = 4,$$

$$i_7 = i_8 = 1, \;\; i_3 = i_6 = 2,$$

$$\begin{aligned}
& a_{181}\delta_{11}\delta_{22}\epsilon_{3421} + a_{184}\delta_{11}\delta_{22}\epsilon_{2341} + a_{188}\delta_{11}\delta_{22}\epsilon_{2341} + \\
& a_{196}\delta_{11}\delta_{22}\epsilon_{3421} + a_{199}\delta_{11}\delta_{22}\epsilon_{2341} + a_{203}\delta_{11}\delta_{22}\epsilon_{2341} + \\
& a_{220}\delta_{22}\delta_{11}\epsilon_{1342} + a_{250}\delta_{22}\delta_{11}\epsilon_{1234} + a_{288}\delta_{22}\delta_{11}\epsilon_{1234} = 0 \\
& \Rightarrow -a_{181} - a_{184} - a_{188} - a_{196} - a_{199} - a_{203} + a_{220} + a_{250} + a_{288} = 0.
\end{aligned} \tag{109}$$

(158) (236) $$i_1 = 1, i_2 = 2, \; i_4 = 3, \; i_7 = 4,$$

$$i_5 = i_8 = 1, \;\; i_3 = i_6 = 2,$$

$$\begin{aligned}
& a_{151}\delta_{11}\delta_{22}\epsilon_{3241} + a_{153}\delta_{11}\delta_{22}\epsilon_{2341} + a_{159}\delta_{11}\delta_{22}\epsilon_{2341} + \\
& a_{196}\delta_{11}\delta_{22}\epsilon_{3124} + a_{199}\delta_{11}\delta_{22}\epsilon_{2314} + a_{203}\delta_{11}\delta_{22}\epsilon_{2314} + \\
& a_{217}\delta_{22}\delta_{11}\epsilon_{1324} + a_{249}\delta_{22}\delta_{11}\epsilon_{1234} + a_{287}\delta_{22}\delta_{11}\epsilon_{1234} = 0 \\
& \Rightarrow a_{151} - a_{153} - a_{159} + a_{196} + a_{199} + a_{203} - a_{217} + a_{249} + a_{287} = 0.
\end{aligned} \tag{110}$$

(157) (236) $$i_1 = 1, i_2 = 2, \; i_4 = 3, \; i_8 = 4,$$

$$i_5 = i_7 = 1, \;\; i_3 = i_6 = 2,$$

$$\begin{aligned}
& a_{151}\delta_{11}\delta_{22}\epsilon_{3214} + a_{153}\delta_{11}\delta_{22}\epsilon_{2314} + a_{157}\delta_{11}\delta_{22}\epsilon_{2314} + \\
& a_{181}\delta_{11}\delta_{22}\epsilon_{3124} + a_{184}\delta_{11}\delta_{22}\epsilon_{2314} + a_{188}\delta_{11}\delta_{22}\epsilon_{2314} + \\
& a_{216}\delta_{22}\delta_{11}\epsilon_{1324} + a_{248}\delta_{22}\delta_{11}\epsilon_{1234} + a_{286}\delta_{22}\delta_{11}\epsilon_{1234} = 0 \\
& \Rightarrow -a_{151} + a_{153} + a_{157} + a_{181} + a_{184} + a_{188} - a_{216} + a_{248} + a_{286} = 0.
\end{aligned} \tag{111}$$

(148) (236) $$i_1 = 1, i_2 = 2, \; i_5 = 3, \; i_7 = 4,$$

$$i_4 = i_8 = 1, \;\; i_3 = i_6 = 2,$$

$$\begin{aligned}
&a_{136}\delta_{11}\delta_{22}\epsilon_{3241} + a_{138}\delta_{11}\delta_{22}\epsilon_{2341} + a_{142}\delta_{11}\delta_{22}\epsilon_{2341} + \\
&a_{196}\delta_{11}\delta_{22}\epsilon_{1324} + a_{199}\delta_{11}\delta_{22}\epsilon_{2134} + a_{203}\delta_{11}\delta_{22}\epsilon_{2134} + \\
&a_{214}\delta_{22}\delta_{11}\epsilon_{1324} + a_{247}\delta_{22}\delta_{11}\epsilon_{1234} + a_{285}\delta_{22}\delta_{11}\epsilon_{1234} = 0 \\
&\Rightarrow a_{136} - a_{138} - a_{142} - a_{196} - a_{199} - a_{203} - a_{214} + a_{247} + a_{285} = 0.
\end{aligned} \tag{112}$$

(147) (236) $$i_1 = 1, i_2 = 2, \, i_5 = 3, \, i_8 = 4,$$

$$i_4 = i_7 = 1, \;\; i_3 = i_6 = 2,$$

$$\begin{aligned}
&a_{136}\delta_{11}\delta_{22}\epsilon_{3214} + a_{138}\delta_{11}\delta_{22}\epsilon_{2314} + a_{142}\delta_{11}\delta_{22}\epsilon_{2314} + \\
&a_{181}\delta_{11}\delta_{22}\epsilon_{1324} + a_{184}\delta_{11}\delta_{22}\epsilon_{2134} + a_{188}\delta_{11}\delta_{22}\epsilon_{2134} + \\
&a_{213}\delta_{22}\delta_{11}\epsilon_{1324} + a_{246}\delta_{22}\delta_{11}\epsilon_{1234} + a_{284}\delta_{22}\delta_{11}\epsilon_{1234} = 0 \\
&\Rightarrow -a_{136} + a_{138} + a_{142} - a_{181} - a_{184} - a_{188} - a_{213} + a_{246} + a_{284} = 0.
\end{aligned} \tag{113}$$

(145) (236) $$i_1 = 1, i_2 = 2, \, i_7 = 3, \, i_8 = 4,$$

$$i_4 = i_5 = 1, \;\; i_3 = i_6 = 2,$$

$$\begin{aligned}
&a_{136}\delta_{11}\delta_{22}\epsilon_{1234} + a_{138}\delta_{11}\delta_{22}\epsilon_{2134} + a_{142}\delta_{11}\delta_{22}\epsilon_{2134} + \\
&a_{151}\delta_{11}\delta_{22}\epsilon_{1234} + a_{153}\delta_{11}\delta_{22}\epsilon_{2134} + a_{157}\delta_{11}\delta_{22}\epsilon_{2134} + \\
&a_{211}\delta_{22}\delta_{11}\epsilon_{1234} + a_{245}\delta_{22}\delta_{11}\epsilon_{1234} + a_{283}\delta_{22}\delta_{11}\epsilon_{1234} = 0 \\
&\Rightarrow a_{136} - a_{138} - a_{142} + a_{151} - a_{153} - a_{157} + a_{211} + a_{245} + a_{283} = 0.
\end{aligned} \tag{114}$$

(178) (235) $$i_1 = 1, i_2 = 2, \, i_4 = 3, \, i_6 = 4,$$

$$i_7 = i_8 = 1,\;\; i_3 = i_5 = 2,$$

$$\begin{aligned}
&a_{181}\delta_{11}\delta_{22}\epsilon_{3241} + a_{183}\delta_{11}\delta_{22}\epsilon_{2341} + a_{187}\delta_{11}\delta_{22}\epsilon_{2341} + \\
&a_{196}\delta_{11}\delta_{22}\epsilon_{3241} + a_{198}\delta_{11}\delta_{22}\epsilon_{2341} + a_{202}\delta_{11}\delta_{22}\epsilon_{2341} + \\
&a_{220}\delta_{22}\delta_{11}\epsilon_{1324} + a_{240}\delta_{22}\delta_{11}\epsilon_{1234} + a_{282}\delta_{22}\delta_{11}\epsilon_{1234} = 0 \\
&\Rightarrow a_{181} - a_{183} - a_{187} + a_{196} - a_{198} - a_{202} - a_{220} + a_{240} + a_{282} = 0.
\end{aligned} \tag{115}$$

(168) (235) $$i_1 = 1, i_2 = 2,\; i_4 = 3,\; i_7 = 4,$$

$$i_6 = i_8 = 1,\;\; i_3 = i_5 = 2,$$

$$\begin{aligned}
&a_{166}\delta_{11}\delta_{22}\epsilon_{3241} + a_{168}\delta_{11}\delta_{22}\epsilon_{2341} + a_{172}\delta_{11}\delta_{22}\epsilon_{2341} + \\
&a_{196}\delta_{11}\delta_{22}\epsilon_{3214} + a_{198}\delta_{11}\delta_{22}\epsilon_{2314} + a_{202}\delta_{11}\delta_{22}\epsilon_{2314} + \\
&a_{219}\delta_{22}\delta_{11}\epsilon_{1324} + a_{239}\delta_{22}\delta_{11}\epsilon_{1234} + a_{281}\delta_{22}\delta_{11}\epsilon_{1234} = 0 \\
&\Rightarrow a_{166} - a_{168} - a_{172} - a_{196} + a_{198} + a_{202} - a_{219} + a_{239} + a_{281} = 0.
\end{aligned} \tag{116}$$

(167) (235) $$i_1 = 1, i_2 = 2,\; i_4 = 3,\; i_8 = 4,$$

$$i_6 = i_7 = 1,\;\; i_3 = i_5 = 2,$$

$$\begin{aligned}
&a_{166}\delta_{11}\delta_{22}\epsilon_{3214} + a_{168}\delta_{11}\delta_{22}\epsilon_{2314} + a_{172}\delta_{11}\delta_{22}\epsilon_{2314} + \\
&a_{181}\delta_{11}\delta_{22}\epsilon_{3214} + a_{183}\delta_{11}\delta_{22}\epsilon_{2314} + a_{187}\delta_{11}\delta_{22}\epsilon_{2314} + \\
&a_{218}\delta_{22}\delta_{11}\epsilon_{1324} + a_{238}\delta_{22}\delta_{11}\epsilon_{1234} + a_{280}\delta_{22}\delta_{11}\epsilon_{1234} = 0 \\
&\Rightarrow -a_{166} + a_{168} + a_{172} - a_{181} + a_{183} + a_{187} - a_{218} + a_{238} + a_{280} = 0.
\end{aligned} \tag{117}$$

(148) (235) $$i_1 = 1, i_2 = 2,\; i_6 = 3,\; i_7 = 4,$$

$$i_4 = i_8 = 1, \;\; i_3 = i_5 = 2,$$

$$\begin{aligned}
&a_{136}\delta_{11}\delta_{22}\epsilon_{2341} + a_{137}\delta_{11}\delta_{22}\epsilon_{2341} + a_{141}\delta_{11}\delta_{22}\epsilon_{2341} + \\
&a_{196}\delta_{11}\delta_{22}\epsilon_{1234} + a_{198}\delta_{11}\delta_{22}\epsilon_{2134} + a_{202}\delta_{11}\delta_{22}\epsilon_{2134} + \\
&a_{214}\delta_{22}\delta_{11}\epsilon_{1234} + a_{237}\delta_{22}\delta_{11}\epsilon_{1234} + a_{279}\delta_{22}\delta_{11}\epsilon_{1234} = 0 \\
&\Rightarrow -a_{136} - a_{137} - a_{141} + a_{196} - a_{198} - a_{202} + a_{214} + a_{237} + a_{279} = 0. \qquad (118)
\end{aligned}$$

(147) (235) $$i_1 = 1, i_2 = 2, \; i_6 = 3, \; i_8 = 4,$$

$$i_4 = i_7 = 1, \;\; i_3 = i_5 = 2,$$

$$\begin{aligned}
&a_{136}\delta_{11}\delta_{22}\epsilon_{2314} + a_{137}\delta_{11}\delta_{22}\epsilon_{2314} + a_{141}\delta_{11}\delta_{22}\epsilon_{2314} + \\
&a_{181}\delta_{11}\delta_{22}\epsilon_{1234} + a_{183}\delta_{11}\delta_{22}\epsilon_{2134} + a_{187}\delta_{11}\delta_{22}\epsilon_{2134} + \\
&a_{213}\delta_{22}\delta_{11}\epsilon_{1234} + a_{236}\delta_{22}\delta_{11}\epsilon_{1234} + a_{278}\delta_{22}\delta_{11}\epsilon_{1234} = 0 \\
&\Rightarrow a_{136} + a_{137} + a_{141} + a_{181} - a_{183} - a_{187} + a_{213} + a_{236} + a_{278} = 0. \qquad (119)
\end{aligned}$$

(146) (235) $$i_1 = 1, i_2 = 2, \; i_7 = 3, \; i_8 = 4,$$

$$i_4 = i_6 = 1, \;\; i_3 = i_5 = 2,$$

$$\begin{aligned}
&a_{136}\delta_{11}\delta_{22}\epsilon_{2134} + a_{137}\delta_{11}\delta_{22}\epsilon_{2134} + a_{141}\delta_{11}\delta_{22}\epsilon_{2134} + \\
&a_{166}\delta_{11}\delta_{22}\epsilon_{1234} + a_{168}\delta_{11}\delta_{22}\epsilon_{2134} + a_{172}\delta_{11}\delta_{22}\epsilon_{2134} + \\
&a_{212}\delta_{22}\delta_{11}\epsilon_{1234} + a_{235}\delta_{22}\delta_{11}\epsilon_{1234} + a_{277}\delta_{22}\delta_{11}\epsilon_{1234} = 0 \\
&\Rightarrow -a_{136} - a_{137} - a_{141} + a_{166} - a_{168} - a_{172} + a_{212} + a_{235} + a_{277} = 0. \qquad (120)
\end{aligned}$$

(178) (234) $$i_1 = 1, i_2 = 2, \; i_5 = 3, \; i_6 = 4,$$

$$i_7 = i_8 = 1, \;\; i_3 = i_4 = 2,$$

$$\begin{aligned}
&a_{181}\delta_{11}\delta_{22}\epsilon_{2341} + a_{182}\delta_{11}\delta_{22}\epsilon_{2341} + a_{186}\delta_{11}\delta_{22}\epsilon_{2341} + \\
&a_{196}\delta_{11}\delta_{22}\epsilon_{2341} + a_{197}\delta_{11}\delta_{22}\epsilon_{2341} + a_{201}\delta_{11}\delta_{22}\epsilon_{2341} + \\
&a_{220}\delta_{22}\delta_{11}\epsilon_{1234} + a_{230}\delta_{22}\delta_{11}\epsilon_{1234} + a_{276}\delta_{22}\delta_{11}\epsilon_{1234} = 0 \\
&\Rightarrow -a_{181} - a_{182} - a_{186} - a_{196} - a_{197} - a_{201} + a_{220} + a_{230} + a_{276} = 0. \qquad (121)
\end{aligned}$$

(168) (234) $$i_1 = 1, i_2 = 2, \; i_5 = 3, \; i_7 = 4,$$

$$i_6 = i_8 = 1, \;\; i_3 = i_4 = 2,$$

$$\begin{aligned}
&a_{166}\delta_{11}\delta_{22}\epsilon_{2341} + a_{167}\delta_{11}\delta_{22}\epsilon_{2341} + a_{171}\delta_{11}\delta_{22}\epsilon_{2341} + \\
&a_{196}\delta_{11}\delta_{22}\epsilon_{2314} + a_{197}\delta_{11}\delta_{22}\epsilon_{2314} + a_{201}\delta_{11}\delta_{22}\epsilon_{2314} + \\
&a_{219}\delta_{22}\delta_{11}\epsilon_{1234} + a_{229}\delta_{22}\delta_{11}\epsilon_{1234} + a_{275}\delta_{22}\delta_{11}\epsilon_{1234} = 0 \\
&\Rightarrow -a_{166} - a_{167} - a_{171} + a_{196} + a_{197} + a_{201} + a_{219} + a_{229} + a_{275} = 0. \qquad (122)
\end{aligned}$$

(167) (234) $$i_1 = 1, i_2 = 2, \; i_5 = 3, \; i_8 = 4,$$

$$i_6 = i_7 = 1, \;\; i_3 = i_4 = 2,$$

$$\begin{aligned}
&a_{166}\delta_{11}\delta_{22}\epsilon_{2314} + a_{167}\delta_{11}\delta_{22}\epsilon_{2314} + a_{171}\delta_{11}\delta_{22}\epsilon_{2314} + \\
&a_{181}\delta_{11}\delta_{22}\epsilon_{2314} + a_{182}\delta_{11}\delta_{22}\epsilon_{2314} + a_{186}\delta_{11}\delta_{22}\epsilon_{2314} + \\
&a_{218}\delta_{22}\delta_{11}\epsilon_{1234} + a_{228}\delta_{22}\delta_{11}\epsilon_{1234} + a_{274}\delta_{22}\delta_{11}\epsilon_{1234} = 0 \\
&\Rightarrow a_{166} + a_{167} + a_{171} + a_{181} + a_{182} + a_{186} + a_{218} + a_{228} + a_{274} = 0. \qquad (123)
\end{aligned}$$

(158) (234) $$i_1 = 1, i_2 = 2, \; i_6 = 3, \; i_7 = 4,$$

$$i_5 = i_8 = 1, \;\; i_3 = i_4 = 2,$$

$$a_{151}\delta_{11}\delta_{22}\epsilon_{2341} + a_{152}\delta_{11}\delta_{22}\epsilon_{2341} + a_{156}\delta_{11}\delta_{22}\epsilon_{2341} +$$
$$a_{196}\delta_{11}\delta_{22}\epsilon_{2134} + a_{197}\delta_{11}\delta_{22}\epsilon_{2134} + a_{201}\delta_{11}\delta_{22}\epsilon_{2134} +$$
$$a_{217}\delta_{22}\delta_{11}\epsilon_{1234} + a_{227}\delta_{22}\delta_{11}\epsilon_{1234} + a_{273}\delta_{22}\delta_{11}\epsilon_{1234} = 0$$
$$\Rightarrow -a_{151} - a_{152} - a_{156} - a_{196} - a_{197} - a_{201} + a_{217} + a_{227} + a_{273} = 0. \tag{124}$$

(157) (234) $$i_1 = 1, i_2 = 2, \; i_6 = 3, \; i_8 = 4,$$

$$i_5 = i_7 = 1, \;\; i_3 = i_4 = 2,$$

$$a_{151}\delta_{11}\delta_{22}\epsilon_{2314} + a_{152}\delta_{11}\delta_{22}\epsilon_{2314} + a_{156}\delta_{11}\delta_{22}\epsilon_{2314} +$$
$$a_{181}\delta_{11}\delta_{22}\epsilon_{2134} + a_{182}\delta_{11}\delta_{22}\epsilon_{2134} + a_{186}\delta_{11}\delta_{22}\epsilon_{2134} +$$
$$a_{216}\delta_{22}\delta_{11}\epsilon_{1234} + a_{226}\delta_{22}\delta_{11}\epsilon_{1234} + a_{272}\delta_{22}\delta_{11}\epsilon_{1234} = 0$$
$$\Rightarrow a_{151} + a_{152} + a_{156} - a_{181} - a_{182} - a_{186} + a_{216} + a_{226} + a_{272} = 0. \tag{125}$$

(156) (234) $$i_1 = 1, i_2 = 2, \; i_7 = 3, \; i_8 = 4,$$

$$i_5 = i_6 = 1, \;\; i_3 = i_4 = 2,$$

$$a_{151}\delta_{11}\delta_{22}\epsilon_{2134} + a_{152}\delta_{11}\delta_{22}\epsilon_{2134} + a_{156}\delta_{11}\delta_{22}\epsilon_{2134} +$$
$$a_{166}\delta_{11}\delta_{22}\epsilon_{2134} + a_{167}\delta_{11}\delta_{22}\epsilon_{2134} + a_{171}\delta_{11}\delta_{22}\epsilon_{2134} +$$
$$a_{215}\delta_{22}\delta_{11}\epsilon_{1234} + a_{225}\delta_{22}\delta_{11}\epsilon_{1234} + a_{271}\delta_{22}\delta_{11}\epsilon_{1234} = 0$$
$$\Rightarrow -a_{151} - a_{152} - a_{156} - a_{166} - a_{167} - a_{171} + a_{215} + a_{225} + a_{271} = 0. \tag{126}$$

4.5.1 Determination of independent linear invariants from the possible members of the basis, of type $\delta_{i_1 i_2}\delta_{i_3 i_4}\epsilon_{i_5 i_6 i_7 i_8}$

In order to find the remaing independent linear invariants for rank 8 we consider these 126 equations and determine that among these 126 equations how many are linearly independent.

Starting from equation (1) our observation is that equations $(1-81)$ are linearly independent because none of them can be written as a linear combination of one another. Now in order to find other linearly independent equations let us move further.

Now using equations (81) and (82) we have

$$-a_{153} - a_{154} - a_{163} + a_{199} + a_{200} - a_{210} + a_{249} + a_{259} + a_{315} = 0$$
$$a_{168} - a_{170} - a_{179} + a_{183} - a_{185} - a_{194} + a_{238} + a_{270} + a_{315} = 0$$
$$\Rightarrow a_{153} + a_{154} + a_{163} + a_{168} - a_{170} - a_{179} + a_{183} - a_{185} - a_{194}$$
$$-a_{199} - a_{200} + a_{210} + a_{238} - a_{249} - a_{259} + a_{270} = 0.$$

By using equations (16), (23), (29) and (36) in the above equation we get

$$a_{153} + a_{154} + a_{157} + a_{158} + a_{168} - a_{170} + a_{172} - a_{174} + a_{183} - a_{185} +$$
$$a_{187} - a_{189} - a_{199} - a_{200} - a_{203} - a_{204} + a_{232} + a_{233} + a_{242} - a_{244} +$$
$$a_{252} - a_{254} - a_{263} = a_{264}. \tag{1a}$$

Using equations (80) and (83) we have

$$a_{153} - a_{155} - a_{164} + a_{184} + a_{185} - a_{195} + a_{248} + a_{269} + a_{314} = 0$$
$$-a_{168} - a_{169} - a_{178} + a_{198} - a_{200} - a_{209} + a_{239} + a_{260} + a_{314} = 0$$
$$\Rightarrow a_{153} - a_{155} - a_{164} + a_{168} + a_{169} + a_{178} + a_{184} + a_{185} - a_{195}$$
$$-a_{198} + a_{200} + a_{209} - a_{239} + a_{248} - a_{260} + a_{269} = 0,$$

by using equations (17), (22), (30) and (35) in the above equation we get

$$-a_{153} + a_{155} - a_{157} + a_{159} - a_{168} - a_{169} - a_{172} - a_{173} - a_{184} - a_{185} -$$
$$a_{188} - a_{189} + a_{198} - a_{200} + a_{202} - a_{204} - a_{232} + a_{234} - a_{242} - a_{243} -$$
$$a_{253} - a_{254} + a_{262} = a_{264}. \tag{2a}$$

Using equations (79) and (84) we have

$$a_{154} + a_{155} - a_{165} + a_{169} + a_{170} - a_{180} + a_{258} + a_{268} + a_{313} = 0$$

$$-a_{183} - a_{184} - a_{193} - a_{198} - a_{199} - a_{208} + a_{240} + a_{250} + a_{313} = 0$$

$$\Rightarrow a_{154} + a_{155} - a_{165} + a_{169} + a_{170} - a_{180} + a_{183} + a_{184} + a_{193}$$

$$+a_{198} + a_{199} + a_{208} - a_{240} - a_{250} + a_{258} + a_{268} = 0,$$

by using equations (18), (24), (28) and (34) in the above equation we get

$$-a_{154} - a_{155} - a_{158} - a_{159} - a_{169} - a_{170} - a_{173} - a_{174} - a_{183} - a_{184} -$$

$$a_{187} - a_{188} - a_{198} - a_{199} - a_{202} - a_{203} - a_{233} - a_{234} - a_{243} - a_{244} -$$

$$a_{252} - a_{253} - a_{262} = a_{263}. \tag{3a}$$

(1a)-(2a)-(3a) gives us

$$a_{153} + a_{154} + a_{157} + a_{158} + a_{168} + a_{169} + a_{172} + a_{173} + a_{183} + a_{184} +$$

$$a_{187} + a_{188} + a_{232} + a_{233} + a_{242} + a_{243} + a_{252} + a_{253} = 0. \tag{4a}$$

Addition of (2a) and (4a) gives us

$$a_{154} + a_{155} + a_{158} + a_{159} + a_{183} - a_{185} + a_{187} - a_{189} + a_{198} - a_{200} +$$

$$a_{202} - a_{204} + a_{233} + a_{234} + a_{252} - a_{254} = a_{264} - a_{262}. \tag{5a}$$

By adding (3a) and (4a) we arrive at

$$a_{153} - a_{155} + a_{157} - a_{159} + a_{168} - a_{170} + a_{172} - a_{174} - a_{198} - a_{199} -$$

$$a_{202} - a_{203} + a_{232} - a_{234} + a_{242} - a_{244} = a_{263} + a_{262}. \tag{6a}$$

Using equations (78) and (85) we have

$$-a_{138} - a_{139} - a_{148} - a_{199} - a_{200} + a_{210} + a_{247} + a_{257} + a_{312} = 0$$

$$a_{167} + a_{170} - a_{177} + a_{182} + a_{185} - a_{192} + a_{228} - a_{270} + a_{312} = 0$$

$$\Rightarrow a_{138} + a_{139} + a_{148} + a_{167} + a_{170} - a_{177} + a_{182} + a_{185} - a_{192}$$

$$+a_{199} + a_{200} - a_{210} + a_{228} - a_{247} - a_{257} - a_{270} = 0,$$

by using equations (10), (21), (27) and (36) in the above equation we get

$$-a_{138} - a_{139} - a_{142} - a_{143} - a_{167} - a_{170} - a_{171} - a_{174} - a_{182} - a_{185} - a_{186} - a_{189} - a_{199} - a_{200} - a_{203} - a_{204} - a_{222} - a_{223} - a_{241} - a_{244} - a_{251} - a_{254} - a_{263} = a_{264}. \tag{7a}$$

Using equations (77) and (86) we have

$$-a_{137} + a_{139} - a_{146} - a_{198} + a_{200} + a_{209} + a_{237} - a_{257} + a_{311} = 0$$

$$a_{152} + a_{155} - a_{162} - a_{182} - a_{185} + a_{192} + a_{226} - a_{269} + a_{311} = 0$$

$$\Rightarrow a_{137} - a_{139} + a_{146} + a_{152} + a_{155} - a_{162} - a_{182} - a_{185} + a_{192} + a_{198} - a_{200} - a_{209} + a_{226} - a_{237} + a_{257} - a_{269} = 0,$$

by using equations (8), (15), (27) and (35) in the above equation we get

$$a_{137} - a_{139} + a_{141} - a_{143} + a_{152} + a_{155} + a_{156} + a_{159} - a_{182} - a_{185} - a_{186} - a_{189} + a_{198} - a_{200} + a_{202} - a_{204} + a_{221} - a_{223} + a_{231} + a_{234} - a_{251} - a_{254} = a_{264} - a_{262}. \tag{8a}$$

Using equations (76) and (87) we have

$$a_{137} + a_{138} - a_{145} + a_{198} + a_{199} + a_{208} - a_{237} - a_{247} + a_{310} = 0$$

$$-a_{152} - a_{155} + a_{162} - a_{167} - a_{170} + a_{177} + a_{225} - a_{268} + a_{310} = 0$$

$$\Rightarrow a_{137} + a_{138} - a_{145} + a_{152} + a_{155} - a_{162} + a_{167} + a_{170} - a_{177} + a_{198} + a_{199} + a_{208} - a_{225} - a_{237} - a_{247} + a_{268} = 0,$$

by using equations (7), (15), (21) and (34) in the above equation we get

$$-a_{137} - a_{138} - a_{141} - a_{142} - a_{152} - a_{155} - a_{156} - a_{159} - a_{167} - a_{170} - a_{171} - a_{174} - a_{198} - a_{199} - a_{202} - a_{203} - a_{221} - a_{222} - a_{231} - a_{234} - a_{241} - a_{244} = a_{263} + a_{262}. \tag{9a}$$

Observation:

$$(7a) - (8a) - (9a) = 0.$$

This means that above three equations are not independent of each other. Infact the the last equation can be obtained from the previous two by subtraction.

By equating equations (5a) and (8a), we arrive at

$$a_{137} - a_{139} + a_{141} - a_{143} + a_{152} - a_{154} + a_{156} - a_{158} - a_{182} - a_{183} - a_{186} - a_{187} + a_{221} - a_{223} + a_{231} - a_{233} - a_{251} - a_{252} = 0. \tag{10a}$$

Now by equating equations (6a) and (9a), we get

$$a_{137} + a_{138} + a_{141} + a_{142} + a_{152} + a_{153} + a_{156} + a_{157} + a_{167} + a_{168} + a_{171} + a_{172} + a_{221} + a_{222} + a_{231} + a_{232} + a_{241} + a_{242} = 0. \tag{11a}$$

Using equations (75) and (88) we have

$$a_{138} - a_{140} - a_{149} - a_{184} - a_{185} + a_{195} + a_{246} + a_{267} + a_{309} = 0$$
$$-a_{167} + a_{169} - a_{176} + a_{197} + a_{200} - a_{207} + a_{229} - a_{260} + a_{309} = 0$$
$$\Rightarrow a_{138} - a_{140} - a_{149} + a_{167} - a_{169} + a_{176} - a_{184} - a_{185} + a_{195} - a_{197} - a_{200} + a_{207} - a_{229} + a_{246} + a_{260} + a_{267} = 0,$$

by using equations (11), (20), (30) and (33) in the above equation we get

$$a_{138} - a_{140} + a_{142} - a_{144} + a_{167} - a_{169} + a_{171} - a_{173} - a_{184} - a_{185} - a_{188} - a_{189} - a_{197} - a_{200} - a_{201} - a_{204} + a_{222} - a_{224} + a_{241} - a_{243} - a_{253} - a_{254} - a_{261} = a_{264}. \tag{12a}$$

Using equations (74) and (89) we have

$$a_{137} + a_{140} - a_{147} - a_{183} + a_{185} + a_{194} + a_{236} - a_{267} + a_{308} = 0$$

$$-a_{152} + a_{154} - a_{161} - a_{197} - a_{200} + a_{207} + a_{227} - a_{259} + a_{308} = 0$$

$$\Rightarrow a_{137} + a_{140} - a_{147} + a_{152} - a_{154} + a_{161} - a_{183} + a_{185} + a_{194}$$

$$+a_{197} + a_{200} - a_{207} - a_{227} + a_{236} + a_{259} - a_{267} = 0,$$

by using equations (9), (14), (29) and (33) in the above equation we get

$$-a_{137} - a_{140} - a_{141} - a_{144} - a_{152} + a_{154} - a_{156} + a_{158} + a_{183} - a_{185} +$$

$$a_{187} - a_{189} - a_{197} - a_{200} - a_{201} - a_{204} - a_{221} - a_{224} - a_{231} + a_{233} +$$

$$a_{252} - a_{254} - a_{261} = a_{264}. \tag{13a}$$

Equating equations (12a) and (13a) we arrive at

$$a_{137} + a_{138} + a_{141} + a_{142} + a_{152} - a_{154} + a_{156} - a_{158} + a_{167} - a_{169} +$$

$$a_{171} - a_{173} - a_{183} - a_{184} - a_{187} - a_{188} + a_{221} + a_{222} + a_{231} - a_{233} +$$

$$a_{241} - a_{243} - a_{252} - a_{253} = 0. \tag{14a}$$

This is not a new equation. It is identical to the equation obtained from (11a)−(4a). Thus the only additional equation is any one of the above two equations for $a_{264} + a_{261}$. We take the form as

$$a_{138} - a_{140} + a_{142} - a_{144} + a_{167} - a_{169} + a_{171} - a_{173} - a_{184} - a_{185} -$$

$$a_{188} - a_{189} - a_{197} - a_{200} - a_{201} - a_{204} + a_{222} - a_{224} + a_{241} - a_{243}$$

$$-a_{253} - a_{254} = a_{264} + a_{261}. \tag{15a}$$

Using equations (73) and (90) we have

$$-a_{137} - a_{138} + a_{145} + a_{183} + a_{184} + a_{193} - a_{236} - a_{246} + a_{307} = 0$$

$$a_{152} - a_{154} + a_{161} + a_{167} - a_{169} + a_{176} - a_{225} - a_{258} + a_{307} = 0$$

$$\Rightarrow a_{137} + a_{138} - a_{145} + a_{152} - a_{154} + a_{161} + a_{167} - a_{169} + a_{176}$$

$$-a_{183} - a_{184} - a_{193} - a_{225} + a_{236} + a_{246} - a_{258} = 0.$$

by using equations (7), (14), (20) and (28) in the above equation we get

$$a_{137} + a_{138} + a_{141} + a_{142} + a_{152} - a_{154} + a_{156} - a_{158} + a_{167} - a_{169} +$$
$$a_{171} - a_{173} - a_{183} - a_{184} - a_{187} - a_{188} + a_{221} + a_{222} + a_{231} - a_{233} +$$
$$a_{241} - a_{243} - a_{252} - a_{253} = 0.$$

This equation is not a new equation. It is identical to equation (14a) which itself is identical to equation (11a) – (4a)

Using equations (72) and (91) we have

$$a_{139} + a_{140} - a_{150} - a_{169} - a_{170} + a_{180} + a_{256} + a_{266} + a_{306} = 0$$
$$-a_{182} + a_{184} - a_{191} - a_{197} + a_{199} - a_{206} + a_{230} - a_{250} + a_{306} = 0$$
$$\Rightarrow a_{139} + a_{140} - a_{150} - a_{169} - a_{170} + a_{180} + a_{182} - a_{184} + a_{191}$$
$$+a_{197} - a_{199} + a_{206} - a_{230} + a_{250} + a_{256} + a_{266} = 0.$$

by using equations (12), (24), (26) and (32) in the above equation we get

$$a_{139} + a_{140} + a_{143} + a_{144} - a_{169} - a_{170} - a_{173} - a_{174} + a_{182} - a_{184} +$$
$$a_{186} - a_{188} + a_{197} - a_{199} + a_{201} - a_{203} + a_{223} + a_{224} - a_{243} - a_{244} +$$
$$a_{251} - a_{253} + a_{261} = a_{263}. \tag{16a}$$

Using equations (71) and (92) we have

$$-a_{137} - a_{140} + a_{147} - a_{168} + a_{170} + a_{179} + a_{235} - a_{266} + a_{305} = 0$$
$$a_{152} + a_{153} - a_{160} + a_{197} - a_{199} + a_{206} - a_{227} - a_{249} + a_{305} = 0$$
$$\Rightarrow a_{137} + a_{140} - a_{147} + a_{152} + a_{153} - a_{160} + a_{168} - a_{170} - a_{179}$$
$$+a_{197} - a_{199} + a_{206} - a_{227} - a_{235} - a_{249} + a_{266} = 0.$$

by using equations (9), (13), (23) and (32) in the above equation we get

$$a_{137} + a_{140} + a_{141} + a_{144} + a_{152} + a_{153} + a_{156} + a_{157} + a_{168} - a_{170} +$$
$$a_{172} - a_{174} + a_{197} - a_{199} + a_{201} - a_{203} + a_{221} + a_{224} + a_{231} + a_{232} +$$
$$a_{242} - a_{244} + a_{261} = a_{263}. \tag{17a}$$

By equating equations (16a) and (17a) we arrive at

$$a_{137} - a_{139} + a_{141} - a_{143} + a_{152} + a_{153} + a_{156} + a_{157} + a_{168} + a_{169} +$$
$$a_{172} + a_{173} - a_{182} + a_{184} - a_{186} + a_{188} + a_{221} - a_{223} + a_{231} + a_{232} +$$
$$a_{242} + a_{243} - a_{251} + a_{253} = 0. \tag{18a}$$

This is not a new equation. It is same as the equation obtained from (10a) – (4a). Thus the only additional equation is any one of the above two equations for $a_{263} - a_{261}$. We take the form as

$$a_{139} + a_{140} + a_{143} + a_{144} - a_{169} - a_{170} - a_{173} - a_{174} + a_{182} - a_{184} +$$
$$a_{186} - a_{188} + a_{197} - a_{199} + a_{201} - a_{203} + a_{223} + a_{224} - a_{243} - a_{244} +$$
$$a_{251} - a_{253} = a_{263} - a_{261}. \tag{19a}$$

In the presence of the above equations we try to obtain a condition similar to (4a), (10a) and (11a). Indeed (5a) and (6a) above, on addition, gives us

$$a_{153} + a_{154} + a_{157} + a_{158} + a_{168} - a_{170} + a_{172} - a_{174} + a_{183} - a_{185} +$$
$$a_{187} - a_{189} - a_{199} - a_{200} - a_{203} - a_{204} + a_{232} + a_{233} + a_{242} - a_{244} +$$
$$a_{252} - a_{254} = a_{264} + a_{263}.$$

By subtracting equation (15a) from it, we get

$$-a_{138} + a_{140} - a_{142} + a_{144} + a_{153} + a_{154} + a_{157} + a_{158} - a_{167} + a_{168} +$$
$$a_{169} - a_{170} - a_{171} + a_{172} + a_{173} - a_{174} + a_{183} + a_{184} + a_{187} + a_{188} +$$
$$a_{197} - a_{199} + a_{201} - a_{203} - a_{222} + a_{224} + a_{232} + a_{233} - a_{241} + a_{242} +$$
$$a_{243} - a_{244} + a_{252} + a_{253} = a_{263} - a_{261}.$$

Equating the above equation and equation (19a), we arrive at

$$a_{138} + a_{139} + a_{142} + a_{143} - a_{153} - a_{154} - a_{157} - a_{158} + a_{167} - a_{168} - 2a_{169} + a_{171} - a_{172} - 2a_{173} + a_{182} - a_{183} - 2a_{184} + a_{186} - a_{187} - 2a_{188} + a_{222} + a_{223} - a_{232} - a_{233} + a_{241} - a_{242} - 2a_{243} + a_{251} - a_{252} - 2a_{253} = 0.$$

Addition of equation (4a) to the above equation, implies that

$$a_{138} + a_{139} + a_{142} + a_{143} + a_{167} - a_{169} + a_{171} - a_{173} + a_{182} - a_{184} + a_{186} - a_{188} + a_{222} + a_{223} + a_{241} - a_{243} + a_{251} - a_{253} = 0. \tag{20a}$$

Thus by now from to (a_{315}) to (a_{305}), we have three independent equations expresing $a_{264}, a_{263}, a_{262}$ in terms of a_{261} and lower ones, three independent equations expressing a_{253}, a_{252} in terms of a_{251} and one equation expressing a_{242} in terms of lower ones. Till now we have found seven independent equations in total. For the sake of convenience, let us rewrite these equations.

$$a_{264} = a_{138} - a_{140} + a_{142} - a_{144} + a_{167} - a_{169} + a_{171} - a_{173} - a_{184} - a_{185} - a_{188} - a_{189} - a_{197} - a_{200} - a_{201} - a_{204} + a_{222} - a_{224} + a_{241} - a_{243} - a_{253} - a_{254} - a_{261}, \tag{1b}$$

$$a_{263} = a_{139} + a_{140} + a_{143} + a_{144} - a_{169} - a_{170} - a_{173} - a_{174} + a_{182} - a_{184} + a_{186} - a_{188} + a_{197} - a_{199} + a_{201} - a_{203} + a_{223} + a_{224} - a_{243} - a_{244} + a_{251} - a_{253} + a_{261}, \tag{2b}$$

$$a_{262} = -a_{137} - a_{138} - a_{141} - a_{142} - a_{152} - a_{155} - a_{156} - a_{159} - a_{167} - a_{170} - a_{171} - a_{174} - a_{198} - a_{199} - a_{202} - a_{203} - a_{221} - a_{222} - a_{231} - a_{234} - a_{241} - a_{244} - a_{263}, \tag{3b}$$

$$a_{253} = -a_{153} - a_{154} - a_{157} - a_{158} - a_{168} - a_{169} - a_{172} - a_{173} - a_{183} - a_{184} - a_{187} - a_{188} - a_{232} - a_{233} - a_{242} - a_{243} - a_{252}, \quad (4b)$$

$$a_{252} = a_{137} - a_{139} + a_{141} - a_{143} + a_{152} - a_{154} + a_{156} - a_{158} - a_{182} - a_{183} - a_{186} - a_{187} + a_{221} - a_{223} + a_{231} - a_{233} - a_{251}, \quad (5b)$$

$$a_{253} = a_{138} + a_{139} + a_{142} + a_{143} + a_{167} - a_{169} + a_{171} - a_{173} + a_{182} - a_{184} + a_{186} - a_{188} + a_{222} + a_{223} + a_{241} - a_{243} + a_{251}, \quad (6b)$$

$$a_{242} = -a_{137} - a_{138} - a_{141} - a_{142} - a_{152} - a_{153} - a_{156} - a_{157} - a_{167} - a_{168} - a_{171} - a_{172} - a_{221} - a_{222} - a_{231} - a_{232} - a_{241}. \quad (7b)$$

Observation:

$$(7b) - (6b) - (5b) = (4b).$$

Thus, we are left with only six independent equations. These equations are (1b), (2b), (3b), (5b), (6b) and (7b).

Using equations (70) and (93) we have

$$a_{137} - a_{139} + a_{146} + a_{168} + a_{169} + a_{178} - a_{235} - a_{256} + a_{304} = 0$$

$$-a_{152} - a_{153} + a_{160} + a_{182} - a_{184} + a_{191} - a_{226} - a_{248} + a_{304} = 0$$

$$\Rightarrow a_{137} - a_{139} + a_{146} + a_{152} + a_{153} - a_{160} + a_{168} + a_{169} + a_{178} - a_{182} + a_{184} - a_{191} + a_{226} - a_{235} + a_{248} - a_{256} = 0.$$

by using equations (8), (13), (22) and (26) in the above equation we get

$$a_{137} - a_{139} + a_{141} - a_{143} + a_{152} + a_{153} + a_{156} + a_{157} + a_{168} + a_{169} + a_{172} + a_{173} - a_{182} + a_{184} - a_{186} + a_{188} + a_{221} - a_{223} + a_{231} + a_{232} + a_{242} + a_{243} - a_{251} + a_{253} = 0.$$

This is not a new equation. This is identical to equation (18a) which itself is identical to (10a) – (4a). Thus we have only six independent equations so far.

Using equations (69) and (94) we have

$$-a_{139} - a_{140} + a_{150} - a_{154} - a_{155} + a_{165} + a_{255} + a_{265} + a_{303} = 0$$
$$a_{182} + a_{183} - a_{190} + a_{197} + a_{198} - a_{205} - a_{230} - a_{240} + a_{303} = 0$$
$$\Rightarrow a_{139} + a_{140} - a_{150} + a_{154} + a_{155} - a_{165} + a_{182} + a_{183} - a_{190}$$
$$+a_{197} + a_{198} - a_{205} - a_{230} - a_{240} - a_{255} - a_{265} = 0.$$

By using equations (12), (18), (25) and (31) in the above equation we get

$$-a_{139} - a_{140} - a_{143} - a_{144} - a_{154} - a_{155} - a_{158} - a_{159} - a_{182} - a_{183} -$$
$$a_{186} - a_{187} - a_{197} - a_{198} - a_{201} - a_{202} - a_{223} - a_{224} - a_{233} - a_{234} -$$
$$a_{251} - a_{252} - a_{261} = a_{262}. \tag{21a}$$

But (2b) – (3b) – (7b) + (4b) gives us

$$a_{139} + a_{140} + a_{143} + a_{144} + a_{154} + a_{155} + a_{158} + a_{159} + a_{182} + a_{183} +$$
$$a_{186} + a_{187} + a_{197} + a_{198} + a_{201} + a_{202} + a_{223} + a_{224} + a_{233} + a_{234} +$$
$$a_{251} + a_{252} = -a_{261} - a_{262}.$$

This equation is exactly the one which we have obtained for a_{303}. Thus uptill now we have only six independent equations.

Using equations (68) and (95) we have

$$-a_{138} + a_{140} + a_{149} - a_{153} + a_{155} + a_{164} + a_{245} - a_{265} + a_{302} = 0$$
$$a_{167} + a_{168} - a_{175} - a_{197} - a_{198} + a_{205} - a_{229} - a_{239} + a_{302} = 0$$
$$\Rightarrow a_{138} - a_{140} - a_{149} + a_{153} - a_{155} - a_{164} + a_{167} + a_{168} - a_{175}$$
$$-a_{197} - a_{198} + a_{205} - a_{229} - a_{239} - a_{245} + a_{265} = 0.$$

By using equations (11), (17), (19) and (31) in the above equation we get

$$a_{138} - a_{140} + a_{142} - a_{144} + a_{153} - a_{155} + a_{157} - a_{159} + a_{167} +$$
$$a_{168} + a_{171} + a_{172} - a_{197} - a_{198} - a_{201} - a_{202} + a_{222} - a_{224} +$$
$$a_{232} - a_{234} + a_{241} + a_{242} - a_{261} = a_{262}. \tag{22a}$$

But (2b) − (3b) − (7b) − (6b) gives us

$$a_{138} - a_{140} + a_{142} - a_{144} + a_{153} - a_{155} + a_{157} - a_{159} + a_{167} +$$
$$a_{168} + a_{171} + a_{172} - a_{197} - a_{198} - a_{201} - a_{202} + a_{222} - a_{224} +$$
$$a_{232} - a_{234} + a_{241} + a_{242} = a_{262} + a_{261}.$$

This equation is exactly the same as equation (22a). i.e., this is not a new equation.

Using equations (67) and (96) we have

$$a_{138} + a_{139} + a_{148} + a_{153} + a_{154} + a_{163} - a_{245} - a_{255} + a_{301} = 0$$
$$-a_{167} - a_{168} + a_{175} - a_{182} - a_{183} + a_{190} - a_{228} - a_{238} + a_{301} = 0$$
$$\Rightarrow a_{138} + a_{139} + a_{148} + a_{153} + a_{154} + a_{163} + a_{167} + a_{168} - a_{175}$$
$$+a_{182} + a_{183} - a_{190} + a_{228} + a_{238} - a_{245} - a_{255} = 0.$$

By using equations (10), (16), (19) and (25) in the above equation we get

$$a_{138} + a_{139} + a_{142} + a_{143} + a_{153} + a_{154} + a_{157} + a_{158} + a_{167} +$$
$$a_{168} + a_{171} + a_{172} + a_{182} + a_{183} + a_{186} + a_{187} + a_{222} + a_{223} +$$
$$a_{232} + a_{233} + a_{241} + a_{242} + a_{251} + a_{252} = 0.$$

But (4a) + (20a) gives us

$$a_{138} + a_{139} + a_{142} + a_{143} + a_{153} + a_{154} + a_{157} + a_{158} + a_{167} +$$
$$a_{168} + a_{171} + a_{172} + a_{182} + a_{183} + a_{186} + a_{187} + a_{222} + a_{223} +$$
$$a_{232} + a_{233} + a_{241} + a_{242} + a_{251} + a_{252} = 0.$$

This is same as equation for a_{301} above. Thus till a_{301} we still have only six independent equations.

Using equations (66) and (97) we have

$$-a_{123} - a_{124} - a_{133} + a_{199} + a_{200} - a_{210} + a_{244} + a_{254} + a_{300} = 0$$

$$a_{166} - a_{170} - a_{174} + a_{181} - a_{185} - a_{189} + a_{218} + a_{270} + a_{300} = 0$$

$$\Rightarrow a_{123} + a_{124} + a_{133} + a_{166} - a_{170} - a_{174} + a_{181} - a_{185} - a_{189}$$

$$-a_{199} - a_{200} + a_{210} + a_{218} - a_{244} - a_{254} + a_{270} = 0.$$

By using equations (4) and (36) in the above equation we arrive at

$$a_{108} + a_{109} + a_{112} + a_{113} + a_{123} + a_{124} + a_{127} + a_{128} + a_{166} -$$

$$a_{170} - a_{174} + a_{181} - a_{185} - a_{189} - a_{199} - a_{200} - a_{203} - a_{204} +$$

$$a_{212} + a_{213} - a_{244} - a_{254} = a_{264} + a_{263}$$

But (1b) + (2b) − (6b) gives us

$$-a_{169} - a_{170} - a_{173} - a_{174} - a_{184} - a_{185} - a_{188} - a_{189} - a_{199} -$$

$$a_{200} - a_{203} - a_{204} - a_{243} - a_{244} - a_{253} - a_{254} = a_{264} + a_{263}.$$

Equating the above two equations for $a_{264} + a_{263}$, we have

$$a_{108} + a_{109} + a_{112} + a_{113} + a_{123} + a_{124} + a_{127} + a_{128} + a_{166} +$$

$$a_{169} + a_{173} + a_{181} + a_{184} + a_{188} + a_{212} + a_{213} + a_{243} + a_{253} = 0. \quad \text{(8b)}$$

This is a new equation. Thus upto a_{300} we have seven independent equations.

Now we find a condition which replace equation (3b).

Now (3b) − (2b) + (6b) gives us

$$-a_{137} - a_{140} - a_{141} - a_{144} - a_{152} - a_{155} - a_{156} - a_{159} - a_{197} -$$

$$a_{198} - a_{201} - a_{202} - a_{221} - a_{224} - a_{231} - a_{234} - a_{261} = a_{262}. \quad \text{(9b)}$$

Equation (9b) replaces equation (3b). Thus uptill now we have seven independent equations namely $(1b), (2b), (9b), (5b), (6b), (7b)$ and $(8b)$. Now for equation (8b) we have

$$-a_{108} - a_{109} - a_{112} - a_{113} - a_{123} - a_{124} - a_{127} - a_{128} - a_{166} - a_{169} - a_{173} - a_{181} - a_{184} - a_{188} - a_{212} - a_{213} = a_{243} + a_{253}.$$

From equation (4b) we have

$$-a_{153} - a_{154} - a_{157} - a_{158} - a_{168} - a_{169} - a_{172} - a_{173} - a_{183} - a_{184} - a_{187} - a_{188} - a_{232} - a_{233} - a_{242} - a_{252} = a_{253} + a_{243}.$$

By equating the above two equations we arrive at

$$a_{108} + a_{109} + a_{112} + a_{113} + a_{123} + a_{124} + a_{127} + a_{128} - a_{153} - a_{154} - a_{157} - a_{158} + a_{166} - a_{168} - a_{172} + a_{181} - a_{183} - a_{187} + a_{212} + a_{213} - a_{232} - a_{233} - a_{242} = a_{252}. \tag{10b}$$

Now equating equation (5b) and equation (10b) we get

$$-a_{108} - a_{109} - a_{112} - a_{113} - a_{123} - a_{124} - a_{127} - a_{128} + a_{137} - a_{139} + a_{141} - a_{143} + a_{152} + a_{153} + a_{156} + a_{157} - a_{166} + a_{168} + a_{172} - a_{181} - a_{182} - a_{186} - a_{212} - a_{213} + a_{221} - a_{223} + a_{231} + a_{232} + a_{242} = a_{251}. \tag{11b}$$

Equation (11b) replaces equation (5b) and equation (10b) replaces equation (6b). Thus till now we have seven independent equations namely $(1b), (2b), (9b), (11b), (10b), (7b)$ and $(8b)$.

Using equations (65) and (98) we have

$$-a_{122} + a_{124} - a_{131} + a_{198} - a_{200} - a_{209} + a_{234} - a_{254} + a_{299} = 0$$
$$a_{151} - a_{155} - a_{159} - a_{181} + a_{185} + a_{189} + a_{216} + a_{269} + a_{299} = 0$$
$$\Rightarrow a_{122} - a_{124} + a_{131} + a_{151} - a_{155} - a_{159} - a_{181} + a_{185}$$
$$+a_{189} - a_{198} + a_{200} + a_{209} + a_{216} - a_{234} + a_{254} + a_{269} = 0.$$

By using equations (2) and (35) in the above equation we get

$$-a_{107} + a_{109} - a_{111} + a_{113} - a_{122} + a_{124} - a_{126} + a_{128} - a_{151} +$$
$$a_{155} + a_{159} + a_{181} - a_{185} - a_{189} + a_{198} - a_{200} + a_{202} - a_{204} -$$
$$a_{211} + a_{213} + a_{234} - a_{254} + a_{262} = a_{264}. \tag{23a}$$

Using equations (64) and (99) we have

$$a_{122} + a_{123} - a_{130} - a_{198} - a_{199} - a_{208} - a_{234} - a_{244} + a_{298} = 0$$
$$-a_{151} + a_{155} + a_{159} - a_{166} + a_{170} + a_{174} + a_{215} + a_{268} + a_{298} = 0$$
$$\Rightarrow a_{122} + a_{123} - a_{130} + a_{151} - a_{155} - a_{159} + a_{166} - a_{170} - a_{174}$$
$$-a_{198} - a_{199} - a_{208} - a_{215} - a_{234} - a_{244} - a_{268} = 0.$$

By using equations (1) and (34) in the above equation we get

$$a_{107} + a_{108} + a_{111} + a_{112} + a_{122} + a_{123} + a_{126} + a_{127} + a_{151} -$$
$$a_{155} - a_{159} + a_{166} - a_{170} - a_{174} - a_{198} - a_{199} - a_{202} - a_{203} +$$
$$a_{211} + a_{212} - a_{234} - a_{244} - a_{262} = a_{263}. \tag{24a}$$

Addition of the above two equations i.e., (23a) + (24a) gives us

$$a_{108} + a_{109} + a_{112} + a_{113} + a_{123} + a_{124} + a_{127} + a_{128} + a_{166} -$$
$$a_{170} - a_{174} + a_{181} - a_{185} - a_{189} - a_{199} - a_{200} - a_{203} - a_{204} +$$
$$a_{212} + a_{213} - a_{244} - a_{254} = a_{264} + a_{263}.$$

This equation is identical to the equation which we have obtained for a_{300}.

From equation (24a) we have

$$a_{107} + a_{108} + a_{111} + a_{112} + a_{122} + a_{123} + a_{126} + a_{127} + a_{151} -$$
$$a_{155} - a_{159} + a_{166} - a_{170} - a_{174} - a_{198} - a_{199} - a_{202} - a_{203} +$$
$$a_{211} + a_{212} - a_{234} - a_{244} = a_{263} + a_{262}.$$

By equating above equation to equation (9a) we arrive at

$$a_{107} + a_{108} + a_{111} + a_{112} + a_{122} + a_{123} + a_{126} + a_{127} + a_{137} +$$
$$a_{138} + a_{141} + a_{142} + a_{151} + a_{152} + a_{156} + a_{166} + a_{167} + a_{171} +$$
$$a_{211} + a_{212} + a_{221} + a_{222} + a_{231} + a_{241} = 0. \tag{12b}$$

This is the eight independent equation. At this stage we have eight independent equations namely (1b), (2b), (9b), (11b), (10b), (7b), (8b) and (12b).

Using equations (63) and (100) we have

$$-a_{121} - a_{124} - a_{128} + a_{197} + a_{200} - a_{207} + a_{224} + a_{254} + a_{297} = 0$$
$$a_{136} - a_{140} - a_{144} + a_{181} - a_{185} - a_{189} + a_{213} + a_{267} + a_{297} = 0$$
$$\Rightarrow a_{121} + a_{124} + a_{128} + a_{136} - a_{140} - a_{144} + a_{181} - a_{185} - a_{189}$$
$$-a_{197} - a_{200} + a_{207} + a_{213} - a_{224} - a_{254} + a_{267} = 0.$$

By using equation (33) in the above equation we get

$$a_{106} + a_{109} + a_{113} + a_{121} + a_{124} + a_{128} + a_{136} - a_{140} - a_{144} +$$
$$a_{181} - a_{185} - a_{189} - a_{197} - a_{200} - a_{201} - a_{204} + a_{213} -$$
$$a_{224} = a_{254} + a_{261} + a_{264}. \tag{13b}$$

From equation (1b) we have

$$a_{138} - a_{140} + a_{142} - a_{144} + a_{167} - a_{169} + a_{171} - a_{173} - a_{184} - a_{185} - a_{188} - a_{189} - a_{197} - a_{200} - a_{201} - a_{204} + a_{222} - a_{224} + a_{241} - a_{243} - a_{253} = a_{254} + a_{261} + a_{264}.$$

Comparison of equation (1b) and (13b) yields

$$a_{106} + a_{109} + a_{113} + a_{121} + a_{124} + a_{128} + a_{136} - a_{138} - a_{142} - a_{167} + a_{169} - a_{171} + a_{173} + a_{181} + a_{184} + a_{188} + a_{213} - a_{222} - a_{241} + a_{243} + a_{253} = 0. \quad (14b)$$

From equation (7b) we have

$$-a_{137} - a_{138} - a_{141} - a_{142} - a_{152} - a_{153} - a_{156} - a_{157} - a_{167} - a_{168} - a_{171} - a_{172} - a_{221} - a_{222} - a_{231} - a_{232} - a_{241} = a_{242}.$$

Using equation (12a) in the above equation we arrive at

$$a_{107} + a_{108} + a_{111} + a_{112} + a_{122} + a_{123} + a_{126} + a_{127} + a_{151} - a_{153} - a_{157} + a_{166} - a_{168} - a_{172} + a_{211} + a_{212} - a_{232} = a_{242}. \quad (15b)$$

Equation (15b) can replace equation (7b).

Let us consider equation (14b)

$$a_{106} + a_{109} + a_{113} + a_{121} + a_{124} + a_{128} + a_{136} - a_{138} - a_{142} - a_{167} + a_{169} - a_{171} + a_{173} + a_{181} + a_{184} + a_{188} + a_{213} - a_{222} - a_{241} + a_{243} + a_{253} = 0.$$

By using equation (8b) in the above equation we get

$$a_{106} - a_{108} - a_{112} + a_{121} - a_{123} - a_{127} + a_{136} - a_{138} - a_{142} - a_{166} - a_{167} - a_{171} - a_{212} - a_{222} = a_{241}. \quad (16b)$$

Equation (12b) may be rewritten as

$$-a_{107} - a_{108} - a_{111} - a_{112} - a_{122} - a_{123} - a_{126} - a_{127} - a_{137} -$$
$$a_{138} - a_{141} - a_{142} - a_{151} - a_{152} - a_{156} - a_{166} - a_{167} - a_{171} -$$
$$a_{211} - a_{212} - a_{221} - a_{222} - a_{231} = a_{241}.$$

Comparison of the above two equations gives us

$$a_{106} + a_{107} + a_{111} + a_{121} + a_{122} + a_{126} + a_{136} + a_{137} + a_{141} +$$
$$a_{151} + a_{152} + a_{156} + a_{211} + a_{221} + a_{231} = 0. \qquad (17b)$$

This equation is a new one and it provides us the ninth independent equation. Also equation (16b) for a_{241} replaces equation (12b). Thus upto a_{297} we have nine independent equations namely (1b), (2b), (9b), (11b), (10b), (15b), (8b), (16b) and (17b).

Using equations (62) and (101) we have

$$a_{121} - a_{123} - a_{127} - a_{197} + a_{199} - a_{206} - a_{224} + a_{244} + a_{296} = 0$$
$$-a_{136} + a_{140} + a_{144} + a_{166} - a_{170} - a_{174} + a_{212} + a_{266} + a_{296} = 0$$
$$\Rightarrow a_{121} - a_{123} - a_{127} + a_{136} - a_{140} - a_{144} - a_{166} + a_{170} + a_{174}$$
$$-a_{197} + a_{199} - a_{206} - a_{212} - a_{224} + a_{244} - a_{266} = 0.$$

By using equation (32) in the above equation we get

$$-a_{106} + a_{108} + a_{112} - a_{121} + a_{123} + a_{127} - a_{136} + a_{140} +$$
$$a_{144} + a_{166} - a_{170} - a_{174} + a_{197} - a_{199} + a_{201} - a_{203} +$$
$$a_{212} + a_{224} = a_{244} - a_{261} + a_{263}.$$

This is not a new equation. Rather this equation is equivalent to (22a) – (13b).

Using equations (61) and (102) we have

$$-a_{121} - a_{122} - a_{126} + a_{197} + a_{198} - a_{205} + a_{224} + a_{234} + a_{295} = 0$$

$$a_{136} - a_{140} - a_{144} + a_{151} - a_{155} - a_{159} + a_{211} + a_{265} + a_{295} = 0$$

$$\Rightarrow a_{121} + a_{122} + a_{126} + a_{136} - a_{140} - a_{144} + a_{151} - a_{155} - a_{159}$$

$$-a_{197} - a_{198} + a_{205} + a_{211} - a_{224} - a_{234} + a_{265} = 0.$$

By using equation (31) in the above equation we get

$$a_{106} + a_{107} + a_{111} + a_{121} + a_{122} + a_{126} + a_{136} - a_{140} -$$

$$a_{144} + a_{151} - a_{155} - a_{159} - a_{197} - a_{198} - a_{201} - a_{202} +$$

$$a_{211} - a_{224} - a_{234} - a_{261} = a_{262}.$$

This is not a new equation. This is same as the one obtained from (9b) + (17b).

Using equations (60) and (103) we have

$$a_{123} - a_{125} - a_{134} + a_{184} + a_{185} - a_{195} + a_{243} + a_{264} + a_{294} = 0$$

$$-a_{166} - a_{169} - a_{173} + a_{196} - a_{200} - a_{204} + a_{219} + a_{260} + a_{294} = 0$$

$$\Rightarrow a_{123} - a_{125} - a_{134} + a_{166} + a_{169} + a_{173} + a_{184} + a_{185} - a_{195}$$

$$-a_{196} + a_{200} + a_{204} - a_{219} + a_{243} - a_{260} + a_{264} = 0.$$

By using equation (30) in the above equation we arrive at

$$-a_{108} + a_{110} + a_{119} - a_{123} + a_{125} + a_{134} - a_{166} - a_{169} -$$

$$a_{173} - a_{184} - a_{185} - a_{188} - a_{189} + a_{196} - a_{200} - a_{204} +$$

$$a_{219} - a_{243} - a_{253} - a_{254} = a_{264}.$$

From equation (1b) we have

$$a_{138} - a_{140} + a_{142} - a_{144} + a_{167} - a_{169} + a_{171} - a_{173} - a_{184} -$$

$$a_{185} - a_{188} - a_{189} - a_{197} - a_{200} - a_{201} - a_{204} + a_{222} - a_{224} +$$

$$a_{241} - a_{243} - a_{253} - a_{254} - a_{261} = a_{264}.$$

Comparison of above two equations implies that

$$a_{108} - a_{110} - a_{119} + a_{123} - a_{125} - a_{134} + a_{138} - a_{140} +$$
$$a_{142} - a_{144} + a_{166} + a_{167} + a_{171} - a_{196} - a_{197} - a_{201} -$$
$$a_{219} + a_{222} - a_{224} + a_{241} = a_{261}. \tag{18b}$$

This is a new equation. Thus upto a_{294}, we have ten independent equations. These are for

$$a_{264},\ a_{263},\ a_{263},\ a_{261},$$
$$a_{253},\ a_{252},\ a_{251},$$
$$a_{242},\ a_{241},$$
$$a_{231}.$$

Now by using equation (16b) in equation (18b) we get

$$a_{106} - a_{110} - a_{112} - a_{119} + a_{121} - a_{125} - a_{127} - a_{134} + a_{136} -$$
$$a_{140} - a_{144} - a_{196} - a_{197} - a_{201} - a_{212} - a_{219} - a_{224} = a_{261}. \tag{19b}$$

This equation can replace equation (18b). Now we keep equations (8b), (15b), (16b), (17b) and (19b) as they are but we can find some replacements for equations (1b), (2b), (9b), (10b) and (11b).

Now by making use of equation (15b) in equation (10b) we get

$$-a_{107} + a_{109} - a_{111} + a_{113} - a_{122} + a_{124} - a_{126} + a_{128} - a_{151} -$$
$$a_{154} - a_{158} + a_{181} - a_{183} - a_{187} - a_{211} + a_{213} - a_{233} = a_{252}. \tag{20b}$$

Now by making use of equation (7b) in equation (11b) we arrive at

$$-a_{108} - a_{109} - a_{112} - a_{113} - a_{123} - a_{124} - a_{127} - a_{128} - a_{138}$$
$$-a_{139} - a_{142} - a_{143} - a_{166} - a_{167} - a_{171} - a_{181} - a_{182} - a_{186}$$
$$-a_{212} - a_{213} - a_{222} - a_{223} - a_{241} = a_{251}.$$

Now by making use of equation (16b) in the above equation we get

$$\begin{aligned}&-a_{106}-a_{109}-a_{113}-a_{121}-a_{124}-a_{128}-a_{136}-a_{139}\\&-a_{143}-a_{181}-a_{182}-a_{186}-a_{213}-a_{223}=a_{251}.\end{aligned}\tag{21b}$$

From equation (9b) we have

$$\begin{aligned}&-a_{137}-a_{140}-a_{141}-a_{144}-a_{152}-a_{155}-a_{156}-a_{159}-a_{197}\\&-a_{198}-a_{201}-a_{202}-a_{221}-a_{224}-a_{231}-a_{234}-a_{261}=a_{262}.\end{aligned}$$

By using equation (18b) in above equation we get

$$\begin{aligned}&-a_{108}+a_{110}+a_{119}-a_{123}+a_{125}+a_{134}-a_{137}-a_{138}-a_{141}\\&-a_{142}-a_{152}-a_{155}-a_{156}-a_{159}-a_{166}-a_{167}-a_{171}+a_{196}\\&-a_{198}-a_{202}+a_{219}-a_{221}-a_{222}-a_{231}-a_{234}-a_{241}=a_{262}.\end{aligned}$$

Now by making use of equations (16b) and (17b) in the above equation we get

$$\begin{aligned}&a_{107}+a_{110}+a_{111}+a_{112}+a_{119}+a_{122}+a_{125}+a_{126}+\\&a_{127}+a_{134}+a_{151}-a_{155}-a_{159}+a_{196}-a_{198}-a_{202}+\\&a_{211}+a_{212}+a_{219}-a_{234}=a_{262}.\end{aligned}\tag{22b}$$

Now, let us consider equation (1b)

$$\begin{aligned}&a_{138}-a_{140}+a_{142}-a_{144}+a_{167}-a_{169}+a_{171}-a_{173}-a_{184}-\\&a_{185}-a_{188}-a_{189}-a_{197}-a_{200}-a_{201}-a_{204}+a_{222}-a_{224}+\\&a_{241}-a_{243}-a_{253}-a_{254}-a_{261}=a_{264}.\end{aligned}$$

Using equations (8b) and (18b) in it we get

$$\begin{aligned}&a_{109}+a_{110}+a_{112}+a_{113}+a_{119}+a_{124}+a_{125}+a_{127}+\\&a_{128}+a_{134}+a_{181}-a_{185}-a_{189}+a_{196}-a_{200}-a_{204}+\\&a_{212}+a_{213}+a_{219}-a_{254}=a_{264}.\end{aligned}\tag{23b}$$

Let us now consider equation (2b)

$$a_{139} + a_{140} + a_{143} + a_{144} - a_{169} - a_{170} - a_{173} - a_{174} + a_{182} -$$
$$a_{184} + a_{186} - a_{188} + a_{197} - a_{199} + a_{201} - a_{203} + a_{223} + a_{224} -$$
$$a_{243} - a_{244} + a_{251} - a_{253} + a_{261} = a_{263}.$$

By making use of equations (6b) and (18b) in it we arrive at

$$a_{108} - a_{110} - a_{119} + a_{123} - a_{125} - a_{134} + a_{166} - a_{170} - a_{174}$$
$$-a_{196} - a_{199} - a_{203} - a_{219} - a_{244} = a_{263}. \tag{24b}$$

Equations (20b), (21b), (22b), (23b)and (24b) replace equations (9b), (10b), (11b), (1b)and (2b) respectively. For the sake of convenience, let us rewrite these ten independent equations.

$$a_{264} = a_{109} + a_{110} + a_{112} + a_{113} + a_{119} + a_{124} + a_{125} + a_{127} +$$
$$a_{128} + a_{134} + a_{181} - a_{185} - a_{189} + a_{196} - a_{200} - a_{204} +$$
$$a_{212} + a_{213} + a_{219} - a_{254}, \tag{1*}$$

$$a_{263} = a_{108} - a_{110} - a_{119} + a_{123} - a_{125} - a_{134} + a_{166} - a_{170} -$$
$$a_{196} - a_{199} - a_{174} - a_{203} - a_{219} - a_{244}, \tag{2*}$$

$$a_{262} = a_{107} + a_{110} + a_{111} + a_{112} + a_{119} + a_{122} + a_{125} + a_{126} +$$
$$a_{127} + a_{134} + a_{151} - a_{155} - a_{159} + a_{196} - a_{198} - a_{202} +$$
$$a_{211} + a_{212} + a_{219} - a_{234}, \tag{3*}$$

$$a_{261} = a_{106} - a_{110} - a_{112} - a_{119} + a_{121} - a_{125} - a_{127} - a_{134} +$$
$$a_{136} - a_{140} - a_{144} - a_{196} - a_{197} - a_{201} - a_{212} -$$
$$a_{219} - a_{224}, \tag{4*}$$

$$a_{253} = -a_{108} - a_{109} - a_{112} - a_{113} - a_{123} - a_{124} - a_{127} - a_{128} - a_{166} - a_{169} - a_{173} - a_{181} - a_{184} - a_{188} - a_{212} - a_{213} - a_{243}, \tag{5*}$$

$$a_{252} = -a_{107} + a_{109} - a_{111} + a_{113} - a_{122} + a_{124} - a_{126} + a_{128} - a_{151} - a_{154} - a_{158} + a_{181} - a_{183} - a_{187} - a_{211} + a_{213} - a_{233}, \tag{6*}$$

$$a_{251} = -a_{106} - a_{109} - a_{113} - a_{121} - a_{124} - a_{128} - a_{136} - a_{139} - a_{143} - a_{181} - a_{182} - a_{186} - a_{213} - a_{223}, \tag{7*}$$

$$a_{242} = a_{107} + a_{108} + a_{111} + a_{112} + a_{122} + a_{123} + a_{126} + a_{127} + a_{151} - a_{153} - a_{157} + a_{166} - a_{168} - a_{172} + a_{211} + a_{212} - a_{232}, \tag{8*}$$

$$a_{241} = a_{106} - a_{108} - a_{112} + a_{121} - a_{123} - a_{127} + a_{136} - a_{138} - a_{142} - a_{166} - a_{167} - a_{171} - a_{212} - a_{222}, \tag{9*}$$

$$a_{231} = -a_{106} - a_{107} - a_{111} - a_{121} - a_{122} - a_{126} - a_{136} - a_{137} - a_{141} - a_{151} - a_{152} - a_{156} - a_{211} - a_{221}. \tag{10*}$$

Uptill now we have found ten independent equations. Now in order to check that other than these ten independent equations, some more independent equations exist or not, we move further.

Now by using equations (59) and (104) we have

$$a_{122} + a_{125} - a_{132} + a_{183} - a_{185} - a_{194} + a_{233} - a_{264} + a_{293} = 0$$

$$-a_{151} - a_{154} - a_{158} - a_{196} + a_{200} + a_{204} + a_{217} + a_{259} + a_{293} = 0$$

$$\Rightarrow a_{122} + a_{125} - a_{132} + a_{151} + a_{154} + a_{158} + a_{183} - a_{185} - a_{194} + a_{196} - a_{200} - a_{204} - a_{217} + a_{233} - a_{259} - a_{264} = 0.$$

By using equation (29) in the above equation we get

$$a_{107} + a_{110} - a_{117} + a_{122} + a_{125} - a_{132} + a_{151} + a_{154} + a_{158} + a_{183} - a_{185} + a_{187} - a_{189} + a_{196} - a_{200} - a_{204} - a_{217} + a_{233} - a_{254} = a_{264} - a_{252}.$$

Now (1*) − (6*) gives us

$$a_{107} + a_{110} + a_{111} + a_{112} + a_{119} + a_{122} + a_{125} + a_{126} + a_{127} + a_{134} + a_{151} + a_{154} + a_{158} + a_{183} - a_{185} + a_{187} - a_{189} + a_{196} - a_{200} - a_{204} + a_{211} + a_{212} + a_{219} + a_{233} - a_{254} = a_{264} - a_{252}.$$

Equating above two equations

$$-a_{111} - a_{112} - a_{117} - a_{119} - a_{126} - a_{127} - a_{132} - a_{134} - a_{211} - a_{212} - a_{217} = a_{219}.$$

Equating the above equation to equation (5)

$$a_{111} + a_{114} + a_{117} + a_{126} + a_{129} + a_{132} + a_{211} + a_{214} + a_{217} = 0.$$

This is not a new equation. It is identical to equation (3).

Using equations (58) and (105) we have

$$-a_{122} - a_{123} + a_{130} - a_{183} - a_{184} - a_{193} - a_{233} - a_{243} + a_{292} = 0$$

$$a_{151} + a_{154} + a_{158} + a_{166} + a_{169} + a_{173} - a_{215} + a_{258} + a_{292} = 0$$

$$\Rightarrow a_{122} + a_{123} - a_{130} + a_{151} + a_{154} + a_{158} + a_{166} + a_{169} + a_{173} + a_{183} + a_{184} + a_{193} - a_{215} + a_{233} + a_{243} + a_{258} = 0.$$

By using equation (28) in the above equation we get

$$-a_{107} - a_{108} + a_{115} - a_{122} - a_{123} + a_{130} - a_{151} - a_{154} - a_{158}$$
$$-a_{166} - a_{169} - a_{173} - a_{183} - a_{184} - a_{187} - a_{188} + a_{215} -$$
$$a_{233} - a_{243} = a_{253} + a_{252}.$$

Now (5*) + (6*) gives us

$$-a_{107} - a_{108} - a_{111} - a_{112} - a_{122} - a_{123} - a_{126} - a_{127} - a_{151} -$$
$$a_{154} - a_{158} - a_{166} - a_{169} - a_{173} - a_{183} - a_{184} - a_{187} - a_{188} -$$
$$a_{211} - a_{212} - a_{233} - a_{243} = a_{253} + a_{252}.$$

By equating the above two equations we arrive at

$$a_{111} + a_{112} + a_{115} + a_{126} + a_{127} + a_{130} + a_{211} + a_{212} + a_{215} = 0.$$

This equation is identical to equation (1). Thus equation for a_{292}, does not yield a new equation.

Using equations (57) and (106) we have

$$a_{121} - a_{125} - a_{129} + a_{182} + a_{185} - a_{192} + a_{223} + a_{264} + a_{291} = 0$$
$$-a_{136} - a_{139} - a_{143} + a_{196} - a_{200} - a_{204} + a_{214} + a_{257} + a_{291} = 0$$
$$\Rightarrow a_{121} - a_{125} - a_{129} + a_{136} + a_{139} + a_{143} + a_{182} + a_{185} - a_{192}$$
$$-a_{196} + a_{200} + a_{204} - a_{214} + a_{223} - a_{257} + a_{264} = 0.$$

By using equation (27) in the above equation we get

$$-a_{106} + a_{110} + a_{114} - a_{121} + a_{125} + a_{129} - a_{136} - a_{139} - a_{143} -$$
$$a_{182} - a_{185} - a_{186} - a_{189} + a_{196} - a_{200} - a_{204} + a_{214} - a_{223} -$$
$$a_{254} = a_{264} + a_{251}.$$

Now (1*) + (7*) gives us

$$-a_{106} + a_{110} + a_{112} + a_{119} - a_{121} + a_{125} + a_{127} + a_{134} - a_{136} -$$
$$a_{139} - a_{143} - a_{182} - a_{185} - a_{186} - a_{189} + a_{196} - a_{200} - a_{204} +$$
$$a_{212} + a_{219} - a_{223} - a_{254} = a_{264} + a_{251}.$$

Equating the above two equations we get

$$a_{112} - a_{114} + a_{119} + a_{127} - a_{129} + a_{134} + a_{212} - a_{214} + a_{219} = 0.$$

This is not a new equation. It is identical to equation (5).

Using equations (56) and (107) we have

$$-a_{121} + a_{123} + a_{127} - a_{182} + a_{184} - a_{191} - a_{223} + a_{243} + a_{290} = 0$$
$$a_{136} + a_{139} + a_{143} - a_{166} - a_{169} - a_{173} - a_{212} + a_{256} + a_{290} = 0$$
$$\Rightarrow a_{121} - a_{123} - a_{127} + a_{136} + a_{139} + a_{143} - a_{166} - a_{169} - a_{173}$$
$$+a_{182} - a_{184} + a_{191} - a_{212} + a_{223} - a_{243} + a_{256} = 0.$$

By using equation (26) in the above equation we get

$$a_{121} - a_{123} - a_{127} + a_{136} + a_{139} + a_{143} - a_{166} - a_{169} - a_{173} + a_{182} -$$
$$a_{184} + a_{191} - a_{212} + a_{223} - a_{243} + a_{106} - a_{108} - a_{112} + a_{186} - a_{188} -$$
$$a_{191} + a_{251} - a_{253} = 0$$
$$\Rightarrow a_{106} - a_{108} - a_{112} + a_{121} - a_{123} - a_{127} + a_{136} + a_{139} + a_{143} - a_{166} - a_{169}$$
$$-a_{173} + a_{182} - a_{184} + a_{186} - a_{188} - a_{212} + a_{223} - a_{243} = a_{253} - a_{251}.$$

Now (5*) − (7*) gives us

$$a_{106} - a_{108} - a_{112} + a_{121} - a_{123} - a_{127} + a_{136} + a_{139} +$$
$$a_{143} - a_{166} - a_{169} - a_{173} + a_{182} - a_{184} + a_{186} - a_{188} -$$
$$a_{212} + a_{223} - a_{243} = a_{253} - a_{251}.$$

This equation is same as the above equation. Thus equation for a_{290}, does not yield a new equation.

By adopting the same procedure shown above we see that upto the equation for a_{271}, we do not find any other independent equation. In all the cases discussed above (i.e., from equation for a_{289} to equation for a_{271}) the resulting equations are equivalent to some of the equations obtained earlier. Also, as mentioned earlier that equation (1) to equation (81) are linearly independent because they can not be written as a linear combination of any other equation. We keep first 36 equations as they are, but the remaining 45 equations (i.e., from equation (37) to equation (81)) may be replaced by new equations which we obtain by using equations (1*) to (10*) in these equations. Here we mention only the resulting 45 equations starting from equation for a_{271} and moving onward to equation for a_{315}. These equations are as follows:

$$\begin{aligned} a_{271} = {} & a_{107} + a_{108} + a_{111} + a_{112} + a_{126} + a_{127} + a_{130} + a_{141} + a_{142} + \\ & a_{145} + a_{151} + a_{152} + a_{156} + a_{166} + a_{167} + a_{171} + a_{211} + a_{212} + \\ & a_{221} + a_{222}, \end{aligned} \tag{11*}$$

$$\begin{aligned} a_{272} = {} & -a_{107} + a_{109} - a_{111} + a_{113} - a_{126} + a_{128} + a_{131} - a_{141} + \\ & a_{143} + a_{146} - a_{151} - a_{152} - a_{156} + a_{181} + a_{182} + a_{186} - a_{211} + \\ & a_{213} - a_{221} + a_{223}, \end{aligned} \tag{12*}$$

$$\begin{aligned} a_{273} = {} & a_{107} + a_{110} + a_{111} + a_{112} + a_{119} + a_{126} + a_{127} + a_{132} + a_{134} + \\ & a_{141} + a_{144} + a_{147} + a_{151} + a_{152} + a_{156} + a_{196} + a_{197} + a_{201} + \\ & a_{211} + a_{212} + a_{219} + a_{221} + a_{224}, \end{aligned} \tag{13*}$$

$$\begin{aligned} a_{274} = {} & -a_{108} - a_{109} - a_{112} - a_{113} - a_{127} - a_{128} + a_{133} - a_{142} - a_{143} + \\ & a_{148} - a_{166} - a_{167} - a_{171} - a_{181} - a_{182} - a_{186} - a_{212} - a_{213} \\ & - a_{222} - a_{223}, \end{aligned} \tag{14*}$$

$$a_{275} = a_{108} - a_{110} - a_{119} + a_{142} - a_{144} + a_{149} + a_{166} + a_{167} + a_{171} - a_{196} - a_{197} - a_{201} - a_{219} + a_{222} - a_{224}, \tag{15*}$$

$$a_{276} = a_{109} + a_{110} + a_{112} + a_{113} + a_{119} + a_{127} + a_{128} + a_{134} + a_{135} + a_{143} + a_{144} + a_{150} + a_{181} + a_{182} + a_{186} + a_{196} + a_{197} + a_{201} + a_{212} + a_{213} + a_{219} + a_{223} + a_{224}, \tag{16*}$$

$$a_{277} = -a_{107} - a_{108} - a_{111} - a_{112} - a_{121} - a_{122} - a_{126} - a_{151} - a_{152} + a_{157} + a_{160} - a_{166} + a_{168} + a_{172} - a_{211} - a_{212} - a_{221} + a_{232}, \tag{17*}$$

$$a_{278} = a_{107} - a_{109} + a_{111} - a_{113} + a_{121} + a_{122} + a_{126} + a_{151} + a_{152} + a_{158} + a_{161} - a_{181} + a_{183} + a_{187} + a_{211} - a_{213} + a_{221} + a_{233}, \tag{18*}$$

$$a_{279} = -a_{107} - a_{110} - a_{111} - a_{112} - a_{119} - a_{121} - a_{122} - a_{126} - a_{127} + a_{129} - a_{134} - a_{151} - a_{152} + a_{159} + a_{162} - a_{196} + a_{198} + a_{202} - a_{211} - a_{212} - a_{219} - a_{221} + a_{234}, \tag{19*}$$

$$a_{280} = a_{108} + a_{109} + a_{112} + a_{113} + a_{127} + a_{128} - a_{133} - a_{157} - a_{158} + a_{163} + a_{166} - a_{168} - a_{172} + a_{181} - a_{183} - a_{187} + a_{212} + a_{213} - a_{232} - a_{233}, \tag{20*}$$

$$a_{281} = -a_{108} + a_{110} + a_{119} + a_{157} - a_{159} + a_{164} - a_{166} + a_{168} + a_{172} + a_{196} - a_{198} - a_{202} + a_{219} + a_{232} - a_{234}, \tag{21*}$$

$$a_{282} = -a_{109} - a_{110} - a_{112} - a_{113} - a_{119} - a_{127} - a_{128} - a_{134} - a_{135} + a_{158} + a_{159} + a_{165} - a_{181} + a_{183} + a_{187} - a_{196} + a_{198} + a_{202} - a_{212} - a_{213} - a_{219} + a_{233} + a_{234}, \tag{22*}$$

$$a_{283} = a_{121} + a_{122} + a_{126} - a_{167} - a_{168} + a_{175} - a_{222} - a_{232}, \quad (23^*)$$

$$a_{284} = a_{108} + a_{109} + a_{112} + a_{113} - a_{121} + a_{123} + a_{127} + a_{166} + a_{167} + a_{173} + a_{176} + a_{181} + a_{184} + a_{188} + a_{212} + a_{213} + a_{222} + a_{243}, \quad (24^*)$$

$$a_{285} = -a_{108} + a_{110} + a_{119} + a_{121} - a_{123} - a_{129} + a_{134} - a_{166} - a_{167} + a_{174} + a_{177} + a_{196} + a_{199} + a_{203} + a_{219} - a_{222} + a_{244}, \quad (25^*)$$

$$a_{286} = -a_{108} - a_{109} - a_{112} - a_{113} - a_{122} - a_{123} - a_{127} - a_{128} - a_{131} - a_{166} + a_{168} - a_{173} + a_{178} - a_{181} - a_{184} - a_{188} - a_{212} - a_{213} + a_{232} - a_{243}, \quad (26^*)$$

$$a_{287} = a_{108} - a_{110} - a_{119} + a_{122} + a_{123} - a_{132} - a_{134} + a_{166} - a_{168} - a_{174} + a_{179} - a_{196} - a_{199} - a_{203} - a_{219} - a_{232} - a_{244}, \quad (27^*)$$

$$a_{288} = a_{109} + a_{110} + a_{112} + a_{113} + a_{119} + a_{127} + a_{128} + a_{134} + a_{135} + a_{173} + a_{174} + a_{180} + a_{181} + a_{184} + a_{188} + a_{196} + a_{199} + a_{203} + a_{212} + a_{213} + a_{219} + a_{243} + a_{244}, \quad (28^*)$$

$$a_{289} = -a_{121} - a_{122} - a_{126} - a_{182} - a_{183} + a_{190} - a_{223} - a_{233} \quad (29^*)$$

$$a_{290} = a_{121} - a_{123} - a_{127} + a_{182} - a_{184} + a_{191} + a_{223} - a_{243}, \quad (30^*)$$

$$a_{291} = -a_{109} - a_{110} - a_{112} - a_{113} - a_{119} - a_{121} - a_{124} - a_{127} - a_{128} + a_{129} - a_{134} - a_{181} - a_{182} + a_{189} + a_{192} - a_{196} + a_{200} + a_{204} - a_{212} - a_{213} - a_{219} - a_{223} + a_{254}, \quad (31^*)$$

$$a_{292} = a_{122} + a_{123} - a_{130} + a_{183} + a_{184} + a_{193} + a_{233} + a_{243} \quad (32^*)$$

$$a_{293} = a_{109} + a_{110} + a_{112} + a_{113} + a_{119} - a_{122} + a_{124} + a_{127} + a_{128} + a_{132} + a_{134} + a_{181} - a_{183} - a_{189} + a_{194} + a_{196} - a_{200} - a_{204} + a_{212} + a_{213} + a_{219} - a_{233} - a_{254}, \tag{33*}$$

$$a_{294} = -a_{109} - a_{110} - a_{112} - a_{113} - a_{119} - a_{123} - a_{124} - a_{127} - a_{128} - a_{181} - a_{184} + a_{189} + a_{195} - a_{196} + a_{200} + a_{204} - a_{212} - a_{213} - a_{219} - a_{243} + a_{254}, \tag{34*}$$

$$a_{295} = a_{121} + a_{122} + a_{126} - a_{197} - a_{198} + a_{205} - a_{224} - a_{234}, \tag{35*}$$

$$a_{296} = -a_{121} + a_{123} + a_{127} + a_{197} - a_{199} + a_{206} + a_{224} - a_{244}, \tag{36*}$$

$$a_{297} = a_{121} + a_{124} + a_{128} - a_{197} - a_{200} + a_{207} - a_{224} - a_{254}, \tag{37*}$$

$$a_{298} = -a_{122} - a_{123} + a_{130} + a_{198} + a_{199} + a_{208} + a_{234} + a_{244}, \tag{38*}$$

$$a_{299} = a_{122} - a_{124} + a_{131} - a_{198} + a_{200} + a_{209} - a_{234} + a_{254}, \tag{39*}$$

$$a_{300} = a_{123} + a_{124} + a_{133} - a_{199} - a_{200} + a_{210} - a_{244} - a_{254}, \tag{40*}$$

$$a_{301} = a_{142} + a_{143} - a_{148} + a_{157} + a_{158} - a_{163} + a_{167} + a_{168} - a_{175} + a_{182} + a_{183} - a_{190} + a_{222} + a_{223} + a_{232} + a_{233}, \tag{41*}$$

$$a_{302} = -a_{142} + a_{144} - a_{149} - a_{157} + a_{159} - a_{164} - a_{167} - a_{168} + a_{175} + a_{197} + a_{198} - a_{205} - a_{222} + a_{224} - a_{232} + a_{234}, \tag{42*}$$

$$a_{303} = -a_{143} - a_{144} - a_{150} - a_{158} - a_{159} - a_{165} - a_{182} - a_{183} + a_{190} - a_{197} - a_{198} + a_{205} - a_{223} - a_{224} - a_{233} - a_{234}, \tag{43*}$$

$$a_{304} = a_{107} + a_{108} + a_{111} + a_{112} + a_{122} + a_{123} + a_{126} + a_{127} + a_{141} - a_{143} - a_{146} + a_{151} + a_{152} - a_{157} - a_{160} + a_{166} - a_{168} + a_{173} - a_{178} - a_{182} + a_{184} - a_{191} + a_{211} + a_{212} + a_{221} - a_{223} - a_{232} + a_{243}, \tag{44*}$$

$$a_{305} = -a_{107} - a_{108} - a_{111} - a_{112} - a_{122} - a_{123} - a_{126} - a_{127} - a_{141} - a_{144} - a_{147} - a_{151} - a_{152} + a_{157} + a_{160} - a_{166} + a_{168} + a_{174} - a_{179} - a_{197} + a_{199} - a_{206} - a_{211} - a_{212} - a_{221} - a_{224} + a_{232} + a_{244}, \tag{45*}$$

$$a_{306} = a_{143} + a_{144} + a_{150} - a_{173} - a_{174} - a_{180} + a_{182} - a_{184} + a_{191} + a_{197} - a_{199} + a_{206} + a_{223} + a_{224} - a_{243} - a_{244}, \tag{46*}$$

$$a_{307} = -a_{107} - a_{108} - a_{111} - a_{112} - a_{122} - a_{123} - a_{126} - a_{127} - a_{141} - a_{142} - a_{145} - a_{151} - a_{152} - a_{158} - a_{161} - a_{166} - a_{167} - a_{173} - a_{176} - a_{183} - a_{184} - a_{193} - a_{211} - a_{212} - a_{221} - a_{222} - a_{233} - a_{243}, \tag{47*}$$

$$a_{308} = a_{107} - a_{109} + a_{111} - a_{113} + a_{122} - a_{124} + a_{126} - a_{128} + a_{141} + a_{144} + a_{147} + a_{151} + a_{152} + a_{158} + a_{161} - a_{181} + a_{183} + a_{189} - a_{194} + a_{197} + a_{200} - a_{207} + a_{211} - a_{213} + a_{221} + a_{224} + a_{233} + a_{254}, \tag{48*}$$

$$a_{309} = a_{108} + a_{109} + a_{112} + a_{113} + a_{123} + a_{124} + a_{127} + a_{128} + a_{142} - a_{144} + a_{149} + a_{166} + a_{167} + a_{173} + a_{176} + a_{181} + a_{184} - a_{189} - a_{195} - a_{197} - a_{200} + a_{207} + a_{212} + a_{213} + a_{222} - a_{224} + a_{243} - a_{254}, \tag{49*}$$

$$a_{310} = a_{107} + a_{108} + a_{111} + a_{112} + a_{122} + a_{123} + a_{126} + a_{127} + a_{141} + a_{142} + a_{145} + a_{151} + a_{152} - a_{159} - a_{162} + a_{166} + a_{167} - a_{174} - a_{177} - a_{198} - a_{199} - a_{208} + a_{211} + a_{212} + a_{221} + a_{222} - a_{234} - a_{244}, \tag{50*}$$

$$\begin{aligned} a_{311} \;=\;& -a_{107}+a_{109}-a_{111}+a_{113}-a_{122}+a_{124}-a_{126}+a_{128}-a_{141}+ \\ & a_{143}+a_{146}-a_{151}-a_{152}+a_{159}+a_{162}+a_{181}+a_{182}-a_{189}- \\ & a_{192}+a_{198}-a_{200}-a_{209}-a_{211}+a_{213}-a_{221}+a_{223}+ \\ & a_{234}-a_{254}, \end{aligned} \tag{51*}$$

$$\begin{aligned} a_{312} \;=\;& -a_{108}-a_{109}-a_{112}-a_{113}-a_{123}-a_{124}-a_{127}-a_{128}-a_{142}- \\ & a_{143}+a_{148}-a_{166}-a_{167}+a_{174}+a_{177}-a_{181}-a_{182}+a_{189}+ \\ & a_{192}+a_{199}+a_{200}-a_{210}-a_{212}-a_{213}-a_{222}-a_{223}+ \\ & a_{244}+a_{254}, \end{aligned} \tag{52*}$$

$$\begin{aligned} a_{313} \;=\;& a_{158}+a_{159}+a_{165}+a_{173}+a_{174}+a_{180}+a_{183}+a_{184}+a_{193}+ \\ & a_{198}+a_{199}+a_{208}+a_{233}+a_{234}+a_{243}+a_{244}, \end{aligned} \tag{53*}$$

$$\begin{aligned} a_{314} =& -a_{108}-a_{109}-a_{112}-a_{113}-a_{123}-a_{124}-a_{127}-a_{128}+a_{157}- \\ & a_{159}+a_{164}-a_{166}+a_{168}-a_{173}+a_{178}-a_{181}-a_{184}+ \\ & a_{189}+a_{195}-a_{198}+a_{200}+a_{209}-a_{212}-a_{213}+a_{232}- \\ & a_{234}-a_{243}+a_{254}, \end{aligned} \tag{54*}$$

$$\begin{aligned} a_{315} =& a_{108}+a_{109}+a_{112}+a_{113}+a_{123}+a_{124}+a_{127}+a_{128}-a_{157}- \\ & a_{158}+a_{163}+a_{166}-a_{168}-a_{174}+a_{179}+a_{181}-a_{183}- \\ & a_{189}+a_{194}-a_{199}-a_{200}+a_{210}+a_{212}+a_{213}-a_{232}- \\ & a_{233}-a_{244}-a_{254}. \end{aligned} \tag{55*}$$

As mentioned earlier that when $d =$ 4 and $r = 8$, we have

$$\frac{8!}{2!.2!.2!.2!}.\frac{1}{4!} \;=\; 105 \text{ isotropic tensors of type } \delta_{i_1 i_2}\delta_{i_3 i_4}\delta_{i_5 i_6}\delta_{i_7 i_8},$$

$$\frac{8!}{2!.2!.4!}.\frac{1}{2!} = 210 \text{ isotropic tensors of type } \delta_{i_1i_2}\delta_{i_3i_4}\epsilon_{i_5i_6i_7i_8},$$

and

$$\frac{8!}{4!.4!}.\frac{1}{2!} = 35 \text{ isotropic tensors of type } \epsilon_{i_1i_2i_3i_4}\epsilon_{i_5i_6i_7i_8}.$$

Thus the number of possible candidates for the membership of a basis for the space of isotropic tensors of rank 8 is 350.

Uptill now, we have found that out of these 350 possible candidates for the membership of the basis for the space of isotropic tensors of rank 8, only 196 isotropic tensors are linearly independent. But we have only considered first 105 isotropic tensors of type $\delta_{i_1i_2}\delta_{i_3i_4}\delta_{i_5i_6}\delta_{i_7i_8}$ and 210 isotropic tensors of type $\delta_{i_1i_2}\delta_{i_3i_4}\epsilon_{i_5i_6i_7i_8}$. We have not yet talked about the last 35 possible candidates of the basis for the space of isotropic tensors of rank 8, which are of the type $\epsilon_{i_1i_2i_3i_4}\epsilon_{i_5i_6i_7i_8}$. We do not know whether these 35 isotropic tensors are linearly independent or not.

In [7], F. Ahmad and M. A. Rashid have used two identities of the form

$$\epsilon_{i_1i_2}\epsilon_{i_3i_4} = \delta_{i_1i_3}\delta_{i_2i_4} - \delta_{i_1i_4}\delta_{i_2i_3},$$

and

$$\begin{aligned}\epsilon_{i_1i_2i_3}\epsilon_{i_4i_5i_6} = {} & \delta_{i_1i_4}\delta_{i_2i_5}\delta_{i_3i_6} - \delta_{i_1i_4}\delta_{i_2i_6}\delta_{i_3i_5} + \\ & \delta_{i_1i_5}\delta_{i_2i_6}\delta_{i_3i_4} - \delta_{i_1i_5}\delta_{i_2i_4}\delta_{i_3i_6} + \\ & \delta_{i_1i_6}\delta_{i_2i_4}\delta_{i_3i_5} - \delta_{i_1i_6}\delta_{i_2i_5}\delta_{i_3i_4}.\end{aligned}$$

By using the above two identities, F. Ahmad and M. A. Rashid have found that among the 12 possible members of the basis for the space of isotropic tensors of rank 4 under $SO(2)$, only 6 are chosen to be linearly independent and remaining 6 members can be written as a linear combination of the 6 independent ones.

In our case i.e., when $d = 4$ and $r = 8$, we may also obtain such kind of identity.

Let us consider the product

$$
\begin{aligned}
\epsilon_{i_1 i_2 i_3 i_4} \epsilon_{i_5 i_6 i_7 i_8} = {} & \delta_{i_1 i_5} \delta_{i_2 i_6} \delta_{i_3 i_7} \delta_{i_4 i_8} - \delta_{i_1 i_5} \delta_{i_2 i_6} \delta_{i_3 i_8} \delta_{i_4 i_7} + \\
& \delta_{i_1 i_5} \delta_{i_2 i_7} \delta_{i_3 i_8} \delta_{i_4 i_6} - \delta_{i_1 i_5} \delta_{i_2 i_7} \delta_{i_3 i_6} \delta_{i_4 i_8} + \\
& \delta_{i_1 i_5} \delta_{i_2 i_8} \delta_{i_3 i_6} \delta_{i_4 i_7} - \delta_{i_1 i_5} \delta_{i_2 i_8} \delta_{i_3 i_7} \delta_{i_4 i_6} + \\
& \delta_{i_1 i_6} \delta_{i_2 i_5} \delta_{i_3 i_7} \delta_{i_4 i_8} - \delta_{i_1 i_6} \delta_{i_2 i_5} \delta_{i_3 i_8} \delta_{i_4 i_7} + \\
& \delta_{i_1 i_6} \delta_{i_2 i_7} \delta_{i_3 i_8} \delta_{i_4 i_5} - \delta_{i_1 i_6} \delta_{i_2 i_7} \delta_{i_3 i_5} \delta_{i_4 i_8} + \\
& \delta_{i_1 i_6} \delta_{i_2 i_8} \delta_{i_3 i_5} \delta_{i_4 i_7} - \delta_{i_1 i_6} \delta_{i_2 i_8} \delta_{i_3 i_7} \delta_{i_4 i_5} + \\
& \delta_{i_1 i_7} \delta_{i_2 i_5} \delta_{i_3 i_6} \delta_{i_4 i_8} - \delta_{i_1 i_7} \delta_{i_2 i_5} \delta_{i_3 i_8} \delta_{i_4 i_6} + \\
& \delta_{i_1 i_7} \delta_{i_2 i_6} \delta_{i_3 i_8} \delta_{i_4 i_5} - \delta_{i_1 i_7} \delta_{i_2 i_6} \delta_{i_3 i_5} \delta_{i_4 i_8} + \\
& \delta_{i_1 i_7} \delta_{i_2 i_8} \delta_{i_3 i_5} \delta_{i_4 i_6} - \delta_{i_1 i_7} \delta_{i_2 i_8} \delta_{i_3 i_6} \delta_{i_4 i_5} + \\
& \delta_{i_1 i_8} \delta_{i_2 i_5} \delta_{i_3 i_6} \delta_{i_4 i_7} - \delta_{i_1 i_8} \delta_{i_2 i_5} \delta_{i_3 i_7} \delta_{i_4 i_6} + \\
& \delta_{i_1 i_8} \delta_{i_2 i_6} \delta_{i_3 i_7} \delta_{i_4 i_5} - \delta_{i_1 i_8} \delta_{i_2 i_6} \delta_{i_3 i_5} \delta_{i_4 i_7} + \\
& \delta_{i_1 i_8} \delta_{i_2 i_7} \delta_{i_3 i_5} \delta_{i_4 i_6} - \delta_{i_1 i_8} \delta_{i_2 i_7} \delta_{i_3 i_6} \delta_{i_4 i_5}.
\end{aligned}
$$

The identities of the form mentioned above indicate that the last 35 members of the basis are dependent on the first 105 members of the form $\delta_{i_1 i_2} \delta_{i_3 i_4} \delta_{i_5 i_6} \delta_{i_7 i_8}$. Thus for $d = 4$ and $r = 8$, we have only $14^2 = 196$ linearly independent isotropic tensors. A particular set is given below

$$
\begin{aligned}
& \delta_{i_1 i_2} \delta_{i_3 i_4} \delta_{i_5 i_6} \delta_{i_7 i_8}, \delta_{i_1 i_2} \delta_{i_3 i_4} \delta_{i_5 i_7} \delta_{i_6 i_8}, \delta_{i_1 i_2} \delta_{i_3 i_4} \delta_{i_5 i_8} \delta_{i_6 i_7}, \delta_{i_1 i_2} \delta_{i_3 i_5} \delta_{i_4 i_6} \delta_{i_7 i_8}, \delta_{i_1 i_2} \delta_{i_3 i_5} \delta_{i_4 i_7} \delta_{i_6 i_8}, \\
& \delta_{i_1 i_2} \delta_{i_3 i_5} \delta_{i_4 i_8} \delta_{i_6 i_7}, \delta_{i_1 i_2} \delta_{i_3 i_6} \delta_{i_4 i_5} \delta_{i_7 i_8}, \delta_{i_1 i_2} \delta_{i_3 i_6} \delta_{i_4 i_7} \delta_{i_5 i_8}, \delta_{i_1 i_2} \delta_{i_3 i_6} \delta_{i_4 i_8} \delta_{i_5 i_7}, \delta_{i_1 i_2} \delta_{i_3 i_7} \delta_{i_4 i_5} \delta_{i_6 i_8}, \\
& \delta_{i_1 i_2} \delta_{i_3 i_7} \delta_{i_4 i_6} \delta_{i_5 i_8}, \delta_{i_1 i_2} \delta_{i_3 i_7} \delta_{i_4 i_8} \delta_{i_5 i_6}, \delta_{i_1 i_2} \delta_{i_3 i_8} \delta_{i_4 i_5} \delta_{i_6 i_7}, \delta_{i_1 i_2} \delta_{i_3 i_8} \delta_{i_4 i_6} \delta_{i_5 i_7}, \delta_{i_1 i_2} \delta_{i_3 i_8} \delta_{i_4 i_7} \delta_{i_5 i_6}, \\
& \delta_{i_1 i_3} \delta_{i_2 i_4} \delta_{i_5 i_6} \delta_{i_7 i_8}, \delta_{i_1 i_3} \delta_{i_2 i_4} \delta_{i_5 i_7} \delta_{i_6 i_8}, \delta_{i_1 i_3} \delta_{i_2 i_4} \delta_{i_5 i_8} \delta_{i_6 i_7}, \delta_{i_1 i_3} \delta_{i_2 i_5} \delta_{i_4 i_6} \delta_{i_7 i_8}, \delta_{i_1 i_3} \delta_{i_2 i_5} \delta_{i_4 i_7} \delta_{i_6 i_8}, \\
& \delta_{i_1 i_3} \delta_{i_2 i_5} \delta_{i_4 i_8} \delta_{i_6 i_7}, \delta_{i_1 i_3} \delta_{i_2 i_6} \delta_{i_4 i_5} \delta_{i_7 i_8}, \delta_{i_1 i_3} \delta_{i_2 i_6} \delta_{i_4 i_7} \delta_{i_5 i_8}, \delta_{i_1 i_3} \delta_{i_2 i_6} \delta_{i_4 i_8} \delta_{i_5 i_7}, \delta_{i_1 i_3} \delta_{i_2 i_7} \delta_{i_4 i_5} \delta_{i_6 i_8}, \\
& \delta_{i_1 i_3} \delta_{i_2 i_7} \delta_{i_4 i_6} \delta_{i_5 i_8}, \delta_{i_1 i_3} \delta_{i_2 i_7} \delta_{i_4 i_8} \delta_{i_5 i_6}, \delta_{i_1 i_3} \delta_{i_2 i_8} \delta_{i_4 i_5} \delta_{i_6 i_7}, \delta_{i_1 i_3} \delta_{i_2 i_8} \delta_{i_4 i_6} \delta_{i_5 i_7}, \delta_{i_1 i_3} \delta_{i_2 i_8} \delta_{i_4 i_7} \delta_{i_5 i_6},
\end{aligned}
$$

$\delta_{i_1i_4}\delta_{i_2i_3}\delta_{i_5i_6}\delta_{i_7i_8}, \delta_{i_1i_4}\delta_{i_2i_3}\delta_{i_5i_7}\delta_{i_6i_8}, \delta_{i_1i_4}\delta_{i_2i_3}\delta_{i_5i_8}\delta_{i_6i_7}, \delta_{i_1i_4}\delta_{i_2i_5}\delta_{i_3i_6}\delta_{i_7i_8}, \delta_{i_1i_4}\delta_{i_2i_5}\delta_{i_3i_7}\delta_{i_6i_8},$

$\delta_{i_1i_4}\delta_{i_2i_5}\delta_{i_3i_8}\delta_{i_6i_7}, \delta_{i_1i_4}\delta_{i_2i_6}\delta_{i_3i_5}\delta_{i_7i_8}, \delta_{i_1i_4}\delta_{i_2i_6}\delta_{i_3i_7}\delta_{i_5i_8}, \delta_{i_1i_4}\delta_{i_2i_6}\delta_{i_3i_8}\delta_{i_5i_7}, \delta_{i_1i_4}\delta_{i_2i_7}\delta_{i_3i_5}\delta_{i_6i_8},$

$\delta_{i_1i_4}\delta_{i_2i_7}\delta_{i_3i_6}\delta_{i_5i_8}, \delta_{i_1i_4}\delta_{i_2i_7}\delta_{i_3i_8}\delta_{i_5i_6}, \delta_{i_1i_4}\delta_{i_2i_8}\delta_{i_3i_5}\delta_{i_6i_7}, \delta_{i_1i_4}\delta_{i_2i_8}\delta_{i_3i_6}\delta_{i_5i_7}, \delta_{i_1i_4}\delta_{i_2i_8}\delta_{i_3i_7}\delta_{i_5i_6},$

$\delta_{i_1i_5}\delta_{i_2i_3}\delta_{i_4i_6}\delta_{i_7i_8}, \delta_{i_1i_5}\delta_{i_2i_3}\delta_{i_4i_7}\delta_{i_6i_8}, \delta_{i_1i_5}\delta_{i_2i_3}\delta_{i_4i_8}\delta_{i_6i_7}, \delta_{i_1i_5}\delta_{i_2i_4}\delta_{i_3i_6}\delta_{i_7i_8}, \delta_{i_1i_5}\delta_{i_2i_4}\delta_{i_3i_7}\delta_{i_6i_8},$

$\delta_{i_1i_5}\delta_{i_2i_4}\delta_{i_3i_8}\delta_{i_6i_7}, \delta_{i_1i_5}\delta_{i_2i_6}\delta_{i_3i_4}\delta_{i_7i_8}, \delta_{i_1i_5}\delta_{i_2i_6}\delta_{i_3i_7}\delta_{i_4i_8}, \delta_{i_1i_5}\delta_{i_2i_6}\delta_{i_3i_8}\delta_{i_4i_7}, \delta_{i_1i_5}\delta_{i_2i_7}\delta_{i_3i_4}\delta_{i_6i_8},$

$\delta_{i_1i_5}\delta_{i_2i_7}\delta_{i_3i_6}\delta_{i_4i_8}, \delta_{i_1i_5}\delta_{i_2i_7}\delta_{i_3i_8}\delta_{i_4i_6}, \delta_{i_1i_5}\delta_{i_2i_8}\delta_{i_3i_4}\delta_{i_6i_7}, \delta_{i_1i_5}\delta_{i_2i_8}\delta_{i_3i_6}\delta_{i_4i_7}, \delta_{i_1i_5}\delta_{i_2i_8}\delta_{i_3i_7}\delta_{i_4i_6},$

$\delta_{i_1i_6}\delta_{i_2i_3}\delta_{i_4i_5}\delta_{i_7i_8}, \delta_{i_1i_6}\delta_{i_2i_3}\delta_{i_4i_7}\delta_{i_5i_8}, \delta_{i_1i_6}\delta_{i_2i_3}\delta_{i_4i_8}\delta_{i_5i_7}, \delta_{i_1i_6}\delta_{i_2i_4}\delta_{i_3i_5}\delta_{i_7i_8}, \delta_{i_1i_6}\delta_{i_2i_4}\delta_{i_3i_7}\delta_{i_5i_8},$

$\delta_{i_1i_6}\delta_{i_2i_4}\delta_{i_3i_8}\delta_{i_5i_7}, \delta_{i_1i_6}\delta_{i_2i_5}\delta_{i_3i_4}\delta_{i_7i_8}, \delta_{i_1i_6}\delta_{i_2i_5}\delta_{i_3i_7}\delta_{i_4i_8}, \delta_{i_1i_6}\delta_{i_2i_5}\delta_{i_3i_8}\delta_{i_4i_7}, \delta_{i_1i_6}\delta_{i_2i_7}\delta_{i_3i_4}\delta_{i_5i_8},$

$\delta_{i_1i_6}\delta_{i_2i_7}\delta_{i_3i_5}\delta_{i_4i_8}, \delta_{i_1i_6}\delta_{i_2i_7}\delta_{i_3i_8}\delta_{i_4i_5}, \delta_{i_1i_6}\delta_{i_2i_8}\delta_{i_3i_4}\delta_{i_5i_7}, \delta_{i_1i_6}\delta_{i_2i_8}\delta_{i_3i_5}\delta_{i_4i_7}, \delta_{i_1i_6}\delta_{i_2i_8}\delta_{i_3i_7}\delta_{i_4i_5},$

$\delta_{i_1i_7}\delta_{i_2i_3}\delta_{i_4i_5}\delta_{i_6i_8}, \delta_{i_1i_7}\delta_{i_2i_3}\delta_{i_4i_6}\delta_{i_5i_8}, \delta_{i_1i_7}\delta_{i_2i_3}\delta_{i_4i_8}\delta_{i_5i_6}, \delta_{i_1i_7}\delta_{i_2i_4}\delta_{i_3i_5}\delta_{i_6i_8}, \delta_{i_1i_7}\delta_{i_2i_4}\delta_{i_3i_6}\delta_{i_5i_8},$

$\delta_{i_1i_7}\delta_{i_2i_4}\delta_{i_3i_8}\delta_{i_5i_6}, \delta_{i_1i_7}\delta_{i_2i_5}\delta_{i_3i_4}\delta_{i_6i_8}, \delta_{i_1i_7}\delta_{i_2i_5}\delta_{i_3i_6}\delta_{i_4i_8}, \delta_{i_1i_7}\delta_{i_2i_5}\delta_{i_3i_8}\delta_{i_4i_6}, \delta_{i_1i_7}\delta_{i_2i_6}\delta_{i_3i_4}\delta_{i_5i_8},$

$\delta_{i_1i_7}\delta_{i_2i_6}\delta_{i_3i_5}\delta_{i_4i_8}, \delta_{i_1i_7}\delta_{i_2i_6}\delta_{i_3i_8}\delta_{i_4i_5}, \delta_{i_1i_7}\delta_{i_2i_8}\delta_{i_3i_4}\delta_{i_5i_6}, \delta_{i_1i_7}\delta_{i_2i_8}\delta_{i_3i_5}\delta_{i_4i_6}, \delta_{i_1i_7}\delta_{i_2i_8}\delta_{i_3i_6}\delta_{i_4i_5},$

$\delta_{i_1i_8}\delta_{i_2i_3}\delta_{i_4i_5}\delta_{i_6i_7}, \delta_{i_1i_8}\delta_{i_2i_3}\delta_{i_4i_6}\delta_{i_5i_7}, \delta_{i_1i_8}\delta_{i_2i_3}\delta_{i_4i_7}\delta_{i_5i_6}, \delta_{i_1i_8}\delta_{i_2i_4}\delta_{i_3i_5}\delta_{i_6i_7}, \delta_{i_1i_8}\delta_{i_2i_4}\delta_{i_3i_6}\delta_{i_5i_7},$

$\delta_{i_1i_8}\delta_{i_2i_4}\delta_{i_3i_7}\delta_{i_5i_6}, \delta_{i_1i_8}\delta_{i_2i_5}\delta_{i_3i_4}\delta_{i_6i_7}, \delta_{i_1i_8}\delta_{i_2i_5}\delta_{i_3i_6}\delta_{i_4i_7}, \delta_{i_1i_8}\delta_{i_2i_5}\delta_{i_3i_7}\delta_{i_4i_6}, \delta_{i_1i_8}\delta_{i_2i_6}\delta_{i_3i_4}\delta_{i_5i_7},$

$\delta_{i_1i_8}\delta_{i_2i_6}\delta_{i_3i_5}\delta_{i_4i_7}, \delta_{i_1i_8}\delta_{i_2i_6}\delta_{i_3i_7}\delta_{i_4i_5}, \delta_{i_1i_8}\delta_{i_2i_7}\delta_{i_3i_4}\delta_{i_5i_6}, \delta_{i_1i_8}\delta_{i_2i_7}\delta_{i_3i_5}\delta_{i_4i_6}, \delta_{i_1i_8}\delta_{i_2i_7}\delta_{i_3i_6}\delta_{i_4i_5},$

$\delta_{i_2i_3}\delta_{i_5i_6}\epsilon_{i_1i_4i_7i_8}, \delta_{i_2i_3}\delta_{i_5i_7}\epsilon_{i_1i_4i_6i_8}, \delta_{i_2i_3}\delta_{i_5i_8}\epsilon_{i_1i_4i_6i_7}, \delta_{i_2i_3}\delta_{i_6i_7}\epsilon_{i_1i_4i_5i_8}, \delta_{i_2i_3}\delta_{i_6i_8}\epsilon_{i_1i_4i_5i_7},$

$\delta_{i_2i_3}\delta_{i_7i_8}\epsilon_{i_1i_4i_5i_6}, \delta_{i_2i_4}\delta_{i_5i_6}\epsilon_{i_1i_3i_7i_8}, \delta_{i_2i_4}\delta_{i_5i_7}\epsilon_{i_1i_3i_6i_8}, \delta_{i_2i_4}\delta_{i_5i_8}\epsilon_{i_1i_3i_6i_7}, \delta_{i_2i_4}\delta_{i_6i_7}\epsilon_{i_1i_3i_5i_8},$

$\delta_{i_2i_4}\delta_{i_6i_8}\epsilon_{i_1i_3i_5i_7}, \delta_{i_2i_4}\delta_{i_7i_8}\epsilon_{i_1i_3i_5i_6}, \delta_{i_2i_5}\delta_{i_3i_4}\epsilon_{i_1i_6i_7i_8}, \delta_{i_2i_5}\delta_{i_4i_6}\epsilon_{i_1i_3i_7i_8}, \delta_{i_2i_5}\delta_{i_4i_7}\epsilon_{i_1i_3i_6i_8},$

$\delta_{i_2i_5}\delta_{i_4i_8}\epsilon_{i_1i_3i_6i_7}, \delta_{i_2i_5}\delta_{i_6i_7}\epsilon_{i_1i_3i_4i_8}, \delta_{i_2i_5}\delta_{i_6i_8}\epsilon_{i_1i_3i_4i_7}, \delta_{i_2i_5}\delta_{i_7i_8}\epsilon_{i_1i_3i_4i_6}, \delta_{i_2i_6}\delta_{i_3i_4}\epsilon_{i_1i_5i_7i_8},$

$\delta_{i_2i_6}\delta_{i_3i_5}\epsilon_{i_1i_4i_7i_8}, \delta_{i_2i_6}\delta_{i_4i_5}\epsilon_{i_1i_3i_7i_8}, \delta_{i_2i_6}\delta_{i_4i_7}\epsilon_{i_1i_3i_5i_8}, \delta_{i_2i_6}\delta_{i_4i_8}\epsilon_{i_1i_3i_5i_7}, \delta_{i_2i_6}\delta_{i_5i_7}\epsilon_{i_1i_3i_4i_8},$

$\delta_{i_2i_6}\delta_{i_5i_8}\epsilon_{i_1i_3i_4i_7}, \delta_{i_2i_6}\delta_{i_7i_8}\epsilon_{i_1i_3i_4i_5}, \delta_{i_2i_7}\delta_{i_3i_4}\epsilon_{i_1i_5i_6i_8}, \delta_{i_2i_7}\delta_{i_3i_5}\epsilon_{i_1i_4i_6i_8}, \delta_{i_2i_7}\delta_{i_3i_6}\epsilon_{i_1i_4i_5i_8},$

$\delta_{i_2i_7}\delta_{i_4i_5}\epsilon_{i_1i_3i_6i_8}, \delta_{i_2i_7}\delta_{i_4i_6}\epsilon_{i_1i_3i_5i_8}, \delta_{i_2i_7}\delta_{i_4i_8}\epsilon_{i_1i_3i_5i_6}, \delta_{i_2i_7}\delta_{i_5i_6}\epsilon_{i_1i_3i_4i_8}, \delta_{i_2i_7}\delta_{i_5i_8}\epsilon_{i_1i_3i_4i_6},$

$\delta_{i_2i_7}\delta_{i_6i_8}\epsilon_{i_1i_3i_4i_5}, \delta_{i_2i_8}\delta_{i_3i_4}\epsilon_{i_1i_5i_6i_7}, \delta_{i_2i_8}\delta_{i_3i_5}\epsilon_{i_1i_4i_6i_7}, \delta_{i_2i_8}\delta_{i_3i_6}\epsilon_{i_1i_4i_5i_7}, \delta_{i_2i_8}\delta_{i_3i_7}\epsilon_{i_1i_4i_5i_6},$

$\delta_{i_2i_8}\delta_{i_4i_5}\epsilon_{i_1i_3i_6i_7}, \delta_{i_2i_8}\delta_{i_4i_6}\epsilon_{i_1i_3i_5i_7}, \delta_{i_2i_8}\delta_{i_4i_7}\epsilon_{i_1i_3i_5i_6}, \delta_{i_2i_8}\delta_{i_5i_6}\epsilon_{i_1i_3i_4i_7}, \delta_{i_2i_8}\delta_{i_5i_7}\epsilon_{i_1i_3i_4i_6},$

$\delta_{i_2i_8}\delta_{i_6i_7}\epsilon_{i_1i_3i_4i_5}, \delta_{i_3i_4}\delta_{i_5i_6}\epsilon_{i_1i_2i_7i_8}, \delta_{i_3i_4}\delta_{i_5i_7}\epsilon_{i_1i_2i_6i_8}, \delta_{i_3i_4}\delta_{i_5i_8}\epsilon_{i_1i_2i_6i_7}, \delta_{i_3i_4}\delta_{i_6i_7}\epsilon_{i_1i_2i_5i_8},$

$$\delta_{i_3i_4}\delta_{i_6i_8}\epsilon_{i_1i_2i_5i_7}, \delta_{i_3i_4}\delta_{i_7i_8}\epsilon_{i_1i_2i_5i_6}, \delta_{i_3i_5}\delta_{i_4i_6}\epsilon_{i_1i_2i_7i_8}, \delta_{i_3i_5}\delta_{i_4i_7}\epsilon_{i_1i_2i_6i_8},$$
$$\delta_{i_3i_5}\delta_{i_4i_8}\epsilon_{i_1i_2i_6i_7}, \delta_{i_3i_5}\delta_{i_6i_7}\epsilon_{i_1i_2i_4i_8}, \delta_{i_3i_5}\delta_{i_6i_8}\epsilon_{i_1i_2i_4i_7}, \delta_{i_3i_5}\delta_{i_7i_8}\epsilon_{i_1i_2i_4i_6},$$
$$\delta_{i_3i_6}\delta_{i_4i_5}\epsilon_{i_1i_2i_7i_8}, \delta_{i_3i_6}\delta_{i_4i_7}\epsilon_{i_1i_2i_5i_8}, \delta_{i_3i_6}\delta_{i_4i_8}\epsilon_{i_1i_2i_5i_7}, \delta_{i_3i_6}\delta_{i_5i_7}\epsilon_{i_1i_2i_4i_8},$$
$$\delta_{i_3i_6}\delta_{i_5i_8}\epsilon_{i_1i_2i_4i_7}, \delta_{i_3i_6}\delta_{i_7i_8}\epsilon_{i_1i_2i_4i_5}, \delta_{i_3i_7}\delta_{i_4i_5}\epsilon_{i_1i_2i_6i_8}, \delta_{i_3i_7}\delta_{i_4i_6}\epsilon_{i_1i_2i_5i_8},$$
$$\delta_{i_3i_7}\delta_{i_4i_8}\epsilon_{i_1i_2i_5i_6}, \delta_{i_3i_7}\delta_{i_5i_6}\epsilon_{i_1i_2i_4i_8}, \delta_{i_3i_7}\delta_{i_5i_8}\epsilon_{i_1i_2i_4i_6}, \delta_{i_3i_7}\delta_{i_6i_8}\epsilon_{i_1i_2i_4i_5},$$
$$\delta_{i_3i_8}\delta_{i_4i_5}\epsilon_{i_1i_2i_6i_7}, \delta_{i_3i_8}\delta_{i_4i_6}\epsilon_{i_1i_2i_5i_7}, \delta_{i_3i_8}\delta_{i_4i_7}\epsilon_{i_1i_2i_5i_6}, \delta_{i_3i_8}\delta_{i_5i_6}\epsilon_{i_1i_2i_4i_7},$$
$$\delta_{i_3i_8}\delta_{i_5i_7}\epsilon_{i_1i_2i_4i_6}, \delta_{i_3i_8}\delta_{i_6i_7}\epsilon_{i_1i_2i_4i_5}, \delta_{i_4i_5}\delta_{i_6i_7}\epsilon_{i_1i_2i_3i_8}, \delta_{i_4i_5}\delta_{i_6i_8}\epsilon_{i_1i_2i_3i_7},$$
$$\delta_{i_4i_5}\delta_{i_7i_8}\epsilon_{i_1i_2i_3i_6}, \delta_{i_4i_6}\delta_{i_5i_7}\epsilon_{i_1i_2i_3i_8}, \delta_{i_4i_6}\delta_{i_5i_8}\epsilon_{i_1i_2i_3i_7}, \delta_{i_4i_6}\delta_{i_7i_8}\epsilon_{i_1i_2i_3i_5},$$
$$\delta_{i_4i_7}\delta_{i_5i_6}\epsilon_{i_1i_2i_3i_8}, \delta_{i_4i_7}\delta_{i_5i_8}\epsilon_{i_1i_2i_3i_6}, \delta_{i_4i_7}\delta_{i_6i_8}\epsilon_{i_1i_2i_3i_5}, \delta_{i_4i_8}\delta_{i_5i_6}\epsilon_{i_1i_2i_3i_7},$$
$$\delta_{i_4i_8}\delta_{i_5i_7}\epsilon_{i_1i_2i_3i_6}, \delta_{i_4i_8}\delta_{i_6i_7}\epsilon_{i_1i_2i_3i_5}, \delta_{i_5i_6}\delta_{i_7i_8}\epsilon_{i_1i_2i_3i_4}, \delta_{i_5i_7}\delta_{i_6i_8}\epsilon_{i_1i_2i_3i_4},$$
$$\delta_{i_5i_8}\delta_{i_6i_7}\epsilon_{i_1i_2i_3i_4}.$$

From these, a set of linearly independent tensors of rank 8 can be constructed. For example $\delta_{i_1i_2}\delta_{i_3i_4}\delta_{i_5i_6}\delta_{i_7i_8}$ leads to $T_{iijjkkll}$ where summation over repeated indices is implied. Similarly $\delta_{i_2i_5}\delta_{i_3i_4}\epsilon_{i_1i_6i_7i_8}$ leads to $\epsilon_{i_1i_6i_7i_8}T_{i_1ijjii_6i_7i_8}$.

4.6 A General formula for the number of independent linear invariants:

So far, we have calculated the number of independent linear invariants of an arbitrary tensor of rank r, with $2 \leq r \leq 8$, by finding the dimension of the space of isotropic tensors. We have not only found the number, but we have also calculated the linearly independent invariants in each case explicitly. However, as the rank increases, the calculations become increasingly complex and it is very difficult to find the independent linear invariants, or merely thier number by using the direct approach. In order to handle such a situation, we use the theory of group representations to derive a formula for finding the number of linear invariants. This general formula does not provide us the independent linear invariants explicitly but it only gives us the number of independent linear invariants.

In this section, we find a general formula to obtain the number of linear invariants of an arbitrary tensor of rank r under $SO(4)$. In order to calculate the number of independent linear invariants of a tensor of an arbitrary rank r we need two things, character and the weight function. An important thing to note here is that, in order to define characters and orthogonality theorems, we require density function. We do not have a direct expression for density function of $SO(4)$. But we can make use of the fact that

$$SO(4) \cong SO(3) \otimes SO(3).$$

Due to this isomorphism we can use the density function of $SO(3)$ to obtain the density function of $SO(4)$.

The matrix representing an orthogonal coordinate transformation, in three dimensions, for a rotation through an angle φ, about z-axis passing through origin is given by

$$\begin{bmatrix} \cos\varphi & -\sin\varphi & 0 \\ \sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

This matrix gives us an irreducible representation of the group $SO(3)$. We are concerned with tensors of rank n. The character of this representation is obviously χ^n. This representation is highly reducible and contains many irreducible representatins with different multiplicities. We are interested in the total number that gives the number of linerly independent linear invariants. For finite groups, we need weight factors for such calculations. Weight factors are basically the numbers of elements in different conjugacy classes divided by the order of the group. For infinite groups, the corresponding number is called the weight function which essentially represents the weight of that particular class. For the rotation group $SO(3)$, this number is $\frac{(1-\cos\varphi)}{2\pi}$,where $0 \leq \varphi \leq 2\pi$. This satisfies

$$\int_0^{2\pi} \frac{1}{2\pi}\left(1-\cos\varphi\right)\, d\varphi = 1.$$

Using the above weight function, the number of linearly independent invariants $I_4(n)$ for a tensor of rank n in four dimensions is given by

$$I_4(n) = \left[\frac{1}{2\pi}\int_0^{2\pi} \chi^n\left(\varphi\right)\left(1-\cos\varphi\right)\, d\varphi\right]^2, \tag{A}$$

where the character χ is still unknown. We are using square of this integral because we are using the fact that $SO(4) \cong SO(3) \otimes SO(3)$. Now for finding the character $\chi\left(\varphi\right)$, we again make use of the fact that $SO(4) \cong SO(3) \otimes SO(3)$. Here $SO(4)$ has six independent generators while $SO(3)$ has three infinitesimal generators, namely $\{J_x, J_y, J_z\}$. For an irreducible representation, a standard convection for choosing a basis for the n-dimensional vector space is given by the following set of $n = 2j+1$ vectors (the value of j is often called the spin of the representation):

$$\left\{|jm> |m = -j, -j+1, -j+2, ..., \ j-2, \ j-1, \ j\right\}.$$

Denoting the (Hermitian) operators on this space that represent the generators, i.e., the basis of the Lie algebra for $SO(3)$, by $J_i^{(j)}$, these vectors are mutual eigenvectors of the matrices $J_z^{(j)}$. In [6,7] for $SO(3)$, the case $j = 1$ has been used. In this case, the vector space is three-dimensional. In this dissertation, we are dealing with $SO(4)$ and using the fact that $SO(4) \cong SO(3) \otimes SO(3)$, so we need value of $j = 1/2$. Here we are actually using the two-valued representation of the rotation group. In this case, the vector space is only two-dimensional, and the representation of the generators is

$$J_x^{\left(\frac{1}{2}\right)} = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad J_y^{\left(\frac{1}{2}\right)} = \frac{1}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad J_z^{\left(\frac{1}{2}\right)} = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Thus, the rotation $R(\varphi, o, o)$ through angle φ about the z-axis is given by

$$\begin{bmatrix} e^{i\varphi/2} & 0 \\ 0 & e^{-i\varphi/2} \end{bmatrix}.$$

The character or trace of this diagonal matrix is given by

$$\chi(\varphi) = 2\cos\frac{\varphi}{2}.$$

By using value of $\chi(\varphi)$ in equation (A) the expression for finding the number of independent linear invariants becomes

$$I_4(n) = \left[\frac{1}{2\pi}\int_0^{2\pi}(2\cos\frac{\varphi}{2})^n\,(1-\cos\varphi)\;d\varphi\right]^2.$$

As mentioned earlier that for $d = 4$, the number of linear invariants is zero for any tensor of odd rank. i.e., $I_4(n) = 0$, if n is an odd integer. So we are left with the case when n is an even integer. As we are concerned only about the tensors of even rank so the above integral may be written as

$$I_4(2n) = \left[\frac{1}{2\pi}\int_0^{2\pi}(2\cos\frac{\varphi}{2})^{2n}\,(1-\cos\varphi)\;d\varphi\right]^2. \qquad \text{(B)}$$

The above integral can be evaluated by using contour integration. Alternatively, we may also evaluate this integral by using Wallis cosine formula.

In this dissertation we evaluate the above integral (B) by using both of the above mentioned methods.

4.6.1 Method 1: (by using Wallis cosine formula)

The number of linear invariants for a tensor of rank r, in four dimensions is given by

$$\begin{aligned} I_4(2n) &= \left[\frac{1}{2\pi}\int_0^{2\pi}(2\cos\frac{\varphi}{2})^{2n}\,(1-\cos\varphi)\;d\varphi\right]^2 \\ &= \left[\frac{1}{2\pi}\int_0^{2\pi}2^{2n}\cos^{2n}\frac{\varphi}{2}\left(2\sin^2\frac{\varphi}{2}\right)\;d\varphi\right]^2 \\ &= \left[\frac{2^{2n+1}}{2\pi}\int_0^{2\pi}\cos^{2n}\frac{\varphi}{2}\left(\sin^2\frac{\varphi}{2}\right)\;d\varphi\right]^2. \end{aligned}$$

Let

$$\begin{aligned} \frac{\varphi}{2} &= x \\ d\varphi &= 2dx, \end{aligned}$$

and

$$\begin{aligned} \varphi \longrightarrow 0, x \longrightarrow 0 \\ \varphi \longrightarrow 2\pi, x \longrightarrow \pi \end{aligned}$$

So, the above integral becomes

$$\begin{aligned} I_4(2n) &= \left[\frac{2^{2n+1}}{2\pi}\int_0^{\pi}\cos^{2n}x.\sin^2 x\;\;(2dx)\right]^2 \\ &= \left[\frac{2^{2n+1}}{\pi}\int_0^{\pi}\cos^{2n}x.\sin^2 x\;dx\right]^2 \\ &= \left[\frac{2^{2n+1}}{\pi}\int_0^{\pi}\cos^{2n}x\left(1-\cos^2 x\right)\;dx\right]^2 \\ &= \left[\frac{2^{2n+1}}{\pi}\int_0^{\pi}\left[\cos^{2n}x-\cos^{2n+2}x\right]\;dx\right]^2 \end{aligned}$$

$$\begin{aligned} I_4(2n) &= \left[\frac{2^{2n+1}}{\pi}\left(\int\limits_0^{\pi}\cos^{2n}xdx - \int\limits_0^{\pi}\cos^{2n+2}x\ dx\right)\right]^2 \\ &= \left[\frac{2^{2n+1}}{\pi}\left(\int\limits_0^{\pi}\cos^{2n}xdx - \int\limits_0^{\pi}\cos^{2n+2}x\ dx\right)\right]^2. \end{aligned} \tag{1}$$

Here we use the property of even function. Let

$$\begin{aligned} f_1(x) &= \cos^{2n}x. \\ f_1(\pi - x) &= [\cos(\pi - x)]^{2n} \\ &= [-\cos x]^{2n} \\ &= \cos^{2n}x, \text{ as we are dealing with tensors of even rank only.} \end{aligned}$$

This implies that

$$f_1(x) = f_1(\pi - x).$$

This shows that $f_1(x)$ is an even function. Thus, we may write

$$f_1(x) = \int\limits_0^{\pi}\cos^{2n}xdx = 2\int\limits_0^{\pi/2}\cos^{2n}xdx.$$

Similarly,

$$f_2(x) = \int\limits_0^{\pi}\cos^{2n+2}xdx = 2\int\limits_0^{\pi/2}\cos^{2n+2}xdx.$$

So, integral (1) becomes

$$I_4(2n) = \left[\frac{2^{2n+2}}{\pi}\left(\int\limits_0^{\pi/2}\cos^{2n}xdx - \int\limits_0^{\pi/2}\cos^{2n+2}x\ dx\right)\right]^2 \tag{2}$$

In order to evaluate these two integrals we use Wallis cosine formula. Now

$$\int\limits_0^{\pi/2}\cos^{2n}xdx = \frac{(2n-1)(2n-3)(2n-5).....5.3.1}{2n(2n-2)(2n-4).......6.4.2}.\frac{\pi}{2}$$

and

$$\int_0^{\pi/2} \cos^{2n+2} x dx = \frac{(2n+1)(2n-1)(2n-3).....5.3.1}{(2n+2)2n(2n-2).......6.4.2}.\frac{\pi}{2}$$

So, equation (2) becomes

$$\begin{aligned}
I_4(2n) &= \left[\frac{2^{2n+2}}{\pi}\left\{\frac{(2n-1)(2n-3)....5.3.1}{2n(2n-2)(2n-4)....6.4.2}.\frac{\pi}{2} - \frac{(2n+1)(2n-1)....5.3.1}{(2n+2)2n(2n-2)....6.4.2}.\frac{\pi}{2}\right\}\right]^2 \\
&= \left[\frac{2^{2n+2}}{\pi}.\frac{\pi}{2}\left\{\frac{(2n-1)(2n-3)(2n-5).....5.3.1}{2n(2n-2)(2n-4).......6.4.2}\right\}\left\{1-\frac{(2n+1)}{(2n+2)}\right\}\right]^2 \\
&= \left[2^{2n+1}\left\{\frac{(2n-1)(2n-3)(2n-5).....5.3.1}{2n(2n-2)(2n-4).......6.4.2}\right\}\left\{\frac{1}{2(n+1)}\right\}\right]^2 \\
&= \left[\frac{2^{2n}}{(n+1)}\left\{\frac{(2n-1)(2n-3)(2n-5).....5.3.1}{2n(2n-2)(2n-4).......6.4.2}\right\}\right]^2 \\
&= \left[\frac{2^{2n}}{(n+1)}\left\{\frac{2n(2n-1)(2n-2)(2n-3)....6.5.4.3.2.1}{[2n(2n-2)(2n-4).......6.4.2]^2}\right\}\right]^2 \\
&= \left[\frac{2^{2n}}{(n+1)}\left\{\frac{(2n)!}{[2^n.n(n-1)(n-2).......3.2.1]^2}\right\}\right]^2 \\
&= \left[\frac{2^{2n}}{(n+1)}\left\{\frac{(2n)!}{2^{2n}.(n!)^2}\right\}\right]^2 \\
&= \left[\frac{2^{2n}}{2^{2n}}\left\{\frac{(2n)!}{(n+1).n!.n!}\right\}\right]^2 \\
&= \left[\frac{(2n)!}{(n+1)!.n!}\right]^2.
\end{aligned}$$

Thus the number of independent linear invariants of an arbitrary tensor of even rank under $SO(4)$ is given by

$$I_4(2n) = \left[\frac{(2n)!}{(n+1)!.n!}\right]^2.$$

4.6.2 Method 2: (by using contour integration)

The number of independent linear invariants for a tensor of rank n, in four dimensions is given by

$$\begin{aligned}
I_4(2n) &= \left[\frac{1}{2\pi} \int_0^{2\pi} (2\cos\frac{\varphi}{2})^{2n} \,(1-\cos\varphi)\; d\varphi \right]^2 \\
&= \left[\frac{1}{2\pi} \int_0^{2\pi} 2^{2n} \left(\cos^2\frac{\varphi}{2}\right)^n (1-\cos\varphi)\; d\varphi \right]^2 \\
&= \left[\frac{2^{2n-1}}{\pi} \int_0^{2\pi} \left(\frac{1+\cos\varphi}{2}\right)^n (1-\cos\varphi)\; d\varphi \right]^2 \\
&= \left[\frac{2^{2n-1}}{2^n\pi} \int_0^{2\pi} (1+\cos\varphi)^n \,(1-\cos\varphi)\; d\varphi \right]^2 \\
&= \left[\frac{2^{n-1}}{\pi} \int_0^{2\pi} (1+\cos\varphi)^n \,(1-\cos\varphi)\; d\varphi \right]^2 .
\end{aligned}$$

In order to evaluate the above integral we make the following substitutions.

Let

$$\begin{aligned}
z &= e^{i\varphi} \\
\frac{dz}{iz} &= d\varphi
\end{aligned}$$

So that

$$\begin{aligned}
\cos\varphi &= \frac{z+z^{-1}}{2} = \frac{1+z^2}{2z}. \\
1+\cos\varphi &= 1+\frac{1+z^2}{2z} = \frac{(1+z)^2}{2z}. \\
1-\cos\varphi &= 1-\frac{1+z^2}{2z} = \frac{2z-1-z^2}{2z}.
\end{aligned}$$

After making these substitutions, the above integral becomes

$$I_4(2n) = \left[\frac{2^{n-1}}{\pi} \oint_c \left\{ \frac{(1+z)^2}{2z} \right\}^n \left\{ \frac{2z-1-z^2}{2z} \right\} \frac{dz}{iz} \right]^2,$$

where C denotes the unit circle with center at the origin

$$\begin{aligned} I_4(2n) &= \left[\frac{2^{n-1}}{2^{n+1}\pi i} \oint_c \frac{1}{z^{n+2}} \ (1+z)^{2n} \left(2z-1-z^2\right) \ dz \right]^2 \\ &= \left[\frac{1}{2^2\pi i} \oint_c \frac{1}{z^{n+2}} \ (1+z)^{2n} \left(2z-1-z^2\right) \ dz \right]^2. \end{aligned}$$

Now by using binomial expansion

$$\begin{aligned} (1+z)^{2n} &= 1 + 2nz + \binom{2n}{2} z^2 + ... + \binom{2n}{n-1} z^{n-1} + \binom{2n}{n} z^n \\ &+ \binom{2n}{n+1} z^{n+1} + ... + \ z^{2n}. \end{aligned}$$

$$\begin{aligned} (1+z)^{2n}\left(2z-1-z^2\right) &= 2z + 4nz^2 + ... + 2\binom{2n}{n-1} z^n + 2\binom{2n}{n} z^{n+1} \\ &+ ... + 2z^{2n+1} - 1 - 2nz - ... - \binom{2n}{n} z^n \\ &- \binom{2n}{n+1} z^{n+1} - ... - z^{2n} - z^2 - 2nz^3 \\ &- \binom{2n}{2} z^4 - ... - \binom{2n}{n-1} z^{n+1} \\ &- \binom{2n}{n} z^{n+2} - .... - z^{2n+2}. \end{aligned}$$

$$\begin{aligned} (1+z)^{2n}\left(2z-1-z^2\right) &= -1 + 2(1-n)z + \left[4n - \binom{2n}{2} - 1\right] z^2 \\ &+ ... + \left[2\binom{2n}{n-1} - \binom{2n}{n} - \binom{2n}{n-2}\right] z^n \\ &+ \left[2\binom{2n}{n} - \binom{2n}{n+1} - \binom{2n}{n-1}\right] z^{n+1} \\ &+ ... - z^{2n} + 2z^{2n+1} - z^{2n+2}. \end{aligned}$$

So, the above integral takes the form

$$I_4(2n) = \left[\frac{1}{2\pi i} \oint_c \frac{1}{2z^{n+2}} \left(\begin{array}{c} -1 + 2z + ... + \left[2\binom{2n}{n-1} - \binom{2n}{n} - \binom{2n}{n-2}\right] z^n \\ + \left[2\binom{2n}{n} - \binom{2n}{n+1} - \binom{2n}{n-1}\right] z^{n+1} + ... - z^{2n+2} \end{array} \right) dz \right]^2 .$$

Now we find the residues of the above integral. The only non-vanishing term here is the coefficient of z^{n+1}. Our integral comes out to be

$$\begin{aligned}
I_4(2n) &= \left[\frac{1}{2} \left\{ 2\binom{2n}{n} - \binom{2n}{n+1} - \binom{2n}{n-1} \right\} \right]^2 \\
&= \left[\frac{1}{2} \left\{ \frac{2\,(2n)!}{(n)!\,(2n-n)!} - \frac{(2n)!}{(n+1)!\,(2n-n-1)!} - \frac{(2n)!}{(n-1)!\,(2n-n+1)!} \right\} \right]^2 \\
&= \left[\frac{(2n)!}{2} \left\{ \frac{2}{(n)!\,(n)!} - \frac{1}{(n+1)!\,(n-1)!} - \frac{1}{(n-1)!\,(n+1)!} \right\} \right]^2 \\
&= \left[\frac{(2n)!}{2} \left\{ \frac{2\,(n+1)}{(n+1)!\,(n)!} - \frac{2}{(n+1)!\,(n-1)!} \right\} \right]^2 \\
&= \left[\frac{(2n)!}{2} \left\{ \frac{2\,(n+1)}{(n+1)!\,(n)!} - \frac{2n}{(n+1)!\,(n)!} \right\} \right]^2 \\
&= \left[\frac{(2n)!}{2\,(n+1)!\,(n)!} \left\{ 2\,(n+1) - 2n \right\} \right]^2 \\
&= \left[\frac{(2n)!}{2\,(n+1)!\,(n)!} \left\{ 2 \right\} \right]^2 \\
&= \left[\frac{(2n)!}{(n+1)!\,(n)!} \right]^2 .
\end{aligned}$$

Thus, the number of linearly independent invariants of a tensor of arbitrary rank under $SO(4)$ comes out to be

$$I_4(2n) = \left[\frac{(2n)!}{(n+1)!\,(n)!} \right]^2 .$$

Since we have calculated our general formula for tensors of even rank that is why we are using $2n$. If we let $r = 2n$, then the general formula may be rewritten as

$$I_4(r) = \left[\frac{(r)!}{\left(\frac{r}{2}+1\right)!\,\left(\frac{r}{2}\right)!} \right]^2 .$$

4.6.3 Theorem:

The number $I_4(r)$ of independent linear invariants of a tensor of rank r in four dimensions is zero if r is an odd integer and if r is an even integer then this number is:

$$I_4(r) = \left[\frac{(r)!}{(\frac{r}{2}+1)!\,(\frac{r}{2})!}\right]^2. \tag{I}$$

Now we verify that the number of independent invariants calculated by finding the dimension of the space of isotropic tensors agrees with the number produced by equation (I).

1. $d = 4, r \;= 2$

$$\begin{aligned} I_4(r) &= \left[\frac{(r)!}{(\frac{r}{2}+1)!\,(\frac{r}{2})!}\right]^2, \\ I_4(2) &= \left[\frac{(2)!}{(\frac{2}{2}+1)!\,(\frac{2}{2})!}\right]^2 \\ &= \left[\frac{(2)!}{(2)!\,(1)!}\right]^2 = 1^2 = 1. \end{aligned}$$

2. $d = 4, r \;= \;4$

$$\begin{aligned} I_4(r) &= \left[\frac{(r)!}{(\frac{r}{2}+1)!\,(\frac{r}{2})!}\right]^2, \\ I_4(4) &= \left[\frac{(4)!}{(\frac{4}{2}+1)!\,(\frac{4}{2})!}\right]^2 \\ &= \left[\frac{(4)!}{(3)!\,(2)!}\right]^2 = 2^2 = 4. \end{aligned}$$

3. $d = 4, r \;=\; 6$

$$\begin{aligned} I_4(r) &= \left[\frac{(r)!}{\left(\frac{r}{2}+1\right)!\left(\frac{r}{2}\right)!}\right]^2, \\ I_4(6) &= \left[\frac{(6)!}{\left(\frac{6}{2}+1\right)!\left(\frac{6}{2}\right)!}\right]^2 \\ &= \left[\frac{(6)!}{(4)!\,(3)!}\right]^2 = 5^2 = 25. \end{aligned}$$

4. $d = 4, r \;=\; 8$

$$\begin{aligned} I_4(r) &= \left[\frac{(r)!}{\left(\frac{r}{2}+1\right)!\left(\frac{r}{2}\right)!}\right]^2, \\ I_4(8) &= \left[\frac{(8)!}{\left(\frac{8}{2}+1\right)!\left(\frac{8}{2}\right)!}\right]^2 \\ &= \left[\frac{(8)!}{(5)!\,(4)!}\right]^2 = (14)^2 = 196. \end{aligned}$$

Chapter 5

Conclusion

In denouement, this work has been *very* challenging, but very interesting as well.

F. Ahmad and M. A. Rashid [6,7] have already studied the number of independent linear invariants under $SO(2)$ as well as $SO(3)$ of a Cartesian tensor of an arbitrary rank r. They have defined a linear form in terms of elements of a tensor. Formulas for finding the number of independent linear invariants for an arbitrary tensor in two and three dimensions have been derived by them and they have also obtained explicit expressions for the simple cases.

In this dissertation, we have found linearly independent invariants of an arbitrary tensor of rank r with $2 \leq r \leq 8$. when $d = 4$. We have not only found the number, but also have calculated the linearly independent invariants in each case explicitly.

However we have seen that for $d = 4$, no isotropic tensor of rank 3 can be constructed in terms of products of $\delta_{i_1 i_2}$ and $\epsilon_{i_1 i_2 i_3 i_4}$. The reason for this is: both $\delta_{i_1 i_2}$ and $\epsilon_{i_1 i_2 i_3 i_4}$ are tensors of even rank. Also we are talking about a rank 3 tensor while $\epsilon_{i_1 i_2 i_3 i_4}$ is a rank 4 tensor. It is obvious that this result must hold for any tensor of odd rank. Thus we concluded that for $d = 4$, we only find the linear invariants of a tensor of even rank because number of independent linear invariants is zero for any tensor of odd rank.

The number of independent linear invariants when $d = 4$ and $r = 2$ is found to be $1^2 = 1$. When $r = 4$, this number increases to $2^2 = 4$. For $r = 6$ and $r = 8$, the number of independent linear invariants are found to be $5^2 = 25$ and $(14)^2 = 196$ repectively.

We ascertained that the calculations for the case, when $d = 4$ and $r = 8$, are quite tedious and for higher ranks i.e., for $r > 8$, these calculations become increasingly complex. So, group theoretic methods have been applied to derive a general formula for finding the number of independent linear invariants for ranks of higher order. The general formula is given by

$$I_4(r) = \left[\frac{(r)!}{\left(\frac{r}{2} + 1\right)! \left(\frac{r}{2}\right)!} \right]^2 .$$

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