Propagation of Plane Waves in Initially-Stressed Elastic Materials

by

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Q. Farooq
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This Work is Dedicated to my
Beloved Parents,
Dearest Elder Brother
Zaheer Iqbal
& **Younger Brother**
Mansoor Iqbal

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Naveed Iqbal

Abstract

In this research work, we study the wave propagation in elastic materials which are initially-stressed. This work is based on the theory of non-linear elasticity and the theory of invariants.

Using the theory of invariants, we derived the general constitutive equations for a hyper-elastic material in the presence of initial stress. The constitutive laws of the material is based on the strain-energy function which also depends on the combined invariants of the right cauchy-Green deformation tensor and the initial stress tensor. In three dimensional case for the compressible materials, there are ten such invariants.

The problem of homogenous plane waves in an initially-stressed compressible half-space is considered. The basic theory of the problem is to study the reflection of plane waves from the boundary of half-space. Using the specific strain-energy function, we deduce the reflection coefficients and then graphically examined the behavior of one or two reflected waves for an incident P wave.

Contents

Chapter 1

Introduction

The research carried out in this dissertation mainly includes study of wave propagation in initially stressed elastic materials. In particular, the problem of reflection of waves is considered in an initially stressed half-space. Linearized theory is used for constitutive equations. We use the theory of infinitesimal deformations superimposed on the finite deformations to obtain the incremental equations for initially stressed deformed hyper-elastic materials. The research is based on the realistic assumption that a material can be internally stressed in its unloaded undeformed (or reference) configuration. In railway tracks, for example, an initial stress is caused by bending of metal because of thermal expansion and/or mechanical processing like reshaping of the metals parts. Internal stresses are imposed on compounds such as rocks due to processes likewise heating, cooling, burial and many past structural deformed earth crust events, etc. In Earth crust, internal and all these types of stresses remain stored inside the material after the compounds are independent of boundary loads. In all above discussed cases, the material is considered to be initially stressed. We can use the term *initial stress* in its broadest sense irrespective of the processes that cause this stress to develop. The term *pre-stress* replaces the term initial stress when there is an associated pre-strain from an unstressed configuration due to an applied load. When an initial stress is present in the absence of body forces and surface traction (applied loads) it is called *residual stress*, as in the definition adopted by Hoger [1]. In the absence of loads we consider a residual stress which maintains equilibrium in the body. *Pre-stresses* and *residual stresses* are examples of initial stresses but they are different in nature.

In this research work, we consider an initially-stressed hyper-elastic material. For such materials, we study the effect of initial stress on wave speed and also the problem of reflection of a homogenous plane wave from an initially-stressed half-space. In [2] the authors (Ogden and Sotiropolus) examined the effect of a homogenous pre-stress and finite deformation on the propagation of plane waves in compressible elastic materials and also the reflection of plane wave from the boundary of a half-space. In this research we follow the notations adopted in [2] however the nature of various constants appearing in this dissertation is considerably different from those in [2]. Biot [3, 4] examined the various effects of wave propagation in an initially stressed material. Also, Tang [5] considered wave motion in an infinite and initially stressed material medium for various special cases and compared his results with the already found results. The existence of residual stress in an elastic body having material symmetry was studied by Hoger [1]. For the development of basic constitutive equations for residually-stressed materials, we refer to [6, 7, 8, 9]. The work done by Hoger [1] was followed by Man and Lu [10] and they presented a generalized results which is much more relevant to Biot's work.

To formulate the problem, we use the theory of invariants as developed in [11]. The effect of the initial stress present in the material is included in the formulation through the components of the elasticity tensor which reduces to the usual elastic constants when the initial stresses reduces to zero.

The basic concepts for this research problem are given in Chapter 2 together with the concept of the elasticity tensor. We present the governing equilibrium equations for finite elasticity when it depends on the initial stress. We present the expressions for the invariants for compressible hyper-elastic material with initial stress (τ). In Chapter 2, we consider the general form of elasticity tensor which depends on the right Cauchy-Green deformation tensor ($\mathbf{C}$) as well as the initial stress tensor (τ). In the case of compressible materials, the detailed expressions for this tensor are given which are particularized for the deformed and the undeformed initially stressed reference configuration.

In Chapter 3, we discuss the propagation of plane waves in an initially stressed compressible half-space. The study of the plane wave propagation in both the cases of a homogenous compressible material in its un-deformed as well as in deformed state is not been carried out yet. We confine this dissertation to the study in the latter case. In this regard, We discuss the reflection of plane wave and derive the reflection coefficients which ensure the presence of one or two reflected waves. For instance, we consider the P (primary) waves and find that only one reflected or two reflected waves may exist, depending on the initial stress components. This theory is applicable to seismic waves which are used to locate the earthquakes.

Chapter 2

Basics of the Theory of Finite Deformations in an Elastic Material

A body **B** is a set and its elements can be put into one-to-one correspondence with the points of a region $\mathcal{B}$ in the three-dimensional Euclidean point space. $\mathcal{B}$ is called a configuration and the elements of **B** are called the material points. A particular but arbitrarily chosen configuration is distinguished as a *reference configuration* and is denoted by $\mathcal{B}_R$. Consider that in this reference configuration the body will be at rest and there are no outer forces present in it. Let the body **B** is deformed quasi-statically from $\mathcal{B}_R$ and occupies a new configuration, denoted by $\mathcal{B}_C$ called as the *deformed configuration* of B. We refer the readers [11, 12, 13, 14] for the basic material covered in this chapter.

2.1 Deformation in an Elastic Material

A deformation is represented by the mapping $\chi : \mathcal{B}_R \to \mathcal{B}_C$ that carries point $\mathbf{X} \in \mathcal{B}_R$ into the point $\mathbf{x} = \boldsymbol{\chi}(\mathbf{X}, t)$ in $\mathcal{B}_C$. Let $\mathbf{x}$ and $\mathbf{X}$ have rectangular Cartesian coordinates x_i and X_A respectively, where i, $A \in \{1, 2, 3\}$, with components

$$x_i = \chi_i(X_A, t), \quad i, A = \{1, 2, 3\}. \tag{2.1.1}$$

We emphasize that Greek and Roman letters are used for the indices associated with the deformed and un-deformed (reference) configuration, respectively.

Let Grad, Div (grad, div) denote the gradient, divergence and curl operator in the reference (current) configuration, i.e. with respect to $\mathbf{X}(\mathbf{x})$. The deformation gradient tensor $\mathbf{F}$ is defined as

$$\mathbf{F}(\mathbf{X}, t) = \operatorname{Grad} \mathbf{x} = \operatorname{Grad} \boldsymbol{\chi}(\mathbf{X}, t), \tag{2.1.2}$$

in component form $F_{iA} = \partial x_i / \partial X_A$.

We define J as

$$J = \det \mathbf{F}. \tag{2.1.3}$$

Let $\mathbf{F}$ be a tensor such that $J > 0$. Then the polar decompositions $\mathbf{F}$ are given by

$$\mathbf{F} = \mathbf{RU} = \mathbf{VR}, \tag{2.1.4}$$

where $\mathbf{U}$ and $\mathbf{V}$ are positive definite, symmetric tensors, called the right stretch and the left stretch tensors, respectively, while $\mathbf{R}$ is proper orthogonal tensor.

$\mathbf{U}$ and $\mathbf{V}$ can be decomposed in the spectral form as

$$\begin{aligned} \mathbf{U} &= \sum_{k=1}^{3} \lambda_k \mathbf{u}^{(k)} \otimes \mathbf{u}^{(k)}, \\ \mathbf{V} &= \sum_{k=1}^{3} \lambda_k \mathbf{v}^{(k)} \otimes \mathbf{v}^{(k)}, \end{aligned} \tag{2.1.5}$$

where $\lambda_k > 0,\ k \in \{1,2,3\}$, are the eigenvalues called the *principal stretches*, and the eigenvectors of $\mathbf{U}$ and $\mathbf{V}$ are $\mathbf{u}^{(k)}$ and $\mathbf{v}^{(k)}$, respectively. $\mathbf{u}^{(k)}$ and $\mathbf{v}^{(k)}$ are the Lagrangian and Eulerian principal axes and $\otimes$ denotes the tensor product.

In terms of the principal stretches λ_k the alternative expressions for $J = \det \mathbf{F}$ are provided by

$$J = \det \mathbf{U} = \det \mathbf{V} = \lambda_1 \lambda_2 \lambda_3. \tag{2.1.6}$$

We also note the vectors $\mathbf{u}^{(k)}$ and $\mathbf{v}^{(k)}$ have the connection

$$\mathbf{v}^{(k)} = \mathbf{R}\mathbf{u}^{(k)}, \quad k = \{1,2,3\}. \tag{2.1.7}$$

The right and left Cauchy-Green deformation tensors, $\mathbf{C}$ and $\mathbf{B}$ are defined by

$$\begin{aligned} \mathbf{C} &= \mathbf{F}^T\mathbf{F} = \mathbf{U}^2, \\ \mathbf{B} &= \mathbf{F}\mathbf{F}^T = \mathbf{V}^2, \end{aligned} \tag{2.1.8}$$

respectively. In the formation of constitutive laws, the tensors $\mathbf{C}$ and $\mathbf{B}$ play important role , in particular through their principal invariants defined (for either $\mathbf{C}$ or $\mathbf{B}$) as

$$I_1 = \mathrm{tr}\,(\mathbf{C}), \quad I_2 = \frac{1}{2}[{I_1}^2 - \mathrm{tr}\,(\mathbf{C}^2)], \quad I_3 = \det(\mathbf{C}). \tag{2.1.9}$$

Above expression can be expressed in terms of principal stretches as

$$I_1 = \lambda_1^2 + \lambda_2^2 + \lambda_3^2, \quad I_2 = \lambda_2^2\lambda_3^2 + \lambda_3^2\lambda_1^2 + \lambda_1^2\lambda_2^2, \quad I_3 = \lambda_1^2\lambda_2^2\lambda_3^2. \tag{2.1.10}$$

The useful principal invariants of $\mathbf{U}$ are provided alternatively by

$$i_1 = \mathrm{tr}\,(\mathbf{U}), \quad i_2 = \frac{1}{2}[{i_1}^2 - \mathrm{tr}\,(\mathbf{U}^2)], \quad i_3 = \det(\mathbf{U}), \tag{2.1.11}$$

or, equivalently, in terms of the principal stretches

$$i_1 = \lambda_1 + \lambda_2 + \lambda_3, \quad i_2 = \lambda_2\lambda_3 + \lambda_3\lambda_1 + \lambda_1\lambda_2, \quad i_3 = \lambda_1\lambda_2\lambda_3. \tag{2.1.12}$$

The connections between I_n and $i_n, n = \{1,2,3\}$ follow from the above expressions and are given by

$$I_1 = i_1^2 - 2i_2, \quad I_2 = i_2^2 - 2i_1 i_3, \quad I_3 = i_3^2. \tag{2.1.13}$$

2.2 Motion in an Elastic Material

The rate of change of position of material particle $\mathbf{X}$ is the velocity $\mathbf{v}$ of a material particle $\mathbf{X}$ is defined by

$$\mathbf{v} \equiv \dot{\mathbf{x}} = \frac{\partial}{\partial t}\boldsymbol{\chi}(\mathbf{X}, t), \tag{2.2.1}$$

The acceleration $\mathbf{a}$ of a material particle $\mathbf{X}$ is

$$\mathbf{a} \equiv \dot{\mathbf{v}} \equiv \ddot{\mathbf{x}} = \frac{\partial^2}{\partial t^2}\boldsymbol{\chi}(\mathbf{X}, t). \tag{2.2.2}$$

In each case a superposed dot denotes the material time derivative.

The velocity $\mathbf{v}$ is a function of $\mathbf{x}$ and t, we define the velocity gradient tensor, denoted by $\mathbf{L}$ as

$$\mathbf{L} = \mathrm{grad}\,\mathbf{v}, \tag{2.2.3}$$

which has components form with respect to the basis $\{\mathbf{e}_i\}$

$$L_{ij} = \frac{\partial v_i}{\partial x_j}. \tag{2.2.4}$$

Using the identity we obtain

$$\mathrm{Grad}\,\mathbf{v} = (\mathrm{grad}\,\mathbf{v})\mathbf{F}, \tag{2.2.5}$$

we can write Eq. (2.2.5) by using Eq. (2.2.3),

$$\mathrm{Grad}\,\mathbf{v} = \mathbf{LF}. \tag{2.2.6}$$

Since $\mathbf{v} \equiv \dot{\mathbf{x}}$ then we have

$$\mathrm{Grad}\,\dot{\mathbf{x}} = \frac{\partial}{\partial t}\mathrm{Grad}\,\mathbf{x} = \dot{\mathbf{F}}. \tag{2.2.7}$$

Hence from Eq. (2.2.6) and (2.2.7), we conclude the important connection

$$\dot{\mathbf{F}} = \mathbf{LF}. \tag{2.2.8}$$

The derivative of the determinant of a tensor we use the result, i.e.

$$\frac{\partial}{\partial t}(\det \mathbf{F}) = (\det \mathbf{F})\mathrm{tr}\,(\mathbf{F}^{-1}\dot{\mathbf{F}}) = J\mathrm{tr}\,(\mathbf{F}^{-1}\dot{\mathbf{F}}), \tag{2.2.9}$$

using the Eq. (2.2.8), we have

$$\frac{\partial}{\partial t}(\det \mathbf{F}) = J\mathrm{tr}\,(\mathbf{L}), \tag{2.2.10}$$

or, we have

$$\dot{J} = J\mathrm{tr}\,(\mathbf{L}) = J\mathrm{div}\,\mathbf{v}, \tag{2.2.11}$$

where the terms used in above equation such as $\mathrm{tr}\,(\mathbf{L}) = L_{ii} = \partial v_i/\partial x_i = \mathrm{div}\,\mathbf{v}$. Here, div is the divergence operator in the deformed configuration. i.e. with respect to $\mathbf{x}$ and $\mathrm{div}\,\mathbf{v}$ measures the rate at which the volume changes during the motion. Also the fact that, $\mathbf{F}\mathbf{F}^{-1} = \mathbf{I}$, and therefore

$$\dot{\overline{\mathbf{F}^{-1}}} = -\mathbf{F}^{-1}\dot{\mathbf{F}}\mathbf{F}^{-1} = -\mathbf{F}^{-1}\mathbf{L}. \tag{2.2.12}$$

2.3 Cauchy Theorem-Field Equation for the Motion of the Continuum

The term stress is used to measure the intensity of force, either on the or within the bounding surface of a body subjected to loads. Using the concept of force we have one of the most important theorems which is Cauchy's Theorem and stated as:

Let $(\mathbf{t}, \mathbf{b})$ *be a system of surface and body forces for a body* $\mathbf{B}$ *during a motion. Then a necessary and sufficient condition that the momentum balance laws be satisfied is that there exists a second-order tensor field* $\mathbf{T}$*, called the Cauchy stress tensor, such that*

- *for each* $\mathbf{n}$ *unit vector and* $\mathbf{T}$ *is independent of* $\mathbf{n}$

$$\mathbf{t}(\mathbf{n}) = \mathbf{T}\mathbf{n}; \tag{2.3.1}$$

- *Spatial tensor field* $\mathbf{T}$ *is symmetric;*
- $\mathbf{T}$ *satisfies the equation of motion*

$$\operatorname{div}\mathbf{T} + \rho\mathbf{b} = \rho\dot{\mathbf{v}}, \tag{2.3.2}$$

where $\mathbf{b}$ *is the body force.*

2.4 Stress Tensors and Equilibrium Equations for a Elastic Materials

We define a stress tensor $\mathbf{S}$ that gives the measure of force per unit area in the *un-deformed configuration.*

The elements of surface area are related by

$$\mathbf{n}da = J\mathbf{F}^{-T}\mathbf{N}dA. \tag{2.4.1}$$

This result is known as *Nanson's formula* which describes how elements of the surface area deform and it also applies to area elements of an arbitrary shape. Where $\mathbf{n}$ and $\mathbf{N}$ be unit normals, and the surface area elements are da and dA in the deformed and reference configuration, respectively.

Using the Nanson's formula Eq. (2.4.1) the traction $\mathbf{t}(\mathbf{n})$ on the area element da in the current configuration as

$$\mathbf{t}da = \mathbf{T}\mathbf{n}da = J\mathbf{T}\mathbf{F}^{-T}\mathbf{N}dA = \mathbf{S}^{T}\mathbf{N}dA, \tag{2.4.2}$$

where the first Piola-Kirchhoff stress tensor $\mathbf{S}^T$ is defined by the expression as

$$\mathbf{S}^{T} = J\mathbf{T}\mathbf{F}^{-T}. \tag{2.4.3}$$

Therefore, the *nominal stress tensor* (second order tensor field) $\mathbf{S}$ also referred as the *engineering stress* is defined as

$$\mathbf{S} = J\mathbf{F}^{-1}\mathbf{T}. \tag{2.4.4}$$

The nominal stress tensor satisfies the equation of motion

$$\mathrm{Div}\mathbf{S} + \rho_r \mathbf{b} = \rho_r \ddot{\boldsymbol{\chi}}, \tag{2.4.5}$$

and in general the nominal stress $\mathbf{S}$ is not symmetric but holds the connection

$$\mathbf{F}\mathbf{S} = \mathbf{S}^T \mathbf{F}^T, \tag{2.4.6}$$

where Div is the divergence operator in the reference configuration. We are considering the case when no body forces are acting which reduces Eq. (2.4.5) to the form

$$\mathrm{Div}\mathbf{S} = \rho_r \ddot{\boldsymbol{\chi}}. \tag{2.4.7}$$

In the deformed configuration Eq. (2.4.7) is of the form

$$\mathrm{div}\,\mathbf{T} = \rho \dot{\mathbf{v}}, \tag{2.4.8}$$

Eqs. (2.4.7) and (2.4.8) can be expressed in the components form as

$$\frac{\partial S_{\alpha i}}{\partial X_\alpha} = \rho_r \dot{\chi}_i, \tag{2.4.9}$$

and

$$\frac{\partial T_{ij}}{\partial x_j} = \rho \dot{v}_i, \tag{2.4.10}$$

respectively.

2.5 Initial Stress in a Material

Generally, it is assumed that the reference configuration $\mathcal{B}_R$ is stress free but in many cases there may be an initial stress present in it. This may be induced in the case of biological tissues, by some manufacturing process, be generated by the process of growth, adaptation or remodeling.

In this dissertation the term initial stress is used in the broadest sense regardless of the process that causes this stress. When there is an associated pre-strain from an unstressed configuration due to an applied load then the term prestress replaces the term initial stress. However, when an initial stress is present in the absence of applied loads called residual stress according to the definition given by Hoger [1].

2.6 Hyperelastic Materials

An elastic body is a *Green elastic material* or a *hyperelastic* if the nominal stress $\hat{\mathbf{S}}(\mathbf{F}, \mathbf{X})$ is the derivative of a scalar function $W(\mathbf{F}, \mathbf{X})$.$W(\mathbf{F}, \mathbf{X})$ represents the work done (per unit volume at $\mathbf{X}$) by the stress in deforming the material from $\mathcal{B}_r$ to $\mathcal{B}_t$ (i.e from $\mathbf{I}$ to $\mathbf{F}$) and is independent of the path taken in deformation space: see [15], i.e.

$$\mathbf{S} = \hat{\mathbf{S}}(\mathbf{F}, \mathbf{X}) = \mathrm{D}W(\mathbf{F}, \mathbf{X}) = \frac{\partial W(\mathbf{F}, \mathbf{X})}{\partial \mathbf{F}}, \tag{2.6.1}$$

where D is the derivative with respect to $\mathbf{F}$. The scalar function W is the strain-energy density function.

In terms of the strain energy function W, Eq. (2.6.1) gives the nominal stress. Using Eq. (2.4.4), the Cauchy stress $\mathbf{T}$ is given by

$$\mathbf{T} = J^{-1}\mathbf{F}\frac{\partial W}{\partial \mathbf{F}}. \tag{2.6.2}$$

In component form, the nominal and Cauchy stresses are represented as

$$S_{\alpha i} = \frac{\partial W}{\partial F_{i\alpha}}, \quad T_{ij} = J^{-1}F_{i\beta}\frac{\partial W}{\partial F_{j\beta}}. \tag{2.6.3}$$

It is convenient to assume that W is measured from the reference configuration, so that

$$W(\mathbf{I}) = 0. \tag{2.6.4}$$

If the reference configuration is stress free then

$$\frac{\partial W}{\partial \mathbf{F}}(\mathbf{I}) = \mathbf{0}, \tag{2.6.5}$$

for an unconstrained material.

2.7 Incremental Equations

Let $\boldsymbol{\chi}$, with $\mathbf{x} = \boldsymbol{\chi}(\mathbf{X})$, be a time-independent deformation and $\boldsymbol{\chi}'$, with $\mathbf{x}' = \boldsymbol{\chi}'(\mathbf{X}, t)$ be a finite time-dependent deformation which is "close" to $\boldsymbol{\chi}$. The displacement, which can be thought of as a perturbation of $\boldsymbol{\chi}$, can be written as

$$\dot{\mathbf{x}} = \mathbf{x}' - \mathbf{x} = \boldsymbol{\chi}'(\mathbf{X}, t) - \boldsymbol{\chi}(\mathbf{X}, t) \equiv \dot{\boldsymbol{\chi}}(\mathbf{X}), \tag{2.7.1}$$

and the gradient which is exact

$$\mathrm{Grad}\,\dot{\boldsymbol{\chi}} = \mathrm{Grad}\;\boldsymbol{\chi}' - \mathrm{Grad}\,\boldsymbol{\chi} \equiv \dot{\mathbf{F}}. \tag{2.7.2}$$

Consider the linear approximation of the stress tensor $\mathbf{S}$. In its exact form the incremental stress tensor $\dot{\mathbf{S}}$ for an unconstrained material is

$$\dot{\mathbf{S}} = \mathbf{S}' - \mathbf{S} = \frac{\partial W}{\partial \mathbf{F}}(\mathbf{F}') - \frac{\partial W}{\partial \mathbf{F}}(\mathbf{F}), \tag{2.7.3}$$

having linear approximation

$$\dot{\mathbf{S}} = \boldsymbol{\mathcal{A}}\dot{\mathbf{F}}, \tag{2.7.4}$$

where $\boldsymbol{\mathcal{A}}$ is the fourth order elasticity tensor, defined by

$$\boldsymbol{\mathcal{A}}(\mathbf{F}, \mathbf{X})[\mathbf{A}] = D\hat{\mathbf{S}}(\mathbf{F}, \mathbf{X})[\mathbf{A}] = \frac{\partial^2 W(\mathbf{F}, \mathbf{X})}{\partial \mathbf{F}^2} \cdot \mathbf{A}, \tag{2.7.5}$$

for each fixed material point $\mathbf{X}$ (see, for example, [14]).

The component form of Eq. (2.7.4) is

$$\dot{S}_{\alpha i} = \mathcal{A}_{\alpha i \beta j} \dot{F}_{j\beta}, \tag{2.7.6}$$

where $\dot{F}_{j\beta} = \dot{x}_{j,\beta}$. Following the equation of motion Eq. (2.4.5) for χ' and subtracting its counterpart equation for χ, we obtain

$$\mathrm{Div}\dot{\mathbf{S}} + \rho_r \dot{\mathbf{b}} = \rho_r \dot{\boldsymbol{\chi}}_{,tt}, \tag{2.7.7}$$

where $\dot{\mathbf{b}}$ and $\dot{\chi}$ are the incremental forms of the body force $\mathbf{b}$ and χ, respectively. The updated elasticity tensor $\boldsymbol{\mathcal{A}}_0$, in terms of $\boldsymbol{\mathcal{A}}$, is

$$\mathcal{A}_{0ijkl} = J^{-1} F_{i\alpha} F_{k\beta} \mathcal{A}_{\alpha j \beta l}. \tag{2.7.8}$$

For the derivation of Eq. (2.7.8) see [15]. The updated nominal stress for an unconstrained material given by

$$\dot{\mathbf{S}}_0 = \boldsymbol{\mathcal{A}}_0 \dot{\mathbf{F}}_0 = \boldsymbol{\mathcal{A}}_0 \boldsymbol{\Gamma}, \tag{2.7.9}$$

where $\dot{\mathbf{F}}_0 = \boldsymbol{\Gamma}$ is the updated incremental form in the reference configuration. In component form, Eq. (2.7.9) can be written as

$$\dot{S}_{0pi} = \mathcal{A}_{0piqj} u_{j,q}, \tag{2.7.10}$$

. The updated counterparts of Eqs. (2.7.7) are

$$\mathrm{Div} \dot{\mathbf{S}}_0 + \rho \dot{\mathbf{b}} = \rho \mathbf{u}_{,tt}, \tag{2.7.11}$$

where ρ is the current density.

2.8 Invariants for an Initially-Stressed Compressible Material

We consider an initially stressed homogeneous hyper-elastic material for which the strain energy function W per unit reference volume depends on the deformation gradient $\mathbf{F}$ and the initial stress $\boldsymbol{\tau}$. By objectivity we can regard W as a function of $\mathbf{F}$ through the right Cauchy-Green deformation tensor. Thus $W = W(\mathbf{C}, \boldsymbol{\tau})$.

When subjected to a rotation $\mathbf{Q}$ in the reference configuration, $\mathbf{C}$ and $\boldsymbol{\tau}$ change to $\mathbf{QCQ}^T$ and $\mathbf{Q}\boldsymbol{\tau}\mathbf{Q}^T$, respectively. The strain energy is invariant under this change if it depends on the ten invariants $I_1, ..., I_{10}$ of the two tensors $\mathbf{C}$ and $\boldsymbol{\tau}$ defined by

$$
\begin{aligned}
&I_1 = \mathrm{tr}(\mathbf{C}), \qquad I_2 = \frac{1}{2}[I_1^2 - \mathrm{tr}(\mathbf{C}^2)], \qquad I_3 = \det(\mathbf{C}),\\
&I_4 = \mathrm{tr}(\boldsymbol{\tau}), \qquad I_5 = \frac{1}{2}[I_4^2 - \mathrm{tr}(\boldsymbol{\tau}^{\mathbf{2}})], \qquad I_6 = \det(\boldsymbol{\tau}),\\
&I_7 = \mathrm{tr}(\mathbf{C}\boldsymbol{\tau}), \qquad I_8 = \mathrm{tr}(\mathbf{C}^2\boldsymbol{\tau}), \qquad I_9 = \mathrm{tr}(\mathbf{C}\boldsymbol{\tau}^{\mathbf{2}}),\\
&I_{10} = \mathrm{tr}(\mathbf{C}^2\boldsymbol{\tau}^{\mathbf{2}}).
\end{aligned} \tag{2.8.1}
$$

Using Eqs. (2.8.1), consider that the initial stress to be independent of the deformation, the first derivatives of these invariants are given by

$$
\begin{aligned}
&\frac{\partial I_1}{\partial F_{i\alpha}} = 2F_{i\alpha}, \quad \frac{\partial I_2}{\partial F_{i\alpha}} = 2(C_{\gamma\gamma}F_{i\alpha} - C_{\alpha\gamma}F_{i\gamma}), \quad \frac{\partial I_3}{\partial F_{i\alpha}} = 2I_3(F^{-1})_{\alpha i},\\
&\frac{\partial I_4}{\partial F_{i\alpha}} = 0, \quad \frac{\partial I_5}{\partial F_{i\alpha}} = 0, \quad \frac{\partial I_6}{\partial F_{i\alpha}} = 0,\\
&\frac{\partial I_7}{\partial F_{i\alpha}} = 2\tau_{\alpha\gamma}F_{i\gamma}, \quad \frac{\partial I_8}{\partial F_{i\alpha}} = 2\tau_{\alpha\delta}C_{\delta\gamma}F_{i\gamma} + 2C_{\alpha\delta}\tau_{\delta\gamma}F_{i\gamma},\\
&\frac{\partial I_9}{\partial F_{i\alpha}} = 2\tau^2_{\alpha\gamma}F_{i\gamma}, \quad \frac{\partial I_{10}}{\partial F_{i\alpha}} = 2\tau^2_{\alpha\delta}C_{\delta\gamma}F_{i\gamma} + 2C_{\alpha\delta}\tau^2_{\delta\gamma}F_{i\gamma}.
\end{aligned} \tag{2.8.2}
$$

from which we obtain the expressions

$$
\begin{aligned}
&F_{p\alpha}\frac{\partial I_1}{\partial F_{i\alpha}} = 2B_{pi}, \quad F_{p\alpha}\frac{\partial I_2}{\partial F_{i\alpha}} = 2(I_1B_{pi} - B_{pq}B_{qi}),\\
&F_{p\alpha}\frac{\partial I_3}{\partial F_{i\alpha}} = 2I_3\mathbf{I}, \quad F_{p\alpha}\frac{\partial I_4}{\partial F_{i\alpha}} = 0, \quad F_{p\alpha}\frac{\partial I_5}{\partial F_{i\alpha}} = 0,\\
&F_{p\alpha}\frac{\partial I_6}{\partial F_{i\alpha}} = 0, \quad F_{p\alpha}\frac{\partial I_7}{\partial F_{i\alpha}} = 2\Sigma_{pi},\\
&F_{p\alpha}\frac{\partial I_8}{\partial F_{i\alpha}} = 2(\Sigma_{pq}B_{qi} + B_{pq}\Sigma_{qi}), \quad F_{p\alpha}\frac{\partial I_9}{\partial F_{i\alpha}} = 2\Sigma_{p\alpha}(B^{-1})_{\alpha q}\Sigma_{qi},\\
&F_{p\alpha}\frac{\partial I_{10}}{\partial F_{i\alpha}} = 2[\Sigma_{p\alpha}(B^{-1})_{\alpha q}\Sigma_{q\gamma}B_{\gamma i} + B_{p\alpha}\Sigma_{\alpha q}(B^{-1})_{q\gamma}\Sigma_{\gamma i}].
\end{aligned} \tag{2.8.3}
$$

Evaluating the Eq. (2.8.2) in the reference configuration, we have the non-zero derivatives as

$$
\begin{aligned}
&\frac{\partial I_1}{\partial F_{i\alpha}} = 2\delta_{i\alpha}, \quad \frac{\partial I_2}{\partial F_{i\alpha}} = 4\delta_{i\alpha}, \quad \frac{\partial I_3}{\partial F_{i\alpha}} = 2\delta_{i\alpha}, \quad \frac{\partial I_7}{\partial F_{i\alpha}} = 2\tau_{\alpha i},\\
&\frac{\partial I_8}{\partial F_{i\alpha}} = 4\tau_{\alpha i}, \quad \frac{\partial I_9}{\partial F_{i\alpha}} = 2\tau^2_{\alpha i}, \quad \frac{\partial I_{10}}{\partial F_{i\alpha}} = 4\tau^2_{\alpha i}.
\end{aligned} \tag{2.8.4}
$$

Now consider that the strain energy function W is a function of $I_1, I_2, ..., I_{10}$. Then

$$\frac{\partial W}{\partial \mathbf{F}} = \sum_{r=1}^{10} W_r \frac{\partial I_r}{\partial \mathbf{F}}, \tag{2.8.5}$$

where $W_r = \partial W / \partial I_r$.

By definition, the nominal stress tensor $\mathbf{S}$ for an initially stressed unconstrained i.e a compressible material is given by

$$\begin{aligned}\mathbf{S} &= 2W_1\mathbf{F}^T + 2W_2(I_1\mathbf{F}^T - \mathbf{F}^T\mathbf{B}) + 2W_3I_3\mathbf{F}^{-1} + 2W_7\boldsymbol{\tau}\mathbf{F}^T \\ &\quad + 2W_8(\boldsymbol{\tau}\mathbf{C}\mathbf{F}^T + \mathbf{C}\boldsymbol{\tau}\mathbf{F}^T) + 2W_9\boldsymbol{\tau}^{\mathbf{2}}\mathbf{F}^T + 2W_{10}(\boldsymbol{\tau}^2\mathbf{C}\mathbf{F}^T + \mathbf{C}\boldsymbol{\tau}^{\mathbf{2}}\mathbf{F}^T).\end{aligned} \tag{2.8.6}$$

The Cauchy stress tensor $\mathbf{T}$ for an initially stressed unconstrained material is given by

$$\begin{aligned}J\mathbf{T} = \mathbf{F}\mathbf{S} &= 2W_1\mathbf{B} + 2W_2\mathbf{B}^* + 2W_3I_3\mathbf{I} + 2W_7\boldsymbol{\Sigma} + 2W_8(\boldsymbol{\Sigma}\mathbf{B} + \mathbf{B}\boldsymbol{\Sigma}) \\ &\quad + 2W_9\boldsymbol{\Sigma}\mathbf{B}^{-1}\boldsymbol{\Sigma} + 2W_{10}(\boldsymbol{\Sigma}\mathbf{B}^{-1}\boldsymbol{\Sigma}\mathbf{B} + \mathbf{B}\boldsymbol{\Sigma}\mathbf{B}^{-1}\boldsymbol{\Sigma}),\end{aligned} \tag{2.8.7}$$

where $\boldsymbol{\Sigma} = \mathbf{F}\boldsymbol{\tau}\mathbf{F}^T$ and $\mathbf{B}^* = I_1\mathbf{B} - \mathbf{B}^2$.

If the above expression is evaluated in the reference configuration, we get the expression for $\boldsymbol{\tau}$ as

$$\boldsymbol{\tau} = 2(W_1 + 2W_2 + W_3)\mathbf{I} + 2(W_7 + 2W_8)\boldsymbol{\tau} + 2(W_9 + 2W_{10})\boldsymbol{\tau}^{\mathbf{2}}, \tag{2.8.8}$$

which suggests to set

$$W_1 + 2W_2 + W_3 = 0, \quad 2(W_7 + 2W_8) = 1, \quad W_9 + 2W_{10} = 0. \tag{2.8.9}$$

2.9 The Elasticity Tensor for an Initially-Stressed Hyperelastic Material

Using Eqs. (2.8.2), the non-zero second derivatives of the invariants are

$$\begin{aligned}
\frac{\partial^2 I_1}{\partial F_{i\alpha} F_{j\beta}} &= 2\delta_{\alpha\beta}\delta_{ij},\\
\frac{\partial^2 I_2}{\partial F_{i\alpha} F_{j\beta}} &= 2I_1\delta_{\alpha\beta}\delta_{ij} + 4F_{i\alpha}F_{j\beta} - 2F_{i\beta}F_{j\alpha} - 2C_{\alpha\beta}\delta_{ij} - 2\delta_{\alpha\beta}B_{ij}, \qquad (2.9.1)\\
\frac{\partial^2 I_3}{\partial F_{i\alpha} F_{j\beta}} &= 4I_3(F^{-1})_{\alpha i}(F^{-1})_{\beta j} - 2I_3(F^{-1})_{\alpha j}(F^{-1})_{\beta i},\\
\frac{\partial^2 I_7}{\partial F_{i\alpha} F_{j\beta}} &= 2\tau_{\alpha\beta}\delta_{ij},\\
\frac{\partial^2 I_8}{\partial F_{i\alpha} F_{j\beta}} &= 2\tau_{\alpha\beta}B_{ij} + 2(\boldsymbol{\tau}\mathbf{C})_{\alpha\beta}\delta_{ij} + 2(\mathbf{C}\boldsymbol{\tau})_{\alpha\beta}\delta_{ij} + 2\delta_{\alpha\beta}\Sigma_{ij}\\
&\quad + 2(\boldsymbol{\tau}\mathbf{F}^T)_{\alpha j}F_{i\beta} + 2(\boldsymbol{\tau}\mathbf{F}^T)_{\beta i}F_{j\alpha},\\
\frac{\partial^2 I_9}{\partial F_{i\alpha} F_{j\beta}} &= 2(\boldsymbol{\tau}^{\mathbf{2}})_{\alpha\beta}\delta_{ij},\\
\frac{\partial^2 I_{10}}{\partial F_{i\alpha} F_{j\beta}} &= 2(\boldsymbol{\tau}^{\mathbf{2}})_{\alpha\beta}B_{ij} + 2(\boldsymbol{\tau}^{\mathbf{2}}\mathbf{C})_{\alpha\beta}\delta_{ij} + 2(\mathbf{C}\boldsymbol{\tau}^{\mathbf{2}})_{\alpha\beta}\delta_{ij}\\
&\quad + 2\delta_{\alpha\beta}(\mathbf{F}\boldsymbol{\tau}^{\mathbf{2}}\mathbf{T})_{ij} + 2(\boldsymbol{\tau}^{\mathbf{2}}\mathbf{F}^T)_{\alpha j}F_{i\beta} + 2(\boldsymbol{\tau}^{\mathbf{2}}\mathbf{F}^T)_{\beta i}F_{j\alpha}. \qquad (2.9.2)
\end{aligned}$$

from the above expressions we obtain

$$\begin{aligned}
F_{p\alpha}F_{q\beta}\frac{\partial^2 I_1}{\partial F_{i\alpha} F_{j\beta}} &= 2B_{pq}\delta_{ij},\\
F_{p\alpha}F_{q\beta}\frac{\partial^2 I_2}{\partial F_{i\alpha} F_{j\beta}} &= 2I_1B_{pq}\delta_{ij} + 4B_{pi}B_{qj} - 2B_{iq}F_{jp} - 2B_{p\gamma}B_{\gamma q}\delta_{ij}\\
&\quad - 2B_{pq}B_{ij}, \qquad (2.9.3)\\
F_{p\alpha}F_{q\beta}\frac{\partial^2 I_3}{\partial F_{i\alpha} F_{j\beta}} &= 4I_3\delta_{ip}\delta_{jq} - 2I_3\delta_{iq}\delta_{jp},\\
F_{p\alpha}F_{q\beta}\frac{\partial^2 I_7}{\partial F_{i\alpha} F_{j\beta}} &= 2\Sigma_{pq}\delta_{ij},\\
F_{p\alpha}F_{q\beta}\frac{\partial^2 I_8}{\partial F_{i\alpha} F_{j\beta}} &= 2\Sigma_{pq}B_{ij} + 2\Sigma_{p\gamma}B_{\gamma q}\delta_{ij} + 2B_{p\gamma}\Sigma_{\gamma q}\delta_{ij} + 2\Sigma_{ij}B_{pq}\\
&\quad + 2\Sigma_{pj}B_{iq} + 2\Sigma_{qi}B_{jp}, \qquad (2.9.4)
\end{aligned}$$

for convince in calculation, we drop the non-linear terms. In the reference configuration Eq. (2.9.1) reduces to

$$\begin{aligned}
\frac{\partial^2 I_1}{\partial F_{i\alpha} F_{j\beta}} &= 2\delta_{\alpha\beta}\delta_{ij}, \\
\frac{\partial^2 I_2}{\partial F_{i\alpha} F_{j\beta}} &= 2\delta_{\alpha\beta}\delta_{ij} + 4\delta_{i\alpha}\delta_{j\beta} - 2\delta_{i\beta}\delta_{j\alpha}, \\
\frac{\partial^2 I_3}{\partial F_{i\alpha} F_{j\beta}} &= 4\delta_{\alpha i}\delta_{\beta j} - 2\delta_{\alpha j}\delta_{\beta i}, \\
\frac{\partial^2 I_7}{\partial F_{i\alpha} F_{j\beta}} &= 2\tau_{\alpha\beta}\delta_{ij}, \\
\frac{\partial^2 I_8}{\partial F_{i\alpha} F_{j\beta}} &= 6\tau_{\alpha\beta}\delta_{ij} + 2\tau_{ij}\delta_{\alpha\beta} + 2\tau_{\alpha j}\delta_{i\beta} + 2\tau_{\beta i}\delta_{j\alpha}, \\
\frac{\partial^2 I_9}{\partial F_{i\alpha} F_{j\beta}} &= 2(\boldsymbol{\tau}^{\mathbf{2}})_{\alpha\beta}\delta_{ij}, \\
\frac{\partial^2 I_{10}}{\partial F_{i\alpha} F_{j\beta}} &= 6(\boldsymbol{\tau}^{\mathbf{2}})_{\alpha\beta}\delta_{ij} + 2(\boldsymbol{\tau}^{\mathbf{2}})_{ij}\delta_{\alpha\beta} + 2(\boldsymbol{\tau}^{\mathbf{2}})_{\alpha j}\delta_{i\beta} + 2(\boldsymbol{\tau}^{\mathbf{2}})_{\beta i}\delta_{j\alpha}. \qquad (2.9.5)
\end{aligned}$$

where $\boldsymbol{\Sigma} = \boldsymbol{\tau} = \mathbf{S}$ in the reference configuration. From Eqs. (2.8.5)

$$\boldsymbol{\mathcal{A}} = \frac{\partial^2 W}{\partial \mathbf{F} \partial \mathbf{F}} = \sum_{r=1}^{N} W_r \frac{\partial^2 I_r}{\partial \mathbf{F} \partial \mathbf{F}} + \sum_{r,s=1}^{N} W_{rs} \frac{\partial I_r}{\partial \mathbf{F}} \otimes \frac{\partial I_s}{\partial \mathbf{F}}, \qquad (2.9.6)$$

where $W_{rs} = \partial^2 W / \partial I_r \partial I_s$.

The updated elasticity tensor in its component form is given by

$$\mathcal{A}_{0piqj} = J^{-1} \left(\sum_{r=1}^{N} W_r F_{p\alpha} F_{q\beta} \frac{\partial^2 I_r}{\partial F_{i\alpha} \partial F_{j\beta}} + \sum_{r,s=1}^{N} W_{rs} F_{p\alpha} F_{q\beta} \frac{\partial I_r}{\partial F_{i\alpha}} \frac{\partial I_s}{\partial F_{j\beta}} \right). \qquad (2.9.7)$$

Therefore, taking $N = 8$ in Eq. (2.9.7), we have for a compressible material

$$\begin{aligned}
J\mathcal{A}_{0piqj} &= 2(W_1 + I_1 W_2)B_{pq}\delta_{ij} + 2W_2[2B_{pi}B_{qj} - B_{iq}B_{jp} - \delta_{ij}B_{p\gamma}B_{\gamma q} \\
&- B_{pq}B_{ij}] + 2W_3 I_3(2\delta_{ip}\delta_{jq} - \delta_{iq}\delta_{jp}) + 2W_7\Sigma_{pq}\delta_{ij} + 2W_8[\Sigma_{pq}B_{ij} \\
&+ \Sigma_{p\gamma}B_{\gamma q}\delta_{ij} + B_{p\gamma}\Sigma_{\gamma q}\delta_{ij} + \Sigma_{ij}B_{pq} + \Sigma_{pj}B_{iq} + \Sigma_{qi}B_{jp}] + 4W_{11}B_{ip}B_{jq} \\
&+ 4W_{22}(I_1 B_{ip} - B_{i\gamma}B_{\gamma p})(I_1 B_{jq} - B_{j\delta}B_{\delta q}) + 4W_{33}I_3^2\delta_{ip}\delta_{jq} \\
&+ 4W_{12}[2I_1 B_{ip}B_{jq} - B_{ip}B_{j\delta}B_{\delta q} - B_{jq}B_{i\gamma}B_{\gamma p}] + 4W_{13}I_3(B_{ip}\delta_{jq} + B_{jq}\delta_{ip}) \\
&+ 4W_{17}(B_{ip}\Sigma_{jq} + B_{jq}\Sigma_{ip}) + 4W_{18}[B_{ip}(\Sigma_{j\delta}B_{\delta q} + B_{j\delta}\Sigma_{\delta q}) + (\Sigma_{i\gamma}B_{\gamma p} \\
&+ B_{i\gamma}\Sigma_{\gamma p})B_{jq}] + 4W_{23}I_3[I_1(B_{ip}\delta_{jq} + B_{jq}\delta_{ip}) - \delta_{ip}B_{j\delta}B_{\delta q} - B_{i\gamma}B_{\gamma p}\delta_{jq}] \\
&+ 4W_{27}[(I_1 B_{ip} - B_{i\gamma}B_{\gamma p})\Sigma_{jq} + \Sigma_{ip}(I_1 B_{jq} - B_{j\delta}B_{\delta q})] + 4W_{28}[(I_1 B_{ip} \\
&- B_{i\gamma}B_{\gamma p})(\Sigma_{j\delta}B_{\delta q} + B_{j\delta}\Sigma_{\delta q}) + (\Sigma_{i\gamma}B_{\gamma p} + B_{i\gamma}\Sigma_{\gamma p})(I_1 B_{jq} - B_{j\delta}B_{\delta q})] \\
&+ 4W_{37}I_3[\delta_{ip}\Sigma_{jq} + \delta_{jq}\Sigma_{ip}] + 4W_{38}I_3[\delta_{ip}(\Sigma_{j\delta}B_{\delta q} + B_{j\delta}\Sigma_{\delta q}) + (\Sigma_{i\gamma}B_{\gamma p} \\
&+ B_{i\gamma}\Sigma_{\gamma p})\delta_{jq}] + 4W_{77}\Sigma_{ip}\Sigma_{jq} + 4W_{78}[\Sigma_{ip}(\Sigma_{j\delta}B_{\delta q} + B_{j\delta}\Sigma_{\delta q}) + (\Sigma_{i\gamma}B_{\gamma p} \\
&+ B_{i\gamma}\Sigma_{\gamma p})\Sigma_{jq}] + 4W_{88}(\Sigma_{i\gamma}B_{\gamma p} + B_{i\gamma}\Sigma_{\gamma p})(\Sigma_{j\delta}B_{\delta q} + B_{j\delta}\Sigma_{\delta q}), \qquad (2.9.8)
\end{aligned}$$

which is a expression already noted in [11].

We consider a compressible material the elastic response of which is described by a general strain energy function $W(\mathbf{C}, \boldsymbol{\tau})$. Let this material be subject to a general pure homogeneous pre-strain such that $\lambda_1, \lambda_2, \lambda_3$ are the principal stretches corresponding the principal axes x_1, x_2 and x_3 respectively. In the refence configuration $\mathbf{F} = \mathbf{I}$ and use of Eq. (2.8.9) in Eq. (2.9.8), the components of the elasticity tensor $\boldsymbol{\mathcal{C}}$, in the reference configuration for an unconstrained compressible material are given by $\mathcal{C}_{piqj}$

$$\begin{aligned}
\mathcal{C}_{piqj} = \mathcal{A}_{0piqj} &= \alpha_1(\delta_{ij}\delta_{pq} + \delta_{iq}\delta_{jp} - \delta_{ip}\delta_{jq}) + \alpha_2\delta_{ip}\delta_{jq} + \delta_{ij}\tau_{pq} + \alpha_3(\delta_{ij}\tau_{pq} \\
&+ \delta_{pq}\tau_{ij} + \delta_{iq}\tau_{jp} + \delta_{jp}\tau_{iq}) + \alpha_4(\delta_{ip}\tau_{jq} + \delta_{jq}\tau_{ip}) + \alpha_5\tau_{ip}\tau_{jq}. \qquad (2.9.9)
\end{aligned}$$

Here, we have defined

$$\alpha_1 = 2(W_1 + W_2), \quad \alpha_2 = 2(W_2 + W_3) + 4(W_{11} + 4W_{12} + 2W_{13} + 4W_{22} + 4W_{23} + W_{33}),$$
$$\alpha_3 = 2W_8, \quad \alpha_4 = 4(W_{17} + 2W_{18} + 2W_{27} + 4W_{28} + W_{37} + 2W_{38}),$$
$$\alpha_5 = 4(W_{77} + 4W_{78} + 4W_{88}), \tag{2.9.10}$$

evaluated in the reference configuration. When $\boldsymbol{\tau} = \mathbf{0}$, Eq. (2.9.9) gives

$$\mathcal{C}_{piqj} = \alpha_1(\delta_{pq}\delta_{ij} + \delta_{iq}\delta_{jp} - \delta_{ip}\delta_{jq}) + \alpha_2\delta_{ip}\delta_{jq}, \tag{2.9.11}$$

which is the classical expression of fourth order elasticity tensor in the linear theory of elasticity. Following from Eq. (2.9.8), various expressions for elastic modulli in this case are given by, for $i \neq j$,

$$\begin{aligned}
J\mathcal{A}_{0iiii} =\ & 2W_1\lambda_i^2 + 2W_2\lambda_i^2(I_1 - \lambda_i^2) + 2W_3I_3 + 2W_7\Sigma_{ii} + 12W_8\lambda_i^2\Sigma_{ii} \\
& + 4W_{11}\lambda_i^4 + 4W_{22}\lambda_i^4(I_1 - \lambda_i^2)^2 + 8W_{12}\lambda_i^4(I_1 - \lambda_i^2) + 8W_{13}I_3\lambda_i^2 \\
& + 8W_{17}\lambda_i^2\Sigma_{ii} + 16W_{18}\lambda_i^4\Sigma_{ii} + 8W_{23}I_3\lambda_i^2(I_1 - \lambda_i^2) + 8W_{27}\lambda_i^2(I_1 \\
& - \lambda_i^2)\Sigma_{ii} + 16W_{28}\lambda_i^4(I_1 - \lambda_i^2)\Sigma_{ii} + 4W_{33}I_3^2 + 8W_{37}I_3\Sigma_{ii} \\
& + 16W_{38}I_3\lambda_i^2\Sigma_{ii} + 4W_{77}\Sigma_{ii}^2 + 16W_{78}\lambda_i^2\Sigma_{ii}^2 + 16W_{88}\lambda_i^4\Sigma_{ii}^2,
\end{aligned} \tag{2.9.12}$$

$$\begin{aligned}
J\mathcal{A}_{0iijj} =\ & 4W_2\lambda_i^2\lambda_j^2 + 4W_3I_3 + 4W_{11}\lambda_i^2\lambda_j^2 + 4W_{22}\lambda_i^2\lambda_j^2(I_1 - \lambda_i^2)(I_1 - \lambda_j^2) \\
& + 4W_{12}\lambda_i^2\lambda_j^2(2I_1 - \lambda_i^2 - \lambda_j^2) + 4W_{13}I_3(\lambda_i^2 + \lambda_j^2) + 4W_{17}(\lambda_i^2\Sigma_{ii} \\
& + \lambda_j^2\Sigma_{jj}) + 8W_{18}\lambda_i^2\lambda_j^2(\Sigma_{ii} + \Sigma_{jj}) + 4W_{23}I_3[I_1(\lambda_i^2 + \lambda_j^2) - \lambda_i^4 \\
& - \lambda_j^4] + 4W_{27}[\lambda_i^2(I_1 - \lambda_i^2)\Sigma_{jj} + \lambda_j^2(I_1 - \lambda_j^2)\Sigma_{ii}] + 8W_{28}\lambda_i^2\lambda_j^2[(I_1 \\
& - \lambda_i^2)\Sigma_{jj} + (I_1 - \lambda_j^2)\Sigma_{ii}] + 4W_{33}I_3^2 + 4W_{37}I_3(\Sigma_{ii} + \Sigma_{jj}) \\
& + 8W_{38}(\lambda_i^2\Sigma_{ii} + \lambda_j^2\Sigma_{jj}) + 4W_{77}\Sigma_{ii}\Sigma_{jj} + 8W_{78}(\lambda_i^2 + \lambda_j^2)\Sigma_{ii}\Sigma_{jj} \\
& + 16W_{88}\lambda_i^2\lambda_j^2\Sigma_{ii}\Sigma_{jj},
\end{aligned} \tag{2.9.13}$$

$$\begin{aligned}
J\mathcal{A}_{0ijij} =\ & 2W_1\lambda_i^2 + 2W_2(I_1\lambda_i^2 - \lambda_i^2\lambda_j^2 - \lambda_i^4) + 2W_7\Sigma_{ii} + 2W_8(\lambda_j^2\Sigma_{ii} \\
& + 2\lambda_i^2\Sigma_{ii} + \lambda_i^2\Sigma_{jj}) + 4W_{77}\Sigma_{ij}^2 + 8W_{78}(\lambda_i^2 + \lambda_j^2)\Sigma_{ij}^2 \\
& + 4W_{88}(\lambda_i^2 + \lambda_j^2)^2\Sigma_{ij}^2,
\end{aligned} \tag{2.9.14}$$

$$
\begin{aligned}
J\mathcal{A}_{0ijji} &= -2W_2\lambda_i^2\lambda_j^2 - 2W_3I_3 + 2W_8(\lambda_j^2\Sigma_{ii} + \lambda_i^2\Sigma_{jj}) + 4W_{77}\Sigma_{ij}^2 \\
&+ 8W_{78}(\lambda_i^2 + \lambda_j^2)\Sigma_{ij}^2 + 4W_{88}(\lambda_i^2 + \lambda_j^2)^2\Sigma_{ij}^2, \qquad (2.9.15) \\
J\mathcal{A}_{0ijjj} &= 2[W_7 + W_8(3\lambda_j^2 + \lambda_i^2) + 2W_{17}\lambda_j^2 + 2W_{18}\lambda_j^2(\lambda_i^2 + \lambda_j^2) \\
&+ 2W_{27}\lambda_j^2(I_1 - \lambda_j^2) + 2W_{28}\lambda_j^2(I_1 - \lambda_j^2)(\lambda_i^2 + \lambda_j^2) + 2W_{37}I_3 \\
&+ 2W_{38}I_3(\lambda_i^2 + \lambda_j^2)]\Sigma_{ij} + 4[W_{77} + W_{78}(3\lambda_j^2 + \lambda_i^2) \\
&+ 2W_{88}\lambda_j^2(\lambda_i^2 + \lambda_j^2)]\Sigma_{ij}\Sigma_{jj}, \qquad (2.9.16) \\
J\mathcal{A}_{0iiij} &= 2[2W_8\lambda_i^2 + 2W_{17}\lambda_i^2 + 2W_{18}\lambda_i^2(\lambda_i^2 + \lambda_j^2) + 2W_{27}\lambda_i^2(I_1 - \lambda_i^2) \\
&+ 2W_{28}\lambda_i^2(I_1 - \lambda_i^2)(\lambda_i^2 + \lambda_j^2) + 2W_{37}I_3 + 2W_{38}I_3(\lambda_i^2 + \lambda_j^2)]\Sigma_{ij} \\
&+ 4[W_{77} + W_{78}(3\lambda_i^2 + \lambda_j^2) + 2W_{88}\lambda_i^2(\lambda_i^2 + \lambda_j^2)]\Sigma_{ij}\Sigma_{ii}, \qquad (2.9.17) \\
J\mathcal{A}_{0iikj} &= J\mathcal{A}_{0iijk} = J\mathcal{A}_{0jkii} = J\mathcal{A}_{0kjii} = 4[W_{17}\lambda_i^2 + W_{18}\lambda_i^2(\lambda_j^2 + \lambda_k^2) \\
&+ W_{27}\lambda_i^2(I_1 - \lambda_i^2) + W_{28}\lambda_i^2(I_1 - \lambda_i^2)(\lambda_j^2 + \lambda_k^2) + W_{37}I_3 \\
&+ W_{38}I_3(\lambda_j^2 + \lambda_k^2)]\Sigma_{jk} + 4[W_{77} + W_{78}(2\lambda_i^2 + \lambda_j^2 + \lambda_k^2) \\
&+ 2W_{88}\lambda_i^2(\lambda_j^2 + \lambda_k^2)]\Sigma_{jk}\Sigma_{ii}, \qquad (2.9.18) \\
J\mathcal{A}_{0ijki} &= J\mathcal{A}_{0ijik} = J\mathcal{A}_{0ikji} = J\mathcal{A}_{0kiij} = 2W_8\lambda_i^2\Sigma_{jk} + 4[W_{77} + W_{78}(2\lambda_i^2 \\
&+ \lambda_j^2 + \lambda_k^2) + W_{88}(\lambda_i^2 + \lambda_j^2)(\lambda_i^2 + \lambda_k^2)]\Sigma_{ij}\Sigma_{ik}, \qquad (2.9.19) \\
J\mathcal{A}_{0ikjk} &= J\mathcal{A}_{0jkik} = 2[W_7 + W_8(\lambda_i^2 + \lambda_j^2 + \lambda_k^2)]\Sigma_{ij} + 4[W_{77} + W_{78}(\lambda_i^2 \\
&+ \lambda_j^2 + 2\lambda_k^2) + W_{88}(\lambda_i^2 + \lambda_k^2)(\lambda_j^2 + \lambda_k^2)]\Sigma_{ik}\Sigma_{jk}. \qquad (2.9.20)
\end{aligned}
$$

Let $\tau_{ii}, (i = 1, 2, 3)$ denote the normal initial stress components and $\tau_{ij}, i \neq j \neq k$, $(i, j \in \{1, 2, 3\})$ denote the shear components of the initial stress. When $\lambda_i = 1$, $i = 1, 2, 3$ and $\Sigma_{ij} = \tau_{ij}$ then Eqs. (2.9.12-2.9.20) reduces in the reference configuration

$$
\begin{aligned}
\mathcal{A}_{0iiii} &= 2W_1 + 4W_2 + 2W_3 + 2W_7\tau_{ii} + 12W_8\tau_{ii} + 4W_{11} + 16W_{22} \\
&+ 16W_{12} + 8W_{13} + 8W_{17}\tau_{ii} + 16W_{18}\tau_{ii} + 16W_{23} + 16W_{27}\tau_{ii} \\
&+ 32W_{28}\tau_{ii} + 4W_{33} + 8W_{37}\tau_{ii} + 16W_{38}\tau_{ii} + 4W_{77}\tau_{ii}^2 + 16W_{78}\tau_{ii}^2 \\
&+ 16W_{88}\tau_{ii}^2, \qquad (2.9.21)
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{A}_{0iijj} &= 4W_2 + 4W_3 + 4W_{11} + 16W_{12} + 8W_{13} + 8W_{17}(\tau_{ii} + \tau_{jj}) \\
&+ 8W_{18}(\tau_{ii} + \tau_{jj}) + 16W_{23} + 8W_{27}(\tau_{ii} + \tau_{jj}) + 16W_{28}(\tau_{ii} + \tau_{jj}) \\
&+ 4W_{33} + 4W_{37}(\tau_{ii} + \tau_{jj}) + 8W_{38}(\tau_{ii} + \tau_{jj}) + 4W_{77}\tau_{ii}\tau_{jj} \\
&+ 16W_{78}\tau_{ii}\tau_{jj} + 16W_{88}\tau_{ii}\tau_{jj}, \qquad (2.9.22)\\
\mathcal{A}_{0ijij} &= 2W_1 + 2W_2 + 2W_7\tau_{ii} + 2W_8(3\tau_{ii} + \tau_{jj}) + 4W_{77}\tau_{ij}^2 + 16W_{78}\tau_{ij}^2 \\
&+ 16W_{88}\tau_{ij}^2, \qquad (2.9.23)\\
\mathcal{A}_{0ijji} &= -2W_2 - 2W_3 + 2W_8(\tau_{ii} + \tau_{jj}) + 4W_{77}\tau_{ij}^2 + 16W_{78}\tau_{ij}^2 \\
&+ 16W_{88}\tau_{ij}^2, \qquad (2.9.24)\\
\mathcal{A}_{0ijjj} &= 2[W_7 + 4W_8 + 2W_{17} + 4W_{18} + 4W_{27} + 8W_{28} + 2W_{37} \\
&+ 4W_{38}]\tau_{ij} + 4[W_{77} + 4W_{78} + 4W_{88}]\tau_{ij}\tau_{jj}, \qquad (2.9.25)\\
\mathcal{A}_{0iiij} &= 2[2W_8 + 2W_{17} + 4W_{18} + 4W_{27} + 8W_{28} + 2W_{37} + 4W_{38}]\tau_{ij} \\
&+ 4[W_{77} + 4W_{78} + 4W_{88}]\tau_{ij}\tau_{ii}, \qquad (2.9.26)\\
\mathcal{A}_{0iikj} &= \mathcal{A}_{0iijk} = \mathcal{A}_{0jkii} = \mathcal{A}_{0kjii} = 4[W_{17} + 2W_{18} + 2W_{27} + 4W_{28} \\
&+ W_{37} + 2W_{38}]\tau_{jk} + 4[W_{77} + 4W_{78} + 4W_{88}]\tau_{jk}\tau_{ii}, \qquad (2.9.27)\\
\mathcal{A}_{0ijki} &= \mathcal{A}_{0ijik} = \mathcal{A}_{0ikji} = \mathcal{A}_{0kiij} = 2W_8\tau_{jk} + 4[W_{77} + 4W_{78} \\
&+ 4W_{88}]\tau_{ij}\tau_{ik}, \qquad (2.9.28)\\
\mathcal{A}_{0ikjk} &= \mathcal{A}_{0jkik} = 2[W_7 + 3W_8]\tau_{ij} + 4[W_{77} + 4W_{78} + 4W_{88}]\tau_{ik}\tau_{jk}, \qquad (2.9.29)
\end{aligned}
$$

When $\tau = 0$ then Eqs. (2.9.21-2.9.29) become

$$
\begin{aligned}
\mathcal{C}_{iiii} &= 2W_1 + 4W_2 + 2W_3 + 4W_{11} + 16W_{22} + 16W_{12} + 8W_{13} + 16W_{23} \\
&+ 4W_{33}, \qquad (2.9.30)\\
\mathcal{C}_{iijj} &= 4W_2 + 4W_3 + 4W_{11} + 16W_{12} + 8W_{13} + 16W_{23} + 4W_{33}, \qquad (2.9.31)\\
\mathcal{C}_{ijij} &= 2W_1 + 2W_2, \qquad (2.9.32)\\
\mathcal{C}_{ijji} &= -2W_2 - 2W_3, \qquad (2.9.33)\\
\mathcal{C}_{ijjj} &= \mathcal{C}_{iiij} = \mathcal{C}_{iikj} = \mathcal{C}_{ijki} = \mathcal{C}_{ikjk} = 0, \qquad (2.9.34)
\end{aligned}
$$

which when used in appropriate strain energy function from the linear theory of elasticity, can reduce to the expression in the classical theory.

Chapter 3

Propagation of Plane Waves in Initially-Stressed Elastic Materials

In this chapter, we study the phenomenon of wave propagation in an initially stressed compressible material. Using the theory of non-linear elasticity presented in chapter 2 and the theory of invariants, we derive the general constitutive equations for a hyper-elastic material in the presence of initial stress. Here we follow the approach as in [2]. The constitutive law of the material is based on a strain-energy function (defined per unit reference volume) which in turn depends on the combined invariants of the right cauchy-Green deformation tensor and the initial stress tensor. For a compressible material, there are ten such independent invariants in the general three-dimensional case. The effect of initial stress is studied on the wave speed of homogenous plane waves in a compressible half-space.

3.1 The Effect of Initial Stress on the Propagation of Homogenous Plane Wave in a Homogeneously Deformed Infinite Medium

Consider an initially stressed medium with initial stress $\boldsymbol{\tau}$ whose elastic response is characterized by the strain energy function $W(\mathbf{C}, \boldsymbol{\tau})$. We consider incremental motions in an infinite medium subject to homogeneous deformation and homogeneous initial stress. The equation of motion for a compressible material is given by

$$\mathcal{A}_{0piqj} u_{j,pq} = \rho u_{i,tt}, \tag{3.1.1}$$

where ρ is the density of the material in the deformed configuration.
For $i = 1$

$$\begin{aligned} &\mathcal{A}_{01111} u_{1,11} + 2\mathcal{A}_{02111} u_{1,12} + \mathcal{A}_{02121} u_{1,22} + \mathcal{A}_{01112} u_{2,11} \\ &+(\mathcal{A}_{01122} + \mathcal{A}_{02112}) u_{2,12} + \mathcal{A}_{02221} u_{2,22} = \rho u_{1,tt}, \end{aligned} \tag{3.1.2}$$

for $i = 2$

$$\begin{aligned} &\mathcal{A}_{01112} u_{1,11} + (\mathcal{A}_{01122} + \mathcal{A}_{02112}) u_{1,12} + \mathcal{A}_{02221} u_{1,22} \\ &+\mathcal{A}_{01212} u_{2,11} + 2\mathcal{A}_{01222} u_{2,12} + \mathcal{A}_{02222} u_{2,22} = \rho u_{2,tt}. \end{aligned} \tag{3.1.3}$$

Here we use a connection

$$\mathcal{A}_{0ijji} = \mathcal{A}_{0iijj} = \mathcal{A}_{0ijij} - T_{ii}, \tag{3.1.4}$$

which can be easily verified by using Eq. (2.8.7) and Eqs. (2.9.12-2.9.20). After using the above connection, Eqs. (3.1.2) and (3.1.3) reduces to

$$\alpha_{11} u_{1,11} + 2\beta_1 u_{2,11} + \beta_2 u_{1,12} + \delta u_{2,12} + \gamma_2 u_{1,22} + \beta_3 u_{2,22} = \rho u_{1,tt}, \tag{3.1.5}$$

$$\beta_1 u_{1,11} + \gamma_1 u_{2,11} + \delta u_{1,12} + 2\beta_4 u_{2,12} + \beta_3 u_{1,22} + \alpha_{22} u_{2,22} = \rho u_{2,tt}, \tag{3.1.6}$$

where $\alpha_{11}, \alpha_{22}, \gamma_1, \gamma_2, \delta$ are material constants and $,t$ indicating differentiation with respect to t and

$$\begin{aligned}
\alpha_{ii} &= J\mathcal{A}_{0iiii}, \quad \beta_1 = J\mathcal{A}_{01112}, \quad \beta_2 = J\mathcal{A}_{02111}, \\
\beta_3 &= J\mathcal{A}_{02221}, \quad \beta_4 = J\mathcal{A}_{01222}, \quad \gamma_1 = J\mathcal{A}_{01212}, \\
\gamma_2 &= J\mathcal{A}_{02121}, \quad \delta = \alpha_{12} + \gamma_2 - T_{22} = \alpha_{12} + \gamma_1 - T_{11}.
\end{aligned} \tag{3.1.7}$$

Let we assume that $\tau_{ij} = 0, i \neq j$ which implies $\Sigma_{ij} = 0, i \neq j$ so that $\beta_1 = \beta_2 = \beta_3 = \beta_4 = 0$ then Eqs. (3.1.5,3.1.6) reduce to

$$\alpha_{11}u_{1,11} + \delta u_{2,12} + \gamma_2 u_{1,22} = \rho u_{1,tt}, \tag{3.1.8}$$

$$\gamma_1 u_{2,11} + \delta u_{1,12} + \alpha_{22}u_{2,22} = \rho u_{2,tt}. \tag{3.1.9}$$

On elimination of either u_1 or u_2 from Eqs. (3.1.8) and (3.1.9) it is seen that u_1 and u_2 each satisfy the same equation, namely

$$\begin{aligned}
&\alpha_{11}\gamma_1 u_{i,1111} + (\alpha_{11}\alpha_{22} + \gamma_1\gamma_2 - \delta^2)u_{i,1122} + \alpha_{22}\gamma_2 u_{i,2222} \\
&= \rho(\alpha_{11} + \gamma_1)u_{i,11tt} + \rho(\alpha_{22} + \gamma_2)u_{i,22tt} - \rho^2 u_{i,tttt},
\end{aligned} \tag{3.1.10}$$

for $i \in (1, 2)$.

Equation (3.1.10) seems to be exactly of the form of Eq. (2.7) of [2]. However the two equations are entirely different in nature as the various material constants in this case depend on the initial stress. Taking the special case of $\tau = 0$ we get the same values of the various constants for an isotropic material as in [2].

3.2 Homogeneous Plane Waves

Consider an incremental plane wave of the form

$$(u_1, u_2) = A(m_1, m_2)\exp[ik(n_1x_1 + n_2x_2 - ct)], \tag{3.2.1}$$

where k is the wave number and c is the wave speed, (n_1, n_2) are the components of the wave normal vector, (m_1, m_2) the components of the polarization vector and A is an arbitrary constant giving the measure of the amplitude of the wave.

Using Eq. (3.2.1) in Eqs. (3.1.8) and (3.1.9) gives the *propagation condition (propagation equation)*, for a compressible material

$$\mathbf{Q}(\mathbf{n})\mathbf{m} = \rho c^2 \mathbf{m}, \tag{3.2.2}$$

where $\mathbf{n} = (n_1, n_2)$, $\mathbf{m} = (m_1, m_2)$ and $\mathbf{Q}(\mathbf{n})$ is the so-called acoustic tensor (see, for example, [14]). It depends on $\mathbf{n}$ and is defined in its component form as

$$Q_{ij}(\mathbf{n}) = \mathcal{A}_{0piqj} n_p n_q. \tag{3.2.3}$$

$$\mathbf{Q}(\mathbf{n}) = \begin{bmatrix} \alpha_{11}n_1^2 + 2\beta_2 n_1 n_2 + \gamma_2 n_2^2 & \beta_1 n_1^2 + \delta n_1 n_2 + \beta_3 n_2^2 \\ \beta_1 n_1^2 + \delta n_1 n_2 + \beta_3 n_2^2 & \gamma_1 n_1^2 + 2\beta_4 n_1 n_2 + \alpha_{22} n_2^2 \end{bmatrix} \tag{3.2.4}$$

For a particular choice of $\mathbf{n}$ it determines possible wave speeds and polarizations corresponding to plane waves propagating in that direction. The wave speeds are determined by the *characteristic equation*

$$\det(\mathbf{Q}(\mathbf{n}) - \rho c^2 \mathbf{I}) = 0, \tag{3.2.5}$$

where $\mathbf{I}$ is the (two-dimensional) identity matrix, then the above equation can be written in matrix form as follows.

$$\det(\begin{bmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{bmatrix} - \rho c^2 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}) = 0, \tag{3.2.6}$$

From Eq. (3.2.6) we obtain an explicit quadratic for ρc^2, namely

$$(\rho c^2)^2 - [(Q_{11} + Q_{22})]\rho c^2 + Q_{11}Q_{22} - Q_{12}Q_{21} = 0, \tag{3.2.7}$$

where

$$\begin{aligned} Q_{11} &= \alpha_{11}n_1^2 + 2\beta_2 n_1 n_2 + \gamma_2 n_2^2, \quad Q_{12} = \beta_1 n_1^2 + \delta n_1 n_2 + \beta_3 n_2^2, \\ Q_{21} &= \beta_1 n_1^2 + \delta n_1 n_2 + \beta_3 n_2^2, \quad Q_{22} = \gamma_1 n_1^2 + 2\beta_4 n_1 n_2 + \alpha_{22} n_2^2. \end{aligned} \tag{3.2.8}$$

Since we assumed that $\tau_{ij} = 0, i \neq j$ which implies $\Sigma_{ij} = 0, i \neq j$ so that $\beta_1 = \beta_2 = \beta_3 = \beta_4 = 0$ then Eq. (3.2.8) reduce to

$$Q_{11} = \alpha_{11} n_1^2 + \gamma_2 n_2^2, \quad Q_{12} = Q_{21} = \delta n_1 n_2, \quad Q_{22} = \gamma_1 n_1^2 + \alpha_{22} n_2^2. \quad (3.2.9)$$

Equivalently, by substituting Eq. (3.2.1) into Eq. (3.1.10) we obtain an explicit quadratic for ρc^2, namely

$$(\rho c^2)^2 - [(\alpha_{11} + \gamma_1) n_1^2 + (\alpha_{22} + \gamma_2) n_2^2] \rho c^2 + \alpha_{11}\gamma_1 n_1^4 + 2\beta n_1^2 n_2^2$$
$$+ \alpha_{22}\gamma_2 n_2^4 = 0, \quad (3.2.10)$$

where

$$2\beta = \alpha_{11}\alpha_{22} + \gamma_1\gamma_2 - \delta^2. \quad (3.2.11)$$

Equation (3.2.10) yields two positive solutions for ρc^2 if and only if the strong ellipticity inequalities hold. These inequalities is followed from [2] which are

$$\alpha_{11} > 0, \quad \alpha_{22} > 0, \quad \gamma_1 > 0, \quad \gamma_2 > 0 \quad (3.2.12)$$

and

$$\beta > -(\alpha_{11}\alpha_{22}\gamma_1\gamma_2)^{1/2}. \quad (3.2.13)$$

From Eq. (3.2.2) we also have

$$m_2 = \xi m_1, \quad (3.2.14)$$

where

$$\xi = \frac{\rho c^2 - \alpha_{11} n_1^2 - \gamma_2 n_2^2}{\delta n_1 n_2} = \frac{\delta n_1 n_2}{\rho c^2 - \gamma_1 n_1^2 - \alpha_{22} n_2^2}, \quad (3.2.15)$$

Equation (3.2.15) is an alternative statement of Eq. (3.2.10).

We may rewrite Eq. (3.2.10 as a quadratic for n_1^2 by using $n_1^2 + n_2^2 = 1$. Thus,

$$(\alpha_{11}\gamma_1 + \alpha_{22}\gamma_2 - 2\beta) n_1^4 + [2\beta - 2\alpha_{22}\gamma_2 - \rho c^2(\alpha_{11}\gamma_1) + \rho c^2(\alpha_{22}\gamma_2)] n_1^2$$
$$+ (\alpha_{22} - \rho c^2)(\gamma_2 - \rho c^2) = 0. \quad (3.2.16)$$

From above equation we see that if material properties are such that

$$2\beta = \alpha_{11}\gamma_1 + \alpha_{22}\gamma_2, \tag{3.2.17}$$

then Eq. (3.2.15) may be simplified to give

$$\begin{aligned}&[(\alpha_{22} - \rho c^2)(\gamma_2 - \rho c^2) - (\alpha_{11} - \rho c^2)(\gamma_1 - \rho c^2)]n_1^2 \\ &= (\alpha_{22} - \rho c^2)(\gamma_2 - \rho c^2).\end{aligned} \tag{3.2.18}$$

In general, Eq. (3.2.2) does not admit pure longitudinal or pure transverse waves. Exceptions are as follows.

(a) *Longitudinal waves.* A longitudinal wave may propagate in a principal direction with speed given by $\rho c^2 = \alpha_{11}$ for $n_1 = 1$ or $\rho c^2 = \alpha_{22}$ for $n_1 = 0$. Using Eq. (3.2.2) where $\mathbf{m} = \mathbf{n}$, a longitudinal wave may also propagate in a direction defined by

$$n_1^2 = \frac{\alpha_{22} - \gamma_2 - \delta}{\alpha_{11} + \alpha_{22} - \gamma_1 - \gamma_2 - 2\delta} \tag{3.2.19}$$

with wave speed given by

$$\rho c^2 = \frac{\alpha_{11}\alpha_{22} - (\gamma_1 + \delta)(\gamma_2 + \delta)}{\alpha_{11} + \alpha_{22} - \gamma_1 - \gamma_2 - 2\delta}. \tag{3.2.20}$$

The existence of such a wave requires, in particular, that the inequality holds

$$(\alpha_{11} - \gamma_1 - \delta)(\alpha_{22} - \gamma_2 - \delta) > 0. \tag{3.2.21}$$

(b) *Shear waves.* As in (a) a shear wave may propagate in a principal direction. In this case the wave speeds are given by $\rho c^2 = \gamma_1$ for $n_1 = 1$ or $\rho c^2 = \gamma_2$ for $n_1 = 0$. For $\mathbf{m} = -\mathbf{n}$, using Eq. (3.2.2) a shear wave may also propagate in the direction defined by Eq. (3.2.19), but with speed given by

$$\rho c^2 = \frac{(\alpha_{11} - \delta)(\alpha_{22} - \delta) - \gamma_1\gamma_2}{\alpha_{11} + \alpha_{22} - \gamma_1 - \gamma_2 - 2\delta}. \tag{3.2.22}$$

The results obtained in Eq. (3.2.19-3.2.22) were obtained in [15] and, more recently, in [16]; see also [17]. Cases (a)and (b) apply in respect of a general form of strain-energy function.

The strong ellipticity condition, for an arbitrary choice of non-zero $\mathbf{m}$ and $\mathbf{n}$ for compressible materials in deformed configuration is given by

$$Q_{ij}m_im_j = \mathcal{A}_{0piqj}n_pn_qm_im_j > 0, \quad \text{for all non-zero } \mathbf{m}, \mathbf{n}. \tag{3.2.23}$$

Taking the scalar product of Eq. (3.2.2), we find

$$\rho c^2 = [\mathbf{Q}(\mathbf{n})\mathbf{m}] \cdot \mathbf{m} = \mathcal{A}_{0piqj}n_pn_qm_im_j. \tag{3.2.24}$$

The above equation holds for the compressible materials. The strong ellipticity condition Eq. (3.2.23) thus guarantees positive values for ρc^2. However, c can be either negative or positive.

Using Eq. (2.9.8) in Eq. (3.2.3), we get for the compressible material

$$\begin{aligned}
\mathbf{Q}(\mathbf{n}) = {} & [2(W_1 + W_2I_1)B^{(\mathbf{n})} - 2W_2B^{2(\mathbf{n})} + 2W_7\Sigma^{(\mathbf{n})} + 4W_8(\Sigma B)^{(\mathbf{n})}]\mathbf{I} \\
& - 2(W_2B^{(\mathbf{n})} - W_8\Sigma^{(\mathbf{n})})\mathbf{B} + 2W_8B^{(\mathbf{n})}\mathbf{\Sigma} + 2(W_3I_3 + 2W_{33}I_3^2)\mathbf{n} \otimes \mathbf{n} \\
& + 2(W_2 + 2W_{11} + 4W_{12}I_1)\mathbf{Bn} \otimes \mathbf{Bn} + 2(W_8 + 2W_{17})(\mathbf{Bn} \otimes \mathbf{\Sigma n} \\
& + \mathbf{\Sigma n} \otimes \mathbf{Bn}) + 4W_{22}\mathbf{B}^*\mathbf{n} \otimes \mathbf{B}^*\mathbf{n} - 4W_{12}(\mathbf{Bn} \otimes \mathbf{B}^2\mathbf{n} + \mathbf{B}^2\mathbf{n} \otimes \mathbf{Bn}) \\
& + 4I_3(W_{13} + W_{23}I_1)(\mathbf{Bn} \otimes \mathbf{n} + \mathbf{n} \otimes \mathbf{Bn}) + 4W_{18}[\mathbf{Bn} \otimes (\mathbf{\Sigma B} \\
& + \mathbf{B\Sigma})\mathbf{n} + (\mathbf{\Sigma B} + \mathbf{B\Sigma})\mathbf{n} \otimes \mathbf{Bn}] - 4W_{23}I_3(\mathbf{n} \otimes \mathbf{B}^2\mathbf{n} + \mathbf{B}^2\mathbf{n} \otimes \mathbf{n}) \\
& + 4W_{27}(\mathbf{B}^*\mathbf{n} \otimes \mathbf{\Sigma n} + \mathbf{\Sigma n} \otimes \mathbf{B}^*\mathbf{n}) + 4W_{28}[\mathbf{B}^*\mathbf{n} \otimes (\mathbf{\Sigma B} + \mathbf{B\Sigma})\mathbf{n} \\
& + (\mathbf{\Sigma B} + \mathbf{B\Sigma})\mathbf{n} \otimes \mathbf{B}^*\mathbf{n}] + 4W_{37}I_3(\mathbf{n} \otimes \mathbf{\Sigma n} + \mathbf{\Sigma n} \otimes \mathbf{n}) \\
& + 4W_{38}I_3[\mathbf{n} \otimes (\mathbf{\Sigma B} + \mathbf{B\Sigma})\mathbf{n} + (\mathbf{\Sigma B} + \mathbf{B\Sigma})\mathbf{n} \otimes \mathbf{n}] \\
& + 4W_{77}\mathbf{\Sigma n} \otimes \mathbf{\Sigma n} + 4W_{78}[\mathbf{\Sigma n} \otimes (\mathbf{\Sigma B} + \mathbf{B\Sigma})\mathbf{n} \\
& + (\mathbf{\Sigma B} + \mathbf{B\Sigma})\mathbf{n} \otimes \mathbf{\Sigma n}] + 4W_{88}(\mathbf{\Sigma B} + \mathbf{B\Sigma})\mathbf{n} \otimes (\mathbf{\Sigma B} + \mathbf{B\Sigma})\mathbf{n},
\end{aligned} \tag{3.2.25}$$

where we have defined $B^{(\mathbf{n})} = \mathbf{n} \cdot \mathbf{Bn}$, $\Sigma^{(\mathbf{n})} = \mathbf{n} \cdot \mathbf{\Sigma n}$, $B^{2(\mathbf{n})} = \mathbf{n} \cdot \mathbf{B}^2\mathbf{n}$ and $(\Sigma B)^{(\mathbf{n})} = \mathbf{n} \cdot \mathbf{\Sigma B n}$. Here we use the same notation as given in [11].

In components form $\mathbf{Q}(\mathbf{n})$ is defined as

$$\begin{aligned}
Q_{ij}(\mathbf{n}) &= [2(W_1+W_2I_1)B^{(\mathbf{n})} - 2W_2B^{2(\mathbf{n})} + 2W_7\Sigma^{(\mathbf{n})} + 4W_8(\Sigma B)^{(\mathbf{n})}]\delta_{ij} \\
&- 2(W_2B^{(\mathbf{n})} - W_8\Sigma^{(\mathbf{n})})B_{ij} + 2W_8B^{(\mathbf{n})}\Sigma_{ij} + 2(W_3I_3 + 2W_{33}I_3^2)n_in_j \\
&+ 2(W_2 + 2W_{11} + 4W_{12}I_1)B_{i\alpha}B_{\alpha j}n_in_j + 2(W_8 + 2W_{17})[B_{i\alpha}\Sigma_{\alpha j} \\
&+ \Sigma_{i\alpha}B_{\alpha j}]n_in_j + 4W_{22}(I_1B_{ij} - B_{i\alpha}B_{\alpha j})(I_1B_{ij} - B_{i\gamma}B_{\gamma j})n_in_j \\
&- 8W_{12}B_{i\alpha}B_{\alpha\gamma}B_{\gamma j}n_in_j + 8I_3(W_{13} + W_{23}I_1)B_{ij}n_in_j + 16W_{18}[B_{ij}(\Sigma_{i\alpha}B_{\alpha j} \\
&+ B_{i\alpha}\Sigma_{\alpha j}) + (\Sigma_{i\alpha}B_{\alpha j} + B_{i\alpha}\Sigma_{\alpha j})B_{ij}]n_in_j - 8W_{23}I_3B_{i\alpha}B_{\alpha j}n_in_j \\
&+ 4W_{27}(((I_1B_{i\alpha} - B_{i\gamma}B_{\gamma\alpha})\Sigma_{\alpha j} - (\Sigma_{i\alpha}(I_1B_{\alpha j} - B_{\alpha\gamma}B_{\gamma j})))n_in_j \\
&+ 4W_{28}(I_1B_{ij} - B_{i\gamma}B_{\gamma j})(\Sigma_{i\alpha}B_{\alpha j} + B_{i\alpha}\Sigma_{\alpha j} + \Sigma_{i\alpha}B_{\alpha j} + B_{i\alpha}\Sigma_{\alpha j})(I_1B_{ij} \\
&- B_{i\gamma}B_{\gamma j}))n_1n_j + 8W_{37}I_3\Sigma_{ij}n_in_j + 4W_{38}I_3((\Sigma_{i\alpha}B_{\alpha j} + B_{i\alpha}\Sigma_{\alpha j} \\
&+ \Sigma_{i\alpha}B_{\alpha j} + B_{i\alpha}\Sigma_{\alpha j})n_in_j + 4W_{77}\Sigma_{i\alpha}\Sigma_{\alpha j}n_in_j4W_{78}[\Sigma_{ij}(\Sigma_{i\alpha}B_{\alpha j} \\
&+ B_{i\alpha}\Sigma_{\alpha j} + \Sigma_{i\alpha}B_{\alpha j} + B_{i\alpha}\Sigma_{\alpha j})\Sigma_{ij}]n_in_j + 4W_{88}(\Sigma_{i\alpha}B_{\alpha j} \\
&+ B_{i\alpha}\Sigma_{\alpha j})(\Sigma_{i\gamma}B_{\gamma j} + B_{i\gamma}\Sigma_{\gamma j}), \qquad (3.2.26)
\end{aligned}$$

In the absence of shear initial stress the above expression reduces to

$$\begin{aligned}
Q_{ij}(\mathbf{n}) &= [2(W_1+W_2I_1)B^{(\mathbf{n})} - 2W_2B^{2(\mathbf{n})}]\delta_{ij} - 2W_2B^{(\mathbf{n})}B_{ij} + 2(W_3I_3 \\
&+ 2W_{33}I_3^2)n_in_j + 2(W_2 + 2W_{11} + 4W_{12}I_1)B_{i\alpha}B_{\alpha j}n_in_j + 4W_{22}(I_1B_{ij} \\
&- B_{i\alpha}B_{\alpha j})(I_1B_{ij} - B_{i\gamma}B_{\gamma j})n_in_j - 8W_{12}B_{i\alpha}B_{\alpha\gamma}B_{\gamma j}n_in_j + 8I_3(W_{13} \\
&+ W_{23}I_1)B_{ij}n_in_j - 8W_{23}I_3B_{i\alpha}B_{\alpha j}n_in_j, \qquad (3.2.27)
\end{aligned}$$

In the reference configuration, $B^{2(\mathbf{n})} = B^{(\mathbf{n})} = 1, B^{*(\mathbf{n})} = 2$ and $(\Sigma B)^{(\mathbf{n})} = \tau^{(\mathbf{n})}$, we have

$$\begin{aligned}
\mathbf{Q}(\mathbf{n}) &= (\alpha_1 + (1+\alpha_3)\tau^{(\mathbf{n})})\mathbf{I} + \alpha_2\mathbf{n}\otimes\mathbf{n} + \alpha_3\boldsymbol{\tau} \\
&+ (\alpha_3+\alpha_4)(\mathbf{n}\otimes\mathbf{n}\boldsymbol{\tau} + \boldsymbol{\tau}\mathbf{n}\otimes\mathbf{n}) + \alpha_5\boldsymbol{\tau}\mathbf{n}\otimes\boldsymbol{\tau}\mathbf{n}, \qquad (3.2.28)
\end{aligned}$$

where

$$
\begin{aligned}
\alpha_1 &= 2(W_1 + W_2), \\
\alpha_2 &= 2(W_2 + W_3) + 4(W_{11} + 4W_{12} + 2W_{13} + 4W_{22} + 4W_{23} + W_{33}), \\
\alpha_3 &= 2W_8, \\
\alpha_4 &= 4(W_{17} + 2W_{18} + 2W_{27} + 4W_{28} + W_{37} + 2W_{38}), \\
\alpha_5 &= 4(W_{77} + 4W_{78} + 4W_{88}).
\end{aligned} \tag{3.2.29}
$$

In components form Eq. (3.2.28) is defined as

$$
\begin{aligned}
Q_{ij}(\mathbf{n}) = {} & (\alpha_1 + (1 + \alpha_3)\tau^{(\mathbf{n})})\delta_{ij} + \alpha_2 n_i n_j + \alpha_3 \tau_{ij} \\
& + 2(\alpha_3 + \alpha_4)\tau_{ij} n_i n_j + \alpha_5 (\boldsymbol{\tau})^2_{ij} n_i n_j,
\end{aligned} \tag{3.2.30}
$$

Since we assumed $\tau_{ij} = 0, i \neq j$, implies $\Sigma_{ij} = 0, i \neq j$ then above expression reduce to

$$
Q_{ij}(\mathbf{n}) = \alpha_1 \delta_{ij} + \alpha_2 n_i n_j, \tag{3.2.31}
$$

3.2.1 Specific Strain Energy Function W for an Initially-Stressed Compressible Materials

We consider a compressible material whose elastic response is characterized by the strain energy function $W(\mathbf{C}, \boldsymbol{\tau})$ given by

$$
\begin{aligned}
W = {} & \frac{\mu}{2}(I_3^{-2/3} I_1 + I_3 - 3) + (\lambda + \frac{2}{3}\mu)(I_3 - 1)^2 \\
& + \frac{\bar{\mu}}{2}(I_7 - I_4)^2 + \frac{1}{2}(I_7 - I_4),
\end{aligned} \tag{3.2.32}
$$

where μ, λ are Lame's parameters, as appeared in the classical linear theory of elasticity and $\bar{\mu}$ is a material constant with dimensions of $(stress)^{-1}$. The above model is motivated through the definition of a Neo-Hookean linear elasticity model. When $\tau = 0$, Eq. (3.2.32) behaves nearly like Neo-Hookean classical solid. Using Eq.

(3.2.32), in the deformed configuration, the required derivatives of W with respect to various invariants are

$$\begin{aligned}
W_1 &= \frac{\mu}{2} I_3^{-2/3}, \qquad W_{13} = \frac{-\mu}{3} I_3^{-5/3}, \\
W_3 &= \frac{\mu}{2}(\frac{-2}{3} I_3^{-5/3} I_1 + 1) + 2(\lambda + \frac{2}{3}\mu)(I_3 - 1), \\
W_{33} &= \frac{5\mu}{9} I_3^{-8/3} I_1 + 2(\lambda + \frac{2}{3}\mu), \quad W_4 = -\bar{\mu}(I_7 - I_4) - \frac{1}{2}, \\
W_7 &= \bar{\mu}(I_7 - I_4) + \frac{1}{2}, \qquad W_{77} = \bar{\mu},
\end{aligned} \tag{3.2.33}$$

which in the reference configuration reduce to

$$\begin{aligned}
& W_1 = \frac{\mu}{2}, \quad W_{13} = \frac{-\mu}{3}, \quad W_3 = \frac{-\mu}{2}, \quad W_{33} = \frac{9\mu}{3} + 2\lambda, \\
& W_4 = -\frac{1}{2}, \quad W_7 = \frac{1}{2}, \quad W_{77} = \bar{\mu}.
\end{aligned} \tag{3.2.34}$$

Using Eq. (3.2.33) in Eq. (3.2.25), we have

$$\begin{aligned}
\mathbf{Q}(\mathbf{n}) = {} & [\mu I_3^{-2/3} B^{(\mathbf{n})} + [2\bar{\mu}(I_7 - I_4) + 1]\Sigma^{(\mathbf{n})}]\mathbf{I} + [\frac{14}{9}\mu I_3^{-2/3} I_1 \\
& + 8(\lambda + \frac{2}{3}\mu) I_3^2 + [\mu - 4(\lambda + \frac{2}{3}\mu)] I_3]\mathbf{n} \otimes \mathbf{n} - \frac{4}{3}\mu I_3^{-2/3}(\mathbf{Bn} \otimes \mathbf{n} \\
& + \mathbf{n} \otimes \mathbf{Bn}) + 4\bar{\mu}\boldsymbol{\Sigma}\mathbf{n} \otimes \boldsymbol{\Sigma}\mathbf{n},
\end{aligned} \tag{3.2.35}$$

for a compressible material in the deformed configuration. Also, using Eq. (3.2.34) in Eq. (3.2.28) gives

$$\mathbf{Q}(\mathbf{n}) = (\mu + \tau^{(\mathbf{n})})\mathbf{I} + (\frac{35}{9}\mu + 4\lambda)\mathbf{n} \otimes \mathbf{n} + 4\bar{\mu}\boldsymbol{\tau}\mathbf{n} \otimes \boldsymbol{\tau}\mathbf{n}, \tag{3.2.36}$$

for a compressible material in the reference configuration.

From Eq. (3.2.24), it follows for a compressible material in the deformed configuration that

$$\begin{aligned}
\rho c^2 = {} & \mu I_3^{-2/3} B^{(\mathbf{n})} + [2\bar{\mu}(I_7 - I_4) + 1]\Sigma^{(\mathbf{n})} + [\frac{14}{9}\mu I_3^{-2/3} I_1 + 8(\lambda + \frac{2}{3}\mu) I_3^2 \\
& + [\mu - 4(\lambda + \frac{2}{3}\mu)] I_3](\mathbf{n} \cdot \mathbf{m})^2 - \frac{4}{3}\mu I_3^{-2/3}(\mathbf{m} \cdot \mathbf{Bn})(\mathbf{n} \cdot \mathbf{m}) \\
& + 4\bar{\mu}(\mathbf{m} \cdot \boldsymbol{\Sigma}\mathbf{n})^2,
\end{aligned} \tag{3.2.37}$$

which in the reference configuration reduces to

$$\rho c^2 = \mu + \tau^{(\mathbf{n})} + (\frac{35}{9}\mu + 4\lambda)(\mathbf{n}\cdot\mathbf{m})^2 + 4\bar{\mu}(\mathbf{m}\cdot\boldsymbol{\Sigma}\mathbf{n})^2. \tag{3.2.38}$$

For the particular choice $\boldsymbol{\tau} = \tau\mathbf{n}\otimes\mathbf{n}$, we have $\boldsymbol{\Sigma} = \tau\mathbf{F}\mathbf{n}\otimes\mathbf{F}\mathbf{n}$. Equation (3.2.37) thus reduces to

$$\begin{aligned}\rho c^2 =& \mu I_3^{-2/3}B^{(\mathbf{n})} + \tau[2\bar{\mu}(I_7 - I_4) + 1](\mathbf{n}\cdot\mathbf{F}\mathbf{n})^2\\ &+ [\frac{14}{9}\mu I_3^{-2/3}I_1 + 8(\lambda + \frac{2}{3}\mu)I_3^2 + (\mu - 4(\lambda + \frac{2}{3}\mu))I_3](\mathbf{n}\cdot\mathbf{m})^2\\ &- \frac{4}{3}\mu I_3^{-2/3}(\mathbf{m}\cdot\mathbf{B}\mathbf{n})(\mathbf{n}\cdot\mathbf{m}) + 4\bar{\mu}\tau(\mathbf{m}\cdot\mathbf{F}\mathbf{n})^2(\mathbf{n}\cdot\mathbf{F}\mathbf{n})^2,\end{aligned} \tag{3.2.39}$$

For arbitrary $\mathbf{m}$ and $\mathbf{n}$, a real speed exists if

$$\begin{aligned}&\mu I_3^{-2/3}B^{(\mathbf{n})} + \tau[2\bar{\mu}(I_7 - I_4) + 1](\mathbf{n}\cdot\mathbf{F}\mathbf{n})^2 + [\frac{14}{9}\mu I_3^{-2/3}I_1\\ &+ 8(\lambda + \frac{2}{3}\mu)I_3^2 + (\mu - 4(\lambda + \frac{2}{3}\mu))I_3](\mathbf{n}\cdot\mathbf{m})^2 - \frac{4}{3}\mu I_3^{-2/3}(\mathbf{m}\cdot\mathbf{B}\mathbf{n})(\mathbf{n}\cdot\mathbf{m})\\ &+ 4\bar{\mu}\tau(\mathbf{m}\cdot\mathbf{F}\mathbf{n})^2(\mathbf{n}\cdot\mathbf{F}\mathbf{n})^2 > 0.\end{aligned} \tag{3.2.40}$$

In deformed configuration after using Eq. (3.1.7) and Eq. (3.2.33) in Eqs. (2.9.12-2.9.20), we get

$$\begin{aligned}\alpha_{11} =& \mu I_3^{-2/3}\lambda_1^2 + \mu(\frac{-2}{3}I_3^{-5/3}I_1 + 1) + 2(\lambda + \frac{2}{3}\mu)(I_3 - 1))I_3\\ &+ 2(\bar{\mu}(I_7 - I_4) + \frac{1}{2})\Sigma_{11} + 8(\frac{-\mu}{3}I_3^{-5/3})I_3\lambda_1^2 + 4(\frac{5\mu}{9}I_3^{-8/3}I_1\\ &+ 2(\lambda + \frac{2}{3}\mu))I_3^2 + 4\bar{\mu}\Sigma_{11}^2,\end{aligned} \tag{3.2.41}$$

$$\begin{aligned}\alpha_{12} =& [2\mu(\frac{-2}{3}I_3^{-5/3}I_1 + 1) + 8(\lambda + \frac{2}{3}\mu)(I_3 - 1)]I_3 + 4(\frac{-\mu}{3}I_3^{-5/3})I_3(\lambda_1^2\\ &+ \lambda_2^2) + 4(\frac{5\mu}{9}I_3^{-8/3}I_1 + 2(\lambda + \frac{2}{3}\mu))I_3^2 + 4(\bar{\mu})\Sigma_{11}\Sigma_{22},\end{aligned} \tag{3.2.42}$$

$$\begin{aligned}\alpha_{22} =& \mu I_3^{-2/3}\lambda_2^2 + \mu(\frac{-2}{3}I_3^{-5/3}I_1 + 1) + 2(\lambda + \frac{2}{3}\mu)(I_3 - 1))I_3\\ &+ 2(\bar{\mu}(I_7 - I_4) + \frac{1}{2})\Sigma_{22} + 8(\frac{-\mu}{3}I_3^{-5/3})I_3\lambda_2^2 + 4(\frac{5\mu}{9}I_3^{-8/3}I_1\\ &+ 2(\lambda + \frac{2}{3}\mu))I_3^2 + 4\bar{\mu}\Sigma_{22}^2,\end{aligned} \tag{3.2.43}$$

$$\beta_1 = 4\bar{\mu}\Sigma_{11}\Sigma_{12}, \tag{3.2.44}$$

$$\beta_2 = [2\bar{\mu}(I_7 - I_4) + 1]\Sigma_{21} + 4\bar{\mu}\Sigma_{21}\Sigma_{11}, \tag{3.2.45}$$

$$\beta_3 = 4\bar{\mu}\Sigma_{21}\Sigma_{22}, \tag{3.2.46}$$

$$\beta_4 = [2\bar{\mu}(I_7 - I_4) + 1]\Sigma_{12} + 4\bar{\mu}\Sigma_{12}\Sigma_{22}, \tag{3.2.47}$$

$$\gamma_1 = \mu I_3^{-2/3}\lambda_1^2 + [2\bar{\mu}(I_7 - I_4) + 1]\Sigma_{11} + 4\bar{\mu}\Sigma_{12}^2, \tag{3.2.48}$$

$$\gamma_2 = \mu I_3^{-2/3}\lambda_2^2 + [2\bar{\mu}(I_7 - I_4) + 1]\Sigma_{22} + 4\bar{\mu}\Sigma_{21}^2, \tag{3.2.49}$$

$$\begin{aligned} \delta = {} & [\mu(\frac{-2}{3}I_3^{-5/3}I_1 + 1) + 4(\lambda + \frac{2}{3}\mu)(I_3 - 1)]I_3 + 4(\frac{-\mu}{3}I_3^{-5/3})I_3(\lambda_1^2 \\ & + \lambda_2^2) + 4(\frac{5\mu}{9}I_3^{-8/3}I_1 + 2(\lambda + \frac{2}{3}\mu))I_3^2 + 4\bar{\mu}(\Sigma_{12}^2 + \Sigma_{11}\Sigma_{22}). \end{aligned} \tag{3.2.50}$$

Since we assumed $\tau_{ij} = 0, i \neq j$, implies $\Sigma_{ij} = 0, i \neq j$ then above expressions reduce

$$\begin{aligned} \alpha_{11} = {} & \mu I_3^{-2/3}\lambda_1^2 + \mu(\frac{-2}{3}I_3^{-5/3}I_1 + 1) + 2(\lambda + \frac{2}{3}\mu)(I_3 - 1))I_3 \\ & + 2(\bar{\mu}(I_7 - I_4) + \frac{1}{2})\Sigma_{11} + 8(\frac{-\mu}{3}I_3^{-5/3})I_3\lambda_1^2 + 4(\frac{5\mu}{9}I_3^{-8/3}I_1 \\ & + 2(\lambda + \frac{2}{3}\mu))I_3^2 + 4\bar{\mu}\Sigma_{11}^2, \end{aligned} \tag{3.2.51}$$

$$\begin{aligned} \alpha_{12} = {} & [2\mu(\frac{-2}{3}I_3^{-5/3}I_1 + 1) + 8(\lambda + \frac{2}{3}\mu)(I_3 - 1)]I_3 + 4(\frac{-\mu}{3}I_3^{-5/3})I_3(\lambda_1^2 \\ & + \lambda_2^2) + 4(\frac{5\mu}{9}I_3^{-8/3}I_1 + 2(\lambda + \frac{2}{3}\mu))I_3^2 + 4(\bar{\mu})\Sigma_{11}\Sigma_{22}, \end{aligned} \tag{3.2.52}$$

$$\begin{aligned} \alpha_{22} = {} & \mu I_3^{-2/3}\lambda_2^2 + \mu(\frac{-2}{3}I_3^{-5/3}I_1 + 1) + 2(\lambda + \frac{2}{3}\mu)(I_3 - 1))I_3 \\ & + 2(\bar{\mu}(I_7 - I_4) + \frac{1}{2})\Sigma_{22} + 8(\frac{-\mu}{3}I_3^{-5/3})I_3\lambda_2^2 + 4(\frac{5\mu}{9}I_3^{-8/3}I_1 \\ & + 2(\lambda + \frac{2}{3}\mu))I_3^2 4\bar{\mu}\Sigma_{22}^2, \end{aligned} \tag{3.2.53}$$

$$\beta_1 = \beta_2 = \beta_3 = \beta_4 = 0, \tag{3.2.54}$$

$$\gamma_1 = \mu I_3^{-2/3}\lambda_1^2 + [2\bar{\mu}(I_7 - I_4) + 1]\Sigma_{11}, \tag{3.2.55}$$

$$\gamma_2 = \mu I_3^{-2/3}\lambda_2^2 + [2\bar{\mu}(I_7 - I_4) + 1]\Sigma_{22}, \tag{3.2.56}$$

$$\begin{aligned} \delta = {} & [\mu(\frac{-2}{3}I_3^{-5/3}I_1 + 1) + 4(\lambda + \frac{2}{3}\mu)(I_3 - 1)]I_3 + 4(\frac{-\mu}{3}I_3^{-5/3})I_3(\lambda_1^2 \\ & + \lambda_2^2) + 4(\frac{5\mu}{9}I_3^{-8/3}I_1 + 2(\lambda + \frac{2}{3}\mu))I_3^2 + 4\bar{\mu}\Sigma_{11}\Sigma_{22}. \end{aligned} \tag{3.2.57}$$

In the reference configuration after using Eq. (3.2.34) in Eqs. (2.9.21-2.9.29) the above material constants reduce to

$$
\begin{aligned}
\alpha_{11} &= 4(\bar{\mu})\tau_{11}^2 + \tau_{11} + \frac{28\mu}{3} + 8\lambda, \quad \alpha_{12} = \frac{22\mu}{3} + 8\lambda + 4(\bar{\mu})\tau_{11}\tau_{22}, \\
\alpha_{22} &= 4(\bar{\mu})\tau_{22}^2 + \tau_{22} + \frac{28\mu}{3} + 8\lambda, \quad \beta_1 = 4(\bar{\mu})\tau_{11}\tau_{12}, \\
\beta_2 &= \tau_{12} + 4(\bar{\mu})\tau_{11}\tau_{12}, \quad \beta_3 = 4(\bar{\mu})\tau_{12}\tau_{22}, \\
\beta_4 &= \tau_{12} + 4(\bar{\mu})\tau_{12}\tau_{22}, \quad \gamma_1 = \mu + \tau_{11} + 4(\bar{\mu})\tau_{12}^2, \\
\gamma_2 &= \mu + \tau_{22} + 4(\bar{\mu})\tau_{12}^2, \quad \delta = \bar{\mu}(\tau_{12}^2 + \tau_{11}\tau_{22}) + \frac{25\mu}{3} + 8\lambda.
\end{aligned} \tag{3.2.58}
$$

For $\tau_{ij} = 0, i \neq j$, the above expressions reduce to

$$
\begin{aligned}
\alpha_{11} &= 4(\bar{\mu})\tau_{11}^2 + \tau_{11} + \frac{28\mu}{3} + 8\lambda, \quad \alpha_{12} = \frac{22\mu}{3} + 8\lambda + 4(\bar{\mu})\tau_{11}\tau_{22}, \\
\alpha_{22} &= 4(\bar{\mu})\tau_{22}^2 + \tau_{22} + \frac{28\mu}{3} + 8\lambda, \quad \beta_1 = \beta_2 = \beta_3 = \beta_4 = 0, \\
\gamma_1 &= \mu + \tau_{11}, \quad \gamma_2 = \mu + \tau_{22}, \quad \delta = 4\bar{\mu}\tau_{11}\tau_{22} + \frac{25\mu}{3} + 8\lambda.
\end{aligned} \tag{3.2.59}
$$

3.3 Reflection from a Plane Boundary

We consider the half-space $x_2 < 0$ in the finitely deformed configuration and note that the boundary $x_2 = 0$ is subjected to the normal traction T_{22} in this configuration. Using Eq. (2.7.9) the incremental traction per unit area of the boundary is $\dot{\mathbf{S}}_0^T \boldsymbol{\nu}$, where $\boldsymbol{\nu}$ is the unit outward normal to the boundary. The component form of $\dot{\mathbf{S}}^T \boldsymbol{\nu}$ as follows

$$
\dot{S}_{0pi}\nu_k = \mathcal{A}_{0piqj} u_{j,q} \nu_k. \tag{3.3.1}
$$

We take the incremental boundary condition on $x_2 = 0$ to correspond to vanishing incremental dead load. Thus, the boundary conditions may be expressed as

$$
\dot{S}_{021} = \mathcal{A}_{02111} u_{1,1} + \mathcal{A}_{02121} u_{1,2} + \mathcal{A}_{02112} u_{2,1} + \mathcal{A}_{02122} u_{2,2} = 0 \quad \text{on } x_2 = 0, \tag{3.3.2}
$$

$$
\dot{S}_{022} = \mathcal{A}_{01122} u_{1,1} + \mathcal{A}_{02221} u_{1,2} + \mathcal{A}_{02212} u_{2,1} + \mathcal{A}_{02222} u_{2,2} = 0 \quad \text{on } x_2 = 0. \tag{3.3.3}
$$

After using Eq. (3.1.7) above equations becomes

$$
\dot{S}_{021} = \gamma_2 u_{1,2} + (\gamma_2 - T_{22}) u_{2,1} = 0 \quad \text{on } x_2 = 0, \tag{3.3.4}
$$

$$
\dot{S}_{022} = \alpha_{12} u_{1,1} + \alpha_{22} u_{2,2} = 0 \quad \text{on } x_2 = 0. \tag{3.3.5}
$$

Under the above boundary conditions it was shown by [18] that the underlying deformation is stable to incremental disturbances of the considered type provided the inequality

$$(\frac{\gamma_1\gamma_2}{\alpha_{11}\alpha_{22}})^{\frac{1}{2}}(\alpha_{11}\alpha_{22}-\alpha_{12}^2)+\gamma_1\gamma_2-(\gamma_2-T_{22})^2>0 \tag{3.3.6}$$

We now consider a homogeneous plane wave of the form Eq. (3.2.1) propagating in the half-space $x_2<0$. We write Eq. (3.2.1) as

$$\mathbf{u}=A\mathbf{m}\exp[ik(\mathbf{n}.\mathbf{x}-ct)], \tag{3.3.7}$$

where $\mathbf{u}=(u_1,u_2),\mathbf{m}=(m_1,m_2),\mathbf{n}=(n_1,n_2)$ and $\omega=kc$ is the frequency. Equation (3.3.7) may represent a quasi-P wave or a quasi-SV wave or, for a specific direction, a P wave or an SV wave.

3.3.1 The Case of Two Reflected Waves

Let $\mathbf{m}^+=\mathbf{m},\mathbf{n}^+=\mathbf{n}$ be identified with the incident wave. Let $\mathbf{m}^-,\mathbf{n}^-$ be the corresponding values associated with the reflected wave and $\mathbf{m}',\mathbf{n}'$ be the values associated with the second reflected wave. Here we have followed the same notation for $\mathbf{m}$ and $\mathbf{n}$ as given in [2]. Hence, we have

$$n_1'=n_1^+=n_1^-=n_1. \tag{3.3.8}$$

which is the statement of Snell's law.

We also have $n_2^-=-n_2$. The total displacement in $x_2<0$ may now be written as

$$\begin{aligned}\mathbf{u}&=A\mathbf{m}^+\exp[ik(\mathbf{n}^+.\mathbf{x}-ct)]+AR\mathbf{m}^-\exp[ik(\mathbf{n}^-.\mathbf{x}-ct)]\\&+AR'\mathbf{m}'\exp[ik(\mathbf{n}'.\mathbf{x}-ct)]\end{aligned} \tag{3.3.9}$$

where R,R' are the reflection coefficients k',c' are respectively the wave number and wave speed associated with the second reflected wave, so that $k'c'=\omega$.

From Eqs. (3.2.14)and (3.2.15), we note that a change in sign of n_2 is reflected in a change in sign of ξ and change in sign of m_2 is reflected in a change in sign of ξ. Let we take $\mathbf{m}^- = (m_1, -m_2)$ since any sign difference can be vanished by R. Similarly, we have for the second reflected wave ,$m_2^{'} = \xi^{'} m_1^{'}$, with $\xi^{'}$ defined analogously to ξ in Eq. (3.2.14), with $c, \mathbf{n}$ replaced by $c^{'}, \mathbf{n}^{'}$.

$$\xi^{'} = (\rho c^2 - \alpha_{11} n_1^2 - \gamma_2 n_2^{'2})/\delta n_1 n_2^{'}. \tag{3.3.10}$$

Since $\mathbf{m}$ and $\mathbf{m}^{'}$ are unit vectors we set

$$m_1 = 1/(1+\xi^2)^{\frac{1}{2}}, \quad m_1^{'} = 1/(1+\xi^{'2})^{\frac{1}{2}}. \tag{3.3.11}$$

3.3.2 Reflection Coefficients

Using Eq. (3.3.9) into the boundary conditions Eq. (3.3.4, 3.3.5) we get the reflection coefficients $R, R^{'}$. After some simplification, this leads to

$$R = \frac{n_2^{'} p^{'} q - n_2 p q^{'}}{n_2^{'} p^{'} q + n_2 p q^{'}}, \tag{3.3.12}$$

$$R^{'} = -\frac{2pq}{n_2^{'} p^{'} q + n_2 p q^{'}} \frac{m_1 n_2^{'}}{m_1^{'}}, \tag{3.3.13}$$

where $p, q, p^{'}, q^{'}$ are defined by

$$p = \delta \alpha_{12} n_1^2 + \alpha_{22}(\rho c^2 - \alpha_{11} n_1^2 - \gamma_2 n_2^2), \tag{3.3.14}$$

$$q = \gamma_2 \alpha_{12} n_2^2 + (\delta - \alpha_{12})(\rho c^2 - \alpha_{11} n_1^2), \tag{3.3.15}$$

$$p^{'} = \delta \alpha_{12} n_1^2 + \alpha_{22}(\rho c^2 - \alpha_{11} n_1^2 - \gamma_2 n_2^{'2}), \tag{3.3.16}$$

$$q^{'} = \gamma_2 \alpha_{12} n_2^{'2} + (\delta - \alpha_{12})(\rho c^2 - \alpha_{11} n_1^2), \tag{3.3.17}$$

and $\frac{m_1 n_2^{'}}{m_1^{'}}$ is given by

$$\frac{m_1 n_2^{'}}{m_1^{'}} = n_2 \frac{(\delta^2 n_1^2 n_2^{'2} + (\rho c^2 - \alpha_{11} n_1^2 - \gamma_2 n_2^{'2})^2)^{\frac{1}{2}}}{(\delta^2 n_1^2 n_2^2 + (\rho c^2 - \alpha_{11} n_1^2 - \gamma_2 n_2^2)^2)^{\frac{1}{2}}}. \tag{3.3.18}$$

If we use the connection

$$n_2 = n_1 \tan\theta \tag{3.3.19}$$

then Eq. (3.2.10) becomes

$$\begin{aligned}&(\alpha_{11}\gamma_1 + 2\beta\tan^2\theta + \alpha_{22}\gamma_2\tan^4\theta)n_1^4 - \rho c^2[\alpha_{11} + \gamma_1 + (\alpha_{22}\\ &+\gamma_2)\tan^2\theta]n_1^2 + (\rho c^2)^2 = 0,\end{aligned} \tag{3.3.20}$$

which gives an equation for n_1 in terms of the angle θ which defines the direction of the wave normal.

Taking derivative of Eq. (3.2.10) with respect to $\frac{\gamma_2 n_2^2}{\rho c^2}$ and multiplying both sides by n_2^2 we get

$$2\beta n_1^2 n_2^2 + (\alpha_{22}\gamma_2(n_2^4 + n_2^2 n_2^{'2})) - (\alpha_{22} + \gamma_2)n_2^2\rho c^2 = 0, \tag{3.3.21}$$

Subtracting Eq. (3.3.21) from Eq. (3.2.10) we get

$$\alpha_{22}\gamma_2 n_2^2 n_2^{'2} = (\alpha_{11}n_1^2 - \rho c^2)(\gamma_1 n_1^2 - \rho c^2), \tag{3.3.22}$$

which gives $n_2^{'}$, indirectly in terms of θ. Following this procedure we allows R, $R^{'}$ explicitly as a function of θ. On use of the connection

$$n_2^{'} = n_1 \tan\theta^{'}, \tag{3.3.23}$$

and using the notation

$$\zeta = \tan\theta, \quad \zeta^{'} = \tan\theta^{'}, \tag{3.3.24}$$

then Eq. (3.3.20) can be written as

$$(\alpha_{11}\gamma_1 + 2\beta\zeta^2 + \alpha_{22}\gamma_2\zeta^4)n_1^4 - \rho c^2[\alpha_{11} + \gamma_1 + (\alpha_{22} + \zeta^2]n_1^2 + (\rho c^2)^2 = 0, \tag{3.3.25}$$

and Eq. (3.3.22) becomes

$$\zeta^2\zeta^{'2} = \frac{(\alpha_{11}n_1^2 - \rho c^2)(\gamma_1 n_1^2 - \rho c^2)}{\alpha_{22}\gamma_2 n_1^4}, \tag{3.3.26}$$

After using connection Eq. (3.3.19) and Eq. (3.3.23), Eqs. (3.3.12-3.3.18) become

$$R = \frac{\zeta' p' q - \zeta p q'}{\zeta' p' q + \zeta p q'}, \tag{3.3.27}$$

$$R' = -\frac{2pq}{\zeta' p' q + \zeta p q'} \frac{m_1 \zeta'}{m_1'}, \tag{3.3.28}$$

where p, q, p', q' are defined by

$$p = \delta\alpha_{12} n_1^2 + \alpha_{22}(\rho c^2 - \alpha_{11} n_1^2 - \gamma_2 n_1^2 \zeta^2), \tag{3.3.29}$$

$$q = \gamma_2 \alpha_{12} n_1^2 \zeta^2 + (\delta - \alpha_{12})(\rho c^2 - \alpha_{11} n_1^2), \tag{3.3.30}$$

$$p' = \delta\alpha_{12} n_1^2 + \alpha_{22}(\rho c^2 - \alpha_{11} n_1^2 - \gamma_2 n_1^2 \zeta'^2), \tag{3.3.31}$$

$$q' = \gamma_2 \alpha_{12} n_1^2 \zeta'^2 + (\delta - \alpha_{12})(\rho c^2 - \alpha_{11} n_1^2), \tag{3.3.32}$$

and $\frac{m_1 \zeta'}{m_1'}$ is given by

$$\frac{m_1 \zeta'}{m_1'} = \zeta \sqrt{\frac{\delta^2 n_1^4 \zeta'^2 + (\rho c^2 - \alpha_{11} n_1^2 - \gamma_2 n_1^2 \zeta'^2)^2}{\delta^2 n_1^4 \zeta^2 + (\rho c^2 - \alpha_{11} n_1^2 - \gamma_2 n_1^2 \zeta^2)^2}}. \tag{3.3.33}$$

3.4 Numerical Results and Discussion

Considering the special class of material we have $2\beta = \alpha_{11}\gamma_2 + \alpha_{22}\gamma_1$ then the Eq. (3.2.10) decouples in the form

$$(\gamma_1 n_1^2 + \gamma_2 n_2^2 - \rho c^2)(\alpha_{11} n_1^2 + \alpha_{22} n_2^2 - \rho c^2) = 0 \tag{3.4.1}$$

3.4.1 Incident P wave for the Reference Configuration

For an incident P using Eq. (3.4.2), we get

$$\alpha_{11} n_1^2 + \alpha_{22} n_2^2 = \rho c^2. \tag{3.4.2}$$

After using Eq. (3.4.2) in Eq. (3.3.26) we get

$$\zeta' = -\sqrt{\frac{\alpha_{22}\zeta^2 + \alpha_{11} - \gamma_1}{\gamma_2}}. \tag{3.4.3}$$

For $\zeta^2\zeta^{'2} > 0$ following inequalities hold $(\alpha_{11}n_1^2 - \rho c^2) < 0$ and $(\gamma_1 n_1^2 - \rho c^2) < 0$ or $\alpha_{22}n_2^2 > 0$ and $\zeta^2 > \frac{\gamma_1 - \alpha_{11}}{\alpha_{22}}$.

Hence, in Eqs. (3.3.27) and (3.3.28) $p, q, p^{'}, q^{'}$ are defined as

$$p = \delta\alpha_{12}n_1^2 - \alpha_{22}\gamma_2 n_1^2\zeta^2 + \alpha_{22}^2 n_2^2, \tag{3.4.4}$$

$$q = \gamma_2\alpha_{12}n_1^2\zeta^2 + \delta\alpha_{22}n_2^2 - \alpha_{12}\alpha_{22}n_2^2, \tag{3.4.5}$$

$$p^{'} = \delta\alpha_{12}n_1^2 - \alpha_{22}\gamma_2 n_1^2\zeta^{'2} + \alpha_{22}^2 n_2^2, \tag{3.4.6}$$

$$q^{'} = \gamma_2\alpha_{12}n_1^2\zeta^{'2} + \delta\alpha_{22}n_2^2 - \alpha_{12}\alpha_{22}n_2^2, \tag{3.4.7}$$

and $\frac{m_1\zeta^{'}}{m_1^{'}}$ is given by

$$\frac{m_1\zeta^{'}}{m_1^{'}} = \zeta\sqrt{\frac{\delta^2 n_1^4\zeta^{'2} + (\alpha_{22}n_2^2 - \gamma_2 n_1^2\zeta^{'2})^2}{\delta^2 n_1^4\zeta^2 + (\alpha_{22}n_2^2 - \gamma_2 n_1^2\zeta^2)^2}}. \tag{3.4.8}$$

3.4.2 Vanishing of the Reflection Coefficient $R^{'}$

From Eqs. (3.3.4,3.3.5) and Eq. (3.3.9) we get

$$\gamma_2[m_1n_2(1-R) + m_1^{'}n_2^{'}R^{'}] + (\gamma_2 - T_{22})[m_2n_1(1-R) + m_2^{'}n_1R^{'}] = 0, \tag{3.4.9}$$

$$\alpha_{12}[m_1n_1(1+R) + m_1^{'}n_1R^{'}] + \alpha_{22}[(1+R)m_2n_2 + m_2^{'}n_2^{'}R^{'}] = 0. \tag{3.4.10}$$

The above expressions are used to vanish the reflection coefficient $R^{'}$. Firstly, we note that for the angle $\theta = 0$ or $\theta = \frac{\pi}{2}$, $R^{'} = 0$. More generally it is easy to see from the boundary conditions that if $R^{'} = 0$ then either $R = 1$ (see 3.1)with

$$\alpha_{12}m_1n_1 + \alpha_{22}m_2n_2 = 0, \tag{3.4.11}$$

or $R = -1$ (see 3.2) with

$$\gamma_2 m_1n_2 + (\gamma_2 - T_{22})m_2n_1 = 0. \tag{3.4.12}$$

Generally, these conditions depend on the angle of incidence and the principal stress components. However, in the first case (of $R = 1$), for $\tau_{22} = 0$, we get an incident

SV wave for which $\alpha_{12} = \alpha_{22}$, if the material properties allow for such a wave to exist. It may be noted that the behavior $R = 1$ exists for large values and $R = -1$ for very small values of stress components. For the second case (of $R = -1$), an incident P-wave is admissible if $T_{22} = 2\gamma_2$ and if the material properties allow such a wave.

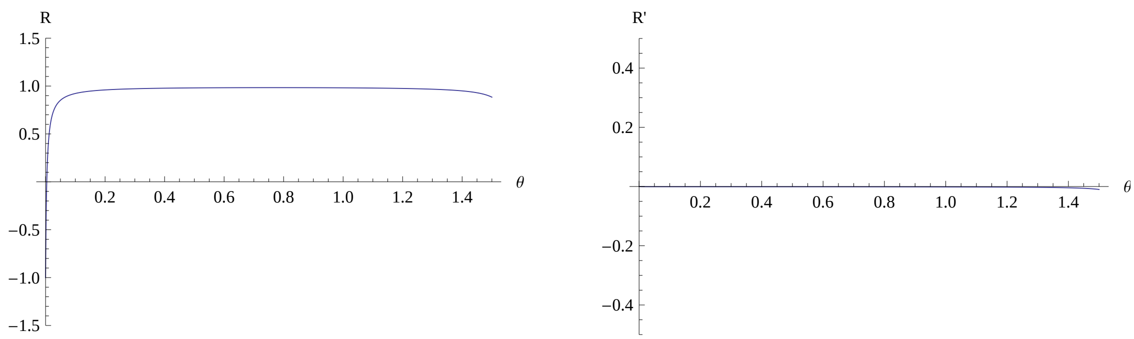

Figure 3.1: Plot of the reflection coefficients R and $R^{'}$ from Eqs. (3.3.27) and (3.3.28) for a specific material with $\lambda = 10$, $\mu = 5$, $\bar{\mu} = 100$ and $\tau_{11} = 300$.

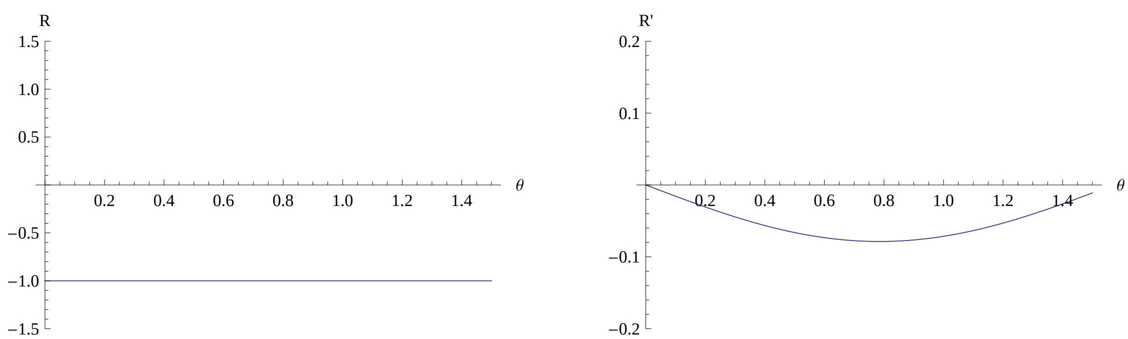

Figure 3.2: Plot of the reflection coefficients R and $R^{'}$ from Eqs. (3.3.27) and (3.3.28) for a specific material with $\lambda = 80$, $\mu = 1$, $\bar{\mu} = 1$, $\tau_{22} = 0.05$ and $\tau_{11} = 0.05$.

From the boundary conditions, after specialized with $n_1 = 1$ (grazing incidence) and either of $m_1 = 1$ or $m_2 = 1$ we see that neither P wave nor SV wave propagate parallel to the boundary and generate a wave which propagate away from the boundary. However, only a P wave may propagate parallel to the boundary. This wave satisfy the boundary conditions if the properties of the material are such like $\alpha_{12} = 0$. Similarly, only an SV wave may propagate parallel to the boundary and this wave satisfy the boundary condition if $T_{22} = \gamma_2$.

Graphical representation of Eqs. (3.3.27, 3.3.28) are given in Figs (3.3-3.13) for various specific values of the material constants. The behavior of two reflected waves is shown. An reflected SV wave accompanies a reflected P-wave in most of the cases.

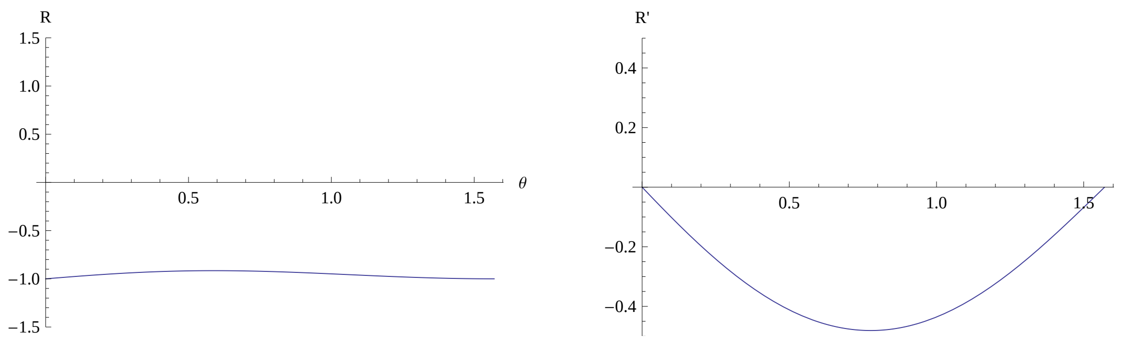

Figure 3.3: Plot of the reflection coefficients R and R' from Eqs. (3.3.27) and (3.3.28) for a specific material with $\lambda = 5$, $\mu = 5$, $\bar{\mu} = 10$ and $\tau_{11} = 2$.

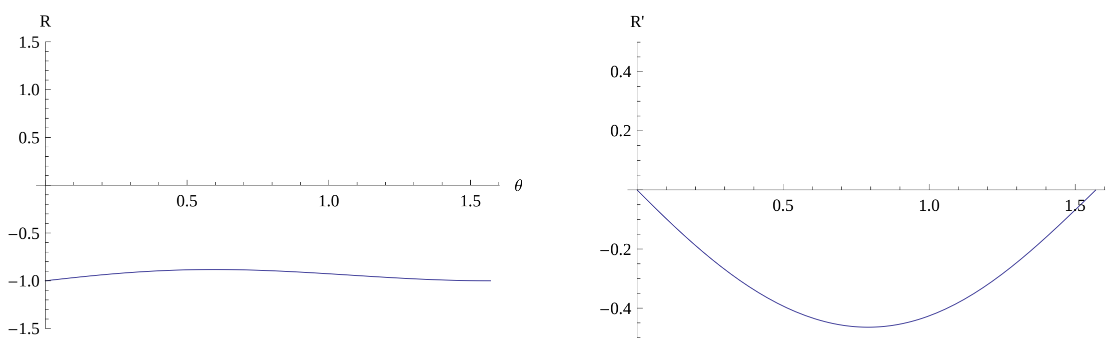

Figure 3.4: Plot of the reflection coefficients R and R' from Eqs. (3.3.27) and (3.3.28) for a specific material with $\lambda = 50$, $\mu = 50$, $\bar{\mu} = 10$ and $\tau_{11} = 2$.

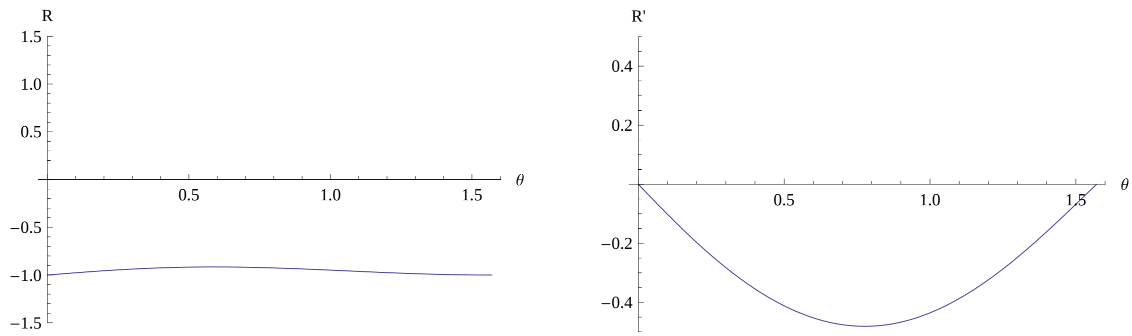

Figure 3.5: Plot of the reflection coefficients R and $R^{'}$ from Eqs. (3.3.27) and (3.3.28) for a specific material with $\lambda = 100$, $\mu = 100$, $\bar{\mu} = 10$ and $\tau_{11} = 2$.

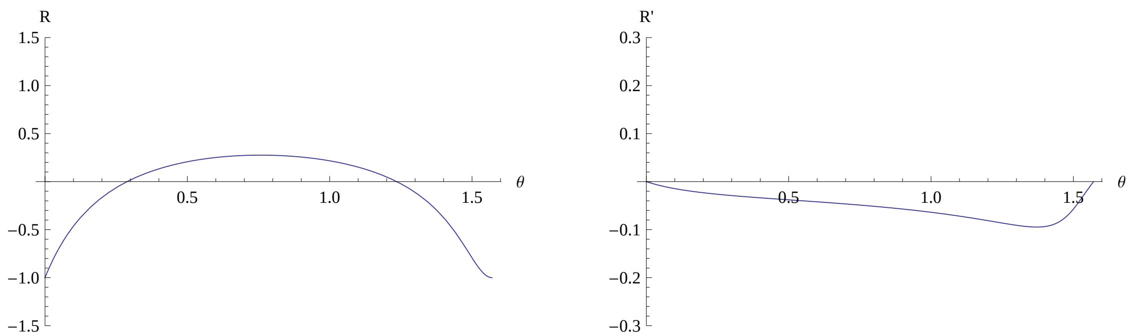

Figure 3.6: Plot of the reflection coefficients R and $R^{'}$ from Eqs. (3.3.27) and (3.3.28) for a specific material with $\lambda = 10$, $\mu = 5$, $\bar{\mu} = 5$ and $\tau_{11} = 20$.

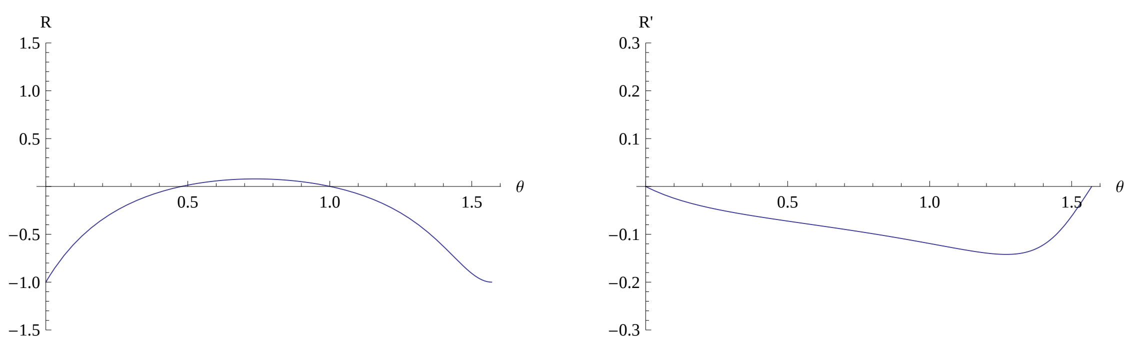

Figure 3.7: Plot of the reflection coefficients R and $R^{'}$ from Eqs. (3.3.27) and (3.3.28) for a specific material with $\lambda = 25$, $\mu = 20$, $\bar{\mu} = 5$ and $\tau_{11} = 20$.

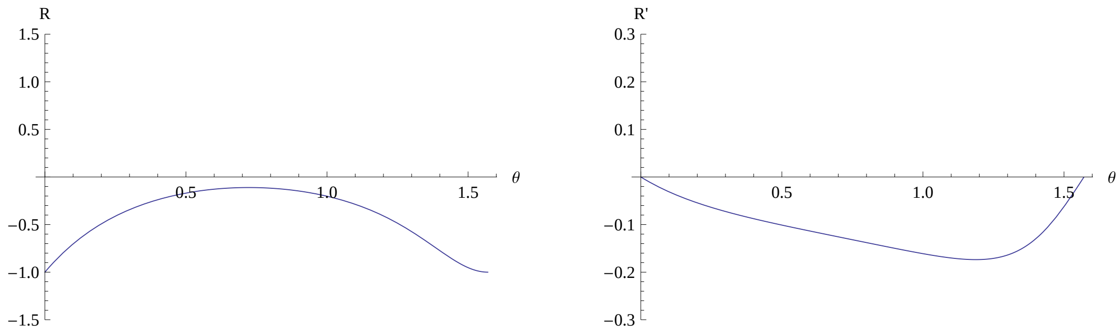

Figure 3.8: Plot of the reflection coefficients R and $R^{'}$ from Eqs. (3.3.27) and (3.3.28) for a specific material with $\lambda = 50$, $\mu = 40$, $\bar{\mu} = 5$ and $\tau_{11} = 20$.

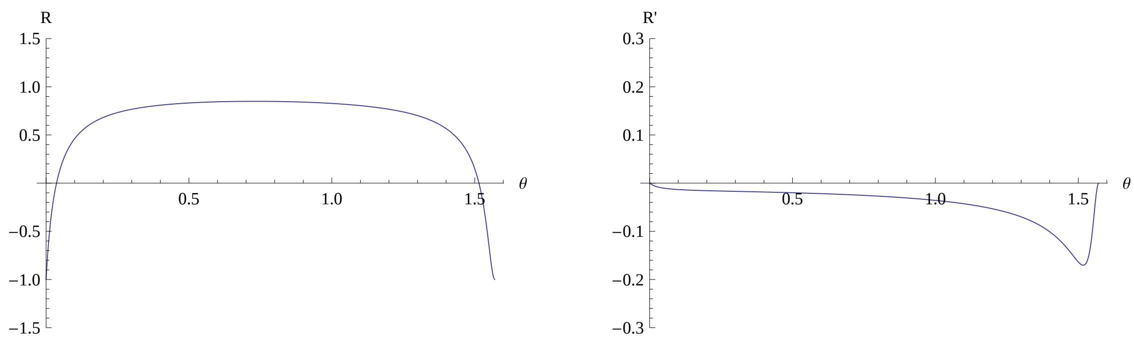

Figure 3.9: Plot of the reflection coefficients R and $R^{'}$ from Eqs. (3.3.27) and (3.3.28) for a specific material with $\lambda = 1$, $\mu = 10$, $\bar{\mu} = 2$ and $\tau_{11} = 100$.

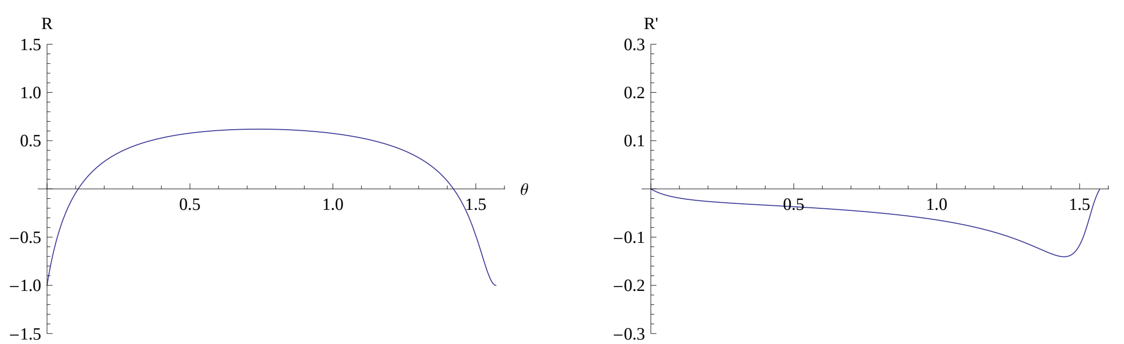

Figure 3.10: Plot of the reflection coefficients R and $R^{'}$ from Eqs. (3.3.27) and (3.3.28) for a specific material with $\lambda = 20$, $\mu = 40$, $\bar{\mu} = 2$ and $\tau_{11} = 100$.

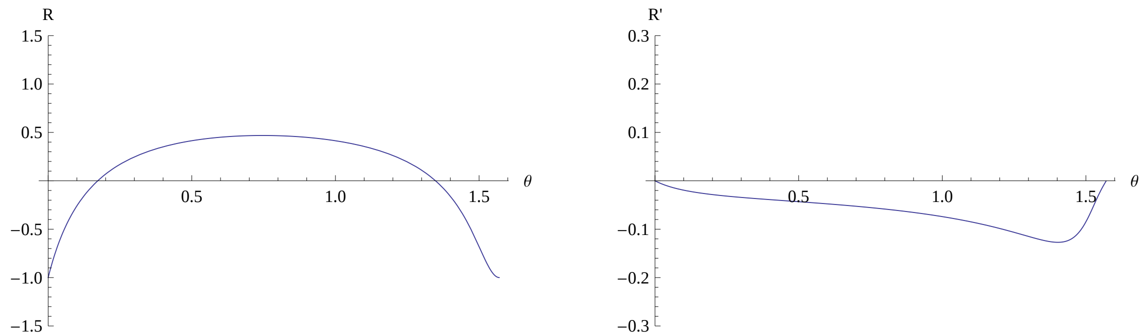

Figure 3.11: Plot of the reflection coefficients R and $R^{'}$ from Eqs. (3.3.27) and (3.3.28) for a specific material with $\lambda = 50$, $\mu = 60$, $\bar{\mu} = 2$ and $\tau_{11} = 100$.

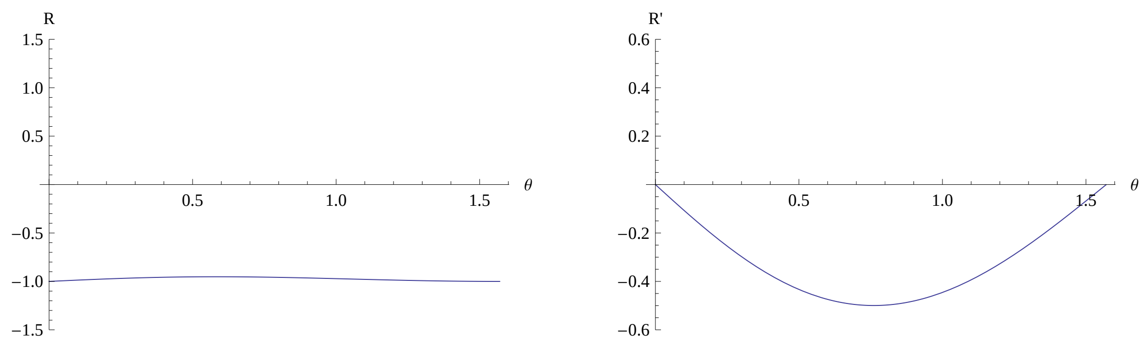

Figure 3.12: Plot of the reflection coefficients R and $R^{'}$ from Eqs. (3.3.27) and (3.3.28) for a specific material with $\lambda = 100$, $\mu = 100$, $\bar{\mu} = 10$ and $\tau_{11} = 0$.

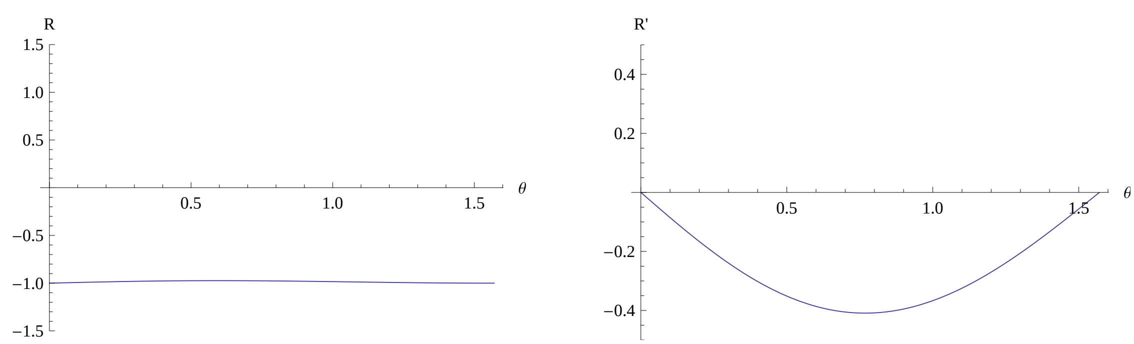

Figure 3.13: Plot of the reflection coefficients R and $R^{'}$ from Eqs. (3.3.27) and (3.3.28) for a specific material with $\lambda = 10$, $\mu = 5$, $\bar{\mu} = 5$ and $\tau_{11} = 0$.

Chapter 4

Conclusion and Future Work

We study the effect of initial stress on waves compressible hyperelastic materials. In particular, we study the problem of reflection of plane waves from the boundary of a initially stressed half-space. It is found that for an incident P wave, one reflected P wave and/or reflected SV wave may exist. We find the conditions under which a P wave is reflected as only a P wave, an SV wave is reflected as only an SV wave and the case of two reflected waves for an incident P wave.

The mathematical expressions are supported using graphs produced for various combinations of material constants and principal stress components.

In future, we may analyze incident SV waves and carry out the study in same manner. The discussion on the wave speed and acoustic tensor $\mathbf{Q}(\mathbf{n})$ may be extended for general expressions of initial stress τ depending on the vector $\mathbf{n}$.

Finally, it is intended that this work will be submitted in very near future for publication in a well reputed Mathematics/Engineering journal.

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