Laplace Transform on Time Scale

by

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Dedicated to My Loving Parents

For their endless love, support and encouragement

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Abstract

Stefan Hilger introduced the theory of time scale in his PhD research work in 1988 [13]. Time scale is an arbitrary non-empty closed subset of real numbers [14]. This theory unifies continuous and discrete analysis to deal with both rd-continuous and ld-continuous functions.

My thesis deals with the Laplace transform on time scale by using different time scales. We consider two papers of Martin Bohner [7] and Gusein Sh. Guseinov [8]. These papers give us definitions of Laplace transform, convolution, inverse Laplace transform and some results about these topics on isolated time scale, $\mathbb{T} = h\mathbb{Z} = \{hk : k \in \mathbb{Z}\}$ where $h > 0$ and $\mathbb{T} = q^{\mathbb{N}_0} = \{q^k : k \in \mathbb{N}_0\}$ where $q > 1$ by using delta operator.

Finally, we extend the work of Martin Bohner and Gusein Sh. Guseinov on their papers by using nabla operator instead of delta operator. Nabla calculus for time scales was introduced by F. M. Atici and G.S. Guseinov [3]. We also give definitions of Laplace transform, convolution, inverse Laplace transform and some results about these topics on isolated time scale, $\mathbb{T} = h\mathbb{Z} = \{hk : k \in \mathbb{Z}\}$ where $h > 0$ and $\mathbb{T} = q^{\mathbb{N}_0} = \{q^k : k \in \mathbb{N}_0\}$ where $q > 1$ by using nabla operator.

Contents

Chapter 1

Introduction to Time Scale

1.1 Basic Definitions of Time Scale with Delta Operator

An arbitrary non-empty closed subset of real numbers is known as time scale, which is denoted by $\mathbb{T}$. Natural numbers, integers, nonnegative integers, real numbers are well known examples of time scales. The cantor set and $[0,2]\cup[4,6]$, $[0,2]\cup\mathbb{N}$ are also the examples of time scales. Open intervals, complex numbers, Rational numbers and irrational numbers are not time scales.

Definition 1.1.1. Suppose $\mathbb{T}$ denotes a time scale. For $s \in \mathbb{T}$, the **forward jump operator** $\sigma : \mathbb{T} \to \mathbb{T}$ is defined by

$$\sigma(s) = \inf\{t \in \mathbb{T} : t > s\},$$

on the other hand, the **backward jump operator** $\rho : \mathbb{T} \to \mathbb{T}$ is defined by

$$\rho(s) = \sup\{t \in \mathbb{T} : t < s\}.$$

In the above definition, substitute $\inf \emptyset = \sup \mathbb{T}$ and $\sup \emptyset = \inf \mathbb{T}$, where $\emptyset$ denotes the empty set. If $\sigma(s) > s$, then s is **right-scattered**, whereas if $\rho(s) < s$, then s is **left-scattered**. Points that are left-scattered and right-scattered at the same time are called **isolated**. Also, if $\sigma(s) = s$ and $s < \sup \mathbb{T}$, then s is called **right-dense** and if $\rho(s) = s$ and $s > \inf \mathbb{T}$, then s is called **left-dense**. Points that are right-dense and left-dense at the same time are called **dense**. Now, the **graininess function** $\mu : \mathbb{T} \to [0, \infty)$ is defined as

$$\mu(s) = \sigma(s) - s.$$

The set $\mathbb{T}^k$ is derived from the time scale $\mathbb{T}$ as follows: If $\mathbb{T}$ has a left-scattered maximum m, then $\mathbb{T}^k = \mathbb{T} - \{m\}$. Otherwise, $\mathbb{T}^k = \mathbb{T}$. In summary,

$$\mathbb{T}^k = \begin{cases} \mathbb{T}\backslash(\rho(\sup \mathbb{T}), \sup \mathbb{T}] & \text{if} \quad \sup \mathbb{T} < \infty \\ \mathbb{T} & \text{if} \quad \sup \mathbb{T} = \infty. \end{cases}$$

1.2 Examples with Delta Operator

Example 1.2.1. Consider three examples $\mathbb{T}=\mathbb{R}$, $\mathbb{T}=\mathbb{Z}$ and $\mathbb{T}=a_m=\{\frac{1}{m}: m\in\mathbb{N}\}\cup\{0\}$.
(a) If $\mathbb{T}=\mathbb{R}$, then for any $s\in\mathbb{R}$

$$\sigma(s)=\inf\{t\in\mathbb{R}: t>s\}=\inf(s,\infty)=s,$$

in similar way

$$\rho(s)=\sup\{t\in\mathbb{R}: t<s\}=\sup(-\infty,s)=s.$$

Thus every point $s\in\mathbb{R}$ is dense. The graininess function μ becomes

$$\mu(s)=0 \qquad \text{for all} \qquad s\in\mathbb{T}.$$

(b) If $\mathbb{T}=\mathbb{Z}$, then for any $s\in\mathbb{Z}$

$$\sigma(s)=\inf\{t\in\mathbb{Z}: t>s\}=\inf\{s+1,s+2,s+3,...\}=s+1,$$

in the same way

$$\rho(s)=\sup\{t\in\mathbb{Z}: t<s\}=\sup\{...s-3,s-2,s-1\}=s-1.$$

Thus every point $s\in\mathbb{Z}$ is isolated. The graininess function μ changes into

$$\mu(s)=1 \qquad \text{for all} \qquad s\in\mathbb{T}.$$

In above two cases, the graininess function is constant. Now we will discuss third case for which the graininess function is not constant.
(c) If $\mathbb{T}=a_m=\{\frac{1}{m}: m\in\mathbb{N}\}\cup\{0\}$, then for any $s\in a_m$

$$\sigma(0)=\inf\{t\in a_m: t>0\}=\inf(0,1]=0,$$

similarly

$$\sigma(1)=\inf\{t\in a_m: t>1\}=\inf\{\emptyset\}=\sup\mathbb{T}=1.$$

Now, we find $\sigma(s)$ where $s=\frac{1}{m}$. As we know

$$\begin{gathered} m+1>m>m-1 \\ \frac{1}{m+1}<\frac{1}{m}<\frac{1}{m-1} \\ \sigma\left(\frac{1}{m}\right)=\inf\left\{t\in a_m: t>\frac{1}{m}\right\}=\frac{1}{m-1}. \end{gathered}$$

$$\sigma(s)=\begin{cases} 0 & \text{if} \quad s=0, \\ \frac{1}{m-1} & \text{if} \quad s=\frac{1}{m}, \\ 1 & \text{if} \quad s=1. \end{cases}$$

Similarly

$$\rho(s)=\begin{cases}0 & \text{if} \quad s=0,\\ \frac{1}{m+1} & \text{if} \quad s=\frac{1}{m},\\ 1 & \text{if} \quad s=1.\end{cases}$$

Thus the points $s=0,1$ are dense and all others points of $\mathbb{T}$ are isolated. The graininess function in these cases are

$$\mu(s)=\begin{cases}0 & \text{if} \quad s=0,\\ \frac{1}{m(m-1)} & \text{if} \quad s=\frac{1}{m},\\ 0 & \text{if} \quad s=1.\end{cases}$$

1.3 Delta Differentiation

Consider $f:\mathbb{T}\to\mathbb{R}$ and define the **delta derivative** of f at $s\in\mathbb{T}^k$.

Definition 1.3.1. [4, 15] Assume a function $f:\mathbb{T}\to\mathbb{R}$ and let $s\in\mathbb{T}^k$. Then define $f^{\triangle}(s)$ is the number (if exists) with the property that any $\epsilon>0$, there is a neighborhood $\mathbf{U}$ of s (that is, $\mathbf{U}=(s-\delta,s+\delta)\bigcap\mathbb{T}$ for some $\delta>0$) such that

$$|[f(\sigma(s))-f(t)]-f^{\triangle}(s)[\sigma(s)-t]|\le\epsilon|\sigma(s)-t| \quad \text{for all} \quad t\in\mathbf{U},$$

where $f^{\triangle}(s)$ called **delta** (or **Hilger**) **derivative** of f at s.

Example 1.3.2. (a) Let a function $f:\mathbb{T}\to\mathbb{R}$ is defined by $f(s)=\gamma$ for all $s\in\mathbb{T}$, where $\gamma\in\mathbb{R}$ is constant, then $f^{\triangle}(s)\equiv 0$. This is true because for any $\epsilon>0$

$$|[f(\sigma(s))-f(t)]-0.[\sigma(s)-t]|=|\gamma-\gamma|=0\le\epsilon|\sigma(s)-t|,$$

holds for all $t\in\mathbb{T}$.
(b) Consider a function $f:\mathbb{T}\to\mathbb{R}$ is defined by $f(s)=s$ for all $s\in\mathbb{T}$, then $f^{\triangle}(s)\equiv 1$. This is clear since for any $\epsilon>0$

$$|[f(\sigma(s))-f(t)]-1.[\sigma(s)-t]|=|\sigma(s)-t-(\sigma(s)-t)|=0\le\epsilon|\sigma(s)-t|,$$

holds for all $t\in\mathbb{T}$.

Theorem 1.3.3. *[9, 11] Assume a function $f:\mathbb{T}\to\mathbb{R}$ and let $s\in\mathbb{T}^k$. Then:*
(a) If f is delta differentiable at s, then f is continuous at s.
(b) If f is continuous at s and s is right-scattered, then f is delta differentiable at s with

$$f^{\triangle}(s)=\frac{f(\sigma(s))-f(s)}{\mu(s)}.$$

(c) If s is right-dense, then f is delta differentiable at s if and only if

$$\lim_{t \to s} \frac{f(s) - f(t)}{s - t},$$

exists as a finite number. In this case

$$f^{\triangle}(s) = \lim_{t \to s} \frac{f(s) - f(t)}{s - t}.$$

(d) If f is delta differentiable at s, then

$$f(\sigma(s)) = f(s) + \mu(s) f^{\triangle}(s).$$

Example 1.3.4. Again consider the cases $\mathbb{T} = \mathbb{R}$ and $\mathbb{T} = \mathbb{Z}$.
(a) If $\mathbb{T} = \mathbb{R}$, then by Theorem (1.3.3)(c) implies that the function $f : \mathbb{R} \to \mathbb{R}$ is **delta differentiable** at the point $s \in \mathbb{R}$ if and only if

$$f'(s) = \lim_{t \to s} \frac{f(s) - f(t)}{s - t},$$

exists, that is, if and only if f is differentiable at s. In this case

$$f^{\triangle}(s) = \lim_{t \to s} \frac{f(s) - f(t)}{s - t} = f'(s),$$

by Theorem (1.3.3)(c).
(b) If $\mathbb{T} = \mathbb{Z}$, then by Theorem (1.3.3)(b) implies that the function $f : \mathbb{Z} \to \mathbb{R}$ is **delta differentiable** at $s \in \mathbb{Z}$ with

$$f^{\triangle}(s) = \frac{f(\sigma(s)) - f(s)}{\mu(s)} = \frac{f(s+1) - f(s)}{1} = \Delta f(s),$$

where Δ denotes the usual forward difference operator.

1.4 Delta Integration and Continuity

Definition 1.4.1. [2] Assume a function $F : \mathbb{T} \to \mathbb{R}$ is called delta antiderivative of the function $f : \mathbb{T} \to \mathbb{R}$ such that $F^{\triangle}(s) = f(s)$, for all $s \in \mathbb{T}^k$. Then the **integral** of f is defined by

$$\int_b^s f(\tau) \triangle \tau = F(s) - F(b) \quad \text{for all } \ b, s \in \mathbb{T}.$$

Definition 1.4.2. A function $p : \mathbb{T} \to \mathbb{C}$ is said to be **regressive** if

$$1 + \mu(s) p(s) \neq 0 \quad \text{for all } \ s \in \mathbb{T}.$$

Definition 1.4.3. A function $f : \mathbb{T} \to \mathbb{R}$ is called **rd-continuous** if it is continuous at each right dense points in $\mathbb{T}$ and $lim_{t \to s^-} f(t)$ exists for all left-dense points $s \in \mathbb{T}$.

Theorem 1.4.4. *[12] Assume the points $a, b \in \mathbb{T}$ and a function $f : \mathbb{T} \to \mathbb{R}$ is rd-continuous.*
(a) Let $\mathbb{T} = \mathbb{R}$, then
$$\int_a^c f(s)\Delta(s) = \int_a^c f(s)ds.$$
(b) If time scale $\mathbb{T}$ consists of only isolated points, then
$$\int_a^c f(s)\Delta(s) = \begin{cases} \sum_{s\in[a,c)} f(s)\mu(s) & if \quad a < c, \\ 0 & if \quad a = c, \\ -\sum_{s\in[c,a)} f(s)\mu(s) & if \quad a > c. \end{cases}$$

1.5 Basic Definitions of Time Scale with Nabla Operator

Let $\mathbb{T}$ denotes time scale with the backward jump operator ρ and the nabla differentiable operator ∇. Then the **graininess function** $\nu : \mathbb{T} \to [0, \infty)$ is defined by
$$\nu(s) = s - \rho(s).$$
The set $\mathbb{T}_k$ is derived from the time scale $\mathbb{T}$ as follows: If $\mathbb{T}$ has a right-scattered minimum m, then $\mathbb{T}_k = \mathbb{T} - \{m\}$. Otherwise, $\mathbb{T}_k = \mathbb{T}$.

1.6 Examples with Nabla Operator

Example 1.6.1. Consider three examples $\mathbb{T} = \mathbb{R}$, $\mathbb{T} = \mathbb{Z}$ and $\mathbb{T} = a_m = \{\frac{1}{m} : m \in \mathbb{N}\} \cup \{0\}$.
(a) If $\mathbb{T} = \mathbb{R}$, then for any $s \in \mathbb{R}$
$$\rho(s) = \sup\{t \in \mathbb{R} : t < s\} = \sup(-\infty, s) = s.$$
Thus every point $s \in \mathbb{R}$ is dense. The graininess function ν becomes
$$\nu(s) = 0 \qquad \text{for all} \quad s \in \mathbb{T}.$$
(b) If $\mathbb{T} = \mathbb{Z}$, then for any $s \in \mathbb{Z}$
$$\rho(s) = \sup\{t \in \mathbb{Z} : t < s\} = \sup\{...s - 3, s - 2, s - 1\} = s - 1.$$
Thus every point $s \in \mathbb{Z}$ is isolated. The graininess function ν changes into
$$\nu(s) = 1 \qquad \text{for all} \quad s \in \mathbb{T}.$$
In above two cases, the graininess function is constant. Now, we will discuss third case for which the graininess function is not constant.

(c) If $\mathbb{T} = a_m = \{\frac{1}{m} : m \in \mathbb{N}\} \cup \{0\}$, then for any $s \in a_m$

$$\rho(s) = \begin{cases} 0 & \text{if} \quad s = 0, \\ \frac{1}{m+1} & \text{if} \quad s = \frac{1}{m}, \\ 1 & \text{if} \quad s = 1. \end{cases}$$

Thus, the points $s = 0, 1$ are dense and all others points of $\mathbb{T}$ are isolated. The graininess function in these cases are

$$\nu(s) = \begin{cases} 0 & \text{if} \quad s = 0, \\ \frac{1}{m(m+1)} & \text{if} \quad s = \frac{1}{m}, \\ 0 & \text{if} \quad s = 1. \end{cases}$$

1.7 Nabla Differentiation

Consider $f : \mathbb{T} \to \mathbb{R}$ and define the **nabla derivative** of f at $s \in \mathbb{T}_k$.

Definition 1.7.1. [1] Assume a function $f : \mathbb{T} \to \mathbb{R}$ and let $s \in \mathbb{T}_k$. Then define $f^{\nabla}(t)$ is the number (if exists) with the property that given any $\epsilon > 0$, there is a neighborhood $\mathbf{U}$ of s (that is, $\mathbf{U} = (s - \delta, s + \delta)$) such that

$$|[f(\rho(s)) - f(t)] - f^{\nabla}(s)[\rho(s) - t]| \leq \epsilon|\rho(s) - t| \quad \text{for all} \quad t \in \mathbf{U},$$

where $f^{\nabla}(s)$ called the **nabla derivative** of f at s.

Example 1.7.2. (a) Consider a function $f : \mathbb{T} \to \mathbb{R}$ is defined by $f(s) = \gamma$ for all $s \in \mathbb{T}$, where $\gamma \in \mathbb{R}$ is constant, then $f^{\nabla}(s) \equiv 0$. This is true because for any $\epsilon > 0$

$$|f(\rho(s)) - f(t) - 0.[\rho(s) - t]| = |\gamma - \gamma| = 0 \leq \epsilon|\rho(s) - t|,$$

holds for all $t \in \mathbb{T}$.

(b) Consider a function $f : \mathbb{T} \to \mathbb{R}$ is defined by $f(s) = s$ for all $s \in \mathbb{T}$, then $f^{\nabla}(s) \equiv 1$. This is clear since for any $\epsilon > 0$

$$|f(\rho(s)) - f(t) - 1.[\rho(s) - t]| = |\rho(s) - t - (\rho(s) - t)| = 0 \leq \epsilon|\rho(s) - t|,$$

holds for all $t \in \mathbb{T}$.

Theorem 1.7.3. *[9, 11] Consider a function $f : \mathbb{T} \to \mathbb{R}$ and let $s \in \mathbb{T}_k$. Then:*
(a) If f is nabla differentiable at s, then f is continuous at s.
(b) If f is continuous at s and s is left-scattered, then f is nabla differentiable at s with

$$f^{\nabla}(s) = \frac{f(s) - f(\rho(s))}{\nu(s)}.$$

(c) If s is left-dense, then f is nabla differentiable at s if and only if

$$\lim_{t\to s}\frac{f(s)-f(t)}{s-t},$$

exists as a finite number. In this case

$$f^{\nabla}(s)=\lim_{t\to s}\frac{f(s)-f(t)}{s-t}.$$

(d) If f is nabla differentiable at s, then

$$f^{\rho}(s)=f(\rho(s))=f(s)-\nu(s)f^{\nabla}(s).$$

Example 1.7.4. Consider also previous two cases $\mathbb{T}=\mathbb{R}$ and $\mathbb{T}=\mathbb{Z}$.
(a) First assume $\mathbb{T}=\mathbb{R}$, then by Theorem (1.7.3)(c) implies that the function $f:\mathbb{R}\to\mathbb{R}$ is **nabla differentiable** at the point $s\in\mathbb{R}$ if and only if

$$f'(s)=\lim_{t\to s}\frac{f(s)-f(t)}{s-t},$$

exists, that is if and only if f is differentiable at s. In this case

$$f^{\nabla}(s)=\lim_{t\to s}\frac{f(s)-f(t)}{s-t}=f'(s),$$

by Theorem (1.7.3)(c). Thus, for $\mathbb{T}=\mathbb{R}$, $f'(s)=f^{\nabla}(s)=f^{\Delta}(s)$.
(b) Now consider $\mathbb{T}=\mathbb{Z}$, then by Theorem (1.7.3)(b) implies that $f:\mathbb{Z}\to\mathbb{R}$ is **nabla differentiable** at the point $s\in\mathbb{Z}$ with

$$f^{\nabla}(s)=\frac{f(s)-f(\rho(s))}{\nu(s)}=\frac{f(s)-f(s-1)}{1}=\nabla f(s),$$

where ∇ is the usual backward difference operator.

1.8 Nabla Integration and Continuity

Definition 1.8.1. [1] Assume a function $F:\mathbb{T}\to\mathbb{R}$ is called a nabla antiderivative of the function $f:\mathbb{T}\to\mathbb{R}$ such that $F^{\nabla}(s)=f(s)$ for all $s\in\mathbb{T}_k$. Then the **integral** of f is defined as

$$\int_b^s f(\tau)\nabla\tau=F(s)-F(b)\quad\text{for all }\ b,s\in\mathbb{T}.$$

Definition 1.8.2. A function $q:\mathbb{T}\to\mathbb{C}$ is said to be **regressive** if

$$1+\nu(s)q(s)\neq 0\quad\text{for all }\ s\in\mathbb{T}.$$

Definition 1.8.3. A function $f:\mathbb{T}\to\mathbb{R}$ is called **ld-continuous** if it is continuous at each left dense points in $\mathbb{T}$ and $lim_{t\to s^+}f(t)$ exists for all right-dense points $s\in\mathbb{T}$.

Theorem 1.8.4. *[12] Assume the points $a, c \in \mathbb{T}$ and a function $f : \mathbb{T} \to \mathbb{R}$ is ld-continuous.*

(a) Let $\mathbb{T} = \mathbb{R}$, then

$$\int_a^c f(s)\nabla(s) = \int_a^c f(s)ds.$$

(b) If time scale $\mathbb{T}$ consists of only isolated points, then

$$\int_a^c f(s)\nabla(s) = \begin{cases} \sum_{s\in(a,c]} f(s)\nu(s) & if \quad a < c, \\ 0 & if \quad a = c, \\ -\sum_{s\in(c,a]} f(s)\nu(s) & if \quad a > c. \end{cases}$$

Chapter 2

Laplace Transform with Isolated Time Scale by using Delta Operator

2.1 The Exponential Function and Laplace Transform by using Delta Operator

This section is based on generalized exponential function and Laplace transform for an arbitrary time scale $\mathbb{T}$ by using delta operator.

The set of all rd-continuous and regressive functions $p : \mathbb{T} \to \mathbb{C}$ denoted by $\mathcal{R}$. Suppose $p \in \mathcal{R}$ and $t \in \mathbb{T}$, where t is any fixed element. Then the initial value problem is

$$y^{\triangle}(s) = p(s)y(s), \quad y(t) = 1, \tag{2.1.1}$$

has unique solution on $\mathbb{T}$. This solution is said to be **exponential function** and is denoted by $e_p(s,t)$.

Assume $p \in \mathcal{R}$, then **exponential function** is defined as

$$e_p(s,t) = exp \int_t^s log \frac{1 + \mu(\tau)p(\tau)}{\mu(\tau)} \triangle(\tau) \quad \text{for} \quad s,t \in \mathbb{T}.$$

Throughout assume that s_r denote real numbers, for all $r \in \mathbb{N}_0$ such that

$$\lim_{r\to\infty} s_r = \infty \qquad \text{and} \qquad u_r = s_{r+1} - s_r > 0 \quad \text{for all} \quad r \in \mathbb{N}_0, \tag{2.1.2}$$

while throughout assume that

$$\lim_{r\to\infty} s_r = \infty \quad \text{and} \quad u = \inf_{r\in\mathbb{N}_0} u_r > 0, \qquad \text{where} \qquad u_r = s_{r+1} - s_r \quad \text{for} \quad r \in \mathbb{N}_0, \tag{2.1.3}$$

holds. For example, the numbers

$$s_r = hr, \;\; r \in \mathbb{N}_0 \quad \text{and} \quad s_r = q^r, \;\; r \in \mathbb{N}_0,$$

where $h > 0$ and $q > 1$, respectively satisfy previous assumption (2.1.3), while

$$s_r = \sqrt{r}, \quad r \in \mathbb{N}_0 \qquad \text{and} \qquad s_m = \ln m, \quad m \in \mathbb{N},$$

do not satisfy previous assumption (2.1.3).

Suppose z denotes complex number, then

$$z \neq -\frac{1}{u_r} \qquad \text{for all} \qquad r \in \mathbb{N}_0. \tag{2.1.4}$$

Then solution of the problem (2.1.1) is $e_z(s_r, s_m)$

$$y(s_{r+1}) = (1 + u_r z)y(s_r), \quad y(s_m) = 1, \quad m, r \in \mathbb{N}_0,$$

satisfies

$$e_z(s_r, s_m) = \prod_{k=m}^{r-1}(1 + u_k z) \quad \text{if} \quad r \geq m, \tag{2.1.5}$$

and

$$e_z(s_r, s_m) = \frac{1}{\prod_{k=r}^{m-1}(1 + u_k z)} \quad \text{if} \quad r \leq m,$$

for $m = r$, the product are understood to be 1.

Assume that $\sup\mathbb{T} = \infty$ and fix $s_0 \in \mathbb{T}$. Also assume that z denotes complex constant that is regressive. Thus $e_z(., s_0)$ is well defined on $\mathbb{T}$. Assume $x : [s_0, \infty)_{\mathbb{T}} \to \mathbb{C}$ is locally Δ-integrable function, that is, this function is Δ-integrable over each compact subinterval of $[s_0, \infty)_{\mathbb{T}}$. Then the **Laplace transform** of x is defined below [7, 10]

$$\mathcal{L}\{x\}(z) = \int_{s_0}^{\infty} \frac{x(s)}{e_z(\sigma(s), s_0)} \Delta s \quad \text{for} \ z \in \mathcal{D}\{x\}, \tag{2.1.6}$$

where $\mathcal{D}\{x\}$ consists of all those complex numbers $z \in \mathcal{R}$ for which improper integral exists.

More general form of Laplace transform is given below.

Definition 2.1.1. Assume (2.1.2) holds. If $x : \{s_r : r \in \mathbb{N}_0\} \to \mathbb{C}$ represents a function, then the **Laplace transform** of this function is defined in this way

$$\tilde{x}(z) = \mathcal{L}\{x\}(z) = \sum_{r=0}^{\infty} \frac{u_r x(s_r)}{\prod_{k=0}^{r}(1 + u_k z)}, \tag{2.1.7}$$

for those complex values of $z \in \mathbb{C}$ satisfying (2.1.4) for which this series converges.

Recall previous assumptions (2.1.3) and (2.1.4). Define

$$P_r(z) = \prod_{k=0}^{r}(1 + u_k z), \qquad r \in \mathbb{N}_0, \tag{2.1.8}$$

represent a polynomial in z whose degree is r+1. Consider

$$P_r(z) - P_{r-1}(z) = \prod_{k=0}^{r}(1+u_k z) - \prod_{k=0}^{r-1}(1+u_k z), \quad r \in \mathbb{N}_0,$$

$$P_r(z) - P_{r-1}(z) = \prod_{k=0}^{r-1}(1+u_k z)[1+u_r z - 1], \quad r \in \mathbb{N}_0,$$

$$P_r(z) - P_{r-1}(z) = z u_r P_{r-1}(z), \qquad r \in \mathbb{N}_0. \tag{2.1.9}$$

It is easily verified that

$$\frac{1}{P_{r-1}(z)} - \frac{1}{P_r(z)} = z\frac{u_r}{P_r(z)}, \qquad r \in \mathbb{N}_0, \tag{2.1.10}$$

holds, where $P_{-1}(z) = 1$.

The numbers $\alpha_r = -u_r^{-1}$, $r \in \mathbb{N}_0$, are contained in $[-u^{-1}, 0)$. For any positive number $\delta > 0$ and $r \in \mathbb{N}_0$, then we set

$$D_\delta = \mathbb{C} \backslash \bigcup_{r=0}^{\infty} D_\delta^r \quad \text{where} \quad D_\delta^r = \{z \in \mathbb{C} : |z - \alpha_r| < \delta\}, \ \ r \in \mathbb{N}_0, \tag{2.1.11}$$

so that D_δ is closed domain of $\mathbb{C}$, where $\mathbb{C}$ is the complex plane and the distance of the points of D_δ are not less than δ from this set $\{\alpha_r : r \in \mathbb{N}_0\}$.

Lemma 2.1.2. *[7] Assume* (2.1.3), (2.1.4), (2.1.8) *and* (2.1.11) *holds. For any $z \in D_\delta$, then*

$$|P_r(z)| \geq (\delta u)^{r+1} \quad \textit{and} \quad |P_r(z)| \geq \delta(\delta u)^r u_r \quad \textit{for all} \quad r \in \mathbb{N}_0. \tag{2.1.12}$$

Moreover,

$$\lim_{r\to\infty} P_r(z) = \infty \quad \textit{for all} \quad z \in D_\delta \quad \textit{provided} \quad \delta > u^{-1}. \tag{2.1.13}$$

Proof. For any $z \in D_\delta$ and $r \in \mathbb{N}_0$, we have

$$\begin{aligned} |P_r(z)| \ &= \ \left|\prod_{k=0}^{r}(1+u_k z)\right| \\ &= \left|\prod_{k=0}^{r} u_k(u_k^{-1} + z)\right| \\ &= \left|\prod_{k=0}^{r}(u_k(z-\alpha_k))\right| \\ &= u_r\left(\prod_{k=0}^{r-1} u_k\right)\left(\prod_{k=0}^{r}|z-\alpha_k|\right) \\ &\geq \ u_k u^k \delta^{k+1} \\ &= \delta(\delta u)^k u_k. \end{aligned}$$

The proof of second statement in (2.1.12) is complete. The proof of first statement in (2.1.12) is as follows

$$\begin{aligned}|P_r(z)| &= \left(\prod_{k=0}^{r} u_k\right)\left(\prod_{k=0}^{r} |z-\alpha_k|\right)\\ |P_r(z)| &\geq (u^{r+1})(\delta^{r+1})\\ |P_r(z)| &\geq (\delta u)^{r+1}.\end{aligned}$$

Thus proof of (2.1.13) follows from (2.1.12). □

Example 2.1.3. If $x(s_r) = 1$ then its Laplace transform

$$\mathcal{L}\{1\}(z) = \frac{1}{z},$$

and Laplace transform of exponential function is defined below

$$\mathcal{L}\{e_\alpha\}(z) = \frac{1}{z-\alpha}.$$

For any $z \in D_\delta$, with $\delta > u^{-1}$, by using (2.1.7), (2.1.8), (2.1.10) and (2.1.13)

$$\begin{aligned}\mathcal{L}\{1\}(z) &= \sum_{r=0}^{\infty} \frac{u_r}{P_r(z)}\\ &= \frac{1}{z}\sum_{r=0}^{\infty}\left[\frac{1}{P_{r-1}(z)} - \frac{1}{P_r(z)}\right]\\ &= \frac{1}{z}\lim_{m\to\infty}\left[1 - \frac{1}{P_m(z)}\right] = \frac{1}{z}.\end{aligned}$$

The Laplace transform of the second function, by using (2.1.5) and (2.1.8)

$$e_\alpha(s_r) = \prod_{k=0}^{r-1}(1+u_k\alpha) = P_{r-1}(\alpha) \quad \text{for} \quad r \in \mathbb{N}_0.$$

It follows that

$$\begin{aligned}\mathcal{L}\{e_\alpha\}(z) = \tilde{e}_\alpha(z) &= \sum_{r=0}^{\infty} \frac{u_r e_\alpha(s_r)}{P_r(z)}\\ &= \sum_{r=0}^{\infty} \frac{u_r P_{r-1}(\alpha)}{P_r(z)}\\ &= \sum_{r=0}^{\infty} \frac{u_r \prod_{k=0}^{r-1}(1+u_k\alpha)}{\prod_{k=0}^{r}(1+u_k z)}\\ &= \sum_{r=0}^{\infty} \frac{u_r \prod_{k=0}^{r-1}(1+u_k\alpha)}{(1+u_r z)\prod_{k=0}^{r-1}(1+u_k z)}\\ &= \sum_{r=0}^{\infty} \frac{u_r}{1+u_r z}\prod_{k=0}^{r-1}\frac{1+u_k\alpha}{1+u_k z}\end{aligned}$$

$$= \sum_{r=0}^{\infty} \frac{u_r}{1+u_r z} \prod_{k=0}^{r-1} \frac{\alpha-\alpha_k}{z-\alpha_k}. \tag{2.1.14}$$

Since the numbers α_k, where $k \in \mathbb{N}_0$, are contained in $[-u^{-1}, 0)$, thus there exists a sufficiently large number denoted by $R_0 > 0$, such that

$$\left| \frac{\alpha-\alpha_k}{z-\alpha_k} \right| \leq \frac{1}{2} \quad \text{for all} \quad |z| \geq R_0 \quad \text{and} \quad k \in \mathbb{N}_0. \tag{2.1.15}$$

Thus, the series (2.1.14) converges for the values $|z| \geq R_0$, because

$$\left| \frac{u_r}{1+u_r z} \right| = \frac{1}{|z-\alpha_r|} \leq \frac{1}{\delta}$$

is bounded. Next, the Laplace transform of this function by using (2.1.10)

$$\begin{aligned}
\tilde{e}_\alpha(z) &= \sum_{r=0}^{\infty} \frac{u_r P_{r-1}(\alpha)}{P_r(z)} \\
&= \frac{u_0}{P_0(z)} + \sum_{r=1}^{\infty} \frac{u_r P_{r-1}(\alpha)}{P_r(z)} \\
&= \frac{u_0}{P_0(z)} + \frac{1}{z} \sum_{r=1}^{\infty} \left[\frac{P_{r-1}(\alpha)}{P_{r-1}(z)} - \frac{P_{r-1}(\alpha)}{P_r(z)} \right] \\
&= \frac{u_0}{P_0(z)} + \frac{1}{z} \sum_{r=1}^{\infty} \left[\frac{(1+u_{r-1}\alpha)P_{r-2}(\alpha)}{P_{r-1}(z)} - \frac{P_{r-1}(\alpha)}{P_r(z)} \right] \\
&= \frac{u_0}{P_0(z)} + \frac{1}{z} \sum_{r=1}^{\infty} \left[\frac{P_{r-2}(\alpha)}{P_{r-1}(z)} + \frac{\alpha u_{r-1} P_{r-2}(\alpha)}{P_{r-1}(z)} - \frac{P_{r-1}(\alpha)}{P_r(z)} \right] \\
&= \frac{u_0}{P_0(z)} + \frac{1}{z} \sum_{r=1}^{\infty} \left[\frac{P_{r-2}(\alpha)}{P_{r-1}(z)} - \frac{P_{r-1}(\alpha)}{P_r(z)} \right] + \frac{\alpha}{z} \sum_{r=1}^{\infty} \frac{u_{r-1} P_{r-2}(\alpha)}{P_{r-1}(z)} \\
&= \frac{u_0}{P_0(z)} + \frac{1}{zP_0(z)} - \frac{1}{z} \lim_{m\to\infty} \frac{P_{m-1}(\alpha)}{P_m(z)} + \frac{\alpha}{z} \tilde{e}_\alpha(z) \\
&= \frac{1}{z} + \frac{\alpha}{z} \tilde{e}_\alpha(z),
\end{aligned}$$

where, the fact used that

$$\lim_{m\to\infty} \frac{P_{m-1}(\alpha)}{P_m(z)} = 0,$$

because of

$$\begin{aligned}
\frac{P_{m-1}(\alpha)}{P_m(z)} &= \frac{P_{m-1}(\alpha)}{(1+u_m z)P_{m-1}(z)} \\
&= \frac{1}{1+u_m z} \prod_{k=0}^{m-1} \frac{\alpha-\alpha_k}{z-\alpha_k}
\end{aligned}$$

and (2.1.15). Thus, the final equality is

$$\tilde{e}_\alpha(z) = \frac{1}{z} + \frac{\alpha}{z} \tilde{e}_\alpha(z).$$

Hence

$$\tilde{e}_\alpha(z) = \frac{1}{z-\alpha}.$$

Theorem 2.1.4. *[7] Assume* (2.1.3) *holds. If* $x : \{s_r : r \in \mathbb{N}_0\} \to \mathbb{C}$ *is a function and satisfies this condition*

$$|x(s_r)| \le CR^r \quad \textit{for all} \quad r \in \mathbb{N}_0, \tag{2.1.16}$$

where R *and* C *are constants which are positive, then the series converges uniformly which is in* (2.1.7)*, with respect to* z *in region* D_δ *with* $\delta > Ru^{-1}$*.*

Proof. By Lemma (2.1.2) and (2.1.16), the general term of the series in (2.1.7)

$$\left|\frac{u_r x(s_r)}{P_r(z)}\right| \le \frac{u_r C R^r}{\delta(\delta u)^r u_r} = \frac{C}{\delta}\left(\frac{R}{\delta u}\right)^r \quad \text{for } r \in \mathbb{N}_0 \text{ and } z \in D_\delta.$$

The series

$$\sum_{r=0}^{\infty}\left(\frac{R}{\delta u}\right)^r$$

converges if $\delta > Ru^{-1}$. This completes the proof. □

$\mathcal{A}_\delta$ denotes a class of function $x : \{s_r : r \in \mathbb{N}_0\} \to \mathbb{C}$ for which the Laplace transform exists and it satisfying the following condition

$$\sum_{r=0}^{\infty}(\delta u)^{-r}|x(s_r)| < \infty. \tag{2.1.17}$$

Theorem 2.1.5. *[7] Assume* (2.1.3) *holds. Let* $x : \{s_r : r \in \mathbb{N}_0\} \to \mathbb{C}$ *is a function and define* $x^\triangle : \{s_r : r \in \mathbb{N}_0\} \to \mathbb{C}$ *is another function, by*

$$x^\triangle(s_r) = \frac{x(s_r + u_r) - x(s_r)}{u_r}.$$

Suppose that $x \in \mathcal{A}_\delta$*, then* $x^\triangle \in \mathcal{A}_\delta$ *too, and*

$$\mathcal{L}\{x^\triangle\}(z) = z\tilde{x}(z) - x(s_0). \tag{2.1.18}$$

Moreover, we have that $x^{\triangle\triangle} \in \mathcal{A}_\delta$

$$\mathcal{L}\{x^{\triangle\triangle}\}(z) = z^2\tilde{x}(z) - zx(s_0) - x^\triangle(s_0). \tag{2.1.19}$$

Proof. Consider

$$\begin{aligned}\sum_{r=0}^{\infty}(\delta u)^{-r}|x^\triangle(s_r)| &= \sum_{r=0}^{\infty}(\delta u)^{-r}\frac{|x(s_{r+1}) - x(s_r)|}{u_r} \\ &\le u^{-1}\sum_{r=0}^{\infty}(\delta u)^{-r}[|x(s_{r+1})| + |x(s_r)|] \\ &= \delta\sum_{r=0}^{\infty}(\delta u)^{-r-1}|x(s_{r+1})| + u^{-1}\sum_{r=0}^{\infty}(\delta u)^{-r}|x(s_r)| < \infty.\end{aligned}$$

This shows that $x^{\triangle} \in \mathcal{A}_{\delta}$. Now by using the definition of Laplace transform (2.1.7)

$$\begin{aligned}
\mathcal{L}\{x^{\triangle}\}(z) &= \sum_{r=0}^{\infty} \frac{u_r x^{\triangle}(s_r)}{P_r(z)} \\
&= \sum_{r=0}^{\infty} \frac{x(s_{r+1}) - x(s_r)}{P_r(z)} \\
&= \sum_{r=0}^{\infty} \frac{x(s_{r+1})}{P_r(z)} - \sum_{r=0}^{\infty} \frac{x(s_r)}{P_r(z)}
\end{aligned}$$

$$\begin{aligned}
&= \sum_{r=0}^{\infty} \frac{x(s_{r+1})}{P_{r+1}(z)}(1 + u_{r+1}z) - \sum_{r=0}^{\infty} \frac{x(s_r)}{P_r(z)} \\
&= \sum_{r=0}^{\infty} \frac{x(s_{r+1})}{P_{r+1}(z)} - \sum_{r=0}^{\infty} \frac{x(s_r)}{P_r(z)} + z\sum_{r=0}^{\infty} \frac{u_{r+1}x(s_{r+1})}{P_{r+1}(z)} \\
&= -\frac{x(s_0)}{P_0(z)} + z\left[\tilde{x}(z) - \frac{u_0 x(s_0)}{P_0(z)}\right] \\
&= -\frac{(1+u_0 z)x(s_0)}{P_0(z)} + z\tilde{x}(z) \\
&= -x(s_0) + z\tilde{x}(z).
\end{aligned}$$

The proof of the second statement (2.1.19) is obtained by applying the first statement (2.1.18).

$$\begin{aligned}
\mathcal{L}\{x^{\triangle\triangle}\}(z) &= z\tilde{x}^{\triangle}(z) - x^{\triangle}(s_0) \\
&= z[z\tilde{x}(z) - x(s_0)] - x^{\triangle}(s_0) \\
&= z^2\tilde{x}(z) - zx(s_0) - x^{\triangle}(s_0).
\end{aligned}$$

□

Theorem 2.1.6. *[7] (**Initial and Final Value Theorem**). Assume* (2.1.3) *holds. Then:*

(a) Assume $x \in \mathcal{A}_{\delta}$ and for some $\delta > 0$, then

$$x(s_0) = \lim_{z\to\infty}\{z\tilde{x}(z)\}. \tag{2.1.20}$$

(b) Assume $x \in \mathcal{A}_{\delta}$ and for all $\delta > 0$, then

$$\lim_{r\to\infty} x(s_r) = \lim_{z\to 0}\{z\tilde{x}(z)\}. \tag{2.1.21}$$

Proof. Let $x \in \mathcal{A}_{\delta}$ for some $\delta > 0$. It follows by definition of Laplace transform (2.1.7)

$$\tilde{x}(z) = \frac{u_0 x(s_0)}{1+u_0 z} + \frac{u_1 x(s_1)}{(1+u_0 z)(1+u_1 z)} + \frac{u_2 x(s_2)}{(1+u_0 z)(1+u_1 z)(1+u_2 z)} + ...$$

and

$$(1+u_0 z)\tilde{x}(z) = u_0 x(s_0) + \frac{u_1 x(s_1)}{(1+u_1 z)} + \frac{u_2 x(s_2)}{(1+u_1 z)(1+u_2 z)} + ...$$

Hence

$$\lim_{z\to\infty} \tilde{x}(z) = 0 \ \text{ and } \ \lim_{z\to\infty} \{(1+u_0 z)\tilde{x}(z)\} = u_0 x(s_0)$$

$$\lim_{z\to\infty} [\tilde{x}(z) + u_0 z\tilde{x}(z)] = u_0 x(s_0)$$

$$u_0 \lim_{z\to\infty} z\tilde{x}(z) = u_0 x(s_0)$$

$$\lim_{z\to\infty} z\tilde{x}(z) = x(s_0).$$

This completes the proof of (2.1.20). To show (2.1.21), let $x \in \mathcal{A}_\delta$ for all $\delta > 0$. By using the equality which is obtained in the proof of Theorem (2.1.5)

$$\sum_{r=0}^{\infty} \frac{x(s_{r+1}) - x(s_r)}{P_r(z)} = z\tilde{x}(z) - x(s_0). \tag{2.1.22}$$

By using Lemma (2.1.2)

$$\lim_{z\to 0} P_r(z) = 1 \ \text{ for any } \ r \in \mathbb{N}_0.$$

To arrive at (2.1.21), applying limit at (2.1.22), then

$$-x(s_0) + \lim_{r\to\infty} x(s_r) = \lim_{z\to 0}(z\tilde{x}(z)) - \lim_{z\to 0} x(s_0).$$

Hence

$$\lim_{r\to\infty} x(s_r) = \lim_{z\to 0}(z\tilde{x}(z)).$$

The proof of second statement is complete. □

2.2 The Convolution by using Delta Operator

We are using two basic concepts of shift. These concepts are introduced in [6].

Assume a function $f : [s_0, \infty)_{\mathbb{T}} \to \mathbb{C}$, **shift** (or **delay**) of this function is denoted by $\hat{f}(s,t)$ and is defined as solution of the problem

$$\begin{aligned} \hat{f}^{\Delta_s}(s, \sigma(t)) = -\hat{f}^{\Delta_t}(s,t), \quad t, s \in \mathbb{T}, \quad s_0 \le t \le s, \\ \hat{f}(s, s_0) = f(s), \quad s \in \mathbb{T}, \quad s_0 \le s. \end{aligned} \tag{2.2.1}$$

For given function $f, g : [s_0, \infty)_{\mathbb{T}} \to \mathbb{C}$, **convolution** of $f * g$ is defined as

$$(f * g)(s) = \int_{s_0}^{s} \hat{f}(s, \sigma(t)) g(t) \Delta(t), \quad s \in \mathbb{T}, \quad s_0 \le s. \tag{2.2.2}$$

In this section assume only (2.1.2). For a given $f : \{s_r : r \in \mathbb{N}_0\} \to \mathbb{C}$, consider the **shifting problem** (2.2.1)

$$\begin{aligned} \hat{f}^{\Delta s}(s_r, s_{m+1}) = -\hat{f}^{\Delta t}(t_r, t_m) \qquad m, r \in \mathbb{N}_0, \qquad r \ge m, \\ \hat{f}(s_r, s_0) = f(s_r), \quad r \in \mathbb{N}_0. \end{aligned}$$

$$u_m\left[\hat{f}(s_{r+1},s_{m+1})-\hat{f}(s_r,s_{m+1})\right]+u_r\left[\hat{f}(s_r,s_{m+1})-\hat{f}(s_r,s_m)\right]=0,$$
$$m,r\in\mathbb{N}_0,\qquad r\geq m, \tag{2.2.3}$$
$$\hat{f}(s_r,s_0)=f(s_r),\quad r\in\mathbb{N}_0.$$

Theorem 2.2.1. *[7] Assume* (2.1.2) *holds. For an arbitrary function* $f:\{s_r:r\in\mathbb{N}_0\}\to\mathbb{C}$*, shifting problem* (2.2.3) *has unique solution.*

Proof. Setting $\hat{f}(s_r,s_m)=\hat{f}_{r,m}$ for convenience, let us rewrite (2.2.3)

$$u_m(\hat{f}_{r+1,m+1}-\hat{f}_{r,m+1})+u_r(\hat{f}_{r,m+1}-\hat{f}_{r,m})=0,\ \ m,r\in\mathbb{N}_0,\ \ r\geq m, \tag{2.2.4}$$

$$\hat{f}_{r,0}=f(s_r),\quad r\in\mathbb{N}_0, \tag{2.2.5}$$

where $\hat{f}_{r,m}$ is defined for $m,r\in\mathbb{N}_0$ with $m\leq r$ is a desired solution. Suppose $\hat{f}_{r,m}$ is a solution of (2.2.4), (2.2.5). Then there are two cases, $m=r$ and $m<r$. For $m=r$, putting $m=r$ in (2.2.4)

$$u_r(\hat{f}_{r+1,r+1}-\hat{f}_{r,r+1})+u_r(\hat{f}_{r,r+1}-\hat{f}_{r,r})=0,\ \text{ for all }\ r\in\mathbb{N}_0,$$
$$\hat{f}_{r+1,r+1}-\hat{f}_{r,r}=0\ \text{ for all }\ r\in\mathbb{N}_0,$$
$$\hat{f}_{r+1,r+1}=\hat{f}_{r,r}\ \text{ for all }\ r\in\mathbb{N}_0.$$

Note that $\hat{f}_{r+1,r+1}$ is constant for all values of $r\in\mathbb{N}_0$, and since by (2.2.5) $\hat{f}_{0,0}=f(s_0)$, then

$$\hat{f}_{r,r}=f(s_0)\ \text{ for all }\ r\in\mathbb{N}_0. \tag{2.2.6}$$

Furthermore, it is enough to show that (2.2.4) has a unique solution satisfying (2.2.5) and (2.2.6). For this, now discuss the case when $m<r$. For any $i\in\mathbb{N}_0$, let us set $\mathbb{N}_i=[i,\infty)\cap\mathbb{N}_0$. Put $m=r-1$ with $r\in\mathbb{N}_1$ in (2.2.4), then

$$u_{r-1}(\hat{f}_{r+1,r}-\hat{f}_{r,r})+u_r(\hat{f}_{r,r}-\hat{f}_{r,r-1})=0,\ \text{ for all }\ r\in\mathbb{N}_1.$$

Putting the value of $\hat{f}_{r,r}$ from (2.2.6), then

$$u_{r-1}(\hat{f}_{r+1,r}-f(s_0))+u_r(f_(s_0)-\hat{f}_{r,r-1})=0,\quad \text{for all }\ r\in\mathbb{N}_1,$$
$$u_{r-1}\hat{f}_{r+1,r}-f(s_0)(u_{r-1}-u_r)-u_r\hat{f}_{r,r-1}=0,\quad \text{for all }\ r\in\mathbb{N}_1,$$
$$u_{r-1}\hat{f}_{r+1,r}=(u_{r-1}-u_r)f(s_0)+u_r\hat{f}_{r,r-1},\qquad \text{for all }\ r\in\mathbb{N}_1.$$

Hence

$$\hat{f}_{r+1,r}=\left(1-\frac{u_r}{u_{r-1}}\right)f(s_0)+\frac{u_r}{u_{r-1}}\hat{f}_{r,r-1},\qquad \text{for all }\ r\in\mathbb{N}_1, \tag{2.2.7}$$

by (2.2.5)

$$\hat{f}_{1,0}=f(s_1). \tag{2.2.8}$$

Now by using (2.2.8), we are able to find the value of $\hat{f}_{r+1,r}$ from (2.2.7) recursively in unique way for all $r \in \mathbb{N}_0$. Next, put $m = r - 2$ with $r \in \mathbb{N}_2$ in (2.2.4), then

$$u_{r-2}(\hat{f}_{r+1,r-1} - \hat{f}_{r,r-1}) + u_r(\hat{f}_{r,r-1} - \hat{f}_{r,r-2}) = 0, \quad \text{for all } r \in \mathbb{N}_2,$$
$$u_{r-2}\hat{f}_{r+1,r-1} + (u_r - u_{r-2})\hat{f}_{r,r-1} - u_r\hat{f}_{r,r-2} = 0, \quad \text{for all } r \in \mathbb{N}_2,$$
$$u_{r-2}\hat{f}_{r+1,r-1} = (u_{r-2} - u_r)\hat{f}_{r,r-1} + u_r\hat{f}_{r,r-2} = 0, \quad \text{for all } r \in \mathbb{N}_2.$$

Hence

$$\hat{f}_{r+1,r-1} = \left(1 - \frac{u_r}{u_{r-2}}\right)\hat{f}_{r,r-1} + \frac{u_r}{u_{r-2}}\hat{f}_{r,r-2}, \quad \text{for all } r \in \mathbb{N}_2, \tag{2.2.9}$$

by (2.2.5)

$$\hat{f}_{2,0} = f(s_2). \tag{2.2.10}$$

In (2.2.9), the term $\hat{f}_{r,r-1}$ is known for all $r \in \mathbb{N}_1$ from the first step. Then, by using (2.2.10), we are able to find the value of $\hat{f}_{r+1,r-1}$ from (2.2.9) recursively in unique way for all $r \in \mathbb{N}_1$. Repeating this technique, put $m = r - i$ for $r \in \mathbb{N}_i$ in (2.2.4), then

$$u_{r-i}(\hat{f}_{r+1,r-i+1} - \hat{f}_{r,r-i+1}) + u_r(\hat{f}_{r,r-i+1} - \hat{f}_{r,r-i}) = 0, \quad \text{for all } r \in \mathbb{N}_i,$$
$$u_{r-i}\hat{f}_{r+1,r-i+1} + (u_r - u_{r-i})\hat{f}_{r,r-i+1} - u_r\hat{f}_{r,r-i} = 0, \quad \text{for all } r \in \mathbb{N}_i,$$
$$u_{r-i}\hat{f}_{r+1,r-i+1} = (u_{r-i} - u_r)\hat{f}_{r,r-i+1} + u_r\hat{f}_{r,r-i} = 0, \quad \text{for all } r \in \mathbb{N}_i.$$

Hence

$$\hat{f}_{r+1,r-i+1} = \left(1 - \frac{u_r}{u_{r-i}}\right)\hat{f}_{r,r-i+1} + \frac{u_r}{u_{r-i}}\hat{f}_{r,r-i}, \quad \text{for all } r \in \mathbb{N}_i, \tag{2.2.11}$$

by (2.2.5)

$$\hat{f}_{i,0} = f(s_i). \tag{2.2.12}$$

In (2.2.9), the term $\hat{f}_{r,r-i+1}$ is known for all $r \in \mathbb{N}_{i-1}$ from the previous step. Then, by using (2.2.12), we are able to find the value of $\hat{f}_{r+1,r-i+1}$ from (2.2.11) recursively in unique way for all $r \in \mathbb{N}_{i-1}$. Since we take $i \in \mathbb{N}$ arbitrarily, so $\hat{f}_{r,m}$ is constructed uniquely in this way for all $m, r \in \mathbb{N}_0$ with $m \leq r$. □

Definition 2.2.2. Assume (2.1.2) holds, suppose two function $f, g : \{s_r : r \in \mathbb{N}_0\} \to \mathbb{C}$ and let $\hat{f}$ is the solution of the previous problem (2.2.3). Then **convolution** of f and g is denoted by $f * g$ and is defined by $(f * g)(s_0) = 0$ and

$$(f * g)(s_r) = \sum_{k=0}^{r-1} u_k \hat{f}(s_r, s_{k+1}) g(s_k), \quad r \in \mathbb{N}_0.$$

The following theorem is constructed with the help of the classical theorem of Titchmarsh [18, 19] for usual continuous convolution. This theorem is discrete analogue for usual continuous convolution.

Theorem 2.2.3. *[7] Assume* (2.1.2) *holds and* $f, g : \{s_r : r \in \mathbb{N}_0\} \to \mathbb{C}$ *are two functions. If convolution of* f *and* g *on* $\{s_r : r \in \mathbb{N}_0\}$ *is identically zero, then at least one of the functions* f *and* g *is identically zero on* $\{s_r : r \in \mathbb{N}_0\}$*.*

Proof. Suppose that $f * g$ is identically zero on $\{s_r : r \in \mathbb{N}_0\}$. Then by definition (2.2.2) of convolution and by using the notation $\hat{f}(s_r, s_m) = \hat{f}_{r,m}$, we have

$$\begin{aligned}
&u_0\hat{f}_{1,1}g(s_0) = 0,\\
&u_0\hat{f}_{2,1}g(s_0) + u_1\hat{f}_{2,2}g(s_1) = 0,\\
&u_0\hat{f}_{3,1}g(s_0) + u_1\hat{f}_{3,2}g(s_1) + u_2\hat{f}_{3,3}g(s_2) = 0,\\
&\quad .\\
&\quad .\\
&\quad .\\
&u_0\hat{f}_{r,1}g(s_0) + u_1\hat{f}_{r,2}g(s_1) + ... + u_{r-1}\hat{f}_{r,r}g(s_{r-1}) = 0,
\end{aligned} \tag{2.2.13}$$

where $r \in \mathbb{N}$. It is enough to show that if one of the function is not identically zero, then other function must be identically zero. Assume that $f : \{s_r : r \in \mathbb{N}_0\}$ is not identically zero, then we have to show that $g : \{s_r : r \in \mathbb{N}_0\}$ is identically zero. Suppose $f(s_m)$ with $m \in \mathbb{N}_0$ be the first of the values of $f(s_0)$, $f(s_1)$,... that is different from zero. Thus

$$f(s_0) = f(s_1) = ... = f(s_{m-1}) = 0 \;\text{ and }\; f(s_m) \neq 0. \tag{2.2.14}$$

For showing $g(s_r) = 0$ for all $r \in \mathbb{N}_0$, consider all the values of $m \in \mathbb{N}_0$ in (2.2.14) separately. Assume $m = 0$ in (2.2.14), then by (2.2.14) $f(s_0) \neq 0$. Assume (2.2.13) as a homogeneous system of linear equations $A_0y_0 = 0$ with $y_0 = (g(s_0), g(s_1), ..., g(s_{r-1}))^T$. The determinant of A_0 (triangular matrix) is

$$\prod_{k=0}^{r-1} u_k\hat{f}_{k+1,k+1} = [f(s_0)]^r \prod_{k=0}^{r-1} u_k,$$

by using (2.2.6). Hence the determinant of A_0 is different from zero by the assumption $f(s_0) \neq 0$. Thus A_0 is invertible matrix and $A_0y_0 = 0$ implies $y_0 = 0$, that is $g(s_0) = g(s_1) = ... = g(s_{r-1}) = 0$. Since $r \in \mathbb{N}$ is arbitrary, so we get g is identically zero on $\{s_r : r \in \mathbb{N}_0\}$.

Assume $m = 1$ in (2.2.14), then

$$f(s_0) = 0 \;\text{ and }\; f(s_1) \neq 0. \tag{2.2.15}$$

In this case, (2.2.6) implies that

$$\hat{f}_{r,r} = f(s_0) = 0 \quad \text{for all} \quad r \in \mathbb{N}_0, \tag{2.2.16}$$

and (2.2.13) becomes

$$\begin{aligned}
&u_0\hat{f}_{2,1}g(s_0)=0,\\
&u_0\hat{f}_{3,1}g(s_0)+u_1\hat{f}_{3,2}g(s_1)=0,\\
&u_0\hat{f}_{4,1}g(s_0)+u_1\hat{f}_{4,2}g(s_1)+u_2\hat{f}_{4,3}g(s_2)=0,\\
&\quad .\\
&\quad .\\
&\quad .\\
&u_0\hat{f}_{r,1}g(s_0)+u_1\hat{f}_{r,2}g(s_1)+...+u_{r-2}\hat{f}_{r,r-1}g(s_{r-2})=0.
\end{aligned} \tag{2.2.17}$$

Next, since $f(s_0)=0$, then from (2.2.7)

$$\hat{f}_{r+1,r}=\frac{u_r}{u_{r-1}}\hat{f}_{r,r-1},\ \ r\in\mathbb{N}_1.$$

Iterating previous equation and taking into account (2.2.5), then

$$\hat{f}_{r+1,r}=\frac{u_r}{u_0}\hat{f}_{1,0}=\frac{u_r}{u_0}f(s_1),\ \ r\in\mathbb{N}_0. \tag{2.2.18}$$

Let us consider the system (2.2.17) as before as a system $A_1y_1=0$, the determinant of A_1 is equal to

$$\prod_{k=0}^{r-2}u_k\hat{f}_{k+2,k+1}=[f(s_1)]^{r-1}\prod_{k=0}^{r-2}\frac{u_{k+1}}{u_0},$$

and this is different from zero by (2.2.15). Therefore $y_1=0$, that is, $g(s_0)=g(s_1)=...=g(s_{r-2})=0$ and since $r\in\mathbb{N}$ is arbitrary, so we get g is identically zero on $\{s_r:r\in\mathbb{N}_0\}$.

Assume $m=2$ in (2.2.14), then

$$f(s_0)=f(s_1)=0\ \text{ and }\ f(s_2)\neq 0. \tag{2.2.19}$$

In this case, (2.2.16) and (2.2.18) still hold. Besides, from $f(s_1)=0$, (2.2.18) yields

$$\hat{f}_{r+1,r}=0\ \text{ for all }\ r\in\mathbb{N}_0. \tag{2.2.20}$$

Therefore (2.2.13) becomes

$$\begin{aligned}
&u_0\hat{f}_{3,1}g(s_0)=0,\\
&u_0\hat{f}_{4,1}g(s_0)+u_1\hat{f}_{4,2}g(s_1)=0,\\
&u_0\hat{f}_{5,1}g(s_0)+u_1\hat{f}_{5,2}g(s_1)+u_2\hat{f}_{5,3}g(s_2)=0,\\
&\quad .\\
&\quad .\\
&\quad .\\
&u_0\hat{f}_{r,1}g(s_0)+u_1\hat{f}_{r,2}g(s_1)+...+u_{r-3}\hat{f}_{r,r-2}g(s_{r-3})=0.
\end{aligned} \tag{2.2.21}$$

Next, since $\hat{f}_{r,r-1} = 0$ for $r \in \mathbb{N}_1$ by (2.2.20), then by (2.2.9)

$$\hat{f}_{r+1,r-1} = \frac{u_r}{u_{r-2}} \hat{f}_{r,r-2}, \quad r \in \mathbb{N}_2.$$

Iterating the last equation, then

$$\hat{f}_{r+1,r-1} = \frac{u_r u_{r-1}}{u_1 u_0} \hat{f}_{2,0} = \frac{u_r u_{r-1}}{u_1 u_0} f(s_2), \quad r \in \mathbb{N}_1.$$

Writing the system (2.2.21) again as $A_2 y_2 = 0$, the determinant of A_2 is equal to

$$\prod_{k=0}^{r-3} u_k \hat{f}_{k+3,k+1} = [f(s_2)]^{r-2} \prod_{k=0}^{r-3} \frac{u_{k+2} u_{k+1}}{u_1 u_0},$$

and this is different from zero by (2.2.19). Then $y_2 = 0$, that is $g(s_0) = g(s_1) = ... = g(s_{r-3}) = 0$ and since $r \in \mathbb{N}$ is arbitrary, so we get that g is identically zero on $\{s_r : r \in \mathbb{N}_0\}$. We observe that one can discuss the system $A_m y_m = 0$ and argue in this way for any value of $m \in \mathbb{N}_0$ in (2.2.14) in order to obtain that $g(s_r) = 0$ for all $r \in \mathbb{N}_0$. □

Theorem 2.2.4. *[7] (**Convolution Theorem**). Assume* (2.1.2) *holds and* $f, g : \{s_r : r \in \mathbb{N}_0\} \to \mathbb{C}$ *are two functions such that* $\mathcal{L}\{f\}(z)$, $\mathcal{L}\{g\}(z)$, $\mathcal{L}\{f * g\}(z)$ *exist for a given* $z \in \mathbb{C}$ *satisfying* (2.1.4)*. Then, at the point* z,

$$\mathcal{L}\{f * g\}(z) = \mathcal{L}\{f\}(z). \mathcal{L}\{g\}(z) \tag{2.2.22}$$

Proof. For our convenience, we set

$$e_{r,m}(z) = e_z(s_r, s_m) \text{ and } \hat{f}_{r,m} = \hat{f}(s_r, s_m).$$

Then by (2.1.4)

$$e_{r,r}(z) = 1 \text{ for all } r \in \mathbb{N}_0 \tag{2.2.23}$$

$$e_{r+1,m}(z) = (1 + u_r z) e_{r,m}(z) \text{ for all } r, m \in \mathbb{N}_0 \text{ with } m \leq r, \tag{2.2.24}$$

$$e_{r,m+1}(z) = \frac{e_{r,m}(z)}{(1 + u_m z)} \text{ for all } r, m \in \mathbb{N}_0 \text{ with } m + 1 \leq r, \tag{2.2.25}$$

and shifting problem (2.2.3) can be rewritten as (2.2.4), (2.2.5). By using definition of Laplace transform (2.1.7) and the definition of convolution (2.2.2)

$$\begin{aligned}\mathcal{L}\{f * g\}(z) &= \sum_{r=1}^{\infty} \frac{u_r (f * g)(s_r)}{e_{r+1,0}(z)} = \sum_{r=1}^{\infty} \frac{u_r}{e_{r+1,0}(z)} \sum_{k=0}^{r-1} u_k \hat{f}_{r,k+1} g(s_k) \\ &= \sum_{k=0}^{\infty} u_k g(s_k) \sum_{r=k+1}^{\infty} \frac{u_r \hat{f}_{r,k+1}}{e_{r+1,0}(z)}.\end{aligned}$$

Substituting here

$$e_{r+1,0}(z) = e_{r+1,k+1}(z) e_{k+1,0}(z),$$

we get that

$$\mathcal{L}\{f * g\}(z) = \sum_{k=0}^{\infty} \frac{u_k g(s_k)}{e_{k+1,0}(z)} \sum_{r=k+1}^{\infty} \frac{u_r \hat{f}_{r,k+1}}{e_{r+1,k+1}(z)}.$$

Hence

$$\mathcal{L}\{f * g\}(z) = \mathcal{L}\{g\}(z) \sum_{r=k+1}^{\infty} \frac{u_r \hat{f}_{r,k+1}}{e_{r+1,k+1}(z)}. \tag{2.2.26}$$

For our convenience, we set

$$\Phi_m = \sum_{r=m}^{\infty} \frac{u_r \hat{f}_{r,m}}{e_{r+1,m}(z)}, \quad m \in \mathbb{N}_0. \tag{2.2.27}$$

The target is to show that Φ_m is independent of the value $m \in \mathbb{N}_0$, then

$$\sum_{r=k+1}^{\infty} \frac{u_r \hat{f}_{r,k+1}}{e_{r+1,k+1}(z)} = \sum_{r=0}^{\infty} \frac{u_r \hat{f}_{r,0}}{e_{r+1,0}(z)} = \sum_{r=0}^{\infty} \frac{u_r f(s_r)}{e_{r+1,0}(z)} = \mathcal{L}\{f\}(z),$$

thus yields (2.2.22) by using (2.2.26). Now, the remaining part is to show that Φ_m does not depend on the value of $m \in \mathbb{N}_0$. Now putting $e_{r,m} = e_{r,m}(z)$ and by using (2.2.4) and (2.2.23), (2.2.24) and (2.2.25)

$$\begin{aligned}
\Phi_{m+1} &= \sum_{r=m+1}^{\infty} \frac{u_r \hat{f}_{r,m+1}}{e_{r+1,m+1}} = \sum_{r=m+1}^{\infty} \frac{u_r \hat{f}_{r,m} + u_m \hat{f}_{r,m+1} - u_m \hat{f}_{r+1,m+1}}{e_{r+1,m+1}} \\
&= \sum_{r=m+1}^{\infty} \frac{u_r \hat{f}_{r,m}}{e_{r+1,m+1}} - u_m \sum_{r=m+1}^{\infty} \left[\frac{\hat{f}_{r+1,m+1}}{e_{r+1,m+1}} - \frac{\hat{f}_{r,m+1}}{e_{r,m+1}} + \frac{\hat{f}_{r,m+1}}{e_{r,m+1}} - \frac{\hat{f}_{r,m+1}}{e_{r+1,m+1}} \right] \\
&= \sum_{r=m+1}^{\infty} \frac{u_r \hat{f}_{r,m}}{e_{r+1,m+1}} - u_m \sum_{r=m+1}^{\infty} \left[\frac{\hat{f}_{r+1,m+1}}{e_{r+1,m+1}} - \frac{\hat{f}_{r,m+1}}{e_{r,m+1}} + \frac{\hat{f}_{r,m+1}(1+u_r z)}{e_{r+1,m+1}} - \frac{\hat{f}_{r,m+1}}{e_{r+1,m+1}} \right] \\
&= \sum_{r=m+1}^{\infty} \frac{u_r \hat{f}_{r,m}}{e_{r+1,m+1}} - u_m \sum_{r=m+1}^{\infty} \left[\frac{\hat{f}_{r+1,m+1}}{e_{r+1,m+1}} - \frac{\hat{f}_{r,m+1}}{e_{r,m+1}} + \frac{\hat{f}_{r,m+1}}{e_{r+1,m+1}} - \frac{\hat{f}_{r,m+1}}{e_{r+1,m+1}} + \frac{u_r z \hat{f}_{r,m+1}}{e_{r+1,m+1}} \right] \\
&= \sum_{r=m+1}^{\infty} \frac{u_r \hat{f}_{r,m}}{e_{r+1,m+1}} - u_m \sum_{r=m+1}^{\infty} \left[\frac{\hat{f}_{r+1,m+1}}{e_{r+1,m+1}} - \frac{\hat{f}_{r,m+1}}{e_{r,m+1}} \right] - u_m \sum_{r=m+1}^{\infty} \frac{\hat{f}_{r,m+1}}{e_{r+1,m+1}} u_r z \\
&= \sum_{r=m+1}^{\infty} \frac{u_r \hat{f}_{r,m}}{e_{r+1,m}} (1+u_m z) + u_m \frac{\hat{f}_{m+1,m+1}}{e_{m+1,m+1}} - u_m \sum_{r=m+1}^{\infty} \frac{\hat{f}_{r,m+1}}{e_{r+1,m+1}} u_r z \\
&= \sum_{r=m+1}^{\infty} \frac{u_r \hat{f}_{r,m}}{e_{r+1,m}} (1+u_m z) + u_m \frac{\hat{f}_{m,m}}{e_{m+1,m}} (1+u_m z) - u_m \frac{\hat{f}_{m,m}}{e_{m+1,m}} (1+u_m z) \\
&\quad + u_m \hat{f}_{m+1,m+1} - u_m z \sum_{r=m+1}^{\infty} \frac{\hat{f}_{m,m+1}}{e_{r+1,m+1}} u_r \\
&= (1+u_m z)\Phi_m - u_m \frac{\hat{f}_{m,m}}{e_{m+1,m}} (1+u_m z) + u_m \hat{f}_{m+1,m+1} - u_m z \Phi_{m+1} \\
&= (1+u_m z)\Phi_m - u_m \frac{\hat{f}_{m,m}}{1+u_m z} (1+u_m z) + u_m \hat{f}_{m+1,m+1} - u_m z \Phi_{m+1}
\end{aligned}$$

$$
\begin{aligned}
&= (1+u_m z)\Phi_m - u_m \hat{f}_{m,m} + u_m \hat{f}_{m+1,m+1} - u_m z\Phi_{m+1} \\
&= (1+u_m z)\Phi_m - u_m \hat{f}_{m,m} + u_m \hat{f}_{m,m} - u_m z\Phi_{m+1} \\
&= (1+u_m z)\Phi_m - u_m z\Phi_{m+1},
\end{aligned}
$$

by using the fact that $\hat{f}_{r,r} = f(s_0)$ for all $r \in \mathbb{N}_0$. Consequently

$$(1+u_m z)\Phi_{m+1} = (1+u_m z)\Phi_m.$$

Hence $\Phi_{m+1} = \Phi_m$ as $1 + u_m z \neq 0$ under condition (2.1.4). □

2.3 The Inverse Laplace Transform by using Delta Operator

Theorem 2.3.1. *[7] (**Uniqueness Theorem**). Assume* (2.1.3) *holds and suppose $x : \{s_r : r \in \mathbb{N}_0\} \to \mathbb{C}$ is a function in the space $\mathcal{A}_\delta$, that is x satisfies* (2.1.17)*. Further, let $\tilde{x}(z)$ denotes the Laplace transform of x which is defined by* (2.1.7) *for $z \in D_\delta$. If $\tilde{x}(z) \equiv 0$ for $z \in D_\delta$, then $x(s_r) = 0$ for all $r \in \mathbb{N}_0$.*

Proof. By using the definition of Laplace transform (2.1.7), we have

$$\frac{u_0 x(s_0)}{1+u_0 z} + \frac{u_1 x(s_1)}{(1+u_0 z)(1+u_1 z)} + \frac{u_2 x(s_2)}{(1+u_0 z)(1+u_1 z)(1+u_2 z)} + ... \equiv 0, \tag{2.3.1}$$

for $z \in D_\delta$. Multiplying (2.3.1) by $1 + u_0 z$ and then passing the limit as $|z| \to \infty$, then we get $x(s_0) = 0$. Use $x(s_0) = 0$ in (2.3.1) and get

$$\frac{u_1 x(s_1)}{(1+u_0 z)(1+u_1 z)} + \frac{u_2 x(s_2)}{(1+u_0 z)(1+u_1 z)(1+u_2 z)} + ... \equiv 0.$$

Now multiplying this equation by $(1+u_0 z)(1+u_1 z)$ and then passing the limit as $|z| \to \infty$ to obtain $x(s_1) = 0$. By repeating this technique, we find $x(s_0) = x(s_1) = x(s_2) = ... = 0$. □

Theorem (2.4.1) gives that inverse of Laplace transform exists. The inverse Laplace transform can be found out by using the formula stated in the following theorem.

Theorem 2.3.2. *[7] (**Inverse Laplace Transform**). Assume* (2.1.3) *holds, suppose $x \in \mathcal{A}_\delta$ and let $\tilde{x}(z)$ denotes its Laplace transform which is defined by* (2.1.7)*. Then*

$$x(s_r) = \frac{1}{2\Pi i} \int_\Gamma \tilde{x}(z) \prod_{k=0}^{r-1} (1+u_k z) dz \quad \text{for} \quad r \in \mathbb{N}_0, \tag{2.3.2}$$

where Γ denotes any positive oriented closed curve in D_δ that contains all points $\alpha_k = -u_k^{-1}$ for $k \in \mathbb{N}_0$.

Proof. Integrating the equality

$$\tilde{x}(z) = \frac{u_0 x(s_0)}{1+u_0 z} + \frac{u_1 x(s_1)}{(1+u_0 z)(1+u_1 z)} + \frac{u_2 x(s_2)}{(1+u_0 z)(1+u_1 z)(1+u_2 z)} + ..., \tag{2.3.3}$$

over the curve Γ with respect to z and then we can take term-by-term integration, and get that

$$\int_\Gamma \tilde{x}(z)dz = u_0x(s_0)\int_\Gamma \frac{dz}{1+u_0z} + u_1x(s_1)\int_\Gamma \frac{dz}{(1+u_0z)(1+u_1z)} + u_2x(s_2)\int_\Gamma \frac{dz}{(1+u_0z)(1+u_1z)(1+u_2z)} + ...$$

Next

$$\int_\Gamma \frac{dz}{1+u_0z} = \frac{1}{u_0}\int_\Gamma \frac{dz}{z-\alpha_0} = \frac{2\pi i}{u_0},$$

and

$$\int_\Gamma \frac{dz}{\prod_{k=0}^{r-1}(1+u_kz)} = 0 \quad \text{for all} \quad r \in \mathbb{N}-\{1\},$$

because $P(z)$ denotes the polynomial whose degree greater than or equal to two and if Γ be any closed contour containing all the roots of the polynomial $P(z)$, then

$$\int_\Gamma \frac{dz}{P(z)} = 0.$$

Therefore

$$x(s_0) = \frac{1}{2\pi i}\int_\Gamma \tilde{x}(z)dz.$$

Now multiplying (2.3.3) by $1+u_0z$ and then integrating over Γ with respect to z, then

$$\int_\Gamma (1+u_0z)\tilde{x}(z)dz = u_0x(s_0)\int_\Gamma dz + u_1x(s_1)\int_\Gamma \frac{dz}{1+u_1z} + u_2x(s_2)\int_\Gamma \frac{dz}{(1+u_1z)(1+u_2z)} + ...$$

Next,

$$\int_\Gamma dz = 0, \quad \int_\Gamma \frac{dz}{1+u_1z} = \frac{1}{u_1}\int_\Gamma \frac{dz}{z-\alpha_1} = \frac{2\pi i}{u_1},$$

and

$$\int_\Gamma \frac{dz}{\prod_{k=1}^{r}(1+u_kz)} = 0 \quad \text{for all} \; r \in \mathbb{N}-\{1\}.$$

Therefore

$$x(s_1) = \frac{1}{2\pi i}\int_\Gamma \tilde{x}(z)(1+u_0z)dz.$$

Repeating this technique, we are able to obtain this formula (2.3.2) for an arbitrary $r \in \mathbb{N}_0$. □

2.4 Examples

The time scale which used in these examples are defined in [16, 17].

Example 2.4.1. Let $s_r = hr$, $r \in \mathbb{N}_0$, where $h > 0$ is any fixed real number. In this case

$$u_r = s_{r+1} - s_r = (r+1)h - rh = h \quad \text{for all} \quad r \in \mathbb{N}_0.$$

Note that (2.1.3) holds with $u = h$. For a function $x : \{hr : r \in \mathbb{N}_0\} \to \mathbb{R}$, its Laplace transform (2.1.7) becomes

$$\mathcal{L}\{x\}(z) = \tilde{x}(z) = h \sum_{r=0}^{\infty} \frac{x(rh)}{(1+hz)^{r+1}}.$$

The inversion formula (2.3.2) takes the form

$$x(rh) = \frac{1}{2\Pi i} \int_{\Gamma} \tilde{x}(z)(1+hz)^r dz, \quad r \in \mathbb{N}_0,$$

where Γ denotes positive oriented curve that encloses the point $-\frac{1}{h}$.

Example 2.4.2. Let $s_r = q^r$, $r \in \mathbb{N}_0$, where $q > 1$ is any fixed real number. Then

$$u_r = s_{r+1} - s_r = q^{r+1} - q^r = (q-1)q^r = q'q^r \quad \text{for all} \quad r \in \mathbb{N}_0, \quad \text{where} \quad q' = q - 1.$$

Notice that (2.1.3) holds with $u = q'$. For a function $x : \{q^r : r \in \mathbb{N}_0\} \to \mathbb{R}$, its Laplace transform (2.1.7) becomes

$$\mathcal{L}\{x\}(z) = \tilde{x}(z) = q' \sum_{r=0}^{\infty} \frac{q^r x(q^r)}{\prod_{k=0}^{r}(1+q'q^k z)}.$$

The inversion formula (2.3.2) takes the form

$$x(q^r) = \frac{1}{2\Pi i} \int_{\Gamma} \tilde{x}(z) \prod_{k=0}^{r-1} (1+q'q^k z) dz, \quad r \in \mathbb{N}_0,$$

where Γ denotes positive oriented curve that contains all points $-(q'q^k)^{-1}$ with $k \in \mathbb{N}_0$.

Chapter 3

The h-Laplace and q-Laplace Transform in Time Scale

3.1 The Exponential Function and The Laplace Transform

This section consist definition of exponential function for an arbitrary time scale $\mathbb{T}$ and some properties regarding exponential functions are also discussed. All these properties are given in [6, 10].

The set $\mathcal{R}$ under the operation of addition $\oplus$ is an abelian group where $\oplus$ is defined by

$$(p \oplus g)(s) = p(s) + g(s) + \mu(s)p(s)g(s) \quad \text{for all} \;\; s \in \mathbb{T}.$$

For any $g \in \mathcal{R}$, $\ominus g$ denotes the additive inverse of g, defined by

$$(\ominus g)(s) = -\frac{g(s)}{1+\mu(s)g(s)} \quad \text{for all} \;\; s \in \mathbb{T}.$$

The subtraction $\ominus$ on $\mathcal{R}$ is defined by

$$\begin{aligned}
(p \ominus g)(s) &= (p + (\ominus g))(s) = p(s) + (\ominus(g))(s) + \mu(s)p(s)(\ominus g)(s) \\
&= p(s) - \frac{g(s)}{1+\mu(s)g(s)} - \frac{\mu(s)p(s)g(s)}{1+\mu(s)g(s)} \\
&= \frac{p(s) + \mu(s)p(s)g(s) - g(s) - \mu(s)p(s)g(s)}{1+\mu(s)g(s)} \\
&= \frac{p(s) - g(s)}{1+\mu(s)g(s)}, \quad \text{for all} \;\; s \in \mathbb{T}.
\end{aligned}$$

Similarly, the following properties directly follows from the definition

$$\begin{aligned}
g \ominus g &= 0, \\
\ominus(\ominus g) &= g, \\
\ominus(p \ominus g) &= g \ominus p, \\
\ominus(p \oplus g) &= (\ominus p) \oplus (\ominus g).
\end{aligned}$$

$(\mathcal{R}, \oplus)$ is said to be **regressive group**.

Let $p \in \mathcal{R}$ and fix a point $s_0 \in \mathbb{T}$. Then the initial value problem

$$y^{\triangle} = p(s)y, \quad y(s_0) = 1, \tag{3.1.1}$$

has a unique solution on $\mathbb{T}$.

Definition 3.1.1. If $p \in \mathcal{R}$ and $s_0 \in \mathbb{T}$, then (3.1.1) has a unique solution is said to be exponential function which is denoted by $e_p(., s_0)$.

Few properties of exponential functions are stated in the form of following theorem.

Theorem 3.1.2. *[6, 10] If $p, g \in \mathcal{R}$, then*

(1) $e_0(s,t) \equiv 1$ *and* $e_p(s,s) \equiv 1$;

(2) $e_p(\sigma(s),t) = [1+\mu(s)p(s)]e_p(s,t)$ *and* $e_p(t,\sigma(s)) = \dfrac{e_p(s,t)}{1+\mu(s)p(s)}$;

(3) $e_p(s,t) = \dfrac{1}{e_p(t,s)} = e_{\ominus p}(t,s)$;

(4) $e_p(s,t)e_p(t,w) = e_p(s,w)$;

(5) $e_p(s,t)e_g(s,t) = e_{p\oplus g}(s,t)$;

(6) $\dfrac{e_p(s,t)}{e_g(s,t)} = e_{p\ominus g}(s,t)$;

(7) $(e_p(.,t))^{\triangle}(s) = p(s)e_p(s,t)$ *and* $(e_p(s,.))^{\triangle}(s) = -p(s)e_p(t,\sigma(s))$;

(8) $(\dfrac{1}{e_p(.,t)})^{\triangle}(s) = -\dfrac{p(s)}{e_p(\sigma(s),t)}$.

If $\mathbb{T} = \mathbb{R}$ then $\sigma(s) = s$, $\mu(s) = 0$ and $y^{\triangle} = \acute{y}$ is usual derivative. Thus, in this case $p : \mathbb{T} \to \mathbb{C}$ be any function and $s, s_0 \in \mathbb{T}$, then

$$e_p(s,s_0) = exp\left\{\int_{s_0}^{s} p(\tau)d\tau\right\}.$$

In particular, α is any complex constant, then

$$e_\alpha(s,s_0) = e^{\alpha(s-s_0)}.$$

Assume that $\sup\mathbb{T} = \infty$ and fix $s_0 \in \mathbb{T}$. Suppose that z is complex constant which is regressive, then $\ominus z \in \mathcal{R}$ is also regressive. Thus $e_{\ominus z}$ is well defined on time scale $\mathbb{T}$.

Definition 3.1.3. Assume $x : [s_0, \infty)_{\mathbb{T}} \to \mathbb{C}$ is locally Δ-integrable function, that is, it is Δ-integrable over each compact subinterval of $[s_0, \infty)_{\mathbb{T}}$. Then **Laplace transform** of x is defined as

$$\mathcal{L}\{x\}(z) = \int_{s_0}^{\infty} x(s)e_{\ominus z}(\sigma(s),s_0)\Delta t \text{ for } z \in \mathcal{D}\{x\}, \tag{3.1.2}$$

where $\mathcal{D}\{x\}$ consists of all those complex numbers $z \in \mathcal{R}$ for which the improper Δ-integral exists.

Definition 3.1.4. Assume $f : [s_0, \infty)_{\mathbb{T}} \to \mathbb{C}$ is a function, its **shift** denoted by $\hat{f}(s,t)$ and defined to be the solution of problem

$$\begin{aligned} \hat{f}^{\Delta_s}(s,\sigma(t)) &= -\hat{f}^{\Delta_t}(s,t), \quad t,s \in \mathbb{T}, \quad s_0 \le t \le s, \\ \hat{f}(s,s_0) &= f(s), \quad s \in \mathbb{T}, \quad s_0 \le s. \end{aligned} \tag{3.1.3}$$

Definition 3.1.5. Assume two functions $f, g : [s_0, \infty)_{\mathbb{T}} \to \mathbb{C}$, then **convolution** of these two functions $f * g$ is defined as

$$(f * g)(s) = \int_{s_0}^{s} \hat{f}(s,\sigma(t))g(t)\Delta(t), \quad s \in \mathbb{T}, \quad s_0 \le s. \tag{3.1.4}$$

3.2 The h-Laplace Transform in Time Scale

In this section, consider time scale $\mathbb{T}$ which is introduced in [16, 17].

$$\mathbb{T} = h\mathbb{Z} = \{hk : k \in \mathbb{Z}\},$$

where $\mathbb{Z}$ denotes integers set and h denotes fixed positive real number. Then

$$\sigma(s) = s + h \quad and \quad \mu(s) = h.$$

For a function $g : h\mathbb{Z} \to \mathbb{C}$, we have

$$g^{\triangle}(s) = \frac{g(s+h) - g(s)}{h} \quad \text{for all} \quad s \in h\mathbb{Z}.$$

Therefore for any complex number z, the initial value problem

$$y^{\triangle} = zy, \quad s \in \mathbb{T}, \quad y(s_0) = 1,$$

is transformed as

$$\begin{aligned} \frac{y(s+h) - y(s)}{h} &= zy(s) \\ y(s+h) - y(s) &= hzy(s), \end{aligned}$$

and the final initial value problem has the form

$$y(s+h) = (1 + hz)y(s), \quad s \in h\mathbb{Z}, \quad y(s_0) = 1.$$

Next

$$\begin{aligned} e_z(s,s_0) &= \prod_{k=s_0}^{s} 1 + (s_{k+1} - s_k)z = \prod_{k=s_0}^{s} (1 + hz) \\ &= [1 + (s_1 - s_0)z][1 + (s_2 - s_1)z][1 + (s_3 - s_2)z]... \\ &= (1 + hz)(1 + hz)(1 + hz)... \end{aligned}$$

Hence $e_z(s, s_0)$ has (for $z \neq -1/h$) the form

$$e_z(s, s_0) = (1 + hz)^{\frac{s-s_0}{h}} \quad \text{for all} \quad s \in h\mathbb{Z}.$$

Next

$$\ominus z = -\frac{z}{1 + \mu(s)z} = -\frac{z}{1 + hz},$$

so the initial value problem

$$y^{\Delta} = (\ominus z)(s)y, \quad s \in \mathbb{T}, \;\; y(s_0) = 1$$

is transformed as

$$\begin{aligned}
\frac{y(s+h) - y(s)}{h} &= -\frac{z}{1 + hz} y(s) \\
y(s+h) - y(s) &= -\frac{hz}{1 + hz} y(s) \\
y(s+h) &= \left(1 - \frac{hz}{1 + hz}\right) y(s),
\end{aligned}$$

and the final initial value problem has the form

$$y(s+h) = \left(\frac{1}{1 + hz}\right) y(s), \quad s \in h\mathbb{Z}, \quad y(s_0) = 1.$$

Since

$$e_z = (1 + hz)^{\frac{s-s_0}{h}}.$$

By using previous equation, we find the value of $e_{\ominus z}$

$$\begin{aligned}
e_{\ominus z} &= [1 + h(\ominus z)]^{\frac{s-s_0}{h}} \\
&= \left[1 + h\left(\frac{-z}{1 + \mu(s)z}\right)\right]^{\frac{s-s_0}{h}} \\
&= \left(1 - \frac{hz}{1 + hz}\right)^{\frac{s-s_0}{h}} \\
&= \left(\frac{1}{1 + hz}\right)^{\frac{s-s_0}{h}}.
\end{aligned}$$

Hence $e_{\ominus z}(s, s_0)$ has ($z \neq -\frac{1}{h}$) the form

$$e_{\ominus z} = (1 + hz)^{-\frac{s-s_0}{h}}.$$

Any function $x : [s_0, \infty)_{h\mathbb{Z}} \to \mathbb{C}$, by using (4.2.14), its Laplace transform is

$$\begin{aligned}\mathcal{L}\{x\}(z) = \tilde{x(z)} &= \sum_{s\in[s_0,\infty)_{h\mathbb{Z}}} \mu(s)\, x(s)\, e_{\ominus z}(\sigma(s), s_0)\\ &= \sum_{s\in[s_0,\infty)_{h\mathbb{Z}}} h\, x(s)\, e_{\ominus z}(s+h, s_0)\\ &= h \sum_{s\in[s_0,\infty)_{h\mathbb{Z}}} x(s)\, (1+hz)^{\left(-\frac{s+h-s_0}{h}\right)}\\ &= h \sum_{s\in[s_0,\infty)_{h\mathbb{Z}}} x(s+s_0)\, (1+hz)^{\left(-\frac{s+h}{h}\right)}\\ &= h \sum_{k=0}^{\infty} \frac{x(kh+k_0h)}{(1+hz)^{k+1}}\\ &= \frac{h}{1+hz} \sum_{k=0}^{\infty} \frac{x(kh+k_0h)}{(1+hz)^{k}},\end{aligned}$$

where we are using $k_0 = s_0/h$ so $k_0 \in \mathbb{Z}$. The following definition is in the case when $s_0 \neq 0$.

Definition 3.2.1. If a function $x : h\mathbb{N}_0 \to \mathbb{C}$, then **h-Laplace transform** of x is given by

$$\mathcal{L}\{x\}(z) = \tilde{x(z)} = \frac{h}{1+hz} \sum_{k=0}^{\infty} \frac{x(kh)}{(1+hz)^{k}}, \tag{3.2.1}$$

for $z \neq -\frac{1}{h}$ for which this series converges.

Setting

$$h_* = -\frac{1}{h}, \tag{3.2.2}$$

then (3.2.1) takes the form

$$\begin{aligned}\tilde{x(z)} = \mathcal{L}\{x\}(z) &= \frac{h}{1+hz} \sum_{k=0}^{\infty} \frac{x(kh}{(1+hz)^{k}}\\ &= \frac{1}{z+\frac{1}{h}} \sum_{k=0}^{\infty} \frac{x(kh)}{h^k \left(\frac{1+hz}{h}\right)^k}\\ &= \frac{1}{z+\frac{1}{h}} \sum_{k=0}^{\infty} \frac{x(kh)}{h^k \left(z+\frac{1}{h}\right)^k}.\end{aligned}$$

Finally

$$\mathcal{L}\{x\}(z) = \tilde{x(z)} = \frac{1}{z-h_*} \sum_{k=0}^{\infty} \frac{x(kh)}{h^k (z-h_*)^k}. \tag{3.2.3}$$

Theorem 3.2.2. *[8] (**Shifting Theorem**) If*

$$\tilde{x}(z) = \mathcal{L}\{x(kh)\}(z) \quad for \quad |z-h_*| > A,$$

where A be any real number, then

$$\mathcal{L}\{x(kh+h)\}(z) = (1+hz)\tilde{x}(z) - hx(0), \tag{3.2.4}$$

and

$$\mathcal{L}\{x(kh+2h)\}(z) = (1+hz)^2\tilde{x}(z) - h(1+hz)x(0) - hx(h). \tag{3.2.5}$$

Proof.

$$\begin{aligned}
\mathcal{L}\{x(kh+h)\}(z) &= \frac{1}{z-h_*}\sum_{k=0}^{\infty}\frac{x(kh+h)}{h^k(z-h_*)^k} \\
&= \frac{1}{z-h_*}\sum_{k=0}^{\infty}\frac{x(kh)}{h^{k-1}(z-h_*)^{k-1}} \\
&= h\sum_{k=1}^{\infty}\frac{x(kh)}{h^k(z-h_*)^k} \\
&= hx(0) + h\sum_{k=1}^{\infty}\frac{x(kh)}{h^k(z-h_*)^k} - hx(0) \\
&= h\sum_{k=0}^{\infty}\frac{x(kh)}{h^k(z-h_*)^k} - hx(0) \\
&= h(z-h_*)\left[\frac{1}{z-h_*}\sum_{k=0}^{\infty}\frac{x(kh)}{h^k(z-h_*)^k}\right] - hx(0) \\
&= h\left(z+\frac{1}{h}\right)\tilde{x}(z) - hx(0),
\end{aligned}$$

Hence

$$\mathcal{L}\{x(kh+h)\}(z) = (1+hz)\tilde{x}(z) - hx(0).$$

Now

$$\begin{aligned}
\mathcal{L}\{x(kh+2h)\}(z) &= (1+hz)\mathcal{L}\{x(kh+h)\}(z) - hx(h) \\
&= (1+hz)[(1+hz)\tilde{x}(z) - hx(0)] - hx(h) \\
&= (1+hz)^2\tilde{x}(z) - hx(h) - h(1+hz)x(0).
\end{aligned}$$

□

Theorem 3.2.3. *[8] (**Initial and Final Value Theorem**).*
(a) If $\tilde{x}(z)$ exists for $|z-h_*| > A$, then

$$x(0) = \lim_{z\to\infty}\{z\tilde{x}(z)\}.$$

(b) If $\tilde{x}(z)$ exists for $|z-h_*| > h^{-1}$ and $z\tilde{x}(z)$ is analytic at $z=0$, then

$$\lim_{k\to\infty} x(kh) = \lim_{z\to 0}\{z\tilde{x}(z)\}.$$

Proof. By using (3.2.1)

$$\begin{aligned} z\tilde{x}(z) &= \frac{hz}{1+hz}\sum_{k=0}^{\infty}\frac{x(kh)}{(1+hz)^k} \\ &= \frac{1}{1+\frac{1}{hz}}\sum_{k=0}^{\infty}\frac{x(kh)}{(1+hz)^k} \\ &= \frac{1}{1+\frac{1}{hz}}x(0) + \frac{1}{1+\frac{1}{hz}}\sum_{k=1}^{\infty}\frac{x(kh)}{(1+hz)^k}. \end{aligned}$$

Applying limit at $z \to \infty$

$$\lim_{z\to\infty}[z\tilde{x}(z)] = x(0).$$

For the proof of second part

$$\mathcal{L}\{x(kh+h) - x(kh)\}(z) = \frac{1}{z-h_*}\sum_{k=0}^{\infty}\frac{x(kh+h)-x(kh)}{h^k(z-h_*)^k}.$$

By using $h_* = -\frac{1}{h}$

$$\mathcal{L}\{x(kh+h) - x(kh)\}(z) = \frac{h}{1+hz}\sum_{k=0}^{\infty}\frac{x(kh+h)-x(kh)}{(1+hz)^k}. \tag{3.2.6}$$

Now by using the previous shifting Theorem (3.2.2)

$$\begin{aligned} \mathcal{L}\{x(kh+h) - x(kh)\}(z) &= \mathcal{L}\{x(kh+h)\}(z) - \mathcal{L}\{x(kh)\}(z) \\ &= (1+hz)\tilde{x}(z) - hx(0) - \tilde{x}(z) \\ &= hz\tilde{x}(z) - hx(0). \end{aligned}$$

By using this value in (3.2.6)

$$\begin{aligned} hz\tilde{x}(z) - hx(0) &= \frac{h}{1+hz}\sum_{k=0}^{\infty}\frac{x(kh+h)-x(kh)}{(1+hz)^k} \\ (1+hz)z\tilde{x}(z) - (1+hz)x(0) &= \sum_{k=0}^{\infty}\frac{x(kh+h)-x(kh)}{(1+hz)^k} \\ (1+hz)[z\tilde{x}(z) - x(0)] &= \sum_{k=0}^{\infty}\frac{x(kh+h)-x(kh)}{(1+hz)^k}. \end{aligned}$$

Hence for any $r \in \mathbb{N}_0$

$$(1+hz)[z\tilde{x}(z) - x(0)] - \sum_{k=0}^{r}\frac{x(kh+h)-x(kh)}{(1+hz)^k} = \sum_{k=r+1}^{\infty}\frac{x(kh+h)-x(kh)}{(1+hz)^k}. \tag{3.2.7}$$

Consider

$$
\begin{aligned}
\sum_{k=0}^{r}\frac{x(kh+h)-x(kh)}{(1+hz)^k} &= \sum_{k=0}^{r}\frac{x(kh+h)}{(1+hz)^k}-\sum_{k=0}^{r}\frac{x(kh)}{(1+hz)^k}\\
&= -x(0)+\sum_{k=0}^{r}\frac{x(kh+h)}{(1+hz)^k}-\sum_{k=1}^{r}\frac{x(kh)}{(1+hz)^k}\\
&= -x(0)+\left[\sum_{j=1}^{r}\frac{x(jh)}{(1+hz)^{j-1}}-\sum_{j=1}^{r}\frac{x(jh)}{(1+hz)^j}\right]+\frac{x(rh+h)}{(1+hz)^r}\\
&= -x(0)+\sum_{j=1}^{r}x(jh)\left[\frac{1}{(1+hz)^{j-1}}-\frac{1}{(1+hz)^j}\right]+\frac{x(rh+h)}{(1+hz)^r}\\
&= \sum_{k=0}^{r}\frac{x(kh+h)-x(kh)}{(1+hz)^k} = -x(0)+hz\sum_{j=1}^{r}\frac{x(jh)}{(1+hz)^j}+\frac{x(rh+h)}{(1+hz)^r}.
\end{aligned}
$$

By using this value in (3.2.7)

$$
\begin{aligned}
\sum_{k=r+1}^{\infty}\frac{x(kh+h)-x(kh)}{(1+hz)^k} &= (1+hz)[z\tilde{x}(z)-x(0)]+x(0)\\
&\quad -\frac{x(rh+h)}{(1+hz)^r}-hz\sum_{j=1}^{r}\frac{x(jh)}{(1+hz)^j}.
\end{aligned}
\tag{3.2.8}
$$

Now by choosing sufficiently large non-negative value of $r\in\mathbb{N}_0$, we can make the absolute value of right-hand side of equation (3.2.8) less then for any value of $\epsilon>0$, uniformly with respect to z in a small neighbourhood of $z=0$. Then by passing the limit in (3.2.8) as $z\to 0$. These reasoning completes the proof of second part. □

In case $\mathbb{T}=h\mathbb{Z}$, the shifting problem (3.1.3) has the form

$$
\begin{aligned}
&\hat{f}^{\Delta_s}(s,t+h)+\hat{f}^{\Delta_t}(s,t)=0, \quad t,s\in h\mathbb{Z},\ \ s_0\le t\le s,\\
&\hat{f}(s,s_0)=f(s), \quad s\in h\mathbb{Z},\ \ s_0\le s,\\
&\frac{\hat{f}(s+h,t+h)-\hat{f}(s,t+h)}{h}+\frac{\hat{f}(s,t+h)-\hat{f}(s,t)}{h}=0, \quad t,s\in h\mathbb{Z},\ \ s_0\le t\le s,\\
&\hat{f}(s,s_0)=f(s), \quad s\in h\mathbb{Z}, \quad s_0\le s,\\
&\hat{f}(s+h,t+h)=\hat{f}(s,t), \quad t,s\in h\mathbb{Z},\ \ s_0\le t\le s,\\
&\hat{f}(s,s_0)=f(s), \quad s\in h\mathbb{Z},\ \ s_0\le s.
\end{aligned}
$$

where $f:[s_0,\infty)_{h\mathbb{Z}}\to\mathbb{C}$ is a given function. Then unique solution of this problem is

$$\hat{f}(s,t)=\hat{f}(s-t+s_0).$$

therefore the convolution of f, g is denoted by $f*g$, where $f,g:h\mathbb{N}_0\to\mathbb{C}$, is defined by using (3.1.4)

$$(f*g)(s)=h\sum_{t\in[s_0,s)_{h\mathbb{Z}}}f(s-t-h+s_0)g(t), \quad \text{for } s\in h\mathbb{Z}, \quad s_0\le s.$$

For $s_0=0$, this formula convert in the following definition.

Definition 3.2.4. The **convolution** of these two functions $f, g : h\mathbb{N}_0 \to \mathbb{C}$, is defined as

$$(f * g)(s) = h \sum_{t\in[0,s)_{h\mathbb{N}_0}} f(s-t-h)g(t), \quad \text{for } \ s \in h\mathbb{N}_0,$$

that is

$$(f * g)(kh) = h \sum_{m=0}^{k-1} f(kh-mh-h)g(mh), \quad \text{for } \ k \in \mathbb{N}_0,$$

where

$$(f * g) = 0,$$

and $\mathbb{N}_0$ denotes the set of natural numbers includes zero.

Theorem 3.2.5. *[8] (**Convolution Theorem**). If $\mathcal{L}\{f\}(z)$ exists for $|z-h_*| > A$ and $\mathcal{L}\{g\}(z)$ exists for $|z-h_*| > B$, then*

$$\mathcal{L}\{f * g\}(z) = \mathcal{L}\{f\}(z)\mathcal{L}\{g\}(z) \quad \textit{for } \ |z-h_*| > max\{A, B\},$$

where A, B are any real numbers.

Proof. For $|z-h_*| > max\{A, B\}$, then

$$\begin{aligned}\mathcal{L}\{f\}(z)\mathcal{L}\{g\}(z) &= \frac{1}{z-h_*}\left[\sum_{j=0}^{\infty}\frac{f(jh)}{h^j(z-h_*)^j}\right]\frac{1}{z-h_*}\left[\sum_{m=0}^{\infty}\frac{g(mh)}{h^m(z-h_*)^m}\right] \\ &= \frac{h}{z-h_*}\sum_{j=0}^{\infty}\sum_{m=0}^{\infty}\frac{f(jh)g(mh)}{h^{j+m+1}(z-h_*)^{j+m+1}},\end{aligned}$$

and if $j+m+1 = k$, then

$$\mathcal{L}\{f\}(z)\mathcal{L}\{g\}(z) = \frac{h}{z-h_*}\sum_{j=0}^{\infty}\sum_{k=j+1}^{\infty}\frac{f(jh)g(kh-jh-h)}{h^k(z-h_*)^k}.$$

Interchanging the order of summation

$$\begin{aligned}\mathcal{L}\{f\}(z)\mathcal{L}\{g\}(z) &= \frac{1}{z-h_*}\sum_{k=1}^{\infty}\left[h\sum_{j=0}^{k-1}f(jh)g(kh-jh-h)\right]\frac{1}{h^k(z-h_*)^k} \\ &= \frac{1}{z-h_*}\sum_{k=1}^{\infty}\left[h\sum_{m=0}^{k-1}f(kh-mh-h)g(mh)\right]\frac{1}{h^k(z-h_*)^k} \\ &= \frac{1}{z-h_*}\sum_{k=0}^{\infty}\left[h\sum_{m=0}^{k-1}f(kh-mh-h)g(mh)\right]\frac{1}{h^k(z-h_*)^k} \\ &= \mathcal{L}\{f * g\}(z).\end{aligned}$$

□

Now consider the inverse problem, in which $\tilde{x}(z)$ is given and find $x(s)$. For existence of well defined inverse transform, uniqueness property must hold, that is, if there are two functions x and y for which $\tilde{x}(z) = \tilde{y}(z)$, then $x(s) = y(s)$.

Suppose $x : h\mathbb{N}_0 \to \mathbb{C}$ is a function and R is defined in (3.2.9) and h_* is defined in (3.2.2). Let $R < \infty$.

Lemma 3.2.6. *[8] For each $A > R/h$, the series* (3.2.3) *converges uniformly in the region $|z - h_*| \geq A$.*

Proof. From $A > R/h$, it follows that there exists $\epsilon > 0$ such that

$$A > \frac{R + \epsilon}{h}.$$

Next, for this ϵ we are able to find an integer $m > 0$ by (3.2.9) such that

$$|x(kh)| \leq (R + \epsilon)^k \quad \text{for all} \quad k \geq m.$$

Then for the value of $|z - h_*| \geq A$,

$$\begin{aligned}
\left| \sum_{k=m}^{\infty} \frac{x(kh)}{h^k |z - h_*|^k} \right| &\leq \sum_{k=m}^{\infty} \frac{|x(kh)|}{h^k |z - h_*|^k} \\
&\leq \sum_{k=m}^{\infty} \frac{(R + \epsilon)^k}{h^k A^k} \\
&= \left(1 - \frac{R + \epsilon}{hA}\right)^{-1} \left(\frac{R + \epsilon}{hA}\right)^m \\
&\to 0 \quad as \quad m \to \infty.
\end{aligned}$$

This shows that (3.2.3) in the region $|z - h_*| \geq A$ is uniformly convergent . □

Theorem 3.2.7. *[8] (**Uniqueness Theorem**). Consider $\tilde{x}(z)$ which is defined in* (3.2.3)*. If $\tilde{x}(z) = 0$ for $|z - h_*| > R/h$, then $x(s) = 0$ for $s \in h\mathbb{N}_0$, where*

$$R = \lim_{k \to \infty} \sup(|x(kh)|)^{\frac{1}{k}}, \tag{3.2.9}$$

where R may depend on h.

Proof.

$$x(0) + \frac{x(h)}{h(z - h_*)} + \frac{x(2h)}{h^2(z - h_*)^2} + ... = 0 \quad \text{for} \quad |z - h_*| > \frac{R}{h}. \tag{3.2.10}$$

Passing the limit in (3.2.10) as $|z| \to \infty$ (due to uniform convergence which is proved in previous lemma, we can apply the limit in (3.2.10)) term by term, by this process we get $x(0) = 0$. Now multiplying the remaining part of (3.2.10) by $z - h_*$ and passing the limit as $|z| \to \infty$ to obtain x(h)=0. By repeating this process, we obtain $x(0) = x(h) = x(2h) = ... = 0$. □

Theorem 3.2.8. *[8] Let $\tilde{x}(z)$ is defined by* (3.2.3) *and A is any real number such that $A > R/h$. Then*

$$x(kh) = \frac{h^k}{2\pi i}\int_\Gamma (z-h_*)^k \tilde{x}(z)dz \quad \textit{for } k \in \mathbb{N}_0, \tag{3.2.11}$$

where Γ is the positively oriented curve $\{z \in \mathbb{C} : |z - h_*| = A\}$.

Proof. For any $j \in \mathbb{N}_0$, By using (3.2.3),

$$(z-h_*)^j \tilde{x}(z) = \sum_{k=0}^{\infty} \frac{x(kh)}{h^k}(z-h_*)^{j-k-1}.$$

Integrating this equation both sides over the circle Γ , and integrate this term under the sum sign by the uniform convergence of the series, then

$$\int_\Gamma (z-h_*)^j \tilde{x}(z)dz = \sum_{k=0}^{\infty} \frac{x(kh)}{h^k}\int_\Gamma (z-h_*)^{j-k-1}dz.$$

As, we know

$$\int_\Gamma (z-h_*)^{j-k-1}dz = \begin{cases} 2\pi i & \text{if } \ k=j, \\ 0 & \text{if } \ k \neq j, \end{cases}$$

then

$$\int_\Gamma (z-h_*)^j \tilde{x}(z)dz = 2\pi i \frac{x(jh)}{h^j}$$

This theorem concludes that

$$\mathcal{L}^{-1}\{\tilde{x}\}(kh) = \frac{h^k}{2\pi i}\int_\Gamma (z-h_*)^k \tilde{x}(z)dz \quad \text{for } \ k \in \mathbb{N}_0.$$

□

3.3 The q-Laplace Transform in Time Scale

Consider the time scale in the whole section which is defined in [17, 20].

$$\mathbb{T} = q^{\mathbb{N}_0} = \{q^k : k \in \mathbb{N}_0\} = \{1, q, q^2, q^3, ...\},$$

where $q > 1$ is a fixed number. In which,

$$\sigma(s) = qs \quad \textit{and} \quad \mu(s) = (q-1)s.$$

Let a function $g : q^{\mathbb{N}_0} \to \mathbb{C}$, then its $\triangle$-derivative is defined as

$$g^{\triangle}(s) = \frac{g(qs) - g(s)}{(q-1)(s)} \quad \text{for all } \ s \in q^{\mathbb{N}_0}. \tag{3.3.1}$$

So, for any $z \in \mathbb{C}$, the initial value problem is

$$y^{\triangle}(s) = zy(s), \quad y(t) = 1, \quad s, t \in \mathbb{T},$$

then

$$\begin{aligned}\frac{y(qs)-y(s)}{(q-1)s} &= zy(s)\\ y(qs) &= z(q-1)sy(s)+y(s)\\ y(qs) &= y(s)(z(q-1)s+1),\end{aligned}$$

the initial value problem takes the form

$$y(qs)=(1+\acute{q}zs)y(s) \quad y(t)=1, \;\; s,t\in q^{\mathbb{N}_0}, \tag{3.3.2}$$

where $q-1=\acute{q}$.

By putting $s=q^r$ and $t=q^m$ with $m,r\in\mathbb{N}_0$, then $e_z(s,t)$

$$\begin{aligned}e_z(q^r,q^m) &= \prod_{k=m}^{r-1}[1+u_k z] \quad if \;\; r\geq m,\\ &= \prod_{k=m}^{r-1}[1+(q^{k+1}-q^k)z],\\ &= \prod_{k=m}^{r-1}[1+q^k(q-1)z].\end{aligned}$$

Hence

$$e_z(q^r,q^m)=\prod_{k=m}^{r-1}(1+\acute{q}q^k z) \quad \text{if} \;\; r\geq m, \tag{3.3.3}$$

and similarly

$$e_z(q^r,q^m)=\frac{1}{\prod_{k=r}^{m-1}(1+\acute{q}q^k z)} \quad \text{if} \;\; r\leq m, \tag{3.3.4}$$

for $m=r$, the product are understood to be 1. Now assume that

$$z\neq -\frac{1}{\acute{q}q^k} \;\; \text{for all} \;\; k\in\mathbb{N}_0. \tag{3.3.5}$$

Since

$$(\ominus z)(s)=-\frac{z}{1+\mu(s)z}=-\frac{1}{1+(q-1)sz}=-\frac{1}{1+\acute{q}sz},$$

the initial value problem is

$$y^{\triangle}(s)=(\ominus z)sy(s), \;\; y(t)=1, \;\; s,t\in\mathbb{T}.$$

Now from equation (3.3.2)

$$\begin{aligned}y(qs) &= y(s)(1+\ominus z\acute{q}s)\\ &= y(s)\left[1-\frac{\acute{q}sz}{1+\acute{q}sz}\right]\\ &= y(s)\left[\frac{1+\acute{q}sz-\acute{q}sz}{1+\acute{q}sz}\right]\end{aligned}$$

So the initial value problem becomes

$$y(qs) = \frac{1}{1+\acute{q}sz} y(s), \;\; y(t) = 1, \;\; s,t \in q^{\mathbb{N}_0}.$$

So $e_{\ominus z}(q^r, q^m)$, where $m, r \in \mathbb{N}_0$ and z satisfies (3.3.5), takes the form

$$e_{\ominus z}(q^r, q^m) = \frac{1}{\prod_{k=m}^{r-1}(1+\acute{q}q^k z)} \;\; \text{if } r \geq m, \tag{3.3.6}$$

and

$$e_{\ominus z}(q^r, q^m) = \prod_{k=r}^{m-1}(1+\acute{q}q^k z) \;\; \text{if } \; r \leq m.$$

By comparing (3.3.3), (3.3.4)

$$e_{\ominus z}(q^r, q^m) = \frac{1}{e_z(q^r, q^m)}.$$

Taking the general definition of Laplace transform (4.2.14) and (3.3.6), for any function $x : [s_0, \infty)_{q^{\mathbb{N}_0}} \to \mathbb{C}$ with $s_0 \in q^{\mathbb{N}_0}$, then Laplace transform of x is defined as

$$\mathcal{L}\{x\}(z) = \tilde{x(z)} = \sum_{s \in [s_0,\infty)_{q^{\mathbb{N}_0}}} \mu(s)\; x(s)\; e_{\ominus z}(qs, s_0) x(s),$$

put $t = q^r$

$$\mathcal{L}\{x\}(z) = \tilde{x(z)} = (q-1)\sum_{r=r_0}^{\infty} q^r e_{\ominus z}(q^{r+1}, q^{r_0}) x(q^r),$$

now by using the value of $e_{\ominus z}(q^{r+1}, q^{r_0})$

$$\tilde{x(z)} = \mathcal{L}\{x\}(z) = \acute{q}\sum_{r=r_0}^{\infty} \frac{q^r x(q^r)}{\prod_{k=r_0}^{r}(1+\acute{q}q^k z)}.$$

The following definition is in the case when $s_0 \neq 0$.

Definition 3.3.1. If a function $x : q^{\mathbb{N}_0} \to \mathbb{C}$, then **q-Laplace transform** of x is defined as

$$\mathcal{L}\{x\}(z) = \tilde{x(z)} = \acute{q}\sum_{r=0}^{\infty} \frac{q^r x(q^r)}{\prod_{k=0}^{r}(1+\acute{q}q^k z)}, \tag{3.3.7}$$

for $z \neq -\frac{1}{\acute{q}q^k}$ for which the series is convergent, where $\acute{q} = q - 1$.

Setting

$$P_r(z) = \prod_{k=0}^{r}(1+\acute{q}q^k z), \quad r \in \mathbb{N}_0, \tag{3.3.8}$$

which is a polynomial in z whose degree is $r+1$. Consider

$$\begin{aligned} P_r(z) - P_{r-1}(z) &= \prod_{k=0}^{r}(1+\acute{q}q^k z) - \prod_{k=0}^{r-1}(1+\acute{q}q^k z) \\ &= \prod_{k=0}^{r-1}(1+\acute{q}q^k z)(1+\acute{q}q^r z - 1). \end{aligned}$$

Hence

$$P_r(z) - P_{r-1}(z) = \acute{q}q^r z P_{r-1}(z), \quad r \in \mathbb{N}_0, \tag{3.3.9}$$

and similarly

$$\frac{1}{P_{r-1}(z)} - \frac{1}{P_r(z)} = z\frac{\acute{q}q^r}{P_r(z)}, \quad r \in \mathbb{N}_0, \tag{3.3.10}$$

also holds, where $P_{-1}(z) = 1$.

The numbers $\alpha_k = -\frac{1}{\acute{q}q^k}$, where $\acute{q} = q - 1$ and $k \in \mathbb{N}_0$, are contained in $[-(q-1)^{-1}, 0)$ and approaching to zero as $k \to \infty$. For any positive number $\delta > 0$ and $k \in \mathbb{N}_0$, we set

$$D_\delta^k = \{z \in \mathbb{C} : |z - \alpha_k| < \delta\}$$

and

$$\xi_\delta = \mathbb{C} \backslash \bigcup_{k=0}^{\infty} D_\delta^k = \{z \in \mathbb{C} : |z - \alpha_k| \geq \delta\}, \text{ for all } k \in \mathbb{N}_0,$$

so that ξ_δ is a closed domain of the complex plane $\mathbb{C}$ and the distance of the points are not less than δ from the set $\{\alpha_k : k \in \mathbb{N}_0\}$.

Lemma 3.3.2. *[8] For any $z \in \xi_\delta$, we have*

$$|P_r(z)| \geq (\acute{q}\delta)^{r+1} q^{\frac{r(r+1)}{2}}, \quad r \in \mathbb{N}_0 \bigcup \{-1\}. \tag{3.3.11}$$

Therefore, for an arbitrary positive number R, a positive integer $r_0 = r_0(R, \delta, q)$ exists, such that

$$|P_r(z)| \geq R^{r+1} \quad \text{for all } r \geq r_0 \text{ and } z \in \xi_\delta. \tag{3.3.12}$$

In particular,

$$\lim_{r \to \infty} P_r(z) = \infty \quad \text{for all } z \in \xi_\delta. \tag{3.3.13}$$

Proof. For any $z \in \xi_\delta$, then

$$\begin{aligned} |P_r(z)| &= \left| \prod_{k=0}^{r} (1 + \acute{q}q^k z) \right| \\ &= \left| \prod_{k=0}^{r} \acute{q}q^k (z - \alpha_k) \right| \end{aligned}$$

Since $|z - \alpha_k| \geq \delta$, then

$$\begin{aligned} |P_r(z)| &\geq \prod_{k=0}^{r} \acute{q}q^k \delta \\ &= (\acute{q}\delta)^{r+1} \prod_{k=0}^{r} q^k \\ &= (\acute{q}\delta)^{r+1} q^{\frac{r(r+1)}{2}}. \end{aligned}$$

Hence (3.3.11) holds. Now consider

$$|P_r(z)| \geq (\acute{q}\delta q^{\frac{r}{2}})^{r+1}$$

On the other hand, since $q > 1$, for any number $R > 0$, we can choose a positive integer $r_0 = r_0(R, \delta, q)$ such that

$$\begin{aligned}
\acute{q}\delta q^{\frac{r}{2}} &\geq R, \quad \text{for all } \; r \geq r_0 \\
|P_r(z)| &\geq (\acute{q}\delta q^{\frac{r}{2}})^{r+1} \geq R^{r+1} \\
|P_r(z)| &\geq R^{r+1}.
\end{aligned}$$

The proof of (3.3.12) is complete. □

Example 3.3.3. Let us show that

$$\mathcal{L}\{1\}(z) = \frac{1}{z} \quad \textit{and} \quad \mathcal{L}\{e_\alpha(z) = \frac{1}{z-\alpha}.$$

First find q-Laplace transform of $x(s) = 1$, by using (3.3.7), (3.3.8) (3.3.10) and (3.3.13)

$$\begin{aligned}
\mathcal{L}\{1\}(z) &= \acute{q}\sum_{r=0}^{\infty} \frac{q^r}{P_r(z)} \\
&= \frac{1}{z}\sum_{r=0}^{\infty}\left[\frac{1}{P_{r-1}(z)} - \frac{1}{P_r(z)}\right] \\
&= \frac{1}{z}\lim_{m\to\infty}\left[1 - \frac{1}{P_m(z)}\right] = \frac{1}{z}.
\end{aligned}$$

The q-Laplace transform of the second function, by using (3.3.3) and (3.3.8)

$$e_\alpha(q^r) = \prod_{k=0}^{r-1}(1 + q^k\alpha) = P_{r-1}(\alpha) \quad \text{for all} \quad r \in \mathbb{N}_0.$$

It follows that

$$\begin{aligned}
\tilde{e}_\alpha(z) = \mathcal{L}\{e_\alpha\}(z) &= \acute{q}\sum_{r=0}^{\infty} \frac{q^r e_\alpha(q^r)}{P_r(z)} \\
&= \acute{q}\sum_{r=0}^{\infty} \frac{q^r P_{r-1}(\alpha)}{P_r(z)} \\
&= \acute{q}\sum_{r=0}^{\infty} \frac{q^r}{1+\acute{q}q^r z}\prod_{k=0}^{r-1}\frac{1+\acute{q}q^k\alpha}{1+\acute{q}q^k z} \\
&= \acute{q}\sum_{r=0}^{\infty} \frac{q^r}{1+\acute{q}q^r z}\prod_{k=0}^{r-1}\frac{\frac{1+\acute{q}q^k\alpha}{\acute{q}q^k}}{\frac{1+\acute{q}q^k z}{\acute{q}q^k}} \\
&= \acute{q}\sum_{r=0}^{\infty} \frac{q^r}{1+\acute{q}q^r z}\prod_{k=0}^{r-1}\frac{\alpha + \frac{1}{\acute{q}q^k}}{z + \frac{1}{\acute{q}q^k}}
\end{aligned}$$

$$= \acute{q}\sum_{r=0}^{\infty}\frac{q^r}{1+\acute{q}q^r z}\prod_{k=0}^{r-1}\frac{\alpha-\alpha_k}{z-\alpha_k}. \tag{3.3.14}$$

Since the positive numbers α_k where $k \in \mathbb{N}_0$, are contained in $[-(q-1)^{-1}, 0]$, there is a sufficiently large positive number $R_0 > 0$, such that

$$\left|\frac{\alpha-\alpha_k}{z-\alpha_k}\right| \leq \frac{1}{2} \quad \text{for all } |z| \geq R_0 \quad \text{and} \quad k \in \mathbb{N}_0. \tag{3.3.15}$$

Therefore the series (3.3.14) converges for $|z| \geq R_0$, next the Laplace transform of this function by using (3.3.10)

$$\begin{aligned}
\tilde{e}_\alpha(z) &= \acute{q}\sum_{r=0}^{\infty}\frac{q^r P_{r-1}(\alpha)}{P_r(z)} \\
&= \frac{\acute{q}}{P_0(z)} + \acute{q}\sum_{r=1}^{\infty}\frac{q^r P_{r-1}(\alpha)}{P_r(z)} \\
&= \frac{\acute{q}}{P_0(z)} + \frac{1}{z}\sum_{r=1}^{\infty}\left[\frac{P_{r-1}(\alpha)}{P_{r-1}(z)} - \frac{P_{r-1}(\alpha)}{P_r(z)}\right] \\
&= \frac{\acute{q}}{P_0(z)} + \frac{1}{z}\sum_{r=1}^{\infty}\left[\frac{(1+\acute{q}q_{r-1}\alpha)P_{r-2}(\alpha)}{P_{r-1}(z)} - \frac{P_{r-1}(\alpha)}{P_r(z)}\right] \\
&= \frac{\acute{q}}{P_0(z)} + \frac{1}{z}\sum_{r=1}^{\infty}\left[\frac{P_{r-2}(\alpha)+\acute{q}q_{r-1}\alpha P_{r-2}(\alpha)}{P_{r-1}(z)} - \frac{P_{r-1}(\alpha)}{P_r(z)}\right] \\
&= \frac{\acute{q}}{P_0(z)} + \frac{1}{z}\sum_{r=1}^{\infty}\left[\frac{P_{r-2}(\alpha)}{P_{r-1}(z)} - \frac{P_{r-1}(\alpha)}{P_r(z)} + \frac{\acute{q}q_{r-1}\alpha P_{r-2}(\alpha)}{P_{r-1}(z)}\right] \\
&= \frac{\acute{q}}{P_0(z)} + \frac{1}{z}\sum_{r=1}^{\infty}\left[\frac{P_{r-2}(\alpha)}{P_{r-1}(z)} - \frac{P_{r-1}(\alpha)}{P_r(z)}\right] + \frac{\alpha\acute{q}}{z}\sum_{r=1}^{\infty}\frac{q^{r-1}P_{r-2}(\alpha)}{P_{r-1}(z)} \\
&= \frac{\acute{q}}{P_0(z)} + \frac{1}{z}\left[\frac{1}{P_0(z)} - \frac{P_0(\alpha)}{P_1(z)} + \frac{P_0(\alpha)}{P_1(z)} - \frac{P_1(\alpha)}{P_2(z)} + \ldots - \lim_{m\to\infty}\frac{P_{m-1}(\alpha)}{P_m(z)}\right] \\
&+ \frac{\alpha}{z}\left[\acute{q}\sum_{r=1}^{\infty}\frac{q^{r-1}P_{r-2}(\alpha)}{P_{r-1}(z)}\right] \\
&= \frac{\acute{q}}{P_0(z)} + \frac{1}{zP_0(z)} - \frac{1}{z}\lim_{m\to\infty}\frac{P_{m-1}(\alpha)}{P_m(z)} + \frac{\alpha}{z}\tilde{e}_\alpha(z) \\
&= \frac{\acute{q}}{P_0(z)} + \frac{1}{zP_0(z)} + \frac{\alpha}{z}\tilde{e}_\alpha(z),
\end{aligned}$$

by using the fact that

$$\lim_{m\to\infty}\frac{P_{m-1}(\alpha)}{P_m(z)} = 0,$$

because

$$\begin{aligned}
\frac{P_{m-1}(\alpha)}{P_m(z)} &= \frac{P_{m-1}(\alpha)}{(1+\acute{q}q^m z)P_{m-1}(z)} \\
&= \frac{1}{1+\acute{q}q^m z}\prod_{k=0}^{m-1}\frac{\alpha-\alpha_k}{z-\alpha_k},
\end{aligned}$$

by using (3.3.14)

$$\frac{P_{m-1}(\alpha)}{P_m(z)} \leq \frac{1}{1+\acute{q}q^m z}(1/2)^m.$$

Apply the limit as $m \to \infty$, then $(1/2)^m \to 0$, hence

$$\lim_{m\to\infty} \frac{P_{m-1}(\alpha)}{P_m(z)} = 0.$$

Thus the q-Laplace of exponential function takes the form

$$\begin{aligned}
\tilde{e}_\alpha(z) &= \frac{1+\acute{q}z}{zP_0(z)} + \frac{\alpha}{z}\tilde{e}_\alpha(z)\\
\tilde{e}_\alpha(z) &= \frac{P_0(z)}{zP_0(z)} + \frac{\alpha}{z}\tilde{e}_\alpha(z)\\
\tilde{e}_\alpha(z) &= \frac{1}{z} + \frac{\alpha}{z}\tilde{e}_\alpha(z)\\
\left(1-\frac{\alpha}{z}\right)\tilde{e}_\alpha(z) &= \frac{1}{z}\\
\tilde{e}_\alpha(z) &= \frac{1}{z-\alpha}.
\end{aligned}$$

Theorem 3.3.4. *[8] Suppose the function $x : q^{\mathbb{N}_0} \to \mathbb{C}$ satisfies the condition*

$$|x(q^r)| \leq CR^r \quad \textit{for all} \quad r \in \mathbb{N}_0, \tag{3.3.16}$$

where R and C are constants which are positive then the series converges uniformly which is in (3.3.7) *with respect to z in region ξ_δ.*

Proof. By Lemma (3.3.3), R is given in (3.3.16), then choose a natural number $r_0 \in \mathbb{N}$ such that

$$|P_r(z)| \geq [q(1+R)]^{r+1} \quad \text{for all} \quad r \geq r_0 \quad \text{and} \quad z \in \xi_\delta.$$

The general term of the series in (3.3.7) is given by

$$\left|\frac{q^r x(q^r)}{P_r(z)}\right| \leq \frac{C}{q(1+R)}\left(\frac{R}{1+R}\right)^r \quad \text{for all} \quad r \geq r_0 \quad \text{and} \quad z \in \xi_\delta.$$

The series

$$\sum_{r=0}^{\infty}\left(\frac{R}{1+R}\right)^r,$$

converges if $1+R > R$. This completes the proof. □

$\mathcal{A}_\delta$ denotes a class of functions $x : \{q^r : r \in \mathbb{N}_0\} \to \mathbb{C}$ for which the q-Laplace transform exists and it satisfying the following condition

$$\sum_{r=0}^{\infty}(\delta\acute{q})^{-r} q^{-\frac{r(r-1)}{2}}|x(q^r)| < \infty. \tag{3.3.17}$$

Theorem 3.3.5. *[8] Let $x : \{q^r : r \in \mathbb{N}_0\} \to \mathbb{C}$ be a function and x^Δ denote the q-derivative of x, which is defined in* (3.3.1)*. Suppose that $x \in \mathcal{A}_\delta$, then*

$$\mathcal{L}\{x^\Delta\}(z) = z\tilde{x}(z) - x(1), \tag{3.3.18}$$

and

$$\mathcal{L}\{x^{\Delta\Delta}\}(z) = z^2\tilde{x}(z) - zx(1) - x^\Delta(1). \tag{3.3.19}$$

Proof. Definition of q-Laplace transform (3.3.7) implies that

$$\begin{aligned}
\mathcal{L}\{x^\Delta\}(z) &= \acute{q}\sum_{r=0}^{\infty}\frac{q^r x^\Delta(q^r)}{P_r(z)}\\
&= \acute{q}\sum_{r=0}^{\infty}\frac{q^r\left(\frac{x(q^{r+1})-x(q^r)}{q^{r+1}-q^r}\right)}{P_r(z)}\\
&= \acute{q}\sum_{r=0}^{\infty}\frac{q^r\left(\frac{x(q^{r+1})-x(q^r)}{q^r(q-1)}\right)}{P_r(z)}\\
&= \sum_{r=0}^{\infty}\frac{x(q^{r+1})-x(q^r)}{P_r(z)}\\
&= \sum_{r=0}^{\infty}\frac{x(q^{r+1})}{P_r(z)} - \sum_{r=0}^{\infty}\frac{x(q^r)}{P_r(z)}\\
&= \sum_{r=0}^{\infty}\frac{x(q^{r+1})}{P_{r+1}(z)}(1+\acute{q}q^{r+1}z) - \sum_{r=0}^{\infty}\frac{x(q^r)}{P_r(z)}\\
&= \sum_{r=0}^{\infty}\frac{x(q^{r+1})}{P_{r+1}(z)} - \sum_{r=0}^{\infty}\frac{x(q^r)}{P_r(z)} + \acute{q}z\sum_{r=0}^{\infty}\frac{q^{r+1}x(q^{r+1})}{P_{r+1}(z)}\\
&= -\frac{x(q^0)}{P_0(z)} - \sum_{r=1}^{\infty}\frac{x(q^r)}{P_r(z)} + \sum_{r=0}^{\infty}\frac{x(q^{r+1})}{P_{r+1}(z)} + \acute{q}z\sum_{r=0}^{\infty}\frac{q^{r+1}x(q^{r+1})}{P_{r+1}(z)}\\
&= -\frac{x(q^0)}{P_0(z)} + \acute{q}z\sum_{r=0}^{\infty}\frac{q^{r+1}x(q^{r+1})}{P_{r+1}(z)} + \frac{z\acute{q}x(q^0)}{P_0(z)} - \frac{z\acute{q}x(q^0)}{P_0(z)}\\
&= -\frac{x(q^0)}{P_0(z)} + z\acute{q}\sum_{r=0}^{\infty}\frac{q^r x(q^r)}{P_r(z)} - \frac{z\acute{q}x(q^0)}{P_0(z)}\\
&= z\tilde{x}(z) - \frac{z\acute{q}x(q^0)}{P_0(z)} - \frac{x(q^0)}{P_0(z)}\\
&= z\tilde{x}(z) - \left[\frac{x(q^0)(1+z\acute{q})}{P_0(z)}\right]\\
&= z\tilde{x}(z) - \frac{x(q^0)P_0(z)}{P_0(z)}\\
&= z\tilde{x}(z) - x(1).
\end{aligned}$$

The proof of the second statement (3.3.19) is obtained by applying the first statement

(3.3.18).

$$\mathcal{L}\{x^{\Delta\Delta}\}(z) = z\tilde{x}^{\Delta}(z) - x^{\Delta}(1)$$
$$= z[z\tilde{x}(z) - x(1)] - x^{\Delta}(1)$$
$$= z^2\tilde{x}(z) - zx(1) - x^{\Delta}(1).$$

□

Theorem 3.3.6. *[8] (**Initial and Final Value Theorem**).*
(a) If $x \in \mathcal{A}_\delta$ for some positive real number $\delta > 0$, then

$$x(1) = \lim_{z\to\infty}\{z\tilde{x}(z)\}. \tag{3.3.20}$$

(b) If $x \in \mathcal{A}_\delta$ for all positive real number $\delta > 0$, then

$$\lim_{r\to\infty} x(q^r) = \lim_{z\to 0}\{z\tilde{x}(z)\}. \tag{3.3.21}$$

Proof. Let $x \in \mathcal{A}_\delta$ for some positive real number $\delta > 0$. This follows by definition of q-Laplace transform

$$\tilde{x}(z) = \acute{q}\sum_{r=0}^{\infty}\frac{q^r x(q^r)}{\prod_{k=0}^{r}(1+\acute{q}q^k z)}$$
$$\tilde{x}(z) = \frac{\acute{q}x(1)}{1+\acute{q}z} + \frac{\acute{q}qx(q)}{(1+\acute{q}z)(1+\acute{q}qz)} + ...$$

and

$$(1+\acute{q}z)\tilde{x}(z) = \acute{q}x(1) + \frac{\acute{q}qx(q)}{(1+\acute{q}qz}) + ...$$

Hence

$$\lim_{z\to\infty}\tilde{x}(z) = 0 \ \ and \ \ \lim_{z\to\infty}\{(1+\acute{q}z)\tilde{x}(z)\} = \acute{q}x(1)$$
$$\lim_{z\to\infty}\{\tilde{x}(z) + \acute{q}z\tilde{x}(z)\} = \acute{q}x(1)$$
$$\lim_{z\to\infty}\tilde{x}(z) + \acute{q}\lim_{z\to\infty}\{z\tilde{x}(z)\} = \acute{q}x(1)$$
$$\acute{q}\lim_{z\to\infty}\{z\tilde{x}(z)\} = \acute{q}x(1)$$
$$\lim_{z\to\infty}\{z\tilde{x}(z)\} = x(1).$$

By taking term-by-term limit because of uniform convergence of series in ξ_δ. This completes the proof of (3.3.20).

To show (3.3.21), let $x \in \mathcal{A}_\delta$ for all $\delta > 0$. By using the equality which was obtained in proof of the Theorem (3.3.5)

$$\sum_{r=0}^{\infty}\frac{x(q^{r+1}) - x(q^r)}{P_r(z)} = z\tilde{x}(z) - x(1).$$

Further, by using the argument as in proof of the Theorem (3.2.3)(b) such that

$$\lim_{z\to 0} P_r(z) = 1 \text{ for any } r \in \mathbb{N}_0.$$

To arrive at (3.3.21), applying limit at (3.3), then

$$\begin{aligned}
&\sum_{r=0}^{\infty} \frac{\lim_{z\to 0}[x(q^{r+1}) - x(q^r)]}{\lim_{z\to 0} P_r(z)} = \lim_{z\to 0}[z\tilde{x}(z) - x(1)]\\
&\sum_{r=0}^{\infty}[x(q^{r+1}) - x(q^r)] = \lim_{z\to 0}[z\tilde{x}(z) - x(1)]\\
&- x(q^0) + x(q^1) - x(q^1) + x(q^2) - ... + \lim_{r\to\infty} x(q^{r+1}) = \lim_{z\to 0}[z\tilde{x}(z) - x(1)]\\
&- x(q^0) + \lim_{r\to\infty} x(q^r) = \lim_{z\to 0}[z\tilde{x}(z)] - x(1)\\
&- x(1) + \lim_{r\to\infty} x(q^r) = \lim_{z\to 0}[z\tilde{x}(z)] - x(1)\\
&\lim_{r\to\infty} x(q^r) = \lim_{z\to 0}[z\tilde{x}(z)].
\end{aligned}$$

This completes the proof of second statement. □

For the case when $\mathbb{T} = q^{\mathbb{N}_0}$, then the shifting problem (3.1.3) with the value of $t_0 = 1$ is defined below

$$\begin{aligned}
\hat{f}^{\Delta_s}(s, qt) &= -\hat{f}^{\Delta_t}(s, t), \quad t, s \in q^{\mathbb{N}_0}, \quad s_0 \le t \le s,\\
\hat{f}(s, 1) &= f(s), \quad s \in q^{\mathbb{N}_0}, \quad s_0 \le s.\\
\frac{\hat{f}(qs, qt) - \hat{f}(s, qt)}{(q-1)s} + \frac{\hat{f}(s, qt) - \hat{f}(s, t)}{(q-1)t} &= 0, \quad t, s \in q^{\mathbb{N}_0}, \quad s_0 \le t \le s,\\
\hat{f}(s, 1) &= f(s), \quad s \in q^{\mathbb{N}_0}, \quad s_0 \le s.
\end{aligned}$$

$$\begin{aligned}
t[\hat{f}(qs, qt) - \hat{f}(s, qt)] + s[\hat{f}(s, qt) - \hat{f}(s, t)] &= 0, \quad t, s \in q^{\mathbb{N}_0}, \quad s_0 \le t \le s,\\
\hat{f}(s, 1) &= f(s), \quad s \in q^{\mathbb{N}_0}, \quad s_0 \le s.
\end{aligned} \tag{3.3.22}$$

Definition 3.3.7. Let two functions $f, g : q^{\mathbb{N}_0} \to \mathbb{C}$, then **convolution** of these functions is denoted by $f * g$ and is defined by

$$(f * g)(q^r) = (q-1)\sum_{k=0}^{r-1} q^k \hat{f}(q^r, q^{k+1})g(q^k),$$

where $(f * g)(q^0) = 0$ with $r \in \mathbb{N}_0$.

Theorem 3.3.8. *[8] (**Convolution Theorem**). Consider $\mathcal{L}\{f\}(z), \mathcal{L}\{g\}(z)$ and $\mathcal{L}\{f * g\}(z)$ exist for given complex number $z \in \mathbb{C}$. Then at the point z*

$$\mathcal{L}\{f * g\}(z) = \mathcal{L}\{f\}(z)\mathcal{L}\{g\}(z). \tag{3.3.23}$$

Proof. For our convenience, we set

$$e_{r,m}(z) = e_z(q^r, q^m) \quad \text{and} \quad \hat{f}_{r,m} = \hat{f}(q^r, q^m).$$

By using (3.3.3)

$$e_{r,r}(z) = 1 \quad \text{for all} \quad r \in \mathbb{N}_0, \tag{3.3.24}$$

$$e_{r+1,m}(z) = (1 + \acute{q}q^r z)e_{r,m}(z) \quad \text{for } r, m \in \mathbb{N}_0, \quad r \geq m, \tag{3.3.25}$$

$$e_{r,m+1}(z) = \frac{e_{r,m}(z)}{1 + \acute{q}q^m z} \quad \text{for } r, m \in \mathbb{N}_0, \quad r \geq m+1, \tag{3.3.26}$$

Put $s = q^r$ and $t = q^m$ with $r \geq m$ in equation (3.3.22), then

$$\begin{aligned} &q^m[\hat{f}_{r+1,m+1} - \hat{f}_{r,m+1}] + q^r[\hat{f}_{r,m+1} - \hat{f}_{r,m}] = 0, \quad r \geq m \geq 0, \\ &\hat{f}_{r,0} = f(q^r), \quad r \in \mathbb{N}_0. \end{aligned} \tag{3.3.27}$$

By using two definitions (3.3.7) and (3.3.9), we obtained

$$\begin{aligned} \mathcal{L}\{f * g\}(z) &= (q-1)\sum_{r=1}^{\infty} \frac{q^r (f*g)(q^r)}{e_{r+1,0}(z)} \\ &= (q-1)^2 \sum_{r=1}^{\infty} \frac{q^r}{e_{r+1,0}(z)} \sum_{k=0}^{r-1} q^k \hat{f}_{r,k+1} g(q^k) \\ &= (q-1)^2 \sum_{k=0}^{\infty} q^k g(q^k) \sum_{r=k+1}^{\infty} \frac{q^r \hat{f}_{r,k+1}}{e_{r+1,0}(z)}. \end{aligned}$$

By substituting in the last equation

$$e_{r+1,0}(z) = e_{r+1,k+1}(z)e_{k+1,0}(z),$$

then

$$\mathcal{L}\{f * g\}(z) = (q-1)^2 \sum_{k=0}^{\infty} \frac{q^k g(q^k)}{e_{k+1,0}(z)} \sum_{r=k+1}^{\infty} \frac{q^r \hat{f}_{r,k+1}}{e_{r+1,k+1}(z)}$$

$$\mathcal{L}\{f * g\}(z) = (q-1)\mathcal{L}\{g\}(z) \sum_{r=k+1}^{\infty} \frac{q^r \hat{f}_{r,k+1}}{e_{r+1,k+1}(z)}. \tag{3.3.28}$$

For our convenience, let us set

$$\Phi_m = \sum_{r=m}^{\infty} \frac{q^r \hat{f}_{r,m}}{e_{r+1,m}(z)}, \quad m \in \mathbb{N}_0. \tag{3.3.29}$$

The target is to show that Φ_m is independent the value of $m \in \mathbb{N}_0$, then

$$\begin{aligned} (q-1)\sum_{r=k+1}^{\infty} \frac{q^r \hat{f}_{r,k+1}}{e_{r+1,k+1}(z)} &= (q-1)\sum_{r=0}^{\infty} \frac{q^r \hat{f}_{r,0}}{e_{r+1,0}(z)} \\ &= (q-1)\sum_{r=0}^{\infty} \frac{q^r f(q^r)}{e_{r+1,0}(z)} = \mathcal{L}\{f\}(z), \end{aligned}$$

by using this value in (3.3.28), we get (3.3.23)

$$\mathcal{L}\{f * g\}(z) = \mathcal{L}\{f\}(z)\mathcal{L}\{g\}(z).$$

Now, for showing Φ_m does not depend on the value of $m \in \mathbb{N}_0$. Now putting $e_{r,m}(z) = e_{r,m}$ and by using (3.3.27) and (3.3.24), (3.3.25) and (3.3.26)

$$\Phi_{m+1} = \sum_{r=m+1}^{\infty} \frac{q^r \hat{f}_{r,m+1}}{e_{r+1,m+1}} = \sum_{r=m+1}^{\infty} \frac{q^r \hat{f}_{r,m} + q^m \hat{f}_{r,m+1} - q^m \hat{f}_{r+1,m+1}}{e_{r+1,m+1}}$$

$$\Phi_{m+1} = \sum_{r=m+1}^{\infty} \frac{q^r \hat{f}_{r,m}}{e_{r+1,m+1}} + q^m \sum_{r=m+1}^{\infty} \left[\frac{\hat{f}_{r,m+1} - \hat{f}_{r+1,m+1}}{e_{r+1,m+1}}\right]$$

$$\Phi_{m+1} = \sum_{r=m+1}^{\infty} \frac{q^r \hat{f}_{r,m}}{e_{r+1,m+1}} - q^m \sum_{r=m+1}^{\infty} \left[\frac{\hat{f}_{r+1,m+1}}{e_{r+1,m+1}} - \frac{\hat{f}_{r,m+1}}{e_{r,m+1}} + \frac{\hat{f}_{r,m+1}}{e_{r,m+1}} - \frac{\hat{f}_{r,m+1}}{e_{r+1,m+1}}\right]$$

$$\Phi_{m+1} = \sum_{r=m+1}^{\infty} \frac{q^r \hat{f}_{r,m}}{e_{r+1,m+1}} - q^m \sum_{r=m+1}^{\infty} \left[\frac{\hat{f}_{r+1,m+1}}{e_{r+1,m+1}} - \frac{\hat{f}_{r,m+1}}{e_{r,m+1}} + \frac{\hat{f}_{r,m+1}(1+\acute{q}q^r z)}{e_{r+1,m+1}} - \frac{\hat{f}_{r,m+1}}{e_{r+1,m+1}}\right]$$

$$\Phi_{m+1} = \sum_{r=m+1}^{\infty} \frac{q^r \hat{f}_{r,m}}{e_{r+1,m+1}} - q^m \sum_{r=m+1}^{\infty} \left[\frac{\hat{f}_{r+1,m+1}}{e_{r+1,m+1}} - \frac{\hat{f}_{r,m+1}}{e_{r,m+1}} + \frac{\hat{f}_{r,m+1}}{e_{r+1,m+1}} + \frac{(\acute{q}q^r z)\hat{f}_{r,m+1}}{e_{r+1,m+1}} - \frac{\hat{f}_{r,m+1}}{e_{r+1,m+1}}\right]$$

$$\Phi_{m+1} = \sum_{r=m+1}^{\infty} \frac{q^r \hat{f}_{r,m}}{e_{r+1,m+1}} - q^m \sum_{r=m+1}^{\infty} \left[\frac{\hat{f}_{r+1,m+1}}{e_{r+1,m+1}} - \frac{\hat{f}_{r,m+1}}{e_{r,m+1}}\right] - q^m \sum_{r=m+1}^{\infty} \frac{(\acute{q}q^r z)\hat{f}_{r,m+1}}{e_{r+1,m+1}}$$

$$\Phi_{m+1} = \sum_{r=m+1}^{\infty} \frac{q^r \hat{f}_{r,m}}{e_{r+1,m+1}} + q^m \frac{\hat{f}_{m+1,m+1}}{e_{m+1,m+1}} - q^m \sum_{r=m+1}^{\infty} \frac{\hat{f}_{r,m+1}}{e_{r+1,m+1}}(\acute{q}q^r z)$$

$$\Phi_{m+1} = \sum_{r=m+1}^{\infty} \frac{q^r \hat{f}_{r,m}}{e_{r+1,m}}(1+\acute{q}q^m z) + q^m \frac{\hat{f}_{m+1,m+1}}{e_{m+1,m+1}} - q^m \sum_{r=m+1}^{\infty} \frac{\hat{f}_{r,m+1}}{e_{r+1,m+1}}(\acute{q}q^r z)$$

$$\Phi_{m+1} = \sum_{r=m+1}^{\infty} \frac{q^r \hat{f}_{r,m}}{e_{r+1,m}}(1+\acute{q}q^m z) + q^m \frac{\hat{f}_{m,m}}{e_{m+1,m}}(1+\acute{q}q^r z) - q^m \frac{\hat{f}_{m,m}}{e_{m+1,m}}(1+\acute{q}q^r z)$$
$$+ q^m \frac{\hat{f}_{m+1,m+1}}{1} - \acute{q}q^m z \sum_{r=m+1}^{\infty} \frac{q^m \hat{f}_{m,m+1}}{e_{r+1,m+1}}$$

$$\Phi_{m+1} = \sum_{r=m}^{\infty} \frac{q^r \hat{f}_{r,m}}{e_{r+1,m}}(1+\acute{q}q^m z) - \frac{q^m \hat{f}_{m,m}}{e_{m+1,m}}(1+\acute{q}q^m z) + q^m \hat{f}_{m+1,m+1} - \acute{q}q^m z\Phi_{m+1}$$

$$\Phi_{m+1} = (1+\acute{q}q^m z)\Phi_m - q^m \frac{\hat{f}_{m,m}}{e_{m,m}} + q^m \hat{f}_{m+1,m+1} - \acute{q}q^m z\Phi_{m+1}$$

$$(1+\acute{q}q^m z)\Phi_{m+1} = (1+\acute{q}q^m z)\Phi_m - q^m \hat{f}_{m,m} + q^m \hat{f}_{m+1,m+1}$$

$$(1+\acute{q}q^m z)\Phi_{m+1} = (1+\acute{q}q^m z)\Phi_m,$$

by using the fact that $\hat{f}_{r,r} = f(1)$ for all $r \in \mathbb{N}_0$. Hence $\Phi_{m+1} = \Phi_m$ as $1+\acute{q}q^m z \neq 0$ under condition (3.3.5). □

Now discuss some result of inverse q-Laplace transform.

Theorem 3.3.9. *[8] (**Uniqueness Theorem**). Assume a function $x : q^{\mathbb{N}_0} \to \mathbb{C}$ in the space $\mathcal{A}_\delta$, that is x satisfies* (3.3.17)*. Further, suppose $\tilde{x}(z)$ denotes the q-Laplace transform of x which is defined by* (3.3.7) *for $z \in \xi_\delta$. If $\tilde{x}(z) \equiv 0$ for complex value of $z \in \xi_\delta$, then $x(q^r) = 0$ for all non-negative integers $r \in \mathbb{N}_0$.*

Proof. Definition of q-Laplace transform (3.3.7) implies that

$$\frac{x(q^0)}{1+\acute{q}z} + \frac{qx(q)}{(1+\acute{q}z)(1+\acute{q}qz)} + \frac{q^2x(q)}{(1+\acute{q}z)(1+\acute{q}qz)(1+\acute{q}q^2z)} + ... \equiv 0, \tag{3.3.30}$$

for $z \in \xi_\delta$. Multiplying (3.3.30) by $1+\acute{q}z$ and then passing to the limit as $|z| \to \infty$, we get $x(q^0) = 0$. Use $x(q^0) = 0$ in (3.3.30) and get

$$\frac{qx(q)}{(1+\acute{q}z)(1+\acute{q}qz)} + \frac{q^2x(q)}{(1+\acute{q}z)(1+\acute{q}qz)(1+\acute{q}q^2z)} + ... \equiv 0.$$

Now multiplying this equation by $(1+q^0z)(1+q^1z)$ and then passing the limit as $|z| \to \infty$, then we get $x(q^1) = 0$. By repeating this technique, we find $x(q^0) = x(q^1) = x(q^2) = ... = 0$. □

Theorem (3.3.9) gives that inverse of Laplace transform exists. The next theorem gives the integral formula for the inverse Laplace transform.

Theorem 3.3.10. *[8] (**Inverse Laplace Transform**). Let $x \in \mathcal{A}_\delta$ and $\tilde{x}(z)$ denotes its q-Laplace transform which is defined by* (3.3.7)*. Then*

$$x(q^r) = \frac{1}{2\Pi i}\int_\Gamma \tilde{x}(z)\prod_{k=0}^{r-1}(1+\acute{q}q^kz)dz \quad \textit{for } r \in \mathbb{N}_0, \tag{3.3.31}$$

where Γ denotes any positive oriented closed curve in ξ_δ that have encloses all points $\alpha_k = -(\acute{q}q^k)^{-1}$ for $k \in \mathbb{N}_0$.

Proof. For any $j \in \mathbb{N}_0$, by using (3.3.7)

$$\tilde{x}(z)\prod_{k=0}^{j-1}(1+\acute{q}q^kz) = \acute{q}\sum_{r=0}^{j-1}q^rx(q^r)\prod_{\substack{k=0\\k\neq r}}^{j-1}(1+\acute{q}q^kz) + \frac{\acute{q}q^jx(q^j)}{1+\acute{q}q^jz} + \acute{q}\sum_{r=j+1}^{\infty}\frac{q^rx(q^r)}{\prod_{k=j}^{r}(1+\acute{q}q^kz)} \tag{3.3.32}$$

Integrating the equality over the curve Γ with respect to z and then we can take term-by-term integration and get that

$$\int_\Gamma \tilde{x}(z)\prod_{k=0}^{j-1}(1+\acute{q}q^kz)dz = \acute{q}\sum_{r=0}^{j-1}q^rx(q^r)\int_\Gamma\prod_{\substack{k=0\\k\neq r}}^{j-1}(1+\acute{q}q^kz)dz + \acute{q}q^jx(q^j)\int_\Gamma\frac{dz}{1+\acute{q}q^jz}$$
$$+\acute{q}\sum_{r=j+1}^{\infty}q^rx(q^r)\int_\Gamma\frac{dz}{\prod_{k=j}^{r}(1+\acute{q}q^kz)}.$$

Since

$$\int_\Gamma \prod_{\substack{k=0\\k\neq r}}^{j-1}(1+\acute{q}q^k z)dz = 0 \quad \text{for } \ j \geq 0,$$

$$\int_\Gamma \frac{dz}{1+\acute{q}q^j z} = \frac{2\pi i}{\acute{q}q^j} \quad \text{for } \ j \geq 0,$$

$$\int_\Gamma \frac{dz}{\prod_{k=j}^{r}(1+\acute{q}q^k z)} = 0 \quad \text{for } \ j \geq 0, \ \ r \geq j+1,$$

By putting these values in the previous equation

$$\int_\Gamma \tilde{x}(z)\prod_{k=0}^{j-1}(1+\acute{q}q^k z)dz = 2\pi i x(q^j).$$

We have used these result from the theory of complex function: If $P(z)$ is any polynomial of degree greater then two and Γ denotes any positive contour that contains all roots of the polynomial $P(z)$, then

$$\int_\Gamma \frac{dz}{P(z)} = 0.$$

This completes the proof. □

Chapter 4

Laplace Transform on Time Scale with Nabla Operator

4.1 Introduction

Remark 4.1.1. *(a) Every continuous function is both ld-continuous and rd-continuous but it is not necessary every ld or rd-continuous function is continuous.*
(b) It is not necessary every ld-continuous function is rd-continuous and its vice versa.

Example: Consider the time scale

$$\mathbb{P}_{1,1} = \bigcup_{k=0}^{\infty}[2k, 2k+1].$$

then

$$\sigma(s) = \begin{cases} s \quad \text{if} \quad \bigcup_{k=0}^{\infty}[2k, 2k+1) \\ s+1 \quad \text{if} \quad \bigcup_{k=0}^{\infty}\{2k+1\} \end{cases}$$

and

$$\mu(s) = \begin{cases} 0 \quad \text{if} \quad s \in \bigcup_{k=0}^{\infty}[2k, 2k+1) \\ 1 \quad \text{if} \quad s \in \bigcup_{k=0}^{\infty}\{2k+1\} \end{cases}$$

and

$$\rho(s) = s \quad \text{if } s \in \bigcup_{k=0}^{\infty}[2k, 2k+1]$$

and

$$\nu(s) = 0 \quad \text{if } s \in \bigcup_{k=0}^{\infty}[2k, 2k+1]$$

where

$$f(s) = \sigma(s) \quad \text{for all } s \in \mathbb{T}$$

This function is continuous at right dense points and its limit exists at left dense points, so it is rd-continuous function but not ld-continuous, since it is not continuous at left dense points. This example is used for both parts of this remark.

Remark 4.1.2. *Every nabla and delta integrable function are not given same result always.*

Consider $\mathbb{T} = \{1, 2, 3, 4, 5\}$ and

$$f(s) = s \quad \text{for all} \quad s \in \mathbb{T},$$

Then

$$\begin{aligned}\int_1^5 f(s)\nabla(s) &= \sum_{s\in(1,5]} f(s)\nu(s) \\ &= \sum_{s\in(1,5]} (s)(1) \\ &= \sum_{s\in(1,5]} s \\ &= 2+3+4+5 = 14.\end{aligned}$$

and

$$\begin{aligned}\int_1^5 f(s)\Delta(s) &= \sum_{s\in[1,5)} f(s)\mu(s) \\ &= \sum_{s\in[1,5)} (s)(1) \\ &= \sum_{s\in[1,5)} s \\ &= 1+2+3+4 = 10.\end{aligned}$$

4.2 Exponential Function and Laplace Transform by using Nabla Operator

In this section we define a generalized exponential function and Laplace transform for an arbitrary time scale $\mathbb{T}$ by using nabla operator.

The set of all ld-continuous and regressive functions $q : \mathbb{T} \to \mathbb{C}$ denoted by $\mathcal{S}$. Suppose $q \in \mathcal{S}$ and fix $s \in \mathbb{T}$. Then initial value problem is

$$y^{\nabla}(s) = q(s)y(s), \quad y(t) = 1 \tag{4.2.1}$$

has a unique solution on $\mathbb{T}$. This solution is said to be **exponential function** and denoted by $e_q(s,t)$.

If $q \in \mathcal{S}$, then the **exponential function** is defined by

$$e_q(s,t) = exp \int_t^s log \frac{1+\nu(\tau)q(\tau)}{\nu(\tau)} \bigtriangledown (\tau) \quad \text{for} \quad s,t \in \mathbb{T}.$$

Throughout let us consider s_r are real numbers for all $r \in \mathbb{N}$ such that

$$\lim_{r\to\infty} s_r = \infty \quad \text{and} \quad v_r = s_r - s_{r-1} > 0 \quad \text{for all} \quad r \in \mathbb{N} \tag{4.2.2}$$

while throughout assume that

$$\lim_{r\to\infty} s_r = \infty \quad \text{and} \quad v = \inf_{r\in\mathbb{N}} v_r > 0, \quad \text{where} \quad v_r = s_r - s_{r-1} \quad \text{for} \quad r \in \mathbb{N} \tag{4.2.3}$$

holds. For example, the numbers

$$s_r = hr, \ \ r \in \mathbb{N} \quad \text{and} \quad s_r = q^r, \ \ r \in \mathbb{N},$$

where $h > 0$ and $q > 1$, respectively satisfy the assumption (4.2.3), while

$$s_m = \sqrt{m}, \quad m \in \mathbb{N} \qquad \text{and} \qquad s_m = \ln m, \quad m \in \mathbb{N},$$

do not satisfy the assumption (4.2.3).

Suppose z be the complex number such that

$$z \neq -\frac{1}{v_r} \qquad \text{for all} \qquad r \in \mathbb{N}. \tag{4.2.4}$$

Then solution of the problem (4.2.1) is $e_z(s_r, s_m)$

$$y(s_{r+1}) = (1 + v_r z) y(s_r), \quad y(s_m) = 1, \quad m, r \in \mathbb{N}_0,$$

satisfies

$$e_z(s_r, s_m) = \prod_{k=m+1}^{r} (1 + v_k z) \quad \text{if} \quad r \geq m, \tag{4.2.5}$$

and

$$e_z(s_r, s_m) = \frac{1}{\prod_{k=m+1}^{r} (1 + v_k z)} \quad \text{if} \quad r \leq m,$$

for $m = r$, the product are understood to be 1.

Assume that $\sup\mathbb{T} = \infty$ and fix $s_0 \in \mathbb{T}$. Also assume that z denotes complex constant that is regressive. Thus $e_z(., s_0)$ is well defined on $\mathbb{T}$. Assume $x : [s_0, \infty)_{\mathbb{T}} \to \mathbb{C}$ is locally ∇-integrable function, that is, this function is ∇-integrable over each compact subinterval of $[s_0, \infty)_{\mathbb{T}}$. Then Laplace transform of x is defined below

$$\mathcal{L}\{x\}(z) = \int_{s_0}^{\infty} \frac{x(s)}{e_z(\rho(s_{r+1}), s_0)} \nabla s \ \text{ for } \ z \in \mathcal{E}\{x\}, \tag{4.2.6}$$

where $\mathcal{E}\{x\}$ consists of all those complex numbers $z \in \mathcal{S}$ for which improper integral exists.

After solving this improper integral by using the help of M. Bohner, G.Sh. Guseinov [5], we are able to make more general form of Laplace transform which is given below.

Definition 4.2.1. Assume (4.2.2) holds. If $x : \{s_r : r \in \mathbb{N}\} \to \mathbb{C}$ represents a function and its **Laplace transform** is defined in this way

$$\mathcal{L}\{x\}(z) = \tilde{x}(z) = \sum_{r=1}^{\infty} \frac{v_r x(s_r)}{\prod_{k=1}^{r} (1 + v_k z)}, \tag{4.2.7}$$

for those complex values of $z \in \mathbb{C}$ satisfying (4.2.4) for which the series converges.

Recall previous assumptions (4.2.3) and (4.2.4). Define

$$Q_r(z) = \prod_{k=1}^{r}(1+v_k z), \qquad r \in \mathbb{N}, \tag{4.2.8}$$

represent a polynomial in z whose degree is r.

$$Q_r(z) - Q_{r-1}(z) = \prod_{k=1}^{r}(1+v_k z) - \prod_{k=1}^{r-1}(1+v_k z), \quad r \in \mathbb{N},$$

$$Q_r(z) - Q_{r-1}(z) = \prod_{k=1}^{r-1}(1+v_k z)[1+v_r z - 1], \quad r \in \mathbb{N},$$

$$Q_r(z) - Q_{r-1}(z) = z v_r Q_{r-1}(z), \qquad r \in \mathbb{N}. \tag{4.2.9}$$

It is easily verified that

$$\frac{1}{Q_{r-1}(z)} - \frac{1}{Q_r(z)} = z\frac{v_r}{Q_r(z)}, \qquad r \in \mathbb{N}, \tag{4.2.10}$$

hold, where $Q_0(z) = 1$.

The numbers $\beta_r = -v_r^{-1}$, $r \in \mathbb{N}$, are contained in the interval $[-v^{-1}, 0)$. For any positive real number $\delta > 0$ and $r \in \mathbb{N}$, we set

$$E_\delta = \mathbb{C}\backslash \bigcup_{r=1}^{\infty} E_\delta^r \quad \text{where} \quad E_\delta^r = \{z \in \mathbb{C} : |z - \beta_r| < \delta\}, \ \ r \in \mathbb{N}, \tag{4.2.11}$$

so that E_δ is closed domain of $\mathbb{C}$, and the distance of the points of E_δ are not less than δ from this set $\{\beta_r : r \in \mathbb{N}\}$.

Lemma 4.2.2. *Assume* (4.2.3)*,* (4.2.4)*,* (4.2.8) *and* (4.2.11) *holds. For any $z \in E_\delta$, we have*

$$|Q_r(z)| \geq (\delta v)^r \quad \text{and} \quad |Q_r(z)| \geq \delta(\delta v)^{r-1} v_r \quad \text{for all} \quad r \in \mathbb{N}. \tag{4.2.12}$$

Moreover,

$$\lim_{r\to\infty} Q_r(z) = \infty \quad \text{for all} \quad z \in E_\delta \quad \text{provided} \quad \delta > v^{-1}. \tag{4.2.13}$$

Proof. For any $z \in E_\delta$ and $r \in \mathbb{N}$, we have

$$\begin{aligned}
|Q_r(z)| &= \left|\prod_{k=1}^{r}(1+v_k z)\right| \\
&= \left|\prod_{k=1}^{r} v_k(v_k^{-1} + z)\right| \\
&= \left|\prod_{k=1}^{r}(v_k(z-\beta_k))\right| \\
&= v_r\left(\prod_{k=1}^{r-1} v_k\right)\left(\prod_{k=1}^{r}|z-\beta_k|\right) \\
&\geq v_r v^{r-1}\delta^r \\
&= \delta(\delta v)^{r-1} v_r.
\end{aligned}$$

The proof of second statement in (4.2.12) is complete. The proof of first statement in (4.2.12) is as follows

$$\begin{aligned}|Q_r(z)| &= \left(\prod_{k=1}^{r} v_k\right)\left(\prod_{k=1}^{r} |z-\beta_k|\right)\\ |Q_r(z)| &\geq (v^r)(\delta^r)\\ |Q_r(z)| &\geq (\delta v)^r.\end{aligned}$$

The proof of (4.2.13) follows from (4.2.12). By using (4.2.3), we have

$$\lim_{r\to\infty} s_r = \infty \quad \text{for all} \quad r \in \mathbb{N}$$

implies that

$$\lim_{r\to\infty} s_{r-1} = \infty,$$

thus

$$\lim_{r\to\infty} (s_r - s_{r-1}) = \infty,$$

$$\lim_{r\to\infty} v_r = \infty.$$

From this, we have

$$\begin{aligned}&\lim_{r\to\infty} \delta(\delta v)^{r-1} v_r = \infty\\ &\lim_{r\to\infty} Q_r(z) = \infty \quad \text{for all} \quad z \in E_\delta \quad \text{provided} \quad \delta > v^{-1}.\end{aligned}$$

□

Example 4.2.3. If $x(s_r) = 1$ then its Laplace transform

$$\mathcal{L}\{1\}(z) = \frac{1}{z}$$

For $z \in E_\delta$, with $\delta > v^{-1}$, using (4.2.7), (4.2.8), (4.2.10) and (4.2.13),

$$\begin{aligned}\mathcal{L}\{1\}(z) &= \sum_{r=1}^{\infty} \frac{v_r}{Q_r(z)}\\ &= \frac{1}{z}\sum_{r=1}^{\infty}\left[\frac{1}{Q_{r-1}(z)} - \frac{1}{Q_r(z)}\right]\\ &= \frac{1}{z}\sum_{r=1}^{\infty}\left[\frac{1}{Q_0(z)} - \frac{1}{Q_1(z)} + \frac{1}{Q_1(z)} - \frac{1}{Q_2(z)} + ...\right]\\ &= \frac{1}{z}\lim_{m\to\infty}\left[1 - \frac{1}{Q_m(z)}\right] = \frac{1}{z}.\end{aligned}$$

Theorem 4.2.4. *Consider* (4.2.3). *If* $x : \{s_r : r \in \mathbb{N}\} \to \mathbb{C}$ *is a function and satisfies this condition*

$$|x(s_r)| \leq CS^{r-1} \quad \textit{for all} \quad r \in \mathbb{N}, \tag{4.2.14}$$

where S *and* C *are constants which are positive, then the series converges uniformly which is in* (4.2.7), *with respect to* z *in region* E_δ *with* $\delta > Sv^{-1}$.

Proof. By Lemma (4.2.2) and (4.2.14), the general term of the series in (4.2.7), we have

$$\left|\frac{v_r x(s_r)}{Q_r(z)}\right| \le \frac{v_r C S^{r-1}}{\delta(\delta v)^{r-1} v_r} = \frac{C}{\delta}\left(\frac{S}{\delta v}\right)^{r-1} \quad \text{for } r \in \mathbb{N} \text{ and } z \in E_\delta.$$

The series

$$\sum_{r=1}^{\infty}\left(\frac{S}{\delta v}\right)^{r-1}$$

converges if $\delta > Sv^{-1}$. This completes the proof. □

$\mathcal{G}_\delta$ denotes a class of function $x : \{s_r : r \in \mathbb{N}\} \to \mathbb{C}$ for which the Laplace transform exists and it satisfying the following condition

$$\sum_{r=1}^{\infty}(\delta v)^{-r}|x(s_r)| < \infty. \tag{4.2.15}$$

Theorem 4.2.5. *Assume* (4.2.3) *holds. Then:*
Assume $x \in \mathcal{G}_\delta$ *for some positive real number* $\delta > 0$, *then*

$$x(s_1) = \lim_{z\to\infty}\{z\tilde{x}(z)\}. \tag{4.2.16}$$

Proof. Let $x \in \mathcal{G}_\delta$ for some positive real number $\delta > 0$. It follows by definition of Laplace transform (4.2.7)

$$\tilde{x}(z) = \sum_{r=1}^{\infty}\frac{v_r x(s_r)}{Q_r(z)} = \sum_{r=1}^{\infty}\frac{v_r x(s_r)}{\prod_{k=1}^{r}(1+v_k z)}$$

$$\tilde{x}(z) = \frac{v_1 x(s_1)}{1+v_1 z} + \frac{v_2 x(s_2)}{(1+v_1 z)(1+v_2 z)} + \frac{v_3 x(s_3)}{(1+v_1 z)(1+v_2 z)(1+v_3 z)} + ...$$

and

$$(1+v_1 z)\tilde{x}(z) = v_1 x(s_1) + \frac{v_2 x(s_2)}{(1+v_2 z)} + \frac{v_3 x(s_3)}{(1+v_2 z)(1+v_3 z)} + ...$$

Hence

$$\lim_{z\to\infty}\tilde{x}(z) = 0 \text{ and } \lim_{z\to\infty}\{(1+v_1 z)\tilde{x}(z)\} = v_1 x(s_1)$$

$$\lim_{z\to\infty}[\tilde{x}(z) + v_1 z\tilde{x}(z)] = v_1 x(s_1)$$

$$v_1 \lim_{z\to\infty} z\tilde{x}(z) = v_1 x(s_1)$$

$$\lim_{z\to\infty} z\tilde{x}(z) = x(s_1).$$

This completes the proof of (4.2.16). □

4.3 The Convolution by using Nabla Operator

Assume a function $f : (s_0, \infty)_{\mathbb{T}} \to \mathbb{C}$, its **shift** denoted by $\hat{f}(s,t)$ and is defined as solution of the problem

$$\begin{aligned} \hat{f}^{\nabla_s}(s,\rho(t)) &= -\hat{f}^{\nabla_t}(s,t), \quad t,s \in \mathbb{T}, \quad s_1 \le t \le s, \\ \hat{f}(s,s_1) &= f(s), \quad s \in \mathbb{T}, \quad s_1 \le s. \end{aligned} \tag{4.3.1}$$

For given function $f, g : (s_0, \infty)_{\mathbb{T}} \to \mathbb{C}$, then **convolution** of these two functions denoted by $f * g$ and is defined by

$$(f * g)(s) = \int_{s_1}^{s} \hat{f}(s, \rho(t))g(t)\nabla(t), \quad s \in \mathbb{T}, \quad s_1 \leq s. \tag{4.3.2}$$

Assume only (4.2.2) in this section. For given function $f : \{s_r : r \in \mathbb{N}\} \to \mathbb{C}$, consider the **shifting problem** (4.3.1)

$$\begin{aligned} \hat{f}^{\nabla s}(s_r, s_{m-1}) &= -\hat{f}^{\nabla t}(t_r, t_m) \qquad m, r \in \mathbb{N}, \qquad r \geq m, \\ \hat{f}(s_r, s_1) &= f(s_r), \quad r \in \mathbb{N}. \end{aligned}$$

$$\begin{aligned} v_m \left[\hat{f}(s_r, s_{m-1}) - \hat{f}(s_{r-1}, s_{m-1})\right] + v_r \left[\hat{f}(s_r, s_m) - \hat{f}(s_r, s_{m-1})\right] &= 0, \qquad m, r \in \mathbb{N}, \qquad r \geq m, \\ \hat{f}(s_r, s_1) &= f(s_r), \quad r \in \mathbb{N}. \end{aligned} \tag{4.3.3}$$

Definition 4.3.1. Assume (4.2.2) holds, let two functions $f, g : \{s_r : r \in \mathbb{N}\} \to \mathbb{C}$, and let $\hat{f}$ is the solution of the problem (4.3.3). Then **convolution** of these two functions f and g is denoted by $f * g$ and is defined by

$$(f * g)(s_r) = \sum_{k=1}^{r} v_k \hat{f}(s_r, s_{k-1})g(s_k), \quad r \in \mathbb{N}.$$

Chapter 5

Conclusion

This thesis is concerned with Laplace transform on time scale. The main objective of our study is to define Laplace transform on time scale by using nabla operator.

In chapter 1, we give a brief introduction of time scale $\mathbb{T}$. This chapter consists of definitions, related examples and theorems which we have used in the next chapters. Chapter 2 deals with Laplace transform on isolated time scale by using delta operator. Similarly, chapter 3 deals with Laplace transform on two different time scales $\mathbb{T} = h\mathbb{Z} = \{hk : k \in \mathbb{Z}\}$ where $h > 0$ and $\mathbb{T} = q^{\mathbb{N}_0} = \{q^k : k \in \mathbb{N}_0\}$ where $q > 1$, by using delta operator. Chapter 4 serves as essential and introductry material on nabla operator. After this we give some definitions related to Laplace transform and prove some related results on isolated time scale $\mathbb{T}$ by using nabla operator.

Martin Bohner and Gusein Sh. Guseinov has given the concept of Laplace transform with different time scales by using delta operator. In our work, we have given the concept of Laplace transform on isolated time scales $\mathbb{T}$ by using nabla operator. In future, this work can be extended on Laplace transform with time scale $\mathbb{T} = h\mathbb{Z} = \{hk : k \in \mathbb{Z}\}$ where $h > 0$ and $\mathbb{T} = q^{\mathbb{N}_0} = \{q^k : k \in \mathbb{N}_0\}$ where $q > 1$, by using nabla operator. If we consider that all the problems of Laplace transform [7, 8] can be solved by using nabla operator then this work can be extended in such a way that we use diamond operator instead of nabla operator.

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