Stanley Depth of Quotient of Monomial Ideals

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by

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Dedicated to

My Loving parents and Uncle for their Support, Trust and Encouragement.

Abstract

In this thesis, we discuss the Stanley depth of the quotient of some monomial ideals. We discuss the Stanley depth of the quotient of irreducible monomial ideals and quotient of complete intersection monomial ideals. We also discuss about some sharp bounds for the Stanley depth and Stanley's conjecture for such quotients.

As our contribution to this area, we find a lower bound for the Stanley depth of the quotient of monomial maximal ideal and a monomial ideal. Also we find a lower bound for the Stanley depth of the quotient of maximal ideal and a class of ideals having large number of generators. Furthermore, we find bounds for the Stanley depth of the quotient of some other classes of monomial ideals.

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Nayab

Introduction

Richard. P. Stanley (1982) introduced the idea of what is now called the Stanley depth of a finitely generated $\mathbb{Z}^n$-graded module over a $\mathbb{Z}^n$ commutative ring, in paper [20]. This is geometric invariant of a module that relates to an algebraic invariant of the module called simply the depth. Stanley conjectured that $depth(M) \leq sdepth(M)$, for all finitely generated $\mathbb{Z}^n$-graded modules M, but it still remains largely open. In [9], Herzog, Vladoiu and Zheng showed that Stanley depth of monomial ideal can be computed in finite steps by partitioning a finite poset associated to the monomial ideal into intervals. The difficulty of computing Stanley depths is one of the main obstacles for verifying the Stanley's conjecture. It is still practically very difficult to find the Stanley depth for modules even for monomial ideals, if the method of Herzog is applied.

This thesis consists of four chapters. First chapter gives the detailed overview of definitions, results, and examples, related to abstract algebra and commutative algebra, that are fundamental to the development later in this project.

Second chapter review the fundamentals of the theory of Stanley depth and Stanley decomposition. Here we recall the principle results related to the depth and Stanley depth of some multigraded S-modules, where S is a polynomial ring in n variables over a field K.

In third chapter, Stanley depth of the quotient of two irreducible monomial ideals are computed and it is shown that Stanley's conjecture holds for these type of monomial ideals. Theorem 3.2.4 says that there exist some sharp bounds for Stanley depth of quotient of two complete intersection monomial ideals, and Corollary 3.2.6 says that Stanley's conjecture holds for these type of quotients.

In chapter four, we find a lower bound for the Stanley depth of the quotient of monomial maximal ideal and a monomial ideal. Also we find a lower bound for the Stanley depth of the quotient of maximal ideal and a class of ideals having large number of generators. Furthermore, we prove that there exists some good bounds for the Stanley depth of quotient of two monomial ideals when these ideals are the intersections of prime ideals generated by pairwise disjoint set of variables.

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Chapter 1

Preliminaries

This chapter consist of a brief introduction of basic concepts of abstract algebra and commutative algebra. It gives us a detailed overview of definitions, results, and examples related to them. These concepts will be helpful in the results of next chapters.

Definition 1.0.1. A ring R is a nonempty set together with two binary operations " $+$ " and " $\cdot$ " (called addition and multiplication) satisfying the following axioms:

(1) $(R, +)$ is an abelian group,

(2) R is associative with respect to " $\cdot$ ",

(3) For all $a,\ b,\ c \in R$, the left distributive law, $a \cdot (b + c) = (a \cdot b) + (a \cdot c)$ and the right distributive law $(a + b) \cdot c = (a \cdot c) + (b \cdot c)$ hold.

Definition 1.0.2. A ring R is said to be commutative, if multiplication in R is commutative that is for all $a, b \in R$, $a \cdot b = b \cdot a$.

Definition 1.0.3. A ring R is said to have an identity, if it contains identity with respect to multiplication.

Proposition 1.0.1 ([7])**.** *Let R is a ring with additive identity* 0*, then for any $x, y \in R$, we have*

(1) *$0x = x0 = 0$,*

(2) $x(-y) = (-x)y = -(xy)$,

(3) $(-x)(-y) = xy$.

Example 1.0.1. $(\mathbb{Z}, +, \cdot)$, $(\mathbb{Q}, +, \cdot)$, $(\mathbb{R}, +, \cdot)$ and $(\mathbb{C}, +, \cdot)$, are some examples of ring.

Example 1.0.2. The quotient group $\mathbb{Z}/n\mathbb{Z}$ is a commutative ring with identity (the element $\bar{1}$) under the operations of addition and multiplication of residue classes.

Example 1.0.3. Let H denote the set of real quaternion, then it has elements of the form $a+bi+cj+dk$, where $a, b, c, d \in \mathbb{R}$, where addition is defined componentwise by $(a_1+b_1i+c_1j+d_1k)+(a_2+b_2i+c_2j+d_2k) = (a_1+a_2)+(b_1+b_2)i+(c_1+c_2)j+(d_1+d_2)k$ and multiplication is defined by expanding $(a_1+b_1i+c_1j+d_1k)(a_2+b_2i+c_2j+d_2k)$, using the distributive law and simplifying, $i^2 = j^2 = k^2 = -1$, $i \cdot j = -j \cdot i = k$, $j \cdot k = -k \cdot j = i$, $k \cdot i = -i \cdot k = j$ (where the real number coefficients commute with i, j *and* k). H is a non-commutative ring with identity $1 + 0i + 0j + 0k = 1$.

Definition 1.0.4. A commutative ring R with identity $1 \neq 0$ is called a division ring (or skew field), if for every non-zero element $x \in R$, there exist $y \in R$ such that $xy = yx = 1$.

Example 1.0.4. The set of rational numbers $\mathbb{Q}$, and set of real numbers $\mathbb{R}$ are division rings.

Definition 1.0.5. Let R be a ring with unity $1 \neq 0$. An element x of R is called a unit in R, if there exist some y in R such that $xy = yx = 1$.

Theorem 1.0.2 ([7])**.** *In the ring $\mathbb{Z}/n\mathbb{Z}$, all those elements which are relatively prime to n are units in $\mathbb{Z}/n\mathbb{Z}$.*

Example 1.0.5. The units of set of integers $\mathbb{Z}$ are ± 1.

Example 1.0.6. All non-zero elements of the ring $\mathbb{Z}_5$ are units, because all non-zero elements of $\mathbb{Z}_5$ are relatively prime to 5.

Definition 1.0.6. A commutative division ring is called a field.

Remark 1.0.1 ([7])**.** If R is a commutative ring with unity $1 \neq 0$, and every non-zero element of R is a unit. Then R is a field.

Example 1.0.7. $(\mathbb{Q}, +, \cdot)$, $(\mathbb{R}, +, \cdot)$ and $(\mathbb{C}, +, \cdot)$ are fields.

Example 1.0.8. Let D be a rational number, which is not a perfect square in $\mathbb{Q}$, and define

$$F = \{a + b\sqrt{D} : a, b \in \mathbb{Q}\}.$$

Then F is a field.

Definition 1.0.7. Let R ba a ring, a non-zero element $x \in R$ is called a zero divisor, if there exist a non-zero element $y \in R$ such that either $xy = 0$ or $yx = 0$.

Theorem 1.0.3 ([7])**.** *In the ring $\mathbb{Z}/n\mathbb{Z}$, the zero divisors are precisely those non-zero elements that are not relatively prime to n.*

Corollary 1.0.4 ([7])**.** *If p is a prime. Then $\mathbb{Z}/p\mathbb{Z}$ has no zero divisors.*

Example 1.0.9. $\mathbb{Z}_3$ contain no zero divisors, because 3 is a prime number.

Definition 1.0.8. A commutative ring R with unity $1 \neq 0$ without zero divisors is called an integral domain.

Proposition 1.0.5 ([7])**.** *Assume x, y and z are elements of any ring R with x not a zero divisor. If $xy = xz$, then either $x = 0$ or $y = z$. In particular, if x, y and z are any elements in an integral domain and $xy = xz$, then either $x = 0$ or $y = z$.*

Theorem 1.0.6. *Every field F is an integral domain.*

Theorem 1.0.7 ([7])**.** *Any finite integral domain is a field.*

Corollary 1.0.8 ([7])**.** *If p is a prime, then $\mathbb{Z}_p$ is a field.*

Example 1.0.10. $\mathbb{Z}_{19}$ contain no zero divisors, because 19 is prime. So $\mathbb{Z}_{19}$ is an integral domain, also $\mathbb{Z}_{19}$ is finite. So from Corollary 3.1.1, we have $\mathbb{Z}_{19}$ is a field.

Definition 1.0.9. Let R be a ring. A nonempty subset S of R is called a subring. If

(1) S is an additive subgroup of R,

(2) S is closed under multiplication.

Example 1.0.11. (a) $\mathbb{Z}$ is a subring of $\mathbb{R}$.

(b) $n\mathbb{Z}$ is a subring of $\mathbb{Z}$ and $\{0, 2\}$ is a subring of $\mathbb{Z}_4$.

Example 1.0.12. Let D be a square free integer. Then the set

$$H = \{a + b\sqrt{D} : a, b \in \mathbb{Z}\}$$

form a subring of the field $F = \{a + b\sqrt{D} : a, b \in \mathbb{Q}\}$.

Remark 1.0.2. (1) A subring of a ring with identity can be a ring without identity.

(2) A subring of a ring without identity can be a ring with identity.

(3) A subring of a ring with identity can be a ring with same identity.

(4) A subring of a ring with identity can be ring with different identity.

Example 1.0.13. (1) $2\mathbb{Z}$ is a subring of $\mathbb{Z}$. The ring $\mathbb{Z}$ has the identity 1, but $2\mathbb{Z}$ has no identity.

(2) $\mathbb{Q}$ and $\mathbb{Z}$ both have identities which are same.

Definition 1.0.10. The polynomial ring $R[x]$ in variable x with coefficients from a ring commutative R is the set of all polynomials

$$a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0$$

with $n \geq 0$ and each $a_i \in R$. If $a_n \neq 0$, then the polynomial is of degree n, $a_n x^n$ is the leading term and a_n is the leading coefficient. $R[x]$ is a commutative ring with identity same as the identity of ring R. The polynomial ring in the variables $x_1, x_2, \ldots, x_n$ with coefficients in R is denoted by $R[x_1, x_2, \ldots, x_n]$, and is defined inductively by

$$R[x_1, x_2, \ldots, x_n] = R[x_1, x_2, \ldots, x_{n-1}][x_n].$$

This definition means that we can consider polynomials in n variables with coefficients in R simply as polynomials in one variable x_n, but now with coefficients that are themselves polynomials in $n-1$ variables.

Proposition 1.0.9 ([7]). *Let R be an integral domain and let $p(x)$ and $q(x)$ be a nonzero elements of $R[x]$. Then*

(1) *degree $p(x)q(x) =$ degree $p(x) +$ degree $q(x)$.*

(2) *The units of $R[x]$ are just the units of R.*

(3) *$R[x]$ is an integral domain.*

Remark 1.0.3. If S is a subring of ring R, then $S[x]$ is a subring of $R[x]$.

Definition 1.0.11. Let R_1 and R_2 be rings. A ring homomorphism is a map $\varphi : R_1 \to R_2$ satisfying:

(a) $\varphi(x+y) = \varphi(x) + \varphi(y)$, for all $x, y \in R_1$.

(b) $\varphi(xy) = \varphi(x)\varphi(y)$, for all $x, y \in R_1$.

The kernel of the ring homomorphism φ, denoted by $ker(\varphi)$ is the set

$$ker(\varphi) = \{r \in R_1 : \varphi(r) = 0_{R_2}\}.$$

A bijective ring homomorphism is called an isomorphism.

Example 1.0.14. The map $\psi : \mathbb{Z} \to \mathbb{Z}/2\mathbb{Z}$ defined by

$$\psi(n) = \begin{cases} 0, \text{if } n \text{ is even;} \\ 1, \text{if } n \text{ is odd.} \end{cases}$$

is a ring homomorphism.

Example 1.0.15. For $n \in \mathbb{Z}$, the map $\psi_n : \mathbb{Z} \to \mathbb{Z}$, defined by $\psi_n(x) = nx$ is not a ring homomorphism, if n is not 0 and 1.

Proposition 1.0.10. *Let R_1 and R_2 be rings and let $\psi : R_1 \to R_2$ be a ring homomorphism.*

(1) *The $Im(\psi)$ is a subring of S.*

(2) *The $ker(\psi)$ is a subring of R. Furthermore, if $x \in ker(\psi)$, then rx and $xr \in ker(\psi)$ for every $r \in R_1$, that is $ker(\psi)$ is closed under multiplication by elements from R_1.*

Definition 1.0.12. Let I be a subset of the ring R, and consider the following three properties.

(1) I is an additive subgroup of R.

(2) If $x \in I$ and $r \in R$, then $rx \in I$, In other words $rI \subset I$ for every $r \in R$.

(3) If $x \in I$ and $r \in R$, then $xr \in I$, In other words $Ir \subset I$ for every $r \in R$.

If (1) and (2) holds, I is said to be a left ideal of R. If (1) and (3) holds, I is said to be a right ideal of R. If all three properties hold, I is said to be an ideal (or two sided ideal) of R.

Remark 1.0.4. (1) Every ideal is a subring.

(2) A subring in general is not an ideal. For instance $\mathbb{Z}$ is a subring of $\mathbb{R}$, but not an ideal of $\mathbb{R}$. Because $2 \in \mathbb{Z}$ and $(3/5) \in \mathbb{R}$, but $(2)(3/5) = (6/5) \notin \mathbb{Z}$.

Proposition 1.0.11 ([7]). *Let R be a ring. A non-empty subset I of ring R is an ideal if and only if for all $x, y \in I$ and $r \in R$, $x - y \in I$, $xr \in I$ and $rx \in I$.*

Remark 1.0.5. Let $\psi : R \to S$ be a ring homomorphism, then $ker(\psi)$ is an ideal of R.

Proposition 1.0.12. *Every proper ideal is the kernel of a ring homomorphism.*

Proposition 1.0.13. *Let I be an ideal of R. Then*

(1) *$I = R$ iff I contains a unit.*

(2) *Assume R is commutative. Then R is a field if and only if its only ideals are 0 and R.*

Definition 1.0.13. Let I be a proper ideal of the ring R, since I is a subgroup of the additive group R, so clearly I is a normal subgroup of R, and we can form the quotient ring R/I, consisting of cosets $r+I, r \in R$, we define multiplication of cosets in the natural way

$$(r_1+I)(r_2+I) = r_1r_2+I.$$

Example 1.0.16. $n\mathbb{Z}$ is an ideal of $\mathbb{Z}$ for any $n \in \mathbb{Z}$, and clearly $n\mathbb{Z}$ is a normal subgroup of $\mathbb{Z}$, because it is an additive subgroup of additve group $\mathbb{Z}$. The associated quotient ring is $\mathbb{Z}/n\mathbb{Z}$.

Theorem 1.0.14 ([7]). ***(First Isomorphism Theorem for Rings)*** *If $f : R \to S$ is a ring homomorphism, then $ker f$ is an ideal of R and image of f is a subring of S, and $Im(f)$ is isomorphic to $R/ker f$, that is*

$$Im(f) \cong R/ker f.$$

Theorem 1.0.15 ([7]). ***(Second Isomorphism Theorem for Rings)*** *Let I be an ideal of the ring R and let S be a subring of R. Then $S+I = \{x+y \mid x \in S, y \in I\}$ is a subring of R, $S \cap I$ is an ideal of S and*

$$S/(S \cap I) \cong (S+I)/I.$$

Theorem 1.0.16 ([7]). ***(Third Isomorphism Theorem for Rings)*** *Let I and J be ideals of R with $J \subseteq I$. Then I/J is an ideal of R/J, and*

$$(R/J)/(I/J) \cong R/I.$$

Theorem 1.0.17. ***(Correspondence Theorem for Rings)*** *Let I be an ideal of the ring R, then the canonical map $\phi : R \to R/I$ set up a one-to-one correspondence between*

(1) *The set of all subrings of R containing I and the set of all subrings of R/I.*

(2) *The set of all ideals of R containing I and the set of all ideals of R/I.*

Definition 1.0.14. Let I and J be ideals of R. Then

(1) Denote the sum of I and J by $(I, J) = I + J$ and define as $(I, J) = I + J = \{x + y \mid x \in I, y \in J\}$.

(2) Denote the product of I and J by IJ and define to be the set of all finite sums of elements of the form xy with $x \in I$ and $y \in J$.

Example 1.0.17. (1) Let $I = 6\mathbb{Z}$ and $J = 8\mathbb{Z}$, then $I + J$ consists elements of the form $6x + 8y$ with $x, y \in \mathbb{Z}$. Since every such integer is divisible by 2, so the ideal $I + J$ is contained in $2\mathbb{Z}$, that is $I + J \subseteq 2\mathbb{Z}$. On the other hand $2 = 6(3) + 8(-2)$, shows that the ideal $I + J$ contains $2\mathbb{Z}$, that is $2\mathbb{Z} \subseteq I + J$. Therefore we have $6\mathbb{Z} + 8\mathbb{Z} = 2\mathbb{Z}$. In general $m\mathbb{Z} + n\mathbb{Z} = d\mathbb{Z}$, where d is the greatest common divisor of m and n.

(2) The product IJ contains of all finite sum of elements of the form $(6x)(8y)$ with $x, y \in \mathbb{Z}$ and $IJ = 48\mathbb{Z}$.

Definition 1.0.15. Let I be an ideal in R, I is said to be a principal ideal, if it is generated by a single element and it is written as $I = (x)$, where $x \in R$.

Example 1.0.18. Let $R = \mathbb{Z}$, then $n\mathbb{Z}$, where $n \in \mathbb{Z}$ is an ideal of $\mathbb{Z}$, $n\mathbb{Z}$ is a principal ideal and is generated by n, and is denoted by (n).

Definition 1.0.16. Let R is a ring with unity and A be any subset of R. (A) is said to be the ideal generated by A, if it is the smallest ideal of R containing A. If A is a finite set, then (A) is called a finitely generated ideal.

Definition 1.0.17. An ideal generated by a finite set is called a finitely generated ideal.

Definition 1.0.18. Let R be a ring. An ideal M in R is called maximal ideal. If $M \neq R$ and the only ideals containing M are M and R.

Proposition 1.0.18 ([7]). *In a ring with identity, every proper ideal is contained in a maximal ideal.*

Proposition 1.0.19. *Assume R is commutative. The ideal M is a maximal ideal if and only if the quotient ring R/M is a field.*

Example 1.0.19. Let n be an non-negative integer, The ideal $n\mathbb{Z}$ of $\mathbb{Z}$ is a maximal ideal if and only if $\mathbb{Z}/n\mathbb{Z}$ is a field, and $\mathbb{Z}/n\mathbb{Z}$ is a field if and only if n is a prime. Thus if n is prime, then $n\mathbb{Z}$ is a maximal ideal of $\mathbb{Z}$.

Definition 1.0.19. Let R be a commutative ring. If $I \neq R$ is an ideal in R. Then I is called a prime ideal. If $ab \in I$, where $a, b \in R$, then either $a \in I$ or $b \in I$.

Proposition 1.0.20 ([7])**.** *Let R be a commutative ring. Then the ideal I is a prime ideal in R if and only if the quotient ring R/I is an integral domain.*

Corollary 1.0.21 ([7])**.** *In commutative ring, a maximal ideal is a prime ideal.*

Example 1.0.20. (1) Since $\mathbb{Z}[x]/(x) \cong \mathbb{Z}$, and $\mathbb{Z}$ is an integral domain, so $\mathbb{Z}[x]/(x)$ is an integral domain and from Proposition (3.1.2), we have (x) is a prime ideal in $\mathbb{Z}[x]$.

(2) The principle ideals generated by prime in $\mathbb{Z}$ are both prime and maximal ideals.

Definition 1.0.20. Let R be a commutative ring. A proper ideal P of R is called primary ideal for $a, b \in R$, if whenever ab is an element of P, then either $a \in P$ or $b^n \in P$ for some integer $n > 0$.

Definition 1.0.21. Let I be an ideal in a commutative ring R. Then the radical of I, is denoted by rad$I$, is the collection of elements in R some power of which lie in I, that is

$$rad(I) = \{r \in R \mid r^k \in I, \text{ for some } k \geq 1\}.$$

Remark 1.0.6. (1) Every prime ideal is primary.

(2) Let P is an ideal, whose radical is a maximal ideal. Then P is a primary ideal.

Definition 1.0.22. A partially ordered set Q is said to satisfy the ascending chain condition (ACC), if every strictly ascending sequence of elements eventually terminates. Equivalently, given any sequence

$$q_1 \leq q_2 \leq q_3 \leq \dots,$$

there exists a positive integer n such that

$$q_n = q_{n+1} = q_{n+2} = \dots$$

Definition 1.0.23. A partially ordered set Q is said to satisfy the descending chain condition (DCC), if every strictly descending sequence of elements eventually terminates, that is there is no infinite descending chain. Equivalently, every descending sequence

$$q_1 \geq q_2 \geq q_3 \geq \dots,$$

of elements of Q eventually terminates.

Definition 1.0.24. A commutative ring R is said to be Noetherian ring, if it satisfy the ascending chain condition on ideals with respect to inclusion, that is whenever

$$I_0 \subseteq I_1 \subseteq I_2 \subseteq \dots,$$

is an increasing chain of ideals of R, then there is a positive integer r such that $I_k = I_r$ for all $k \geq r$.

Proposition 1.0.22 ([7]). (1) *Let R is a Noetherian ring and I is an ideal of R, then the quotient R/I is a Noetherian ring.*

(2) *Any homomorphic image of a Noetherian ring is Noetherian.*

Theorem 1.0.23 ([7]). *Let R be a commutative ring, then the following are equivalent:*

(1) *R is a Noetherian ring.*

(2) *Every nonempty set of ideals of R contains a maximal element under inclusion.*

(3) *Every ideal of R is finitely generated.*

Example 1.0.21. (1) Every Principle ideal domain is Noetherian, because every ideal of Principle ideal domain is finitely generated. In particular $\mathbb{Z}$ is Noetherian.

(2) The ring $\mathbb{Z}[x_1, x_2, x_3, \ldots]$ is not Noetherian. Since the ideal $(x_1, x_2, x_3, \ldots)$ cannot be generated by any finite set.

Remark 1.0.7. A Noetherian ring may have arbitrarily long ascending chains of ideals and may have infinitely long descending chains of ideals. For example, $\mathbb{Z}$ has the infinite descending chain

$$(2) \supset (4) \supset (8) \supset \ldots$$

that is, a Noetherian ring need not satisfy the descending chain condition on ideals.

Theorem 1.0.24 (Hilbert's Basis Theorem)**.** *If R is a Noetherian ring, then $R[x]$ is also a Noetherian ring.*

Corollary 1.0.25. *If $R[x_1]$ is a Noetherian ring, then $R[x_1][x_2] = R[x_1, x_2]$ is also a Noetherian ring, and by induction, we can say that $R[x_1, x_2, \ldots, x_n]$ is also Noetherian ring.*

Definition 1.0.25. A ring that has a unique maximal ideal is called a Local ring.

Definition 1.0.26. Let R be a commutative ring. A non empty set M is said to be an R-module (or a module over R), if it satisfies the following conditions:

(1) M is an abelian group under a binary operation of addition.

(2) An action of R on M (that is, a map $R \times M \to M$) denoted by rm, such that for all $r_1, r_2 \in R$ and $m, n \in M$.

  (a) $(r_1 + r_2)m = r_1m + r_2m$.

  (b) $(r_1r_2)m = r_1(r_2m)$.

  (c) $r_1(m + n) = r_1m + r_1n$.

  (d) $1m = m$.

If R is a field F, then the axioms of an R-module coincide with the axioms of a vector space over F.

Example 1.0.22. (1) Let $R = \mathbb{Z}$, let G be any abelian group (finite or infinite) and write operation of G as "+", for any $n \in \mathbb{Z}$ and $g \in G$ define

$$ng = \begin{cases} g + g + \ldots + g \ (n \ times) & \text{if } n > 0; \\ 0 & \text{if } n = 0; \\ -g - g - \ldots - g \ (-n \ times) & \text{if } n < 0. \end{cases}$$

Here 0 is the additive identity of group G. This definition of an action of the integers on G makes G into a $\mathbb{Z}$-module. Thus every abelian group is a $\mathbb{Z}$-module.

(2) If R is any ring and n a natural number. Then the cartesian product R^n is module over R, if we use component wise operations.

Definition 1.0.27. Let R be a ring. An additive subgroup N of the R-module M is called a submodule of M, if whenever $r \in R$ and $n \in N$, then $rn \in N$.

Remark 1.0.8. Submodules of M are subsets of M, which are themselves modules under the restricted operations.

Proposition 1.0.26 ([7])**.** *Let R be a commutative ring and let M be an R-module. A subset M_1 of M is a submodule of M if and only if*

(1) $M_1 \neq \phi$, *and*

(2) $x - y \in M_1$ *and* $xr \in M_1$, *for all* $x, y \in M_1$ *and* $r \in R$.

Definition 1.0.28. Let R be a ring and let M and N be R-modules. A map $\phi : M \to N$ is an R-module homomorphism, if it satisfies the following conditions:

(1) $\phi(a + b) = \phi(a) + \phi(b)$, for all $a, b \in M$, and

(2) $\phi(ra) = r\phi(a)$, for all $r \in R$, $a \in M$.

Definition 1.0.29. An R-module homomorphism is an isomorphism (of R-modules), if it is both injective and surjective. If $\phi : M \to N$ is an R-module isomorphism, then the modules M and N are said to be isomorphic and is denoted by $M \cong N$.

Definition 1.0.30. If $\phi : M \to N$ is an R-module homomorphism, then kernel of ϕ is the set

$$ker(\phi) = \{m \in M : \phi(m) = 0\}.$$

Image of ϕ is the set

$$\phi(M) = \{n \in N : n = \phi(m),\ for\ some\ m \in M\}.$$

Example 1.0.23. If R is a ring and $M = R$ is a module over itself, then R-module homomorphism (from R to itself) need not be ring homomorphisms and ring homomorphisms need not be R-module homomorphisms. For example when $R = \mathbb{Z}$. The $\mathbb{Z}$-module homomorphism $\phi : r \mapsto 2r$ is not a ring homomorphism, because $\phi(mn) = 2mn \neq \phi(m)\phi(n) = 4mn$. When $R = F[x]$, the ring homomorphism $\psi : f(x) \mapsto f(x^2)$ is not an $F[x]$-module homomorphism, because $x^2 = \psi(x) = \psi(x.1) = x\psi(1) = x$.

Theorem 1.0.27 ([7])**.** ***(First Isomorphism Theorem For Modules)*** *Let M_1, M_2 be R-modules and let $\varphi : M_1 \to M_2$ be an R-module homomorphism. Then $ker\varphi$ is a submodule of M_1 and*

$$M_1/ker\varphi \cong \varphi(M_1).$$

Theorem 1.0.28 ([7])**.** ***(Second Isomorphism Theorem for Modules)*** *Let M_1, M_2 be submodules of the R-module M. Then*

$$(M_1 + M_2))/M_2 \cong M_1/(M_1 \cap M_2).$$

Theorem 1.0.29 ([7])**.** ***(Third Isomorphism Theorem for Modules)*** *Let M be an R-module, and let M_1, M_2 be submodules of M with $M_1 \subseteq M_2$. Then*

$$(M/M_1)/(M_2/M_1) \cong M/M_2.$$

Definition 1.0.31. If M is an R-module and if $M_1, M_2, \ldots, M_s$ are submodules of M, then M is said to be the direct sum of $M_1, \ldots, M_s$, if every element $m \in M$ can be written in a unique manner as $m = m_1 + m_2 + \ldots + m_s$, where $m_1 \in M_1, m_2 \in M_2, \ldots, m_s \in M_s$.

Definition 1.0.32. An R-module M is said to be cyclic, if there is an element $m_0 \in M$ such that every $m \in M$ is of the form $m = rm_0$, where $r \in R$.

Definition 1.0.33. An R-module M is said to be finitely generated, if there exist elements $m_1, m_2, \ldots, m_s \in M$ such that every m in M is of the form $m = r_1m_1 + r_2m_2 + \ldots + r_sm_s$, where $r_1, r_2, \ldots, r_s \in R$. The set $\{m_1, m_2, \ldots, m_s\}$ is referred as a generating set for M.

Definition 1.0.34. An R-module M is said to be Noetherian R-module or to satisfy the ascending chain condition on its submodules with respect to inclusion, if every increasing chain of submodules stops that

$$M_0 \subseteq M_1 \subseteq M_2 \subseteq \ldots,$$

is an increasing chain of submodules of M, then there is a positive integer r such that $M_k = M_r$ for all $k \geq r$.

Example 1.0.24. (1) The integers, considered as a module over the ring of integers is a Noetherian module.

(2) Every field over itself is a Noetherian module.

Proposition 1.0.30 ([21])**.** *If M is a finitely generated R-module over a Noetherian ring R, then M is a Noetherian module.*

Proposition 1.0.31 ([21])**.** *If R is a Noetherian ring and I is an ideal of R, then R/I and R^n are Noetherian R-modules. In particular any sub-modules of R^n is finitely generated.*

Definition 1.0.35. Let M be an R-module. The annihilator of M is given by

$$Ann_R(M) = \{x \in R : xM = 0\}.$$

Example 1.0.25. Let $M = \mathbb{Z}/5\mathbb{Z}$. Then $Ann(\mathbb{Z}/5\mathbb{Z}) = 5\mathbb{Z} = 0$.

Definition 1.0.36. Let R be a commutative ring and M be an R-module. Then an element $r \neq 0$ of R is called a zero divisor in the module M if there is an $m \in M$ such that $rm = mr = 0$.

Definition 1.0.37. Let R be a commutative ring and M be an R-module. We say that a non-zero element $r \in R$ is an M-regular element, if for any $m \in M$, $rm = 0$ implies $m = 0$. In other words r is not a zero divisor on M.

Definition 1.0.38. A sequence $a = (a_1, a_2, \ldots, a_n)$ of elements of a ring R is called an M-regular sequence or simply M-sequence, if it satisfy the following axioms:

(1) a_i is an $M/(a_1, a_2, \ldots, a_{i-1})M$-regular element for $i = 1, 2, \ldots, n$.

(2) $M/aM \neq 0$.

Example 1.0.26. If $S = K[x_1, x_2, \ldots, x_n]$ be a module over itself, then $x_1, x_2, \ldots, x_n$ is a regular sequence on S.

Definition 1.0.39. Let (R, m) be a Noetherian local ring and M a finite R-module. Then the common length of all maximal M- regular sequence in m is called the depth of M, written as $depth(M)$.

Proposition 1.0.32. *An R-module M has a depth zero, if and only if every element in m is a zero divisor on M.*

1.1 Monomial ideals

Let K be a field and let $S = K[x_1, x_2, \ldots, x_n]$ be the polynomial ring in n indeterminates x_i over K. Let $\mathbb{R}^n_+$ denote the set of those vectors $b = (b_1, b_2, \ldots, b_n) \in \mathbb{Z}^n$ with each $b_i \geq 0$ and $\mathbb{Z}^n_+ = \mathbb{R}^n_+ \cap \mathbb{Z}^n$. Any product $x_1^{b_1} \ldots x_n^{b_n}$ with $b_i \in \mathbb{Z}_+ \cup \{0\}$ is called a monomial. A monomial $w = x_1^{b_1} \ldots x_n^{b_n}$ can be written in the form $w = x^b$, with $b = (b_1, b_2, \ldots, b_n) \in \mathbb{Z}^n_+$. Let W denote the set of all monomials of S, then W form a K-basis of S. In other words, any polynomial $f \in S$ can be written as a linear combination of monomials with coefficients from field K in a unique way. Write

$$g = \sum_{w \in W} b_w w \quad \textit{with } b_w \in K.$$

Then we call the set

$$supp(f) = \{w \in W : b_w \neq 0\},$$

the support of f. And the set

$$supp(w) = \{x_j : x_j|w\},$$

is called the support of monomial w.

Definition 1.1.1. An ideal $I \subset S$ is called a monomial ideal, if it is generated by monomials.

Theorem 1.1.1. *The set W of monomials belonging to $I \subset S$ is a K-basis of I.*

Proposition 1.1.2. *Let $v_1, \ldots, v_m$ be a monomial system of generators of the monomial ideal $I \subset S$. Then the monomial u belongs to I if and only if there exists a monomial w such that $u = wvi$ for some i.*

Proposition 1.1.3. *Every monomial ideal $I \subset S$ has a unique minimal monomial set of generators. Usually $G(I)$ denote the unique minimal set of monomial generators of the monomial ideal I. Where by $|G(I)|$, we mean the minimum number of monomials in I.*

Example 1.1.1. Let $S = K[x_1, x_2, x_3, x_4]$. Then $I = (x_1^2x_2^2, x_2^2x_3^3, x_1^2x_4^2)$ is the monomial ideal. Here $G(I) = \{x_1^2x_2^2, x_2^2x_3^3, x_1^2x_4^2\}$.

Definition 1.1.2. A monomial $x_1^{\alpha_1}x_2^{\alpha_2}\ldots x_m^{\alpha_m}$ is called squarefree. If $\alpha_i's$ are 0 or 1.

Definition 1.1.3. A monomial ideal $I \subset S$ is called a squarefree monomial ideal, if I is generated by squarefree monomials.

Example 1.1.2. Let $S = K[x_1, x_2, x_3, x_4]$. Then $I = (x_1x_2x_3, x_1x_3x_4, x_2x_4)$ is a square free monomial ideal.

Definition 1.1.4. The radical of an ideal $I \subset S$ in a commutative ring R is denoted by $Rad(I)$ or $\sqrt{I}$ and is defined as

$$\sqrt{I} = \{r \in R : r^n \in I,\ for\ some\ integer\ n > 0\}.$$

Proposition 1.1.4 ([8])**.** *Let J be a monomial ideal. Then $\{\sqrt{v} : v \in G(J)\}$ is a set of generators of $\sqrt{J}$.*

Definition 1.1.5. An ideal J is called radical ideal, if $J = rad(J)$.

Corollary 1.1.5 ([8])**.** *A monomial ideal J is a radical ideal, that is $J = \sqrt{J}$, if and only if J is a squarefree monomial ideal.*

Example 1.1.3. (1) The radical of the ideal $4\mathbb{Z}$ is $2\mathbb{Z}$.

(2) In general the radical of $n\mathbb{Z}$ is $r\mathbb{Z}$, where r is the product of all prime factors of n.

(3) Let $S = K[x_1, x_2, x_3, x_4, x_5]$. The radical of ideal $I = (x_1^2x_2^2, x_3^2x_4^2, x_5^2)$ is $\sqrt{I} = (x_1x_2, x_3x_4, x_5)$.

Definition 1.1.6. A monomial ideal I is called p-primary ideal, if its radical is equal to p.

Definition 1.1.7. A monomial ideal I is called irreducible, If it cannot be written non-trivially as a proper intersection of two other monomial ideals, that is if

$$I = P \cap Q, \quad then \quad I = P \quad or \quad I = Q.$$

Corollary 1.1.6. *Let $I \subset S = K[x_1, \ldots, x_n]$ be a monomial ideal. Then I is irreducible, if and only if it is generated by pure power of the variables.*

Theorem 1.1.7. *In a Noetherian ring, every irreducible ideal is primary.*

Proposition 1.1.8. *The irreducible ideal $(x_{j1}^{b_1}, \ldots, x_{jk}^{b_k})$ is $(x_{j1}, \ldots, x_{jk})$ -primary.*

Remark 1.1.1. Converse of the above proposition is not true. For instance Let

$$I = (x_1^2, x_2^2, x_1x_2).$$

Then I is (x_1, x_2)- primary. And

$$I = (x_1, x_2^2) \cap (x_1^2, x_2),$$

where

$$(x_1^2, x_2^2, x_1x_2) \subset (x_1, x_2^2),$$

and

$$(x_1^2, x_2^2, x_1x_2) \subset (x_1^2, x_2).$$

So I is not irreducible.

Proposition 1.1.9 ([8]). *Let I and J be monomial ideals. Then $I \cap J$ is a monomial ideal, and its set of generators is defined as*

$$\{lcm(u, v) : u \in G(I), v \in G(J)\}.$$

Definition 1.1.8. Let $I, J \subset S$ be two monomial ideals. The set

$$I : J = \{f \in S | \; fh \in I \; for \; all \; h \in J\},$$

is an ideal and is called the colon ideal of I with respect to J.

Proposition 1.1.10. *Let I and J be monomial ideals. Then $I : J$ is a monomial ideal, and*

$$I : J = \bigcap_{w \in G(J)} I : (w).$$

Moreover $\{v/gcd(v, w) \mid v \in G(I)\}$ is a set of generators of $I : (w)$.

Definition 1.1.9. A monomial prime ideal $I \subset S$ is an ideal, which is generated by variables not of its powers.

Corollary 1.1.11. *A squarefree monomial ideal is an intersection of monomial prime ideals.*

Example 1.1.4. Let $S = K[x_1, x_2, x_3]$ and $I = (x_1, x_3)$. Then I is a monomial prime ideal, because it is generated by variables not of its powers.

Definition 1.1.10. Let R be a ring and $I \subset R$ is an ideal. A prime ideal P is called a minimal prime ideal I, if $I \subset R$ and there is no prime ideal containing I, which is properly contained in P. We denote the set of minimal prime ideals of I by $Min(I)$.

Definition 1.1.11. A presentation of an ideal J as an intersection $J = \bigcap_{i=1}^{m} P_i$ of ideals is called irredundant, if none of the ideals P_i can be omitted in this presentation.

Theorem 1.1.12. *Let $J \subset S = K[x_1, \ldots, x_n]$ be a monomial ideal. Then $J = \bigcap_{i=1}^{m} P_i$, where each P_i is generated by pure powers of the variables. Moreover, an irredundant presentation of this form is unique.*

Lemma 1.1.13 ([8]). *Suppose J has irredundant presentation $J = P_1 \cap P_2 \cap \ldots \cap P_m$ as an intersection of prime ideals. Then*

$$Min(J) = \{P_1, P_2, \ldots, P_m\}.$$

Corollary 1.1.14 ([8]). *Let $J \subset S$ be a squarefree monomial ideal. Then*

$$J = \bigcap_{p \in Min(J)} p.$$

And each $p \in Min(J)$ is a monomial prime ideal.

Definition 1.1.12. Let $G(I) = \{u_1, u_2, \ldots, u_m\}$. Then I is called a complete intersection ideal if and only if

$$supp(u_i) \cap supp(u_j) = \emptyset \quad for\ all\ i \neq j.$$

Example 1.1.5. Let $S = K[x_1, x_2, \ldots, x_7]$ and $I = (x_1^2x_2^2, x_3x_5^2, x_4^2, x_6^2x_7^2)$. Then I is a complete intersection monomial ideal.

Definition 1.1.13. A presentation of an ideal I as intersection $I = \bigcap_{i=1}^{s} P_i$, where each P_i is a primary ideal is called a primary decomposition of I.

Theorem 1.1.15. *In Noetherian ring, every ideal can be written as an intersection of finite number of primary ideals.*

Definition 1.1.14. Let M be an R-module, then a prime ideal P is called an associated prime ideal of M, if there is an injective morphism of R-modules:

$$\varphi : R/P \hookrightarrow M.$$

The set of associated primes of M is denoted by $Ass_R(M)$.

Definition 1.1.15. Let R be a Noetherian ring and M a finitely generated R-module. A prime ideal $P \subset R$ is called an associated prime ideal of M, if there exists a non-zero element $m \in M$ such that $P = Ann(m)$.

Corollary 1.1.16. *The associated prime ideals of a monomial ideal are monomial prime ideals.*

Example 1.1.6. Let $I = (x_1^3, x_2^3, x_1^2x_3^2, x_1x_2x_3^2, x_2^2x_3^2)$. Then

$$I = (x_1^3, x_2^3, x_3^2) \cap (x_1^2, x_2) \cap (x_1, x_2^2),$$

is irredundant presentation as intersection of irreducible ideals, and $Ass(I) = \{(x_1, x_2, x_3), (x_1, x_2)\}$.

Corollary 1.1.17. *Let $I \subset S$ be a monomial ideal, and let $q \in Ass(I)$. Then there exists a monomial u such that $q = I : u$.*

Definition 1.1.16. Let $(G, +)$ is an abelian semi-group. Then a (commutative) ring R is called a graded ring or G-graded. If there is a family of subgroups of R, $\{R_g\}_{g \in G}$, such that:

$$R = \bigoplus_{g \in G} R_g = R_0 \oplus R_1 \oplus R_2 \oplus R_3 \oplus \dots \quad (as\ a\ group),$$

such that $R_gR_h \subset R_{g+h}$, for all $g, h \in G$.

Definition 1.1.17. An element of any factor R_k of the decomposition are called homogenous element of degree k and R_k is called the homogenous component of R of degree k.

Definition 1.1.18. Let R is G-graded ring. Then an R-module M is called G-graded module, if M is the direct sum of additive subgroups:

$$M = \bigoplus_{g \in G} M_g = M_0 \oplus M_1 \oplus M_2 \oplus M_3 \oplus \dots,$$

such that $R_gM_h \subset M_{g+h}$, for all $g, h \in G$.

Example 1.1.7. (1) Every ring R itself is a graded ring by letting $R_0 = R$ and $R_n = 0$ for all $n \neq 0$.

(2) Let $G = \mathbb{Z}$ and $S = K[x] = \bigoplus_{n \in \mathbb{Z}} Kx^n$, where $Kx^n = 0$ if $n < 0$. Then S is $\mathbb{Z}$-graded, because

$$S = S_0 \oplus S_1 \oplus S_2 \oplus S_3 \oplus \dots,$$

where $S_0 = K$, $S_1 = \{kx : k \in K\}$, $S_2 = \{kx^2 : k \in K\}$, ...

Example 1.1.8. Let R be a ring and $x_1, \dots, x_n$ is the variables of R. For $m = (m_1, \dots, m_d) \in \mathbf{N}^d$, let $x^m = x_1^{m_1} \dots x_n^{m_d}$. Then the polynomial ring $S = R[x_1, x_2, \dots, x_n]$ is a graded ring, where

$$S_n = \left\{ \sum_{m \in \mathbf{N}^d} r_m x^m \mid r_m \in R \textit{ and } m_1 + ... + m_d = n \right\}.$$

This is called the standard grading on the polynomial ring $S = R[x_1, \dots, x_n]$, where $S_0 = R$.

Example 1.1.9. Every graded ring is a graded module over itself.

Proposition 1.1.18. *Let R be a graded ring, M a graded R-module and N a submodule of M. Then M/N is a graded R-module.*

Example 1.1.10. Let $u \in \mathbb{Z}^n$, then $f \in S$ is said to be homogenous of degree u. If it is in the form cx^u, where $c \in K$. The polynomial ring $S = K[x_1, x_2, \dots, x_n]$ is a $\mathbb{Z}^n$-graded ring with graded components

$$S_u = \begin{cases} Kx^u, & u \in \mathbb{Z}^n_+; \\ 0, & \text{otherwise.} \end{cases}$$

Definition 1.1.19. An S-module M is called $\mathbb{Z}^n$-graded module, if

$$M = \bigoplus_{u \in \mathbf{Z}^n} M_u,$$

such that $S_u M_v \subset M_{u+v}$, for all $u, v \in \mathbb{Z}^n$.

Example 1.1.11. Let $S = K[x_1, x_2, x_3, x_4]$ be a polynomial ring over a field K, clearly S is graded ring with maximal ideal $m = (x_1, x_2, x_3, x_4)$. Let $I = (x_1x_3^2, x_2^2x_3, x_2^2x_4^2) \subset S$ be an ideal, and $M = S/I$. Then $x_1 - x_4$, $x_1 - x_2 - x_3$ is a maximal regular sequence on M. Hence $depth(M) = 2$.

Definition 1.1.20. Let R be a commutative ring. The chain of prime ideals of the form

$$P_0 \subsetneq P_1 \subsetneq P_2 \subsetneq \dots \subsetneq P_n \tag{1.1.1}$$

has lenght n. The krull dimension of a commutative ring R is the supremum of the lenghts of all chains of prime ideals as mentioned in (1.1.1).

(1) A field K has krull dimension 0.

(2) A principle ideal domain that is not a field has a krull dimension 1.

(3) $K[x_1, x_2, \ldots, x_n]$ has krull dimension n.

Example 1.1.12. The ring of integers $\mathbb{Z}$ has krull dimension 1, because it is a principle ideal domain, which is not a field.

Definition 1.1.21. Height of a prime ideal P is define as

$$ht(P) = max\{k : \text{there exist a chain of prime ideals } P_0 \subset P_1 \subset P_2 \subset \ldots \subset P_k = P\},$$

and Height of any ideal I is defined as

$$ht(I) = min\{ht(P) : \ P \text{ is a prime ideal and } I \subset P\}.$$

Definition 1.1.22. Let R be a ring. The krull dimension of R-module M is

$$dim(M) = dim(R/ann(M)).$$

And the co-dimension of M is

$$codim(M) = dim(R) - dim(M).$$

Let M be a finitely generated module over the Noetherian ring R. Then $Ass(M)$ is finite and

$$dim(M) = sup\{dim(R/P_i) : P_i \in Ass(M)\}.$$

Definition 1.1.23. Let $(R, +, \cdot)$ be a ring. A finite or infinite sequence

$$\ldots M_0 \xrightarrow{f_1} M_1 \xrightarrow{f_2} M_2 \xrightarrow{f_3} \ldots \xrightarrow{f_n} M_n \ldots, \tag{1.1.2}$$

of R-modules and R-module homomorphisms is called exact, if the image of each homomorphism is equal to the kernel of the next, that is

$$Im(f_k) = ker(f_{k+1}).$$

Definition 1.1.24. The sequence (1.1.2) of definition 1.1.23 is a short exact sequence, if it is finite and of the form

$$0 \to M_0 \xrightarrow{f_1} M_1 \xrightarrow{f_2} M_2 \to 0.$$

Where f_1 is monomorphism and f_2 is epimorphism.

Remark 1.1.2. If N is a submodule of R-module M. Then

$$0 \to N \xrightarrow{d_1} M \xrightarrow{d_2} M/N \to 0,$$

is a short exact sequence.

Example 1.1.13.

$$0 \to \mathbb{Z} \xrightarrow{2} \mathbb{Z} \to \mathbb{Z}/2\mathbb{Z} \to 0,$$

is a short exact sequence. Because the map 2 from $\mathbb{Z}$ to $\mathbb{Z}$ is monomorphism, and the map $\varphi : \mathbb{Z} \to \mathbb{Z}/2\mathbb{Z}$ is epimorphism.

Chapter 2

Stanley decomposition and the Stanley depth

This chapter includes the discussion about Stanley decomposition, Stanley depth and depth of $\mathbb{Z}^n$-graded S-modules, where S is the polynomial ring in n variables over a field. We also discuss about a conjecture of R. P. Stanley given in [20]. Some results related to Stanley depth and Stanley's conjecture obtained in recent years are also discussed.

2.1 Stanley decomposition and the Stanley depth

Definition 2.1.1. Let K be a field and $S = K[x_1, x_2, \ldots, x_n]$ a polynomial ring over K in n indeterminates $x_1, x_2, \ldots, x_n$. Let M be a finitely generated multigraded (that is $\mathbb{Z}^n$-graded) S-module. Let $v \in M$ be a homogenous element in M and $Z \subseteq \{x_1, x_2, \ldots, x_n\}$. We denote by $vK[Z]$ the K-subspace of M, which is generated by all elements vw, where w is a monomial in $K[Z]$. The $\mathbb{Z}^n$-graded K-subspace $vK[Z] \subset M$ is called a Stanley space of dimension $|Z|$, if $vK[Z]$ is a free $K[Z]$-module. Where $|Z|$ denotes the number of elements of Z. A Stanley decomposition (vector space decomposition) of M is a presentation of the K-vector space M as a finite direct sum of Stanley spaces

$$\mathcal{D} : M = \bigoplus_{i=1}^{s} v_i K[Z_i].$$

The number

$$sdepth(\mathcal{D}) = min\{|Z_i| : i = 1, 2, \ldots, s\},$$

is called the Stanley depth of decomposition and the number

$$sdepth(M) = max\{sdepth(\mathcal{D}) : \mathcal{D} \text{ is a Stanley decomposition of M}\},$$

is called the Stanley depth of M.

Conjecture 2.1.1. *Stanley conjectured in [20], that*

$$sdepth(M) \geq depth(M),$$

for all finitely generated $\mathbb{Z}^n$-graded S-modules M. This conjecture has been proved in several special cases, but it is still open in general.

2.2 The method of Herzog

Let $J \subset I \subset S$ be monomial ideals. Then I/J is a $\mathbb{Z}^n$-graded. Now we will discuss the method of Herzog as described in [9]. By using this method, we can compute Stanley depth of monomial ideal (I/J) by using posets. Suppose I is generated by monomials $x^a = x_1^{a_1}, \ldots, x_n^{a_n}$ and J is generated by $x^b = x_1^{b_1}, \ldots, x_n^{b_n}$. We choose $g \in \mathbb{N}^n$ such that $a_i \leq g$ and $b_j \leq g$ for all i and j. The set $\mathcal{P}^g_{I/J}$ is the characteristic poset of I/J with respect to g, where g is the join of all a_i, b_j define as $g = (a \vee b)(i) = \max\{a(i), b(j)\}$. For this g, the poset has the least number of elements and we denote it by simply $\mathcal{P}_{I/J}$. In order to describe the Stanley decomposition of I/J from a partition of $\mathcal{P}^g_{I/J}$. We shall need the following notation:

For each $b \in \mathcal{P}^g_{I/J}$, we set $Z_b = \{x_j : b(j) = g(j)\}$, we also introduce the function

$$\rho : \mathcal{P}^g_{I/J} \to Z_{\geq 0}, \qquad c \longmapsto \rho(c).$$

Where

$$\rho(c) = |\{j : c(j) = g(j)\}| = |Z_c|.$$

Then we have the following result.

Theorem 2.2.1. *[9, Theorem 2.1] Let $\rho : \mathcal{P}^g_{I/J} = \bigcup_{j=1}^{s}[c_j, d_j]$ be a partition of $\mathcal{P}^g_{I/J}$. Then*

$$\mathcal{D}(\mathcal{P}) : I/J = \bigoplus_{j=1}^{s}\left(\bigoplus_{c} x^c K[Z_{d_j}]\right),$$

is a Stanley decomposition of I/J, where the inner direct sum is taken over all $c \in [c_i,\ d_i]$, for which $c(i) = c_j(i)$, for all i with $x_i \in Z_{d_j}$. Moreover,

$$sdepth\mathcal{D}(\mathcal{P}) = min\{\rho(d_j) : j = 1, \ldots, s\}$$

Example 2.2.1. Let $I = (x_1^2, x_2^2) \subset S = K[x_1, x_2]$, if we consider $g = (2, 2)$. The characteristic poset of I with respect to g is

$$\rho_I^{(2,2)} = \{(2, 0), (0, 2), (2, 1), (1, 2), (2, 2)\}.$$

Partition of posets $\rho_I^{(2,2)}$ is :

$$\mathbb{P}_1 = [(2, 0), (2, 0)] \cup [(0, 2), (0, 2)] \cup [(1, 2), (1, 2)] \cup [(2, 1), (2, 2)].$$

We have for $[(2, 0), (2, 0)]$, $Z_{d_i} = \{x_j : d_i(j) = g(j)\} = \{x_1\}$, so all elements in the interval $[(2, 0), (2, 0)]$, whose first term is same as $c(1) = 2$ is $(2, 0)$, and for $[(2, 0), (2, 0)]$ inner direct sum is $(x_1^2K[x_1])$. For $[(0, 2), (0, 2)]$ inner direct sum is $(x_2^2K[x_2])$, for $[(1, 2), (1, 2)]$ inner direct sum is $(x_1x_2^2K[x_2])$ and for $[(2, 1), (2, 2)]$ inner direct sum is $(x_1^2x_2K[x_1, x_2])$. Hence Stanley decomposition for partition is

$$\mathcal{D} : I = x_1^2K[x_1] \bigoplus x_2^2K[x_2] \bigoplus x_1x_2^2K[x_2] \bigoplus x_1^2x_2K[x_1, x2].$$

And

$$sdepth_S(I) \geq 1.$$

Example 2.2.2. Let $S = K[x_1, x_2]$, $I = (x_1^3x_2^5, x_1^4x_2^4, x_1^6x_2)$ and $J = (x_1^5x_2^6,\ x_1^7x_2^2)$, Then

$$I/J = \{x_1^3x_2^5, x_1^4x_2^5, x_1^4x_2^6, x_1^4x_2^4, x_1^5x_2^4, x_1^6x_2^4, x_1^4x_2^5, x_1^3, x_2^6, x_1^6x_2, x_1^6x_2^2, x_1^5x_2^5, x_1^6x_2^5, x_1^8x_2, x_1^4x_2^7\}.$$

The characteristic poset of I/J with respect to $g = (8, 7)$ is

$\mathcal{P}_{I/J}^{(8,7)} = \{(3,5),(4,5),(3,6),(4,6),(4,4),(5,4),(6,4),(6,1),(6,2),(5,5),(6,5),(8,1),(4,7)\}.$

Partition of the posets $\mathcal{P}_{I/J}^{(8,7)}$ is

$\mathcal{P}_1 : \mathcal{P}_{I/J}^{(8,7)} = [(3,5),(4,7)] \cup [(5,4),(6,5)] \cup [(6,1),(8,1)] \cup [(4,4),(4,4)] \cup$

$[(6,2),(6,2)] \cup [(6,4),(6,4)].$

Stanley decomposition for partition is

$\mathcal{D}(\mathcal{P}_1) := (x_1^3x_2^5K[x_2] \bigoplus x_1^4x_2^5K[x_2]) \bigoplus (x_1^5x_2^4K \bigoplus x_1^5x_2^5K \bigoplus x_1^6x_2^5K) \bigoplus (x_1^6x_2K[x_1]) \bigoplus$

$(x_1^4x_2^4K) \bigoplus (x_1^6x_2^2K) \bigoplus (x_1^6x_2^4K).$

Stanley depth of the decomposition is

$$sdepth\mathcal{D}(\mathcal{P}_1) = 0.$$

Another partition of $\mathcal{P}_{I/J}^{(8,7)}$ is

$\mathcal{P}_2 : \mathcal{P}_{I/J}^{(8,7)} = [(3,5),(4,7)] \cup [(6,1),(8,1)] \cup [(5,4),(5,5)] \cup$

$[(4,4),(4,5)] \cup [(6,4),(6,4)].$

And corresponding Stanley decomposition is

$\mathcal{D}(\mathcal{P}_2) := (x_1^3x_2^5K[x_2] \bigoplus x_1^4x_2^5K[x_2]) \bigoplus (x_1^6x_2K[x_1]) \bigoplus (x_1^5x_2^4K \bigoplus x_1^5x_2^5K) \bigoplus$

$(x_1^4x_2^4K) \bigoplus (x_1^4x_2^5K) \bigoplus (x_1^6x_2^4K).$

Stanley depth of the other decomposition is

$$sdepth\mathcal{D}(\mathcal{P}_2) = 0.$$

Hence by Cocoa,

$$sdepth_S(I/J) = 0$$

2.3 Values and bounds for the Stanley depth of monomial ideals

Proposition 2.3.1. *[16, Proposition 1.3] Let $I \subset S$ be a monomial ideal. Then*

$$sdepth_S(I : u) \geq sdepth_S(I) \quad for\ each\ monomial\ u \notin I.$$

Theorem 2.3.2. *[2, Theorem 2.2] Let $S = K[x_1, x_2, \ldots, x_n]$ be a polynomial ring over a field K. If $m = (x_1, x_2, \ldots, x_n)$ be the maximal ideal in S, then*

$$sdepth(m) = \left\lceil \frac{n}{2} \right\rceil.$$

Theorem 2.3.3. *[3, Theorem 1.3] Let $b_1, b_2, \ldots, b_n$ be some positive integers, Then*

$$sdepth((x_1^{b_1}, \ldots, x_n^{b_n})) = sdepth((x_1, \ldots, x_n)) = \left\lceil \frac{n}{2} \right\rceil.$$

In particular,

$$sdepth((x_1^{b_1}, \ldots, x_m^{b_m})) = n - m + \left\lceil \frac{m}{2} \right\rceil \qquad for\quad any \quad 1 \leq m \leq n.$$

Theorem 2.3.4. *[3, Theorem 2.1] Let $J \subset S$ be a complete intersection monomial ideal. Then*

$$sdepth(J) = sdepth(\sqrt{J}).$$

Lemma 2.3.5. *[3, Lemma 2.2] Let $J^{'} \subset S[x_{n+1}]$ be a monomial ideal, we consider the homomorphism $\psi : S[x_{n+1}] \to S$, $\psi(x_i) = x_i$ for $i \leq n$ and $\psi(x_{n+1}) = 1$. Let $I = \psi(I^{'})$, then*

$$sdepth(I^{'}) \leq sdepth(I) + 1.$$

Lemma 2.3.6. *[5, Lemma 1.1] Let $v, w \in S$ be two monomials and $Y, Z \subset x_1, \ldots, x_n$, such that supp$(v) \subset Y$ and supp$(w) \subset Z$. Then*

$$vK[Y] \cap wK[Z] = lcm(v, w)K[Y \cap Z].$$

Proposition 2.3.7. *[5, Proposition 3.1] The following statements are equivalent:*

(1) *For any integer $n \geq 1$ and any monomial ideal $J \subset S = K[x_1, \ldots, x_n]$. Stanley conjecture holds for J, that is*

$$sdepth_S(J) \geq depth_S(J).$$

(2) *For any integer $n \geq 1$, and any monomial ideals $I, J \subset S = K[x_1, \ldots, x_n]$. If $sdepth_S(I + J) \geq depth_S(I + J)$, then*

$$sdepth_S(J) \geq depth_S(J).$$

Theorem 2.3.8. *[17, Theorem 4.2] Let $J = \bigcap_{k=1}^{4} P_k$ be a reduced intersection of four monomial prime ideals of S. Then Stanleys Conjecture holds for J.*

Theorem 2.3.9. *[23, Theorem 2.4] Let $J \subset S = K[x_1, x_2, \ldots, x_n]$ be a monomial ideal minimally generated by m elements. Then*

$$sdepth(J) = n - \left\lfloor \frac{m}{2} \right\rfloor.$$

Theorem 2.3.10. *[14, Theorem 2.3] Let J be a monomial ideal of S with $|G(J)| = m$. Then*

$$sdepth(J) \geq max \left\{1, n - \left\lfloor \frac{m}{2} \right\rfloor \right\}.$$

Lemma 2.3.11. *[14, Lemma 2.4] Let M be a $\mathbb{Z}^n$-graded S-module, and suppose that M_1 and M_2 be its two submodules. Let*

$$0 \to M_1 \to M \to M_2 \to 0,$$

be an exact sequence. Then

$$sdepth(M) \geq min \{sdepth(M_1), sdepth(M_2)\}.$$

Theorem 2.3.12. *[10, Theorem 2.8] Let P and $P^{'}$ be two primary monomial ideals with $\sqrt{P} = (x_1, \dots, x_s)$ and $\sqrt{P'} = (x_{s+1}, \dots, x_n)$, where $s \geq 2$ and $n \geq 4$. Then*

$$sdepth(P \cap P^{'}) \leq \frac{n+2}{2}.$$

Corollary 2.3.13. *[10, Corollary 2.10] Let P and $P^{'}$ be two irreducible monomial ideals with $\sqrt{P} = (x_1, \dots, x_s)$ and $\sqrt{P'} = (x_{s+1}, \dots, x_n)$. Suppose that n is even. Then*

$$sdepth(P \cap P^{'}) = \begin{cases} \frac{n}{2} + 1, & \text{if } s \text{ is odd;} \\ \frac{n}{2} \text{ or } \frac{n}{2} + 1, & \text{if } s \text{ is even.} \end{cases}$$

Lemma 2.3.14. *[10, Lemma 2.11] Let $I \subset S$ be a monomial ideal, and let $I^{'} = (I, x_{n+1})$ be a monomial ideal of $S^{'} = S[x_{n+1}]$. Then*

$$sdepth_S(I) \leq sdepth_{S'}(I^{'}) \leq sdepth_S(I) + 1.$$

Proposition 2.3.15. *[10, Proposition 2.13] Let P and $P^{'}$ be two primary monomial ideals with $\sqrt{P} = (x_1, \dots, x_s)$ and $\sqrt{P'} = (x_{t+1}, \dots, x_n)$, where $1 < t \leq s < n$, $n \geq 4$. Then*

$$sdepth(P \cap P^{'}) \leq \frac{n+s-t+2}{2}.$$

Proposition 2.3.16. *[10, Proposition 2.16] Let P and $P^{'}$ be two primary monomial ideals with $\sqrt{P} = (x_1, \dots, x_s)$ and $\sqrt{P'} = (x_{t+1}, \dots, x_n)$, where $1 < t \leq s < n$. Then*

$$sdepth(P \cap P^{'}) \leq min\left\{n - \left\lfloor \frac{s}{2} \right\rfloor, n - \left\lfloor \frac{n-s}{2} \right\rfloor\right\}.$$

Theorem 2.3.17. *[10, Theorem 2.19] Let P and $P^{'}$ be two primary monomial ideals with $\sqrt{P} = (x_1, \dots, x_s)$ and $\sqrt{P'} = (x_{t+1}, \dots, x_p)$, where $1 < t \leq s < p \leq n$, $n \geq 4$. Then*

$$sdepth(P \cap P^{'}) \leq min\left\{\frac{2n+s-p-t+2}{2}, n - \left\lfloor \frac{s}{2} \right\rfloor, n - \left\lfloor \frac{p-s}{2} \right\rfloor\right\}.$$

Theorem 2.3.18. *[13, Theorem 2.2] Let $I = \bigcap_{i=1}^{s} P_i$ be a monomial ideal in S such that each P_i is irreducible and $G(\sqrt{P_i}) \cap G(\sqrt{P_j}) = \emptyset$ for all $i \neq j$. Then*

$$sdepth(I) = sdepth(\sqrt{I}).$$

Theorem 2.3.19. *[13, Theorem 2.6] Let $I = \bigcap_{i=1}^{r} Q_i$ be a monomial ideal in S, where each Q_i is a monomial prime ideal and $\sum_{i=1}^{r} Q_i = m$, where m is the maximal ideal. Suppose $G(Q_i) \cap G(Q_j) = \emptyset$ for all $i \neq j$. Then*

$$sdepth(I) \leq \frac{n+r}{2}.$$

Corollary 2.3.20. *[13, Corollary 2.7] Let $I = \bigcap_{i=1}^{r} Q_i$ be a monomial ideal in S such that $G(\sqrt{Q_i}) \cap G(\sqrt{Q_j}) = \emptyset$ for all $i \neq j$, $ht(Q_i) = d_i$, $\sum_{i=1}^{r} \sqrt{Q_i} = m$ and $A = \{Q_i : ht(Q_i) \text{ is odd}\}$. Then*

$$\frac{n+|A|}{2} \leq sdepth(I) \leq \left\lfloor \frac{n+r}{2} \right\rfloor.$$

Corollary 2.3.21. *[13, Corollary 2.10] Let $I = \bigcap_{i=1}^{r} Q_i$ be a monomial ideal in S such that each Q_i is irreducible and $G(\sqrt{Q_i}) \cap G(\sqrt{Q_j}) = \emptyset$ for all $i \neq j$ and $\sum_{i=1}^{r} Q_i = m$. Suppose that r is odd and $|A| = r-1$. Then*

$$sdepth(I) = \frac{n+r-1}{2}.$$

Lemma 2.3.22. *[18, Lemma 1.2] Let $S = K[x_1, \ldots, x_n]$ and $I \subset S' = K[x_1, \ldots, x_k]$, $J \subset S'' = K[x_{k+1}, \ldots, x_n]$, where $1 < k < n$ be monomial ideals. Then*

$$sdepth_S(IS \cap JS) \geq sdepth_{S'}(I) + sdepth_{S''}(J).$$

Lemma 2.3.23. *[12, Lemma 3.1] Let $S = K[x_1, \ldots, x_n]$ be a polynomial ring and $I' \subset S' = K[x_{n+1}]$, x_{n+1} being a new variable. If $I' \cap S \neq (0)$, then*

$$sdepth_S(I' \cap S) \geq sdepth_{S[x_{n+1}]}(I') - 1.$$

Lemma 2.3.24. *[19, Lemma 4.1] Let P, $P' \subset S = K[x_1, \ldots.., x_n]$ be two non-zero irreducible monomial ideals such that $\sqrt{P} = \{x_1, \ldots, x_r\}$, $\sqrt{P'} = \{x_{r+1}, \ldots, x_n\}$ for some integer r with $1 \leq r \leq n$. Then*

$$sdepth(P \cap P') \geq \left\lceil \frac{r}{2} \right\rceil + \left\lceil \frac{n-r}{2} \right\rceil \geq n/2.$$

Lemma 2.3.25. *[1, Lemma 2.1] Let $I \subset S = K[x_1, x_2, \ldots, x_n]$ be a monomial ideal, Janet's algorithm gives a Stanley decomposition of I.*

2.4 Values and bounds for the Stanley depth of quotient ideals

Lemma 2.4.1. *[9, Lemma 3.6] Let $J \subset I$ be monomial ideals of S, and let $S^{'} = S[x_{n+1}]$ be the polynomial ring over S in the variable x_{n+1}. Then*

$$depth(IS^{'}/JS^{'}) = depth(I/J) + 1.$$

$$sdepth(IS^{'}/JS^{'}) = sdepth(I/J) + 1.$$

Proposition 2.4.2. *[4, Proposition 1.2] Let $J \subset S$ be a monomial ideal. Then*

$$sdepth(S/J) \geq n - G(J).$$

Theorem 2.4.3. *[4, Theorem 1.4] Let $I \subset S$ be a monomial ideal which is not principle. Assume $I = uI^{'}$, where $u \in S$ is a monomial and $I^{'} = (I : u)$. Then*

(1) $sdepth(S/I) = sdepth(S/I^{'})$.

(2) $sdepth(I) = sdepth(I^{'})$.

Corollary 2.4.4. *[4, Corollary 3.3] If $I \subset K[x_1, x_2, x_3]$ is a monomial ideal, then*

$$sdepth(I) \geq sdepth(S/I) + 1.$$

In particular, if

$$sdepth(I) = 1, \quad then \quad depth(I) = 1.$$

Theorem 2.4.5. *[22, Theorem 1.1] Let $J \subset S = K[x_1, x_2, \ldots, x_n]$ be a monomial ideal and $v \in S$ be a monomial regular on S/J. Then*

$$sdepth(S/(J, v)) = sdepth(S/J) - 1.$$

In particular, J is a Stanley ideal if and only if (J, v) is a Stanley ideal.

Theorem 2.4.6. *[10, Theorem 2.1] Let $J \subset I$ be two monomial ideals of S and $\sqrt{I}$ and $\sqrt{J}$ be the radical ideals of I and J respectively. Then*

$$sdepth_S(I/J) \leq sdepth_S(\sqrt{I}/\sqrt{J}).$$

Corollary 2.4.7. *[10, Corollary 2.2] Let $J \subset S$ be a monomial ideal and $\sqrt{J}$ be its radical. Then*

$$sdepth(S/J) \leq sdepth(S/\sqrt{J}).$$

And

$$sdepth(J) \leq sdepth(\sqrt{J}).$$

Corollary 2.4.8. *[10, Corollary 2.3] Let $J \subset I$ be two monomial ideals of S, and let $\sqrt{I}$ and $\sqrt{J}$ be the radical ideals of I and J, respectively. If $sdepth(I/J) = dim(I/J)$, then*

$$sdepth(\sqrt{I}/\sqrt{J}) = dim(\sqrt{I}/\sqrt{J}).$$

Theorem 2.4.9. *[13, Theorem 3.1] Let $S = K[x_1, \ldots, x_n]$ be a polynomial ring and $P_1, P_2, \ldots, P_r$ monomial irreducible ideals of S such that $G(\sqrt{P_i}) \cap G(\sqrt{P_j}) = \emptyset$ for all $i \neq j$. Let $ht(P_i) = d_i$ and $\sum_{i=1}^{r} d_i = n$. If $I = P_1 \cap P_2 \cap ... \cap P_r$, then*

$$sdepth(S/I) \geq min\left\{n - d_1, min_{2\leq i\leq r}\left\{\left\lceil \frac{d_1}{2}\right\rceil + \ldots + \left\lceil \frac{d_{i-1}}{2}\right\rceil + d_{i+1} + \ldots + d_k\right\}\right\}.$$

Corollary 2.4.10. *[13, Corollary 3.4] Let $d_1 \geq d_2 = \ldots = d_r$, then*

$$sdepth(S/I) \geq \left\lceil \frac{d_1}{2}\right\rceil + \left\lceil \frac{d_2}{2}\right\rceil + \ldots + \left\lceil \frac{d_{r-1}}{2}\right\rceil.$$

Lemma 2.4.11. *[19, Lemma 1.2] Let P be a primary monomial ideal in $S = K[x_1, \ldots, x_n]$. Then*

$$sdepth(S/P) = dim(S/P) = depth(S/P).$$

Lemma 2.4.12. *[19, Lemma 1.3] Let I, J be two monomial ideals in a polynomial ring $S = K[x_1, \ldots, x_n]$. Then*

$$\begin{aligned} sdepth(S/(I \cap J)) \geq & max\{min\{sdepth(S/I), sdepth(I/(I \cap J))\}, \\ & min\{sdepth(S/J), sdepth(J/(I \cap J))\}\}. \end{aligned}$$

Corollary 2.4.13. *[3, Corollary 2.4] If I, $J \subset S$ are two monomial ideals and $|G(J)| = m$. Then*

$$sdepth((I + J)/I) \geq sdepth(S/I) - \left\lfloor \frac{m}{2}\right\rfloor.$$

Lemma 2.4.14. *[1, Lemma 2.3] If $I \subset S = K[x_1, x_2, \ldots, x_n]$ is a square free monomial ideal, Janet's algorithm gives a square free Stanley decomposition of S/I recursively.*

Lemma 2.4.15. *[16, Lemma 2.3] Let $S^* = K[x_1, \ldots, x_{n-1}]$ be a polynomial ring in $n-1$ variables over a field K. Let $I, J \subset S^*$, $J \subset I$, $I \neq J$ be two monomial ideals, $T = (I + x_nJ)S$ such that*

(1) $depth_{S^*}(S^*/J) = depth_S(S/T) - 1$.

(2) $depth_{S^*}(J) \geq 1 + depth_{S^*}(S^*/J)$.

(3) $sdepth_{S^*}(I/J) \geq depth_{S^*}(I/J)$. *Then*

$$sdepth_S(T) \geq 1 + depth_S(S/T).$$

Chapter 3

Stanley depth of quotient of monomial complete intersection ideals

In this chapter we discuss in detail the results of paper [6]. In this paper Stanley depth of quotient of irreducible monomial ideals and quotient of complete intersection monomial ideals $J \subset I \subset S$ are discussed. If $J \subsetneq I \subset S$ are complete intersection monomial ideals, then some sharp bounds exist for $sdepth_S(I/J)$, and also Stanley's conjecture holds for quotients of complete intersection monomial ideals.

3.1 The Case Of irreducible ideals

Lemma 3.1.1. *[6, Lemma 1.1] Let b be a positive integer, and $J \subsetneq I \subset S^* = K[x_2, \dots, x_n]$ be monomial ideals. Then*

$$(x_1^b, I)/(x_1^b, J) \cong \bigoplus_{i=0}^{b-1} x_1^i (I/J),$$

as $\mathbb{Z}^n$-graded, S^*-modules. Moreover

$$sdepth_S((x_1^b, I)/(x_1^b, J)) = depth_S^*(I/J).$$

Proof. let $v \in (x_1^b, I)/(x_1^b, J)$ be a monomial. Then $v = x_1^i \cdot v'$, for some nonnegative integer i and some monomial $v' \in S^*$. Since $v \notin (x_1^b, J)$, it follows that $i < b$ and also $v' \notin J$. On the other hand, since $v \in (x_1^b, I)$ and $v \notin x_1^b S$, it follows that $v' \in I$. Therefore, $v \in \bigoplus_{i=0}^{b-1} x_1^i (I/J)$, so we have

$$((x_1^b, I)/(x_1^b, J)) \subseteq \bigoplus_{i=0}^{b-1} x_1^i (I/J). \tag{3.1.1}$$

Conversely, let us take a monomial $v' \in (I/J)$ and an integer $0 \leq i < b$. Since $v' \notin J$, so $v' \notin (x_1^b, J)$ also $v' \in I$, so $v' \in (x_1^b, I)$. Thus $v' \in (x_1^b, I)/(x_1^b, J)$, since $(x_1^b, I)/(x_1^b, J) \subseteq S/(x_1^b, J)$ is an ideal of $S/(x_1^b, J)$. So by definition of ideal, for $v' \in (x_1^b, I)/(x_1^b, J)$ and $x_1^i \in S/(x_1^b, J)$, we have $x_1^i v' \in (x_1^b, I)/(x_1^b, J)$. Thus

$$\bigoplus_{i=0}^{b-1} x_1^i (I/J) \subseteq (x_1^b, I)/(x_1^b, J). \tag{3.1.2}$$

Hence from equation (3.1.1) and (3.1.2) , we have

$$(x_1^b, I)/(x_1^b, J) \cong \bigoplus_{i=0}^{b-1} x_1^i (I/J).$$

By this decomposition clearly

$$sdepth_S(x_1^b, I)/(x_1^b, J) \geq sdepth_{S^*}(I/J). \tag{3.1.3}$$

Also

$$I/J = (x_1^b, I)/(x_1^b, J) \cap (S^*/J), \tag{3.1.4}$$

via the natural injection $I/J \hookrightarrow (x_1^b, I)/(x_1^b, J)$. In order to prove the other inequality, we consider a Stanley decomposition of $v_i K[Z_i] = \bigoplus_{j=1}^{s} v_i K[Z_i]$. Note that $v_i K[Z_i] \cap S^* = 0$, if $x_1 \in supp(v_i)$ or $x_1 | v$, otherwise $v_i K[Z_i] \cap S^* = v_i K[Z_i]$. Thus from equation (3.1.4), we have

$$I/J = \bigoplus_{x_1 \notin supp(v_i)} v_i K[Z_i].$$

And thus

$$sdepth_{S^*}(I/J) \geq sdepth_S((x_1^b, I)/(x_1^b, J)). \tag{3.1.5}$$

Hence from equation (3.1.3) and (3.1.5), we have

$$sdepth_S((x_1^b, I)/(x_1^b, J)) = sdepth_{S^*}(I/J)$$

. ∎

Corollary 3.1.2. *[6, Corollary 1.2] Let* $0 \leq p < q$ *be integer. Then*

$$sdepth_S((x_1, x_2, \ldots, x_q)/(x_1, x_2, \ldots, x_p)) = q - p - \lfloor \frac{q-p}{2} \rfloor.$$

Proof. We use induction on p.

(1) If $p = 0$, then by [2, Theorem 2.2], we have

$$sdepth(x_1, x_2, \ldots, x_q) = \left\lceil \frac{q}{2} \right\rceil = q - \left\lfloor \frac{q}{2} \right\rfloor.$$

(2) The case $p = q$ is trivial.

(3) Now we assume $1 \leq p < q$. Let $S' = K[x_2, x_3, \ldots, x_q]$ and denote $I = (x_2, x_3, \ldots, x_q) \subset S'$ and $J = (x_2, x_3, \ldots, x_p) \subset S'$. Then According to Lemma 3.1.1

$$(x_1, I)/(x_1, J) \cong (I/J).$$

And thus by induction hypothesis

$$sdepth_S(x_1, I)/(x_1, J) = sdepth_{S'}(I/J) = (q-1)-(p-1)-\left\lfloor \frac{(q-1)-(p-1)}{2} \right\rfloor.$$

Thus

$$sdepth_S((x_1, x_2, \ldots, x_q)/(x_1, x_2, \ldots, x_p)) = q - p - \left\lfloor \frac{q-p}{2} \right\rfloor.$$

If we denote $S^{**} = K[x_{p+1}, ..., x_q]$, note that the above corollary follow also from the isomorphism of multigraded S^{**}-modules.

$$(x_1, ..., x_q)/(x_1, \ldots, x_p) \cong (x_{p+1}, \ldots, x_q).$$

∎

Lemma 3.1.3. *[6, Lemma 1.3] Let $1 \leq a < b$ be integers, and $I \subset S^* = K[x_2, x_3, \ldots, x_n]$ be a monomial ideal. Then*

$$(x_1^a, I)/(x_1^b, I) \cong \bigoplus_{i=a}^{b-1} x_1^i (S^*/I),$$

as $\mathbb{Z}^n$-graded S^*-modules. Moreover

$$sdepth_S((x_1^a, I)/(x_1^b, I)) = sdepth_{S^*}(S^*/I).$$

Proof. Let $u \in (x_1^a, I)/(x_1^b, I)$. Then we write $u = x_1^i \cdot u^{'}$, for some integer $a \leq i < b$ and some monomial $u^{'} \in S^*$. Since $u \notin (x_1^b, I)$, it follows that $i < b$ and $u^{'} \notin I$. Also $u \in (x_1^a, I)$, so $a \leq i < b$ and $u^{'} \in S^*$. Therefore $u^{'} \in (S^*/I)$, so

$$((x_1^a, I)/(x_1^b, I)) \subseteq \bigoplus_{i=a}^{b-1} x_1^i (S^*/I). \tag{3.1.6}$$

Conversely, if we take an arbitrary monomial $u^{'} \in (S^*/I)$ and $a \leq i < b$. Then clearly, $u^{'} \in (x_1^a, I)/(x_1^b, I)$. Since $(x_1^a, I)/(x_1^b, I) \subseteq S/(x_1^b, I)$ is an ideal. So by definition of ideal for $u^{'} \in (x_1^a, I)/(x_1^b, I)$ and $x_1^i \in S/(x_1^b, I)$, we have $x_1^i.u^{'} \in (x_1^a, I)/(x_1^b, I)$. Thus

$$\bigoplus_{i=a}^{b-1} x_1^i (S^*/I) \subseteq (x_1^a, I)/(x_1^b, I). \tag{3.1.7}$$

Hence from equation (3.1.6) and (3.1.7), we have

$$(x_1^a, I)/(x_1^b, I) \cong \bigoplus_{i=a}^{b-1} x_1^i (S^*/I).$$

The above isomorphism implies

$$sdepth_S(x_1^a, I)/(x_1^b, I) \geq sdepth_{S^*}(S^*/I). \tag{3.1.8}$$

Also note that

$$x_1^a(S^*/I) = (x_1^a, I)/(x_1^b, I) \cap x_1^a(S^*/I), \tag{3.1.9}$$

via natural injection $x_1^a S^*/I \hookrightarrow (x_1^a, I)/(x_1^b, I)$. In order to prove other in-equality, let us consider Stanley decomposition of $(x_1^a, I)/(x_1^b, I) = \bigoplus_{i=1}^{r} u_i K[Z_i]$. Note that

$u_iK[Z_i] \cap S^* = 0$, if $x_1|u$ or $x_1 \in supp(u_i)$, otherwise, $u_iK[Z_i] \cap S^* = u_iK[Z_i]$. Thus from equation (3.1.9), we have

$$S^*/I = \bigoplus_{x_1 \notin supp(u_i)} u_iK[Z_i].$$

Clearly

$$sdepth_{S^*}(S^*/I) \geq sdepth_S(x_1^a, I)/(x_1^b, I). \tag{3.1.10}$$

Hence from equation (3.1.8) and (3.1.10), we have

$$sdepth_S(x_1^a, I)/(x_1^b, I) = sdepth_{S^*}(S^*/I).$$

□

Lemma 3.1.4. *[6, Lemma 1.4] Let* $1 \leq a < b$ *be integers and* $J \subset I \subset S^* = K[x_2, \ldots, x_n]$ *be monomial ideals. Then*

$$sdepth_S((x_1^a, I)/(x_1^b, J) \geq min\left\{sdepth_S^*(I/J), sdepth_S^*(S^*/J)\right\}.$$

Proof. Since $(x_1^a, J)/(x_1^b, J) \subseteq (x_1^a, I)/(x_1^b, J)$ is a $\mathbb{Z}^n$-graded submodule. So

$$0 \to (x_1^a, J)/(x_1^b, J) \to (x_1^a, I)/(x_1^b, J) \to ((x_1^a, I)/(x_1^b, J))/((x_1^a, J)/(x_1^b, J)) \to 0,$$

is a short exact sequence. Thus we have

$$(x_1^a, I)/(x_1^b, J) \cong ((x_1^a, I)/(x_1^b, J))/((x_1^a, J)/(x_1^b, J)) \bigoplus (x_1^a, J)/(x_1^b, J).$$

Now by the third isomorphism theorem, we have

$$(x_1^a, I)/(x_1^b, J) \cong (x_1^a, I)/(x_1^a, J) \bigoplus (x_1^a, J)/(x_1^b, J), \tag{3.1.11}$$

as $\mathbb{Z}^n$-graded, S^*-modules. Now by [14, Lemma 2.4], we have

$$sdepth_S(x_1^a, I)/(x_1^b, J) \geq min\left\{sdepth_S(x_1^a, I)/(x_1^a, J), sdepth_S(x_1^a, J)/(x_1^b, J)\right\}. \tag{3.1.12}$$

By lemma 3.1.1, we have

$$sdepth_S(x_1^a, I)/(x_1^a, J) = sdepth_{S^*}(I/J).$$

Also by lemma 3.1.3, we have

$$sdepth_S(x_1^a, J)/(x_1^b, J) = sdepth_{S^*}(S^*/J).$$

Hence from equation (3.1.12), we have

$$sdepth_S(x_1^a, I)/(x_1^b, J) \geq min\{sdepth_{S^*}(I/J), sdepth_{S^*}(S^*/J)\}.$$

□

Theorem 3.1.5. *[6, Theorem 1.5] Let $0 \leq p \leq q$ be integers. Let $a_i \geq 1$, for $1 \leq i \leq q$ and $b_i \geq a_i$, for $1 \leq i \leq p$, be some integers. Then*

$$sdepth_S((x_1^{a_1}, \ldots, x_q^{a_q})/(x_1^{b_1}, \ldots, x_p^{b_p})) = q - p - \left\lfloor \frac{q-p}{2} \right\rfloor.$$

And thus

$$sdepth_S((x_1^{a_1}, \ldots, x_q^{a_q})/(x_1^{b_1}, \ldots, x_p^{b_p})) \geq depth_S((x_1^{a_1}, \ldots, x_q^{a_q})/(x_1^{b_1}, \ldots, x_p^{b_q})).$$

Proof. We use induction on p.

(1) Let $p = 0$. Then by [3, Theorem 1.3], we have

$$sdepth_S((x_1^{a_1}, x_2^{a_2}, \ldots, x_q^{a_q})) = \left\lceil \frac{q}{2} \right\rceil = q - \left\lfloor \frac{q}{2} \right\rfloor, \quad \textit{as required.}$$

(2) Let $p = q$. Then $(x_1^{a_1}, \ldots, x_q^{a_q})/(x_1^{b_1}, \ldots, x_p^{b_p})$ is a finite K- vector space, and thus its Stanley depth is 0.

(3) Now assume $1 \leq p \leq q$. We denote $S^* = K[x_2, x_3, \ldots, x_q]$, $I = (x_2^{a_2}, \ldots, x_q^{a_q}) \subset S^*$, and $J = (x_2^{b_2}, \ldots, x_p^{b_p}) \subset S^*$. By induction hypothesis, we have

$$sdepth_{S^*}(I/J) = (q-1) - (p-1) - \left\lfloor \frac{(q-1)-(p-1)}{2} \right\rfloor.$$

Thus

$$sdepth_{S^*}(I/J) = q - p - \left\lfloor \frac{q-p}{2} \right\rfloor \qquad (3.1.13)$$

On the other hand by [22, Theorem 1.1], or by [9, lemma 3.6], we have $sdepth_{S^*}(S^*/J) = sdepth(S^*) - (p-1)$. So

$$sdepth_{S^*}(S^*/J) = (q-1) - (p-1) = q - p.$$

Now according to Lemma 3.1.4, we have

$$sdepth_S((x_1^{a_1}, \ldots, x_q^{a_q})/(x_1^{b_1}, \ldots, x_p^{b_p})) \geq min\left\{sdepth_{S^*}(I/J), sdepth_{S^*}(S^*/J)\right\}.$$

Thus

$$sdepth_S((x_1^{a_1}, \ldots, x_q^{a_q})/(x_1^{b_1}, \ldots, x_p^{b_p})) \geq sdepth_{S^*}(I/J) = q - p - \left\lfloor \frac{q-p}{2} \right\rfloor. \tag{3.1.14}$$

In order to prove other inequality, we will consider two cases:

case(a) If $a_1 = b_1$, so by Lemma 3.1.1, we have

$$sdepth_S((x_1^{a_1}, ..., x_q^{a_q})/(x_1^{b_1}, ..., x_p^{b_p})) = sdepth_{S^*}(I/J) = q - p - \left\lfloor \frac{q-p}{2} \right\rfloor.$$

case(b) Assume $a_1 < b_1$, we denote $a = a_1, b = b_1$. By short exact sequence

$$0 \to (x_1^a, J)/(x_1^b, J) \to (x_1^a, I)/(x_1^b, J) \to ((x_1^a, I)/(x_1^b, J))/((x_1^a, J)/(x_1^b, J)) \to 0.$$

We consider decomposition

$$(x_1^a, I)/(x_1^b, J) \cong (x_1^a, I)/(x_1^a, J) \bigoplus (x_1^a, J)/(x_1^b, J).$$

Now by Lemma 3.1.1, and Lemma 3.1.3, we have

$$(x_1^a, I)/(x_1^b, J) \cong \bigoplus_{i=0}^{a-1} x_1^i(I/J) \bigoplus \bigoplus_{i=a}^{b-1} x_1^i(S^*/J). \tag{3.1.15}$$

Also

$$I/J = (x_1^a, I)/(x_1^b, J) \cap (S^*/J), \tag{3.1.16}$$

via natural injection $I/J \hookrightarrow (x_1^a, I)/(x_1^b, J)$. Now we consider a Stanley decomposition of $(x_1^a, I)/(x_1^b, J) = \bigoplus_{j=1}^{r} v_j K[Z_j]$, so from equation (3.1.16), we have

$$I/J = \bigoplus_{j=1}^{r} (v_j K[Z_j] \cap S^*/J). \tag{3.1.17}$$

Note that $v_j K[Z_j] \cap S^* = 0$, if $x_1 \in supp(v_j)$, otherwise $v_j K[Z_j] \cap S^* = v_j K[Z_j]$. Thus from equation (3.1.17), we have

$$I/J = \bigoplus_{x_1 \notin supp(v_j)} v_j K[Z_j].$$

Clearly, $sdepth_{S^*}(I/J) \geq sdepth_S(x_1^a, I)/(x_1^b, J)$. Thus

$$q - p - \lfloor \frac{q-p}{2} \rfloor \geq sdepth_S(x_1^a, I)/(x_1^b, J). \tag{3.1.18}$$

Hence from equation (3.1.14), and (3.1.18), we have

$$sdepth_S(x_1^a, I)/(x_1^b, J) = q - p - \lfloor \frac{q-p}{2} \rfloor.$$

In order to complete the proof, notice that $depth_S(x_1^{a_1}, \dots, x_q^{a_q})/(x_1^{b_1}, \dots, x_p^{b_p}) = 1$, if $q > p$ and 0, if $p = q$.

□

3.2 The case of complete intersection ideals

Lemma 3.2.1. *[6, Lemma 2.1] Let $1 \leq p < q$ be integer, $J \subsetneq I \subset S^* = K[x_1, x_2, \dots, x_p]$ be distinct monomial ideals, and let $J_1 \subset S^{**} = K[x_{p+1}, \dots, x_n]$ be a monomial ideal. Then*

$$sdepth_S(I, J_1)/(J, J_1) \geq sdepth_{S^{**}}(S^{**}/J_1) + sdepth_{S^*}(I/J).$$

Proof. Let $v \in (I, J_1)/(J, J_1)$ be a monomial. Then we write $v = v^{'}.v^{''}$, where $v^{'} \in S^*$ and $v^{''} \in S^{**}$. Since $v \in (I, J_1)$, it follows that $v \in IS$ or $v \in J_1 S$. Thus

$v^{'} \in I$ and $v^{''} \in S^{**}$. On the other hand, since $v \notin (J, J_1)$, it follows that $v \notin JS$ or $v \notin J_1S$. Therefore we get $v^{'} \notin J$ and $v^{''} \notin J_1$. Thus $v^{'} \in (I/J)$ and $v^{''} \in (S^{**}/J_1)$. So $v \in (I/J).(S^{**}/J_1)$. Therefore

$$(I, J_1)/(J, J_1) \subseteq (I/J).(S^{**}/J_1) \tag{3.2.1}$$

Conversely, Let $v^{'} \in (I/J)$, and $v^{''} \in (S^{**}/J_1)$. Since $v^{'} \in I$, so $v^{'} \in (I, J_1)$. Also $v^{'} \notin J$, so $v^{'} \notin (J, J_1)$. Thus $v^{'} \in (I, J_1)/(J, J_1)$. Since $(I, J_1)/(J, J_1) \subseteq S/(J, J_1)$ is an ideal, so by definition of ideal for $v^{'} \in (I, J_1)/(J, J_1)$ and $v^{''} \in S/(J, J_1)$, we have $v^{'}.v^{''} \in (I, J_1)/(J, J_1)$. Therefore

$$(I/J).(S^{**}/J_1) \subseteq (I, J_1)/(J, J_1). \tag{3.2.2}$$

Hence from equation (3.2.1) and (3.2.2), we have

$$(I, J_1)/(J, J_1) = (I/J).(S^{**}/J_1). \tag{3.2.3}$$

Now, let us consider two Stanley decompositions $I/J = \bigoplus_{i=1}^{s} u_i K[Y_i]$, and $S^{**}/J_1 = \bigoplus_{j=1}^{r} w_j K[Z_j]$. It follows from equation (3.2.3), that

$$(I, J_1)/(J, J_1) = \bigoplus_{i=1}^{s} \bigoplus_{j=1}^{r} u_i w_j K[Y_i \cup Z_j],$$

is a Stanley decomposition, and thus

$$sdepth_S(I, J_1)/(J, J_1) \geq sdepth_{S^{**}}(S^{**}/J_1) + sdepth_{S^*}(I/J).$$

□

Lemma 3.2.2. *[6, Lemma 2.2] Let* $1 \leq p < q$ *be integer,* $J \subsetneq I \subset S^* = K[x_1, \ldots, x_p]$ *be monomial ideals and* $J_1 \subset I_1 \subsetneq S^{**} = K[x_{p+1}, \ldots, x_q]$ *be other monomial ideals. Then*

$$\begin{aligned} sdepth_S(I, I_1)/(J, J_1) \geq & min\{sdepth_{S^*}\left(\frac{S^*}{I}\right) + sdepth_{S^{**}}\left(\frac{I_1}{J_1}\right), \\ & sdepth_{S^{**}}\left(\frac{S^{**}}{J_1}\right) + sdepth_{S^*}\left(\frac{I}{J}\right)\}. \end{aligned}$$

Proof. Since $(I, J_1)/(J, J_1) \subseteq (I, I_1)/(J, J_1)$ is a $\mathbb{Z}^n$-graded submodule. So

$$0 \to (I, J_1)/(J, J_1) \to (I, I_1)/(J, J_1) \to ((I, J_1)/(J, J_1))/((I, I_1)/(J, J_1)) \to 0,$$

is a short exact sequence. By third isomorphism theorem, we have

$$0 \to (I, J_1)/(J, J_1) \to (I, I_1)/(J, J_1) \to (I, J_1)/(I, I_1) \to 0,$$

is a short exact sequence. And by [22, Theorem 1.1], we have

$$sdepth_S(I, I_1)/(J, J_1) \geq min\{sdepth_S(I, J_1)/(J, J_1), sdepth_S(I, J_1)/(I, I_1)\}. \tag{3.2.4}$$

Now from Lemma 3.2.1, we have

$$\begin{aligned} sdepth_S(I, I_1)/(J, J_1) \geq & min\{sdepth_{S^{**}}\left(\frac{S^*}{I_1}\right) + sdepth_{S^*}\left(\frac{I}{J}\right), \\ & sdepth_{S^*}\left(\frac{S^*}{I}\right) + sdepth_{S^{**}}\left(\frac{I_1}{J_1}\right)\}. \end{aligned}$$

□

If $w \in S$. Then $supp(w) = \{x_i : x_i | w\}$ denote the support of monomial w.

Lemma 3.2.3. *[6, Lemma 2.3] Let $u_1, \ldots, u_p \in S$ and $v_1, \ldots, v_p \in S$ be two regular sequences of monomials such that $u_j | v_j$ and $v_i \neq u_i$ for some index i. Then*

$$sdepth_S((u_1, \ldots, u_p)/(v_1, \ldots, v_p)) = n - p.$$

Moreover, $(u_1, \ldots, u_p)/(v_1, \ldots, v_p)$ has a Stanley decomposition with all its Stanley spaces of dimension $n - p$.

Proof. We use induction on $p \geq 1$.

(1) If $p = 1$. By the short exact sequence,

$$0 \to S/(I : u) \to S/I \to S/(I, u) \to 0.$$

We have

$$(I, u)/I \cong S/(I : u). \tag{3.2.5}$$

Let $u = u_1$ and $I = v_1$. So from (3.2.5), we have

$$(u_1)/(v_1) \cong S/(v_1/u_1).$$

And therefore by [22, Theorem 1.1], we have

$$sdepth_S((u_1/(v_1)) \cong sdepth(S) - 1 = n - 1.$$

(2) Now assume $p > 1$. We apply Lemma 3.2.2 for $I = (u_1, \ldots, u_{p-1})$, $I_1 = (u_p)$ and $J = (v_1, \ldots, v_{p-1})$, $J_1 = (v_p)$. So we get

$$sdepth_S((u_1, \ldots, u_p)/(v_1, \ldots, v_p)) \geq min\{sdepth_{S^*}\left(\frac{S^*}{I}\right) + sdepth_{S^{**}}\left(\frac{I_1}{J_1}\right), \\ sdepth_{S^{**}}\left(\frac{S^{**}}{I_1}\right) + sdepth_{S^*}\left(\frac{I}{J}\right)\}. \quad (3.2.6)$$

Since, $sdepth(S/J) = sdepth_{S^*}(S^*/J) + sdepth(S^{**})$, and also by [22, Theorem 1.1], we have

$$sdepth_{S^{**}}(J_1/I_1) = sdepth(S^{**}) - 1.$$

So from equation (3.2.6), we have

$$sdepth_S((u_1, \ldots, u_p)/(v_1, \ldots, v_p)) \geq min\{sdepth_S(S/J) - 1, sdepth_S(S/I_1) - 1\}.$$

Thus

$$sdepth_S((u_1, \ldots, u_p)/(v_1, \ldots, v_p)) \geq n - (p-1) - 1 = n - p. \quad (3.2.7)$$

In order to prove the opposite inequality, let$uK[Z]$ be a Stanley space of $(u_1, \ldots, u_p)/(v_1, \ldots, v_p)$. Since $v_iS \cap uK[Z] = (0)$, it follows that there exist an index j_i such that $x_{j_i} \notin Z$. Since $v_1, \ldots, v_p$ is a regular sequence, so their support are disjoint and therefore, $\{x_{j1}, \ldots, x_{j_p}\}$ is a set of p variables, which do not belong to Z. Thus $|Z| \leq n - p$, so

$$sdepth_S((u_1, ..., u_p)/(v_1, ..., v_p)) \leq n - p. \quad (3.2.8)$$

Hence from equation (3.2.7) and (3.2.8), we have

$$sdepth_S((u_1, \ldots, u_p)/(v_1, \ldots, v_p)) = n - p.$$

Note that the inequality " $\leq$ " also follow from the inequalities

$$\begin{aligned} sdepth_S((u_1,\dots,u_p)/(v_1,\dots,v_p)) \leq & dim((u_1,\dots,u_p)/(v_1,\dots,v_p) \\ & \leq dim(S/(v_1,\dots,v_p)) = n-p. \end{aligned}$$

□

Theorem 3.2.4. *[6, Theorem 2.4] Let $J \subsetneq I \subset S$ be complete intersection monomial ideals. Assume I is generated by q monomials and J is generated by p monomials. Then*

$$n-p \geq sdepth(I/J) \geq n-p-\left\lfloor \frac{q-p}{2} \right\rfloor.$$

Proof. Assume $I = (u_1,\dots,u_q)$ and $J = (v_1,\dots,v_p)$. Since $v_1,\dots,v_p$ is a regular sequence on S, so their support are disjoint. If we take $uK[Z]$ be a Stanley space of I/J. It follows as in proof of Lemma 3.2.3 that $|Z| \leq n-p$, and thus

$$n-p \geq sdepth_S(I/J). \tag{3.2.9}$$

In order to prove the second inequality, we use induction on p.

(1) If $p=0$, then by [23, Theorem 2.4], we have

$$sdepth_S(I/J) = sdepth_S(I) = n - \left\lfloor \frac{q}{2} \right\rfloor.$$

(2) Let $p=q$. Then by 3.2.3, we have

$$sdepth_S(I/J) = n-p.$$

(3) Now assume that $1 \leq p < q$. Since $J \subset I$, so we can assume that $u_1|v_1$. Note that $u_1 \nmid v_i$ for all $i > 1$, because if $p \geq 2$ and also $u_1|v_2$. Then $supp(v_1) \cap supp(v_2) \supseteq supp(u_1)$. Which is contradiction to the fact that v_1 and v_2 is a regular sequence. Thus by using induction, we may assume that $u_j|v_j,\ for\ all\ 1 \leq j \leq p$. We denote $I_1 = (u_1,\dots,u_p)$ and $I_2 = (u_{p+1},\dots,u_q)$. Since $I_1/J \subseteq (I_1,I_2)/J$ is a $\mathbb{Z}^n$-graded submodule, so

$$0 \to I_1/J \to (I_1,I_2)/J \to ((I_1,I_2)/J)/(I_1/J) \to 0,$$

is a short exact sequence. Thus

$$(I_1, I_2)/J = (I_1, I_2)/I_1 \bigoplus I_1/J.$$

We have

$$sdepth_S(I_1, I_2)/J \geq min\{sdepth(I_1, I_2)/I_1), sdepth(I_1/J)\}. \tag{3.2.10}$$

Using [5, Corollary 2.4], we have

$$sdepth_S((I_1, I_2)/I_1) \geq sdepth_S(S/I_1) - \left\lfloor \frac{q-p}{2} \right\rfloor.$$

Also by [22, Theorem(1.1)], we have from above

$$sdepth_S((I_1, I_2)/J) \geq n - p - \left\lfloor \frac{q-p}{2} \right\rfloor. \tag{3.2.11}$$

On the other hand if $J \subsetneq I_1$, so by Lemma 3.2.3 we have,

$$sdepth_S(I_1/J) = n - p. \tag{3.2.12}$$

Put equation (3.2.11) and (3.2.12) in equation (3.2.10), we have

$$sdepth_S((I_1, I_2)/J) \geq min\{n - p - \left\lfloor \frac{q-p}{2} \right\rfloor, n - p\}.$$

Thus

$$sdepth_S((I_1, I_2)/J) \geq n - p - \left\lfloor \frac{q-p}{2} \right\rfloor. \tag{3.2.13}$$

Hence from equation (3.2.9) and (3.2.13), we have

$$n - p \geq sdepth_S(I/J) \geq n - p - \left\lfloor \frac{q-p}{2} \right\rfloor.$$

□

Corollary 3.2.5. *[6, Corollary 2.5] Let $J \subsetneq I \subset S$ be monomial complete intersection ideals. Assume I is generated by $q = p + 1$ monomials and J is generated by p monomials. Then*

$$sdepth_S(I/J) = n - p.$$

Corollary 3.2.6. *[6, Corollary 2.6] If $J \subsetneq I \subset S$ be complete intersection monomial ideals, then*

$$sdepth_S(I/J) \geq depth_S(I/J).$$

Proof. It is enough to notice that $depth_S(I/J) = n-q+1$ if $q > q$, or $depth_S(I/J) = n-q$, if $q = p$ and then apply Theorem 3.2.5. $\square$

Lemma 3.2.7. *[6, Lemma 2.7] Let $1 \leq p < q$ be integer, $J \subsetneq I \subset S^* = K[x_1, \ldots, x_p]$ be distinct monomial ideals and let $u \in S^{**} = K[x_{p+1}, \ldots, x_q]$ be a monomial. Then*

$$sdepth_S(I,u)/(J,u) = sdepth_S(IS/JS) - 1.$$

Proof. Let $v \in (I,u)/(J,u)$. Then we write $v = v^{'}.v^{''}$, where $v^{'} \in (I/J)$ and $v^{''} \in S^{**}$. Since $v \in (I,u)$, so $v \in IS$ and $v \in uS$, and thus $v^{'} \in I$ and $v^{''} \in S^{**}$. Also since $v \notin (J,u)$, so $v \notin JS$ and $v \notin uS$. Thus $v^{'} \notin J$ and $v^{''} \notin (u)$. So we have $v^{'} \in (I/J)$ and $v^{''} \in S^{**}/(u)$. Thus $v \in (I/J).(S^{**}/(u))$ and

$$(I,u)/(J,u) \subseteq (I/J).(S^{**}/(u)). \tag{3.2.14}$$

Conversely, If we take arbitrary monomials $v^{'} \in (I/J)$ and $v^{''} \in S^{**}/(u)$. Since $v^{'} \in I$, so $v^{'} \in (I,u)$. Also $v^{'} \notin J$, so $v^{'} \notin (J,u)$. Thus $v^{'} \in (I,u)/(J,u)$. Since $(I,u)/(J,u) \subseteq S/(J,u)$ is an ideal of $S/(J,u)$, so by definition of ideal, for $v^{''} \in (I,u)/(J,u)$ and $v^{''} \in S/(J,u)$, we have $v^{'}.v^{''} \in (I,u)/(J,u)$, so

$$(I/J).(S^{**}/(u)) \subseteq (I,u)/(J,u) \tag{3.2.15}$$

From equation (3.2.14) and (3.2.15), we have

$$(I,u)/(J,u) = (I/J).(S^{**}/(u)). \tag{3.2.16}$$

Hence by Lemma 3.2.1, we have

$$sdepth_S(I,u)/(J,u) \geq sdepth_{S^*}(I/J) + sdepth_{S^{**}}(S^{**}/(u)). \tag{3.2.17}$$

Since, $sdepth_{S^{**}}(S^{**}/(u)) = sdepth(S^{**}) - 1 = q - p - 1$. Hence from equation (3.2.16), we have

$$sdepth_S(I,u)/(J,u) \geq sdepth_{S^*}(I/J) + n - m - 1.$$

Thus

$$sdepth_S(I,u)/(J,u) \geq sdepth_S(IS/JS) - 1. \tag{3.2.18}$$

In order to prove the other in-equality, Let $(I,u)/(J,u) = \bigoplus_{j=1}^{r} v_j K[Z_j]$ be a Stanley decomposition with its Stanley depth equal with $sdepth_S((I,u)/(J,u))$. Also $I/J = ((I,u)/(J,u)) \cap (S^*/J)$, via natural injection $I/J \hookrightarrow (I,u)/(J,u)$, so

$$I/J = (\bigoplus_{j=1}^{r} v_j K[Z_j]) \cap (S^*/J) = \bigoplus_{j=1}^{r} (v_j K[Z_j] \cap (S^*/J)). \tag{3.2.19}$$

If $v_j \notin S^*$, then $v_j K[Z_j] \cap (S^*/J) = \{0\}$, and if $v_j \in S^*$, then $v_j K[Z_j] \cap (S^*/J) = v_j K[Z_j \backslash \{x_{p+1}, \ldots, x_q\}]$. Note that $\{x_{p+1}, \ldots, x_q\} \notin Z_j$, because $uS \cap v_j K[Z_j] = \{0\}$. Thus $|Z_j \backslash \{x_{p+1}, \ldots, x_q\}| \geq |Z_j| - \{q-p-1\}$. Hence from equation (3.2.19), we have

$$sdepth_{S^*}(I/J) \geq sdepth_S(I,u)/(J,u) - q + p + 1.$$

It follows that

$$sdepth_S(I,u)/(J,u) \leq sdepth_S(IS/JS) - 1. \tag{3.2.20}$$

Hence from equation (3.2.18) and (3.2.20), we have

$$sdepth_S(I,u)/(J,u) = sdepth_S(IS/JS) - 1.$$

□

Lemma 3.2.8. *[6, Lemma 2.8] Let $1 \leq p < q$ be integer, $I \subset S^* = K[x_1, \ldots, x_p]$ be a monomial ideal and let $u, v \in S^{**} = K[x_{p+1}, \ldots, x_q]$ be distinct monomials with $u|v$. Then*

$$sdepth_S(I,u)/(I,v) = sdepth_S(S/IS) - 1.$$

Proof. Let $w \in (I,u)/(I,v)$ be a monomial. Then we write $w = w^{'}.w^{''}$, where $w^{'} \in S^*$ and $w^{''} \in S^{**}$. Since $w \in (I,u)$, so $w \in IS$ and $w \in uS$, thus $w^{'} \in S^*$ and $w^{''} \in (u)$. Also Since $w \notin (I,v)$, so $w \notin IS$ and $w \notin vS$. Thus $w^{'} \notin I$ and $w^{''} \notin (v)$, so we get $w^{'} \in (S^*/I)$ and $w^{''} \in (u)/(v)$. Thus

$$(I,u)/(I,v) \subseteq (S^*/I).((u)/(v)) \tag{3.2.21}$$

Conversely, let us take an arbitrary monomials, $w^{'} \in (S^*/I)$, and $w^{''} \in (u)/(v)$. Since $w^{''} \in (u)$, so $w^{''} \in (I, u)$, also since $w^{''} \notin (v)$, so $w^{''} \notin (I, v)$. Therefore $w^{''} \in (I, u)/(I, v)$, since $(I, u)/(I, v) \subseteq S/(I, v)$ is an ideal, so by definition of ideal for $w^{'} \in S/(I, v)$ and $w^{''} \in (I, u)/(I, v)$, we have $w^{'}.w^{''} \in (I, u)/(I, v)$. Therefore

$$(S^*/I).((u)/(v)) \subseteq (I, u)/(I, v). \tag{3.2.22}$$

Hence from equation (3.2.21) and (3.2.22), we have

$$(I, u)/(I, v) \cong (S^*/I).((u)/(v)).$$

By Lemma 3.2.1 ,we have

$$sdepth_S(I, u)/(I, v) \geq sdepth_{S^*}(S^*/I) + sdepth_{S^{**}}((u)/(v)). \tag{3.2.23}$$

Since, $sdepth_{S^{**}}((u)/(v)) = sdepth(S^{**}) - 1 = n - m - 1$. Hence from equation (3.2.23), we have

$$sdepth_S(I, u)/(I, v) \geq sdepth_S(S/IS) - 1. \tag{3.2.24}$$

In order to prove other inequality, note that

$$u(S^*/I) = (I, u)/(I, v) \cap u(S^*/I), \tag{3.2.25}$$

via natural injection $u(S^*/I) \hookrightarrow (I, u)/(I, v)$. Thus by using similar argument as in proof of Lemma 3.2.7 ,we have

$$sdepth_S(I, u)/(I, v) \leq sdepth_S(S/SI) - 1. \tag{3.2.26}$$

Hence from equation (3.2.24) and (3.2.26), we have

$$sdepth_S(I, u)/(I, v) = sdepth_S(S/IS) - 1.$$

□

Theorem 3.2.9. *[6, Theorem 2.9] Let $u_1, \ldots, u_q \in S$ and $v_1, \ldots, v_p \in S$ be two regular sequences with $u_i|v_i$ for all $1 \leq i \leq p$, where $q \geq p$ are positive integers. We consider the monomial ideals $I = (u_1, \ldots, u_q) \subset S$ and $J = (v_1, \ldots, v_p) \subset S$. We also assume that $u_{p+1}, \ldots, u_q$ is a regular sequence on S/J. Then*

$$sdepth(I/J) = n - p - \left\lfloor \frac{q-p}{2} \right\rfloor.$$

Proof. We use induction on p.

(a) If $p = 0$. Then by [23], and [9, Lemma 3.6], we have

$$sdepth_S(I/J) = sdepth_S(J) = n - \left\lfloor \frac{q}{2} \right\rfloor.$$

(b) If $p = q$. Then by Lemma 3.2.2, we get

$$sdepth_S(I/J) = n - p.$$

(c) Now assume $1 \leq p < q$, we denote $I_1 = (u_2, \ldots, u_q)$ and $J_1 = (v_2, v_3, \ldots, v_p)$. By induction hypothesis, we have

$$sdepth_S(I_1/J_1) = n - (p-1) - \left\lfloor \frac{(q-1)-(p-1)}{2} \right\rfloor.$$

Thus

$$sdepth_S(I_1/J_1) = n - p - \left\lfloor \frac{q-p}{2} \right\rfloor + 1. \tag{3.2.27}$$

Here we take two cases:

Cases(1) If $u_1 = v_1$, then by Lemma 3.2.7, we have

$$sdepth_S(I_1, u_1)/(J_1, v_1) = sdepth_S(I_1S/J_1S) - 1.$$

Now from equation (3.2.27), we have

$$sdepth_S(I/J) = n - p - \lfloor \frac{q-p}{2} \rfloor + 1 - 1.$$

Thus

$$sdepth_S(I/J) = n - p - \left\lfloor \frac{q-p}{2} \right\rfloor. \tag{3.2.28}$$

Cases(2) If $u_1 \neq v_1$. By Theorem 3.2.6, we have

$$sdepth_S(I/J) \geq n - p - \lfloor \frac{q-p}{2} \rfloor.$$

Note that this inequality also deduced from Lemma 3.2.7, and Lemma 3.2.8 using the decomposition

$$I/J = (I_1, u_1)/(J_1, u_1) \bigoplus (J_1, u_1)/(J_1, v_1).$$

We have

$$sdepth_S(I/J) \geq min\{sdepth_S(I_1,u_1)/(J_1,u_1), sdepth_S(J_1,u_1)/(J_1,v_1)\}. \tag{3.2.29}$$

From equation (3.2.28), we have

$$sdepth_S(I/J) \geq min\{n-p-\left\lfloor \frac{q-p}{2} \right\rfloor, sdepth(S/I_1S)-1\}$$

$$sdepth_S(I/J) \geq min\{n-p-\left\lfloor \frac{q-p}{2} \right\rfloor, n-p\}$$

$$sdepth_S(I/J) \geq n-p-\left\lfloor \frac{q-p}{2} \right\rfloor. \tag{3.2.30}$$

In order to prove the other inequality, we consider a Stanley decomposition $I/J = \bigoplus_{j=1}^{r} w_j K[Z_j]$ with its Stanley depth equal to $sdepth(I/J)$. Since by hypothesis, $u_2, u_3, \ldots, u_q$ is a regular sequence on $S/(v_1)$, so by reordering of variables, we may assume that $supp(v_1) = \{x_{m+1}, x_{m+2}, \ldots, x_n\}$ and $u_2, u_3, \ldots, u_q \in S^* = K[x_1, x_2, \ldots, x_m]$, where $1 \leq m < n$ is an integer. Thus I_1 and J_1 are the extension in S for some monomial ideals in $\bar{I}_1$, $\bar{J}_1 \subset S^*$ generated by the same monomials as I_1 and J_1. Note that

$$(\bar{I}_1/\bar{J}_1) = (I/J) \cap (S^*/\bar{J}_1),$$

where we regard $S^*/\bar{J}_1$ as a submodule of S/J. Using the same argument as in the proof of Lemma 3.2.7, we get

$$sdepth_{S^*}(\bar{I}_1/\bar{J}_1) \geq sdepth_S(I/J) - n + m + 1. \tag{3.2.31}$$

On the other hand, by induction hypothesis, we have

$$sdepth_{S^*}(\bar{I}_1/\bar{J}_1) = m-p+1-\left\lfloor \frac{q-p}{2} \right\rfloor.$$

So from equation (3.2.30), we have

$$sdepth_S(I/J) \leq n-p-\left\lfloor \frac{q-p}{2} \right\rfloor. \tag{3.2.32}$$

Hence from (3.2.30) and (3.2.32), we have

$$sdepth_S(I/J) = n-p-\left\lfloor \frac{q-p}{2} \right\rfloor.$$

□

Remark 3.2.1. [6, Remark 2.10] Note that the hypothesis $u_{p+1}, u_{p+2}, \ldots, u_q$ is a regular sequence on S/I from the Theorem 3.2.9 is essential in order to have the equality. Take for instance, $I = (x_1, x_2, x_3) \subset S$ and $J = (x_1x_2x_3)$. Then

$$I/J = x_1K[x_1, x_2] \oplus x_2K[x_2, x_3] \oplus x_3K[x_1, x_3],$$

is a Stanley decomposition for I/J, and therefore

$$sdepth_S(I/J) = 2 > 3 - 1 - \left\lfloor \frac{3-1}{2} \right\rfloor = 1.$$

Proposition 3.2.10. *[6, Proposition 2.11]*
Let $J \subset I \subset S$ be monomial ideals and let $v \in S$ be a monomial. Then either $(J : v) = (I : v)$, either

$$sdepth_S(I : v)/(J : v) \geq sdepth_S(I/J).$$

Proof. It is enough to consider the case $v = x_1$ and to assume that $(J : x_1) \subsetneq (I : x_1)$. Firstly, note that $x_1(I : x_1) = I \cap (x_1)$ and $x_1(J : x_1) = J \cap (x_1)$. Therefore we have

$$(I : x_1)/(J : x_1) \cong (I \cap (x_1))/(J \cap (x_1)) \cong (I/J) \cap (x_1),$$

as multigraded K-vector spaces. Let $I/J = \bigoplus_{i=1}^{r} u_iK[Z_i]$ be a Stanley decomposition for I/J. It follows that

$$(I/J) \cap (x_1) = \bigoplus_{i=1}^{r} (u_iK[Z_I] \cap x_1S).$$

One can easily see that, if $x_1 \notin supp(u_i) \cup Z_i$, then $u_iK[Z_i] \cap x_1S = \{0\}$. Otherwise, we claim that $u_iK[Z_i] \cap x_1S = LCM(u_i, x_1)K[Z_i]$. Indeed, if $x_1|u_i$, then $u_iK[Z_i] \subset x_1S$ and the previous equality holds. If $x_1 \nmid u_i$, then $x_1 \in Z_i$ and $LCM(u_i, x_1) = x_1u_i$. Obviously, we get

$$x_1u_iK[Z_i] \subseteq u_iK[Z_i] \cap x_1S. \tag{3.2.33}$$

For the other inclusion, choose $v \in u_iK[Z_i] \cap x_1S$ a monomial. It follows that $v \in u_iK[Z_i]$ and $x_1|v$, so $x_1u_i|v$, since $x_1 \nmid u_i$. Therefore $v \in x_1u_iK[Z_i]$, so

$$u_iK[Z_i] \cap x_1S \subseteq x_1u_iK[Z_i]. \tag{3.2.34}$$

Hence from equation (3.2.33) and (3.2.34), we have

$$u_i K[Z_i] \cap x_1 S = x_1 u_i K[Z_i].$$

By our assumption that $(J : x_1) \subsetneq (I : x_1)$, there exist some i such that $u_i K[Z_i] \cap x_1 S \neq \{0\}$. Thus we obtain a Stanley decomposition for $(I/J) \cap (x_1)$ with its Stanley depth $\geq$ than the Stanley depth of the given decomposition for I/J.

$\square$

Chapter 4

Bounds for the Stanley depth of quotient of some monomial ideals

In this chapter, we show that lower bound exist for quotient of monomial ideal (m/I), where m is a maximal ideal and $I \subset S = K[x_1, \ldots, x_n]$ is any ideal. We also give some good bounds for quotient of monomial ideals $J \subsetneq I \subset S = K[x_1, \ldots, x_n]$, where I and J is the intersection of some prime ideals.

4.1 Bounds for the quotient of maximal ideals

Let $I \subsetneq m \subset S = K[x_1, \ldots, x_n]$ be monomial ideals, where $m = (x_1, \ldots, x_n)$ is a maximal ideal. Let us suppose

$$m \cap x_n^k S' = x_n^k m_k.$$

Where $m_k \subset S' = K[x_1, \ldots, x_{n-1}]$ be a monomial ideals, and we have an inclusion of the form

$$m_0 \subseteq m_1 \subseteq m_2 \subseteq \ldots$$

Also let us suppose

$$I \cap x_n^k S' = x_n^k I_k.$$

Where $I_k \subset S'$ be a monomial ideals and we have an inclusion of the form

$$I_0 \subseteq I_1 \subseteq I_2 \subseteq \ldots$$

Lemma 4.1.1. *Let m be a maximal ideal and $I \subset S = K[x_1, \ldots, x_n]$ be a monomial ideal. Then*

$$m/I \cong m_0/I_0 \oplus x_n(S^{'}/I_1) \oplus \ldots \oplus x_n^{q-1}(S^{'}/I_{q-1}) \oplus \ldots$$

Proof. Let us consider

$$m/I \cap x_n^k S^{'} = x_n^k(m_k/I_k).$$

Then by Janet's algorithm, we have a decomposition in form

$$m/I \cong (m_0/I_0) \oplus x_n(m_1/I_1) \oplus x_n^2(m_2/I_2) \oplus \ldots \oplus x_n^{q-1}(m_{q-1}/I_{q-1}) \oplus \ldots \quad (4.1.1)$$

Since m contain variable x_n. So $m_1 = m_2 = m_3 \ldots = S^{'}$. Hence from equation (4.1.1), we have

$$m/I \cong m_0/I_0 \oplus x_n(S^{'}/I_1) \oplus \ldots \oplus x_n^{q-1}(S^{'}/I_{q-1}) \oplus \ldots$$

□

Example 4.1.1. Let $m = (x_1, x_2, x_3, x_4)$ is a maximal ideal and $I = (x_1^2x_2, x_2x_3^2, x_1^2x_4^2)$ is a monomial ideal on $S = K[x_1, x_2, x_3, x_4]$. Then by using Lemma 3.1.1, we have

$$q = degx_4(I) = 2$$

Also by Janet's algorithm , we have
$m_0 = (x_1, x_2, x_3)$, $m_1 = S^{'}$, $m_2 = S^{'}$, $m_3 = S^{'}$ and $I_0 = (x_1^2x_2, x_2x_3^2)$, $I_1 = (x_1^2x_2, x_2x_3^2)$, $I_2 = (x_1^2, x_2x_3^2)$, $I_3 = (x_1^2, x_2x_3^2)$. Clearly

$$m_0 \subseteq m_1 \subseteq m_2 \subseteq \ldots$$

and

$$I_0 \subseteq I_1 \subseteq I_2 \subseteq \ldots$$

Thus we have a decomposition of the form

$$m/I \cong (x_1, x_2, x_3)/(x_1^2x_2, x_2x_3^2) \oplus x_4(S^{'}/(x_1^2x_2, x_2x_3^2)) \oplus x_4^2(S^{'}/(x_1^2, x_2x_3^2))[x_4].$$

Proposition 4.1.2. *Let I be a monomial ideal and $m = (x_1, \dots, x_n) \subset S = K[x_1, \dots, x_n]$ a maximal ideal. Then*

$$sdepth_S(m/I) \geq n - |G(I)| - \left\lfloor \frac{n - |G(I)|}{2} \right\rfloor.$$

Proof. By induction and by Lemma 4.1.1, we have

$$sdepth_S(m/I) \geq n - |G(I)| - \left\lfloor \frac{n - |G(I)|}{2} \right\rfloor.$$

$\square$

Example 4.1.2. Let $m = (x_1, x_2, x_3, x_4)$ is a maximal ideal and $I = (x_1x_2, x_3x_4)$ is monomial ideal. Then

$$m/I = \{x_1, x_2, x_3, x_4, x_1x_3, x_1x_4, x_2x_3, x_2x_4\}.$$

Corresponding Stanley decompositions of m/I is as follow:

$$\mathcal{D}_1 : m/I = x_1K[x_1, x_4] \oplus x_2K[x_2, x_3] \oplus x_3K[x_1, x_3] \oplus x_4K[x_2, x_4]$$

$$\mathcal{D}_2 : m/I = x_1K[x_1, x_3] \oplus x_2K[x_2, x_4] \oplus x_3K[x_2, x_3] \oplus x_4K[x_1, x_4].$$

So by cocoa

$$sdepth_S(m/I) = 2.$$

And by using Proposition 4.1.2, we have

$sdepth_S(m/I) \geq 4 - 2 - \left\lfloor \frac{4-2}{2} \right\rfloor$. Thus

$$sdepth_S(m/I) \geq 1.$$

Which is a good Lower bound to $sdepth(m/I)$.

4.2 Stanley depth of quotient of intersection of monomial ideals

Lemma 4.2.1. *Let $S = K[x_1, \dots, x_n]$ be a polynomial ring and P_1, P_2, P_3, P_4 a monomial prime ideals such that $G(P_i) \cap G(P_j) = \emptyset$, for all $i \neq j$. Let $ht(P_i) = r_i$, $\sum_{i=1}^{4} r_i = n$, $J = P_1 \cap P_2 \cap P_3$ and $I = P_1 \cap P_2 \cap P_3 \cap P_4$, where $J \subsetneq I \subset S$. Then*

$$sdepth_S(I/J) \geq \left\lceil \frac{r_1}{2} \right\rceil + \left\lfloor \frac{r_2}{2} \right\rfloor + \left\lceil \frac{r_3}{2} \right\rceil.$$

Proof. By Second isomorphism theorem, we have

$$(P_1 \cap P_2 \cap P_3)/(P_1 \cap P_2 \cap P_3 \cap P_4) \cong (P_1 \cap P_2 \cap P_3 + P_4)/P_4. \tag{4.2.1}$$

And also
$(P_1 \cap P_2 \cap P_3 + P_4)/P_4 \cong (P_1 \cap P_2 \cap P_3) \cap K[x_j : x_j \notin G(P_4)]$.

Now by using [18, Lemma 1.2], we have

$$\begin{aligned} sdepth_S(P_1 \cap P_2 \cap P_3) \cap K[x_j : x_j \notin G(P_4)] \geq & sdepth_{S'}(P_1 \cap P_2 \cap P_3) + \\ & sdepth_{S''}(K[x_j : x_j \notin G(P_4)]). \end{aligned}$$

Again by using [18, Lemma 1.2], we have

$$\begin{aligned} sdepth_S(P_1 \cap P_2 \cap P_3) \cap K[x_j : x_j \notin G(P_4)] \geq & sdepth_{S_1}(P_1) + \ldots + sdepth_{S_4}(P_3) + \\ & sdepth_{S''}(K[x_j : x_j \notin G(P_4)]). \end{aligned}$$

Now by using [9, Lemma 3.6], we have

$$sdepth_S(P_1 \cap P_2 \cap P_3) \cap K[x_j : x_j \notin G(P_4)] \geq \left\lceil \frac{r_1}{2} \right\rceil + \left\lfloor \frac{r_2}{2} \right\rfloor + \left\lceil \frac{r_3}{2} \right\rceil.$$

Hence from equation (4.2.1), we have

$$sdepth_S(P_1 \cap P_2 \cap P_3)/(P_1 \cap P_2 \cap P_3 \cap P_4) \geq \left\lceil \frac{r_1}{2} \right\rceil + \left\lfloor \frac{r_2}{2} \right\rfloor + \left\lceil \frac{r_3}{2} \right\rceil.$$

□

Proposition 4.2.2. *Let $S = K[x_1, \dots, x_n]$ be a polynomial ring $\boldsymbol{m}$ a maximal ideal on S and $P_1, P_2, \dots, P_5$ a monomial prime ideals of S such that $G(P_i) \cap G(P_j) = \emptyset$, for all $i \neq j$. Let $ht(P_i) = r_i$, $\sum_{i=1}^{5} r_i = n$, if $I = P_1 \cap P_2 \cap \dots \cap P_5$. Then*

$$sdepth_S(m/I) \geq min\{n - r_1 - \left\lfloor \frac{n - r_1}{2} \right\rfloor, min\{\left\lceil \frac{r_1}{2} \right\rceil + r_3 + r_4 + r_5, \left\lceil \frac{r_1}{2} \right\rceil + \left\lfloor \frac{r_2}{2} \right\rfloor + r_4 + r_5, \left\lfloor \frac{r_1}{2} \right\rfloor + \left\lfloor \frac{r_2}{2} \right\rfloor + \left\lfloor \frac{r_3}{2} \right\rfloor + r_5, \left\lfloor \frac{r_1}{2} \right\rfloor + \left\lfloor \frac{r_2}{2} \right\rfloor + \left\lfloor \frac{r_3}{2} \right\rfloor + \left\lfloor \frac{r_4}{2} \right\rfloor\}\}.$$

Proof. As a K-Linear space m/I is isomorphic to the direct sum of some multigraded modules as

$$m/I \cong m/P_1 \oplus P_1/(P_1 \cap P_2) \oplus \dots \oplus (P_1 \cap \dots \cap P_4)/(P_1 \cap \dots \cap P_5).$$

$$\begin{aligned} sdepth_S(m/I) \geq & min\{sdepth(m/P_1), sdepth(P_1/P_1 \cap P_2), \dots, \\ & sdepth(P_1 \cap \dots \cap P_4)/(P_1 \cap \dots \cap P_5)\}. \end{aligned}$$

Now By [6, Corollary 1.2], we have

$$sdepth_S(m/I) = n - r_1 - \left\lfloor \frac{n - r_1}{2} \right\rfloor. \tag{4.2.2}$$

And also by Lemma 4.2.1, we have

$$sdepth_S(P_1 \cap \dots \cap P_4)/(P_1 \cap \dots \cap \dots \cap P_5) \geq \left\lceil \frac{r_1}{2} \right\rceil + \left\lceil \frac{r_2}{2} \right\rceil + \left\lfloor \frac{r_3}{2} \right\rfloor + \left\lfloor \frac{r_4}{2} \right\rfloor. \tag{4.2.3}$$

Put equations (4.2.2) and (4.2.3) in above inequality, we have

$$sdepth_S(m/I) \geq min\{n - r_1 - \left\lfloor \frac{n - r_1}{2} \right\rfloor, min\{\left\lceil \frac{r_1}{2} \right\rceil + r_3 + r_4 + r_5, \left\lceil \frac{r_1}{2} \right\rceil + \left\lfloor \frac{r_2}{2} \right\rfloor + r_4 + r_5, \left\lfloor \frac{r_1}{2} \right\rfloor + \left\lfloor \frac{r_2}{2} \right\rfloor + \left\lfloor \frac{r_3}{2} \right\rfloor + r_5, \left\lfloor \frac{r_1}{2} \right\rfloor + \left\lfloor \frac{r_2}{2} \right\rfloor + \left\lfloor \frac{r_3}{2} \right\rfloor + \left\lfloor \frac{r_4}{2} \right\rfloor\}\}.$$

□

Lemma 4.2.3. *Let $S = K[x_1, \dots, x_n]$ be a polynomial ring and P_1, P_2, P_3, P_4 be a monomial prime ideals such that $G(P_1) \cap G(P_j) = \emptyset$, for all $i \neq j$. Let $ht(P_i) = r_i$, $\sum_{i=1}^{4} r_i = n$, if $J = P_1 \cap P_2 \cap P_3 \cap P_4$ and $I = P_1 \cap P_2 \cap P_3$, where $J \subsetneq I \subset S$. Then*

$$sdepth_S(I/J) \leq \left\lceil \frac{r_1}{2} \right\rceil + r_3 + r_4.$$

Proof. Let $u \in S$ be a monomial. Then by [6, Proposition 2.11], we have

$$sdepth_S(P_1 \cap P_2 \cap P_3)/(P_1 \cap P_2 \cap P_3 \cap P_4) \leq sdepth_S(P_1 \cap P_2)/(P_1 \cap P_2 \cap P_3).$$

Again applying [6, Proposition 2.11], we have

$$sdepth_S(P_1 \cap P_2)/(P_1 \cap P_2 \cap P_3) \leq sdepth_S(P_1/P_1 \cap P_2). \tag{4.2.4}$$

By second isomorphism theorem, we have

$$(P_1/P_1 \cap P_2) \cong (P_1 + P_2)/P_2.$$

Also

$$(P_1 + P_2)/P_2 \cong P_1 \cap K[x_j : x_j \notin G(P_1)].$$

So we have

$$(P_1/P_1 \cap P_2) \cong P_1 \cap K[x_j : x_j \notin G(P_1)]. \tag{4.2.5}$$

By [9, Lemma 3.6] and [18, Lemma 1.2], we have

$$sdepth_S(P_1 \cap K[x_j : x_j \notin G(P_1)] = \left\lceil \frac{r_1}{2} \right\rceil + r_3 + r_4.$$

So from equation (4.2.5), we have

$$sdepth_S(P_1/P_1 \cap P_2) = \left\lceil \frac{r_1}{2} \right\rceil + r_3 + r_4.$$

Hence from equation (4.2.4), we have

$$sdepth_S(P_1 \cap P_2 \cap P_3)/(P_1 \cap P_2 \cap P_3 \cap P_4) \leq \left\lceil \frac{r_1}{2} \right\rceil + r_3 + r_4.$$

□

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