Surface Waves in Orthotropic Elastic Medium with Voids

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Noureen Tayyab

Dedicated

to

My Beloved Parents

Contents

Abstract

In this thesis, propagation of surface waves in orthotropic elastic half space, with and without voids is discussed. Surface wave solutions are obtained by solving the governing equations for both cases (with and without voids). These solutions satisfy the boundary conditions and yield the frequency equations. Various graphs are plotted in each case for illustration purposes and analysis of the solution. The case of Rayleigh wave in orthotropic material with voids is a new problem and discussed in detail. Graphs for different choices of elastic constants are drawn between non dimensional speed and non dimensional wave number. The results established are valid for waves of smaller wave number. It is found that the speed of Rayleigh waves is affected considerably due to the presence of voids. The speed of Love waves is, however, found to remain unaffected by voids as well as rotation.

Chapter 1

Introduction

Waves are everywhere and have intrinsic effect on human life. Most of the information that we receive comes to us in the form of waves. We can cook with the help of waves, talk to others and see things all because of waves. Earthquakes are detected and studied by observing the properties of waves that they create. Waves are transmitted through the Earth to detect oil and gas deposits and to study the Earth's geological structure. Properties of materials are determined through the behavior of waves transmitted from them. Non destructive testing is the most efficient and economic technique to check the cracks and discontinuities in materials or parts of a system without destroying the material. In other words, when the inspection or test is completed the part or material can still be used. This technique is also used to ensure the quality of materials. In this testing waves are produced in the material and their behavior is studied. In recent years, elastic waves transmitted through the human body have been used for medical diagnosis and therapy.

Waves are classified as mechanical and electromagnetic waves. Mechanical waves require a medium to propagate while the electromagnetic waves do not. Here in this thesis we are only concerned with the former one. Mechanical waves are disturbances in a deformable medium originated by the forced motion of a portion of the medium, which propagates from its source point to other positions and transfer energy without transferring the medium's particles.

The history of the study of wave propagation is long and fascinating. The notion of linear elasticity was established by the English scientist Robert Hooke in 1660, but not in a way that was expressible in terms of stress and strain. He observed that for many materials the displacement under a load was proportional to the force applied. But the main incentive for the early work on elastic waves was the belief which remained until the middle of nineteenth century that light is a wave which can propagate through a special medium called as elastic aether. The idea was proved wrong later but it helps in developing the theory known as elasticity theory now a days (Achenbach, 1973). Applications of elastic waves in various fields such as geophysics was also a stimulus for scientists and mathematicians to study the waves. The names which made contributions of lasting significance in the field were Poisson, Cauchy, Lame, Stokes, Christoffel, Lamb and many others. A detailed discussion of their work is given in the historical introduction to Love's treatise of the mathematical theory of elasticity (Love, 1944). Rayleigh (1885) and Love (1944, 1911) also made remarkable additions to the theory in the later part of nineteenth century.

The existence of Rayleigh waves in elastic isotropic half space was first noted by Rayleigh (1885). He considered the plane waves in an elastic isotropic half space and assumed that the amplitude of these waves decreases with depth. He found that plane waves propagating in this case are non dispersive. It was predicted by him that these waves may play an important role in the earthquakes and this was found true later. Love (1911) considered transverse waves of decaying amplitude in isotropic half space covered with an isotopic layer and the dispersive behavior of these waves was noted.

The linear theory of elastic materials with voids or pores is a generalization of the classical theory of elasticity. The classical theory is found inadequate for describing the behavior of materials having a distribution of pores. In classical theory, mechanical behavior of materials is studied without considering the effect of micro structure of material. However some discrepancies were observed between theoretical and experimental work, indicating that micro structure might be important. For example, discrepancies were found in the stress concentrations in the area of holes, cracks, and particularly in materials consisting of grains and pores. Therefore the theory of elastic materials with voids was established and void volume is taken as a separate kinematic variable. The theory has applications in the study of geological materials like rocks and soil, synthetic materials like ceramics and pressed powders, and biological structure like bones. In the limiting case when void volume tends to zero, the theory reduces to the classical theory of elasticity.

Goodman and Cowin (1972) introduced a continuum theory for granular materials like sand, grains, powder etc. The theory is developed from the formal arguments of continuum mechanics. The concept of distributed body was introduced, which represents a continuum model for granular as well as porous materials like rocks, soil, sponge etc. The key idea which serves as foundation for this theory is the representation of bulk density of the material as the product of matrix density (ratio of mass and volume without pores of the material) and the volume fraction field (the ratio of the volume occupied by voids to the bulk volume at a point of the material). This idea was later used by Nunziato and Cowin (1979) to develop a nonlinear theory of elastic material with voids. Cowin and Nunziato (1983) introduced a linear theory of elastic material with voids which helps in the mathematical study of the mechanical behavior of porous solids. They considered many applications of the linear theory by investigating the response of the materials to homogeneous deformations, pure bending of beams and small amplitudes of acoustic waves.

Puri and Cowin (1985) studied the behavior of plane waves in an elastic material with voids. They found that due to presence of voids there exist three plane waves. Out of the three waves one is transverse and two are longitudinal. It is explained that the transverse wave is same as we encounter in classical elasticity and is not effected by voids. Among the two longitudinal waves one is same as that of linear elasticity while the other is a result of void pores present in the material. The coupling of the equations of motion makes the waves dispersive in nature.

Thermal effects on linear elastic material with voids are studied by Iesan (1986). He derived basic field equations and discussed the condition for propagation of acceleration waves in homogeneous isotropic medium with voids.

Propagation of Love waves in an elastic layer with void pores is discussed by Dey et al (2004). Where it was reported that two types of Love waves can be transmitted through such a material. One of them is the same as discovered by Love (1911) and the second is the result of voids. But the results he presented are found erroneous.

Tomar and Singh (2005) cosidered the problem of transmission of longitudinal waves through a plane interface between two dissimilar porous elastic solid half spaces. It was observed that the presence of the voids influences the reflection and transmission parameters of the waves only for the case of low frequency incident longitudinal wave. Whereas for high frequency incident longitudinal wave, the results coincides with the results of classical elasticity, showing that there is no effect of presence of voids in the media.

Singh and Tomar (2006) studied the problem of reflection and transmission of transverse waves at a plane interface between two different porous elastic solid half spaces. It was reported that contrary to longitudinal waves, presence of voids effect significantly the reflection and transmission of transverse waves. They found that for high frequency incident wave, the transverse wave corresponding to the change in void-volume disappears completely. However, for low frequency incident wave, this wave exists.

Tomar and Ogden (2014) presented a mathematical study of two dimensional wave propagation in rotating elastic porous media. They explored the existence of three waves, one transverse and two longitudinal. All these waves are found to be coupled. This coupling is the consequence of rotation and porosity. In the absence of rotation transverse wave propagate without any influence of porosity and depicts the same properties as that of classical elasticity. However longitudinal waves remain coupled and show significant effects of voids on their propagation.

A. M. Abd-Alla et al (2015) found that Love waves in fibre-reinforced viscoelastic media with voids are not influenced by the presence of voids.

In this thesis, propagation of surface waves in orthotropic elastic half space with voids is studied and effect of porosity of material on the speed of surface waves is explored. In Particular, Love and Rayleigh waves are considered. Chapter wise summary of the thesis is given below.

Chapter 2 introduces the reader to the basic concepts of elasticity. The notion of stress, strain, their relationship, and effect of crystal symmetries on elastic stiffness tensors are revised. Equation of wave propagation is derived. Types of waves and some wave parameters like wave number, phase velocity etc are defined. This chapter also contains a brief review of propagation of Rayleigh and Love waves in isotropic elastic half space.

In chapter 3, mathematical expression for the speed of Love waves propagating in orthotropic elastic half space is derived. A brief introduction to the theory of linear elastic materials with voids as proposed by Cowin and Nunziato (1983), is given. Affect of rotation and porosity on the speed of Love waves in orthotropic elastic half space is studied. It is found that rotation of half space and porosity of material do not affect the speed of Love waves.

In chapter 4, propagation of Rayleigh waves in orthotropic elastic half space with and without voids is discussed. Approximate secular equation is established in case of porous orthotropic elastic half space. Contrary to Love waves, significant impact of porosity is noticed on speed of Rayleigh waves. Various numerical values of the elastic constants and void parameters are used to illustrate the effects of voids on the speed of Rayleigh waves. The results established are valid for waves of smaller wave number.

In chapter 5 all the results found throughout the thesis are concluded briefly.

Chapter 2

Basics of Elasticity

The purpose of this chapter is to make the reader familiar with some of the fundamental concepts of the theory of elasticity. Use of tensor notations in elasticity theory is frequent, so tensors are briefly discussed in Section 2.1. Section 2.2 deals with the very basic concepts of stress, strain, Hook's law, effect of crystal symmetry on elasticity constants, and equation of motion etc. The last section gives a review of the propagation of Love waves and Rayleigh waves in an isotropic material.

2.1 Fundamentals of tensors

Tensors are simple mathematical objects that can be used to describe physical properties of materials. These provide a natural and concise mathematical framework for formulating and solving problems in areas such as elasticity, fluid mechanics, and general relativity etc. The rank (or order) of a tensor is defined by the number of independent directions required to describe it. These are a mere generalization of scalars and vectors; a scalar is a zero rank tensor, and a vector is a first rank tensor. Properties that require one direction (as in the case of a first rank tensor) can be fully described by a 3×1 column vector, and properties that require two directions (as in the case of second rank tensor), can be described by 9 numbers, as a 3×3 matrix.

In mathematical terms, an n^{th} rank tensor in an m dimensional space is a mathematical object that has n indices and m^n components and obeys certain transformation rules. More specifically, a first order tensor is a linear operator that sends vectors to scalars and a second order tensor is a linear operator that sends vectors to vectors. Similarly a third order tensor is a linear operator that sends vectors to second order tensors. In the same fashion one can say that a tensor of order n is a linear mapping which maps a vector to a tensor of order $n-1$. Components of an nth rank tensor transform from one basis to another as

$$T'_{j_1 j_2 \ldots j_n} = Q_{i_1 j_1} Q_{i_2 j_2} \ldots Q_{i_n j_n} T_{i_1 i_2 \ldots i_n}, \tag{2.1.1}$$

where $Q_{i_1 j_1} Q_{i_2 j_2} \ldots Q_{i_n j_n}$ are elements of transformation matrix. Following are some elementary definitions which will be used in the subsequent chapters.

Definition 1. Transpose of a tensor $\mathbf{T}$ in an Euclidean vector space $\mathbf{V}$ is a function $\mathbf{T}^t$ defined as

$$(\mathbf{T}^t \mathbf{u}) \cdot \mathbf{v} = \mathbf{u} \cdot \mathbf{T}\mathbf{v}, \qquad \text{for any } \mathbf{u},\mathbf{v} \in \mathbf{V}. \tag{2.1.2}$$

Definition 2. A tensor $\mathbf{T}$ is symmetric if

$$\mathbf{T}^t = \mathbf{T}, \tag{2.1.3}$$

and antisymmetric if

$$\mathbf{T}^t = -\mathbf{T}. \tag{2.1.4}$$

Definition 3. The Kronecker delta δ_{ij} is a second rank symmetric tensor defined as

$$\delta_{ij} = \begin{cases} 1, & \text{if } i = j, \\ 0, & \text{if } i \neq j. \end{cases} \tag{2.1.5}$$

On contracting the index i where i ranges from 1 to n, one gets

$$\delta_{ii} = \delta_{11} + \delta_{22} + \ldots + \delta_{nn} = 1 + 1 \ldots + 1 = n. \tag{2.1.6}$$

Definition 4. The Levi-Civita or permutation tensor of rank three ϵ_{ijk} is an antisymmetric tensor defined as

$$\epsilon_{ijk} = \begin{cases} 1, & \text{for even permutations of ijk,} \\ -1, & \text{for odd permutations of ijk,} \\ 0, & \text{otherwise.} \end{cases} \tag{2.1.7}$$

Definition 5. A tensor is called isotropic tensor if its components do not change with the change in coordinate system.
The Kronecker delta δ_{ij} defined in Definition 3 is an isotropic tensor of rank two. And permutation tensor ϵ_{ijk} given in Definition 4 is an example of a third rank isotropic tensor.

2.2 Foundations of elasticity

Foundations of elasticity are laid upon the concept of *continuum approximation*. This is an idealization of matter as continuous material, that is, atoms and molecules are distributed continuously, so that one can think of material properties; for example, density, as a continuous function of position and time. The continuum material possesses two properties that it can be subdivide sufficiently many times and all sub-divisions have identical properties. The continuum approximation can not be used on the nanometer scale, but gives very good results on a scale larger than the gap between particles.

2.2.1 Strain, stress and their relationship

Strain and stress are the imperative concepts in the theory of elasticity. The following detail is about these notions and their relationship.
Strain:
When a force, either body force or surface force, is applied to a material, it effects every point of the material. This effect or amount of deformation experienced by the body compared to its original size and shape is defined as strain. Since the shape of an object is characterized by the relative positions of its particles, in order to

analyze deformation one has to focus on the change in displacements of neighboring particles. Let $\mathbf{u}(\mathbf{x}, t)$ be the displacement vector. In three dimensional linear case the strain at point P can be determined by the second rank strain tensor S_{ij}

$$S_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right). \tag{2.2.1}$$

It is evident from above equation that strain tensor S_{ij} is symmetric.

Stress:

In an orthonormal frame of reference, stress tensor T_{ik} is defined as

$$T_{ik} = \lim_{\Delta s_k \to 0}\left(\frac{\Delta F_i}{\Delta s_k}\right), \tag{2.2.2}$$

where ΔF_i is the i^{th} component of force acting on surface element Δs_k by the medium in positive direction. In the notation T_{ik}, the index i specifies the direction of force acting on a surface element perpendicular to the $k-$axis. It can be proved easily that the stress tensor is symmetric. The component of stress acting perpendicular to the surface element is called as *normal stress*, and the one tangential to the surface is said to be *shear stress*.

Relationship between stress and strain:

An elastic medium is the one which returns to its initial state after the removal of external forces. This returns to the initial state due to internal stresses. This means that stress causes strain or stress is a function of strain and vice versa. Also if there is no stress there is no strain and vice versa. In an elastic medium there is one to one relation between stress and strain. It is known experimentally that the elastic behaviour of most substances can be described adequately for small deformations with the help of first order term in the Taylor expansion of the function

$$T_{ij}\left(S_{kl}\right) = T_{ij}\left(0\right) + \left(\frac{\partial T_{ij}}{\partial S_{kl}}\right)_{S_{kl}=0} S_{kl} + \frac{1}{2}\left(\frac{\partial^2 T_{ij}}{\partial S_{kl}\partial S_{mn}}\right)_{S_{kl}=0} S_{kl}S_{mn} + \cdots . \tag{2.2.3}$$

The assumption if there is no stress, there is no strain, and vice versa, means $T_{ij}(0) = 0$. By ignoring the higher order terms, Eq. (2.2.3) reduces to

$$T_{ij} = C_{ijkl}S_{kl}, \tag{2.2.4}$$

where $C_{ijkl} = \left(\frac{\partial T_{ij}}{\partial S_{kl}}\right)_{S_{kl}=0}$.

The fourth rank tensor C_{ijkl} is called elastic stiffness tensor. This describes the most general linear relationship between the the second rank tensors T_{ij} and S_{kl}. This proportionality between stress and strain was first introduced by Robert Hooke in seventeenth century, for the simple case of a stretched string.

The elastic stiffness tensor C_{ijkl} has 81 components. Due to the symmetries of stress and strain tensors, the elastic stiffness tensor possesses two symmetries that are $C_{ijkl} = C_{jikl}$, and $C_{ijkl} = C_{jilk}$. In addition to these, the stored energy function also imposes a symmetry condition on stiffness tensor which is $C_{ijkl} = C_{klij}$. These three symmetries reduce the number of independent components of C_{ijkl} from 81 to 21.

It is conventional to use Voigt notation or two index representation of C_{ijkl} in which a pair of indices corresponds to a single index in the following manner.

$$(11) \longleftrightarrow 1, (22) \longleftrightarrow 2, (33) \longleftrightarrow 3,$$

$$(23) = (32) \longleftrightarrow 4, (13) = (31) \longleftrightarrow 5, (21) = (12) \longleftrightarrow 6.$$

Thus independent elastic constants are labeled by only two indices α and β ranging from 1 to 6 that is $C_{\alpha\beta} = C_{ijkl}$, where $\alpha \longleftrightarrow ij$ and $\beta \longleftrightarrow kl$.

2.2.2 Crystal symmetries and elastic stiffness tensor

A material is called elastically homogeneous if the components of elastic stiffness tensor C_{ijkl} are constants. Material is said to be isotropic if its properties remain same in every direction. On the other hand properties of an anisotropic medium are direction dependent. Anisotropy is a result of crystalline structure of solids. Crystal structure of solids is an ordered arrangement of atoms and molecules in a periodic manner. The smallest unit which on repetition generates the entire crystal is called a unit cell.

On the basis of symmetries of unit cells the structures of crystals are classified into seven groups called crystal systems. The seven crystal systems, listed in order of increasing symmetry, are: triclinic, monoclinic, orthorhombic, trigonal, tetragonal, hexagonal, and cubic.

Symmetry conditions of crystal systems reduce the number of independent components of tensor describing the physical properties of that crystal system. Thus the 21 independent elastic stiffness constants can be reduced further by considering the symmetry conditions found in different crystal structures. For example in the case of an isotropic material, these are reduced from 21 to 2. How these components reduce in isotropic and orthorhombic crystals is briefly discussed here.

As in isotropic crystals, physical properties do not depend upon the direction, that is they do not depend upon choice of reference frame. Particularly it means that elastic stiffness constant C_{ijkl} is not affected by the transformations of reference frame. It is well known that the tensor δ_{ij} remains invariant under all transformations. In order to get isotropic form of C_{ijkl}, every component of C_{ijkl} should be expressed in terms of components of the tensor δ_{ij}. Moreover due to the symmetry $\delta_{ij} = \delta_{ji}$ there are only three distinct fourth rank isotropic tensors: $\delta_{ij}\delta_{kl}$, $\delta_{ik}\delta_{jl}$, $\delta_{il}\delta_{jk}$. Therefore the isotropic form of elastic stiffness tensor can be obtained by writing it as a linear combination of all fourth rank isotropic tensors, that is,

$$C_{ijkl} = \lambda\delta_{ij}\delta_{kl} + \mu(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}). \tag{2.2.5}$$

The components of C_{ijkl} for isotropic crystals can be obtained from Eq. (2.2.5), and are given in the following matrix

$$C_{\alpha\beta} = \begin{bmatrix} \lambda+2\mu & \lambda & \lambda & 0 & 0 & 0 \\ \lambda & \lambda+2\mu & \lambda & 0 & 0 & 0 \\ \lambda & \lambda & \lambda+2\mu & 0 & 0 & 0 \\ 0 & 0 & 0 & \mu & 0 & 0 \\ 0 & 0 & 0 & 0 & \mu & 0 \\ 0 & 0 & 0 & 0 & 0 & \mu \end{bmatrix}. \tag{2.2.6}$$

The constants λ and μ are known as Lame's constants, named after the French mathematician G. Lame.

In orthotropic or orthorhombic crystals there are three mutually perpendicular axes all of different lengths. Unit cell for this crystal system is given in Fig. 2.1. The crystals belong to this system possess one center of symmetry and three 2-fold axis of symmetry. If an object is rotated about a line by an angle of 180 degrees and this leaves the object invariant then the object is said to have a two-fold axis of symmetry.

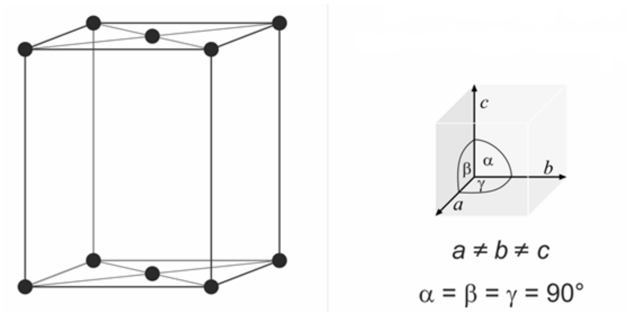


Figure 2.1: Unit cell for orthotropic crystals.

Choose a as two-fold axis of symmetry, then the transformation matrix specifying the change due to rotation is

$$Q = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{2.2.7}$$

From Eq. (2.1.1), one can write

$$C'_{ijkl} = Q_{pi}Q_{qj}Q_{rk}Q_{sl}C_{pqrs}, \tag{2.2.8}$$

C'_{ijkl} are components of elastic stiffness tensor after rotation or in new basis and Q_{ij} are components from Eq. (2.2.7). Applying the symmetry transformations to the tensor gives

$$C'_{11} = C'_{1111} = (-1)(-1)(-1)(-1)C_{1111} = C_{11}, \tag{2.2.9}$$

and

$$C'_{14} = C'_{1123} = (-1)(-1)(-1)(1)C_{1123} = -C_{14}. \tag{2.2.10}$$

But due to symmetry $C'_{14} = C_{14}$, therefore the result is

$$C_{14} = C_{41} = 0. \tag{2.2.11}$$

Similarly for C_{15}

$$C'_{15} = C'_{1113} = (-1)(-1)(-1)(1)C_{1113} = -C_{15}. \tag{2.2.12}$$

But due to symmetry $C'_{15} = C_{15}$, that means

$$C_{15} = C_{51} = 0. \tag{2.2.13}$$

All other components can be calculated in the same way. It is noted that all the components in which index $3, 2$ or 1 appears an odd number of time will vanish. As mentioned earlier there are three 2-fold axis so applying the same argument to the remaining axes leads to the matrix given in Eq. (2.2.14).

Symmetry conditions employed by orthorhombic crystal system reduce the components of C_{ijkl} from 21 to 12. Out of these 12 only 9 are independent. The following matrix encapsulate the results for orthorhombic case

$$C_{\alpha\beta} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & 0 & 0 & 0 \\ C_{12} & C_{22} & C_{23} & 0 & 0 & 0 \\ C_{13} & C_{23} & C_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & C_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & C_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & C_{66} \end{bmatrix}. \tag{2.2.14}$$

A detailed discussion on crystal symmetries and their effect on elastic stiffness constants is given in (Royer and Dieulesaint, 1996). Wood, aluminum, barium sodium niobate are some examples of orthotropic materials.

2.2.3 Types of elastic waves and wave parameters

A wave is a disturbance or oscillation that travels through space and matter, accompanied by transfer of energy. Few wave parameters describing the properties of waves are reviewed here briefly.

Wavelength and wave number

The wavelength of a wave is the distance between two peaks. Wave number is a reciprocal of a wavelength and is denoted by letter k.

Frequency

The frequency f of a wave is the number of waves produced by a source in one second.

Amplitude

The amplitude of a wave is its maximum disturbance from its undisturbed position.

Phase velocity

Phase velocity c is the rate at which a phase of wave propagates. Phase is position of a point at some instant on wave measured as an angle. A complete cycle is defined as 360° of phase.

Angular frequency

Angular frequency ω is 2π times frequency f of a wave that is

$$\omega = 2\pi f.$$

It is measured in radians per second.

There is a large variety of elastic waves that can propagate through solids. Their classification depends upon how the motion of the particles of the solid is related to direction of wave propagation and on the boundary conditions. Some common types of elastic waves in solids are longitudinal or primary or P-waves, transverse or shear or S-waves, and surface waves. A discussion on these waves is reviewed from (Royer and Dieulesaint, 1996) and the various terms are given briefly as follows.

Longitudinal or P-waves

These waves corresponds to the situation when direction of particle displacement and wave propagation are parallel. These waves can travel through solids, liquids, and gases.

Transverse or S-waves

The waves for which particle displacement direction is perpendicular to the wave propagation direction are called transverse or S-waves. Such type of waves can travel through solids only.

Surface waves

Waves traveling near the surface or boundary of a solid material and characterized by a decay in the amplitude as they move away from the surface are called as surface waves. There are many types of surface waves out of which the most important are Rayleigh and Love waves. A detailed discussion on these two waves is given in Section 2.3.

Dispersive and non-dispersive waves

If the speed of waves depends upon wave number then the waves are said to be dispersive. If speed of waves is independent of wave number then waves are non dispersive which means waves of any wave number can propagate at the same speed.

2.2.4 Wave propagation equation

The equation of motion comes from the fundamental law of dynamics that is famous as Newton's second law of motion $\mathbf{F} = m\mathbf{a}$, where $\mathbf{F}$ is force causing an acceleration $\mathbf{a}$ in a body of mass m. Consider a solid in stress such that some disturbance is propagating through it. Change in displacement at some arbitrary point in solid is given by $\mathbf{u}$ and components of a force at some point due to stress T are given by

$$F_i = \frac{\partial T_{ij}}{\partial x_j}, \qquad i, j = 1, 2, 3, \tag{2.2.15}$$

where T_{ij} are the components of stress tensor T as defined in Eq. (2.2.2). According to Newton's second law, this force gives rise to the acceleration $\frac{\partial^2 u_i}{\partial t^2}$ along the i^{th} axis for the unit volume mass ρ. In the absence of body forces the equation of motion will be

$$\frac{\partial T_{ij}}{\partial x_j} = \rho \frac{\partial^2 u_i}{\partial t^2}, \tag{2.2.16}$$

By using Hook's law given in Eq. (2.2.4), the equation of motion takes the form

$$C_{ijkl} \frac{\partial^2 u_l}{\partial x_k \partial x_j} = \rho \frac{\partial^2 u_i}{\partial t^2}. \tag{2.2.17}$$

This is a set of three second order partial differential equations, which govern the wave motion for the three dimensional case. By making use of Eq. (2.2.4) and Eq. (2.2.5), Hook's law can be written in the form

$$T_{ij} = \lambda S_{kk} \delta_{ij} + 2\mu S_{ij}. \tag{2.2.18}$$

Thus, for a homogenous linear isotropic elastic material, equation of motion becomes

$$(\lambda + \mu) \nabla (\nabla \cdot \mathbf{u}) + \mu \nabla^2 \mathbf{u} = \rho \ddot{\mathbf{u}}. \tag{2.2.19}$$

2.3 Surface waves in isotropic elastic media

Waves generated during earthquakes and artificial explosions propagating along the Earth's surface are called as surface waves or sometimes seismic surface waves.

Rayleigh waves and Love waves are two most important types of surface waves.

Surface waves traveling along the free surface of an elastic half space were predicted by Rayleigh (1885). In the later years these waves were named after him as *Rayleigh waves*. Rayleigh waves result due to an elliptical motion of particles. They produce both a vertical and horizontal component of motion in the direction of wave propagation. Love (1911) proved the existence of transverse waves whose amplitude decay with depth, in an elastic half space covered with an isotropic elastic layer. These waves were also named after their discoverer as *Love waves*. Love waves are produced due to the side to side motion of ground. The particle motion in these waves is transverse and parallel to the surface.

The simplest medium in which Rayleigh waves can propagate is a homogeneous isotropic half-space and are non-dispersive in nature. The simplest model in which Love waves can propagate consists of a homogeneous isotropic layer on a homogeneous isotropic half-space. Love waves depict dispersive nature in this model, that is their velocities are dependent on wave number. Propagation of Rayleigh waves and Love waves in an isotropic elastic half space are discussed in (Achenbach, 1973) and briefly reviewed here.

2.3.1 Love waves

Consider a homogeneous elastic isotropic half space covered with layer of another isotropic material of thickness H, having material properties specified by Lame's constants μ, μ^B and mass densities ρ, ρ^B in half space and the layer, respectively, as shown in Fig. 2.2.

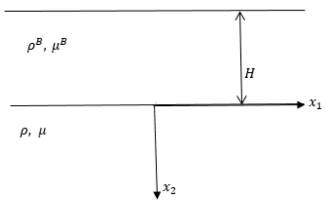


Figure 2.2: Isotropic elastic half space covered with an isotropic layer.

Dispersion relation for Love waves is as follows

$$\tan\left[\sqrt{\left(\frac{c}{c_T^B}\right)^2 - 1}kH\right] - \frac{\mu}{\mu^B}\left[\frac{\sqrt{1-\left(\frac{c}{c_T}\right)^2}}{\sqrt{\left(\frac{c}{c_T^B}\right)^2 - 1}}\right] = 0, \tag{2.3.1}$$

where c is phase velocity, $c_T = \sqrt{\frac{\mu}{\rho}}$ is transverse wave speed in half space, $c_T^B = \sqrt{\frac{\mu^B}{\rho^B}}$ is transverse wave speed in layer, and k is wave number. It is observed from Eq. (2.3.1) that speed and wave number are related. Hence Love waves are dispersive in nature.

The left-hand side of Eq. (2.3.1) is negative for $c = c_T^B$, and positive for $c = c_T$. It is noticeable that a real root can be found in the interval $c_T^B < c \leq c_T$, and no real root will exist if $c_T^B > c_T$.

Consider kH as an independent variable then for $kH = 0$ phase velocity c is same as c_T. It is observed from the graph given in Fig. 2.3 that with the increase in kH, the phase velocity c decreases. As the number $\sqrt{(\frac{c}{c_T^B})^2 - 1} * kH$ approaches π, 2π,... the phase velocity c approaches c_T. The Lowest modes of Love waves are shown in the following graph plotted against dimensionless wave number versus dimensionless phase velocity.

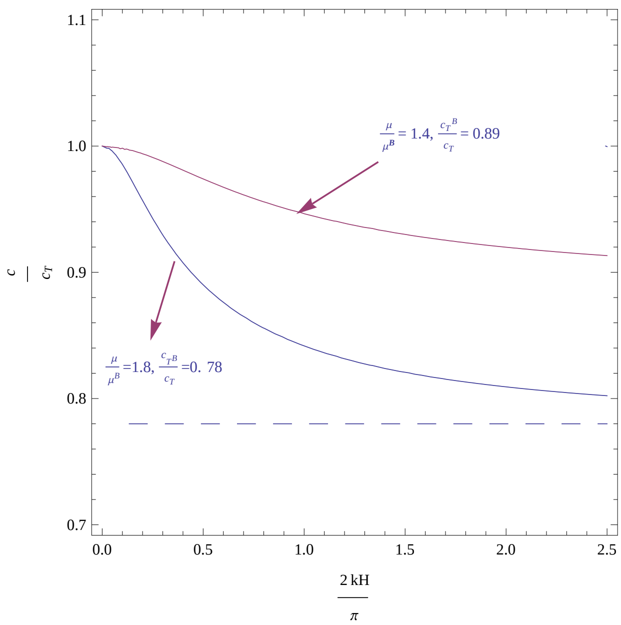


Figure 2.3: Phase velocity for lowest mode of Love waves.

2.3.2 Rayleigh waves

The existence of Rayleigh waves for two dimensional case of plane waves in an isotropic material is reviewed here. Consider that plane waves are propagating in x_1 direction along the surface of an elastic half space. Positive direction of x_2 is taken downward. It is assumed that motion is taking place in x_1x_2-plane only. Figure 2.4 illustrates the geometry of the problem.

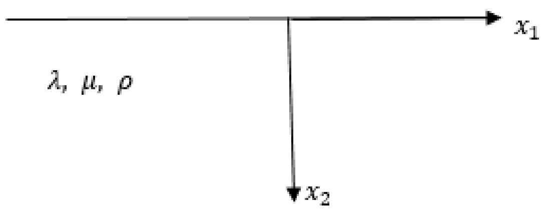


Figure 2.4: Isotropic elastic half space.

As the motion is considered in x_1x_2-plane only so the components of displacement vector $\mathbf{u}$ are

$$u_1 = A\exp^{-kbx_2}\exp(ik(x_1 - ct)), \tag{2.3.2a}$$
$$u_2 = B\exp^{-kbx_2}\exp(ik(x_1 - ct)), \tag{2.3.2b}$$

where A and B represents amplitudes of waves, k is wave number, c is phase velocity, and b is positive real constant. Note that $i = \sqrt{-1}$ throughout the thesis. Substitution of Eqs. (2.3.2a)-(2.3.2b) into Eq. (2.2.19) and consideration of boundary condition of vanishing stresses

$$T_{21} = \mu\left(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}\right), \tag{2.3.3a}$$
$$T_{22} = \lambda\left(\frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2}\right) + 2\mu\left(\frac{\partial u_2}{\partial x_2}\right), \tag{2.3.3b}$$

at $x_2 = 0$ yields the following well known expression for the phase velocity of Rayleigh waves

$$4\left[\sqrt{\left(1 - \frac{c^2}{c_L^2}\right)\left(1 - \frac{c^2}{c_T^2}\right)}\right] = \left(2 - \frac{c^2}{c_T^2}\right)^2. \tag{2.3.4}$$

In Eq. (2.3.4), $c_L^2 = \frac{\lambda+2\mu}{\rho}$ is longitudinal wave velocity and $c_T^2 = \frac{\mu}{\rho}$ is transverse wave velocity in isotropic material. It is interesting to note that in Eq. (2.3.4) wave number k does not appear, which means that Rayleigh waves in an isotropic elastic half space are non dispersive in nature.

It is observed that speed of these waves vary depending on the density and the elastic properties of the material they pass through. Thus the consideration of material's elastic anisotropy and non-homogeneity of half space opens up an extensive field of study. Searching for the conditions of wave existence in such media and analyzing their properties because of their application in various branches of engineering and also in some applied sciences such as geophysics has attracted the interest of many researchers. In the coming chapter a discussion on the existence of Love and Rayleigh waves in orthotropic medium is given. Effect of porosity of the medium on speed of these waves is also analyzed which constitutes of some novel work.

Chapter 3

Love Waves in Orthotropic Elastic Materials

Surface waves are very important from the scientific and practical points of view. It is a known fact that energy of surface waves is confined to the region very near to the surface, due to this property they are widely used in sensors. Love wave sensors are highly sensitive micro acoustic devices which are especially suited for sensing in liquids. These devices are used for density and viscosity measurements. Long period Love waves are used for studying earthquake mechanism and to explore internal structure of Earth.

In this chapter, a study on Love waves in an orthotropic medium with and without void pores is carried out. In Section 3.1, formulae for Love waves' speed in an orthotropic medium is derived and graphs are drawn between dimensionless speed and dimensionless wave number. The effect of rotation on phase velocity of Love waves in an orthotropic material is discussed in Section 3.2. Linear theory of elasticity for porous media is presented in Section 3.3. In Section 3.4 effect of voids on speed of Love waves is discussed. It is found that speed of Love waves is not influenced by rotation and presence of voids.

3.1 Love waves at the boundary between an orthotropic elastic half space and an orthotropic layer

Consider Love waves propagating in x_1 direction along the surface of an orthotropic elastic half space $x_2 \geq 0$ with material properties C_{44}, C_{55} and mass density ρ. Positive direction of x_2 is taken downward throughout the thesis. In this case the non vanishing component of displacement is $u_3(x_1, x_2, t)$. So displacement vector can be written as $\mathbf{u} = (0, 0, u_3(x_1, x_2, t))$. By using Eq. (2.2.14) and Eq. (2.2.17) the governing equation of motion obtained is as follows

$$C_{55}\frac{\partial^2 u_3}{\partial x_1^2} + C_{44}\frac{\partial^2 u_3}{\partial x_2^2} = \rho\frac{\partial^2 u_3}{\partial t^2}. \tag{3.1.1}$$

For a surface wave, Eq. (3.1.1) assumes a solution of the form

$$u_3 = A\exp(-gx_2)\exp[ik(x_1 - ct)], \tag{3.1.2}$$

where A is the amplitude, c is the phase velocity of the wave, and g is a positive real constant. Substitution of Eq. (3.1.2) in Eq. (3.1.1) gives

$$g = k\sqrt{\frac{1}{C_{44}}(C_{55} - \rho c^2)}. \tag{3.1.3}$$

For a free surface, boundary condition at $x_2 = 0$ is

$$T_{32} = C_{44}\left[\frac{\partial u_3}{\partial x_2}\right] = 0. \tag{3.1.4}$$

The boundary condition given in Eq. (3.1.4) can be satisfied only if either $A = 0$ or $g = 0$. Both these cases do not represent a surface wave. Now it is considered that half space is covered with another orthotropic elastic material having thickness H with material properties C_{44}^B, C_{55}^B and mass density ρ^B as shown in Fig. 3.1.

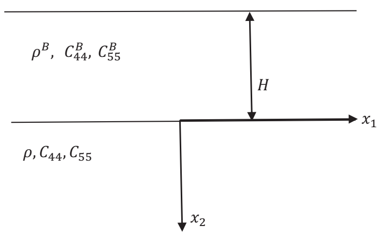


Figure 3.1: Orthotropic elastic half space covered with another orthotropic layer.

In layer, the equation of motion is

$$C_{55}^B \frac{\partial^2 u_3^B}{\partial x_1^2} + C_{44}^B \frac{\partial^2 u_3^B}{\partial x_2^2} = \rho^B \frac{\partial^2 u_3^B}{\partial t^2}, \tag{3.1.5}$$

and a solution assumed in layer is

$$u_3{}^B = f(x_2) \exp[ik(x_1 - ct)], \tag{3.1.6}$$

where $f(x_2)$ is an arbitrary function. Use of Eq. (3.1.6) in Eq. (3.1.5) yields

$$u_3^B = [R_1 \sin(q_B x_2) + R_2 \cos(q_B x_2)] \exp[ik(x_1 - ct)], \tag{3.1.7}$$

where

$$q_B = k\sqrt{\frac{1}{C_{44}^B} (\rho^B c^2 - C_{55}^B)}. \tag{3.1.8}$$

Consideration of the condition of vanishing shear stress $T_{32}^B = 0$, at free surface $x_2 = -H$ gives

$$R_1 \cos(q_B H) + R_2 \sin(q_B H) = 0. \tag{3.1.9}$$

The condition of continuity of displacement and shear stress at $x_2 = 0$ results in

$$A = R_2, \tag{3.1.10}$$

$$-C_{44} g A - C_{44}^B q_B R_1 = 0. \tag{3.1.11}$$

For a non trivial solution determinant of above mentioned three equations in A, R_1 and R_2 should vanish. As a result, the phase velocity c of Love waves is given by

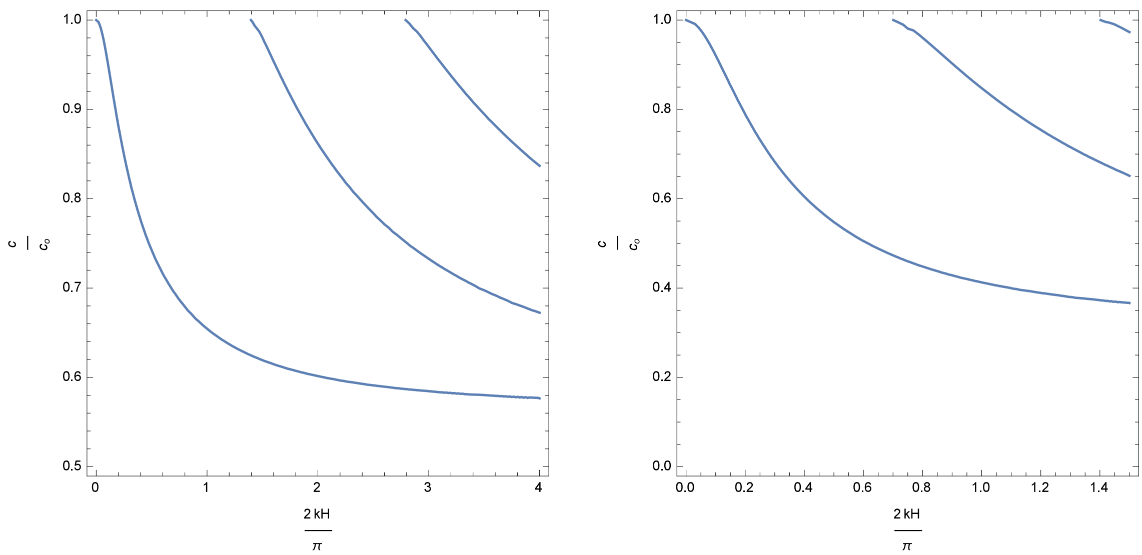
$$\tan\left[\sqrt{\frac{\rho^B c^2}{C_{44}^B}-\frac{C_{55}^B}{C_{44}^B}}kH\right]-\frac{C_{44}}{C_{44}^B}\left[\frac{\sqrt{\frac{C_{55}}{C_{44}}-\frac{\rho c^2}{C_{44}}}}{\sqrt{\frac{\rho^B c^2}{C_{44}^B}-\frac{C_{55}^B}{C_{44}^B}}}\right]=0. \quad (3.1.12)$$

For simplification consider

$$\begin{aligned}(c_o)^2=\frac{C_{55}}{\rho},\tau=\frac{\rho c_o^2}{C_{44}},\zeta=\frac{C_{44}}{C_{44}^B},\\ (c_o^B)^2=\frac{C_{55}^B}{\rho^B},\tau^B=\frac{\rho^B(c_o^B)^2}{C_{44}^B},\end{aligned} \quad (3.1.13)$$

where c_o^B is transverse wave speed in orthotropic layer and c_o is the transverse wave speed in orthotropic half space. By introducing these notations, Eq. (3.1.12) becomes

$$\tan\left[\sqrt{\tau^B((\frac{c}{c_o^B})^2-1)}kH\right]-\zeta\left[\frac{\sqrt{\tau(1-(\frac{c}{c_o})^2)}}{\sqrt{\tau^B((\frac{c}{c_o^B})^2-1)}}\right]=0. \quad (3.1.14)$$



(a) $\frac{c_o^B}{c_o}=0.56,\zeta=0.96,\tau=1.009,\tau^B=0.96.$

(b) $\frac{c_o^B}{c_o}=0.31,\zeta=2.87,\tau=1.009,\tau^B=0.88.$

Figure 3.2: Plots drawn for dimensionless wave speed $\mathbf{\frac{c}{c_o}}$ against dimensionless wave number $\mathbf{\frac{2kH}{\pi}}$ for orthotropic medium with different parameters.

It is observed from Eq. (3.1.14) that Love waves are dispersive in nature and a real root will exist in the interval $c_o^B < c \leq c_o$, since $\tau > 0$ and $\tau^B > 0$. Considering kH as an independent variable, it is observed that $c = c_o$ for $kH = 0$. Love modes are shown in Fig. 3.2 for different set of parameters of orthotropic medium. It can be seen from the Fig. 3.2 that phase velocity decreases with the increase in kH.

3.2 Acceleration in rotating frame

Before discussing the effect of rotation on speed of Love waves, acceleration of a particle in a rotating frame is reviewed. For this two frame of references are considered: one whose coordinates are fixed, another rotating with constant angular velocity $\mathbf{\Omega}$ in counterclockwise direction along an axis of rotation. Velocity of a particle moving in rotating frame of reference when observed from fixed frame will be

$$\dot{\mathbf{u}}^{'} = \dot{\mathbf{u}} + \mathbf{\Omega} \times \mathbf{u}, \tag{3.2.1}$$

where $\dot{\mathbf{u}}^{'}$ and $\dot{\mathbf{u}}$ are the velocities of particle in fixed and rotating frame of reference respectively. Here the superimposed dot represents the time derivative. For calculating acceleration of particle Eq. (3.2.1) is differentiated with respect to time

$$\ddot{\mathbf{u}}^{'} = \ddot{\mathbf{u}} + 2\mathbf{\Omega} \times \dot{\mathbf{u}} + \mathbf{\Omega} \times (\mathbf{\Omega} \times \mathbf{u}). \tag{3.2.2}$$

The terms $\mathbf{\Omega} \times (\mathbf{\Omega} \times \mathbf{u})$, $\mathbf{2\Omega} \times \dot{\mathbf{u}}$ in Eq. (3.2.2) are called centripetal acceleration and Coriolis acceleration, respectively. Centripetal acceleration is the acceleration of an object moving in a circle and is directed towards the center of the circle. When an object simultaneously rotates about a point and moves relative to that point, an acceleration results from this. This acceleration is called Coriolis acceleration. In index notation, Eq. (3.2.2) is

$$\ddot{u}_i^{'} = \ddot{u}_i + \Omega_j u_j \Omega_i - \Omega^2 u_i + 2\epsilon_{ijk}\Omega_j \dot{u}_k. \tag{3.2.3}$$

3.2.1 Impact of rotation on speed of Love waves

According to Schoenberg and Censore (1973), the equation of motion (2.2.16) for a medium rotating with constant angular velocity $\mathbf{\Omega}$, in the absence of body forces, get the form

$$\frac{\partial T_{ij}}{\partial x_j} = \rho[\ddot{u}_i + \Omega_j u_j \Omega_i - \Omega^2 u_i + 2\epsilon_{ijk}\Omega_j \dot{u}_k]. \tag{3.2.4}$$

To study the affect of rotation on speed of Love waves, it is considered that Love waves are propagating in a rotating elastic homogeneous orthotropic half space covered with a layer of another orthotropic material in x_1 direction. Elastic constant C_{44}, C_{55} and C^B_{44}, C^B_{55} specify material properties in half space and in layer respectively. If rotation is considered about coordinate axes, three possible cases are as follows.

If x_1 axis or x_2 axis is taken as the axis of rotation then angular velocity vector $\mathbf{\Omega}$ is $\mathbf{\Omega} = (\Omega, 0, 0)$ or $\mathbf{\Omega} = (0, \Omega, 0)$. Figure 3.3(a) and Fig. 3.3(b) represent the case when x_1 and x_2 axes are taken as axis of rotation, respectively.

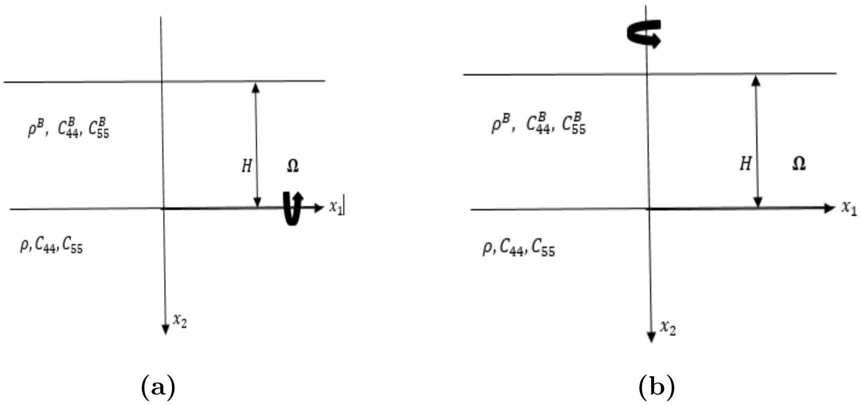


Figure 3.3: Rotating orthotropic elastic half space covered with another orthotropic layer.

The following system of governing equations model the situation in both the cases

$$2\rho\Omega\frac{\partial u_3}{\partial t} = 0, \tag{3.2.5a}$$

$$C_{55}\frac{\partial^2 u_3}{\partial x_1^2} + C_{44}\frac{\partial^2 u_3}{\partial x_2^2} = \rho\left(\frac{\partial^2 u_3}{\partial t^2} - \Omega^2 u_3\right). \tag{3.2.5b}$$

Analysis of Eq. (3.2.5a) reveals that displacement can't be the function of time or is constant, this condition reduces the Eq. (3.2.5b) to the form

$$C_{55}\frac{\partial^2 u_3}{\partial x_1^2} + C_{44}\frac{\partial^2 u_3}{\partial x_2^2} = -\rho\Omega^2 u_3. \tag{3.2.6}$$

From Eq. (3.2.6) it is noticed that time derivative is not involved in the equation which mean there is no phase velocity and hence no wave.

Figure (3.4) corresponds to the situation if rotation is taken about x_3 axis, that is about the axis parallel to the direction of particle's displacement.

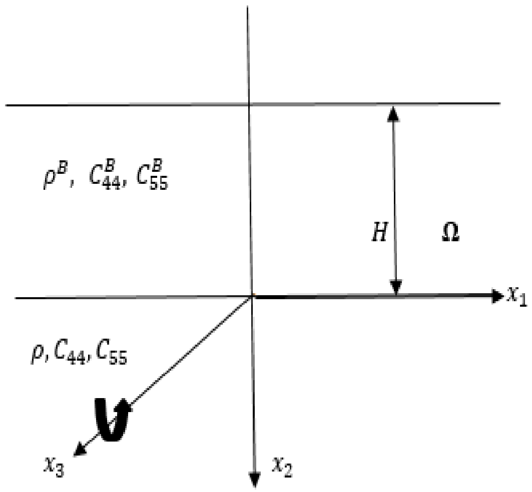


Figure 3.4: Rotating orthotropic elastic half space covered with another orthotropic layer.

The following equation models the situation

$$C_{55}\frac{\partial^2 u_3}{\partial x_1^2} + C_{44}\frac{\partial^2 u_3}{\partial x_2^2} = \rho\frac{\partial^2 u_3}{\partial t^2}. \tag{3.2.7}$$

It is observed that rotational terms do not appear in Eq. (3.2.7), and this is the same case as discussed in Section 3.1, which means that there will be no effect of rotation on speed of Love waves if rotation axis is parallel to displacement direction. It is concluded that speed of Love waves is not effected by rotation of half space.

3.3 Theory of linear elastic materials with voids

Porous materials occur every where and influence our lives. Some examples of porous media are animal fur, sandstone, bones, skin, wood, building material such as sand, cement etc. Applications of porous media in real life are countless, for example porous materials are used in heat transfer devices and also used as sound absorber in acoustics. Here in this section, a brief discussion on elastic theory of porous media is presented.

The nonlinear and linear theories of elastic material with voids were established by Nunziato and Cowin (1979) and Cowin and Nunziato (1983) respectively. The basic assumption of the theory is that voids contain nothing of mechanical and energetic significance and bulk density ρ of the material is product of matrix density Γ and volume fraction distribution function ν.

$$\rho = \Gamma\nu. \tag{3.3.1}$$

Matrix density Γ is the ratio of mass and volume without pores of the material. The restriction on void volume fraction distribution function ν is $0 < \nu \leq 1$, where limit $\nu = 0$ is associated with the absence of material while $\nu = 1$ corresponds to absence of voids in elastic material. It is noticeable that for an incompressible solid, the matrix density Γ is constant, while the bulk density ρ can vary due to the change in void volume fraction ϕ, where this change can be written as $\phi = \nu(x,t) - \nu_0(x,t)$. $\nu_0(x,t)$ is volume fraction in reference configuration. In linear theory, the change in volume fraction ϕ is taken as an independent kinematic variable in addition to classical displacement vector $\mathbf{u}$.

For a linear elastic continuum with voids the governing equations of motion are the balance of linear momentum and balance of equilibrated force as proposed by Cowin and Nunziato (1983). The balance of linear momentum results in

$$T_{ij,j} + \rho F_i = \rho \ddot{u}_i, \tag{3.3.2}$$

where T_{ij} is symmetric stress tensor and F_i represents components of body force.

The balance of equilibrated force results in

$$h_{i,i} + g + \rho l = \rho \bar{k} \ddot{\phi}, \tag{3.3.3}$$

where h_i is equilibrated stress vector, g is intrinsic equilibrated body force, l is extrinsic equilibrated body force, $\bar{k}$ is equilibrated inertia.

In linear theory of elastic material with voids the constitutive relations, relating different voids and stress parameters as given in Cowin and Nunziato (1983) are

$$T_{ij} = C_{ijkl}S_{kl} + D_{ijk}\phi_{,k} + B_{ij}\phi + T^R_{ij}, \tag{3.3.4}$$

$$h_i = A_{ij}\phi_{,j} + D_{ijk}S_{jk} + f_i\phi + h^R_i, \tag{3.3.5}$$

$$g = -\varpi\dot{\phi} - \xi\phi - B_{ij}S_{ij} - f_i\phi_{,i} + g^R, \tag{3.3.6}$$

where C_{ijkl} represents elastic stiffness tensor, T^R_{ij} is stress tensor in reference configuration, $D_{ijk}, B_{ij}, f_i, \varpi, \xi$ and A_{ij} are void parameters. All these quantities are functions of ν_0 and its gradient. $h^R_i$and g^R are equilibrated stress vector and intrinsic equilibrated body force in reference state, respectively. According to Cowin and Nunziato (1983) in reference state S_{ij}, ϕ and $\phi_{,i}$ should vanish. And it is required that T^R_{ij}, h^R_i, and g^R satisfy the conditions for mechanical equilibrium in reference state in the absence of body forces, that is

$$T^R_{ij,j} = 0, \qquad h^R_{i,i} + g^R = 0. \tag{3.3.7}$$

It is assumed that T^R_{ij}, h^R_i, and g^R vanish if and only if the gradient of ν_0 vanishes

$$(\nu_0)_{,i} = 0 \quad \Leftrightarrow \quad T^R_{ij} = 0, h^R_i = 0, g^R = 0. \tag{3.3.8}$$

If material possesses center of symmetry, then the tensors D_{ijk} and f_i are identically zero. According to (Ranjeesh and Kumar, 2011) if material is orthotropic then A_{ij}, and B_{ij} can be written as

$$A_{ij} = A_i\delta_{ij}, \qquad B_{ij} = B_i\delta_{ij}. \tag{3.3.9}$$

By imposing above mentioned conditions and assumptions the constitutive relations given in Eqs. (3.3.4), (3.3.5), and (3.3.6) take the simplified form

$$T_{ij} = C_{ijkl}S_{kl} + B_{ij}\phi. \tag{3.3.10}$$

$$h_i = A_{ij}\phi_{,j}. \tag{3.3.11}$$

$$g = -\varpi\dot{\phi} - \xi\phi - B_{ij}S_{ij}. \tag{3.3.12}$$

Therefore in the absence of body forces F_i and extrinsic equilibrated body force l Eq. (3.3.2) and Eq. (3.3.3) take the form

$$C_{ijkl}u_{k,jl} + B_i\phi_{,i} = \rho\ddot{u}_i, \tag{3.3.13}$$

$$A_i\phi_{,ii} - \varpi\dot{\phi} - \xi\phi - B_iu_{i,i} = \rho\bar{k}\ddot{\phi}. \tag{3.3.14}$$

3.4 Love waves in orthotropic elastic medium with voids

To study the influence of voids on speed of Love waves in orthotropic material, consider a homogeneous orthotropic elastic half space with continuous distribution of void pores, covered with a layer of another porous orthotropic medium. It is assumed that wave is propagating in x_1 direction, therefore the only non vanishing component of displacement is $u_3(x_1, x_2, t)$. Geometry of the problem is shown in Fig. 3.5.

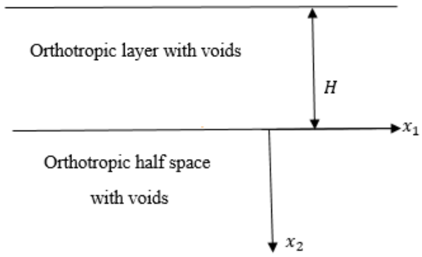


Figure 3.5: Orthotropic elastic half space with voids covered with porous orthotropic layer.

By using Eq. (2.2.14) and the conditions of above mentioned problem, Eqs. (3.3.13), and (3.3.14) reduces to the following system

$$B_1 \frac{\partial \phi}{\partial x_1} = 0, \tag{3.4.1a}$$

$$B_2 \frac{\partial \phi}{\partial x_2} = 0, \tag{3.4.1b}$$

$$C_{55} \frac{\partial^2 u_3}{\partial x_1^2} + C_{44} \frac{\partial^2 u_3}{\partial x_2^2} = \rho \frac{\partial^2 u_3}{\partial t^2}, \tag{3.4.1c}$$

$$A_1 \frac{\partial^2 \phi}{\partial x_1^2} + A_2 \frac{\partial^2 \phi}{\partial x_2^2} - \varpi \frac{\partial \phi}{\partial t} - \xi \phi = \rho \bar{k} \frac{\partial^2 \phi}{\partial t^2}, \tag{3.4.1d}$$

where C_{55}, C_{44} are material constants and $B_1, B_2, A_1, A_2, \phi, \varpi, \xi, \bar{k}$ are parameters due to the porosity of the material. To solve the system of four partial differential equations given in Eqs. (3.4.1a)-(3.4.1d), a solution of the form

$$u_3 = A \exp(-g x_2) \exp[ik(x_1 - ct)], \tag{3.4.2}$$

is considered. Following form of void volume fraction ϕ is assumed (A. M. Abd-Alla et al, 2015)

$$\phi = \bar{\phi}(x_2) \exp[ik(x_1 - ct)]. \tag{3.4.3}$$

Substitution of Eqs. (3.4.2) and (3.4.3) in Eqs. (3.4.1a)-(3.4.1d) result in

$$g = k \sqrt{\frac{1}{C_{44}} (C_{55} - \rho c^2)}, \tag{3.4.4a}$$

$$\phi = 0. \tag{3.4.4b}$$

In an orthotropic layer with voids the governing equations are

$$B_1^{'}\frac{\partial \phi^{'}}{\partial x_1} = 0, \tag{3.4.5a}$$

$$B_2^{'}\frac{\partial \phi^{'}}{\partial x_2} = 0, \tag{3.4.5b}$$

$$C_{55}^{'}\frac{\partial^2 u_3^{'}}{\partial x_1^2} + C_{44}^{'}\frac{\partial^2 u_3^{'}}{\partial x_2^2} = \rho^{'}\frac{\partial^2 u_3^{'}}{\partial t^2}, \tag{3.4.5c}$$

$$A_1^{'}\frac{\partial^2 \phi^{'}}{\partial x_1^2} + A_2^{'}\frac{\partial^2 \phi^{'}}{\partial x_2^2} - \varpi^{'}\frac{\partial \phi^{'}}{\partial t} - \xi^{'}\phi^{'} = \rho^{'}\bar{k}^{'}\frac{\partial^2 \phi^{'}}{\partial t^2}, \tag{3.4.5d}$$

where $C_{55}^{'}, C_{44}^{'}$ are material constants in layer. The constants $B_1^{'}, B_2^{'}, A_1^{'}, A_2^{'}, \phi^{'}, \varpi^{'}$, $\xi^{'}, \bar{k}^{'}$ specify the porosity of the material in layer. In layer solution take the form

$$u_3^{'} = \bar{u_3}(x_2)\exp[ik(x_1 - ct)], \tag{3.4.6a}$$

$$\phi^{'} = \bar{\phi}^{'}(x_2)\exp[ik(x_1 - ct)]. \tag{3.4.6b}$$

Use of Eqs. (3.4.6a) and (3.4.6b) in Eqs. (3.4.5a)-(3.4.5d) result in

$$\phi^{'} = 0, \tag{3.4.7a}$$

$$u_3^{'} = [C_1\sin(q^{'}x_2) + C_2\cos(q^{'}x_2)]\exp[ik(x_1 - ct)], \tag{3.4.7b}$$

where

$$q^{'} = k\sqrt{\frac{\rho^{'}c^2 - C_{55}^{'}}{C_{44}^{'}}}. \tag{3.4.8}$$

It is observed that results obtained above are the same as explored in Section 3.1, Eqs. (3.1.3) and (3.1.8), where the propagation of Love waves in orthotropic medium without voids is discussed. Therefore it is concluded that speed of Love waves is not affected by voids.

Chapter 4

Rayleigh Waves in Orthotropic Elastic Materials

Vibrations produced during civil engineering work are often dominated by surface waves, the propagation of which is strongly effected by site conditions. So surface waves specially Rayleigh waves are widely used for materials characterization, and to discover the mechanical and structural properties of objects.

Since Rayleigh waves propagate near the free boundary of a solid, so their energy is concentrated in the vicinity of the surface which make these waves sensitive to surface discontinuities. Non destructive testing using Rayleigh waves is technique for the inspection of defects in engineering components such as high temperature turbine rotors, where cracks are expected to form on the surface of the component. Also, these are one of the seismic waves that are produced inside the Earth by earthquakes. Because of many applications it is worthwhile to study Rayleigh waves and analyze wave properties under different conditions.

In this chapter propagation of Rayleigh waves in orthotropic elastic half space with and without voids is discussed in detail. Section 4.1 is focused on the derivation of formula for Rayleigh wave speed in an orthotropic medium. Section 4.2 is about the effect of porosity on speed of Rayleigh waves. Numerical results and discussions are given in Section 4.3.

4.1 Rayleigh waves in orthotropic elastic half space

Consider Rayleigh waves propagating in x_1 direction along the surface of an orthotropic elastic half space as shown in Fig. 4.1. It is assumed that motion is taking place in x_1x_2-plane only and x_2 is taken positive in downward direction. As motion is in x_1x_2-plane so displacement vector is $\mathbf{u} = (u_1(x_1, x_2, t), u_2(x_1, x_2, t), 0)$.

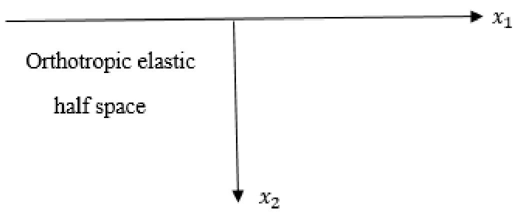


Figure 4.1: Orthotropic elastic half space.

The situation is modeled by the following governing equations obtained from Eq. (2.2.14) and Eq. (2.2.17)

$$C_{11}\frac{\partial^2 u_1}{\partial x_1^2} + C_{12}\frac{\partial^2 u_2}{\partial x_1 \partial x_2} + C_{66}\left[\frac{\partial^2 u_1}{\partial x_2^2} + \frac{\partial^2 u_2}{\partial x_1 \partial x_2}\right] = \rho\frac{\partial^2 u_1}{\partial t^2}, \tag{4.1.1a}$$
$$C_{22}\frac{\partial^2 u_2}{\partial x_2^2} + C_{12}\frac{\partial^2 u_1}{\partial x_1 \partial x_2} + C_{66}\left[\frac{\partial^2 u_2}{\partial x_1^2} + \frac{\partial^2 u_1}{\partial x_1 \partial x_2}\right] = \rho\frac{\partial^2 u_2}{\partial t^2}, \tag{4.1.1b}$$

where $C_{11}, C_{22}, C_{12}, C_{66}, \rho$ are elastic constants specifying material properties. Rearrangement of terms in Eqs. (4.1.1a)-(4.1.1b) and dividing by C_{66} yields

$$\alpha\frac{\partial^2 u_1}{\partial x_1^2} + \gamma\frac{\partial^2 u_2}{\partial x_1 \partial x_2} + \frac{\partial^2 u_1}{\partial x_2^2} = \frac{1}{c_1^2}\frac{\partial^2 u_1}{\partial t^2}, \tag{4.1.2a}$$
$$\beta\frac{\partial^2 u_2}{\partial x_2^2} + \gamma\frac{\partial^2 u_1}{\partial x_1 \partial x_2} + \frac{\partial^2 u_2}{\partial x_1^2} = \frac{1}{c_1^2}\frac{\partial^2 u_2}{\partial t^2}, \tag{4.1.2b}$$

where $\alpha = \frac{C_{11}}{C_{66}}$, $\beta = \frac{C_{22}}{C_{66}}$, $\gamma = \frac{C_{12}+C_{66}}{C_{66}}$, $c_1^2 = \frac{C_{66}}{\rho}$. To solve the above system, a solution of the following form is assumed

$$u_1(x_1,x_2,t) = A\exp(-kbx_2)\exp(ik(x_1-ct)), \tag{4.1.3a}$$

$$u_2(x_1,x_2,t) = B\exp(-kbx_2)\exp(ik(x_1-ct)), \tag{4.1.3b}$$

where A, B are amplitudes of waves, k is wave number, b is some positive real number, and c is the phase velocity of the wave. Use of Eqs. (4.1.3a)-(4.1.3b) in Eqs. (4.1.2a)-(4.1.2b) results in

$$A\left[\frac{c^2}{c_1^2}+b^2-\alpha\right]-ibB\gamma = 0, \tag{4.1.4a}$$

$$B\left[\frac{c^2}{c_1^2}-1+b^2\beta\right]-ibA\gamma = 0. \tag{4.1.4b}$$

For non trivial solution of A and B, we must have

$$\begin{vmatrix} \frac{c^2}{c_1^2}+b^2-\alpha & -ib\gamma \\ -ib\gamma & \frac{c^2}{c_1^2}-1+b^2\beta \end{vmatrix} = 0. \tag{4.1.5}$$

The consequent result is

$$Rb^4+Sb^2+T = 0, \tag{4.1.6}$$

where

$$R = \beta, \tag{4.1.7a}$$

$$S = (\frac{c^2}{c_1^2}-1)+\beta(\frac{c^2}{c_1^2}-\alpha)+\gamma^2, \tag{4.1.7b}$$

$$T = (\frac{c^2}{c_1^2}-\alpha)(\frac{c^2}{c_1^2}-1). \tag{4.1.7c}$$

Let b_1^2 and b_2^2 be the two roots of the Eq. (4.1.6). The values of b_1^2 and b_2^2 as calculated from Mathematica are

$$b_1^2 = \frac{\left(\beta' y^2+\alpha'\right)+\sqrt{\left(\beta' y^2+\alpha'\right)^2-4\beta\left(y^2-\alpha\right)\left(y^2-1\right)}}{2\beta}, \tag{4.1.8a}$$

$$b_2^2 = \frac{\left(\beta' y^2+\alpha'\right)-\sqrt{\left(\beta' y^2+\alpha'\right)^2-4\beta\left(y^2-\alpha\right)\left(y^2-1\right)}}{2\beta}, \tag{4.1.8b}$$

where

$$\beta' = 1 + \beta, \tag{4.1.9a}$$

$$\alpha' = -1 - \alpha\beta + \gamma^2, \tag{4.1.9b}$$

$$y = \frac{c}{c_1}. \tag{4.1.9c}$$

To ensure exponential decay, b_1^2 and b_2^2 should be positive and real. Since we have two values of b_1 and b_2 so b can be written as their linear combination, that is

$$u_1 = [A_1 \exp(-kb_1x_2) + A_2 \exp(-kb_2x_2)] \exp(ik(x_1 - ct)), \tag{4.1.10a}$$

$$u_2 = [B_1 \exp(-kb_1x_2) + B_2 \exp(-kb_2x_2)] \exp(ik(x_1 - ct)). \tag{4.1.10b}$$

The ratios $(\frac{B}{A})$ corresponding to b_1 and b_2 can be written as

$$\frac{B_1}{A_1} = i\alpha_1, \qquad \frac{B_2}{A_2} = i\alpha_2, \tag{4.1.11}$$

where

$$\alpha_1 = \frac{\alpha - b_1^2 - y^2}{b_1\gamma}, \tag{4.1.12a}$$

$$\alpha_2 = \frac{b_2\gamma}{y^2 - 1 + b_2^2\beta}. \tag{4.1.12b}$$

By using Eq. (4.1.11), u_1 and u_2 take the form

$$u_1 = [A_1 \exp(-kb_1x_2) + A_2 \exp(-kb_2x_2)] \exp(ik(x_1 - ct)), \tag{4.1.13a}$$

$$u_2 = [i\alpha_1A_1 \exp(-kb_1x_2) + i\alpha_2A_2 \exp(-kb_2x_2)] \exp(ik(x_1 - ct)). \tag{4.1.13b}$$

From Eq. (2.2.4) and Eq. (2.2.14) the expression for shear stress and normal stress T_{21} and T_{22} in an orthotropic material are respectively

$$T_{21} = C_{66}\left[\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}\right], \tag{4.1.14a}$$

$$T_{22} = C_{12}\left[\frac{\partial u_1}{\partial x_1}\right] + C_{22}\left[\frac{\partial u_2}{\partial x_2}\right]. \tag{4.1.14b}$$

Boundary conditions of vanishing stresses at $x_2 = 0$, yields

$$A_1 (b_1 + \alpha_1) + A_2 (b_2 + \alpha_2) = 0, \tag{4.1.15a}$$

$$A_1 (\delta - b_1\alpha_1) + A_2 (\delta - b_2\alpha_2) = 0, \tag{4.1.15b}$$

where $\delta = \frac{C_{12}}{C_{22}}$. For non trivial solution, determinant of the above system should vanish that is

$$\begin{vmatrix} b_1 + \alpha_1 & b_2 + \alpha_2 \\ \delta - b_1\alpha_1 & \delta - b_2\alpha_2 \end{vmatrix} = 0. \tag{4.1.16}$$

The consequent outcome is

$$(b_1 + \alpha_1)(\delta - b_2\alpha_2) = (b_2 + \alpha_2)(\delta - b_1\alpha_1). \tag{4.1.17}$$

Equation. (4.1.17) is the formulae for Rayleigh wave speed in an orthotropic material. It is observed that Rayleigh waves are non dispersive in this case. The results reduces to the corresponding Rayleigh equation when the elastic constants of orthotropic medium are replaced with those of an isotropic material. For an isotropic material the values of various parameters are

$$\begin{gathered} \delta = 1 - \left(\frac{2c_T^2}{c_L^2}\right), \alpha = \beta = \frac{c_L^2}{c_T^2}, \gamma = \frac{c_L^2}{c_T^2} - 1, \\ b_1 = \sqrt{1 - \frac{c^2}{c_T^2}}, b_2 = \sqrt{1 - \frac{c^2}{c_L^2}}, \alpha_1 = \frac{1}{\sqrt{1 - \frac{c^2}{c_T^2}}}, \alpha_2 = \frac{b_2\gamma}{b_2^2\beta - b_1^2}, \end{gathered} \tag{4.1.18}$$

where $c_L^2 = \frac{\lambda+2\mu}{\rho}$ and for isotropic material $c_T^2 = \frac{\mu}{\rho}$. By using Eq. (4.1.18) in Eq. (4.1.17) we get the famous Rayleigh wave velocity equation for isotropic material given in Eq. (2.3.4).

4.2 Rayleigh waves in porous orthotropic elastic half space

The effect of porosity on the speed of Rayleigh waves is explored in this section. Propagation of Rayleigh waves is considered in x_1 direction in a homogeneous porous orthotropic elastic half space. It is assumed that motion is taking place in x_1x_2-plane and therefore $u_1(x_1, x_2, t)$ and $u_2(x_1, x_2, t)$ are the nonzero components of displacement. Positive x_2 axis is taken in the downward direction. Figure 4.2 illustrates the situation.

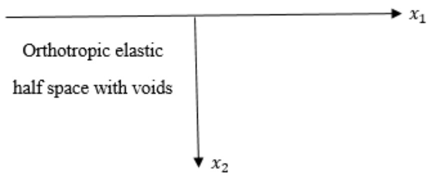


Figure 4.2: Homogeneous orthotropic elastic half space with voids.

The problem is governed by the following equations, deduced from Eqs. (3.3.13) and (3.3.14)

$$C_{11}\frac{\partial^2 u_1}{\partial x_1^2} + C_{12}\frac{\partial^2 u_2}{\partial x_1 \partial x_2} + C_{66}\left[\frac{\partial^2 u_1}{\partial x_2^2} + \frac{\partial^2 u_2}{\partial x_1 \partial x_2}\right] + B_1\frac{\partial \phi}{\partial x_1} = \rho\frac{\partial^2 u_1}{\partial t^2}, \tag{4.2.1a}$$

$$C_{22}\frac{\partial^2 u_2}{\partial x_2^2} + C_{12}\frac{\partial^2 u_1}{\partial x_1 \partial x_2} + C_{66}\left[\frac{\partial^2 u_2}{\partial x_1^2} + \frac{\partial^2 u_1}{\partial x_1 \partial x_2}\right] + B_2\frac{\partial \phi}{\partial x_2} = \rho\frac{\partial^2 u_2}{\partial t^2}, \tag{4.2.1b}$$

$$A_1\frac{\partial^2 \phi}{\partial x_1^2} + A_2\frac{\partial^2 \phi}{\partial x_2^2} - \varpi\frac{\partial \phi}{\partial t} - \xi\phi - \left[B_1\frac{\partial u_1}{\partial x_1} + B_2\frac{\partial u_2}{\partial x_2}\right] = \rho\bar{k}\frac{\partial^2 \phi}{\partial t^2}, \tag{4.2.1c}$$

where $C_{11}, C_{22}, C_{12}, C_{66}, \rho$ are material constants, and $B_1, B_2, A_1, A_2, \phi, \varpi, \xi, \bar{k}$ specify the presence of voids. By rearranging the terms and dividing by C_{66}, Eqs. (4.2.1a)-(4.2.1c) take the form

$$\alpha\frac{\partial^2 u_1}{\partial x_1^2} + \gamma\frac{\partial^2 u_2}{\partial x_1 \partial x_2} + \frac{\partial^2 u_1}{\partial x_2^2} + \lambda_1\frac{\partial \phi}{\partial x_1} = \frac{1}{c_1^2}\frac{\partial^2 u_1}{\partial t^2}, \tag{4.2.2a}$$

$$\beta\frac{\partial^2 u_2}{\partial x_2^2} + \gamma\frac{\partial^2 u_1}{\partial x_1 \partial x_2} + \frac{\partial^2 u_2}{\partial x_1^2} + \lambda_2\frac{\partial \phi}{\partial x_2} = \frac{1}{c_1^2}\frac{\partial^2 u_2}{\partial t^2}, \tag{4.2.2b}$$

$$\lambda_3\frac{\partial^2 \phi}{\partial x_1^2} + \lambda_4\frac{\partial^2 \phi}{\partial x_2^2} - \frac{\varpi}{C_{66}}\frac{\partial \phi}{\partial t} - \frac{\xi}{C_{66}}\phi - \left[\lambda_1\frac{\partial u_1}{\partial x_1} + \lambda_2\frac{\partial u_2}{\partial x_2}\right] = \frac{\bar{k}}{c_1^2}\frac{\partial^2 \phi}{\partial t^2}, \tag{4.2.2c}$$

where $\alpha = \frac{C_{11}}{C_{66}}$, $\beta = \frac{C_{22}}{C_{66}}$, $\gamma = \frac{C_{12}+C_{66}}{C_{66}}$, $\lambda_1 = \frac{B_1}{C_{66}}$, $\lambda_2 = \frac{B_2}{C_{66}}$, $\lambda_3 = \frac{A_1}{C_{66}}$, $\lambda_4 = \frac{A_2}{C_{66}}$, and $c_1^2 = \frac{C_{66}}{\rho}$. To solve the system given in Eqs. (4.2.2a)-(4.2.2c), a solution of following form is assumed

$$u_1(x_1, x_2, t) = f(x_2)\exp(ik(x_1 - ct)), \tag{4.2.3a}$$

$$u_2(x_1, x_2, t) = g(x_2)\exp(ik(x_1 - ct)), \tag{4.2.3b}$$

$$\phi(x_1, x_2, t) = h(x_2)\exp(ik(x_1 - ct)), \tag{4.2.3c}$$

where $f(x_2)$, $g(x_2)$, $h(x_2)$ are arbitrary functions. By using Eqs. (4.2.3a)-(4.2.3c) in Eqs. (4.2.2a)-(4.2.2c) we get

$$\left[D^2 - \left(k^2\alpha - \frac{k^2c^2}{c_1^2}\right)\right] f(x_2) + \gamma ikDg(x_2) + \lambda_1 ikh(x_2) = 0, \tag{4.2.4a}$$

$$\gamma ikDf(x_2) + \left[\beta D^2 - \left(k^2 - \frac{k^2c^2}{c_1^2}\right)\right] g(x_2) + \lambda_2 Dh(x_2) = 0, \tag{4.2.4b}$$

$$-\lambda_1 ikf(x_2) - \lambda_2 Dg(x_2) + \left[\lambda_4 D^2 - \left(\lambda_3 k^2 - \frac{\varpi ikc}{C_{66}} + \frac{\xi}{C_{66}} - \frac{\bar{k}k^2c^2}{c_1^2}\right)\right] h(x_2) = 0, \tag{4.2.4c}$$

where $D = \frac{d}{dx_2}$. Assuming the following notation

$$\sigma_1 = k^2\alpha - \frac{k^2c^2}{c_1^2}, \tag{4.2.5a}$$

$$\sigma_2 = k^2 - \frac{k^2c^2}{c_1^2}, \tag{4.2.5b}$$

$$\sigma_3 = \frac{\lambda_3}{\lambda_4}\left(\lambda_4 k^2\right) - \frac{\varpi ic_1}{\sqrt{\lambda_4}C_{66}}\left(\sqrt{\lambda_4}k\right)\left(\frac{c}{c_1}\right) + \frac{\xi}{C_{66}} - \frac{\bar{k}}{\lambda_4}\left(\lambda_4 k^2\right)\left(\frac{c^2}{c_1^2}\right). \tag{4.2.5c}$$

Equations (4.2.4a)-(4.2.4c) take the form

$$\left[D^2 - \sigma_1\right] f(x_2) + \gamma ikDg(x_2) + \lambda_1 ikh(x_2) = 0, \tag{4.2.6a}$$

$$\gamma ikDf(x_2) + \left[\beta D^2 - \sigma_2\right] g(x_2) + \lambda_2 Dh(x_2) = 0, \tag{4.2.6b}$$

$$-\lambda_1 ikf(x_2) - \lambda_2 Dg(x_2) + \left[\lambda_4 D^2 - \sigma_3\right] h(x_2) = 0. \tag{4.2.6c}$$

For non trivial solution of the above system of equations, we have

$$\begin{vmatrix} D^2 - \sigma_1 & \gamma ikD & \lambda_1 ik \\ \gamma ikD & \beta D^2 - \sigma_2 & \lambda_2 D \\ -\lambda_1 ik & -\lambda_2 D & \lambda_4 D^2 - \sigma_3 \end{vmatrix} = 0. \tag{4.2.7}$$

This implies

$$\big(AD^6 - BD^4 + CD^2 - E\big) = 0,$$

where

$$A = \beta\lambda_4, \tag{4.2.8a}$$

$$B = \sigma_1\beta\lambda_4 + \lambda_4\sigma_2 + \beta\sigma_3 - \lambda_2^2 - \gamma^2k^2\lambda_4, \tag{4.2.8b}$$

$$C = \lambda_4\sigma_1\sigma_2 + \beta\sigma_1\sigma_3 + \sigma_2\sigma_3 - \sigma_1\lambda_2^2 + k^2\left(2\lambda_1\lambda_2\gamma - \beta\lambda_1^2 - \gamma^2\sigma_3\right), \tag{4.2.8c}$$

$$E = \sigma_1\sigma_2\sigma_3 - \lambda_1^2\sigma_2k^2. \tag{4.2.8d}$$

The auxiliary equation can be written as

$$As^3 - Bs^2 + Cs - E = 0. \tag{4.2.9}$$

Let s_1, s_2, s_3 be three positive real roots of Eq. (4.2.9) and by Vieta's formulas, we can write

$$s_1^2 + s_2^2 + s_3^2 = \frac{B}{A}, \tag{4.2.10a}$$

$$s_1^2s_2^2 + s_1^2s_3^2 + s_2^2s_3^2 = \frac{C}{A}, \tag{4.2.10b}$$

$$s_1^2s_2^2s_3^2 = \frac{E}{A}. \tag{4.2.10c}$$

A discussion on positive real roots of cubic equation is given in appendix A. From Eqs. (4.2.10a)-(4.2.10c), the following approximated roots are obtained

$$\frac{s_1^2}{k^2} = \alpha - \frac{c^2}{c_1^2}, \tag{4.2.11a}$$

$$\frac{s_2^2}{k^2} = \frac{1}{\beta}\left(1 - \frac{c^2}{c_1^2}\right), \tag{4.2.11b}$$

$$\frac{s_3^2}{k^2} = \frac{A_1}{A_2} - \frac{\bar{k}}{\lambda_4}\frac{c^2}{c_1^2}. \tag{4.2.11c}$$

The arbitrary functions $f(x_2), g(x_2), h(x_2)$ can be written as linear combination of these roots

$$f(x_2) = \sum_{n=1}^{3} R_n \exp(-s_n x_2), \tag{4.2.12a}$$

$$g(x_2) = \sum_{n=1}^{3} R_{1n} \exp(-s_n x_2), \tag{4.2.12b}$$

$$h(x_2) = \sum_{n=1}^{3} R_{2n} \exp(-s_n x_2), \tag{4.2.12c}$$

where R_n, R_{1n} and R_{2n} are constants. Substitution of Eqs. (4.2.12a)-(4.2.12c) in Eqs. (4.2.6a)-(4.2.6c) yields

$$\left(s_n^2 - \sigma_1\right) R_n - \gamma i k s_n R_{1n} + \lambda_1 i k R_{2n} = 0, \tag{4.2.13a}$$

$$-\gamma i k s_n R_n + \left(\beta s_n^2 - \sigma_2\right) R_{1n} - \lambda_2 s_n R_{2n} = 0, \tag{4.2.13b}$$

$$-\lambda_1 i k R_n + \lambda_2 s_n R_{1n} + \left(\lambda_4 s_n^2 - \sigma_3\right) R_{2n} = 0. \tag{4.2.13c}$$

Equations (4.2.13a), (4.2.13b), and (4.2.13c) can also be written as

$$\left(\frac{s_n^2}{k^2} - \frac{\sigma_1}{k^2}\right) \frac{R_n}{k} - \gamma i \frac{s_n}{k} \frac{R_{1n}}{k} + \lambda_1 i \frac{R_{2n}}{k^2} = 0, \tag{4.2.14a}$$

$$-\gamma i \frac{s_n}{k} \frac{R_n}{k} + \left(\beta \frac{s_n^2}{k^2} - \frac{\sigma_2}{k^2}\right) \frac{R_{1n}}{k} - \lambda_2 \frac{s_n}{k} \frac{R_{2n}}{k^2} = 0, \tag{4.2.14b}$$

$$-\lambda_1 i \frac{R_n}{k} + \lambda_2 \frac{s_n}{k} \frac{R_{1n}}{k} + \left(\lambda_4 \frac{k^2 s_n^2}{k^2} - \sigma_3\right) \frac{R_{2n}}{k^2} = 0. \tag{4.2.14c}$$

It can be assumed that $\bar{R}_n = \frac{R_n}{k}$, $\bar{\sigma}_1 = \frac{\sigma_1}{k^2}, \bar{s}_n = \frac{s_n}{k}, \bar{R}_{1n} = \frac{R_{1n}}{k}, \bar{R}_{2n} = \frac{R_{2n}}{k^2}$, and $\bar{\sigma}_2 = \frac{\sigma_2}{k^2}$. After introducing these notations, Eqs. (4.2.14a), (4.2.14b), and (4.2.14c) take the form

$$\left(\bar{s}_n^2 - \bar{\sigma}_1\right) \bar{R}_n - \gamma i \bar{s}_n \bar{R}_{1n} + \lambda_1 i \bar{R}_{2n} = 0, \tag{4.2.15a}$$

$$-\gamma i \bar{s}_n \bar{R}_n + \left(\beta \bar{s}_n^2 - \bar{\sigma}_2\right) \bar{R}_{1n} - \lambda_2 \bar{s}_n \bar{R}_{2n} = 0, \tag{4.2.15b}$$

$$-\lambda_1 i \bar{R}_n + \lambda_2 \bar{s}_n \bar{R}_{1n} + \left((\lambda_4 k^2) \bar{s}_n^2 - \sigma_3\right) \bar{R}_{2n} = 0. \tag{4.2.15c}$$

From Eq. (4.2.15b)

$$\bar{R}_{2n} = \left(\frac{\beta \bar{s}_n^2 - \bar{\sigma}_2}{\lambda_2 \bar{s}_n}\right) \bar{R}_{1n} - \left(\frac{\gamma i}{\lambda_2}\right) \bar{R}_n. \tag{4.2.16}$$

By using Eq. (4.2.16) in Eq. (4.2.15a), we get

$$\bar{R}_{1n} = M_{1n}\bar{R}_n, \qquad n = 1, 2, 3. \tag{4.2.17}$$

where

$$M_{1n} = \left[\frac{i\lambda_2\bar{s}_n(\bar{s}_n^2 - \bar{\sigma}_1 + \gamma\frac{\lambda_1}{\lambda_2})}{\lambda_1(\beta\bar{s}_n^2 - \bar{\sigma}_2) - \gamma\lambda_2\bar{s}_n^2}\right], \qquad n = 1, 2, 3. \tag{4.2.18}$$

Substitution of Eq. (4.2.17) into Eq. (4.2.15c) results in

$$\bar{R}_{2n} = M_{2n}\bar{R}_n, \qquad n = 1, 2, 3 \tag{4.2.19}$$

where

$$M_{2n} = \left[\frac{i\lambda_1 - \lambda_2\bar{s}_n M_{1n}}{(\lambda_4 k^2)\bar{s}_n^2 - \sigma_3}\right], \qquad n = 1, 2, 3. \tag{4.2.20}$$

For simplification, the arbitrary functions given in Eqs. (4.2.12a)-(4.2.12c) can be redefined as

$$f(x_2) = \sum_{n=1}^{3} \bar{R}_n \exp(-s_n x_2), \tag{4.2.21a}$$

$$g(x_2) = \sum_{n=1}^{3} \bar{R}_{1n} \exp(-s_n x_2), \tag{4.2.21b}$$

$$h(x_2) = \sum_{n=1}^{3} \bar{R}_{2n} \exp(-s_n x_2). \tag{4.2.21c}$$

Hence the expression for displacement and void volume function given in Eqs. (4.2.3a)-(4.2.3c)take the form

$$u_1(x_1, x_2, t) = \bar{R}_n \exp(-s_n x_2)\exp(ik(x_1 - ct)), \tag{4.2.22a}$$

$$u_2(x_1, x_2, t) = M_{1n}\bar{R}_n \exp(-s_n x_2)\exp(ik(x_1 - ct)), \tag{4.2.22b}$$

$$\phi(x_1, x_2, t) = M_{2n}\bar{R}_n \exp(-s_n x_2)\exp(ik(x_1 - ct)). \tag{4.2.22c}$$

Applying the boundary condition of vanishing stresses $T_{21} = T_{22} = 0$ at $x_2 = 0$, gives

$$\sum_{n=1}^{3}(iM_{1n} - \bar{s}_n)\bar{R}_n = 0. \tag{4.2.23}$$

$$C_{12}\sum_{n=1}^{3} i\bar{R}_n - C_{22}\sum_{n=1}^{3}(M_{1n}\bar{s}_n)\bar{R}_n = 0. \tag{4.2.24}$$

Expressions for T_{21} and T_{22} in an orthotropic medium are given in Eqs. (4.1.14a)-(4.1.14b).

Due to the presence of voids, a boundary condition on equilibrated stress vector $\mathbf{h}$ as given in (Tomar and Ogden, 2014) is observed. The condition says $\mathbf{h}\cdot\mathbf{n} = 0$. Using Eq. (3.3.11), the condition becomes $A_i \bigtriangledown \phi \cdot \mathbf{n} = 0$, where $\mathbf{n}$ is the outward unit normal on the boundary. Here in this case unit normal is $\mathbf{n} = (0, 1, 0)$. This condition results in

$$\sum_{n=1}^{3}(M_{2n}\bar{s}_n)\bar{R}_n = 0. \tag{4.2.25}$$

Elimination of constants $\bar{R}_1, \bar{R}_2, \bar{R}_3$ from Eq. (4.2.23), Eq. (4.2.24), and Eq. (4.2.25) gives secular equation for Rayleigh waves in homogeneous orthotropic porous elastic half space, that is

$$\begin{vmatrix} -\bar{s}_1 + iM_{11} & -\bar{s}_2 + iM_{12} & -\bar{s}_3 + iM_{13} \\ iC_{12} - C_{22}M_{11}\bar{s}_1 & iC_{12} - C_{22}M_{12}\bar{s}_2 & iC_{12} - C_{22}M_{13}\bar{s}_3 \\ M_{21}\bar{s}_1 & M_{22}\bar{s}_2 & M_{23}\bar{s}_3 \end{vmatrix} = 0, \tag{4.2.26}$$

where, M_{1n} and M_{2n} are same as given in Eq. (4.2.18) and Eq. (4.2.20). Expansion of determinant given in Eq. (4.2.26) yields

$$\begin{aligned} 0 = {} & (\bar{s}_1)(\bar{s}_2)(\bar{s}_3)\left\{(M_{12}-M_{13})M_{21} + (M_{13}-M_{11})M_{22} + (M_{11}-M_{12})M_{23}\right\} \\ & - w\left\{(M_{13}-M_{12})M_{21}(\bar{s}_1) + (M_{11}-M_{13})M_{22}(\bar{s}_2) + (M_{12}-M_{11})M_{23}(\bar{s}_3)\right\} \\ & + iw\left\{(M_{21}-M_{22})(\bar{s}_1)(\bar{s}_2) + (M_{23}-M_{21})(\bar{s}_1)(\bar{s}_3) + (M_{22}-M_{23})(\bar{s}_2)(\bar{s}_3)\right\} \\ & + i\left\{M_{13}(M_{11}M_{22} - M_{12}M_{21})(\bar{s}_1)(\bar{s}_2)\right\} \\ & + i\left\{M_{12}(M_{13}M_{21} - M_{11}M_{23})(\bar{s}_1)(\bar{s}_3) + M_{11}(M_{12}M_{23} - M_{13}M_{22})(\bar{s}_2)(\bar{s}_3)\right\} \end{aligned} \tag{4.2.27}$$

where $w = \frac{C_{12}}{C_{22}}$. This is the approximated dispersion relation for Rayleigh waves propagating in orthotropic elastic half space.

4.3 Numerical results and discussion

The approximated dispersion relation given in Eq. (4.2.27) is used to compute the non-dimensional speed $\frac{c}{c_1}$ of Rayleigh wave propagating in orthotropic elastic half space with voids. Change in dimensionless speed $\frac{c}{c_1}$ is observed against dimensionless wave number $\sqrt{\lambda_4}k$.

The elastic and void constants listed in Table 4.1 are used for Figures 4.3, 4.4, 4.5 and taken from (Royer and Dieulesaint, 1996) and (Ranjeesh and Kumar, 2011). Numerical values of other non dimensional void parameters $\xi^{'} = \frac{\xi}{C_{66}}$, $\varpi^{'} = \frac{\varpi i c_1}{\sqrt{\lambda_4} C_{66}}$ and $k^{'} = \frac{\bar{k}}{\lambda_4}$ are taken arbitrarily.

Symbol	Value	Unit
C_{11}	23.9×10^{10}	N/m^2
C_{22}	24.7×10^{10}	N/m^2
C_{12}	10.4×10^{10}	N/m^2
C_{66}	7.6×10^{10}	N/m^2
A_1	14.798×10^{-5}	N
A_2	13.9714×10^{-5}	N
B_1	8.52849×10^{10}	N/m^2
B_2	7.41×10^{6}	N/m^2
ρ	5300	kg/m^3

Table 4.1: Elastic constants and void parameters.

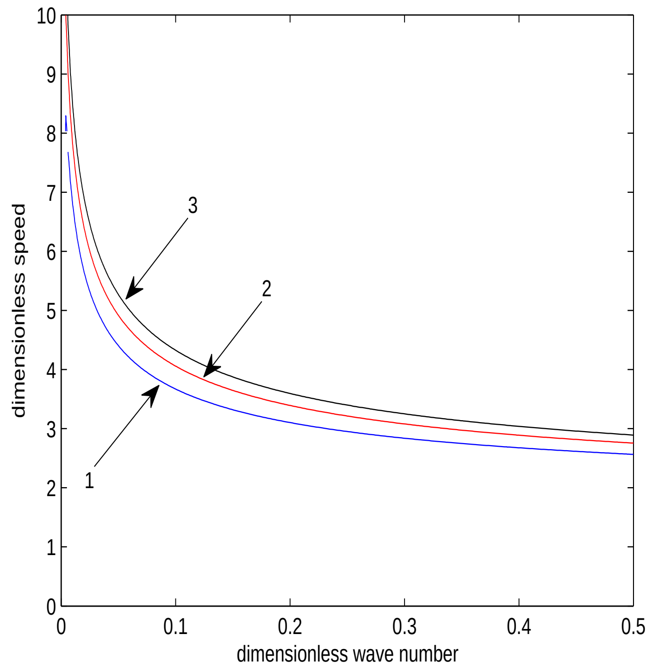


Figure 4.3: Variations of dimensionless wave speed $\frac{\mathbf{c}}{\mathbf{c_1}}$ against dimensionless wave number $\sqrt{\lambda_4}\mathbf{k}$ when $\xi^{'} = 0.7$, $k^{'} = 1.5$ and curve1- $\varpi^{'} = 55$, curve2- $\varpi^{'} = 79$, curve3- $\varpi^{'} = 99$.

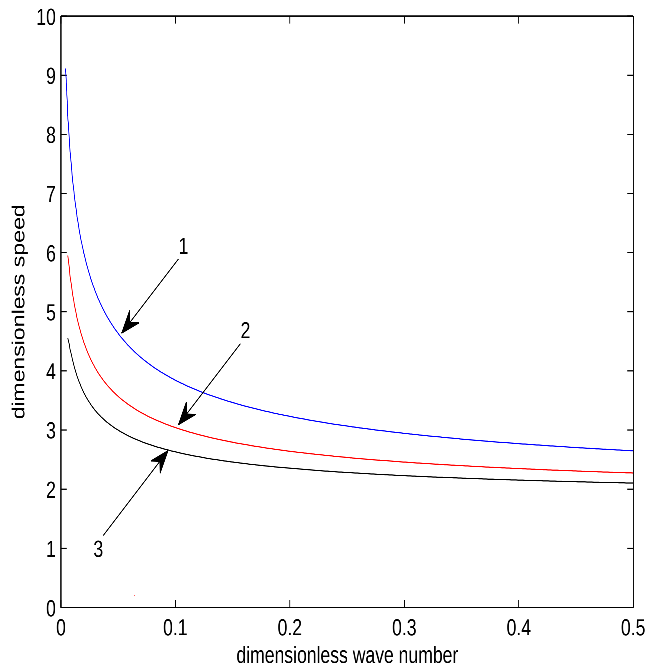


Figure 4.4: Variations of dimensionless wave speed $\frac{\mathbf{c}}{\mathbf{c_1}}$ against dimensionless wave number $\sqrt{\lambda_4}\mathbf{k}$ when $\xi^{'} = 0.7$, $\varpi^{'} = 65$ and curve1- $k^{'} = 1.5$, curve2- $k^{'} = 3$, curve3- $k^{'} = 5$.

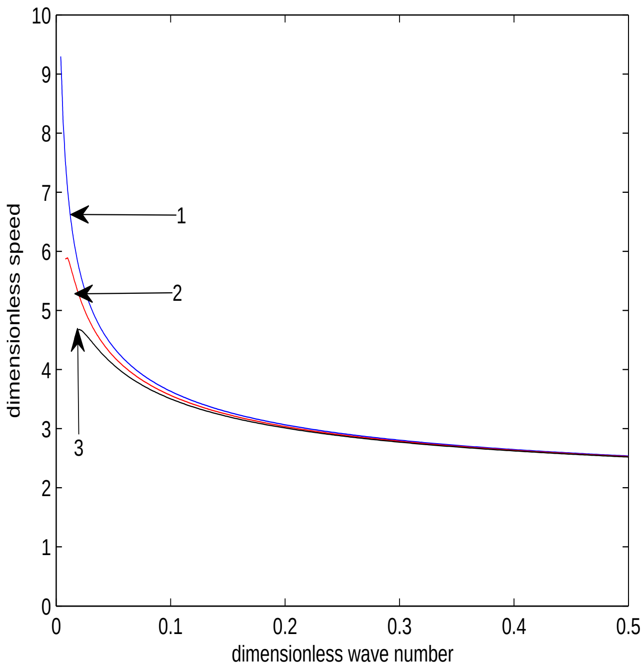


Figure 4.5: Variations of dimensionless wave speed $\frac{\mathbf{c}}{\mathbf{c_1}}$ against dimensionless wave number $\sqrt{\lambda_4}\mathbf{k}$ when $\varpi^{'} = 75$, $k^{'} = 2$ curve1-$\xi^{'} = 0.1$, curve2-$\xi^{'} = 2$, curve3-$\xi^{'} = 3.5$.

Figure 4.3 shows the variations of dimensionless wave speed $\frac{c}{c_1}$ against dimensionless wave number $\sqrt{\lambda_4}k$ for gradual increase in $\varpi^{'}$ and fixed values of $\xi^{'}$ and $k^{'}$. It is noticed that with the gradual increase in $\varpi^{'}$, speed of Rayleigh wave increases.

Figure 4.4 depicts the variation of dimensionless wave speed $\frac{c}{c_1}$ against dimensionless wave number $\sqrt{\lambda_4}k$ when the value of void parameter $k^{'}$ increases gradually and the values of $\xi^{'}$ and $\varpi^{'}$ are kept constant. It is observed that speed decreases with the increase in value of $k^{'}$.

The variation of dimensionless wave speed $\frac{c}{c_1}$ against dimensionless wave number $\sqrt{\lambda_4}k$ when the value of void parameter $\xi^{'}$ increases gradually and the values of $k^{'}$ and $\varpi^{'}$ are kept constant is shown in Fig. 4.5. It is found that speed decreases with the increase in value of $\xi^{'}$.

To check the influence of material constants on speed of Rayleigh waves, two different sets of elastic constants are taken from (Baljeet et al, 2013), and (Royer and Dieulesaint, 1996), and are given in Table 4.2 and Table 4.3 respectively. For figures 4.6, 4.7 and 4.8 constants given in Table 4.2 are used. Whereas constants given in Table 4.3 are used for figures 4.9, 4.10 and 4.11. Same results are observed as in the first case. Therefore it is concluded that with the gradual increase in ϖ', speed of Rayleigh wave increases. Whereas by increasing k' and ξ' speed decreases. Also with the increase in dimensionless wave number $\sqrt{\lambda_4}k$ speed of the wave decreases.

Symbol	Value	Unit
C_{11}	11.65×10^{10}	N/m^2
C_{22}	11.71×10^{10}	N/m^2
C_{12}	7.69×10^{10}	N/m^2
C_{66}	1.98×10^{10}	N/m^2
A_1	14.798×10^{-5}	N
A_2	13.9714×10^{-5}	N
B_1	8.52849×10^{10}	N/m^2
B_2	7.41×10^{6}	N/m^2
ρ	2.19×10^{3}	kg/m^3

Table 4.2: Elastic constants and void parameters.

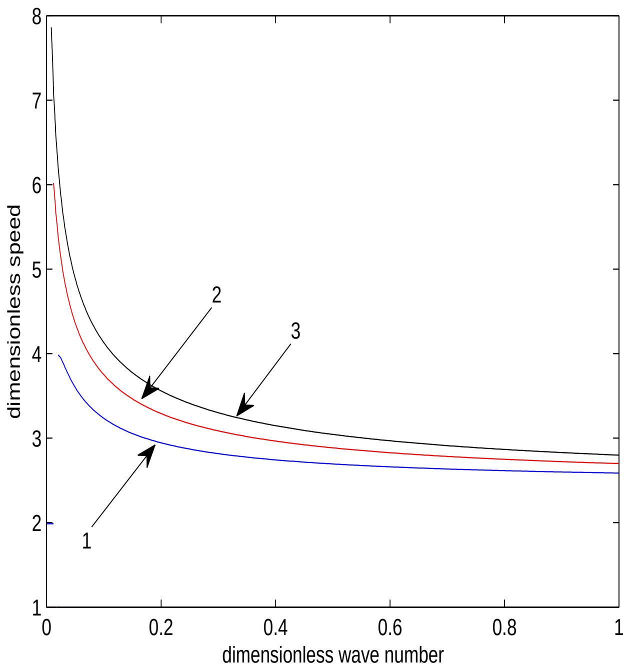


Figure 4.6: Variations of dimensionless wave speed $\frac{\mathbf{c}}{\mathbf{c_1}}$ against dimensionless wave number $\sqrt{\lambda_4}\mathbf{k}$ when $\xi' = 0.2$, $k' = 1$ and curve1- $\varpi' = 5$, curve2- $\varpi' = 10$, curve3- $\varpi' = 15$.

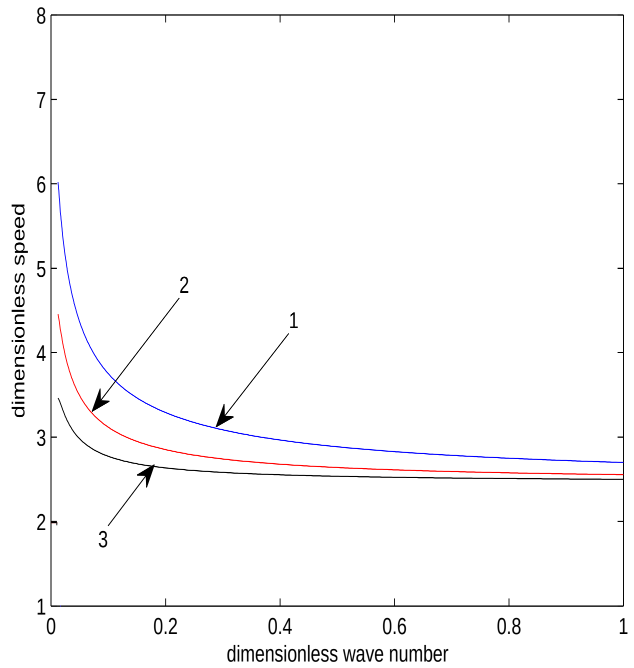


Figure 4.7: Variations of dimensionless wave speed $\frac{\mathbf{c}}{\mathbf{c_1}}$ against dimensionless wave number $\sqrt{\lambda_4}\mathbf{k}$ when $\xi^{'} = 0.2$, $\varpi^{'} = 10$ and curve1- $k^{'} = 1$, curve2- $k^{'} = 2.5$, curve3- $k^{'} = 6$.

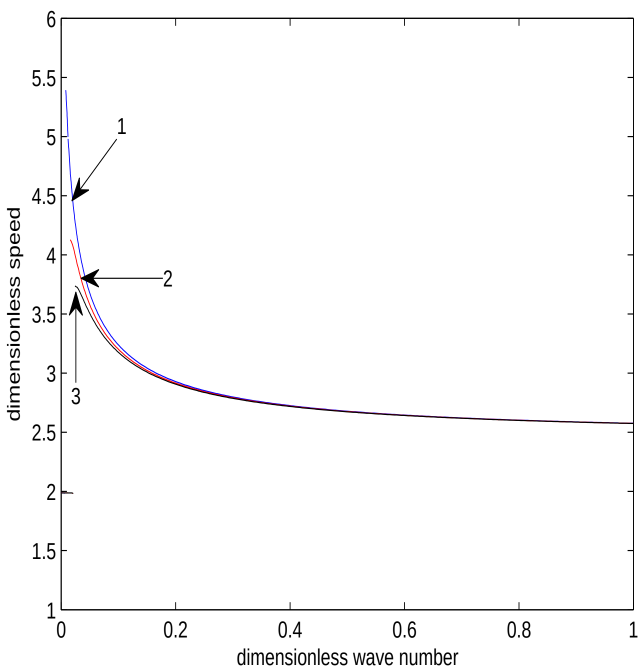


Figure 4.8: Variations of dimensionless wave speed $\frac{\mathbf{c}}{\mathbf{c_1}}$ against dimensionless wave number $\sqrt{\lambda_4}\mathbf{k}$ when $\varpi^{'} = 15$, $k^{'} = 3$ curve1-$\xi^{'} = 0.2$, curve2-$\xi^{'} = 0.5$, curve3-$\xi^{'} = 0.7$.

Symbol	Value	Unit
C_{11}	3.01×10^{10}	N/m^2
C_{22}	5.8×10^{10}	N/m^2
C_{12}	1.61×10^{10}	N/m^2
C_{66}	1.58×10^{10}	N/m^2
A_1	14.798×10^{-5}	N
A_2	13.9714×10^{-5}	N
B_1	8.52849×10^{10}	N/m^2
B_2	7.41×10^{6}	N/m^2
ρ	4640	kg/m^3

Table 4.3: Elastic constants and void parameters.

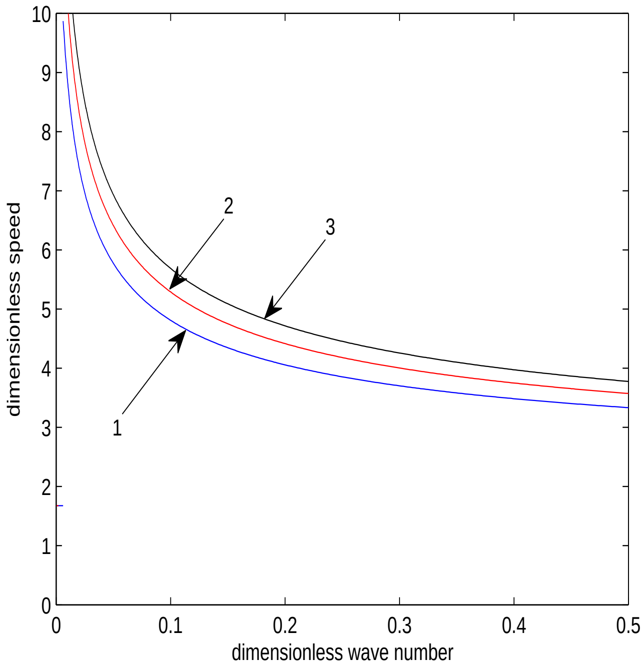


Figure 4.9: Variations of dimensionless wave speed $\frac{\mathbf{c}}{\mathbf{c_1}}$ against dimensionless wave number $\sqrt{\lambda_4}\mathbf{k}$ when $\xi' = 0.5$, $k' = 0.2$ and curve1- $\varpi' = 25$, curve2- $\varpi' = 35$, curve3- $\varpi' = 45$.

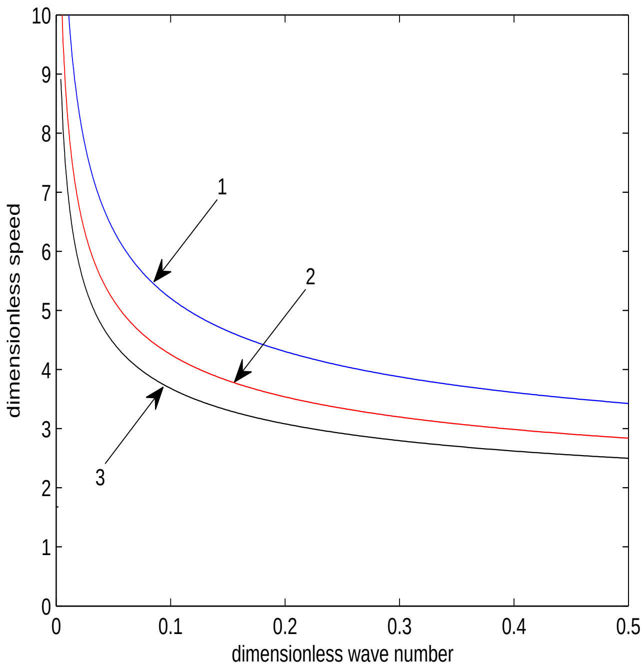


Figure 4.10: Variations of dimensionless wave speed $\frac{\mathbf{c}}{\mathbf{c_1}}$ against dimensionless wave number $\sqrt{\lambda_4}\mathbf{k}$ when $\xi^{'} = 0.5$, $\varpi^{'} = 55$ and curve1- $k^{'} = 0.3$, curve2- $k^{'} = 0.5$, curve3- $k^{'} = 0.7$.

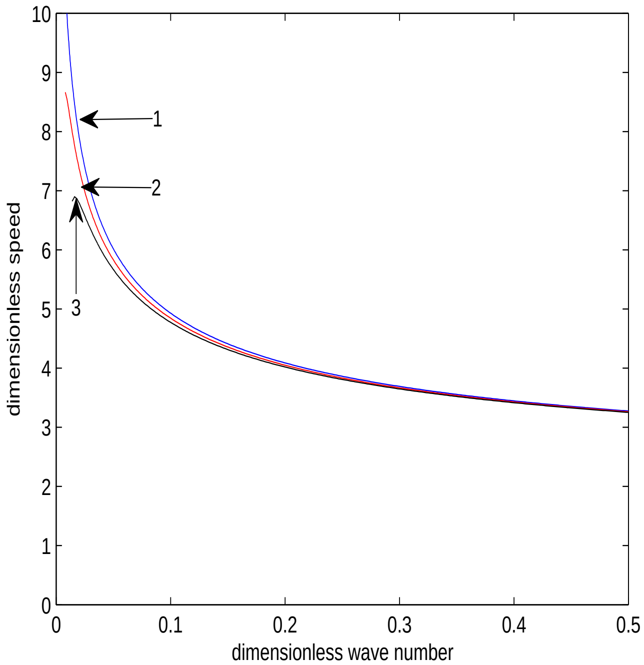


Figure 4.11: Variations of dimensionless wave speed $\frac{\mathbf{c}}{\mathbf{c_1}}$ against dimensionless wave number $\sqrt{\lambda_4}\mathbf{k}$ when $\varpi^{'} = 45$, $k^{'} = 0.3$ curve1-$\xi^{'} = 0.2$, curve2-$\xi^{'} = 1.5$, curve3-$\xi^{'} = 2.5$.

Chapter 5

Conclusion

In this thesis, the problems of propagation of Love and Rayleigh waves in orthotropic elastic half space with and without voids are discussed. The results obtained are summarized in the following lines.

Dispersive nature of Love waves is noted in orthotropic elastic half space. It is found that these waves are not affected by the presence of voids. Speed of these waves is not influenced by rotation of half space as well.

An expression for the speed of Rayleigh waves in orthotropic elastic half space is derived. An approximate frequency equation is obtained for Rayleigh waves propagating in orthotropic elastic half space with voids. It is observed that the relation obtained in this case is dispersive and dispersion is caused by the presence of voids. Numerical results reveals that with the gradual increase in ϖ', speed of Rayleigh wave increases. Whereas by increasing k' and ξ' speed decreases. Also with the increase in dimensionless wave number $\sqrt{\lambda_4}k$ speed of the wave decreases.

Appendix A

Positive real roots of cubic polynomial

A cubic equation has the form

$$ax^3 + bx^2 + cx + d = 0, \quad where \quad a \neq 0. \tag{A.1}$$

All cubic equations have either one real root, or three real roots. Solution of general cubic equation is found by eliminating x^2 term. For this a substitution of following form is made

$$x = t - \frac{b}{3a}. \tag{A.2}$$

Use of Eq. (A.2) in Eq. (A.1) gives depressed cubic equation

$$t^3 + pt + q = 0, \tag{A.3}$$

where $p = \frac{3ac-b^2}{3a^2}$ and $q = \frac{2b^3-9abc+27a^2d}{27a^3}$. If $p < 0$, then all roots of Eq. (A.1) are real, and if $p > 0$ then there will be only one real root.

Also by **Descartes' rule of signs** one can identify the possible number of positive real roots of a polynomial without actually graphing or solving it. The rule states that the number of positive real roots of a polynomial is bounded by the number of changes of sign in its coefficients. If n is the maximum number of positive roots then the number of allowable roots is $n, n-2, n-4, \cdots$.

Example: Consider the polynomial

$$f(x) = 3x^7 + 5x^6 - x^4 - x^3 - x^2 + x - 1. \tag{A.4}$$

Since there are three sign changes, so maximum three positive roots are possible.

For negative roots, starting with a polynomial $f(x)$, write a new polynomial $f(-x)$. For the new polynomial signs of all odd powers reversed, while the signs of the even powers remains unchanged. Then proceed as before to count the number of sign changes n. Then n is the maximum number of negative roots.

Example: Consider the above polynomial again to check the number of negative roots,

$$f(-x) = -3x^7 + 5x^6 - x^4 + x^3 - x^2 - x - 1. \tag{A.5}$$

In above example, there are four sign changes, so there are a maximum of four negative roots.

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