On Existence of Solutions for Impulsive Fractional Differential Equations

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Dedicated to

My Parents

&

Sisters

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Abstract

The theory of fractional impulsive differential equations is used for modeling some real world processes in different areas of science and technology. The motivation behind this work is to establish the existence results of fractional impulsive differential equations. Both initial value problems and boundary value problems have been discussed in this work.

We study some priliminary facts to establish the solution of fractional impulsive differential equations. We prove the existence and uniqueness results of the solution of both initial and boundary value problems. Then we present Ulam's type stability for fractional impulsive differential equations. Finally, we study the existence result of multi-point boundary value problems for fractional order impulsive differential equations. We have also prove that positive solution of this multi-point boundary value problem exist under certain conditions on the nonlinear functions.

Contents

Chapter 1

Introduction

The fractional differential equations have proved to be important tools in the modeling of various processes in the fields of engineering, science, physics and economics. On the other hand, the study of impulsive differential equations is also an important field of research which has been investigated by a number of mathematician in the recent years. These equations better model phenomena and dynamical processes which involves great changes in short times, for example, in Physics, Biotechnology and Robotics. To understand more about the recently used techniques for these type of problems we refer the books [8, 9]. We will study the impulsive fractional differential equations from theoretical aspects of existence and uniqueness of solutions. We will also establish the positive solutions for impulsive fractional multi-point boundary value problems.

In Chapter 1, we describe the preliminary facts which are necessary to establish the existence of solution and also positive solution for impulsive fractional differential equations. In first section, we explain Gamma function, history of fractional calculus and its applications. In second section, we present fractional derivatives, integrals and impulsive differential equations with an application. Then in the last section we give some definitions and results from analysis.

In Chapter 2, we discuss some results for the existence and uniqueness of solutions for impulsive fractional initial value problem

$$\begin{aligned} &{}^cD^{\alpha}y(t) = f(t,y), \quad t \neq t_k, \quad 0 < \alpha \leq 1, \\ &y(t_k+0) - y(t_k) = I_k(y(t_k)), \quad k = 1, 2, \cdots, n, \\ &y(t_0) = y_0, \end{aligned} \tag{1.0.1}$$

and for impulsive fractional boundary value problem

$$\begin{aligned} {}^cD^{\alpha}y(t) = f(t,y(t)) \quad &\text{for} \quad t \in [0,1], \quad 1 < \alpha \leq 2, \\ \Delta y(t_k) &= P_k(y(t_k)), \\ \Delta y'(t_k) &= Q_k(y(t_k)), \\ y(0) =& g(y), \quad y(1) = k, \quad k \in \mathbb{R}. \end{aligned} \tag{1.0.2}$$

where ${}^cD^\alpha$ is the Caputo fractional derivative, $f:[0,1]\times\mathbb{R}\to\mathbb{R}$, is a continuous function, $g: C([0,1],\mathbb{R})\to\mathbb{R}$ a continuous function, $P_k, Q_k:\mathbb{R}\to\mathbb{R}$, $k=1,\cdots,n$, $0=t_0<t_1<\cdots<t_n<t_{n+1}=1$. In first section we will use Schaefer's fixed point theorem to establish the existence result for (1.0.1) then we will discuss its uniqueness, continuity and bounds of solution. In second section of this chapter we will present two results for existence and uniqueness of solution of (1.0.2).

In Chapter 3, we will establish the Ulam's type stability of the following fractional impulsive differential equations

$$\begin{aligned} &{}^cD^\alpha y(t)=f(t,y), \quad t\in J'=J\setminus\{t_1,\cdots,t_m\},\ J=[0,T],\ T>0, \quad 0<\alpha\le 1,\\ &y(t_k+0)-y(t_k)=I_k(y(t_k)), \quad k=1,2,\cdots,n. \end{aligned} \tag{1.0.3}$$

We will also show that equation (1.0.3) is generalized Ulam-Hyers-Rassias stable.

Finally in Chapter 4, we study positive solutions for fractional impulsive boundary value problem

$$D^\alpha y(t)+\lambda g(t,y(t))=0, \quad t\in J,\ t\neq t_k, 1<\alpha\le 2, \tag{1.0.4}$$

$$\Delta y'(t_k)+I_k(y(t_k))=0, \quad k=1,2,\cdots,n \tag{1.0.5}$$

$$ay(0)-by'(0)=\sum_{i=1}^{m-2}a_i y(\xi_i), \quad cy(1)+dy'(1)=\sum_{i=1}^{m-2}b_i y(\xi_i), \tag{1.0.6}$$

where $J=[0,1]$, $t_0=0<t_1<t_2<\cdots<t_n<t_{n+1}=1$, $\Delta y'(t_k)=y'(t_k^+)-y'(t_k^-)$, $y(t_k^+)$ and $y(t_k^-)$ represents the right hand limit and left hand limit of the function $y(t)$ at $t=t_k$, $0<\xi_i<\xi_{i+1}<1$, $\xi_i\neq t_k$ and $i=1,2,\cdots,m-3$. In the first section we construct fixed point operator and Green function. We also establish some interesting properties of the Greens function. We construct a cone and then define a completely continuous map by Green's functions. In the second section of this chapter we establish the necessary conditions for the existence of at least one positive solution. Also we establish the interval for parameter λ for which there exists at least one solution of the boundary value problem (1.0.4)-(1.0.6).

Chapter 2

Preliminaries

2.1 Introduction

In this chapter, we recall some basic definitions and known results which are necessary to establish the existence, uniqueness and stability of solution of a fractional impulsive boundary value problem. The Gamma function, fractional derivatives and integrals, impulsive differential equations, fractional differential equations and Krasnoselskii's cone fixed point theorem are discussed in detail. The basic idea of Krasnoselskii's cone fixed point theorem is to establish the positive solution of boundary value problem. Several examples will be provided to explain the given results.

2.1.1 The Gamma function

One of the basic functions of fractional calculus is Euler's Gamma function, which is actually the generalization of factorial.
For $z > 0$, the Gamma function is defined by the formula

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} dt. \tag{2.1.1}$$

The Gamma function given by (2.1.1) is uniformly convergent for all z in $[a, b]$ where $0 < a \leq b < \infty$, and hence , $\Gamma(z)$ is a continuous function for all $z > 0$.
Properties
Some basic properties of gamma function are:

(i)

$$\Gamma(z+1) = z\Gamma(z) \quad z > 0 \text{ as}, \tag{2.1.2}$$

$$\begin{aligned}\Gamma(z+1)=&\int_0^\infty e^{-t}t^z dt\\ =&\frac{-t^z}{e^t}\Big|_0^\infty + z\int_0^\infty e^{-t}t^{z-1}dt\\ =&z\Gamma(z).\end{aligned}$$

In particular,

$$\Gamma(z+1) = z!. \tag{2.1.3}$$

So Gamma function is also called as generalized factorial function.

(ii)

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin \pi z} \quad 0 < z < 1.$$

(iii)

$$2^{2z-1}\Gamma(z)\Gamma\left(z+\frac{1}{2}\right) = \sqrt{\pi}\Gamma(2z) \quad z > 0.$$

Now to calculate gamma of rational numbers, we put $t = u^2$ in (2.1.1)

$$\Gamma(z) = 2\int_0^\infty \exp(-u^2)u^{2z-1}du, \quad Re(z) > 0.$$

Letting $z = \frac{1}{2}$, we find

$$\Gamma\left(\frac{1}{2}\right) = 2\int_0^\infty \exp(-u^2)du = 2\frac{\sqrt{\pi}}{2} = \sqrt{\pi}.$$

Using (2.1.2), we find

$$\Gamma\left(\frac{3}{2}\right) = \frac{1}{2}\Gamma\left(\frac{1}{2}\right) = \frac{\sqrt{\pi}}{2}.$$

Similarly, we can get the values of $\Gamma\left(\frac{5}{2}\right), \Gamma\left(\frac{7}{2}\right), \cdots, \Gamma\left(\frac{2n+1}{2}\right)$.
To extend the gamma function for negative values of z, (2.1.2) can be rewrite as

$$\Gamma(z) = \frac{\Gamma(z+1)}{z}, \quad z \neq 0, -1, -2, \cdots.$$

For example

$$\Gamma\left(-\frac{1}{2}\right) = \frac{\Gamma\left(\frac{1}{2}\right)}{-\frac{1}{2}} = -2\Gamma\left(\frac{1}{2}\right) = -2\sqrt{\pi},$$

$$\Gamma\left(-\frac{3}{2}\right) = \frac{\Gamma\left(-\frac{1}{2}\right)}{-\frac{3}{2}} = \frac{4}{3}\sqrt{\pi}.$$

2.1.2 History of fractional calculus

Basically, calculus discovered in seventieth century by Issac Newton (1642-1727) and Gottfried Wihelm Leibniz (1646-1716). John Von Neumann's (1903-1957) quoted the importance of calculus as: "...the calculus was the first achievement of modern mathematics and it is difficult to overestimate its importance. Leibniz (1646-1716) first introduced the idea of symbolic method and used the symbol $\frac{d^n y}{dx^n} = D^n y$ for the nth derivative, where $n \geq 0$. The concept of fractional calculus was introduced in 1695. L Hospital (1661-1704) asked Leibniz "What if the order will be $n = \frac{1}{2}$". Leibniz (1695) answered, "It will lead to a paradox, from which one day useful consequences will be drawn". Can the concept of derivatives $D^n y$ be extended so that n is any rational, irrational, or complex number? Leibniz gave the fractional order derivative for non-integer values of n as quoted in [1]:

$$\frac{d^n e^{mx}}{dx^n} = m^n e^{mx}.$$

L.Euler (1730) developed the formula for nth derivative as:

$$\frac{d^n x^m}{dx^n} = m(m-1)\cdots(m-n+1)x^{m-n}.$$

Since

$$\Gamma(m+1) = m(m-1)\cdots(m-n+1)\Gamma(m-n+1).$$

Therefore

$$\frac{d^n x^m}{dx^n} = \frac{\Gamma(m+1)}{\Gamma(m-n+1)} x^{m-n}.$$

Euler gave this relationship for negative and non-integer (rational) values of n. He introduced the generalization of factorials to gamma function. Letting $m = 1$ and $n = \frac{1}{2}$, we have

$$\frac{d^{\frac{1}{2}} x}{dx^{\frac{1}{2}}} = \sqrt{\frac{4x}{\pi}}.$$

Then a few years later, S.F. Lacroix developed the formula for the nth derivative of $y = x^m$, m is some natural number,

$$D^n x^m = \frac{m!}{(m-n)!} x^{m-n}, \quad m \geq n.$$

Lacroix used the Euler Gamma function to generalize the factorial and then he further obtained the formula for the fractional derivative as

$$D^n x^m = \frac{\Gamma(m+1)}{\Gamma(m-n+1)} x^{m-n}, \quad m \geq n.$$

In particular, he calculated the $\frac{1}{2}$ derivative for function $y(x) = x$ as

$$D^{\frac{1}{2}} x = \frac{\Gamma(2)}{\Gamma(\frac{3}{2})} x^{\frac{1}{2}} = 2\sqrt{\frac{x}{\pi}}.$$

On the other hand, in 1832, J. Liouville(1809-1882) provided result for derivatives of integral order

$$D^n e^{ax} = a^n e^{ax},$$

where a is a positive real number and extended in a natural way to the derivative of arbitrary order α as

$$D^\alpha e^{ax} = a^\alpha e^{ax}.$$

Using the series expansion of a function $f(x)$, Liouville derived the formula

$$D^\alpha f(x) = \sum_{n=0}^{\infty} c_n a_n^\alpha e^{a_n x}.$$

where c_n and a_n are real numbers for $n = 1, 2, \cdots$, this formula is known as $Liouville's\ first\ formula$ for fractional derivative where

$$f(x) = \sum_{n=0}^{\infty} c_n e^{a_n x}, \quad Re(a_n) > 0.$$

It can be used as a formula for derivative of arbitrary order α, which may rational, irrational or complex. However, it has a disadvantage that α must be restricted to values where series converges. In order to extend his first formula, Liouville provided another definition of a fractional derivative which was applied to explicit functions of the form $x^{-\beta}$, $\beta > 0$

$$D^\alpha x^{-\beta} = (-1)^\alpha \frac{\Gamma(\alpha+\beta)}{\Gamma(\beta)} x^{-\alpha-\beta}, \quad \beta > 0.$$

This is called the $Liouville's\ second\ definition$ of fractional derivative. He successfully applied both his definitions to problems in potential theory. However, Liouville's first definition is restricted to a certain values of α and his second definition is not useful to wide class of functions.

George Peacock (1833) favored Lacroix definition for fractional derivatives, but other mathematicians favored Liouville's definition. According to Peacock-Lacroix the fractional derivative of a constant gives a result other than zero while the fractional derivative of a constant according to Liouville's gives zero because $\Gamma(0) = \infty$. Two different definitions of a fractional derivative when applied to a constant gave different results.

In 1822, J.B.J.Fourier provided more applicable definition for fractional calculus [1]. He generalized the notion of differentiation for arbitrary function $f(x)$ by introducing his famous formula, given by

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(\xi) d\xi \int_{-\infty}^{\infty} \cos p(x-\xi) dp,$$

Fourier made a remark that

$$D^\alpha f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(\xi) d\xi \int_{-\infty}^{\infty} \cos\left(px - p\xi + \alpha\frac{\pi}{2}\right) dp,$$

and this relationship could serve as a definition of the αth order derivative for non-integer α.

2.1.3 Applications of fractional calculus

The applications of fractional calculus just emerged in last few decades in several diverse areas of sciences, such as physics, bio-sciences, chemistry and engineering, namely in acoustic wave propagation in inhomogeneous porous material, anomalous diffusion, diffusive transport, fluid flow, dynamical processes in self similar structures, dynamics of earthquakes, electromagnetic theory, optics, geology, viscoelasticity, atmospheric physics, probability and statistics, astrophysics, chemical engineering, signal processing, nonlinear control, thermal engineering and chaotic dynamics. The fractional calculus is an evolving field of science. There is a wonderful universe of mathematics staying within the boundaries of integer order differentiation and integration.
Perhaps
"$The\ fractional\ calculus\ is\ the\ calculus\ of\ the\ twenty-first\ century.$"

Example 2.1.1. [2] Viscoelasticity is the main field of the most extensive applications of fractional differential and integral operators. We want to describe the behaviour of certain materials under the influence of external forces. We start with the Newton's law about the relationships between stress $\sigma(t)$ and strain $\varepsilon(t)$, both of which are taken as functions of time t. If we are dealing with viscous liquids, then Newton's law

$$\sigma(t) = \eta D^1 \varepsilon(t),$$

is the tool of our choice. Here η is a material constant and is called as viscosity of the material. Hooke's law

$$\sigma(t) = E D^0 \varepsilon(t),$$

where E is the constant known as the modulus of elasticity of the material. These mathematical models are applicable for ideal solid material and for an ideal fluid. In fact, in real world neither of such material exist. It can be stated that the $zero$ and $first$ order derivative of strain is proportional to the stress for solid and stress for fluid respectively.

Now consider a particular experiment where the strain is changed by taking $\varepsilon(t) = t$ for $t \in [0, T]$ with some $T > 0$. Then stress for elastic solids becomes

$$\sigma(t) = Et,$$

and

$$\sigma(t) = \eta = \text{const},$$

for a viscous liquid. Now summarizing these equations we have

$$\psi_k = \frac{\sigma(t)}{\varepsilon(t)} t^k, \tag{2.1.4}$$

where $\psi_0 = E$ and $\psi_1 = \eta$=const. When $k = 0$ it refers to Hooke's law for solids and for $k = 1$ it corresponds to Newton's law for liquids.

It is common phenomena to find so-called viscoelastic materials that exhibit a behavior somewhere between the pure viscous liquid and pure elastic solid, i.e. where we observe a relationship of the form (2.1.4) with $0 < k < 1$, examples of such materials are polymers. When the case of a constant strain ε is considered, the stress in such a material would change according to the formula

$$\sigma(t) = \text{const} \cdot t^{-k},$$

and after long observations, it converges to zero. Due to this, it once again lies between a viscous liquid for which σ vanishes identically and an elastic solid whose stress σ is a nonzero constant.

As a result of all these 'interpolation' properties, for a viscoelastic material it is also possible to model the relation between stress and strain by an equation of the form

$$\sigma(t) = \nu D^k \varepsilon(t),$$

where ν is a material constant and $k \in (0, 1)$ is the parameter.

2.2 Fractional derivatives and integrals

In this section we study fractional derivatives and integrals with their properties. We will explain two types of impulsive differential equations with examples. Then fractional differential equations will be discussed and in the end an application of impulsive equations is given. We will use Caputo differential operator of arbitrary order α in our work.

Definition 2.2.1. The fractional (arbitrary) order integral of the function $f \in L^1([a,b], \mathbb{R}_+)$ of order $\alpha \in \mathbb{R}_+$ is defined by

$$I_a^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-\tau)^{\alpha-1} f(\tau) d\tau,$$

where Γ is gamma function. When $a = 0$, we write

$$I^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f(\tau) d\tau.$$

Example 2.2.2. Let $f(t) = (t-a)^\gamma$ for some $\gamma > -1$ and $\alpha > 0$. Then

$$\begin{aligned} I_a^\alpha f(t) =& \frac{1}{\Gamma(\alpha)} \int_a^t (\tau - a)^\gamma (t-\tau)^{\alpha-1} d\tau \\ =& \frac{1}{\Gamma(\alpha)} (t-a)^{\alpha+\gamma} \int_0^1 x^\gamma (1-x)^{\alpha-1} dx = \frac{\Gamma(\gamma+1)}{\Gamma(\alpha+\gamma+1)} (t-a)^{\alpha+\gamma}. \end{aligned}$$

Properties

The fractional order integral carries some important properties.

(i) The integration of order $\alpha = 0$ is an identity operator given as:

$$I^0 f(t) = f(t).$$

(ii) From the definition of fractional order integral, it can be seen that fractional integral satisfies semigroup property

$$I^\alpha I^\beta = I^{\alpha+\beta} = I^\beta I^\alpha, \qquad \alpha, \beta \in \mathbb{R}.$$

(iii) Fractional integrals are linear

$$I^\alpha (af(x) + bg(x)) = aI^\alpha f(x) + bI^\alpha g(x).$$

(iv) Its effect on power functions [1].

$$I^\alpha t^\gamma = \frac{\Gamma(\gamma+1)}{\Gamma(\gamma+\alpha+1)} t^{\gamma+\alpha}, \quad \text{where } \alpha > 0, \gamma > -1, t > 0.$$

(v) Defining the following function as:

$$\varphi_\alpha(t) = \frac{t^{\alpha-1}}{\Gamma(\alpha)}, \quad \alpha > 0,$$

then

$$I^\alpha f(t) = \varphi_\alpha(t) * f(t), \quad \alpha > 0,$$

where $*$ is a convolution operator.

Definition 2.2.3. For a function f given on the interval $[a, b]$, the αth Riemann-Liouville fractional order derivative of f is defined by

$${}_aD_t^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dt}\right)^n \int_a^t \frac{f(\tau)d\tau}{(t-\tau)^{\alpha-n+1}},$$

where $n - 1 < \alpha \leq n \in \mathbb{N}$.

This fractional derivative $D^\alpha f$ is not zero for the constant function. For example, for $f(t) = 1$ if $\alpha \notin \mathbb{N}$

$$D^\alpha 1 = \frac{t^{-\alpha}}{\Gamma(1-\alpha)}, \quad \alpha \geq 0, t > 0,$$

for $\alpha \in \mathbb{N}$, $D^\alpha 1 = 0$.

Example 2.2.4. For $f(t) = t^\beta$, we have

$$D_0^\alpha f(x) = \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)} x^{\beta-\alpha} \quad \alpha > 0,\ \beta > -1.$$

Riemann-Liouville definition for derivative has certain limitation when it is used for modeling of real-world phenomena associated with fractional differential equation. Another modified definition of fractional differential operator given by Caputo.

Definition 2.2.5. Suppose that $\alpha > 0$, $t > a$ and $\alpha, a, t \in \mathbb{R}$. The Caputo differential operator of order α is defined as

$$^cD^\alpha h(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t (t-s)^{n-\alpha-1} h^{(n)}(s) ds,$$

where $n - 1 < \alpha \leq n \in \mathbb{N}$.

Properties

Some basic properties of the Caputo operator of fractional calculus are [3]

(i) Interpolation

$$\lim_{\alpha \to n} {}^cD^\alpha h(t) = h^{(n)}(t),$$
$$\lim_{\alpha \to n-1} {}^cD^\alpha h(t) = h^{(n-1)}(t) - h^{(n-1)}(0).$$

(ii) Linearity

$$^cD^\alpha(\lambda f(t) + h(t)) = \lambda D^\alpha f(t) + D^\alpha h(t).$$

(iii) Non-commutative

$$^cD^\alpha D^m h(t) \neq D^m D^\alpha h(t).$$

(iv)

$$I^\alpha\ {}^cD^\alpha h(t) = h(t).$$

Lemma 2.2.6. *[4] Let $\alpha > 0$, then*

$$I^\alpha\ {}^cD^\alpha h(t) = h(t) + c_0 + c_1 t + c_2 t^2 + \cdots + c_{n-1} t^{n-1}, \quad c_i \in \mathbb{R}, \quad i = 0, 1, 2, \cdots, n-1, \quad n = [\alpha] + 1.$$

Lemma 2.2.7. *Let $\alpha > 0$, then*

$$^cD^\alpha h(t) = 0, \quad n = [\alpha] + 1$$

where $[\alpha]$ is a integer value of α, has solutions

$$h(t) = c_0 + c_1 t + c_2 t^2 + \cdots + c_{n-1} t^{n-1}, \quad c_i \in \mathbb{R}, \quad i = 0, 1, 2, \cdots, n-1.$$

In the next chapter we will use the following result which is a consequence of the above Lemma.

Lemma 2.2.8. *Let* $0 < \alpha < 1$ *and let* $h : [0, T] \to \mathbb{R}$ *be continuous. A function* y *is a solution of the fractional integral equation*

$$y(t) = y_0 + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} h(s) ds, \tag{2.2.1}$$

if and only if y *is a solution of the initial value problem for the fractional differential equation*

$$\begin{aligned} {}^cD^\alpha y(t) &= h(t), \qquad t \in [0, T] \\ y(0) &= y_0. \end{aligned} \tag{2.2.2}$$

Proof. Suppose that y is a solution of (2.2.2). By applying Lemma 2.2.6, we have

$$y(t) + c_0 = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} h(s) ds,$$

for some $c_0 \in \mathbb{R}$. Applying the condition $y(0) = y_0$ in above equation then

$$c_0 = -y_0,$$

and

$$y(t) = y_0 + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} h(s) ds.$$

which is the required result. $\square$

2.2.1 Impulsive differential equations

Impulsive differential equations are differential equations involving impulse effects which have wide range of applications and hence became an important mathematical tool in modeling of many real world processes. For example, impulsive interruptions are observed in population dynamics, biology, mechanics, industrial robotics, radio engineering, communication security, control theory and neural networks. In recent years, there have been great studies on the behavior of solutions of impulsive differential equations. The dynamics of many evolution processes from various fields experience a change of state abruptly at certain moments of times like earthquake, shock etc. These abrupt changes are of very short time period and can be neglected in comparison with the duration of the process. Naturally these short-time perturbations act instantaneously as an impulses. Most of the problems related to impulsive differential equations can be solved analytically.

Basically, the impulsive equations consist of two components [6]. A continuous-time differential equation, which shows the continuous part of the solution. This can be ordinary differential equations, integro-differential equations, fractional differential equations, partial differential equations, etc. In particular

$$x' = f(t, x),$$

is a continuous time differential equation and the impulse equation, that defines the jump function on that moment where impulse occurs is given as:

$$\Delta x(t) = I(t, x(t)).$$

This component of the impulsive equations is called a *jump condition*. *Moment of impulses* is defined as the point at which the impulses occur. Moreover the functions, that define the number of impulses, are called *impulsive functions*.

Impulsive differential equations are classified into two types according to the way of moment of the change by impulses are determined, which are as follows [7]:

• Equations with fixed moments of impulse effect(the instant of jump are initially fixed at given points);

• Equations with unfixed(variable) moments of impulse effect(the instant of jump occur on initially given sets i.e.the impulse occurs when certain space-time relations are satisfied on given set).

For better understanding, we will give detailed description of these two components of impulsive differential equations.

First type. *Impulsive differential equations with impulse effect at fixed moment.*
Assume that $\mathbb{R}^n$ be a Euclidean n-space and the points $t_k \in \mathbb{R}$ be fixed are called *moments of impulse* such that $t_{k+1} > t_k$, $k = 0, 1, 2, \cdots$. Also at $t = t_k$, $x(t_k^+)$ and $x(t_k^-)$ represent the right limit and left limit respectively.
Consider

$$x'(t) = f(t, x(t)) \quad \text{for} \quad t \geq t_0,\ t \neq t_k, \tag{2.2.3}$$

$$x(t_k + 0) - x(t_k - 0) = I_k(x(t_k - 0)), \quad t = t_k, \quad \text{for} \quad k = 1, 2, \cdots, \tag{2.2.4}$$

where $I_k : \mathbb{R}^n \to \mathbb{R}^n$ represents the jump of state at each t_k, $x \in \mathbb{R}^n$, $f : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ is a given function and $t_0 < t_1 < \cdots < t_k < t_{k+1} < \cdots$, $k = 1, 2, 3, \cdots$
Equations (2.2.3), (2.2.4) together with the initial condition

$$x(t_0) = x_0. \tag{2.2.5}$$

form a initial value problem for the system of impulsive differential equations.
Generally, the solutions of the impulsive differential equations are piecewise continuous functions having a point of discontinuity at the impulsive moment. Let $x(t) = x(t; t_0, x_0)$ be the solution of initial value problem for the system of impulsive differential equations (2.2.3),(2.2.4),(2.2.5).

The solution of the impulsive differential equations (2.2.3),(2.2.4),(2.2.5) is characterized by the following:

(i) For $t = t_0$ the solution $x(t)$ satisfies the initial condition (2.2.5).

(ii) For $t \in (t_k, t_{k+1}]$, the solution $x(t)$ satisfies the (2.2.3).

(iii) For $t = t_k$ the solution $x(t)$ satisfies the equation (2.2.4).

Furthermore we will assume that
$x(t_k) = x(t_k - 0) = \lim_{t \to t_k - 0} x(t) < \infty$ and $x(t_k + 0) = \lim_{t \to t_k + 0} x(t) < \infty$.

Now we consider the motion of the point (t, x) of the integral curve of the solution of the system of impulsive differential equations (2.2.3),(2.2.4) with initial condition (2.2.5) is performed in the following pattern.

The point (t, x) begins its motion from the point (t_0, x_0), $t_0 \in \mathbb{R}$ of the set $D \subset \mathbb{R} \times \mathbb{R}^n$ and continues to move along the integral curve $(t, x(t))$ explained by the solution of (2.2.3) with initial condition $x(t_0) = x_0$ until moment $t_1 > t_0$ and at that moment the point instantaneously moves from position (t_1, x_1) to position (t_1, x_1^+), where $x_1 = x(t_1)$, $x_1^+ = x_1 + I_1(x_1)$. Then the point goes on moving along the integral curve described by the solution of the corresponding system (2.2.3)-(2.2.4) with initial condition $x(t_1) = x_1^+$ until moment $t_2 > t_1$ at which it jumps and the amount of the jump is determined by the equality (2.2.4) and so on.

We will give some examples to illustrate the behavior of the solutions of the impulsive differential equations with fixed moment of impulses.

Example 2.2.9. Given is the fractional impulsive differential equation

$$D^\alpha x(t) = t, \quad t \neq k, \ k = 1, 2, \cdots \ \text{ and } \ \alpha \in (0, 1], \tag{2.2.6}$$

$$x(k + 0) - x(k - 0) = d. \tag{2.2.7}$$

The solution of the corresponding differential equation is:

$$\begin{aligned} I^\alpha D^\alpha x(t) &= I^\alpha t, \\ x(t) + c =& \frac{1}{\Gamma(\alpha)} \int_0^t (t - s)^{\alpha - 1} s ds, \\ x(t) + c =& \frac{t^{\alpha+1}}{\Gamma(\alpha + 2)}, \end{aligned}$$

where c is some constant whose value can be determined by taking initial point as $x(0) = x_0$ then solution becomes:

$$x(t) = x_0 + \frac{t^{\alpha+1}}{\Gamma(\alpha + 2)},$$

for $t \geq 0$. The solution of the impulsive differential equation (2.2.6),(2.2.7) with initial condition $x(0) = x_0$ is:

$$x(t) = x_0 + \frac{t^{\alpha+1}}{\Gamma(\alpha+2)} + kd,$$

for $t \in (k, k+1]$, $k = 1, 2, \cdots$. The solution is piecewise continuous function, which is increasing for $d > 0$, decreasing for $d < 0$, and for $d = 0$ the solution coincides with the solution of the corresponding differential equation without impulses.

Second type. *Impulsive differential equations with impulse at variable times.*
Systems of impulsive equations with variable moments involve difficult problems than systems with fixed moment of impulses. So solution of the problem at different points will have different points of discontinuity.
Let the sequence of sets $\sigma_k = \{(t, x) \in \mathbb{R} \times \mathbb{R}^n : t = \tau_k(x)\}$, $k = \pm 1, \pm 2, \cdots$ be given.
Consider the impulsive differential equation

$$D^\alpha x = f(t, x) \quad \text{for } t \neq, \tau_k(x),\ k = \pm 1, \pm 2, \cdots, \quad \alpha \in (0, 1). \tag{2.2.8}$$

$$x(t+0) - x(t-0) = I_k(x(t)) \ \text{ for } \ t = \tau_k(x(t)), \quad k = \pm 1, \pm 2, \cdots, \tag{2.2.9}$$

where $x \in \mathbb{R}^n$, $f : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$, $I_k : \mathbb{R}^n \to \mathbb{R}^n$, $k = 1, 2, 3, \cdots$, the functions $\tau_k(x)$ are such that $\tau_k(x) < \tau_{k+1}(x)$ for $k = \pm 1, \pm 2, \cdots$, $\lim_{k \to \infty} \tau_k(x) = \infty$ ($\lim_{k \to -\infty} \tau_k(x) = -\infty$).

The moments of impulsive effect of (2.2.8)-(2.2.9) depend on the functions $t = \tau_k(x)$, for each k that define the sets of impulses. For instance, if the initial value problem for the impulsive system with variable moments has two solutions, these solutions will have different points of discontinuity at different points depending on the solutions. This case is called as *beating*. This is the case when the integral curve of the solution intersects the same set several or infinitely many times. In this case the nonexistence of the solution over the whole given interval is observed.

Now we consider the motion of the point (t, x) of the integral curve of the solution of the system of impulsive differential equations (2.2.8),(2.2.9) performed in the following pattern.

Point (t, x) begins its motion from the point (t_0, x_0), $t_0 \in \mathbb{R}$, in set $D \subset \mathbb{R} \times \mathbb{R}^n$ and continues to move along the integral curve $(t, x(t))$ of the corresponding ordinary differential equation with initial condition $x(t_0) = x_0$ until moment $t_1 > t_0$, at which integral curve meets set σ_{k_1}, i.e. until moment $t_1 > t_0$, at which the equality $t_1 = \tau_{k_1}(x(t_1))$ holds. At this moment the point transfers instantaneously from position (t_1, x_1) to position (t_1, x_1^+), $x_1 = x(t_1)$, $x_1^+ = x_1 + I_{k_1}(x_1)$ and it continues to move on the integral curve of the solution of the corresponding ordinary differential equation with initial condition $x(t_1) = x_1^+$ until moment $t_2 > t_1$, at which the integral curve meets set σ_{k_2} , then point jumps instantaneously, and the amount of the jump is defined by equality (2.2.9) and so on.

Systems of impulsive differential equations with fixed moments of impulse effect can be regarded as a particular case of the systems with variable impulsive perturbations.

We will give examples to illustrate the behavior of the solutions of the impulsive differential equations with impulse effects at variable times.

Example 2.2.10. Consider the initial value problem for the linear impulsive differential equation [8]

$$x' = 0, \quad t \neq \tau(x), \tag{2.2.10}$$

$$x(t+0) - x(t) = x(t), \quad t = \tau(x), \tag{2.2.11}$$

$$x(0) = 1, \tag{2.2.12}$$

where $x \in \mathbb{R}$, $\tau(x) = \arctan x$.
Solution of (2.2.10) and (2.2.12) is $x(t) = 1$. The first impulse occurs where the curve $x(t) = 1$ and $x = \tan t$ intersects i.e. $\tan t = 1 \Rightarrow t_1 = \arctan(1)$. In this way general solution of equation (2.2.10)-(2.2.12) is

$$x(t; 0, 1) = i \text{ for } t_{i-1} < t \leq t_i, \; i = 1, 2, \cdots$$

where $t_i = \arctan(i)$.
Maximum interval on which solution is defined is $[0, \frac{\pi}{2})$, because $\lim_{i\to\infty} \arctan i = \arctan(\infty) = \frac{\pi}{2}$. The integral curve of the solution of the initial value problem for the impulsive equation (2.2.10),(2.2.11),(2.2.12) intersects infinitely many times the curve $\sigma = \{(t, x) \in \mathbb{R}\times\mathbb{R} : t = \arctan x\}$ at moments $\tau_i = \arctan i$, $i = 1, 2, \cdots$, therefore there occurs *beating.*

2.2.2 Fractional differential equations

Fractional differential equation is, in general, an ordinary differential equation of arbitrary (non-integer) order. The results obtained from fractional system are of a more general nature. Fractional differential equation have ability to model complex phenomena. Impulsive fractional differential equations represent a real framework for mathematical modeling to real world problems. Research in fractional differential equations has developed significant potential all over the world because of its extensive applications in engineering and science. In fact, real-world processes generally or most likely are fractional order systems.

Definition 2.2.11. A Fractional differential equation of order α, $0 < \alpha < 1$ is an equation of the form

$$\frac{d^\alpha y}{dt^\alpha} = f(y, t),$$

where $y : \mathbb{R} \to \mathbb{R}^n$ is an α differentiable function in the variable $t \in \mathbb{R}$ and $f(y, t) : \mathbb{R}^n \times \mathbb{R} \to \mathbb{C}^n$ is a complex valued function.

2.2.3 An application of impulsive equations as "predator-prey" model

The interaction between two species as "predator prey" is given by mathematical model by Alfred Lotka and Vito Voltera. Differential system of predator-prey model is given as: [9]

$$\dot{P}_1(t) = P_1(t)[a_1 - bP_2(t)],$$

$$\dot{P}_2(t) = P_2(t)[-a_2 + cP_1(t)].$$

At time t, when $t \geq 0$, $a_1 > 0$ represents the growth rate of prey and $a_2 > 0$ is the death rate of predator. $P_1(t)$ and $P_2(t)$ are the population masses of prey and predator respectively, constants b and c represents their relationship.

In this type of models impulsive effects which exists in the real world has been ignored. For instance, the birth of many species is an annual birth pulse. In population masses of given species, impulsive reductions in fishing or in agriculture can be seen by using poisoned chemicals etc. These factors can greatly effect the growth rate of population. If we consider impulsive factors of these population interaction model than above equation becomes impulsive differential equation as:

$$\dot{P}_1(t) = P_1(t)[a_1 - bP_2(t)], \quad t \neq t_k,$$

$$\dot{P}_2(t) = P_2(t)[-a_2 + cP_1(t)], \quad t \neq t_k,$$

$$P_1(t_k^+) = P_1(t_k), \quad P_2(t_k^+) - P_2(t_k^-) = g_k P_2(t_k).$$

Where $P_2(t_k^-)$ and $P_2(t_k^+)$ are the population masses of predator before and after the impulsive effects respectively and t_k are fixed moments of time, $0 < t_1 < \cdots, \lim_{k\to\infty} t_k = \infty$. At $t = t_k$ the population masses of predator is changed, g_k is a real constant which gives the magnitude of impulsive effect at moment t_k. At moment t_k the population masses increases when $g_k > 0$ and decreases when $g_k < 0$.

With the help of these models, the environmental and other exterior changes which effect the population masses of the predator momentarily can be considered.

2.3 Some results from analysis

In this section, we give some definitions, known results and preliminary facts that will be used to establish the existence of boundary value problem for fractional impulsive differential equations.

Definition 2.3.1. A subset S of a Banach space R is *compact* if every sequence in S has a convergent subsequence.

Note that

• If S is compact then it is closed and bounded.

Definition 2.3.2. A subset S of a Banach space R is called convex, if x,$y \in S$ implies

$$K = \{z \in R \mid z = \alpha x + (1-\alpha)y,\ 0 \leq \alpha \leq 1\} \subset S.$$

K is called a closed segment with boundary points x and y; any other $z \in K$ is called interior point of K.

Example 2.3.3. Let R be a Banach space. Consider $S = \{x \in R : \|x\| \leq 1\}$. Let $x, y \in S$ so $\|x\| \leq 1$ and $\|y\| \leq 1$.

$$\begin{aligned}\|\alpha x + (1-\alpha)y\| &\leq \|\alpha x\| + \|(1-\alpha)y\|,\\ &= \alpha\|x\| + (1-\alpha)\|y\|,\\ &\leq \alpha + 1 - \alpha = 1.\end{aligned}$$

Hence S is a convex set.

Definition 2.3.4. A family $\{f_n(x)\}_{n\in\mathbb{N}}$ of functions defined on some closed interval I is said to be $uniformaly$ *bounded*, if there exists a number $M \geq 0$ such that

$$|f_n(x)| \leq M \quad forall \quad x \in I \quad \text{and for all } f \text{ belonging to the given family.}$$

Example 2.3.5. Consider the family of functions

$$f_n(x) = \sin nx \quad \text{where} \quad x \in \mathbb{R},\ n \in \mathbb{Z}$$

is uniformly bounded as

$$|\sin nx| \leq 1.$$

Definition 2.3.6. A family $\{f_n(x)\}_{n\in\mathbb{N}}$ of functions is said to be *equicontinuous*, if for given $\epsilon > 0$, there is a $\delta > 0$ such that

$$|f_n(x_1) - f_n(x_2)| < \epsilon \ \text{ whenever } \ |x_1 - x_2| < \delta \ \text{ for all } \ f.$$

Example 2.3.7. Let $X = \mathbb{R}$ and $f_n(x) = n$. Then for any $\epsilon > 0$, choose any $\delta > 0$ and we have

$$|f_n(x) - f_n(y)| = |n - n| = 0 < \epsilon,$$

whenever $|x - y| < \delta$ so $\{f_n\}$ is *equicontinuous.*

Example 2.3.8. The family $\{\sin nx : n = 1, 2, 3, \cdots\}$ is not *equicontinuous* on $\mathbb{R}$.

If we can show that functions are not continuous at 0 then they are not continuous on $\mathbb{R}$. Let $x = 0$ pick $\epsilon = \frac{1}{2}$, let $\delta > 0$. Then there is n such that $\frac{\pi}{2n} < \delta$. Let $y = \frac{\pi}{2n}$ then $y < \delta$ but

$$|\sin(nx) - \sin(ny)| = |\sin ny| = \sin n\left(\frac{\pi}{2n}\right) = 1 > \epsilon.$$

So $\sin(nx)$ is not *equicontinuous.*

Definition 2.3.9. $S \subset B$ (Banach space) is said to be relatively compact iff its closure $\overline{S}$ is compact.

Definition 2.3.10. Let B and C be Banach spaces and let $T : B \to C$ be linear operator. T is compact if image $T(S)$ of every bonded set S in B is relatively compact in C.

Theorem 2.3.11. *(**Arzela-Ascoli**) Let K be a compact metric space. A subset S of $C(K)$ is compact if and only if it is bounded and equicontinuous.*

Theorem 2.3.12. *(**Banach Fixed Point Theorem**) Consider a non-empty Banach space X. Let $T : X \to X$ be a contraction mapping on X. Then T has precisely one fixed point.*

Theorem 2.3.13. *(**Schaefer's Fixed Point Theorem**) Let X be a Banach space and $F : X \to X$ completely continuous operator. If the set*

$$E(F) = \{y \in X : y = \lambda F y \text{ for } 0 \leq \lambda \leq 1\},$$

is bounded, then F has fixed points.

Theorem 2.3.14. *Let K be a closed convex nonempty subset of a Banach space X. Let A, B be operators such that*

(i) $Ax + By \in K$ whenever $x, y \in K$.

(ii) A is compact and continuous.

(iii) B is contraction mapping.

Then there exists $z \in A$ such that $z = Az + Bz$.

The next theorem will provide useful integral inequalities which we will use in next chapter to establish the uniqueness, continuity and bounds of solution of fractional impulsive initial value problems.

Theorem 2.3.15. *Assume that following conditions be satisfied:*

(i) Function $\nu(t) \in PC([0,\infty),[0,\infty))$.

(ii) Function $q(t) \in PC([0,\infty),[0,\infty))$ is nondecreasing.

(iii) Function $\mu(t) \in PC([0,\infty),[0,\infty))$ satisfies the inequality

$$\mu(t) \leq q(t)[c + \sum_{0<t_k<t} \beta_k \mu(t_k) + \int_0^t \nu(s)\mu(s)ds], \tag{2.3.1}$$

where $c \geq 0$, $\beta_k \geq 0$, $(k = 1, 2, \cdots)$ are constants.
Then for $t \geq 0$ the inequality

$$\mu(t) \leq c q^2(t) \prod_{0<t_k<t} q(t_k)(1 + \beta_k q(t_k)) e^{\int_0^t \nu(s)q(s)ds}, \tag{2.3.2}$$

holds.

2.3.1 Krasnoselskii's cone fixed point theorem

In [10] a generalization of Krasnoselskii fixed point theorem for cone maps has been developed to establish the existence of multiple solutions in the study of boundary value problems.Krasnoselskii applied his result to establish the existence of periodic solutions of period systems of ordinary differential equations. The main purpose for seeking cone fixed point theorems is to apply them to obtain the existence of solutions of boundary value problems. The proofs of Krasnoselskii's theorem and its generalization, mostly use topological index theory.

Krasnoselskii's theorem consists of two parts. The first part, called the compressive form has very much resemblance to the generalized Brouwer-Schauder theorem. The second part is called the expansive form. The close relationship between Krasnoselskii's theorem and Brouwer-Schauder theorem is that first one is usually stated a cone embedded in a Banach space with a given norm, where the norm functional plays important roles: to define the region of points and state the properties of the images under the defined mapping. In contrast, Brouwer-Schauder theorem is more topological in nature without metric.

Krasnoselskii's theorem certainly be interpreted in a non-metric framework. The norm function is more of a ease rather than a requirement. There are simpler ways to simplify the theorem without using functionals.

Definition 2.3.16. Let X be a Banach space. A cone $K \subset X$ is a closed convex set such that $\lambda K \subset K$ for all $\lambda \geq 0$ and $K \cap \{-K\} = \{0\}$.

For example, $\mathbb{R}^n_+ = \{x \in \mathbb{R}^n : x \geq 0\}$ is a cone. A cone map $T : K \to K$ is a completely continuous map. Any continuous map is completely continuous when X is finite dimensional. A point $x \in K$ is a fixed point of T if $T(x) = x$.

Consider $0 < a < b$ be two given numbers. Now we will find the conditions which shows that T has a fixed point in the annular region $K(a, b) = \{x \in K : a \leq \|x\| \leq b\}$. $K(a, b)$ is in general not convex. We represent $K_a = \{x \in K : \|x\| = a\}$ and $K_b = \{x \in K : \|x\| = b\}$ the inner and outer boundaries, respectively, of $K(a, b)$. Given below is a simplified version of Krasnoselskii's original theorem.

Theorem 2.3.17. *Consider $K(a, b)$, T, K_a, and K_b as defined above.*
*1. **Compressive Form**: T has a fixed point in $K(a, b)$ if*

$$\|T(x)\| \geq \|x\| \quad \text{for all} \quad x \in K_a, \tag{2.3.3}$$

and

$$\|T(x)\| \leq \|x\| \quad \text{for all} \quad x \in K_b. \tag{2.3.4}$$

*2. **Expansive Form**: T has a fixed point in $K(a, b)$ if*

$$\|T(x)\| \leq \|x\| \quad \text{for all} \quad x \in K_a, \tag{2.3.5}$$

and

$$\|T(x)\| \geq \|x\| \quad \textit{for all} \quad x \in K_b. \tag{2.3.6}$$

Note that the conditions (2.3.3)-(2.3.6) are compelled only on points on the two curved boundaries of $K(a, b)$. Inner points and point on the sides of the cone can be shifted in any direction. It cannot be specified that any particular image point $T(x)$ must lie inside $K(a, b)$.

When (2.3.3) (or (2.3.4)) holds, then T is compressive on K_a (or K_b) relating to $K(a, b)$. If there is strict inequality in (2.3.3) (or (2.3.4)), then T is strictly compressive on K_a(or K_b). Similarly when (2.3.5) (or (2.3.6)) holds, then T is expansive on K_a (or K_b), and T is strictly expansive if there is strict inequality in (2.3.5) (or (2.3.6)).

The basic method to apply the cone fixed point theorem is to write the boundary value problem as an integral equation by the use of Green's function to obtain existence results for a problem. The integral operator is a completely continuous cone map then by finding suitable constants a and b so that the hypotheses of the cone theorem are satisfied, then the $K(a, b)$ has a fixed point that is equal to a positive solution of the boundary value problem.

The first direction of extension of Theorem 2.3.18 is to simplify the conditions (2.3.3)-(2.3.6). Krasnoselskii's original result is actually stated with weaker postulates. The compressive form, instead of (2.3.3) and (2.3.4), only require that

$$T(x) - x \notin K \quad \text{for all} \quad x \in K_a,$$

and

$$x - T(x) \notin K \quad \text{for all} \quad x \in K_b.$$

This permits some part of the inner boundary K_a to move forward nearer the origin, and some part of the outer boundary K_b to be pushed away from the origin.

In particular, conditions (2.3.3) and (2.3.4) can be replaced by

$$\exists p \in K \quad \text{such that} \quad x - T(x) \neq \lambda p, \; \forall \lambda \geq 0, \; x \in K_a. \tag{2.3.7}$$

and

$$T(x) \neq \lambda x, \quad \text{for any } \lambda > 1, \; x \in K_b, \tag{2.3.8}$$

and conditions (1.3.3) and (1.3.4) can be exchanged as

$$T(x) \neq \lambda x, \; \text{for any} \quad \lambda > 1, \; x \in K_a. \tag{2.3.9}$$

$$\exists p \in K \quad \text{such that} \quad x - T(x) \neq \lambda p, \; \forall \lambda \geq 0, \; x \in K_b. \tag{2.3.10}$$

In the text, condition (2.3.8) is called the Leray-Schauder condition. By (2.3.8) it means that no point on K_b is pushed by T away from the origin and by (2.3.7) it means that no point on K_a is pushed by T towards the origin in a direction along p; pushing it in the opposite direction away from the origin. There is an obvious asymmetry in the pair of conditions (2.3.7) and (2.3.8), as comparison with (2.3.3) and (2.3.4).

The second extension is to look at regions more broader than $K(a,b)$. Guo presented a result, replaces $K(a,b)$ in Theorem 2.3.18 by the more broader region

$$J = K \cap (\overline{\Omega}_2 \setminus \Omega_1),$$

where $\overline{A}$ denotes the closure of a set A and Ω_1 and Ω_2 are two bounded open sets in X such that $0 \in \Omega_1 \subset \overline{\Omega}_1 \subset \Omega_2$. Boundary of A can also be denoted by ∂A. Assume that the conditions (2.3.3),(2.3.4) or (2.3.5), (2.3.6) hold, but now for points on $K \cap \partial\Omega_1$ and $K \cap \partial\Omega_2$, instead of on K_a and K_b, respectively. The hypotheses that Ω_1 and Ω_2 are open but otherwise arbitrarily means that we can apply the result to fairly general regions J.

The basic method to get multiple solutions to a boundary value problem is to gather two or more annular regions together and apply the alternative forms of Krasnoselskii's theorem to each of the regions to get a fixed point. For instance, consider three positive numbers $0 < a < b < c$, and define the corresponding regions $K(a,b)$ and $K(b,c)$. Let us assume that (2.3.3) and (2.3.4) hold for K_a and K_b, and (2.3.6) holds for K_c (replace b in (2.3.6) by c). Then there exists one fixed point in $K(a,b)$ and one fixed point in $K(b,c)$. There is a option that these two fixed points are one and the same. If this will happen, it must lie on the common boundary K_b. If we have to exclude this condition, we have to make the stronger assumption that T maps K_b strictly away from K_b.

Hence, a third way to extend the cone theorem is to look for more broader ways to construct such gathered-annulus structures. For example, we can use the same inner and outer boundaries K_a and K_c as the example above, but replace K_b by a set of points defined by some given continuous functional. In view of that the conditions (2.3.3)-(2.3.6) will have to be altered.

Chapter 3

Existence and uniqueness of solution of fractional impulsive differential equations

In this chapter, sufficient condition for the existence and uniqueness of solution for fractional impulsive differential equations involving Caputo fractional derivative will be discussed. In the first section, we will study the existence, uniqueness and continuity of solution of following fractional impulsive initial value problem

$$\begin{aligned} &{}^cD^\alpha y(t) = f(t, y), \quad t \neq t_k, \quad 0 < \alpha \leq 1, \\ &y(t_k + 0) - y(t_k) = I_k(y(t_k)), \quad k = 1, 2, \cdots, n, \\ &y(t_0) = y_0. \end{aligned}$$

where $t_0 \in \mathbb{R}$ is a fixed point and $y \in \mathbb{R}^n$. The results presented in section 3.1 are generalizations of the results in [8]; from integer order to fractional order differential equations.

In the second section of this chapter existence and uniqueness of the solution of following fractional impulsive boundary value problem with examples will be discussed.

$$\begin{aligned} {}^cD^\alpha y(t) = f(t, y(t)) \quad &\text{for} \quad t \in [0,1], \quad 1 < \alpha \leq 2, \\ &\Delta y(t_k) = P_k(y(t_k)), \\ &\Delta y'(t_k) = Q_k(y(t_k)), \text{ where } k = 1, 2, \cdots, n \text{ are the number of impulses}, \\ y(0) =& g(y), \qquad y(1) = k \qquad k \in \mathbb{R}. \end{aligned}$$

where ${}^cD^\alpha$ is the Caputo fractional derivative, $f : [0,1] \times \mathbb{R} \to \mathbb{R}$, is a continuous function, $g : C([0,1], \mathbb{R}) \to \mathbb{R}$ a continuous function, $P_k, Q_k : \mathbb{R} \to \mathbb{R}$, $k = 1, \cdots, n$, $0 = t_0 < t_1 < \cdots < t_n < t_{n+1} = 1$. We define $PC([0,1], \mathbb{R}) = \{y : [0,1] \to \mathbb{R} : y \in C([0,1], \mathbb{R})\}$.

Uniqueness of solution of fractional boundary value problem without impulsive condition has been studied in [11] and other uniqueness results with impulsive conditions are discussed in [4]. Existence of at least one solution of fractional boundary value problem under the impulsive conditions has been studied in [12].

3.1 Existence and uniqueness: Initial value problem

In this section we will consider the following impulsive fractional differential equation

$$\begin{aligned} &{}^cD^\alpha y(t) = f(t,y), \quad t \neq t_k, \quad 0 < \alpha \leq 1, \\ &y(t_k + 0) - y(t_k) = I_k(y(t_k)), \\ &y(t_0) = y_0, \end{aligned} \tag{3.1.1}$$

where $t_0 \in \mathbb{R}$ is a fixed point and $y \in \mathbb{R}^n$.
Firstly we will establish its existence of solution by Schaefer's fixed point theorem then we will study uniqueness, continuity and bounds of the solution of (3.1.1) with the help of some integral inequalities as in Theorem 2.3.16. In [8] Snezhana has studied the impulsive differential equation (3.1.1) for $\alpha = 1$. Existence and uniqueness of impulsive initial value problem have also been discussed in [4].

Lemma 3.1.1. *[5] Assume that $y \in PC(J, \mathbb{R})$ satisfies*

$${}^cD^\alpha y(t) = h(t), \qquad t \neq t_k, k = 1, 2, \cdots, n, 1 < \alpha \leq 2, \tag{3.1.2}$$

then

$$y(t) = c_1 + c_2 t + \sum_{0<t_k<t} \Delta y'(t_k)(t - t_k) + I^\alpha h(t). \tag{3.1.3}$$

3.1.1 Existence of solutions

Lemma 3.1.2. *Let $0 < \alpha \leq 1$ and let $f : [0, 1] \to \mathbb{R}$ be continuous. A function $y(t; t_0, y_0)$ is a solution of the fractional integral equation*

$$y(t; t_0, y_0) = y_0 + \frac{1}{\Gamma(\alpha)} \int_{t_0}^t (t - s)^{\alpha-1} f(s, y(s; t_0, y_0))ds + \sum_{k=1}^n I_k(y(t_k; t_0, y_0)). \tag{3.1.4}$$

if and only if $y(t; t_0, y_0)$ is the solution of fractional impulsive IVP

$$\begin{aligned} &{}^cD^\alpha y(t) = f(t,y), \quad t \neq t_k, \quad 0 < \alpha \leq 1, \\ &y(t_k + 0) - y(t_k) = I_k(y(t_k)), \quad k = 1, 2, \cdots, n \\ &y(t_0) = y_0. \end{aligned}$$

Proof. Suppose that $y(t; t_0, y_0)$ is a solution of (3.1.1). By applying Lemma 3.1.1, we have

$$y(t; t_0, y_0) = c_0 + \frac{1}{\Gamma(\alpha)} \int_{t_0}^t (t - s)^{\alpha-1} f(s, y(s; t_0, y_0))ds + \sum_{t_0<t_k<t} \Delta y(t_k),$$

for some $c_0 \in \mathbb{R}$. Applying first condition $y(t_0) = y_0$ in above equation we get $c_0 = y_0$ then

$$y(t; t_0, y_0) = y_0 + \frac{1}{\Gamma(\alpha)} \int_{t_0}^t (t - s)^{\alpha-1} f(s, y(s; t_0, y_0))ds + \sum_{k=1}^n I_k(y(t_k; t_0, y_0)).$$

□

Our next result is based on Schaefer's fixed point theorem.

Theorem 3.1.3. *Assume that:*

(i) The function $f : J = [0,1] \times \mathbb{R} \to \mathbb{R}$ is continuous and there exists a constant $M > 0$ such that

$$f(t, y(t; t_0, y_0)) \leq M \quad \textit{for all } t \in J \quad \textit{and all } y \in \mathbb{R}.$$

(ii) The function $I_k : \mathbb{R} \to \mathbb{R}$ are continuous and there exists a constant $M^* > 0$ such that

$$I_k(y(t_k; t_0, y_0)) \leq M^* \quad \textit{for each } y \in \mathbb{R} \quad \textit{and } k = 1, 2, \cdots, n.$$

Then the initial value problem (3.1.1) *has at least one solution on J.*

Proof. Transform the problem (3.1.1) into a fixed point problem. Consider the operator F :PC(J,$\mathbb{R}$)$\to$PC(J,$\mathbb{R}$) defined by

$$F(y)(t; t_0, y_0) = y_0 + \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} (t-s)^{\alpha-1} f(s, y(s; t_0, y_0)) ds + \sum_{t_0 < t_k < t} I_k(y(t_k; t_0, y_0)).$$

The operator F :PC(J,$\mathbb{R}$)$\to$PC(J,$\mathbb{R}$) is continuous and completely continuous. Now it remains to show that the set

$$E = \{y \in \mathrm{PC}(J, \mathbb{R}) : y(t; t_0, y_0) = \lambda F(y(t; t_0, y_0)) \text{ for } 0 < \lambda < 1\},$$

is bounded.
Let $y \in E$ then $y = \lambda F(y(t; t_0, y_0))$ for some $0 < \lambda < 1$. Thus for each $t \in J$, we have

$$|y(t; t_0, y_0)| \leq \lambda |y_0| + \frac{\lambda}{\Gamma(\alpha)} \int_{t_0}^{t} (t-s)^{\alpha-1} |f(s, y(s; t_0, y_0))| ds + \lambda \sum_{k=1}^{n} |I_k(y(t_k; t_0, y_0))|.$$

This implies by (i) and (ii) that for each $t \in J$

$$\|y(t; t_0, y_0)\| \leq |y_0| + \frac{M(t-t_0)^{\alpha}}{\Gamma(\alpha+1)} + nM^* = R.$$

This shows that the set E is bounded. As a consequence of Schaefer's fixed point theorem, we deduce that F has a fixed point which is a solution of the problem (3.1.1). □

Following will be assumed for the uniqueness, continuity and bounds for solution.

(H1) $f(t, y) \in C([t_0, \infty) \times \mathbb{R}^n, \mathbb{R}^n)$.

(H2) $Z(t, s) \in C([t_0, \infty) \times [0, \infty), [0, \infty))$ and satisfies the inequality $\|f(t, y)\| \leq Z(t, \|y\|)$ for $t \geq t_0$, $y \in \mathbb{R}^n$.

(H3) There exists some functions $Q(t) \in C([0, \infty), [0, \infty))$ and $\lambda(t) \in C([t_0, \infty), [0, \infty))$ such that $\lambda(\mu) > 0$, $\mu > 0$ and $\|f(t, y) - f(t, x)\| \leq \lambda(t) Q(\|y - x\|)$ for $t \geq t_0$ and $x, y \in \mathbb{R}^n$.

(H4) for $(t_0, y_0) \in [0,\infty)\times\mathbb{R}^n$ the initial value problem (3.1.1) has a solution $y(t;t_0,y_0)$, defined for $t \geq t_0$.

(H5) there exists functions $\delta_k \in C([0,\infty),[0,\infty))$, $k = 1,2,\cdots$ such that for $y \in \mathbb{R}^n$ the inequalities $\|I_k(y)\| \leq \delta_k(\|y\|)$, $k = 1,2,\cdots$ hold.

(H6) there exists functions $\gamma_k \in C([0,\infty),[0,\infty))$, $k = 1,2,\cdots$ such that for x,$y \in \mathbb{R}^n$the inequalities $\|I_k(y) - I_k(x)\| \leq \gamma_k(\|y - x\|)$, $k = 1,2,\cdots$ hold.

3.1.2 Uniqueness of solution

Theorem 3.1.4. *Assume that the conditions* $(H1),(H3),(H4)$ *and* $(H6)$ *hold for* $Q(y) = y$, $\gamma_k(y) = \beta_k y$, $\beta_k =$*constant*> 0, $k = 1,2,\cdots$. *Then the initial value problem* (3.1.1) *has a unique solution.*

Proof. Consider the function $\mu(t) = \|y(t;t_0,y_0) - x(t;t_0,y_0)\| \geq 0$, where the functions $y(t;t_0,y_0)$ and $x(t;t_0,y_0)$ are two arbitrary solutions of the initial value problem (3.1.1).
From the integral equation (3.1.4), we have

$$\begin{aligned}\|y(t;t_0,y_0) - x(t;t_0,y_0)\| \leq & \frac{1}{\Gamma(\alpha)}\int_{t_0}^{t}(t-s)^{\alpha-1}\|f(s,y(s;t_0,y_0)) - f(s,x(s;t_0,y_0))\|ds \\ & + \sum_{t_0<t_k<t}\|I_k(y(t_k;t_0,y_0)) - I_k(x(t_k;t_0,y_0))\| \\ & \leq \frac{1}{\Gamma(\alpha)}\int_{t_0}^{t}(1-s)^{\alpha-1}\lambda(s)Q(\|y(s;t_0,y_0) - x(s;t_0,y_0)\|)ds \\ & + \sum_{t_0<t_k<t}\gamma_k(\|y(t_k;t_0,y_0) - x(t_k;t_0,y_0)\|), \\ & = \frac{1}{\Gamma(\alpha)}\int_{t_0}^{t}\lambda(s)\|y(s;t_0,y_0) - x(s;t_0,y_0)\|ds \\ & + \sum_{t_0<t_k<t}\beta_k\|y(t_k;t_0,y_0) - x(t_k;t_0,y_0)\|.\end{aligned}$$

So we get integral inequality

$$\mu(t) \leq \frac{1}{\Gamma(\alpha)}\int_{t_0}^{t}\lambda(s)\mu(s)ds + \sum_{t_0<t_k<t}\beta_k\mu(t_k).$$

Comparing above relation with (2.3.1)(Theorem 2.3.16) we get $c = 0$, $q(t) = 1$ and $\nu(t) = \lambda(t)$ then from (2.3.2) we have $\mu(t) \leq 0$, that proves the equality $\mu(t) = 0$ for $t \geq t_0$, hence both solutions are equal. □

3.1.3 Continuity and bounds of solution

Theorem 3.1.5. *Assume that the conditions* $(H1),(H3),(H4)$ *and* $(H6)$ *hold for* $Q(\mu) = \mu$, $\gamma_k(\mu) = \beta_k\mu$, $\beta_k =$*constant*> 0, $k = 1,2,\cdots$. *Then the initial value problem* (3.1.1) *has a continuous solution.*

Proof. Consider the function $\mu(t) = \|y(t;t_0,y_0) - x(t;t_0,x_0)\| \geq 0$, where the functions $y(t;t_0,y_0)$ and $x(t;t_0,x_0)$ are two arbitrary solutions of the initial value problem (3.1.1).
From the integral equation (3.1.4), we have

$$\begin{aligned}\|y(t;t_0,y_0) - x(t;t_0,x_0)\| \leq & \|y_0 - x_0\| + \frac{1}{\Gamma(\alpha)}\int_{t_0}^{t}(t-s)^{\alpha-1}\|f(s,y(s;t_0,y_0)) - f(s,x(s;t_0,x_0))\|ds \\ & + \sum_{t_0<t_k<t}\|I_k(y(t_k;t_0,y_0)) - I_k(x(t_k;t_0,x_0))\| \\ & \leq \|y_0 - x_0\| + \frac{1}{\Gamma(\alpha)}\int_{t_0}^{t}(1-s)^{\alpha-1}\lambda(s)Q(\|y(s;t_0,y_0) - x(s;t_0,x_0)\|)ds \\ & + \sum_{t_0<t_k<t}\gamma_k(\|y(t_k;t_0,y_0) - x(t_k;t_0,x_0)\|), \\ & = \|y_0 - x_0\| + \frac{1}{\Gamma(\alpha)}\int_{t_0}^{t}\lambda(s)Q(\mu(s))ds + \sum_{t_0<t_k<t}\gamma_k(\mu(t_k)).\end{aligned}$$

Comparing above relation with (2.3.1)(Theorem 2.3.16) we get $c = \|y_0 - x_0\|$, $q(t) = 1$ and $\nu(t) = \lambda(t)$ then from (2.3.2) we have

$$\mu(t) \leq \|y_0 - x_0\| \prod_{0<t_i<t}(1+\beta_k q(t_k))e^{\int_{t_0}^{t}\lambda(s)q(s)ds}, \quad t \geq t_0. \tag{3.1.5}$$

Let $\varepsilon > 0$ be an arbitrary number, $L > t_0$ be a fixed constant. We define a constant $\delta = \delta(\varepsilon) > 0$ by the equality

$$\delta = \varepsilon\left[\prod_{t_0<t_i<L}(1+\beta_k)e^{M(L-t_0)}\right]^{-1},$$

where $M = max\{\lambda(t) : t \in [t_0, L]\} < \infty$.
Then from the inequality (3.1.5) for $t \in [t_0, L]$ follows that $\mu(t) < \varepsilon$ holds. Hence the solution of the initial value problem (3.1.1) is continuous. □

Theorem 3.1.6. *Assume that the conditions* $(H1), (H2), (H4)$ *and* $(H5)$ *holds for* $Z(t,\mu) = L\mu$, $\delta(\mu) = \beta_k\mu$, $L =$*constant*> 0, $\beta_k =$*constant*≥ 0, $k = 1, 2, \cdots$. *Then we will get bounds of the solution of* (3.1.1).

Proof. From the integral equation (3.1.4), we have

$$\begin{aligned}\|y(t;t_0,y_0)\| \leq & \|y_0\| + \frac{1}{\Gamma(\alpha)}\int_{t_0}^{t}(t-s)^{\alpha-1}\|f(s,y(s;t_0,y_0))\|ds + \sum_{t_0<t_k<t}\|I_k(y(t_k;t_0,y_0))\| \\ \leq & \|y_0\| + \frac{1}{\Gamma(\alpha)}\int_{t_0}^{t}(1-s)^{\alpha-1}Z(s,\|y(s;t_0,y_0)\|)ds + \sum_{t_0<t_k<t}\delta_k(\|y(t_k;t_0,y_0)\|), \\ = & \|y_0\| + \frac{1}{\Gamma(\alpha)}\int_{t_0}^{t}(1-s)^{\alpha-1}L\|y(s;t_0,y_0)\|ds + \sum_{t_0<t_k<t}\beta_k\|y(t_k;t_0,y_0)\|.\end{aligned}$$

From equation (2.3.1) we have $c = \|y_0\|$, $q(t) = 1$ and $\nu(t) = L(1-t)^{\alpha-1}$, then from (2.3.2) we get

$$\|y(t;t_0,y_0)\| \le \|y_0\| \prod_{0<t_k<t} (1+\beta_k) e^{\frac{L}{\alpha}[(1-t_0)^\alpha - (1-t)^\alpha]}.$$

□

3.2 Existence and uniqueness: Boundary value problem

A Boundary value problem is a system of differential equations with solution and derivative values specified at more than one point. A function is said to be the solution of boundary value problem if it satisfies the differential equation and the boundary condition. In this section we will consider the following impulsive fractional boundary value problem

$$\begin{aligned} {}^cD^\alpha y(t) &= f(t,y(t)) \quad \text{for} \quad t\in[0,1], \quad 1<\alpha\le 2,\\ \Delta y(t_k) &= P_k(y(t_k)),\\ \Delta y'(t_k) &= Q_k(y(t_k)),\\ y(0) =& g(y), \qquad y(1)=k \qquad k\in\mathbb{R}. \end{aligned} \tag{3.2.1}$$

The boundary conditions with non-classical boundary that link the values of unknown function on the boundary and inside of the domain are called non-local boundary conditions. The analysis of the derivative operator with non-local boundary conditions provides an important tool to study the stability of solutions for general classes of differential equations.

In this section, we present existence results for the problem (3.2.1) involving Caputo fractional derivative. We give two results, one based on Theorem 1.3.15 and another one based on Banach fixed point theorem (Theorem 1.3.13). Two examples are given in this section to demonstrate the application of our main results.

For measurable functions $\nu : J=[0,1]\to\mathbb{R}$, define the norm [12]

$$\|\nu\|_{L^p(J)} = \left(\int_J |\nu(t)|^p dt\right)^{\frac{1}{p}}, \quad 1\le p<\infty.$$

$L^p(J,\mathbb{R})$ denotes the Banach space of all Lebesgue measurable functions $\nu : J=[0,1]\to\mathbb{R}$ with $\|\nu\|_{L^p(J)}<\infty$.

3.2.1 Existence of solutions

Lemma 3.2.1. *Let $1<\alpha\le 2$ and let $f:[0,1]\to\mathbb{R}$ be continuous. A function y is a solution of the fractional integral equation*

$$\begin{aligned} y(t) =& \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} f(s,y(s))ds - \frac{t}{\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-1} f(s,y(s))ds\\ &+(1-t)\sum_{k=1}^n P_k(y(t_k)) + \sum_{k=1}^n G(t,t_k)Q_k(y(t_k)) + (1-t)g(y) + kt, \end{aligned}$$

if and only if y is a solution of the fractional impulsive BVP

$$\begin{aligned} {}^cD^\alpha y(t) &= f(t, y(t)) \quad for \;\; t \in [0,1], \quad 1 < \alpha \le 2, \\ \Delta y(t_k) &= P_k(y(t_k)), \\ \Delta y'(t_k) &= Q_k(y(t_k)), \\ y(0) =& g(y), \qquad y(1) = k \qquad k \in \mathbb{R}. \end{aligned}$$

Proof. Suppose that y is a solution of (3.2.1). By applying Lemma 3.1.1, we have

$$y(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, y(s))ds + \sum_{0<t_k<t} \Delta y(t_k) + \sum_{0<t_k<t} \Delta y'(t_k)(t - t_k) + c_0 + c_1 t,$$

for some c_0,$c_1 \in \mathbb{R}$. Applying first condition $y(0) = g(y)$ in above equation we get $c_0 = g(y)$ then

$$y(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, y(s))ds + \sum_{0<t_k<t} \Delta y(t_k) + \sum_{0<t_k<t} \Delta y'(t_k)(t - t_k) + g(y) + c_1 t.$$

Applying second second condition in above equation then

$$c_1 = k - \frac{1}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} f(s, y(s))ds - \sum_{k=1}^{n} P_k(y(t_k)) - \sum_{k=1}^{n} Q_k(y(t_k))(1 - t_k) - g(y),$$

and

$$\begin{aligned} y(t) =& \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, y(s))ds - \frac{t}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} f(s, y(s))ds + (1-t) \sum_{k=1}^{n} P_k(y(t_k)) \\ &+ \sum_{0<t_k<t} Q_k(y(t_k))(t - t_k) - t \sum_{k=1}^{n} Q_k(y(t_k))(1 - t_k) + (1-t)g(y) + kt, \\ =& \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, y(s))ds - \frac{t}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} f(s, y(s))ds + (1-t) \sum_{k=1}^{n} P_k(y(t_k)) \\ &+ \sum_{k=1}^{n} G(t, t_k) Q_k(y(t_k)) + (1-t)g(y) + kt. \end{aligned}$$

which is the required result. □

Theorem 3.2.2. *Let* $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ *be a continuous function mapping with* $|f(t, y)| \le \nu(t)$, *for all* $(t, y) \in [0,1] \times \mathbb{R}$ *where* $\nu \in L^{\frac{1}{\tau}}([0,1], \mathbb{R})$ *and* $\tau \in (0, \alpha - 1)$. *Assume that:*

(H7) There exists constants $M_1, M_2, M_3 > 0$ *such that* $|g(y_1) - g(y_2)| \le M_1|y_1 - y_2|$, $|P_k(y_1) - P_k(y_2)| \le M_2|y_1 - y_2|$, $|Q_k(y_1) - Q_k(y_2)| \le M_3|y_1 - y_2|$ *for each* $t \in [0,1]$ *and all* y_1,$y_2 \in \mathbb{R}$.

Furthermore if $(M_1 + n(M_2 + M_3)) < 1$, *then the BVP* (3.2.1) *has at least one solution on* $[0,1]$.

Proof. Let us choose

$$r \geq \frac{2\|\nu\|_{L^{\frac{1}{\tau}}(J)}}{\Gamma(\alpha)\left(\frac{\alpha-\tau}{1-\tau}\right)^{1-\tau}},$$

and denote

$$P_r = \{y \in \mathrm{PC}([0,1],\mathbb{R}) : \|y\|_{\mathrm{PC}} \leq r\}.$$

Consider the operators $A, B : C([0,1],\mathbb{R}) \to C([0,1],\mathbb{R})$ defined by

$$(Ay)(t) = (1-t)\sum_{k=1}^{n} P_k(y(t_k)) + \sum_{k=1}^{n} G(t,t_k)Q_k(y(t_k)) + (1-t)g(y) + kt,$$

and

$$(By)(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} f(s,y(s))ds - \frac{t}{\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-1} f(s,y(s))ds.$$

Firstly, we show that the operator $A : C([0,1],\mathbb{R}) \to C([0,1],\mathbb{R})$ is contraction.

$$\begin{aligned}|A(y_1)(t) - A(y_2)(t)| \leq & |g(y_1) - g(y_2)| + \sum_{k=1}^{n} |P_k(y_1(t_k)) - P_k(y_2(t_k))| + \sum_{k=1}^{n} |Q_k(y_1(t_k)) - Q_k(y_2(t_k))|, \\ \leq & M_1\|y_1 - y_2\| + nM_2\|y_1 - y_2\| + nM_3\|y_1 - y_2\|, \\ \leq & (M_1 + n(M_2 + M_3))\|y_1 - y_2\|.\end{aligned}$$

Consequently A is a contraction.

For any $y, z \in P_r$ and $t \in J = [0,1]$, using the estimation condition on f and Holder inequality,

$$\int_0^t |(t-s)^{\alpha-1} f(s,y(s))|ds \leq \left(\int_0^t (t-s)^{\frac{\alpha-1}{1-\tau}} ds\right)^{1-\tau} \left(\int_0^t (\nu(s))^{\frac{1}{\tau}} ds\right)^{\tau} \leq \frac{\|\nu\|_{L^{\frac{1}{\tau}}(J)}}{\left(\frac{\alpha-\tau}{1-\tau}\right)^{1-\tau}},$$

$$t\int_0^1 |(1-s)^{\alpha-1} f(s,y(s))|ds \leq t\left(\int_0^1 (1-s)^{\frac{\alpha-1}{1-\tau}} ds\right)^{1-\tau} \left(\int_0^1 (\nu(s))^{\frac{1}{\tau}} ds\right)^{\tau} \leq \frac{\|\nu\|_{L^{\frac{1}{\tau}}(J)}}{\left(\frac{\alpha-\tau}{1-\tau}\right)^{1-\tau}}.$$

Operator B is uniformly bounded on P_r since

$$\|By\|_{\mathrm{PC}} \leq \frac{2\|\nu\|_{L^{\frac{1}{\tau}}(J)}}{\Gamma(\alpha)\left(\frac{\alpha-\tau}{1-\tau}\right)^{1-\tau}} \leq r.$$

Now we will show that operator B is compact and continuous. Note that B is continuous in view of continuity of f. To prove operator B is compact, let $\Omega = J \times P_r$ we can define $\sup_{(t,x)\in\Omega} |f(t,y)| = f_0$

for any $0 < t_2 < t_1 \leq 1$, we have

$$\begin{aligned}
|(By)(t_2) - (By)(t_1)| =& \Big|\frac{1}{\Gamma(\alpha)}\int_0^{t_2}(t_2-s)^{\alpha-1}f(s,y(s))ds - \frac{t_2}{\Gamma(\alpha)}\int_0^1(1-s)^{\alpha-1}f(s,y(s))ds \\
&-\frac{1}{\Gamma(\alpha)}\int_0^{t_1}(t_1-s)^{\alpha-1}f(s,y(s))ds + \frac{t_1}{\Gamma(\alpha)}\int_0^1(1-s)^{\alpha-1}f(s,y(s))ds\Big| \\
\leq& \Big|\frac{1}{\Gamma(\alpha)}\int_0^{t_2}[(t_2-s)^{\alpha-1}-(t_1-s)^{\alpha-1}]f(s,y(s))ds + \frac{1}{\Gamma(\alpha)}\int_{t_2}^{t_1}(t_1-s)^{\alpha-1}f(s,y(s))ds\Big| \\
&+\Big|\frac{(t_2-t_1)}{\Gamma(\alpha)}\int_0^1(1-s)^{\alpha-1}f(s,y(s))ds\Big| \\
\leq& \frac{f_0}{\Gamma(\alpha+1)}(2(t_1-t_2)^\alpha + t_2^\alpha - t_1^\alpha + (t_1-t_2),
\end{aligned}$$

which tends to zero when $t_1 \to t_2$. This shows that B is equicontinuous, so B is relatively compact on P_r. Hence by Arzela-Ascoli theorem, B is compact on P_r. Therefore, Theorem 2.3.15 implies that problem (3.2.1) has at least one solution on $[0,1]$. This completes the proof. □

Example 3.2.3. Consider the impulsive fractional boundary value problem, where $\alpha = \frac{3}{2}$ and $n = 1$,

$$\begin{aligned}
{}^cD^{\frac{3}{2}}y(t) &= \frac{e^t y}{(9+t)(1+y^2)}, \quad t \neq \frac{1}{3}, \\
\Delta y\left(\frac{1}{3}\right) &= \frac{\|y\left(\frac{1}{3}\right)\|}{2+\|y\left(\frac{1}{3}\right)\|}, \quad \Delta y'\left(\frac{1}{3}\right) = \frac{\|y\left(\frac{1}{3}\right)\|}{3+\|y\left(\frac{1}{3}\right)\|}, \\
g(y) &= \sum_{i=1}^n a_i y(t_i).
\end{aligned}$$

Set $f(t,y) = \frac{e^t y}{(9+t)(1+y^2)}$, $\sum_{i=1}^n a_i < \frac{1}{4}$, where $0 < t_1 < t_2 < \cdots < t_n < 1 a_i$, $i = 1,\cdots,n$ are given positive constants.
Let y,$z \in [0,\infty)$ and $t \in [0,1]$, then clearly,

$$|f(t,y)| \leq \frac{e^t}{(9+t)}.$$

Also

$$|P_k(y_1) - P_k(y_2)| \leq \frac{1}{2}|y_1 - y_2|,$$

$$|Q_k(y_1) - Q_k(y_2)| \leq \frac{1}{3}|y_1 - y_2|,$$

and

$$|g(y_1) - g(y_2)| \leq \sum_{i=1}^n a_i|y_1 - y_2|.$$

Hence the condition (H7) holds with $M_1 = \sum_{i=1}^n a_i$, $M_2 = \frac{1}{2}$ and $M_3 = \frac{1}{3}$.
Now we will check the condition in Theorem 3.2.2 is satisfied with $\alpha = 1.5$, $M_1 = 0.1$, $M_2 = 0.5$ and $M_3 = 0.3$.

$$(M_1 + n(M_2 + M_3)) = 0.9 < 1.$$

Then by the Theorem 3.2.2 the problem (3.2.1) has at least one solution.

3.2.2 Uniqueness of solution

Our next result is based on Banach fixed point theorem.

Theorem 3.2.4. *Assume that:*

(H8) There exists constants $L_1, L_2 > 0$ such that $|f(t,y_1) - f(t,y_2)| \leq L_1|y_1 - y_2|$, $|g(y_1) - g(y_2)| \leq L_2|y_1 - y_2|$ for each $t \in [0,1]$ and all $y_1, y_2 \in \mathbb{R}$.

(H9) There exists a constant $L_3, L_4 > 0$ such that $|P_k(y_1) - P_k(y_2)| \leq L_3|y_1 - y_2|$, $|Q_k(y_1) - Q_k(y_2)| \leq L_4|y_1 - y_2|$ for each $t \in [0,1]$ and all $y_1, y_2 \in \mathbb{R}$.

if

$$\left[\frac{2L_1}{\Gamma(\alpha+1)} + L_2 + n(L_3 + L_4)\right] < 1, \tag{3.2.2}$$

then problem (3.2.1) *has a unique solution on* $[0,1]$.

Proof. Transform the problem (3.2.1) into a fixed point problem. Consider the operator $F : C([0,1],\mathbb{R}) \to C([0,1],\mathbb{R})$ defined by

$$\begin{aligned} F(y)(t) =& \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} f(s,y(s))ds - \frac{t}{\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-1} f(s,y(s))ds \\ &+ (1-t)\sum_{k=1}^{n} P_k(y(t_k)) + \sum_{k=1}^{n} G(t,t_k) Q_k(y(t_k)) + (1-t)g(y) + kt \end{aligned}$$

Clearly, the fixed points of the operator F are solutions of the problem (3.2.1). Now we use Banach contraction principle to prove that F has a fixed point. We shall show that F is a contraction. Let $y_1, y_2 \in C([0,1],\mathbb{R})$. Then, for each $t \in [0,1]$ we have

$$\begin{aligned} |F(y_1)(t) - F(y_2)(t)| \leq& \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}|f(s,y_1(s)) - f(s,y_2(s))|ds \\ &+ \frac{1}{\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-1}|f(s,y_1(s)) - f(s,y_2(s))|ds + |g(y_1) - g(y_2)| \\ &+ \sum_{k=1}^{n} |P_k(y_1(t_k)) - P_k(y_2(t_k))| + \sum_{k=1}^{n} |Q_k(y_1(t_k)) - Q_k(y_2(t_k))|, \\ \leq& \frac{L_1\|y_1 - y_2\|}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}ds + \frac{L_1\|y_1 - y_2\|}{\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-1}ds \\ &+ L_2\|y_1 - y_2\| + nL_3\|y_1 - y_2\| + nL_4\|y_1 - y_2\|, \\ \leq& \frac{2L_1}{\Gamma(\alpha+1)}\|y_1 - y_2\| + L_2\|y_1 - y_2\| + nL_3\|y_1 - y_2\| + nL_4\|y_1 - y_2\|. \end{aligned}$$

Thus

$$\|F(y_1) - F(y_2)\| \leq \left[\frac{2L_1}{\Gamma(\alpha+1)} + L_2 + n(L_3 + L_4)\right]\|y_1 - y_2\|.$$

Consequently F is a contraction. As a consequence of Banach fixed theorem, we deduce that F has a fixed point which is a solution of problem (3.2.1). □

Example 3.2.5. Consider the second impulsive fractional boundary value problem,

$$\begin{aligned} {}^cD^\alpha y(t) = f(t, y(t)) \quad \text{for} \quad t \in [0,1], \quad 1 < \alpha \leq 2, \\ \Delta y(t_k) = P_k(y(t_k)), \\ \Delta y'(t_k) = Q_k(y(t_k)), \\ y(0) = g(y), \qquad y(1) = k \qquad k \in \mathbb{R}, \end{aligned}$$

where $\alpha = 2$ and $n = 1$,

$$\begin{aligned} f(t, y) &= \frac{e^{-t} y}{(9 + e^t)(1 + y^2)}, \quad t \neq \frac{1}{4}, \\ \Delta y\left(\frac{1}{4}\right) &= \frac{\|y\left(\frac{1}{4}\right)\|}{5 + \|y\left(\frac{1}{4}\right)\|}, \quad \Delta y'\left(\frac{1}{4}\right) = \frac{\|y\left(\frac{1}{4}\right)\|}{7 + \|y\left(\frac{1}{4}\right)\|} \\ g(y) &= \sum_{i=1}^{n} a_i y(t_i). \end{aligned}$$

Set $\sum_{i=1}^{n} a_i < \frac{3}{4}$, where $0 < t_1 < t_2 < \cdots < t_n < 1 a_i$, $i = 1, \cdots, n$ are given positive constants.

Let $y_1, y_2 \in C([0,1], \mathbb{R})$ and $t \in [0,1]$. Then

$$\begin{aligned} |f(t, y_1) - f(t, y_2)| &= \frac{e^{-t}}{9 + e^t} \left| \frac{y_1}{(1 + y_1^2)} - \frac{y_2}{(1 + y_2^2)} \right| \\ &= \frac{e^{-t} |y_1 - y_2|}{(9 + e^t)(1 + y_1^2)(1 + y_2^2)}, \\ &\leq \frac{e^{-t}}{(9 + e^t)} |y_1 - y_2|, \\ &\leq \frac{1}{10} |y_1 - y_2|. \end{aligned}$$

also

$$|g(y_1) - g(y_2)| \leq \sum_{i=1}^{n} |y_1 - y_2|.$$

Hence the condition (H8) holds with $L_1 = \frac{1}{10}$ and $L_2 = \sum_{i=1}^{n} a_i$.
Clearly

$$|P_k(y_1) - P_k(y_2)| \leq \frac{1}{5} |y_1 - y_2|,$$

and

$$|Q_k(y_1) - Q_k(y_2)| \leq \frac{1}{7} |y_1 - y_2|.$$

Hence the condition (H9) holds with $L_3 = \frac{1}{5}$ and $L_4 = \frac{1}{7}$.
Now we will check the condition (3.2.2) is satisfied with $\alpha = 2$, $L_1 = 0.1$, $L_2 = 0.5$, $L_3 = \frac{1}{5}$ and

$L_4 = \frac{1}{7}$. By computations, we have

$$\left[\frac{2L_1}{\Gamma(\alpha+1)} + L_2 + n(L_3 + L_4)\right] = \frac{2(0.1)}{\Gamma(3)} + 0.5 + \left(\frac{1}{5} + \frac{1}{7}\right) = 0.9 < 1.$$

Then by the Theorem 3.2.4 the problem (3.2.1) has a unique solution.

Chapter 4

Ulam's stability of impulsive fractional differential equations

Ulam in 1940 discussed the stability of functional equations then he posed the problem "Under what conditions does there exist an additive mapping near an approximately additive mapping"? Hyers for the first time answered Ulam for the Banach spaces in 1941. That's why this type of stability is called Ulam-Hyers stability. Rassias in 1978 considered the different variables and provide a generalization result of the Ulam-Hyers stability of mappings. Ulam-Hyers stability and Ulam-Hyers-Rassias stability have been used by many mathematicians and the analysis of this area has grown rapidly in mathematical analysis as one of the central subject.

In this chapter, we present four Ulam's type stability results for impulsive fractional differential equations. J. Wang has studied Ulam's type stability for impulsive ordinary differential equations in [13] and for impulsive fractional differential equations in [14]. In first section, we will give four Ulam's type stability for fractional impulsive differential equations. Finally in the second section we will present generalized Ulam-Hyers-Rassias stability results with an example.

Consider the following impulsive fractional differential equation

$$\begin{aligned} &{}^cD^\alpha y(t) = f(t,y), \quad t \in J' = J \setminus \{t_1, \cdots, t_m\},\ J = [0,T],\ T > 0, \quad 0 < \alpha \leq 1, \\ &y(t_k + 0) - y(t_k) = I_k(y(t_k)), \quad k = 1, 2, \cdots, n, \end{aligned} \tag{4.0.1}$$

where $f : J \times \mathbb{R} \to \mathbb{R}$ is continuous, $I_k : \mathbb{R} \to \mathbb{R}$ and t_k satisfy $0 = t_0 < t_1 < \cdots < t_n < t_{n+1} = T < +\infty$, $y(t_k^+) = \lim_{\epsilon \to 0^+} y(t_k + \epsilon)$ and $y(t_k^-) = \lim_{\epsilon \to 0^-} y(t_k + \epsilon)$ represents the right and left limits of $y(t)$ at $t = t_k$.

4.1 Preliminaries

In this section, we present Ulam's stability concepts for equation (4.0.1), definitions and some remarks. Consider the Banach spaces $C(J, \mathbb{R})$ of all continuous functions from J into $\mathbb{R}$, $\mathrm{PC}(J, \mathbb{R}) =$

$\{y : J \to \mathbb{R} : y \in C((t_k, t_{k+1}], \mathbb{R}), k = 0, \cdots, n$ and there exist $y(t_k^-)$ and $y(t_k^+), k = 1, \cdots, n$, with $y(t_k^-) = y(t_k)\}$ and $\mathrm{PC}^1(J, \mathbb{R}) = \{y \in \mathrm{PC}(J, \mathbb{R}) : y' \in \mathrm{PC}(J, \mathbb{R})\}$. Let $\epsilon > 0$ and $\phi \in \mathrm{PC}(J, \mathbb{R}^+)$ is nondecreasing function. Assume the following inequalities

$$\begin{aligned} &|{}^cD^\alpha x(t) - f(t, x(t))| \le \epsilon, \quad t \in J', \quad 0 < \alpha \le 1, \\ &|\Delta x(t_k) - I_k(x(t_k))| \le \epsilon, \quad k = 1, 2, \cdots, n, \end{aligned} \tag{4.1.1}$$

$$\begin{aligned} &|{}^cD^\alpha x(t) - f(t, x(t))| \le \phi(t), \quad t \in J', \quad 0 < \alpha \le 1, \\ &|\Delta x(t_k) - I_k(x(t_k))| \le \phi(t), \quad k = 1, 2, \cdots, n, \end{aligned} \tag{4.1.2}$$

and

$$\begin{aligned} &|{}^cD^\alpha x(t) - f(t, x(t))| \le \epsilon\phi(t), \quad t \in J', \quad 0 < \alpha \le 1, \\ &|\Delta x(t_k) - I_k(x(t_k))| \le \epsilon\phi(t), \quad k = 1, 2, \cdots, n, \end{aligned} \tag{4.1.3}$$

Definition 4.1.1. Assume that there exists a real number $c_{f,n} > 0$ such that for each $\epsilon > 0$ and for each solution $x \in \mathrm{PC}^1(J, \mathbb{R})$ of the inequation (4.1.1) there exists a solution $y \in \mathrm{PC}^1(J, \mathbb{R})$ of equation (4.0.1) then equation (4.0.1) is Ulam-Hyers stable if

$$|x(t) - y(t)| \le c_{f,n}\epsilon, \quad t \in J.$$

Definition 4.1.2. Assume that there exists $\theta_{f,n} \in C(\mathbb{R}^+, \mathbb{R}^+)$, $\theta_{f,n}(0) = 0$ such that for each solution $x \in \mathrm{PC}^1(J, \mathbb{R})$ of the inequation (4.1.1) there exists a solution $y \in \mathrm{PC}^1(J, \mathbb{R})$ of equation (4.0.1) then equation (4.0.1) is generalized Ulam-Hyers stable if

$$|x(t) - y(t)| \le \theta_{f,n}(\epsilon), \quad t \in J.$$

Definition 4.1.3. Assume that there exists $c_{f,n,\phi} > 0$ such that for each $\epsilon > 0$ and for each solution $x \in \mathrm{PC}^1(J, \mathbb{R})$ of the inequation (4.1.3) there exists a solution $y \in \mathrm{PC}^1(J, \mathbb{R})$ of equation (4.0.1) then equation (4.0.1) is Ulam-Hyers-Rassias stable with respect to ϕ if

$$|x(t) - y(t)| \le c_{f,n,\phi}\epsilon\phi(t), \quad t \in J.$$

Definition 4.1.4. Assume that there exists $c_{f,n,\phi} > 0$, such that for each solution $x \in \mathrm{PC}^1(J, \mathbb{R})$ of the inequation (4.1.2) there exists a solution $y \in \mathrm{PC}^1(J, \mathbb{R})$ of equation (4.0.1) then equation (4.0.1) is generalized Ulam-Hyers-Rassias stable with respect to ϕ if

$$|x(t) - y(t)| \le c_{f,n,\phi}\phi(t), \quad t \in J.$$

Remark 4.1.5. A function $x \in \mathrm{PC}^1(J, \mathbb{R})$ is a solution of inequation (4.1.1) if and only if there is $h \in \mathrm{PC}(J, \mathbb{R})$ and a sequence h_k, $k = 1, 2, \cdots, n$ depending on x such that

(i) $|h(t)| \le \epsilon$, $t \in J$ and $|h_k| \le \epsilon$, $k = 1, 2, \cdots, n$;

(ii) ${}^{c}D^{\alpha}x(t) = f(t, x(t)) + h(t)$, $t \in J'$, $0 < \alpha \leq 1$;

(iii) $\Delta x(t_k) = I_k(x(t_k)) + h_k$, $k = 1, 2, \cdots, n$.

Remark 4.1.6. [14] If $x \in \mathrm{PC}^1(J, \mathbb{R})$ is a solution of inequation (4.1.1) then x is a solution of the following integral inequality

$$\left| x(t) - x(0) - \sum_{i=1}^{k} I_i(x(t_i)) - \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, x(s)) ds \right| \leq \left(n + \frac{t^{\alpha}}{\Gamma(\alpha+1)} \right) \epsilon, \quad t \in J. \quad (4.1.4)$$

By Remark 4.1.5 we have

$$\begin{aligned} &{}^{c}D^{\alpha}x(t) = f(t, x(t)) + h(t), \quad t \in J', \quad 0 < \alpha \leq 1, \\ &\Delta x(t_k) = I_k(x(t_k)) + h_k, \quad k = 1, 2, \cdots, n. \end{aligned}$$

Then

$$x(t) = x(0) + \sum_{i=1}^{k} I_i(x(t_i)) + \sum_{i=1}^{k} h_i + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, x(s)) ds + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} h(s) ds, \; t \in (t_k, t_{k+1}].$$

It follows that

$$\begin{aligned} \left| x(t) - x(0) - \sum_{i=1}^{k} I_i(x(t_i)) - \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, x(s)) ds \right| &\leq \sum_{i=1}^{n} |h_i| + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} |h(s)| ds \\ &\leq n\epsilon + \frac{\epsilon}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} ds \\ &\leq \left(n + \frac{t^{\alpha}}{\Gamma(\alpha+1)} \right) \epsilon. \end{aligned}$$

4.2 Stability results

In this section we present the stability result of equation (4.0.1) (by definition 4.1.4) with example.

Theorem 4.2.1. *[14] Assume $f : J \times \mathbb{R} \to \mathbb{R}$ is continuous and there exists a constant $M_f > 0$ such that $|f(t, \mu) - f(t, \nu)| \leq M_f |\mu - \nu|$ for each $t \in J$ and all $\mu, \nu \in \mathbb{R}$. Moreover, $I_k : \mathbb{R} \to \mathbb{R}$ and there exists constants $\lambda_k > 0$ such that $|I_k(\mu) - I_k(\nu)| \leq \lambda_k |\mu - \nu|$ for all $\mu, \nu \in \mathbb{R}$ and $k = 1, 2, \cdots, n$. If there exists a $\gamma_\phi > 0$ such that $\frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \phi(s) ds \leq \gamma_\phi \phi(t)$ for each $t \in J$ where $\phi \in PC(J, \mathbb{R}^+)$ is nondecreasing function, then equation (4.0.1) is generalized Ulam-Hyers-Rassias stable with respect to ϕ.*

Example 4.2.2. Consider the following impulsive fractional differential equation

$$\begin{aligned} &{}^{c}D^{\alpha}y(t) = 0, \quad t \in (0, 1] \backslash \left\{ \frac{1}{4} \right\}, \\ &\Delta y\left(\frac{1}{4}\right) = \frac{\left|y\left(\frac{1}{4}\right)\right|^2}{1 + \left|y\left(\frac{1}{4}\right)\right|^2}, \end{aligned} \quad (4.2.1)$$

and the inequations

$$
\begin{aligned}
&|{}^cD^\alpha x(t)| \le \epsilon, \quad t \in (0,1]\backslash\left\{\frac{1}{4}\right\},\\
&\left|\Delta x\left(\frac{1}{4}\right) - \frac{\left|x\left(\frac{1}{4}\right)\right|^2}{1+\left|x\left(\frac{1}{4}\right)\right|^2}\right| \le \epsilon, \quad \epsilon > 0.
\end{aligned} \tag{4.2.2}
$$

Let $x \in \mathrm{PC}([0,1],\mathbb{R})$ be a solution of inequation (4.2.2) then there exists $h \in \mathrm{PC}([0,1],\mathbb{R})$ and $h_1 \in \mathbb{R}$ such that

(i) $|h(t)| \le \epsilon, \quad t \in [0,1], \quad |h_1| \le \epsilon$

(ii) ${}^cD^\alpha x(t) = h(t), \qquad t \in [0,1]\backslash\left\{\frac{1}{4}\right\},$

(iii) $\Delta x\left(\frac{1}{4}\right) = \frac{\left|x\left(\frac{1}{4}\right)\right|^2}{1+\left|x\left(\frac{1}{4}\right)\right|^2} + h_1.$

Then we have

$$
x(t) = x(0) + \chi_{\left(\frac{1}{4},1\right]}(t)\left(\frac{\left|x\left(\frac{1}{4}\right)\right|^2}{1+\left|x\left(\frac{1}{4}\right)\right|^2} + h_1\right) + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}h(s)ds
$$

where $\chi_{\left(\frac{1}{4},1\right]}(t)$ is a characteristic function of $\left(\frac{1}{4},1\right]$.
Assume the unique solution $y(t)$ of (4.2.1) given by

$$
y(t) = x(0) + \chi_{\left(\frac{1}{4},1\right]}(t)\frac{\left|y\left(\frac{1}{4}\right)\right|^2}{1+\left|y\left(\frac{1}{4}\right)\right|^2}.
$$

Then we have

$$
\begin{aligned}
|x(t)-y(t)| =& \left|\chi_{\left(\frac{1}{4},1\right]}(t)\left(\frac{\left|x\left(\frac{1}{4}\right)\right|^2}{1+\left|x\left(\frac{1}{4}\right)\right|^2} - \frac{\left|y\left(\frac{1}{4}\right)\right|^2}{1+\left|y\left(\frac{1}{4}\right)\right|^2} + h_1\right) + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}h(s)ds\right|\\
\le& \chi_{\left(\frac{1}{4},1\right]}(t)\left|x\left(\frac{1}{4}\right) - y\left(\frac{1}{4}\right)\right|^2 + |h_1| + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}h(s)ds\\
\le& \chi_{\left(\frac{1}{4},1\right]}(t)\left|x\left(\frac{1}{4}\right) - y\left(\frac{1}{4}\right)\right|^2 + \epsilon + \frac{\epsilon}{\Gamma(\alpha+1)}, \quad t \in [0,1],
\end{aligned}
$$

which gives

$$
|x(t)-y(t)| \le \left(1+\frac{1}{\Gamma(\alpha+1)}\right)\epsilon + \left(\left(1+\frac{1}{\Gamma(\alpha+1)}\right)\epsilon\right)^2, \quad t \in [0,1].
$$

Hence equation (4.2.1) is generalized Ulam-Hyers stable.

Remark 4.2.3. In a Banach space, if we have a nth order differential equation then we use Green function technique to reduce its order in the Banach space as the following example shows.

Now we shall consider the following fractional differential equation

$$
{}^cD^\alpha y(t) = f(t,y(t)), \quad 1 < \alpha \le 2, \quad t \in [a,b], \tag{4.2.3}
$$

where $a < b < +\infty$ and $f \in C([a,b] \times \mathbb{R})$.

Let us denote the Green function of the following boundary value problem by G

$$
\begin{aligned}
&{}^cD^\alpha x(t) = f(t, x(t)) + h(t), \quad 1 < \alpha \le 2 \\
&x(a) = 0, \quad x(b) = 0.
\end{aligned}
$$

The Green function $G_\alpha : [a,b] \times [a,b] \to \mathbb{R}$ is defined as

$$
G_\alpha(t,s) = \begin{cases} \frac{1}{\Gamma(\alpha)} \left((t-s)^{\alpha-1} + \frac{(t-a)(b-s)^{\alpha-1}}{(a-b)} \right), & s \le t, \\ \frac{(t-a)(b-s)^{\alpha-1}}{\Gamma(\alpha)(a-b)}, & s \ge t. \end{cases} \tag{4.2.4}
$$

Theorem 4.2.4. *Assume $f : J \times \mathbb{R} \to \mathbb{R}$ is continuous and there exists a constant $M_f > 0$ such that $|f(t,\mu) - f(t,\nu)| \le M_f|\mu - \nu|$ for each $t \in J$ and all $\mu, \nu \in \mathbb{R}$. If there exists a $\gamma_\phi > 0$ such that $\int_a^b G_\alpha(t,s)\phi(s)ds \le \gamma_\phi \phi(t)$ for each $t \in J$ where $\phi \in PC(J, \mathbb{R}^+)$ is nondecreasing function, then equation* (4.2.3) *is generalized Ulam-Hyers-Rassias stable with respect to ϕ.*

Proof. Assume that $x \in C^2[a,b]$ be a solution of the following inequation

$$
|{}^cD^\alpha x(t) - f(t, x(t))| \le \phi(t), \quad \forall\ t \in [a,b].
$$

Firstly we remark that x is the solution of the following inequality

$$
\begin{aligned}
\left| x(t) - \left(\frac{b-t}{a-b} \right) y(a) - \left(\frac{t-a}{a-b} \right) y(b) - \int_a^b G_\alpha(t,s) f(s, x(s)) ds \right| &\le \int_a^b G_\alpha(t,s)\phi(s)ds \\
&\le \gamma_\phi \phi(t).
\end{aligned}
$$

Now we take y be the solution of following fractional boundary value problem

$$
\begin{aligned}
&{}^cD^\alpha y(t) = f(t, y(t)), \quad 1 < \alpha \le 2, \\
&y(a) = x(a), \quad y(b) = x(b).
\end{aligned}
$$

So, it is clear that

$$
y(t) = \int_a^b G_\alpha(t,s) f(s, y(s)) ds, \quad t \in [a,b].
$$

So for each $t \in (t_k, t_{k+1}]$, it follows that

$$
\begin{aligned}
|x(t) - y(t)| &\le \left| x(t) - \int_a^b G_\alpha(t,s) f(s, x(s)) ds \right| + \int_a^b G_\alpha(t,s) |f(s, x(s)) - f(s, y(s))| ds \\
&\le \gamma_\phi \phi(t) + M_f \int_a^b G_\alpha(t,s) |x(s) - y(s)| ds.
\end{aligned}
$$

By Theorem 2.3.16, there exists a constant $L_f > 0$ independent of $\gamma_\phi \phi(t)$ such that

$$
|x(t) - y(t)| \le L_f \gamma_\phi \phi(t) = c_{f,\phi} \phi(t).
$$

Thus, equation (4.2.3) is generalized Ulam-Hyers-Rassias stable. The proof is completed. □

Chapter 5

Positive solutions for multi-point fractional impulsive boundary value problem

Multi-point boundary value problems occurs in many areas of engineering applications like in modeling the flow of fluid such as oil, gas and water through multilayer porous medium. The existence of positive solutions for multi-point boundary value problems is one of the important areas of research due to its wide applications in engineering like modeling of physical problems having vibrations in a wire of uniform cross section and consisting of material having different densities, in the elastic stability theory and also its applications in fluid flow.

In this chapter, the study of existence of positive solutions of multi-point boundary value problems for fractional order impulsive differential equations with the theory of fixed point theorem in cones will be discussed. In [16] the existence of multiple positive solutions for the multi-point boundary value problem of second order differential equation with impulse effects has been discussed. Existence results of solutions for three-point impulsive fractional boundary problem have been given in [17]. We refer the reader to [18, 19] for positive solutions of boundary value problem without impulses and to [16, 17] with impulse effect.

M.U. Rehman and P.W. Eloe [5] have developed a simple and general method for constructing fixed point operator for fractional impulsive differential equations. The aim of the chapter is to apply the method of [5] to multi-point fractional impulsive boundary value problems to establish the existence results for positive solutions. The results of this chapter are the original work of the author and her supervisor [20].

Consider the fractional order multi-point impulsive boundary value problem

$$D^{\alpha}y(t)+\lambda g(t,y(t))=0,\quad t\in J,\ t\neq t_k, 1<\alpha\leq 2, \tag{5.0.1}$$

$$\Delta y'(t_k)+I_k(y(t_k))=0,\quad k=1,2,\cdots,n \tag{5.0.2}$$

$$ay(0)-by'(0)=\sum_{i=1}^{m-2}a_i y(\xi_i),\quad cy(1)+dy'(1)=\sum_{i=1}^{m-2}b_i y(\xi_i), \tag{5.0.3}$$

where $J=[0,1]$, $t_0=0<t_1<t_2<\cdots<t_n<t_{n+1}=1$, $\Delta y'(t_k)=y'(t_k^+)-y'(t_k^-)$, $y(t_k^+)$ and $y(t_k^-)$ represents the right hand limit and left hand limit of the function $y(t)$ at $t=t_k$, $0<\xi_i<\xi_{i+1}<1$, $\xi_i\neq t_k$ and $i=1,2,\cdots,m-3$.

For convenience we consider the following assumptions:

(H1) $g\in C(J\times\mathbb{R}^+,\mathbb{R}^+)$, $I_k\in C(\mathbb{R}^+,\mathbb{R}^+)$;

(H2) $\Delta>0$, $\sum_{i=1}^{m-2}\frac{a_i}{\delta}[d+(1-\xi_i)c]<1$, $\sum_{i=1}^{m-2}\frac{b_i}{\delta}(b+a\xi_i)<1$;
where

$$\Delta:=\begin{vmatrix} 1-\sum_{i=1}^{m-2}\frac{a_i}{\delta}[d+(1-\xi_i)c] & -\sum_{i=1}^{m-2}\frac{a_i}{\delta}(b+a\xi_i) \\ -\sum_{i=1}^{m-2}\frac{b_i}{\delta}[d+(1-\xi_i)c] & 1-\sum_{i=1}^{m-2}\frac{b_i}{\delta}(b+a\xi_i) \end{vmatrix}$$

(H3) $a,b,c,d>0$.

5.1 Green's function and its properties

Lemma 5.1.1. *Assume that (H1), (H2) and (H3) hold. Then $y\in PC^1([0,1])\cup C^2(\bar{J})$, $\bar{J}=J\backslash\{t_1,t_2,\cdots,t_n\}$, is a solution of multi-point impulsive boundary value problem* (5.0.1)-(5.0.3) *if and only if y is solution of following impulsive integral equation*

$$\begin{aligned} y(t)=&\lambda\int_0^1 G_\alpha(t,s)g(s,y(s))ds+\sum_{k=1}^{n}H(t,t_k)I_k(y(t_k))+\frac{1}{\delta}(c(1-t)+d)\Big[A(g(.,y(.)))+B(I_k(y(.)))\Big] \\ &+\frac{1}{\delta}(b+at)\Big[C(g(.,y(.)))+D(I_k(y(.)))\Big], \end{aligned} \tag{5.1.1}$$

where

$$G_\alpha(t,s)=\begin{cases} \frac{1}{\delta}(b+at)\left(\frac{c(1-s)^{\alpha-1}}{\Gamma(\alpha)}+\frac{d(1-s)^{\alpha-2}}{\Gamma(\alpha-1)}\right)-\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}, & 0\leq s\leq t\leq 1; \\ \frac{1}{\delta}(b+at)\left(\frac{c(1-s)^{\alpha-1}}{\Gamma(\alpha)}+\frac{d(1-s)^{\alpha-2}}{\Gamma(\alpha-1)}\right), & 0\leq t\leq s\leq 1, \end{cases} \tag{5.1.2}$$

$$H(t,t_k)=\begin{cases} \frac{1}{\delta}(b+at)\Big(c(1-t_k)+d\Big)-(t-t_k), & 0\leq t_k\leq t\leq 1; \\ \frac{1}{\delta}(b+at)\Big(c(1-t_k)+d\Big), & 0\leq t\leq t_k\leq 1, \end{cases} \tag{5.1.3}$$

$$A(g(.,y(.))) := \frac{\lambda}{\Delta}\begin{vmatrix} \sum_{i=1}^{m-2} a_i \int_0^1 G_\alpha(\xi_i,s)g(s,y(s))ds & -\sum_{i=1}^{m-2}\frac{a_i}{\delta}(b+a\xi_i) \\ \sum_{i=1}^{m-2} b_i \int_0^1 G_\alpha(\xi_i,s)g(s,y(s))ds & 1-\sum_{i=1}^{m-2}\frac{b_i}{\delta}(b+a\xi_i) \end{vmatrix}, \tag{5.1.4}$$

$$B(I_k(y(.))) := \frac{1}{\Delta}\begin{vmatrix} \sum_{i=1}^{m-2} a_i \sum_{k=1}^{n} H(\xi_i,t_k)I_k(y(t_k)) & -\sum_{i=1}^{m-2}\frac{a_i}{\delta}(b+a\xi_i) \\ \sum_{i=1}^{m-2} b_i \sum_{k=1}^{n} H(\xi_i,t_k)I_k(y(t_k)) & 1-\sum_{i=1}^{m-2}\frac{b_i}{\delta}(b+a\xi_i) \end{vmatrix}, \tag{5.1.5}$$

$$C(g(.,y(.))) := \frac{\lambda}{\Delta}\begin{vmatrix} 1-\sum_{i=1}^{m-2}\frac{a_i}{\delta}[d+(1-\xi_i)c] & \sum_{i=1}^{m-2} a_i \int_0^1 G_\alpha(\xi_i,s)g(s,y(s))ds \\ -\sum_{i=1}^{m-2}\frac{b_i}{\delta}[d+(1-\xi_i)c] & \sum_{i=1}^{m-2} b_i \int_0^1 G_\alpha(\xi_i,s)g(s,y(s))ds \end{vmatrix}, \tag{5.1.6}$$

$$D(I_k(y(.))) := \frac{1}{\Delta}\begin{vmatrix} 1-\sum_{i=1}^{m-2}\frac{a_i}{\delta}[d+(1-\xi_i)c] & \sum_{i=1}^{m-2} a_i \sum_{k=1}^{n} H(\xi_i,t_k)I_k(y(t_k)) \\ -\sum_{i=1}^{m-2}\frac{b_i}{\delta}[d+(1-\xi_i)c] & \sum_{i=1}^{m-2} b_i \sum_{k=1}^{n} H(\xi_i,t_k)I_k(y(t_k)) \end{vmatrix}, \tag{5.1.7}$$

$$\Delta := \begin{vmatrix} 1-\sum_{i=1}^{m-2}\frac{a_i}{\delta}[d+(1-\xi_i)c] & -\sum_{i=1}^{m-2}\frac{a_i}{\delta}(b+a\xi_i) \\ -\sum_{i=1}^{m-2}\frac{b_i}{\delta}[d+(1-\xi_i)c] & 1-\sum_{i=1}^{m-2}\frac{b_i}{\delta}(b+a\xi_i) \end{vmatrix} \quad \textit{and} \quad \delta := a(c+d)+bc. \tag{5.1.8}$$

Proof. Assume that $y(t)$ is solution of the multi-point boundary value problem (5.0.1)-(5.0.3), then using Lemma 3.1.1, we have

$$y(t) = y(0) + y'(0)t - \sum_{0<t_k<t} I_k(y(t_k))(t-t_k) - \lambda\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}g(s,y(s))ds. \tag{5.1.9}$$

Let $c_1 = y(0)$ and $c_2 = y'(0)$, the first boundary condition in (5.0.3) takes the form

$$ac_1 - bc_2 = \sum_{i=1}^{m-2} a_i y(\xi_i). \tag{5.1.10}$$

Now, from (5.1.9), we have

$$y(1) = c_1 + c_2 - \sum_{k=1}^{n} I_k(y(t_k))(1-t_k) - \lambda\int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}g(s,y(s))ds, \tag{5.1.11}$$

$$y'(1) = c_2 - \sum_{k=1}^{n} I_k(y(t_k)) - \lambda\int_0^1 \frac{(1-s)^{\alpha-2}}{\Gamma(\alpha-2)}g(s,y(s))ds. \tag{5.1.12}$$

Substituting $y(1)$ and $y'(1)$ from (5.1.11) and (5.1.12) in the second equation in (5.0.3),

$$\begin{aligned} cc_1 + (c+d)c_2 =& c\lambda\int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}g(s,y(s))ds + d\lambda\int_0^1 \frac{(1-s)^{\alpha-2}}{\Gamma(\alpha-1)}g(s,y(s))ds \\ &+ c\sum_{k=1}^{n} I_k(y(t_k))(1-t_k) + d\sum_{k=1}^{n} I_k(y(t_k)) + \sum_{i=1}^{m-2} b_i y(\xi_i). \end{aligned} \tag{5.1.13}$$

Solving (5.1.12) and (5.1.13) for c_1 and c_2, we have

$$\begin{aligned} c_1 =& \frac{b}{\delta}\left(c\lambda\int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}g(s,y(s))ds + d\lambda\int_0^1 \frac{(1-s)^{\alpha-2}}{\Gamma(\alpha-1)}g(s,y(s))ds\right) + \frac{b}{\delta}\left(c\sum_{k=1}^{n} I_k(y(t_k))(1-t_k)\right. \\ &\left. + d\sum_{k=1}^{n} I_k(y(t_k))\right) + \frac{(c+d)}{\delta}\sum_{i=1}^{m-2} a_i y(x_i) + \frac{b}{\delta}\sum_{i=1}^{m-2} b_i y(x_i), \end{aligned} \tag{5.1.14}$$

$$\begin{aligned} c_2 =& \frac{a}{\delta}\left(c\lambda\int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}g(s,y(s))ds + d\lambda\int_0^1 \frac{(1-s)^{\alpha-2}}{\Gamma(\alpha-1)}g(s,y(s))ds\right) + \frac{a}{\delta}\left(c\sum_{k=1}^{n} I_k(y(t_k))(1-t_k)\right. \\ &\left. + d\sum_{k=1}^{n} I_k(y(t_k))\right) - \frac{c}{\delta}\sum_{i=1}^{m-2} a_i y(x_i) + \frac{a}{\delta}\sum_{i=1}^{m-2} b_i y(x_i). \end{aligned} \tag{5.1.15}$$

Inserting (5.1.12) and (5.1.13) in (5.1.9), we get

$$\begin{aligned} y(t) =& \frac{\lambda}{\delta}(b+at)\left(c\int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}g(s,y(s))ds + d\int_0^1 \frac{(1-s)^{\alpha-2}}{\Gamma(\alpha-1)}g(s,y(s))ds\right) \\ & - \lambda\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}g(s,y(s))ds + \frac{1}{\delta}(b+at)\left(c\sum_{k=1}^{n} I_k(y(t_k))(1-t_k) + d\sum_{k=1}^{n} I_k(y(t_k))\right) \\ & - \sum_{0<t_k<t} I_k(y(t_k))(t-t_k) + \frac{1}{\delta}[c(1-t)+d]\sum_{i=1}^{m-2} a_i y(\xi_i) + \frac{1}{\delta}(b+at)\sum_{i=1}^{m-2} b_i y(\xi_i), \end{aligned} \tag{5.1.16}$$

or,

$$\begin{aligned} y(t) =& \lambda\int_0^1 G_\alpha(t,s)g(s,y(s))ds + \sum_{k=1}^{n} H(t,t_k)I_k(y(t_k)) + \frac{1}{\delta}[c(1-t)+d]\sum_{i=1}^{m-2} a_i y(\xi_i) \\ & + \frac{1}{\delta}(b+at)\sum_{i=1}^{m-2} b_i y(\xi_i). \end{aligned} \tag{5.1.17}$$

Now from above equation, we have

$$\begin{aligned} \sum_{i=1}^{m-2} a_i y(\xi_i) =& \sum_{i=1}^{m-2} a_i\lambda\int_0^1 G_\alpha(\xi_i,s)g(s,y(s))ds + \sum_{i=1}^{m-2} a_i\sum_{k=1}^{n} H(\xi_i,t_k)I_k(y(t_k)) \\ & + \sum_{i=1}^{m-2}\frac{a_i}{\delta}[d+c(1-\xi_i)]\sum_{i=1}^{m-2} a_i y(\xi_i) + \sum_{i=1}^{m-2}\frac{a_i}{\delta}(b+a\xi_i)\sum_{i=1}^{m-2} b_i y(\xi_i), \end{aligned} \tag{5.1.18}$$

$$\begin{aligned} \sum_{i=1}^{m-2} b_i y(\xi_i) =& \sum_{i=1}^{m-2} b_i\lambda\int_0^1 G_\alpha(\xi_i,s)g(s,y(s))ds + \sum_{i=1}^{m-2} b_i\sum_{k=1}^{n} H(\xi_i,t_k)I_k(y(t_k)) \\ & + \sum_{i=1}^{m-2}\frac{b_i}{\delta}[(d+c(1-\xi_i)]\sum_{i=1}^{m-2} a_i y(\xi_i) + \sum_{i=1}^{m-2}\frac{b_i}{\delta}(b+a\xi_i)\sum_{i=1}^{m-2} b_i y(\xi_i). \end{aligned} \tag{5.1.19}$$

From equation (5.1.18) and (5.1.19), we have

$$\sum_{i=1}^{m-2} a_i y(\xi_i) = \frac{1}{\Delta}\begin{vmatrix} \sum_{i=1}^{m-2} a_i\lambda\int_0^1 G_\alpha(\xi_i,s)g(s,y(s))ds + \sum_{i=1}^{m-2} a_i\sum_{k=1}^{n} H(\xi_i,t_k)I_k(y(t_k)) & -\sum_{i=1}^{m-2}\frac{a_i}{\delta}(b+a\xi_i) \\ \sum_{i=1}^{m-2} b_i\lambda\int_0^1 G_\alpha(\xi_i,s)g(s,y(s))ds + \sum_{i=1}^{m-2} b_i\sum_{k=1}^{n} H(\xi_i,t_k)I_k(y(t_k)) & 1-\sum_{i=1}^{m-2}\frac{b_i}{\delta}(b+a\xi_i) \end{vmatrix}, \tag{5.1.20}$$

$$\sum_{i=1}^{m-2} b_i y(\xi_i) = \frac{\lambda}{\Delta}\begin{vmatrix} 1-\sum_{i=1}^{m-2}\frac{a_i}{\delta}[d+(1-\xi_i)c] & \sum_{i=1}^{m-2} a_i\lambda\int_0^1 G_\alpha(\xi_i,s)g(s,y(s))ds + \sum_{i=1}^{m-2} a_i\sum_{k=1}^{n} H(\xi_i,t_k)I_k(y(t_k)) \\ -\sum_{i=1}^{m-2}\frac{b_i}{\delta}[d+(1-\xi_i)c] & \sum_{i=1}^{m-2} b_i\lambda\int_0^1 G_\alpha(\xi_i,s)g(s,y(s))ds + \sum_{i=1}^{m-2} b_i\sum_{k=1}^{n} H(\xi_i,t_k)I_k(y(t_k)) \end{vmatrix}. \tag{5.1.21}$$

Substituting (5.1.20) and (5.1.21) in (5.1.17) we get (5.1.1).
Conversely, if y satisfies (5.1.1), then one can easily show that y is solution of boundary value problem (5.0.1)-(5.0.3). □

Lemma 5.1.2. *The functions $G_\alpha(t,s)$ and $H(t,t_k)$ defined by* (5.1.2) *and* (5.1.3) *respectively satisfies the following conditions:*

(i) $G_\alpha(t,s) > 0$,

(ii) $H(t,t_k) > 0$,

(iii) $G_\alpha(t,s) \geq \gamma_\alpha G_\alpha(s,s)$ *where* $\gamma_\alpha = \frac{bd}{(b+a)(\frac{c}{\alpha-1}+d)}$,

(iv) $H(t,t_k) \geq \gamma_\alpha H(t_k,t_k)$.

Proof. (i) For $0 \leq s \leq t \leq 1$

$$\begin{aligned} G_\alpha(t,s) =& \frac{1}{\delta}(b+at)\left(\frac{c(1-s)^{\alpha-1}}{\Gamma(\alpha)} + \frac{d(1-s)^{\alpha-2}}{\Gamma(\alpha-1)}\right) - \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \\ \geq& \frac{1}{\delta}b\left(\frac{c(1-s)^{\alpha-1}}{\Gamma(\alpha)} + \frac{d(1-s)^{\alpha-2}}{\Gamma(\alpha-1)}\right) - \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}. \end{aligned}$$

As $0 \leq t-s \leq 1-s$, $\frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} \geq \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}$ and $-1 > -\frac{bc}{\delta}$, therefore

$$G_\alpha(t,s) \geq \frac{bd}{\delta}\frac{(1-s)^{\alpha-2}}{\Gamma(\alpha-1)} > 0.$$

For $0 \leq t \leq s \leq 1$

$$G_\alpha(t,s) = \frac{1}{\delta}(b+at)\left(\frac{c(1-s)^{\alpha-1}}{\Gamma(\alpha)} + \frac{d(1-s)^{\alpha-2}}{\Gamma(\alpha-1)}\right) > 0.$$

So $G_\alpha(t,s) > 0$.

(ii) For $0 \le t_k \le t \le 1$

$$\begin{aligned} H(t,t_k) =& \frac{1}{\delta}(b+at)\Big(c(1-t_k)+d\Big)-(t-t_k) \\ \ge& \frac{1}{\delta}b\Big(c(1-t_k)+d\Big)-(t-t_k) \\ \ge& \frac{1}{\delta}b\Big(c(1-t_k)+d\Big)-\frac{bc}{\delta}(t-t_k) \\ =& \frac{bc}{\delta}(1-t)+\frac{bd}{\delta}>0. \end{aligned}$$

For $0 \le t \le t_k \le 1$

$$H(t,t_k) = \frac{1}{\delta}(b+at)\Big(c(1-t_k)+d\Big)>0.$$

So $H(t,t_k)>0$.

(iii) For $0 \le s \le t \le 1$

$$G_\alpha(t,s) \ge \frac{bd}{\delta}\frac{(1-s)^{\alpha-2}}{\Gamma(\alpha-1)}$$

$$\begin{aligned} G_\alpha(s,s) =& \frac{1}{\delta}(b+as)\left(\frac{c(1-s)^{\alpha-1}}{\Gamma(\alpha)}+\frac{d(1-s)^{\alpha-2}}{\Gamma(\alpha-1)}\right) \\ \le& \frac{b+a}{\delta}\left(\frac{c(1-s)^{\alpha-1}}{\Gamma(\alpha)}+\frac{d(1-s)^{\alpha-2}}{\Gamma(\alpha-1)}\right) \\ \le& \frac{b+a}{\delta}\left(\frac{c}{\alpha-1}+d\right)\frac{(1-s)^{\alpha-2}}{\Gamma(\alpha-1)} \end{aligned}$$

$$\frac{1}{G_\alpha(s,s)} \ge \frac{\delta}{b+a}\left(\frac{1}{\left(\frac{c}{\alpha-1}+d\right)}\right)\frac{\Gamma(\alpha-1)}{(1-s)^{\alpha-2}}$$

$$\frac{G_\alpha(t,s)}{G_\alpha(s,s)} \ge \frac{bd}{(b+a)\left(\frac{c}{\alpha-1}+d\right)}$$

$$G_\alpha(t,s) \ge \frac{bd}{(b+a)\left(\frac{c}{\alpha-1}+d\right)}G_\alpha(s,s).$$

For $0 \le t \le s \le 1$

$$\begin{aligned} G_\alpha(t,s) =& \frac{1}{\delta}(b+at)\left(\frac{c(1-s)^{\alpha-1}}{\Gamma(\alpha)}+\frac{d(1-s)^{\alpha-2}}{\Gamma(\alpha-1)}\right) \\ \ge& \frac{1}{\delta}b\left(\frac{c(1-s)^{\alpha-1}}{\Gamma(\alpha)}+\frac{d(1-s)^{\alpha-2}}{\Gamma(\alpha-1)}\right) \end{aligned}$$

As $s \le 1$ and $\frac{b+as}{b+a} \le 1$. Therefore

$$\begin{aligned} G_\alpha(t,s) \ge& \frac{1}{\delta}\frac{b}{b+a}(b+as)\left(\frac{c(1-s)^{\alpha-1}}{\Gamma(\alpha)}+\frac{d(1-s)^{\alpha-2}}{\Gamma(\alpha-1)}\right) \\ =& \frac{b}{b+a}G_\alpha(s,s). \end{aligned}$$

So $G_\alpha(t,s) \geq \frac{bd}{(b+a)(\frac{c}{\alpha-1}+d)} G_\alpha(s,s)$, because $\frac{bd}{(b+a)(\frac{c}{\alpha-1}+d)} < \frac{b}{b+a}$. As

$$\frac{c}{\alpha-1} > 0$$
$$\frac{1}{\frac{c}{\alpha-1}+d} < \frac{1}{d}$$
$$\frac{bd}{(b+a)(\frac{c}{\alpha-1}+d)} < \frac{b}{b+a}$$

Hence $G_\alpha(t,s) \geq \gamma_\alpha G_\alpha(s,s)$, where $\gamma_\alpha = \frac{bd}{(b+a)(\frac{c}{\alpha-1}+d)}$.

(iv) For $0 \leq t_k \leq t \leq 1$

$$\begin{aligned} H(t,t_k) =& \frac{1}{\delta}(b+at)\Big(c(1-t_k)+d\Big) - (t-t_k) \\ \geq& \frac{1}{\delta} b\Big(c(1-t_k)+d\Big) - (t-t_k) \end{aligned}$$

As $-1 > -\frac{bc}{\delta}$

$$\begin{aligned} H(t,t_k) \geq& \frac{1}{\delta} b\Big(c(1-t_k)+d\Big) - \frac{bc}{\delta}(t-t_k) \\ =& \frac{1}{\delta} bc(1-t) + \frac{bd}{\delta} \\ \geq& \frac{1}{\delta} b(c(1-t_k)+d) \\ \geq& \frac{1}{\delta} b \frac{b+at_k}{b+a}(c(1-t_k)+d) \\ H(t,t_k) \geq& \frac{b}{b+a} H(t_k,t_k). \end{aligned}$$

For $0 \leq t \leq t_k \leq 1$

$$\begin{aligned} H(t,t_k) =& \frac{1}{\delta}(b+at)\Big(c(1-t_k)+d\Big) \\ \geq& \frac{1}{\delta} b\Big(c(1-t_k)+d\Big) \\ \geq& \frac{1}{\delta} b \frac{b+at_k}{b+a}(c(1-t_k)+d) \\ H(t,t_k) \geq& \frac{b}{b+a} H(t_k,t_k). \end{aligned}$$

So $H(t,t_k) \geq \gamma_\alpha H(t_k,t_k)$.

□

Construct a cone $K = \{y \in PC[J, \mathbb{R}^+), y(t) \geq \gamma_\alpha \|y\|\}$. Define operator $T_\lambda : K \to K$ as

$$\begin{aligned}T_\lambda y(t) =&\lambda \int_0^1 G_\alpha(t,s)g(s,y(s))ds + \sum_{k=1}^{n} H(t,t_k)I_k(y(t_k)) + \frac{1}{\delta}(c(1-t)+d)\Big[A(g(.,y(.))) + B(I_k(y(.)))\Big]\\ &+ \frac{1}{\delta}(b+at)\Big[C(g(.,y(.))) + D(I_k(y(.)))\Big].\end{aligned} \tag{5.1.22}$$

The fixed point of operator (5.1.22) are the solutions of the fractional impulsive boundary value problem (5.0.1)-(5.0.3).

Lemma 5.1.3. *Assume that (H1)-(H3) holds, then the operator defined by* (5.1.22) *is completely continuous.*

Proof. For $y \in K$, by Lemma 5.1.1, $T_\lambda y \geq 0$ and $T_\lambda y \in PC[J, \mathbb{R}^+)$ and we have

$$\begin{aligned}T_\lambda y(t) =&\lambda \int_0^1 G_\alpha(t,s)g(s,y(s))ds + \sum_{k=1}^{n} H(t,t_k)I_k(y(t_k)) + \frac{1}{\delta}(c(1-t)+d)\Big[A(g(.,y(.))) + B(I_k(y(.)))\Big]\\ &+ \frac{1}{\delta}(b+at)\Big[C(g(.,y(.))) + D(I_k(y(.)))\Big],\end{aligned}$$

Since $a,b,c,d > 0$ and$1 < \alpha \leq 2, \frac{d}{\left(\frac{c}{\alpha-1}+d\right)} < 1$ and $\frac{b}{b+a} < 1$. So $\frac{bd}{(b+a)\left(\frac{c}{\alpha-1}+d\right)} < 1$

$$\begin{aligned}Ty(t) \geq&\frac{bd}{(b+a)\left(\frac{c}{\alpha-1}+d\right)} \int_0^1 \lambda G_\alpha(s,s)g(s,y(s))ds + \frac{bd}{(b+a)\left(\frac{c}{\alpha-1}+d\right)} \sum_{k=1}^{n} H(t_k,t_k)I_k(y(t_k))\\ &+ \frac{bd}{(b+a)\left(\frac{c}{\alpha-1}+d\right)} \frac{1}{\delta}(c(1-t)+d)\Big[A(g(.,y(.))) + B(I_k(y(.)))\Big]\\ &+ \frac{bd}{(b+a)\left(\frac{c}{\alpha-1}+d\right)} \frac{1}{\delta}(b+at)\Big[C(g(.,y(.))) + D(I_k(y(.)))\Big],\\ =&\frac{bd}{(b+a)\left(\frac{c}{\alpha-1}+d\right)} \Big[\lambda \int_0^1 G_\alpha(s,s)g(s,y(s))ds + \sum_{k=1}^{n} H(t_k,t_k)I_k(y(t_k))\\ &+ \frac{1}{\delta}(c(1-t)+d)\Big[A(g(.,y(.))) + B(I_k(y(.)))\Big] + \frac{1}{\delta}(b+at)\Big[C(g(.,y(.))) + D(I_k(y(.)))\Big]\Big]\\ =&\gamma\|T_\lambda y\|_{PC}.\end{aligned}$$

Thus, $T(K) \subset K$.

Suppose $B \subset K$ is any bounded set, then, for any $y \in B$, there exists $M_o > 0$ such that $0 \leq y(t) \leq M_o$. Thus, by $g \in C(J \times \mathbb{R}^+, \mathbb{R}^+)$, and $I_k \in C(\mathbb{R}^+, \mathbb{R}^+)$, we have

$M_1 = \max\ g(s,y(s)) < +\infty$, $M_2 = \max(I_k y(t_k)) < +\infty$

$$\max_{y\in[0,M_0]} A(g(.,y(.))) \leq \frac{M_1}{\Delta}\begin{vmatrix} \sum_{i=1}^{m-2} a_i\lambda \int_0^1 G_\alpha(\xi_i,s)ds & -\sum_{i=1}^{m-2} \frac{a_i}{\delta}(b+a\xi_i) \\ \sum_{i=1}^{m-2} b_i\lambda \int_0^1 G_\alpha(\xi_i,s)ds & 1-\sum_{i=1}^{m-2} \frac{b_i}{\delta}(b+a\xi_i), \end{vmatrix} = M_1\tilde{A},$$

$$\max_{y\in[0,M_0]} B(I_k(y(.))) \leq \frac{M_2}{\Delta}\begin{vmatrix} \sum_{i=1}^{m-2} a_i \sum_{k=1}^{n} H(\xi_i,t_k)I_k & -\sum_{i=1}^{m-2}\frac{a_i}{\delta}(b+a\xi_i) \\ \sum_{i=1}^{m-2} b_i \sum_{k=1}^{n} H(\xi_i,t_k)I_k & 1-\sum_{i=1}^{m-2}\frac{b_i}{\delta}(b+a\xi_i), \end{vmatrix} = M_2\tilde{B},$$

$$\max_{y\in[0,M_0]} C(f(.,y(.))) \leq \frac{M_1}{\Delta}\begin{vmatrix} 1-\sum_{i=1}^{m-2}\frac{a_i}{\delta}[d+(1-\xi_i)c] & \sum_{i=1}^{m-2} a_i\lambda\int_0^1 G_\alpha(\xi_i,s)ds \\ -\sum_{i=1}^{m-2}\frac{b_i}{\delta}[d+(1-\xi_i)c] & \sum_{i=1}^{m-2} b_i\lambda\int_0^1 G_\alpha(\xi_i,s)ds, \end{vmatrix} = M_1\tilde{C},$$

$$\max_{y\in[0,M_0]} D(I_k(y(.))) \leq \frac{M_2}{\Delta}\begin{vmatrix} 1-\sum_{i=1}^{m-2}\frac{a_i}{\delta}[d+(1-\xi_i)c] & \sum_{i=1}^{m-2} a_i \sum_{k=1}^{n} H(\xi_i,t_k) \\ -\sum_{i=1}^{m-2}\frac{b_i}{\delta}[d+(1-\xi_i)c] & \sum_{i=1}^{m-2} b_i \sum_{k=1}^{n} H(\xi_i,t_k), \end{vmatrix} = M_2\tilde{D}.$$

where $\tilde{A}, \tilde{B}, \tilde{C}$ and $\tilde{D}$ respectively are the determinants appearing in above inequalities.

So, for any $y \in B$, we obtain

$$\begin{aligned}
\|T_\lambda y\| &= \max_{t\in[0,1]} \Big[\lambda\int_0^1 G_\alpha(t,s)g(s,y(s))ds + \sum_{k=1}^{n} H(t,t_k)I_k(y(t_k)) \\
&+\frac{1}{\delta}(c(1-t)+d)\Big[A(g(.,y(.))) + B(I_k(y(.)))\Big] + \frac{1}{\delta}(b+at)\Big[C(g(.,y(.))) + D(I_k(y(.)))\Big]\Big], \\
&\leq \max\Big[\lambda\int_0^1 G_\alpha(s,s)g(s,y(s))ds + \sum_{k=1}^{n} H(t_k,t_k)I_k(y(t_k)) \\
&+\frac{d}{\delta}\Big[A(g(.,y(.))) + B(I_k(y(.)))\Big] + \frac{1}{\delta}(b+a)\Big[C(g(.,y(.))) + D(I_k(y(.)))\Big]\Big], \\
&\leq \frac{M_1\lambda}{\delta\Gamma(\alpha)}(b+a)\left(\frac{c}{\alpha-1}+d\right) + \frac{nM_2}{\delta}(a+b)(c+d) + \frac{d}{\delta}(M_1\tilde{A}+M_2\tilde{B}) \\
&+\frac{(a+b)}{\delta}(M_1\tilde{C}+M_2\tilde{D}) < +\infty.
\end{aligned}$$

Thus, $T_\lambda(B)$ is bounded.

Now we show $T_\lambda(B)$ is equicontinuous. For any $t_1, t_2 \in [0,1]$, $s \in [0,1]$, we have $|G_\alpha(t_1,s) - G_\alpha(t_2,s)| \leq a(t_1-t_2)\left(\frac{c(1-s)^{\alpha-1}}{\Gamma(\alpha)} + \frac{d(1-s)^{\alpha-2}}{\Gamma(\alpha-1)}\right)$, $|H(t_1,t_k)-H(t_2,t_k)| \leq (t_1-t_2)(\frac{ac}{\delta}+\frac{ad}{\delta}-1)$

$$\|T_\lambda y(t_1) - T_\lambda y(t_2)\| \leq \max_{t\in[0,1]} \Big[\lambda \int_0^1 |G_\alpha(t_1,s) - G_\alpha(t_2,s)| g(s,y(s))ds + \sum_{k=1}^{n} |H(t_1,t_k) - H(t_2,t_k)| I_k(y(t_k))$$
$$-\frac{c}{\delta}(t_1-t_2)\Big[A(g(.,y(.))) + B(I_k(y(.)))\Big] + \frac{a}{\delta}(t_1-t_2)\Big[C(g(.,y(.))) + D(I_k(y(.)))\Big]\Big],$$
$$\leq \lambda M_1 \int_0^1 a(t_1-t_2)\left(\frac{c(1-s)^{\alpha-1}}{\Gamma(\alpha)} + \frac{d(1-s)^{\alpha-2}}{\Gamma(\alpha-1)}\right) + nM_2(t_1-t_2)(\frac{ac}{\delta} + \frac{ad}{\delta} - 1)$$
$$-\frac{c}{\delta}(t_1-t_2)(M_1\tilde{A} + M_2\tilde{B}) + \frac{a}{\delta}(t_1-t_2)(M_1\tilde{C} + M_2\tilde{D}),$$
$$=(t_1-t_2)\Big[a\lambda M_1\left(\frac{c}{\Gamma(\alpha+1)} + \frac{d}{\Gamma(\alpha)}\right) + nM_2(\frac{ac}{\delta} + \frac{ad}{\delta} - 1)$$
$$-\frac{c}{\delta}(M_1\tilde{A} + M_2\tilde{B}) + \frac{a}{\delta}(M_1\tilde{C} + M_2\tilde{D})\Big].$$

Thus, $T_\lambda(B)$ is equicontinuous. We conclude that $T_\lambda : K \to K$ is compact.
So $T_\lambda : K \to K$ is completely continuous. □

5.2 Existence of positive solutions

In this section we give some results for the existence of positive solutions. For convenience and simplicity following notations will be used in the sequel.

$$g_0 = \lim_{y\to 0}\inf \min_{t\in[0,1]} \frac{g(t,y)}{y}, I_{k0} = \lim_{y\to 0}\inf \frac{I_k(y)}{y}, I_0 = \max_{1\leq k\leq n}\{I_{k0}\},$$

$$g^0 = \lim_{y\to 0}\sup \max_{t\in[0,1]} \frac{g(t,y)}{y}, I_k^0 = \lim_{y\to 0}\sup \frac{I_k(y)}{y}, I^0 = \max_{1\leq k\leq n}\{I_k^0\},$$

$$g_\infty = \lim_{y\to \infty}\inf \min_{t\in[0,1]} \frac{g(t,y)}{y}, I_{k\infty} = \lim_{y\to \infty}\inf \frac{I_k(y)}{y}, I_\infty = \max_{1\leq k\leq n}\{I_{k\infty}\},$$

$$g^\infty = \lim_{y\to \infty}\sup \max_{t\in[0,1]} \frac{g(t,y)}{y}, I_k^\infty = \lim_{y\to \infty}\sup \frac{I_k(y)}{y}, I^\infty = \max_{1\leq k\leq n}\{I_k^\infty\}.$$

Define constants $Q_1 = \frac{1}{\delta}\left(\frac{\lambda}{\Gamma(\alpha)}(a+b)\left(\frac{c}{\alpha-1} + d\right) + (c+d)\tilde{A} + (a+b)\tilde{C}\right)$,
$Q_2 = \frac{1}{\delta}\left(n(a+b)(c+d) + (c+d)\tilde{B} + (a+b)\tilde{D}\right)$, $Q_3 = \frac{b\gamma_\alpha}{\delta\Gamma(\alpha+1)}(c+\alpha d)$.

Theorem 5.2.1. *Assume that $g^0 > 0$ and $g_\infty < +\infty$, then for*

$$\lambda \in \left(\frac{1}{Q_3 g_\infty} \quad \frac{1}{Q_1 g^0 + Q_2 I^0}\right), \tag{5.2.1}$$

the boundary value problem (5.0.1)-(5.0.3) *has at least one positive solution.*

Proof. By (5.2.1), there exist $\varepsilon > 0$ such that

$$\frac{1}{Q_3(g_\infty - \varepsilon)} \leq \lambda \leq \frac{1}{Q_1(g^0+\varepsilon) + Q_2(I^0+\varepsilon)}. \tag{5.2.2}$$

By definition of g^0, there exists $r_1 > 0$ such that

$$g(t,y) \le (g^0+\varepsilon)y, I_k(y) \le (I_k^0+\varepsilon)y \quad \text{for} \quad 0<y\le r_1, t\in[0,1], k=1,2,...,n. \tag{5.2.3}$$

So for $t \in [0,1]$, $y \in \partial K$,with $\|y\| = r_1$, then from (5.2.2) and(5.2.3) we have

$$\begin{aligned}
\|T_\lambda y\| \le & \lambda \int_0^1 G_\alpha(s,s)g(s,y(s))ds + \sum_{k=1}^{n} H(t_k,t_k)I_k(y(t_k)) + \frac{1}{\delta}(c+d)\Big[A(g(.,y(.))) + B(I_k(y(.)))\Big] \\
& + \frac{1}{\delta}(b+at)\Big[C(g(.,y(.))) + D(I_k(y(.)))\Big], \\
\le & \frac{\lambda(a+b)}{\delta\Gamma(\alpha)}\left(\frac{c}{\alpha-1}+d\right)(g^0+\varepsilon)\|y\| + \frac{1}{\delta}n(a+b)(c+d)(I^0+\varepsilon)\|y\| \\
& + \frac{1}{\delta}(c+d)\left(\tilde{A}(g^0+\varepsilon)+\tilde{B}(I^0+\varepsilon)\right)\|y\| + \frac{1}{\delta}(a+b)\left(\tilde{C}(g^0+\varepsilon)+\tilde{D}(I^0+\varepsilon)\right)\|y\|, \\
= & \frac{1}{\delta}\left(\frac{\lambda(a+b)}{\Gamma(\alpha)}\left(\frac{c}{\alpha-1}+d\right)+(c+d)\tilde{A}+(a+b)\tilde{C}\right)(g^0+\varepsilon)\|y\| + \frac{1}{\delta}(n(a+b)(c+d)+(c+d)\tilde{B} \\
& +(a+b)\tilde{D})(I^0+\varepsilon)\|y\|, \\
= & \left(Q_1(g^0+\varepsilon)+Q_2(I^0+\varepsilon)\right)\|y\| \le \|y\|.
\end{aligned}$$

Hence, for $\Omega_1 = \{y \in PC^1[0,1] : \|y\| \le r_1\}$, we have

$$\|T_\lambda y\| \le \|y\|, \quad \text{for} \quad y \in K \cup \partial\Omega_1. \tag{5.2.4}$$

By definition of g_∞, there exists $\bar{r}_2 > 0$, such that

$$g(t,y) \ge (g_\infty - \varepsilon)y, \quad \text{for} \quad y \ge \bar{r}_2. \tag{5.2.5}$$

Define $r_2 = max\{2r_1, \bar{r}_2\}$. Let $y \in \partial K$ with $\|y\| = r_2$, then by (5.2.2) and (5.2.5) we have

$$\begin{aligned}
\|T_\lambda y\| \ge & \lambda \int_0^1 G_\alpha(t,s)g(s,y(s))ds \\
\ge & \lambda\gamma_\alpha \int_0^1 G_\alpha(s,s)(g_\infty-\varepsilon)yds \\
\ge & \frac{b\lambda\gamma_\alpha}{\delta}\int_0^1 \left(\frac{c(1-s)^{\alpha-1}}{\Gamma(\alpha)} + \frac{d(1-s)^{\alpha-2}}{\Gamma(\alpha-1)}\right)(g_\infty-\varepsilon)yds, \\
= & \frac{b\lambda\gamma_\alpha}{\delta\Gamma(\alpha+1)}(c+\alpha d)(g_\infty-\varepsilon)\|y\| \\
= & \lambda Q_3(g_\infty-\varepsilon)\|y\| \ge \|y\|.
\end{aligned}$$

Thus, for $\Omega_2 = \{y \in PC^1[0,1] : \|y\| \le r_2\}$, we have

$$\|T_\lambda y\| \ge \|y\|, \quad \text{for} \quad y \in K \cup \partial\Omega_2. \tag{5.2.6}$$

From (5.2.4), (5.2.6) and Theorem 2.3.18 we conclude that the operator T_λ has a fixed point $y \in K \cap (\bar{\Omega}_2 \backslash \Omega_1)$, with $r_1 \le \|y\| \le r_2$. □

Corollary 5.2.2. *Assume that* $g^0 = 0 = I^0$ *and* $g_\infty = +\infty$*, then for* $\lambda > 0$*, the boundary value problem* (5.0.1)-(5.0.3) *has at least one positive solution.*

Theorem 5.2.3. *Assume that* $g_0 > 0$ *and* $g^\infty < +\infty$*, then for*

$$\lambda \in \left(\frac{1}{Q_3 g_0} \quad \frac{1}{Q_1 g^\infty + Q_2 I^\infty} \right), \tag{5.2.7}$$

the boundary value problem (5.0.1)-(5.0.3) *has at least one positive solution.*

Proof. By (5.2.7), there exist $\varepsilon > 0$ such that

$$\frac{1}{Q_3(g_0 - \varepsilon)} \le \lambda \le \frac{1}{Q_1(g^\infty + \varepsilon) + Q_2(I^\infty + \varepsilon)}. \tag{5.2.8}$$

By definition of g^∞, there exists $r_1 > 0$ such that

$$g(t,y) \le (g^\infty + \varepsilon)y, I_k(y) \le (I_k^\infty + \varepsilon)y \qquad \text{for} \qquad 0 < y \le r_1, t \in [0,1], k = 1, 2, ..., n. \tag{5.2.9}$$

So for $t \in [0,1]$, $y \in \partial K$,with $\|y\| = r_1$, then from (5.2.8) and(5.2.9) we have

$$\begin{aligned}
\|T_\lambda y\| \le& \lambda \int_0^1 G_\alpha(s,s) g(s,y(s)) ds + \sum_{k=1}^{n} H(t_k,t_k) I_k(y(t_k)) + \frac{1}{\delta}(c+d)\Big[A(g(.,y(.))) + B(I_k(y(.)))\Big] \\
&+ \frac{1}{\delta}(b+at)\Big[C(g(.,y(.))) + D(I_k(y(.)))\Big], \\
\le& \frac{\lambda(a+b)}{\delta\Gamma(\alpha)} \left(\frac{c}{\alpha-1} + d \right) (g^\infty + \varepsilon)\|y\| + \frac{1}{\delta} n(a+b)(c+d)(I^\infty + \varepsilon)\|y\| \\
&+ \frac{1}{\delta}(c+d)\left(\tilde{A}(g^\infty + \varepsilon) + \tilde{B}(I^\infty + \varepsilon) \right) \|y\| + \frac{1}{\delta}(a+b)\left(\tilde{C}(g^\infty + \varepsilon) + \tilde{D}(I^\infty + \varepsilon) \right) \|y\|, \\
=& \frac{1}{\delta}\left(\frac{\lambda(a+b)}{\Gamma(\alpha)} \left(\frac{c}{\alpha-1} + d \right) + (c+d)\tilde{A} + (a+b)\tilde{C} \right) (g^\infty + \varepsilon)\|y\| + \frac{1}{\delta}(n(a+b)(c+d) + (c+d)\tilde{B} \\
&+ (a+b)\tilde{D})(I^\infty + \varepsilon)\|y\|, \\
=& \left(Q_1(g^\infty + \varepsilon) + Q_2(I^\infty + \varepsilon) \right) \|y\| \le \|y\|.
\end{aligned}$$

Hence, for $\Omega_1 = \{y \in PC^1[0,1] : \|y\| \le r_1\}$, we have

$$\|T_\lambda y\| \le \|y\|, \qquad \text{for} \qquad y \in K \cup \partial\Omega_1. \tag{5.2.10}$$

By definition of g_0, there exists $\bar{r}_2 > 0$, such that

$$g(t,y) \ge (g_0 - \varepsilon)y, \qquad \text{for} \qquad y \ge \bar{r}_2. \tag{5.2.11}$$

Define $r_2 = \max\{2r_1, \bar{r}_2\}$. Let $y \in \partial K$ with $\|y\| = r_2$, then by (5.2.8) and (5.2.11) we have

$$
\begin{aligned}
\|T_\lambda y\| \geq & \lambda \int_0^1 G_\alpha(t,s) g(s, y(s)) ds, \\
\geq & \lambda \gamma_\alpha \int_0^1 G_\alpha(s,s)(g_0 - \varepsilon) y ds, \\
\geq & \frac{b\lambda\gamma_\alpha}{\delta} \int_0^1 \left(\frac{c(1-s)^{\alpha-1}}{\Gamma(\alpha)} + \frac{d(1-s)^{\alpha-2}}{\Gamma(\alpha-1)} \right) (g_0 - \varepsilon) y ds, \\
= & \frac{b\lambda\gamma_\alpha}{\delta\Gamma(\alpha+1)} (c + \alpha d)(g_0 - \varepsilon)\|y\|, \\
= & \lambda Q_3 (g_0 - \varepsilon)\|y\| \geq \|y\|.
\end{aligned}
$$

Thus, for $\Omega_2 = \{y \in PC^1[0,1] : \|y\| \leq r_2\}$, we have

$$
\|T_\lambda y\| \geq \|y\|, \qquad \text{for} \qquad y \in K \cup \partial\Omega_2. \tag{5.2.12}
$$

From (5.2.10), (5.2.12) and Theorem 2.3.18 we conclude that the operator T_λ has a fixed point $y \in K \cap (\bar{\Omega}_2 \backslash \Omega_1)$, with $r_1 \leq \|y\| \leq r_2$.

□

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