Nullity of Tripartite Graphs

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Abstract

Graph spectra has its wide range applications in applied chemistry, applied physics and applied mathematics. It is also used in modelling virus propagation and error correcting codes in computer networks. It measures robustness of a network against the spread of viruses. Most work done in graph spectra is related to energy of graphs or nullity of graphs.

We study the nullity of different families of graphs and the structure of graphs within those families which exhibit minimum or maximum nullity. We study nullity of bipartite graphs in detail along with a new concept of expanded paths and expanded cycles. We construct a new class of tripartite graphs by introducing some constraints on the family of tripartite graphs. We obtain the nullity set for this class. We characterize the structure of the graphs of this family with nullity $n-4$ and $n-6$.

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No words are enough to elaborate the efforts put in by the parents for the success of there child, neither their is any matching tribute, so I just pray to Almighty Allah to reciprocate them.

I am thankful to my In Laws for giving me the requisite support for which they were blessed to provide in abundance, especially my husband Dr. Kashif Ali who provided the desired motivation and support.

Qudsia Naureen

Introduction

Spectral graph theory studies properties of a graph in relation to its adjacency matrix as well as other associated matrices. The spectral graph theory prompts to make effective use of linear algebra particularly the well established theory of matrices for the purposes of graph theory. The eigenvalues of a graph G are the eigenvalues of adjacency matrix $A(G)$ and the spectrum of G is the multiset of eigenvalues of G. The nullity of graph G, denoted by $\eta(G)$, is the multiplicity of the eigenvalue zero in the spectrum of G. The graph G is singular if $\eta(G) > 0$ and is non-singular if $\eta(G) = 0$. In Mathematics, nullity of a graph has its direct relevance with singularity of graph.

Nullity has a noteworthy application in chemistry and leading to a new field of research, nowadays referred to as Chemical Graph Theory. In order to explain the role of the nullity of graphs in chemistry, we need to recall a few basic facts from the quantum theory of molecules [21]. The behavior of the electrons in molecules is considered to be responsible for the majority of properties of chemical compounds. This behavior is governed by laws of quantum theory and is described by the so-called Schrodinger equation. Finding the solutions of the Schrodinger equation is one of the main tasks of quantum chemistry. In an early stage of quantum chemistry, during the time when computers were not available, the German theoretical chemist Huckel proposed an approximate method for solving the Schrodinger equation for a special class of organic molecules, the so-called unsaturated conjugated hydrocarbons [11]. Nowadays, this method is known under the name Huckel molecular orbital (HMO) theory. The (approximate) energies $E_1, E_2, ...E_n$ that the electrons may possess are related to the eigenvalues $e_1, e_2, ..e_n$ of a so-called molecular graph as

$$E_j = \alpha + \beta\lambda_j, \qquad j = 1, 2, ..., n$$

where α, β are certain constants and λ_j are eigenvalues of the molecular graph. Because $\alpha < 0$, if the graph eigenvalues are labelled in the usual non-increasing manner as $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ then E_1 is the lowest energy level, E_2 is the second lowest energy level. Each energy level in a molecule can be occupied by at most two electrons. Usually, the total number of electrons to which HMO theory is applied is equal to n, and n is most frequently an even number. Usually, $E_{n/2+1} < E_{n/2}$ or it can be said that, $\lambda_{n/2} > \lambda_{n/2+1}$. The results on this theory were obtained by Pairing theorem [6]. According to it, for the majority of unsaturated conjugated hydrocarbons, the eigenvalues of the molecular graph are paired, so that

$$\lambda_j = -\lambda_{-j+1} \quad \forall j = 1, 2, .., n.$$

Such hydrocarbons to which the Pairing theorem applies are referred to as alternant. By our latest knowledge it can be straightforwardly recognize that an unsaturated conjugated hydrocarbon is alternant if and only if its molecular graph is bipartite.

Thus, the nullity of a molecular graph has a far reaching inference on the expected stability of unsaturated conjugated hydrocarbons. This application of nullity of bipartite graphs inspired us to study the nullity of tripartite graphs family. Nullity of tripartite graphs holds its significance in graph theory as it has explicit relation to singularity of its adjacency matrix. It is possible that nullity of tripartite graphs also produce wide range of applications on chemical graph theory. As observed the nullity problem in tripartite graphs does not follow as an extension to that of the nullity of bipartite graphs, this makes the study of nullity in tripartite graphs interesting. In this direction, we obtain the nullity set of a class of n-vertex tripartite graphs and characterize these tripartite graphs with nullity $n-4$ and $n-6$.

Chapter 1 is devoted to basic definitions and terminologies. It contains preliminaries on graph theory and also describes important graph families. Some prerequisites of matrices required for further work along with the brief history of spectral graph theory and the work presently done on its different aspects are given in this chapter. In Chapter 2, we discuss nullity of different families of graphs like trees, cycles, unicyclic, bipartite graphs, etc. Mainly we review the work on nullity of bipartite graphs [9].

Our main work is presented in Chapter 3. Firstly, we define a new class of tripartite graphs by introducing some constraints on tripartite graphs. Nullity set for this family is obtained. We characterize this family of n$-$vertex tripartite graphs by nullity $n-4$ and $n-6$.

Contents

Chapter 1

Introduction and Prerequisite

This part is devoted for elaborating the essential features of graphs and some basic definitions. It consists of different theoretical terms of graphs and their illustrations. Focus is to explain basic properties of matrices, some common graphs with examples that further describe these concepts in a concise way.

1.1 Graph Theory

Graph is a diagram showing the relation between variable quantities. A relationship in which objects are represented in a very simple, convenient and natural way. *Graph theory* is one of the prominent branch of discrete mathematics progressing at fast pace. It includes combinatorics (dealing with combinations of objects belonging to a finite set in accordance with certain constraints). It possesses numerous applications in operational research, genetics, physical/biological/social sciences, computer science and network systems. Graph theory has become one of the most studied and rapidly growing research area. Many problems in discrete mathematics are being interpreted and solved using graph theory. Wide range of applications provided by graph theory help other branches of science *e.g* X-ray, cryptography, circuit design, data security and communication design.

Leonard Euler, a Swedish mathematician is considered as the originator of graph theory as he presented the very first paper in graph theory which was published in 1936. The famous *Konigsberg Bridge Problem* was solved by *Euler*. He introduced the basic terminologies of graph theory which were very lucid and operational. Since then graph theory has progressed in leaps and bounds. In short a graph prepared with emphasis on precision depicts the economic events of centuries in a specific field on one page sketch, easy to understand in a birds-eye view for everyone. In our endeavour to conjure, we the mathematicians(collectively) will prove that the graph theory has a challenging but promising future in presenting the work done in different scientific fields over decades in a concise way just in a single diagram or figure.

A graph $G(V, E, \Phi)$ is a triple consisting of a vertex set $V(G)$ or V_G an edge set $E(G)$ or E_G and a relation Φ that associates each edge with two vertices (not necessarily distinct). In a straight way, if a finite or infinite number of points are connected by lines the resulting figure is a *graph*, the points or corners are called *vertices* or *nodes*, and the lines are called *edges* or *links*. An *edge set* is a set of unordered pairs of vertices, that is, $e = uv$ or $e = vu$, denotes an edge with vertices u and v. For any specified edge $e = uv$ of graph u and v are end points of edge e. The number of *vertices* (denoted by n) and the number of *edges* (denoted by m) in a graph G determines the *order* and *size* of the graph, respectively.

In a graph G, a vertex v joined to itself by an edge is called a *loop*. If two or more edges of G have same end vertices then these edges are called *parallel edges*. A graph G is a *simple graph* if it does not contain any loop or parallel edge.

1.1.1 Basic Definitions

If distinct edges v and u are incident with a common vertex then they are called *adjacent edges*. If $e = uv$ is an edge of a graph G then u and v are termed as *adjacent* in G, and it implies u and v are connected by the edge e. For any vertex $v \in V(G)$, the set of all vertices that are adjacent to v in G is called the *neighbourhood* of v and is denoted by $N_G(v)$ or simply $N(v)$.

A vertex v having no edge incident to it is called an *isolated vertex*. A vertex whose neighbourhood contains only one vertex is called a *pendent vertex* and an edge associated to it is *pendent edge*. A *null graph or empty graph* of order n is a graph with n vertices and no edges. A graph whose edges and vertices are finite is a *finite graph*, otherwise it is an *infinite graph.* The number of edges incident with a vertex, say v, is called the *degree* of the vertex v, denoted as $d_G(v)$ or $d(v)$. The maximum and minimum degree of a graph G is denoted by $\Delta(G)$ and $\delta(G)$, respectively, and is presented as:

$$\Delta(G) = \max\{d_G(v) : v \in V(G)\},$$

$$\delta(G) = \min\{d_G(v) : v \in V(G)\}.$$

A graph H is a subgraph of G if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. A subgraph H is a *spanning subgraph* if $V(H) = V(G)$. A subgraph F is an *induced subgraph* of G if whenever $v, u \in V(F)$ and $e = uv$ is an edge of G, then e is an edge of F as well. If $X \subseteq V(G)$, then the *subgraph of G induced by* set X is the induced subgraph with vertex set X. This induced subgraph is denoted as $G[X]$.

Example 1.1.1. A graph G and its subgraph H, spanning subgraph F and the subgraph S induced by $\{v_9, v_{10}, v_{11}, v_{12}, v_{13}\}$ are described in Figure 1.1.

1.1.2 Paths and Circuits

A *walk* is a finite sequence of vertices denoted as $W = v_0e_1v_1e_2v_2 \dots v_{n-1}e_nv_n$ where v_0 is origin of the walk, v_n is terminus of the walk and $v_1, \dots, v_{n-1}$ are internal vertices of walk. In a walk W terms are arranged in such a way that vertices and edges are alternate such that for $1 \le i \le n$, the edge e_i has ends v_{i-1} and v_i. The number of edges in the walk describe the length of the walk. If the edges $e_1, e_2, e_3, \dots, e_n$ are distinct, that is, the edges are not repeated, then W is a called a *trail.* A *trivial walk/trail* contains no edges. A walk or trail is *closed* if its initial and terminus points are same. If the vertices $v_0, v_1, v_2, \dots, v_n$ of the walk W are distinct, then W is called a *path* and is denoted by P_n. In other words the path P_n can be described as an alternate sequence of n vertices and $n-1$ edges. A non trivial closed trail in G is called a *cycle* if its origin and internal vertices are distinct. A cycle of length n, that is, with n edges is called $n-$cycle and is denoted by C_n. If n is *even* (*odd)* then C_n is known as *even (odd) cycle.* A 3-cycle is called a triangle. A graph containing no cycle is *acyclic graph.*

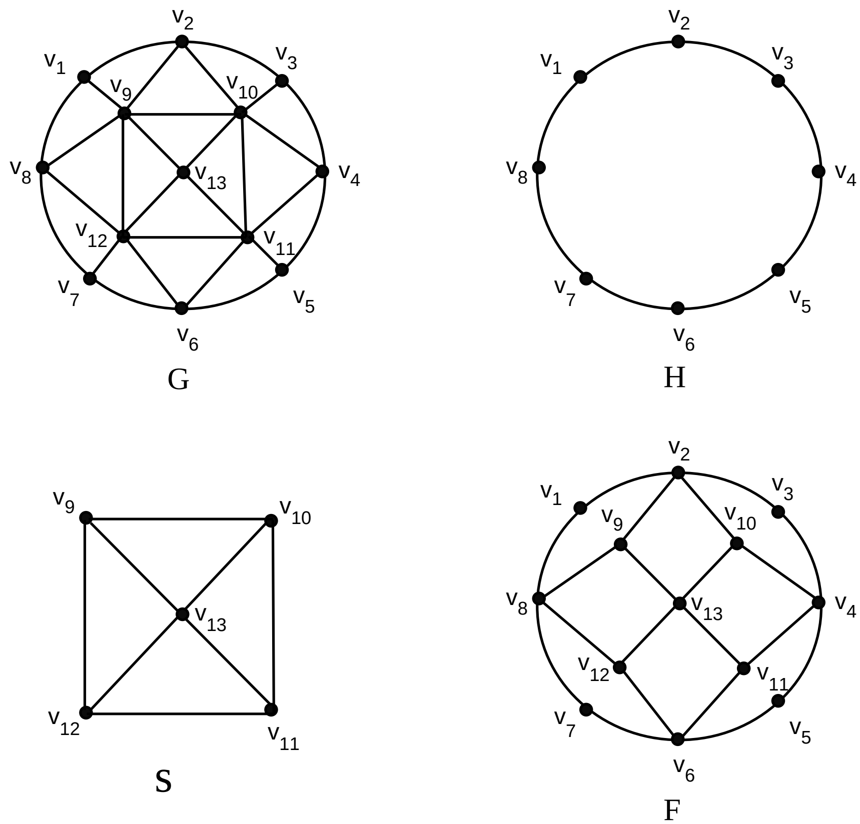

Figure 1.1: A graph G and its subgraph H, spanning subgraph F and induced subgraph S.

Now the definitions of edge and vertex deleted sets are given and for this firstly we describe how setminus works. If $G = (V, E)$ is a graph, then $G \setminus \{e\}$ denotes the subgraph of G having vertex set $V(G)$ and $E(G) \setminus \{e\}$ as edge set, where $e \in E(G)$, and is called *edge deleted subgraph*. It is possible to delete multiple edges or vertices from a graph G and it will still remain as the subgraph of G. So for an edge deleted subgraph denoted by $G \setminus X$, where $X \subseteq E(G)$ the edge set is $E(G) \setminus X$ where vertex set remains $V(G)$. Here in edge deleted subgraph only selected edges are deleted but that is not the case for vertex deleted subgraph. In a vertex deleted subgraph not only the selected vertices are deleted but also the edges incident to them. So, a *vertex deleted subgraph* $G \setminus X$ is defined as subgraph of G with vertex set $V(G) \setminus X$, where $X \subseteq V(G)$ and whose edges are all those of G which are not incident with vertices contained in X.

Example 1.1.2. A graph G and its edge deleted and vertex deleted subgraphs are shown in the Figure 1.2.

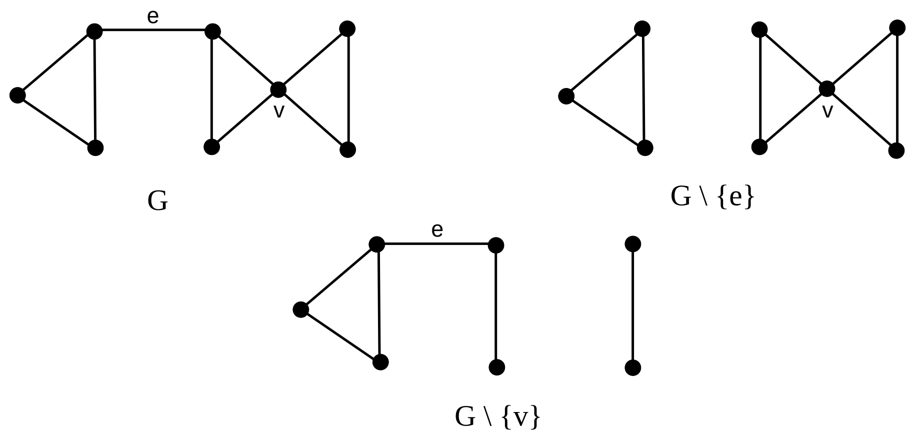

Figure 1.2: A graph G and its vertex and edge deleted subgraphs

For any two graphs G_1 and G_2, if $V(G_1) = V(G_2)$ and $E(G_1) = E(G_2)$ then G_1 and G_2 are equal graphs. The graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ are *isomorphic* if there exists a $1 - 1$ correspondence between the edge sets E_1 and E_2 such a way that if e_1 is an edge with vertices u_1 and v_1 in G_1 then the corresponding edge e_2 in G_2 has its end points the vertices u_2 and v_2 in G_2 which correspond to u_1 and v_1, respectively. Isomorphism is denoted by $\cong$ symbol. The compulsory requirements for two graphs to be isomorphic are:

- Both must have the same number of vertices.
- Both must have same number of edges.
- Both must have the equal number of vertices with the same degree.

Example 1.1.3. In the Figure, 1.3, $G \cong H$ and $G_1 \cong H_1$.

If G_1 and G_2 are disjoint graphs then their union is denoted by $G_1 \cup G_2$ having edge set $E(G_1) \cup E(G_2)$ and vertex set $V(G_1) \cup V(G_2)$. A *join*, denoted by $G_1 + G_2$

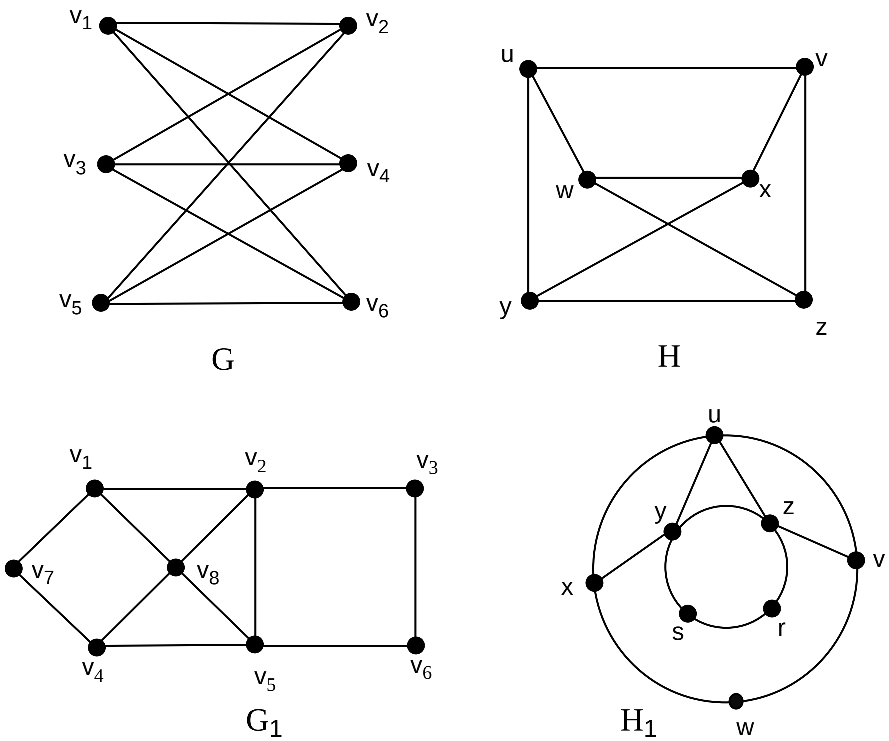

Figure 1.3: Isomorphic graphs

consists of $G_1 \cup G_2$ and all edges joining each vertex of G_1 with each vertex of G_2. For instance the disjoint union of k *copies* of graph G is denoted by kG.

A vertex $u \in V(G)$ is said to be *connected* to a vertex $v \in G$ if there exists a path from u to v, otherwise it is *disconnected*. A graph G is said to be connected if any two of its vertices are connected. A connected subgraph of G such that it is not a proper subgraph of any other connected subgraph of G is called a *component of* G. It implies that a graph is connected if and only if it contains only one component.

1.2 Important Graph Classes

This section presents a brief introduction to some important classes of graphs such as regular graphs, bipartite graphs and some of their properties.

1.2.1 Directed Graphs

If the edges of a graph G are given some direction then these are called *directed edges or arcs* and its edge set is denoted by $A(G)$. A graph with directed edges is called a *directed graph or digraph.* Each arc $'a'$ joins an ordered pair of vertices u and v of G and is denoted by $a = (u, v)$. An arc $a = (u, v)$ is represented by a line (curve) with an arrow sign on it pointing from u to v and is said to be *leaving* vertex u and *entering* the vertex v. The number of arcs leaving a vertex is called *out-degree* of u and is denoted by $d^+(u)$. The number of arcs entering a vertex u is its *in-degree* and is denoted by $d^-(u)$. It is obvious that in a directed graph G, the sum of all in-degrees is equal to the sum of all out-degrees, that is,

$$\sum_{v\in V(G)} d^+(v) = \sum_{v\in V(G)} d^-(v).$$

A sequence $v_0, a_1, v_1, a_2, v_2, \ldots, a_k, v_k$ alternating in vertices and arcs in a directed graph G such that $a_i = (v_{i-1}, v_i)$ for each $1 \leq i \leq k$ is called a *directed walk* in G. A *directed walk* from u to v is an alternating sequence of vertices and arcs beginning from u and ending at v. A vertex v is said to be *reachable* from a vertex u if there is a directed walk from u to v. If both the vertices u and v are reachable from each other then they are called *mutually reachable.* Vertices and arcs may repeat in a walk. A *directed trail* is defined as a directed walk with distinct arcs. A walk with distinct arcs and vertices is a *directed path.* A closed directed trail with distinct vertices (except origin and terminus) is called a *directed cycle.*

An *underlying graph* is a graph obtained by replacing arcs by the non-directed edges of a directed graph. A digraph G is said to be *weakly connected* if its underlying graph is connected. A digraph G is said to be *strongly connected* if every two vertices in G are mutually reachable from each other. A digraph is acyclic if it contains no directed cycle.

1.2.2 Regular Graphs

If degree of each vertex is same in a graph G, that is, $d(v) = k$, for every $v \in V(G)$ then G is called *k-regular graph.* A 3-regular graph is called a *cubic graph.* A simple graph in which every pair of vertices is adjacent is a *complete graph*. In other words, a complete graph K_n of n vertices contains an edge between each pair of distinct vertices. The complete graph K_n is $n-1$ regular graph.

Example 1.2.1. A 4-regular and some cubic graphs are shown in the Figure 1.1

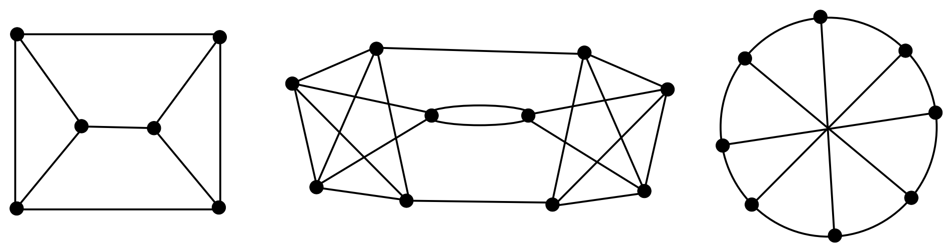

Figure 1.4: Some regular graphs

1.2.3 Bipartite Graphs

A graph G is *bipartite* if its vertex set can be partitioned into two non empty disjoint subsets X and Y in such a manner that each edge of G has one end vertex in X and other in Y. A *bipartite graph* is *complete* if each vertex of X is joined to a every vertex in Y, denoted by K_{n_1,n_2} where $n_1 = |X|$ and $n_2 = |Y|$. A *star* is a complete bipartite graph K_{n_1,n_2} where $n_1 = 1$ or $n_2 = 1$. A graph G is r*-partite* if vertices of G can be partitioned into r subsets V_i, where $1 \leq i \leq r$ and each V_i is a partite set, such that $e = uv$ is an edge of G if u and v belong to different partite sets. A graph G is called *complete* r*-partite* if every two vertices belonging to different partite sets are adjacent. The following theorem gives a strict characterization to bipartite graphs.

Theorem 1.2.1 (Zhang [7])**.** *A nontrivial graph G is bipartite if and only if it has no odd cycle.*

Example 1.2.2. The graphs S, F and H shown in the Figure 1.5 are bipartite graphs. Clearly, S and H are complete bipartite where S is $K_{1,8}$ and H is $K_{2,2}$ graph.

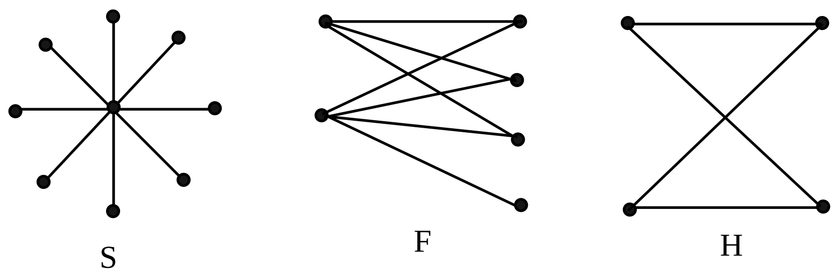

Figure 1.5: Some Bipartite graphs

1.3 Matrix Algebra

Sometimes the structure of graph being complicated requires much effort to obtain information. Social networks can be represented by using matrices. Matrices are used in order to understand such arduous arrangements.

1.3.1 Matrix Structure and Notation

A *matrix* is a rectangular array of quantities or expressions in rows and columns that is treated as a single entity and is manipulated by mathematical operations. The entry in n row and m column of matrix A is denoted by a_{ij}. The order of matrix A having n rows and m columns is given by $n \times m$. So, a matrix A consisting of n rows and m columns is denoted by $A = [a_{ij}]_{n\times m}$. A matrix of order $1\times m$ and $n\times 1$ is called *row* and *column matrix*, respectively. A matrix having same number of rows and columns is called *square matrix*. An equation of the form $a_1x_1+a_2x_2+\ldots+a_nx_n = b$ is called a *linear equation* in the linear variables $x_1, x_2, \ldots, x_n$, respectively. Here $a_1, a_2, \ldots, a_n$ denote the coefficients and b is the constant term of the equation. A finite collection of linear equations in the variables $x_1, x_2, \ldots, x_n$, is called a *system of linear equations* in those variables. An ordered n-tuple $z = (z_1, z_2, \ldots, z_n)$ is the *solution of the system of linear equations* given above if, whenever substituted in place of $x_1, x_2, \ldots, x_n$, respectively, the resulting statements are true. The linear system of equations can be represented in matrix form as $Ax = B$. The space spanned by the vectors $z_1, \ldots, z_n$ is the collection of all linear combinations of these vectors, denoted as $span(z_1, \ldots, z_n)$. The space spanned by the column vectors of

A is therefore *span*(A), which is also known as the column space of A. A vector z is in *span*(A) if it can be expressed as Ac for some vector $c \neq 0$. Similarly, the space spanned by the row vectors of A is *span*(A') and known as the row space of A. The column (*row*) rank of A is the dimension of the column (*row*) space of A. If $A = [a_{ij}]$ is an $m \times n$ matrix, then the elements a_{ii} constitutes *principal diagonal.* For a matrix A of order $n \times m$, its *transpose* A^T is a matrix of order $m \times n$ whose rows are just the columns of A in the same order. If $A = A^T$, that is, $[a_{ij}] = [a_{ji}]$ then matrix A is a *symmetric matrix.* For a matrix $A = [a_{ij}]$ of order $n \times n$, if $Ax = \lambda x$ where x is a non zero vector , then λ is the *eigenvalue* and x is the *eigenvector* of A corresponding to λ. The *rank* of a matrix of order $n \times m$ is the number of linearly independent vectors (those forming either the rows or the columns) in a matrix. The rank of a matrix A is denoted by using the notation rank*(A)*. The matrix rank is defined in terms of number of linearly independent either rows or column because they are always same. These results are presented below in a proper form with theorems.

Theorem 1.3.1 (Spence [22])**.** *The rank of any matrix is the maximum number of its linearly independent columns, that is, the dimension of the subspace generated by its columns.*

Corollary 1.3.2 (Spence [22])**.** *For a matrix A of an order $m \times n$,*

- $\mathrm{rank}(A^T) = \mathrm{rank}(A)$
- *The rows and columns of any matrix generate subspaces of the same dimension, numerically equal to the rank of the matrix.*

Lemma 1.3.3 (Horn [13])**.** *Let A be a matrix. Then, the matrices A^TA and AA^T have the same nonzero eigenvalues.*

1.3.2 Adjacency Matrix of Graph

Let $V(G) = \{v_1, v_2, v_3, \ldots, v_n\}$ be a vertex set of graph G. The *adjacency matrix* of G is an $n \times n$ matrix M with entries $m_{ij} = 1$ when $v_iv_j \in E(G)$ and $m_{ij} = 0$ otherwise. The adjacency matrix of a graph G is denoted by $A(G)$.For vertices v_1, v_2, $A(G)_{v_1v_2} = 1$ shows that contain an edge between them. Observations extracted from the the adjacency matrix of a graph G are stated below.

- The principal diagonal of $A(G)$ has zero entries if graph G does not contain loops.
- The adjacency matrix of a disconnected graph with two components G_1 and G_2 can be partitioned as $A(G) = \left(\begin{array}{c|c} A(G_1) & \mathbf{0} \\ \hline \mathbf{0} & A(G_2) \end{array}\right)$ where $A(G_1)$ is the adjacency matrix of a component G_1 and $A(G_2)$ is the adjacency matrix of a component G_2.
- The structure of adjacency matrix does not accomplish any recognisable definitive relation to parallel edges.

Example 1.3.1. Consider the graph G depicted in Figure 1.6 below along with its adjacency matrix $A(G)$.

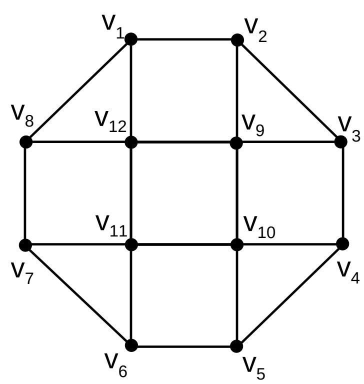

Figure 1.6: Graph G

$$A(G) = \begin{pmatrix} 0&1&0&0&0&0&0&1&0&0&0&1 \\ 1&0&1&0&0&0&0&0&1&0&0&0 \\ 0&1&0&1&0&0&0&0&1&0&0&0 \\ 0&0&1&0&1&0&0&0&0&1&0&0 \\ 0&0&0&1&0&1&0&0&0&1&0&0 \\ 0&0&0&0&1&0&0&0&1&1&0&0 \\ 0&0&0&0&0&1&0&1&0&0&1&0 \\ 1&0&0&0&0&0&1&0&0&0&0&1 \\ 0&1&1&0&0&0&0&0&0&1&0&1 \\ 0&0&0&1&1&0&0&0&1&0&1&0 \\ 0&0&0&0&0&1&1&0&1&0&1&0 \\ 1&0&0&0&0&0&0&1&1&0&1&0 \end{pmatrix}$$

1.4 Spectral Graph Theory

Spectral graph theory elaborates existing specified characteristics of implied antecedent in comparison to other things. In concise it depicts or studies properties of a graph in relation to its adjacency matrix as well as other associated matrices. The spectral graph theory prompts to make effective use of linear algebra particularly the well established theory of matrices for the purposes of graph theory. But the theory of graph spectra is in no way restricted to the theory of matrices. Rather it is proving its importance through its recognized feature and specific way of reasoning. Its legitimacy is evident to distinguish it as complete theory in its own right.

Graph spectra came to forefront when it was put to practical use to fulfill particular purposes in applied chemistry, applied physics, applied mathematics and computer science. Rather motivation for advancement in graph spectra came when it was first incorporated in applied chemistry and physics generating astonishing successful results. One major practical use of graph spectra in applied chemistry is its application in the theory of unsaturated conjugated hydrocarbons. The motive principle for creating first mathematical paper on graph spectra was initiated from the occurrence of membrane vibration problem. The problem was solved through approximation method. The corresponding partial differential equation lead to consider the eigenvalues of graph which represented a discrete model of the membrane.

In statistical physics there appear problems which are easily dealt with by spectra of certain matrices closely related to adjacency matrices. During the last decade it has also been admitted that graph spectra have several significant applications in computer science. The recognition of graph spectra is being widely acknowledged in internet and computer technology and also in so many other fields. Since 1970, it is applied in terms of graph eigenvalues in computer science which is associated with graphs specified as expanders. For further elaborating expanders we can say that a graph possesses advantageous expanding characteristics if each subset of the vertex set of small cardinality has a set of neighbours of large cardinality. Problems such as communication network and error correction codes in computer technologies are helped out and solved through the use of expanders and some related graphs. Expanders can be constructed from graphs with small second largest eigenvalue in modulus. The largest eigenvalue also reveals its vital role in modeling virus propagation in computer networks. The robustness of a network against the spread of viruses is inversely proportional to the largest eigenvalue. This result has been proved in a paper by Wang [25]. The eigenvalues of the graph G are the eigenvalues of $A(G)$ and the *spectrum* of G is the multiset of eigenvalues of G. In spectrum more focus is now on finding nullity of graphs and energy of graphs.

1.4.1 Energy of Graphs

Energy of a simple graph on n vertices with eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ is defined as

$$E(G) = \sum_{i=1}^{n} \mid \lambda_i \mid .$$

The concept of energy of graphs was given by Ivan Gutman [14] in 1978. But the motivation to work on this concept came earlier when Huckel Molecular orbital theory was proposed [11]. Huckel method allows chemists to approximate energies associated with Y electron orbitals in a special class of molecules called conjugated hydrocarbons. As late as 1956, Gunthard and Primas [12] realized that the matrix used in the Huckel method is a first-degree polynomial of the adjacency matrix of a certain graph related to the molecule being studied. With certain assumptions about the molecule, it can be said that its total Y electron energy is the sum of the absolute eigenvalues of this graph. In the last some years not only energy but its different versions are being studied. Some results are presented below which give information about the minimal energy graphs.

Theorem 1.4.1 (Gutman [16]). *Let G be a graph with m edges, then energy of graph $E(G)$ can be given as inequality,*

$$E(G) \geq 2\sqrt{m},$$

with the equality attained if and only if G is a complete bipartite graph with some isolated vertices.

Theorem 1.4.2 (Gutman [16]). *Let G be a graph of order n with no isolated vertices, then*

$$E(G) \geq 2\sqrt{n-1}.$$

The equality holds if and only if G is the star graph S_n.

Incidence matrix of a graph G of n vertices and m edges is $n \times m$ vertex edge matrix whose ij element is equal to 1 if the vertex v_i is incident to the edge e_j, and is equal to 0 otherwise. It is denoted by $I(G)$ or simply I. The *singular values* μ of a real matrix $I(G)$ (not necessarily square) are the square roots of the eigenvalues of the matrix II^T, where I^T denotes the transpose of I. Jooyandeh [18] introduced *incidence energy* denoted by $IE(G)$, is defined as the sum of the singular values of the incidence matrix of G, that is,

$$IE(G) = \sum_{i=1}^{n} \mu_i$$

Lemma 1.4.3 (Jooyandeh [18]). *Let G be a graph with n vertices and m edges. Then*

- *$IE(G) \geq 0$ and equality holds if and only if $m = 0$*
- *if $G_1, G_2, \ldots, G_p$ are all components of G then*

$$IE(G) = \sum_{i=1}^{p} IE(G_i).$$

In 2006, Gutman and Zhou [15] defined the Laplacian energy of a graph as the sum of the absolute deviations, that is, distance from the mean of the eigenvalues of its Laplacian matrix. A *degree matrix* corresponding to a graph G is the diagonal matrix formed from the vertex degrees in the diagonal. The *laplacian matrix* of a simple graph G of order n, is an $n \times n$ symmetric matrix with one row and column for each node defined by $L = D - A$, where D is the degree matrix and A is the adjacency matrix. The element $l_{(ij)}$ of L is -1 if the vertex v_i is adjacent to v_j for $i \neq j$ and 0 otherwise where ij represents the position of entry in the matrix.

1.4.2 Rank of Graphs

The *rank* of a graph G, denoted by $\text{rank}(G)$, is the rank of its adjacency matrix $A(G)$, that is, $\text{rank}(G) = \text{rank}(A(G))$. The *nullity* of graph G, denoted by $\eta(G)$, is the multiplicity of the eigenvalue zero in the spectrum of G. The graph G is *singular* if $\eta(G) > 0$ and is *non-singular* if $\eta(G) = 0$. It is known that $\eta(G) = n - \text{rank}(G)$. The *nullity set* of a family of graphs $\mathtt{F}$ is a set S subset of $\{0, 1, 2, \ldots, n\}$ such that for each $k \in S$ there exists a graph G of order n in $\mathtt{F}$ with $\eta(G) = k$ and conversely, for each graph G of order n in $\mathtt{F}$, $\eta(G) \in S$. Nullity of graphs is further discussed in chapter 2 in detail.

Chapter 2

Nullity of Graphs

This chapter includes the introduction to nullity of graphs and its some main problems. It also highlights the origin, background and applications of nullity of graphs. It also provides nullity of bipartite graphs and its direct impact on other fields.

2.1 Introduction and History

The nullity of a graph has great importance in mathematics because it has relevance to a singularity of adjacency matrix. For a bipartite graph G which is analogous to a variant hydrocarbon, if $\eta(G) > 0$ it depicts that the corresponding molecule is not stable. This problem is not fully resolved. Presently the researchers are mainly concentrating on graphs with fewer edges for example, tree, unicyclic and bicyclic graphs. Attention has not only paid to find nullity of graphs but also to categorize graphs with maximum nullity within each family. It is the consequence of such efforts that it is already a known fact that a graph G of order n with atleast one edge has maximum nullity $n-2$. A brief history of work done on nullity of graphs is given below.

2.1.1 Nullity of Trees

Before proceeding towards the nullity of trees, some definitions are required to be mentioned. An acyclic connected graph is called a *tree* and is denoted by T. The set consisting of all trees of order n is denoted by T_n. A *link* is an edge which is not a loop. A *matching* of graph G is a subset of edge set of G where each edge is a link and every two edges are mutually non adjacent edges of G. A matching in which every vertex of the graph is incident to exactly one edge of the matching is *perfect matching.* A *maximum matching* is a matching with the maximum possible number of edges. Many maximum matchings for a graph can exist. The size of a maximum matching, that is, the maximum number of mutually non adjacent edges of G is called *matching number* and is denoted by $m = m(G)$.

Theorem 2.1.1 (Cvetokovic [4])**.** *For a tree T of order n with maximum matching of size m, we have, $\eta(T) = n - 2m$.*

Corollary 2.1.2 (Jianxi [17])**.** *Let T be a tree of order n. Then $\eta(G) = 0$ if and only if T has a perfect matching.*

By above results it can be easily deduced that nullity of tree can be described explicitly in terms of its matching number. Furthermore, trees which are isomorphic to stars give maximum nullity.

Theorem 2.1.3 (Ellingham [8])**.** *For $T \in T_n$, $\eta(T) \leq n - 2$ and equality holds if and only if $T \cong S_n$.*

2.1.2 Nullity of Uniyclic and Bicyclic Graphs

The way to obtain nullity of a cycle is rather simple and discrete.

Lemma 2.1.4 (Schwenk [23])**.** *Nullity of a cycle C of length n is $\eta(C_n) = 2$ if $n \equiv 0(mod4)$ and $\eta(C_n) = 0$ otherwise.*

A *unicyclic graph* is a simple connected graph whose number of vertices and edges are same. The set consisting of all unicyclic graphs of order n is denoted by U_n. Firstly, Tan and Liu [20] published results on nullity of unicyclic graphs. They found the nullity set for U_n, $n \geq 5$, that is, $\eta(U_n) = \{0, 1, \ldots, n-4\}$. They also characterized the unicyclic graphs with maximum nullity and left open problem of characterizing the graphs with minimum nullity. Recently Guo et. al [10] have found a way to obtain the nullity of unicyclic graphs in terms of their maximum matching numbers.

Theorem 2.1.5 (Guo [10]). *For $G \in U_n$ with a size of matching number m, $\eta(G) = n - 2m - 1$ or $\eta(G) = n - 2m$ or $\eta(G) = n - 2m + 2$.*

It is a well established fact that unicylic graphs with pendent edges give maximum nullity among their family.

A *bicyclic graph* is a simple connected graph whose number of vertices is one more than its number of edges. The set consisting of all bicyclic graphs of order n is denoted by B_n. Jian et. al [17] divided bicyclic graphs into further three categories and obtained the nullity separately for each category. A generalized result taking all bicyclic graphs into account or consideration has not been formulated yet.

2.2 Preliminaries on Nullity

In this section we present theorems related to nullity of graphs and our focus is on bipartite graphs. Most work on nullity of bipartite graphs is done by Fan and Qian [9].

Theorem 2.2.1 (Torgasev [24]). *A graph G is bipartite if and only if its eigenvalues are symmetric with respect to 0, that is, if λ is an eigenvalue with multiplicity k then $-\lambda$ is also an eigenvalue with multiplicity k.*

Theorem 2.2.2 (Cheng [2]). *For a simple graph G on n vertices, $n \geq 2$ and no isolated vertices $\eta(G) = n-2$ if and only if $G \cong K_{n_1,n_2} + kK_1$, where $n_1+n_2+k = n$, $n_1, n_2 > 0$, and $k \geq 0$.*

Theorem 2.2.3 (Cheng [2]). *For a simple graph G on n vertices where $n \geq 3$, $\eta(G) = n - 3$ if and only if $G \cong K_{n_1,n_2,n_3} + kK_1$, where $n_1 + n_2 + n_3 + k = n, n_1, n_2, n_3 > 0$, and $k \geq 0$.*

Lemma 2.2.4 (Cheng [2]). *Suppose $G = \cup_{i=1}^{k} G_i$, $i = 1, 2, \ldots, k$ where G_i are connected components of graph G. Then the nullity of graph G is given by:*

$$\eta(G) = \sum_{i=1}^{k} \eta(G_i) \tag{2.2.1}$$

Lemma 2.2.5 (Collatz [3]). *Let G be a graph containing a vertex v whose degree is 1. The nullity of induced subgraph H obtained by deleting v along with the vertex adjacent to it will be same as nullity of G.*

2.2.1 Expanded Paths and Cycles

Let G be a graph on vertices $v_1, v_2, \ldots, v_n$ and let $x \in \mathbb{R}^n$ be a column vector. Then x can be defined as a function on the vertices of G by mapping each v_i to the value x_i, that is, $x(v_i) = x_i$. Using this function, the eigenvector equation $A(G)x = \lambda x$ can be restated as

$$\lambda x(u) = \sum_{v \epsilon N(u)} x(v) \quad \text{for each} \quad u \in V(G). \tag{2.2.2}$$

The graph G is said to be an *expanded graph* if its vertex set $V(G)$ can be partitioned into $V_1, V_2 \ldots, V_k$, $k \geq 2$, such that $G[V_i]$ is an empty graph, for $1 \leq i \leq k$ and if $G[V_i \cup V_j]$ is a nonempty graph, it is a complete bipartite graph for $1 \leq i, j \leq k$, $i \neq j$. If G is an expanded graph on $V_1, V_2, \ldots, V_k$, each V_i, for $1 \leq i \leq k$, is called an *expanded vertex* of order $|V_i|$. We observe that each simple graph can be viewed as an expanded graph.

The n-vertex graph G is said to be an *expanded path* of length k if its vertex set $V(G)$ can be partitioned into $V_1, \ldots, V_k$, $k \geq 2$, such that

(i) $G[V_i]$ is an empty graph for $1 \leq i \leq k$,

(ii) $G[V_i \cup V_{i+1}]$ is a complete graph for $1 \leq i \leq k-1$,

(iii) $G[V_i \cup V_j]$ is an empty graph for $1 \leq i, j \leq k$ with $j \neq i+1$.

We use the notation $\mathbb{P}_k(V_1, \ldots, V_k)$ to denote an expanded path on $V_1, \ldots, V_k$ of length k. Similarly, an *expanded cycle* of length k, denoted by $\mathbb{C}_k(V_1, \ldots, V_k)$, is obtained from the expanded path $\mathbb{P}_k(V_1, \ldots, V_k)$ by adding edges between each vertex of V_1 and each of V_k. When there is no ambiguity, we simply write $\mathbb{P}_n$ and $\mathbb{C}_n$ to represent an expanded path and an expanded cycle of length n, respectively. An *expanded decomposition* of the graph G is a list of expanded subgraphs such that each edge of G appears in exactly one expanded subgraph in the list.

Theorem 2.2.6 (Cheng [2]). *For a graph G of order n,*

(i) $\eta(G) = n - 2$ *if and only if G is an* an expanded path *of length* 2 *possibly with some isolated vertices,*

(ii) $\eta(G) = n - 3$ *if and only if G is* an expanded cycle *of length* 3 *possibly with some isolated vertices.*

Proof. Firstly, consider an expanded path of length 2. Since such an expanded path forms the structure of complete bipartite graph along with some isolated vertices. So, G can be described as $G \cong K_{n_1,n_2} + kK_1$. By Theorem 2.2.2 result follows. Secondly, for an expanded cycle of length 3, clearly $G \cong K_{n_1,n_2,n_3}$ and its nullity can be established by Theorem 2.2.3. Hence the result. □

The next lemma provides a situation in which nullity of a graph and its subgraph remains same.

Lemma 2.2.7 (Fan [9]). *Let G be a simple graph, which contains two vertices u, w such that $\phi \neq N(u) \subseteq N(w)$. Let H be a subgraph obtained from G by deleting those edges of G which join w and vertices of $N(u)$. Then $\eta(G) = \eta(H)$.*

Proof. Since $N(u) \subseteq N(w)$, this assumption does not allow u and w to be adjacent. It also indicates that $A(G)_{uw} = 0$. Suppose x is an eigenvector of $A(G)$ corresponding to a zero eigenvalue. By using the equation 2.2.2 we can say,

$$\sum_{v \in N(u)} x(v) = 0 \quad \text{and} \quad \sum_{v \in N(w)} x(v) = 0.$$

By using both equations in such a way that common part of their neighbourhood is excluded to form the equality $\sum\limits_{v \in N(w) \setminus N(u)} x(v) = 0$. Now we construct a vector $x^{'}$ as

$$x'(u) = x(u) + x(w) \quad \text{and} \quad x'(v) = x(v) \quad \text{for} \quad v \neq u.$$

For the remaining vertices of G other than u and w, the behaviour of x' is same by its definition $x'(v) = x(v)$ for $v \neq u$. It implies x' is an eigenvector of H corresponding to a zero eigenvalue. Conversely, suppose y is an eigenvector of H corresponding to zero eigenvalue. Then define y' vector as $y'(u) = y(u) - y(w)$. Repeating the same line of actions in the reverse order as above, it leads to the conclusion that y' is an eigenvector of G corresponding to a zero eigenvalue. It implies nullity of graph G and H is same. Hence the result. □

Corollary 2.2.8 (Fan [9]). *An expanded path $\mathbb{P}$ of length $n \geq 2$ and order $\sum_{i=1}^{n} |V_i|$ has nullity $\eta(\mathbb{P}) = \sum_{i=1}^{n} |V_i| - n$, if n is even and $\eta(\mathbb{P}) = \sum_{i=1}^{n} |V_i| - n + 1$, if n is odd.*

Proof. Consider an expanded path $\mathbb{P}$ of length n and suppose $n \geq 2$. We want to find nullity of this expanded path $\mathbb{P}$. This corollary is proved by using induction method separately for even and odd length of paths. Clearly, the result holds for $n = 2$, that is, $\eta(G) = \eta(V_1V_2) = |V_1| + |V_2| - 2$ as by Theorem 3.2.1. Suppose that the required result is true for all $2k$, where k is any natural number, given as

$$\eta(G) = \eta(V_1V_2 \cdots V_{2k}) = |V_1| + |V_2| + \ldots + |V_{2k}| - 2k. \tag{2.2.3}$$

Now we prove it for $n = 2(k+1)$. For $G = V_1V_2 \cdots V_{2k}V_{2k+1}V_{2k+2}$, $N(v_{2k}) = V_{2k-1} \cup V_{2k+1}$ and $N(v_{2k+2}) = V_{2k+1}$ which means $N(v_{2k+2}) \subseteq N(v_{2k})$. By Lemma 2.2.7 deleting those edges joining v_{2k} and $Vm_{2k+1} = N(v_{2k+2})$ the resulting graph will preserve the nullity. By repeating the same operation on remaining vertices $v_i \in V_{2k}$, we get a graph $H = V_1V_2V_3 \cdots V_{2k} \cup V_{2k+1}V_{2k+2}$. It means a disjoint component of a graph can be created. Now finding the nullity of graph H as

$$\eta(H) = \eta(V_1V_2 \cdots V_{2k} \cup V_{2k+1}V_{2k+2}),$$

with the help of Lemma 2.2.4 nullity of disjoint components can be added as given below.

$$\begin{aligned}\eta(H) &= \eta(V_1V_2 \cdots V_{2k}) + \eta(V_{2k+1}V_{2k+2}),\\ &= |V_1| + |V_2| + \ldots + |V_{2k}| - 2k + |V_{2k+1}| + |V_{2k+2}| - 2,\\ &= |V_1| + |V_2| + \ldots + |V_{2k+1}| + |V_{2k+2}| - 2(k+1).\end{aligned}$$

Therefore this result is true for all $2n \in \mathbb{N}$. Suppose that the result is true for all $2k-1$, where k is any natural number greater than 2, that is, $\eta(G) = \eta(V_1V_2 \cdots V_{2k-1}) = |V_1| + |V_2| + \ldots + |V_{2k-1}| - 2(k-1) + 1$. Now we prove it for $n = 2k+1$. For $G =$

$V_1V_2\cdots V_{2k}V_{2k+1}$, $N(v_{2k-1}) = V_{2k-2} \cup V_{2k}$ and $N(v_{2k+1}) = V_{2k}$ implying $N(v_{2k+1}) \subseteq N(v_{2k-1})$. On the same lines by using Lemma 2.2.7 deleting those edges joining v_{2k-1} and $Vm_{2k} = N(v_{2k+1})$, a disjoint component of G is obtained, call it H. Adding nullity of components of G below.

$$\begin{aligned}\eta(H) &= \eta(V_1V_2\cdots V_{2k-1}) + \eta(V_{2k}V_{2k+1}),\\ &= |V_1| + |V_2| + \ldots + |V_{2k-1}| - 2(k-1) + 1 + |V_{2k}| + |V_{2k+1}| - 2,\\ &= |V_1| + |V_2| + \ldots + |V_{2k+1}| - (2k+1) + 1.\end{aligned}$$

Therefore this result is true for all $2(n-1) \in \mathbb{N}$ where $n \geq 2$. □

Lemma 2.2.9 (Fan [9]). *Let G be a simple graph containing an expanded path $\mathbb{P} = V_1, V_2, \cdots V_6$ where V_1 and V_6 are origin and terminus vertices of $\mathbb{P}$, respectively, and $V_2, V_3, \cdots V_5$ are adjacent to no vertex other than in $\mathbb{P}$. Then $\eta(G) = \eta(H) + |V_2| + |V_3| + |V_4| + |V_5| - 4$ where H is obtained by replacing $V_1, V_2, \cdots V_6$ with V_1V_6.*

Proof. Firstly, proving this theorem for specific values of $|V_i|$, that is, $|V_i| = 1$ for $i = 2, 3, \ldots, 5$ and call the resultant graph G_1. Let x be an eigenvector of $A(G_1)$ corresponding to the zero eigenvalue. Let

$$\sum_{v \epsilon V_1} x(v) =: \alpha, \sum_{v \epsilon V_{m_1}} x(v) =: \beta. \tag{2.2.4}$$

For μ_2 : $\sum_{v\epsilon V_1} x(v) + x(\mu_3) = 0,\ \ \alpha + x(\mu_3) = 0,\ \ x(\mu_3) = -\alpha.$
For μ_4 : $x(\mu_3) + x(\mu_5) = 0, \quad x(\mu_5) = \alpha.$
For μ_5 : $\sum_{v\epsilon V_6} x(v) + x(\mu_4) = 0, \quad \beta + x(\mu_4) = 0, \quad x(\mu_4) = -\beta.$
For μ_3 : $x(\mu_2) + x(\mu_4) = 0, \quad x(\mu_2) = \beta.$
Let x' be a subvector of x obtained by eliminating the vertices $\mu_2, \mu_3, \mu_3, \mu_4, \mu_5$ alongwith their coressponding values. Then clearly x' becomes an eigenvector of $A(H)$ corresponding to the zero eigenvalue where $H = V_{m_1}V_{m_6}$.

Conversely, Suppose x' is an eigenvector of $A(H)$ corresponding to the zero eigenvalue and holds equation 2.2.4. Then we can expand this vector to a vector x by including $\mu_2, \mu_3, \mu_4, \mu_5$ vertices into its definition. Then x is an eigenvector of $A(G_1)$ corresponding to the zero eigenvalue. So, nullity of H and G_1 is same. This result can be proved for $i = 2, 3, 4, 5$ by using equation $A(G)x = 0$ as the vertices within Vm_i correspond to it. It means $\eta(G) = \eta(H) + m_2 + m_3 + m_4 + m_5 - 4$. □

Corollary 2.2.10 (Fan [9]). *An expanded cycle $\mathbb{C}$ of length $n \geq 3$ and order $\sum_{i=1}^{n} |V_i|$ has nullity $\eta(\mathbb{C}_n) = \sum_{i=1}^{n} |V_i| - n + 2$, if $n \equiv 0$ mod(4) and $\eta(\mathbb{C}_n) = \sum_{i=1}^{n} |V_i| - n$, if $n \not\equiv 0$ mod(4).*

2.3 Nullity of Bipartite Graphs

The family consisting of all bipartite graphs of order n is denoted by $\mathfrak{B}_\mathfrak{n}$. Firstly we obtain nullity set for this family.

Theorem 2.3.1 (Fan [9]). *The nullity set for $\mathfrak{B}_\mathfrak{n}$ is $S = \{n-2k : k = 0, 1, \ldots, \lfloor n/2 \rfloor\}$.*

Proof. First we prove that for each nullity $l \in S$ (say), there exists a graph G in $\mathfrak{B}_\mathfrak{n}$. For $\eta(G) = n - 2k$ where $k = \{0, 1, \ldots, [n/2]\}$ there exists an expanded path of length $2k(0 \leq k \leq \lfloor n/2 \rfloor)$, that is, $\mathbb{P}_{2k}$ such that $\eta(G) = n - 2k$. Conversely, for a graph G of order n in $\mathfrak{B}_\mathfrak{n}$ $A(G) = \begin{pmatrix} \mathbf{0} & B \\ B^T & \mathbf{0} \end{pmatrix}$.

So $\eta(G) = n - 2(\text{rank}(B))$, where $\text{rank}(B) = k$ (say). It proves S to be the nullity set for $\mathfrak{B}_\mathfrak{n}$. □

Theorem 2.3.2 (Fan [9]). *Suppose G is a bipartite graph of order $n \geq 4$. The nullity of graph G is $n - 4$ if and only if G is a graph H possibly adding some isolated vertices, where H is one of the following graphs:*

- *a union of two disjoint expanded paths of length 2,*
- *an expanded path of length 4,*
- *an expanded path of length 5.*

Proof. First, we show the sufficiency of this theorem. If $H \cong \mathbb{P}_{m_1,m_2} \cup \mathbb{P}_{n_1,n_2}$ where m_i and n_i are order of expanded vertices of first and second path with $i = 1, 2$, respectively. Then $\eta(H) = \eta(\mathbb{P}_2) + \eta(\mathbb{P}_2)$ by Lemma 2.2.4. Since $\eta(\mathbb{P}_2) = n - 2$ so $\eta(H) = n - 4$. If $H \cong \mathbb{P}_4$ or $H \cong \mathbb{P}_5$ then by the Corollary 2.2.8 $\eta(H) = n - 4$.

Now, to prove the necessity we take $A(G) = \begin{pmatrix} \mathbf{0} & B \\ B^T & \mathbf{0} \end{pmatrix}$ into consideration. Here the vertices of bipartition sets X, Y of graph G are used to indicate row index and column index of B submatrix. In other words, for $u \in X$ and $v \in Y$, $B_{uv} = A(G)_{uv}$. For a vertex $u \in X$, its corresponding row vector in B is denoted by B_{uY}. By our supposition $\eta(G) = n - 4$ and we know $\eta(G) = n - rank(A(G))$ and rank$(A(G)) = 2 =$ rank(B). So rank$(B) = 2$, that is, B contains two linearly independent rows, call them B_{uY} and B_{vY}. Obviously, $B_{uY} \neq 0 \neq B_{vY}$ which means $N(u) \neq 0 \neq N(v)$. Then there exists $q \in Y$ such that $B_{uq} \neq B_{vq}$. Without loss of generality, assume $B_{uq} = 1, B_{vq} = 0$. To construct the graph G we consider the following cases:

Case 1 : $N(u) \cap N(v) = \phi$. For any vertex $w \in X$, $B_{wY} = k_1 B_{vY} + k_2 B_{uY}$. Clearly $(k_1, k_2) \subseteq \{(0,0), (0,1), (1,0), (1,1)\}$. These values of k, provide following pairs of vectors and resultant neighbourhoods of w as: $\mathbf{0}$ implies $N(w) = \phi$ or isolated vertex, B_{uY} implies $N(w) = N(u)$, B_{vY} implies $N(w) = N(v)$, $B_{uY} + B_{vY}$ implies $N(w) = N(u) \cup N(v)$. Therefore, G is a graph shown in the Figure 2.1 where U_2 and or U_5 may be empty.

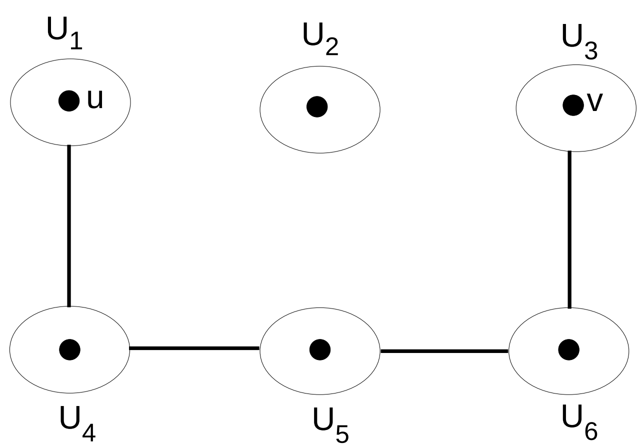

Figure 2.1: Structure of graph for Case 1 of Theorem 2.3.2

Case 2 : $N(u) \cap N(v) \neq \phi$. For any vertex $w \in X$ any dependent row in B is denoted by $B_{wY} = k_1 B_{vY} + k_2 B_{uY}$. Clearly $(k_1, k_2) \neq (1, 1)$. Here $(k_1, k_2) \subseteq$

$\{(0,0),(0,1),(1,0),(1,-1)\}$. These possible paired values of (k_1, k_2) leads to the following vectors and neighbourhoods as $\mathbf{0}$ implies $N(w) = \phi$ or isolated vertex, B_{uY} implies $N(w) = N(u)$, B_{vY} implies $N(w) = N(v)$, $B_{uY} - B_{vY}$ implies $N(w) = N(u) \setminus N(v)$ which means $N(v) \subset N(u)$.

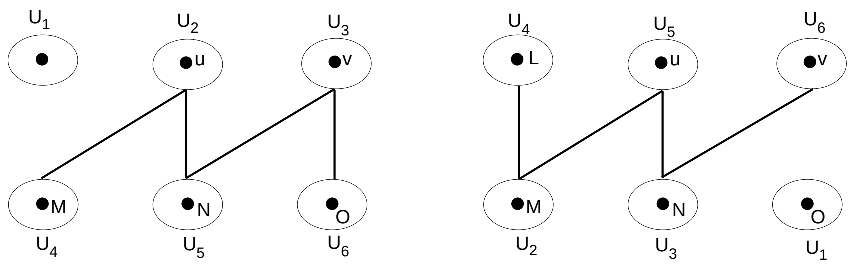

Figure 2.2: Structure of graphs for Case 2 of Theorem 2.3.2

Now we consider both possibilities of neighbourhood of v, sharing only some part or otherwise completely contained in the neighbourhood of u. If $N(v) \nsubseteq N(u)$, then Y can be partitioned into three disjoint subsets M, N, O such that $N(u) = M \cup N$ and $N(u) = N \cup O$ and it means $N(w) \neq N(u) \setminus N(v)$. The reason behind it is that both neighbourhoods share some common vertices instead of one neighbourhood completely lying in the other A graph whose vertices exhibits such a behaviour is shown in the in the left part of Figure 2.2 where U_1 may or may not be empty. For the latter case, suppose $N(v)$ is completely contained in $N(u)$. In such a case Y is denoted by $M \cup N$ where M and N are disjoint and the nature of these subsets can be explained by $N(u) = M \cup N$ and $N(v) = E$. This structure of graph exhibits all possible values of (k_1, k_2) even $(1, -1)$ which was not possible in the former case, since one neighbourhood can exclude the other from it and corresponds to the expanded vertex L of the right graph Figure 2.2. Therefore G is the graph in the right portion of the Figure 2.2 where U_4 and or U_1 may be empty. $\square$

2.3.1 Nullity of Bipartite Graphs of Order ≥ 6

In this section all the graphs whose order is more than 6 are taken into consideration. Definite results are produced regarding their structures and corresponding nullities.

Firstly all the bipartite graphs of order 6 with zero nullity are presented here.

Lemma 2.3.3 (Fan [9])**.** *For a graph G of order 6, G is isomorphic to one of the following graphs shown in the Figure 2.3 if and only if $\eta(G) = 0$.*

Proof. First, we show that all graphs under consideration mentioned in the Figure 2.3 have nullity zero. Consider G_4 and G_5, as they contain isolated vertices by applying Lemma 2.2.5 they can be reduced to expanded path and expanded cycle, respectively. For expanded cycles G_5, G_6 and G_7 their nullity is found to be zero by Corollary 2.2.10. With the help of Corollary 2.2.8 the nullity of $G_1, G_2, \ldots, G_5$ can be obtained. So, by doing these little calculations and applying corollaries, nullity of all graphs under consideration is found to be zero as required.

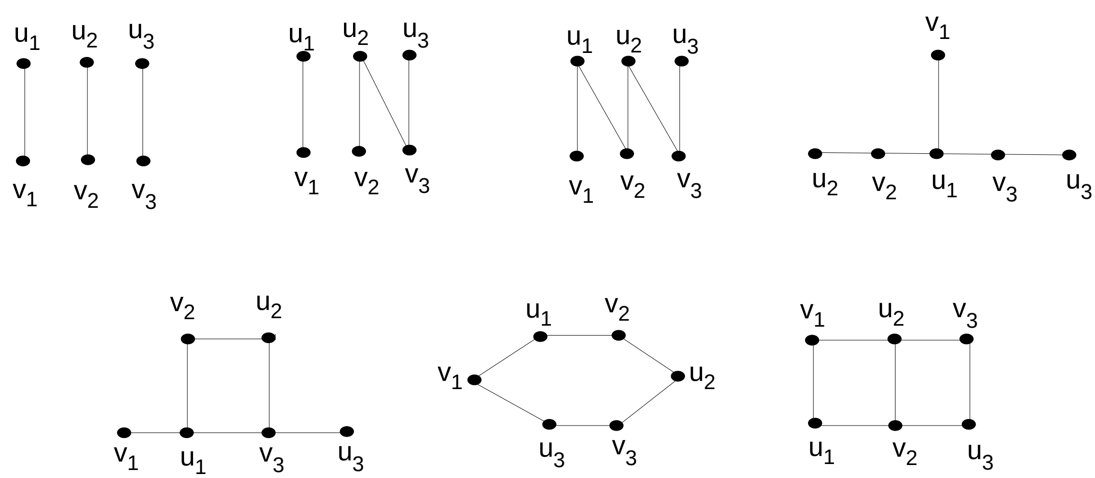

Figure 2.3: Graphs of order 6 with zero nullity of Theorem 2.3.3

Conversely, suppose nullity of graph G is zero, that is, $A(G)$ is a full rank matrix and so rank$(B) = 3$. Therefore, $|X| = 3 = |Y|$. To obtain required results for this non singular graph, the discussion is further divided into cases depending upon the graph containing a vertex of degree 1 or greater than 1.

Case 1 : For a graph G, if there exist $v_1 \in G$ such that $d(v_1) = 1$ and $N(v_1) = u_1$, (say). By using Lemma 2.2.5 delete v_1 and its adjacent vertex u_1. For the resultant subgraph H of G, nullity remains zero. Since G is a bipartite graph so nullity of H

can be written as $n-4$ so by Theorem 2.3.2 H is an expanded path of length 4 or two disjoint expanded paths of length 2. Then G is isomorphic to one of $G_1, G_2, \ldots, G_5$.

Case 2 : Suppose G contains a vertex u such that $|N(u)| = 2$ and no isolated vertex. If G also does not contain any vertex of degree three then it is a cycle of length six. And now consider G which contains vertices such that for any $v_i \in V(G)$, $|N(v_i)| = 2$ or 3. Then G contains even number of vertices whose degree is three. If G has only one vertex of degree three, then it will also contain a pendent vertex and it becomes a graph of case 1. Other than that, by the definition of G, it cannot contain three or five vertices of degree three. A partite set of G cannot have more than one vertex of degree three because in this way all of those vertices will have the same neighbourhood. It produces a graph with nullity more than zero as same rows reduces *rank* which is a contradiction to our assumption. Another way to show that such a graph will have some positive number as nullity is by applying Lemma 2.2.7 repeatedly on vertices of degree three. So, each partite set of G contains exactly one vertex of degree 3. Therefore $G \cong G_7$ or $G \cong G_8$. Hence the result. □

Theorem 2.3.4 (Fan [9]). *For a regular graph G of order $n \geq 6$, $\eta(G) = n - 6$ if and only if G is one of the graphs mentioned below.*

- *a union of three disjoint expanded paths of length* 2
- *an expanded path of length* 6
- *an expanded path of length* 8

where expanded vertex of each graph has same order.

Proof. ($\Longrightarrow$) If G is any of the above given graph then by Lemma 2.2.4 and Corollary 2.2.8 its nullity can be easily calculated and given as $\eta(G) = n - 6$.

($\Longleftarrow$) Suppose $\eta(G) = n - 6$ and $A(G) = \begin{pmatrix} \mathbf{0} & B \\ B^T & \mathbf{0} \end{pmatrix}$, where the partite sets X, Y of graph are represented by the subsets of partition of matrix. By the given nullity, $\text{rank}(A(G)) = 6$ implying $\text{rank}(B) = 3$. Let $B_{u_1Y}, B_{u_2Y}, B_{u_3Y}$ be the linearly independent rows of B corresponding to $u_1, u_2, u_3 \in X$, respectively. By Lemma 2.3.3 there exists a graph of order 6 with zero nullity. It can be observed that it is possible to construct such a subgraph of G and call it H. As both graphs H and G contain same linearly independent rows so by using former graph the latter can be constructed. Let $X' \subseteq X$ and $Y' \subseteq Y$ be the bipartite sets of H. Take $X' = \{u_1, u_2, u_3\}$ and $Y' = \{v_1, v_2, v_3\}$. Suppose the adjacency matrix for the graph H is $\begin{pmatrix} \mathbf{0} & D \\ D^T & \mathbf{0} \end{pmatrix}$, where the row and column indices of matrix D to the vertices of X' and Y', respectively. As H is possibly one of the graphs from Lemma 2.3.3 so constructing their adjacency matrices in the same order as given in the Figure 2.3 as D would be one of the following:

$A(G_1) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, $A(G_2) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$, $A(G_3) = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$,

$A(G_4) = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, $A(G_5) = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$, $A(G_6) = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}$,

$A(G_7) = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}$.

Any dependent vertex $v' \in X$ will correspond to a row of the form

$$B_{u'Y} = k_1 B_{u_1Y} + k_2 B_{u_2Y} + k_3 B_{u_3Y}, \tag{2.3.1}$$

where k_1, k_2, k_3 are real numbers. Now left multiplying the equation with all-ones vector on both sides.

$$B_{u'y_1} + B_{u'y_2} + \ldots + B_{u'y_n} = k_1[B_{u_1'y_1} + B_{u_1'y_2} + \ldots + B_{u_1'y_n}] +$$
$$k_2[B_{u_2'y_1} + B_{u_2'y_2} + \ldots + B_{u_2'y_n}] + k_3[B_{u_3'y_1} + B_{u_3'y_2} + \ldots + B_{u_3'y_n}]$$

Since G is m regular graph (*say*) so its sum of one's in each row is same, that is,

$$m = mk_1 + mk_2 + mk_3,$$

$$k_1 + k_2 + k_3 = 1. \tag{2.3.2}$$

As the independent rows for both matrices B and D are same so from equation 2.3.2, it can be written as

$$D_{u'Y'} = k_1 D_{u_1Y'} + k_2 D_{u_2Y'} + k_3 D_{u_3Y'}.$$

Define a set K as, $K = \{(k_1, k_2, k_3) | D_{u'Y'} = k_1 D_{u_1Y'} + k_2 D_{u_2Y'} + k_3 D_{u_3Y'}, u^{'} \in X\}$ The set containing possible combinations for (k_1, k_2, k_3) with partite sets $(X, Y^{'})$ can be calculated by equation (2.3.2) and stated as

$$K \subseteq \{(1,0,0), (0,1,0), (0,0,1), (1,-1,1), (1,1,-1)\}. \tag{2.3.3}$$

Now efforts are made to derive and figure out the structure of the graph possibly by finding the nature of its vertices and their neighbourhoods by using the independent rows of B. If $N(u_1), N(u_2), N(u_3)$ contain a common vertex, call it $v^{'}$. By equation (2.3.3) every vertex of X is adjacent to $v^{'}$ because $B_{u_1v'} = B_{u_2v'} = B_{u_3v'} = 1$ so by using any possible paired value from K will produce 1 as entry. Since degree of each vertex of G is same so each vertex of Y has an edge with all vertices of X. Such a structure corresponds to a complete bipartite graph which is not possible as by Theorem 2.3.2 its nullity is $n-2$. It indicates that D is not the fifth or seventh matrix. If every dependent vertex $u^{'} \in X$ has neighbourhood of the form that $N(u^{'})$ is exactly same as one of $N(u_1), N(u_2), N(u_3)$ then D will be the sixth matrix and (k_1, k_2, k_3) will take values $(1,0,0), (0,1,0), (0,0,1)$. Here all the dependent vertices with same neighbourhood will represent a same expanded vertex. Therefore, G is an expanded cycle of length 6 and order of each expanded vertex is same as G is regular. It can be seen that G represents one of our required structure of graph. Now assume that D is one of the first four matrices. For such a possibility suppose $v_1 \in N(u_1) \setminus N(u_2) \cup N(u_3)$. Now further making subcases under the assumption that neighbourhoods of u_1, u_2, u_3 are disjoint.

Case 1 : $N(u_1) \cap N(u_2) \neq \phi$. For such a case $(1,1,-1)$ combination is not possible because it will produce 2 entry in the matrix. In the similar manner assumption can be made if $(1,-1,1) \in K$. It implies $K \subset \{(1,0,0), (0,1,0), (0,0,1)\}$. If x contains a vertex q such that q and v_1 has an edge, then $B_{qY} = B_{v_1Y}$. In other words $B_{qp} = B_{v_1p} = 1$ meaning that q and p also has edge between them. It also indicates that u_2 has an edge with p but not with v_1, as v_1 is not contained in $N(u_2)$.

Hence $d(v_1) < d(p)$ which is a contradiction as G is regular. So, $(1, -1, 1) \in K$ and X must contain a vertex r given by

$$B_{rY} = B_{u_1Y} - B_{u_2Y} + B_{u_3Y}. \tag{2.3.4}$$

This dependent row clearly shows that $N(u_2) \subseteq N(u_1) \cup N(u_3)$ and also that $N(r) = N(u_1) \cup N(u_3) \setminus N(u_2)$. Since v_1 is not contained in either of $N(u_2)$ or $N(u_3)$ and by row B_{rY} we can say $v_1 \in N(r) \cap N(u_1)$. Hence their intersection is not empty. From equation 2.3.4 $B_{rp} = B_{u_1p} - B_{u_2p} + B_{u_3p} = 1 - 1 + 1 = 0$ where for $p \in N(u_1) \setminus N(r)$. As it shows that there exist a vertex in $N(u_1)$ but not in $N(r)$ but $d(r) = d(u_1)$ so r must joins some vertex except of $N(u_1)$ and hence in $N(u_3)$. It implies $N(u_3) \cap N(r) \neq 0$. In a similar manner $N(u_2) \cap N(u_3) \neq 0$ since $v_1 \in N(u_1) \setminus N(u_2)$ leading to $N(u_2) \subset N(u_1)$ which will be contradicting the fact that G is a regular graph. Therefore vertex set of such a graph can be partitioned by carefully examining the independent rows, existing intersections of those rows and the possible dependent rows as well the intersections of both dependent and independent rows. $V(G) = \{U_1, U_2, \ldots, U_8\}$ where U_1 : contains v_1, U_2 : contains $v_1 \in N(u_1) \cap N(r)$, U_3 : contains r, U_4 : contains $N(u_3) \cap N(r)$, U_5 : contains u_3, U_6 : contains $N(u_2) \cap N(u_3)$, U_7 : contains u_2, U_8 : contains$N(u_1) \cap N(u_2)$ such that for each U_i and U_j their exists complete mapping for $i - j \equiv 1(mod 8)$ where $i, j = 1, 2, \ldots, 8$ as shown in the Figure 2.4.

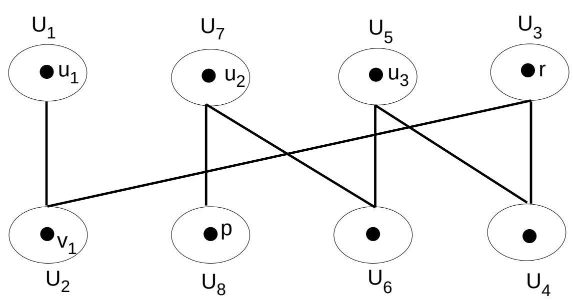

Figure 2.4: Graph for the Case 1 and Case 2 for Theorem 2.3.4

Case 2 : $N(u_1) \cap N(u_3) \neq \phi$. This case represents the same situation as Case 1 but the vertex u_2 is interchanged with u_3 where u_1 remains same. Therefore the graph representing this case will also be same as above graph in Figure 2.4.

Case 3 : $N(u_1) \cap (N(u_2) \cup N(u_3)) = \phi$. Firstly, considering the possible values of K which exist in this case. Assume $(1, -1, 1) \in K$, then X contains a vertex r such that

$$B_{rY} = B_{u_1Y} - B_{u_2Y} + B_{u_3Y}. \tag{2.3.5}$$

It implies that $N(u_2) \subseteq N(u_3)$ and $N(u_3) \subseteq N(u_2)$, that is, $N(u_2) = N(u_3)$ and obviously $N(r) = N(u_1)$. This is because of our supposition under consideration that $N(u_1) \cap (N(u_2) \cup N(u_3)) = \phi$ which clearly indicates that $N(u_1) \cap N(u_2) = \phi$ and $N(u_1) \cap N(u_3) = \phi$. The same relations hold between neighbourhoods of u_1, u_2, u_3 if $(1, 1, -1) \in K$. In both cases the resultant graph is a union of two disjoint expanded paths of length 2. Such a graph has nullity $n - 4$ by Theorem 2.3.2 which is a contradiction. Therefore the possible values for K are $\{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}$. Hence for any vertex $x \in X$, $N(x) = N(u_1), N(u_2)$ or $N(u_1)$. Furthermore if $N(u_2) \cap N(u_3) \neq \phi$ then $N(u_2) = N(u_3)$ and will produce a graph which is a union of two disjoint expanded paths of length 2. It is the same graph obtained above giving contradiction. So $N(u_1), N(u_2), N(u_3)$ are pairwise disjoint. Hence G is a union of three disjoint expanded paths of length 2 with all expanded vertices of same order. Such a graph falls in the category of required graphs. By considering all possibilities for graphs with nullity $n - 6$ every graph structure required for proof is obtained. Hence the result.

2.4 Conclusion

We studied nullity of different types of graphs. We observed from those results that nullity of each family of graphs has its own dependencies. Nullity of trees explicitly depend upon their matching numbers. Nullity of cycles and paths depend on their respective lengths. A lot of useful theorems have been formulated which ease the process of calculations for obtaining nullity such as Lemma 2.2.4 and 2.2.5.

On the other hand, Fan and Qian [9] introduced a complete different approach for finding the nullity of bipartite graphs. They gave a new concept of expanded paths and cycles. Any structure of bipartite graph always fall in the category of expanded path or cycle or their disjoint unions. In this way it becomes a lot easier to find pattern in bipartite graphs. After identifying the patterns and arranging them in a specific order, calculations become less tedious. This is the reason that not only the nullity set of bipartite graphs is obtained but also classes of graphs having same nullity are formulated within the bipartite family.

Chapter 3

Nullity of Tripartite Graphs

This chapter gives a brief introduction to tripartite graphs and work done on them. A special class of tripartite graphs is defined and focus is to study its nullity and obtain useful results on it.

3.1 Background

A lot of work has been done on decomposition of complete tripartite graphs. But presently, no one has produced work regarding the nullity of tripartite graphs. A graph G with m edges is said to be decomposable into the graph H if the edges of G can be partitioned into subsets each of which induces a graph isomorphic to H. Edwards [19] showed that for any tripartite graph G, there is an integer m such that G decomposes $K_{m,m,m}$. Also Cavenagh [5] gave necessary and sufficient conditions for the decomposition of the complete tripartite graph $K_{m,m,m}$ into k-cycles, for any $k \geq 3$.

In the preceding chapter the importance of nullity is described in detail. It is easier to obtain the nullity of graphs with fewer edges because finding some pattern in such graphs is less time consuming. Nullity of bipartite graphs is already obtained. Our focus is to extend the work of nullity to next step by finding the nullity of tripartite graphs. In this chapter we we study nullity for a class of tripartite graphs. As observed the nullity problem in tripartite graphs does not follow as an extension to that of the nullity of bipartite graphs, this makes the study of nullity in tripartite graphs interesting. In this direction, we obtain the nullity set of a class of n-vertex tripartite graphs and characterize these tripartite graphs with nullity $n-4$. We also characterize some tripartite graphs with nullity $n-6$ in this class.

The graph G is *tripartite* if its vertex set can be partitioned into three subsets X, Y and Z such that $G[X]$, $G[Y]$ and $G[Z]$ are empty graphs; such a partition (X, Y, Z) is called a *tripartition*. For any $S \subseteq V(G)$, we denote by $N_X(S)$ the neighbors of S in X. Analogously, we can define $N_Y(S)$ and $N_Z(S)$. Tripartite graphs seem to be an extension to bipartite graphs but that is not the case. As studied in previous chapter, the adjacency matrix of bipartite graph contains only one non zero submatrix. On the other hand the adjacency matrix of tripartite graph contains three non zero submatrices. Such a structure refrains the adjacency matrix of tripartite graph from giving obvious ranks. It is very tough to figure out a pattern in tripartite graphs and that is the very reason that nullity of tripartite graphs is still an unsolved problem. Therefore we tried to make a step which would take us closer to the actual problem. We define a class within tripartite family by adding some constraints so it becomes manageable and then find its nullity. We consider a special class of tripartite graphs defined as follows. Let $\mathcal{T}_n$ be the family of those n-vertex tripartite graphs G, $n \geq 5$, whose tripartition (X, Y, Z) satisfies the following:

$$N_X(Y') \neq X \text{ and } N_Z(Y') \neq Z \;\; \forall \;\; Y' \subseteq Y, \tag{3.1.1}$$

$$G[X \cup Z] \text{ is complete bipartite.} \tag{3.1.2}$$

3.2 Nullity set for $\mathcal{T}_n$

Let $G \in \mathcal{T}_n$ with tripartition (X, Y, Z). The adjacency matrix $A(G)$ of G is defined by

$$A(G) = \begin{array}{c} \\ X \\ Z \\ Y \end{array}\!\!\begin{array}{c} \begin{array}{ccc} X & Z & Y \end{array} \\ \begin{bmatrix} \mathbf{0} & J & C_1 \\ J^t & \mathbf{0} & C_2 \\ C_1^t & C_2^t & \mathbf{0} \end{bmatrix} \end{array},$$

where J and $\mathbf{0}$ respectively denote the matrices with all entries 1 and 0. Let C and B denote the matrices defined as follows.

$$C = \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} \mathbf{0} & J \\ J^t & \mathbf{0} \end{bmatrix}.$$

The matrix $A(G)$ can be viewed as

$$A(G) = \begin{bmatrix} B & C \\ C^t & \mathbf{0} \end{bmatrix}. \tag{3.2.1}$$

Let

$$U = [B \;\; C], \quad L = [C^t \;\; \mathbf{0}]. \tag{3.2.2}$$

Then $A(G)$ can be written as $A(G) = \begin{bmatrix} U \\ L \end{bmatrix}$.

For each $v \in X \cup Z$, we denote by U_v the row of $A(G)$ corresponding to the vertex v. Similarly, for each $v \in Y$, the row of $A(G)$ corresponding to the vertex v is denoted by L_v. Let $S \subseteq X \cup Z$. Then from the matrix $A(G)$, we see that

$$\sum_{v \in S} k_v U_v = [b_1 \; b_2 \; c], \tag{3.2.3}$$

where b_1, b_2 are constant row matrices respectively of dimension $1 \times |X|$ and $1 \times |Z|$, c is row vector of dimension $1 \times |Y|$, and k_v's are real constants. Similarly, for any $Y' \subseteq Y$, we can write

$$\sum_{v \in Y'} k_v^{'} L_v = [c_1 \; c_2 \; \mathbf{0}], \tag{3.2.4}$$

where c_1, c_2 and $\mathbf{0}$ are row vectors respectively of dimension $1 \times |X|$, $1 \times |Z|$ and $1 \times |Y|$, and $k_v^{'}$'s are real constants.

The following result gives information about the rank of a tripartite graph in $\mathcal{T}_n$.

Lemma 3.2.1. *Let $G \in \mathcal{T}_n$ with tripartition (X, Y, Z) and adjacency matrix defined by* (3.2.1). *Then*

$$\operatorname{rank}(G) = \operatorname{rank}(U) + \operatorname{rank}(L), \tag{3.2.5}$$

where U and L are defined by (3.2.2).

Proof. Let S and Y' be arbitrary subsets, respectively of $X \cup Z$ and Y. To prove (3.2.5), it is enough to show that $\sum_{v\in S} k_v U_v \neq \sum_{v\in Y'} k'_v L_v$ whenever $\sum_{v\in S} k_v U_v \neq 0$ and $\sum_{v\in Y'} k'_v L_v \neq 0$, and k_v's and k'_v's are real constants.

We can write $\sum_{v\in S} k_v U_v = [b_1 \; b_2 \; c]$ and $\sum_{v\in Y'} k'_v L_v = [c_1 \; c_2 \; \mathbf{0}]$, where b_1, b_2, c, c_1, c_2 and $\mathbf{0}$ are defined in (3.2.3) and (3.2.4). By condition (3.1.1), there exists a vertex in X which is not adjacent to any vertex in Y. Similarly, there exists a vertex in Z which is not adjacent to any vertex in Y. Thus there are at least two zero columns in C^t corresponding to a vertex in X and to a vertex in Z. That is, there are zero entries in vectors c_1 and c_2. Now, if $\sum_{v\in S} k_v U_v = \sum_{v\in Y'} k'_v L_v$ then $[b_1 \; b_2 \; c] = [c_1 \; c_2 \; \mathbf{0}]$. As b_1 and b_2 are constant vectors, the vectors b_1, b_2, c, c_1, c_2 are all zero vectors. This completes the proof. □

Corollary 3.2.2. *Let $G \in \mathcal{T}_n$ with tripartition (X, Y, Z) and the adjacency matrix $A(G)$ defined by* (3.2.1). *Then* $\operatorname{rank}(G) = 2(1 + \operatorname{rank}(C))$.

Proof. By the construction of the matrix $A(G)$ and by the arguments used in Lemma 3.2.1, we see that $\operatorname{rank}(U) = \operatorname{rank}(B) + \operatorname{rank}(C)$. Since $\operatorname{rank}(B) = 2$ and $\operatorname{rank}(L) = \operatorname{rank}(C) = \operatorname{rank}(C^t)$, we get from (3.2.5) that $\operatorname{rank}(G) = 2(1 + \operatorname{rank}(C))$. □

The next result gives the nullity set of the graphs in $\mathcal{T}_n$.

Theorem 3.2.3. *Let m_1, m_2 and m_3 be positive integers such that $n = m_1+m_2+m_3$. Then for each integer $k \in \{0, 1, \ldots, \min\{m_1+m_3-2, m_2\}\}$, there is a graph $G \in \mathcal{T}_n$ with tripartition (X, Y, Z) such that $|X| = m_1$, $|Y| = m_2$, $|Z| = m_3$ and $\eta(G) = n - 2(k+1)$. Conversely, if $G \in \mathcal{T}_n$ with tripartition (X, Y, Z) then $\eta(G) = n-2(1+k)$, where $k \in \{0, 1, \ldots, \min\{|X| + |Z| - 2, |Y|\}\}$.*

Proof. First, we prove that for each $k \in \{0, 1, \ldots, \min\{m_1 + m_3 - 2, m_2\}\}$, there is a graph $G \in \mathcal{T}_n$ with tripartition (X, Y, Z) such that $|X| = m_1$, $|Y| = m_2$, $|Z| = m_3$ and $\eta(G) = n - 2(k + 1)$. We take three non-empty sets $X = \{x_1, x_2, \ldots, x_{m_1}\}$, $Y = \{y_1, y_2, \ldots, y_{m_2}\}$ and $Z = \{z_1, z_2, \ldots, z_{m_3}\}$. If $k = 0$, we construct a graph $G = P(X, Z) \cup m_2K_1$. Clearly, $G \in \mathcal{T}_n$. By using Lemma 2.2.2, $\eta(G) = n - 2$. If $k > 0$, we consider following two cases.

Case 1. When $k \le m_1 - 1$. Since $k \le m_2$, we construct a tripartite graph G with tripartition (X, Y, Z) that satisfies the following.

(i) $G[X \cup Z] = \mathbb{P}(X, Z)$,

(ii) $|N_X(y_i)| = 1$ and $N_Z(y_i) = \emptyset$ for $1 \le i \le k$,

(iii) $N_X(y_i) \cap N_X(y_j) = \emptyset$ for $i \ne j$ and $1 \le i, j \le k$,

(iv) $d(y_i) = 0$ for $k + 1 \le i \le m_2$.

Then $G \in \mathcal{T}_n$. Moreover, the adjacency matrix of G is given by (3.2.1), where

$$C = \left[\begin{array}{c:c} I_{k\times k} & \mathbf{0}_{k\times(m_2-k)} \\ \hdashline \mathbf{0}_{(m_1+m_3-k)\times k} & \mathbf{0}_{(m_1+m_3-k)\times(m_2-k)} \end{array}\right].$$

Then $\mathrm{rank}(C) = k$. By Corollary 3.2.2, we get $\eta(G) = n - 2(1 + k)$.

Case 2. When $k > m_1 - 1$. Since $k \le m_2$ and $k - (m_1 - 1) \le m_3 - 1$, we construct a tripartite graph G with tripartition (X, Y, Z) that satisfies the following.

(i) $G[X \cup Z] = \mathbb{P}(X, Z)$,

(ii) $|N_X(y_i)| = 1$ for $1 \le i \le m_1 - 1$,

(iii) $|N_Z(y_i)| = 1$ for $m_1 \le i \le k$,

(iv) $N_X(y_i) \cap N_X(y_j) = \emptyset$ for $i \ne j$ and $1 \le i, j \le m_1 - 1$,

(v) $N_Z(y_i) \cap N_Z(y_j) = \emptyset$ for $i \ne j$ and $m_1 \le i, j \le k$,

(vi) $d(y_i) = 0$ for $k + 1 \le i \le m_2$.

Then $G \in \mathcal{T}_n$. Moreover, the adjacency matrix of G is given by (3.2.1), where

$$C = \left[\begin{array}{c:c:c} I_{(m_1-1)\times(m_1-1)} & \mathbf{0}_{(m_1-1)\times(k-m_1+1)} & \mathbf{0}_{(m_1-1)\times(m_2-k)} \\ \hdashline \mathbf{0}_{1\times(m_1-1)} & \mathbf{0}_{1\times(k-m_1+1)} & \mathbf{0}_{1\times(m_2-k)} \\ \hdashline \mathbf{0}_{(k-m_1+1)\times(m_1-1)} & I_{(k-m_1+1)\times(k-m_1+1)} & \mathbf{0}_{(k-m_1+1)\times(m_2-k)} \\ \hdashline \mathbf{0}_{(m_1+m_2-1-k)\times(m_1-1)} & \mathbf{0}_{(m_1+m_2-1-k)\times(k-m_1+1)} & \mathbf{0}_{(m_1+m_2-1-k)\times(m_2-k)} \end{array}\right].$$

Then $\text{rank}(C) = k$. Corollary 3.2.2 gives $\eta(G) = n - 2(1 + k)$.

Conversely, we show that if $G \in \mathcal{T}_n$ with tripartition (X, Y, Z), then $\eta(G) = n - 2(1 + k)$ where $k \in \{0, 1, \dots, \min\{|X| + |Z| - 2, |Y|\}\}$. By Corollary 3.2.2, $\text{rank}(G) = 2(1 + \text{rank}(C))$. By (3.1.1), there are at least two zero rows in C. This implies that $\text{rank}(C) \leq \min\{|X| + |Z| - 2, |Y|\}$. The result is true by setting $\text{rank}(C) = k$. □

From Corollary 3.2.2, for each graph $G \in \mathcal{T}_n$ with $A(G)$ defined by (3.2.1), we can write

$$\eta(G) = n - 2(1 + \text{rank}(C)). \tag{3.2.6}$$

Let $\mathbb{C}_k(\overline{e})$ denote an expanded cycle of length k with an expanded chord $\overline{e}$ joining two non-adjacent expanded vertices of the cycle $\mathbb{C}_k$. We have the following observation.

Lemma 3.2.4. *If $G = \mathbb{C}_5(\overline{e}) \cup kK_1$ is a graph of order n shown in Figure 3.1, $k \geq 0$, then $G \in \mathcal{T}_n$ and $\eta(G) = n - 4$.*

Proof. Let $X = X_1 \cup X'$, $Z = Z_1 \cup Z'$ and $Y = Y_1 \cup Y'$, where Y' is possibly empty. Then we see that the graph G is a tripartite graph with tripartition (X, Y, Z). Moreover, G satisfies (3.1.1) because $N_{X'}(Y) = \emptyset$ and $N_{Z'}(Y) = \emptyset$. Also, $G[X \cup Z] = \mathbb{P}(X, Z)$, that is, G satisfies (3.1.2). Thus $G \in \mathcal{T}_n$. Let $A(G)$ be the adjacency matrix of G defined by (3.2.1). By the construction of G, we see that all rows of C^t are identical and therefore $\text{rank}(C) = 1$. By Corollary 3.2.2, we conclude that $\eta(G) = n - 4$. □

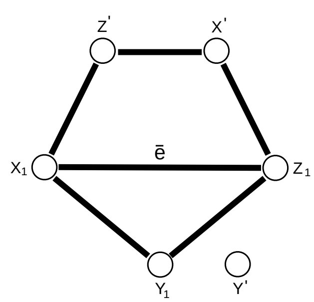

Figure 3.1: An expanded graph $\mathbb{C}_5(\overline{e}) \cup kK_1$

3.3 Characterization of $\mathcal{T}_n$ by Nullity

It is not sufficient to only find the nullity set for a family of graphs. Characterizing some graphs with their nullity in a family give us an insight to the structure of its family. This section is devoted to characterize some graph structures in $\mathcal{T}_n$ with their nullities. The next result is a direct consequence of Lemma 3.2.3.

Theorem 3.3.1. *For a graph $G \in \mathcal{T}_n$ with tripartition (X, Y, Z), $\eta(G) = n - 2$ if and only if $G = \mathbb{P}(X, Z) \cup |Y|K_1$.*

Proof. Let $G \in \mathcal{T}_n$ with tripartition (X, Y, Z) and $\eta(G) = n-2$. Then from equation (3.2.6), we have $\text{rank}(C) = 0$. That is, $d(y) = 0$ for all $y \in Y$. Thus $G = \mathbb{P}(X, Z) \cup |Y|K_1$. Conversely, suppose that $G = \mathbb{P}(X, Z) \cup |Y|K_1$. Using Lemma 2.2.8, we see that $\eta(G) = n - 2$. □

Theorem 3.3.2. *Let $G \in \mathcal{T}_n$ with tripartition (X, Y, Z) and $n \geq 4$. Then $\eta(G) = n - 4$ if and only if G is a graph H possibly with some isolated vertices, where H is an expanded path of length 4 or the expanded graph $\mathbb{C}_5(\overline{e})$.*

Proof. Let $G \in \mathcal{T}_n$ with tripartition (X, Y, Z) and $\eta(G) = n - 4$. Let $A(G)$ be the adjacency matrix of G defined by (3.2.1). Then by (3.2.6), we have $\text{rank}(C) = 1$, that is, $\text{rank}(L) = 1$. This means that there is only one independent row, say, L_{y_0} in L, where $y_0 \in Y$ is the vertex corresponding to L_{y_0}. Then for each $y \in Y$, either $N(y) = N(y_0)$ or $N(y) = \emptyset$. Let $Y_1 \subseteq Y$ is the set of all vertices of Y with non-zero degree. We have the following three cases.

Case 1. When $N_Z(Y_1) = \emptyset$. In this case, $N(Y_1) \subseteq X$. By condition (3.1.1), $N(Y_1) \neq X$. We partition X, Y and Z into Y_1, $N(Y_1)$, Z and $X \setminus N(Y_1)$. Then G can be drawn as an expanded path $\mathbb{P}(Y_1, N(Y_1), Z, X \setminus N(Y_1))$ possibly with some isolated vertices in $Y \setminus Y_1$.

Case 2. When $N_X(Y_1) = \emptyset$. In this case, $N(Y_1) \subseteq Z$. By condition (3.1.1), $N(Y_1) \neq Z$. We partition X, Y and Z into Y_1, $N(Y_1)$, X and $Z \setminus N(Y_1)$. Then G can be drawn as an expanded path $\mathbb{P}(Y_1, N(Y_1), X, Z \setminus N(Y_1))$ possibly with some isolated vertices in $Y \setminus Y_1$.

Case 3. When $N_X(Y_1) \neq \emptyset$ and $N_Z(Y_1) \neq \emptyset$. We can partition X into X_1 and X', such that $X_1 = N_X(Y_1)$ and $X' = X \setminus X_1$. Similarly, we can partition Z into Z_1 and Z', such that $Z_1 = N_Z(Y_1)$ and $Z' = Z \setminus Z_1$. Then, using condition (3.1.2), one can draw the graph G as an expanded graph $\mathbb{C}_5(\overline{e})$ (shown in Figure 3.1). There are possibly some isolated vertices in Y', where $Y' = Y \setminus Y_1$.

Conversely, let G be drawn as an expanded path of length 4 possibly with some

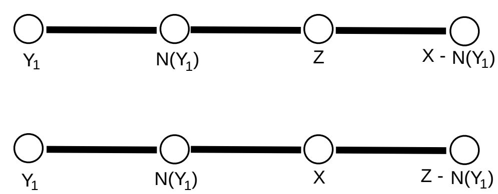

Figure 3.2: Two expanded paths $\mathbb{P}(Y_1, N(Y_1), Z, X \setminus N(Y_1))$ and $\mathbb{P}(Y_1, N(Y_1), X, Z \setminus N(Y_1))$ of length 4

isolated vertices. Then Lemma 2.2.8 yields that $\eta(G) = n - 4$. Furthermore, if G can be drawn as $\mathbb{C}_5(\overline{e})$ with some isolated vertices, then using Lemma 3.2.4, we get $\eta(G) = n - 4$. □

3.4 Some graphs in $\mathcal{T}_n$ with nullity $n - 6$

In this section, we consider some graphs in $\mathcal{T}_n$, $n \geq 6$, with nullity $n-6$. Let $G \in \mathcal{T}_n$ with tripartition (X, Y, Z) and let $X' = X \setminus N_X(Y)$. Note that $X' \neq \emptyset$ by (3.1.1). We assume that

$$G[N_X(Y) \cup Y] = \mathbb{P}(N_X(Y), Y). \tag{3.4.1}$$

The following result gives a characterization of a graph G in $\mathcal{T}_n$, $n \geq 6$ satisfying (3.4.1) and $\eta(G) = n - 6$.

Theorem 3.4.1. *Let $G \in \mathcal{T}_n$ with tripartition (X, Y, Z), $n \geq 6$. Assume that G satisfies* (3.4.1) *and $\eta(G) = n - 6$. Then G has one of the following expanded decomposition.*

(1) $\mathbb{C}_5(\overline{e})$, $\mathbb{P}_2$,

(2) $\mathbb{C}_5(\overline{e})$, $\mathbb{C}_3$, $\mathbb{P}_2$,

(3) $2\mathbb{C}_5(\overline{e})$, $2\mathbb{P}_2$,

(4) $\mathbb{C}_5(\overline{e})$, $\mathbb{C}_3$, $2\mathbb{P}_2$.

Proof. Let $G \in \mathcal{T}_n$ with tripartition (X, Y, Z) satisfying (3.4.1) and $\eta(G) = n - 6$, $n \geq 6$. Let $X' = X \setminus N_X(Y)$ and $Z' = Z \setminus N_Z(Y)$. From (3.1.1), we see that X' and Z' are nonempty. Let $A(G)$ be the adjacency matrix of G defined by (3.2.1). Since $\eta(G) = n - 6$, using Corollary 3.2.2, we have $\text{rank}(L) = 2$. This implies that L has two independent rows, say, L_{y_1} and L_{y_2}, where $y_1, y_2 \in Y$. Using (3.1.1) and (3.4.1), the columns of L corresponding to the vertices of X are constant. Then for each $y \in Y$, either $L_y = L_{y_1}$ or $L_y = L_{y_2}$. Thus we partition Y into Y_1 and Y_2, where

$$Y_1 = \{y \in Y \mid L_y = L_{y_1}\}, Y_2 = \{y \in Y \mid L_y = L_{y_2}\}.$$

Note that $N_Z(y) = N_Z(Y_1)$ for each $y \in Y_1$, and $N_Z(y) = N_Z(Y_2)$ for each $y \in Y_2$. Since $\text{rank}(L) = 2$, either $N_Z(Y_1) \neq \emptyset$ or $N_Z(Y_2) \neq \emptyset$. Without loss of generality, assume that $N_Z(Y_1) \neq \emptyset$ and $N_Z(Y_1) \nsubseteq N_Z(Y_2)$. The following three cases are possible.

Case 1. When $N_Z(Y_1) \cap N_Z(Y_2) = \emptyset$.
If $N_Z(Y_2) = \emptyset$, then $N_Z(Y) = N_Z(Y_1)$. We draw the graph G as an expanded graph on six expanded vertices $X_1 = N_X(Y)$, X', $Z_1 = N_Z(Y)$, Z', Y_1 and Y_2. Here Y' is possibly empty. The graph is shown in Figure 3.3 (i). In this case, we can decompose the graph G into $\mathbb{C}_5(\overline{e})$ and $\mathbb{P}_2$.

If $N_Z(Y_2) \neq \emptyset$, we partition $N_Z(Y)$ into $Z_1 = N_Z(Y_1)$ and $Z_2 = N_Z(Y_2)$. We draw the graph G as an expanded graph on seven expanded vertices $X_1 = N_X(Y)$, X', $Z_1 = N_Z(Y_1)$, $Z_2 = N_Z(Y_2)$, Z', Y_1 and Y_2. The graph is shown in Figure 3.3 (ii). In this case, the graph can be decomposed into $\mathbb{C}_5(\overline{e})$, $\mathbb{C}_3$ and $\mathbb{P}_2$.

Case 2. When $N_Z(Y_1) \cap N_Z(Y_2) \neq \emptyset$ and $N_Z(Y_2) \nsubseteq N_Z(Y_1)$.
Let $Z_1 = N_Z(Y_1) \cap N_Z(Y_2)$, $Z_2 = N_Z(Y_1) \setminus N_Z(Y_2)$ and $Z_3 = N_Z(Y_2) \setminus N_Z(Y_1)$. Then Z', Z_1, Z_2 and Z_3 form a partition of Z. The graph can be drawn as an expanded graph on eight expanded vertices $X_1 = N_X(Y)$, X', Z_1, Z_2, Z_3, Z', Y_1 and Y_2. The graph is shown in Figure 3.4 (i). In this case, the graph G can be decomposed into $2\mathbb{C}_5(\overline{e})$ and $2\mathbb{P}_2$.

Case 3. When $\emptyset \neq N_Z(Y_1) \cap N_Z(Y_2) \neq \emptyset$ and $N_Z(Y_2) \subseteq N_Z(Y_1)$.
We draw the graph G as an expanded graph on seven expanded vertices $X_1 = N_X(Y)$, X', $Z_1 = N_Z(Y_2)$, $Z_2 = N_Z(Y_1) \setminus N_Z(Y_2)$, Z', Y_1 and Y_2. The graph is shown in Figure 3.4 (ii). The graph G can be decomposed into $\mathbb{C}_5(\overline{e})$, $\mathbb{C}_3$ and $2\mathbb{P}_2$. $\square$

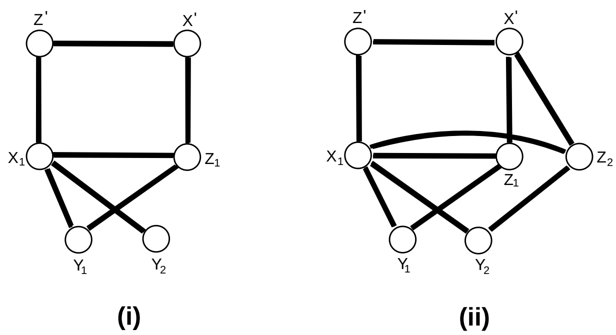

Figure 3.3: Graphs drawn in Case 1 with $N_Z(Y_1) \cap N_Z(Y_2) = \emptyset$

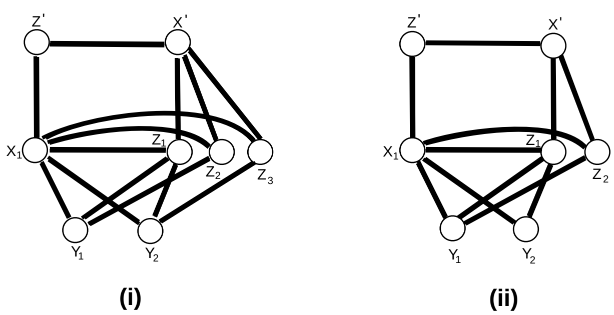

Figure 3.4: Graphs drawn in Case 2 and Case 3 with $N_Z(Y_1) \cap N_Z(Y_2) \neq \emptyset$

3.5 Conclusion

We studied n-vertex tripartite graphs satisfying (3.1.1) and (3.1.2). We obtained the nullity set of this class of n-vertex tripartite graphs and characterized them with nullity $n - 4$. It will be interesting to consider a more general class of n-vertex tripartite graphs and to characterize them with their nullity. In Theorem 3.4.1, we characterized those n-vertex tripartite graphs whose nullity is $n-6$ and that satisfy (3.1.1), (3.1.2) and (3.4.1). We are not sure about the converse of Theorem 3.4.1 and it is left as an open problem.

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