Existence and Stability of Solutions for Fractional Differential Equations

by

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Dedicated to

My Loving Parents

&

Sisters

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Rahat Saeed

Abstract

Fractional calculus has attracted much attention in recent research area due to its use as a modeling tool for real world problems in many different areas of science and technology. In this thesis some priliminary facts based on brief history and definitions related to fractional calculus are discussed. Mainly used functions in fractional calculus such as gamma and Mittag-Leffler functions with their properties are also discussed.

We establish new existence and uniqueness results of the solutions for a coupled system of impulsive fractional differential equations with boundary conditions and also establish related applications in this thesis. These results are established by the application of the Leray-Schauder alternative and the Banach's fixed-point theorem. We review some basic definitions and results of stability for linear and non-linear ordinary differential equations. These results are mainly based on the linearization method, the Liapunov stability method and the Routh-Hurwitz stability criterion. Also we review few important concepts and results on stability of linear and perturbed fractional differential systems. We analyze these stability results with the help of eigenvalues of the system matrix.

Finally, we review the stability of ordinary and fractional delay differential equations. We use linearization and Sturm sequences methods for analyzing the stability of ordinary delay differential equations. To investigate the stability of the system of fractional delay differential equations we use the concept based on the negative real parts of the roots of the characteristic equation of the system and this characteristic equation is obtained, by using the Laplace transform to the system.

Contents

Chapter 1

Preliminaries

1.1 Introduction to Fractional Calculus

In seventieth century, modern calculus was discovered by Issac Newton $(1642-1727)$ and Gottfried Wi-helm Leibniz $(1646-1716)$. "...the calculus was the first achievement of modern mathematics and it is difficult to overestimate its importance." These words quoted by John Von Neumann's $(1903-1957)$ show the importance of calculus. A repeated multiplication of a numerical value is always written in an integer exponent form that provides an easy short notation to it. As in our early education we have learnt $y.y.y.y$ can be written in an easy form as y^4 but we become confused when thinking about the exponents of non integer value $y^{4.3}$ or the transcendental exponent y^{π}. Even one cannot take a step to multiply a quantity or a number by itself 3.4 times, or π times, and yet a definite value exists for these expressions, provable by infinite series expansion, or by calculator in more practical way.

In a similar way we treat the integral and derivative. For any integer value of n, n integration can be done as methodical as multiplication. What if n were not taking an integer value?, a question that can be asked by a curious mind. Similarly, one can easily encounter the differential operators d/dt, d^3/dt^3, d^5/dt^5,etc., but by applying ordinary definition of derivative one cannot solve $d^{\frac{1}{2}}/dt^{\frac{1}{2}}$, $d^{\frac{1}{4}}/dt^{\frac{1}{4}}$,etc. In fact, these were the queries that gave a new dimension in the field of mathematics and also diverted the attentions of mathematicians to a new idea. They started their interest on these queries in 1968, and later on discovered a new subject in the field of mathematics. This new subject is called fractional calculus which flows quit naturally from traditional definitions of elementary calculus. It is a natural generalization of calculus.

Brief Historical Background

Fractional calculus is an ancient subject but studied little. Fractional calculus can be sorted as an applicable mathematics. In the last decade, applied mathematician and scientists found the fractional calculus valuable in various fields: quantitative biology, scattering theory, elasticity, probability, transport theory, diffusion, potential theory and electro chemistry.

Fractional calculus in its origin has the query of the extension of meaning. Two well known examples are, the extension of meaning of real numbers to complex numbers, and the extension of meaning of factorials of integers to factorials of complex numbers. In generalized differentiation and integration we have the question of extension of meaning as: "...can the meaning of derivatives of integral order $d^n f/dt^n$ be extended to have meaning where n is any number irrational, fractional or complex?."

Leibnitz discovered the symbolic notations. Leibniz $(1646 - 1716)$ first invented the idea of symbolic method and for nth derivative used the symbol $\frac{d^n f}{dt^n} = \mathcal{D}^n f$, where $n \geq 0$. The basic concept of fractional calculus was introduced in 1695. In a letter L'Hospital asked Leibnitz " What the order will be if n be $1/2$? " Leibnitz in 1695 replied, " It will lead to a paradox and from this apparent paradox, one day useful consequences will be drawn." And this was the statement which gave birth to fractional calculus. However, by most authors 30th September 1695 is the birth date of fractional calculus.

For non-integer values of n Leibnitz gave the fractional order derivative as;

$$\frac{d^n e^{mt}}{dt^n} = m^n e^{mt},$$

where m is a positive integer.

L. Euler (1730) introduced the formula for nth derivative in the following form;

$$\frac{d^n t^m}{dt^n} = m(m-1)\cdots(m-(n-1))t^{m-n}.$$

As

$$\Gamma(m+1) = m(m-1)\cdots(m-(n-1))\Gamma(m-(n-1)).$$

Thus

$$\frac{d^n t^m}{dt^n} = \frac{\Gamma(m+1)}{\Gamma(m-(n-1))}t^{m-n}.$$

By taking $m = 1$ and $n = 1/2$, Euler obtained the following relation;

$$\frac{d^{1/2}t}{dt^{1/2}} = \sqrt{\frac{4t}{\pi}} = \frac{2}{\sqrt{\pi}}t^{1/2}.$$

J. B. J. Fourier (1820 – 1822) took the first step in the generalization of notation for the differentiation of arbitrary function by using the following relation

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} f(s)ds \int_{-\infty}^{+\infty} \cos(pt - ps)dp.$$

He then gave a remark as

$$\mathcal{D}^n f(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} f(s)ds \int_{-\infty}^{+\infty} \cos\left(pt - ps + \frac{n\pi}{2}\right)dp,$$

where n is a non-integer.

N. H. Abel (1823 – 1826) invented the integral as

$$\psi(t) = \int_0^t \frac{S'(\eta)d\eta}{(t-\eta)^\alpha}.$$

In fact, he solved this integral for an arbitrary α as follows

$$S(t) = \frac{\sin(\pi\alpha)}{\pi}t^\alpha \int_0^1 \frac{\psi(t\tau)}{(1-\tau)^{1-\alpha}}d\tau.$$

J. Liouville (1832) started his approaches with known result made by Leibnitz for integral order derivatives

$$\frac{d^n e^{mt}}{dt^n} = m^n e^{mt}$$

and in a natural way Liouville extended this to derivatives of arbitrary order α;

$$\frac{d^\alpha e^{mt}}{dt^\alpha} = m^\alpha e^{mt}.$$

He used the series expansion $f(t) = \sum_{n=0}^{\infty} c_n e^{m_n t}$ to derive a relation as follows;

$$\mathcal{D}^\alpha f(t) = \sum_{n=0}^{\infty} c_n m_n^\alpha e^{m_n t}.$$

This is known as Liouville's first definition. But it has a disadvantage so that α must be restricted to such values for which the series converges.

Liouville's second definition was applied to explicit functions with the form of $t^{-\beta}, \quad \beta > 0$. He assumed the integral

$$I = \int_0^\infty u^{\beta-1} e^{su} du.$$

Using the transformation $su = t$, then the result obtained was

$$t^{-\beta} = \frac{1}{\Gamma(\beta)} I,$$

then operating $\mathcal{D}^\alpha$ on both sides of the above equation, he obtained

$$\mathcal{D}^\alpha = (-1)^\alpha \frac{\Gamma(\beta+\alpha)}{\Gamma(\beta)} t^{-\beta-\alpha}.$$

G.F.B. Riemann (1847) gave notation with the complementary function $\phi(t)$ as follows

$$\mathcal{D}^{-\alpha} = \frac{1}{\Gamma(\alpha)} \int_c^t (t-\tau)^{\alpha-1} f(\tau) d\tau + \phi(t).$$

A definition which fulfills some criteria given in [26], named in honour of both Riemann and Liouville is

$${}_c\mathcal{D}_t^{-\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_c^t (t-\tau)^{\alpha-1} f(\tau) d\tau.$$

This definition is same as Riemann's definition with no complementary function. This definition gives Riemann's definition when $c = 0$ and for $c = \infty$ it gives Liouville's definition.

1.2 The Gamma Function

Indeed, the Euler's gamma function $\Gamma(z)$ is one of the basic functions of integral calculus, which generalizes the factorial $n!$, where n is non-negative integer.

Definition 1.2.1. The function $\Gamma : (0, \infty) \to \mathbb{R}$, defined by the integral formula

$$\Gamma(z) := \int_0^\infty t^{z-1} \exp(-t) dt, \tag{1.2.1}$$

is called the Euler's gamma function. This improper integral is convergent for all $z \in \mathbb{C}$ with $\mathrm{Re}(z) > 0$.

Properties of Gamma Function:
Gamma function carries some basic properties as follows;

(i) (Functional Equation for Γ)

$$\Gamma(z + 1) := z\Gamma(z), \quad \forall z > 0. \tag{1.2.2}$$

(ii) In particular, for non-negative integer values

$$\Gamma(z + 1) := z!, \quad z = 0, 1, 2 \ldots .$$

So, gamma function is also known as a generalized factorial function.

(iii) (Reflection Formula for Γ)

$$\Gamma(z)\Gamma(1 - z) = \frac{\pi}{\sin \pi z}, \quad 0 < z < 1.$$

(iv) For the gamma function of rational numbers, we substitute $t = u^2$ in (1.2.1)

$$\Gamma(z) = 2 \int_0^\infty u^{2z-1} \exp(-u^2) du, \quad \mathrm{Re}(z) > 0.$$

Taking $z = \frac{1}{2}$, we obtain

$$\Gamma\left(\frac{1}{2}\right) = 2 \int_0^\infty \exp(-u^2) du = 2\frac{\sqrt{\pi}}{2} = \sqrt{\pi}.$$

Using property **(i)**, we obtain

$$\Gamma\Big(\frac{3}{2}\Big) = \frac{1}{2}\Gamma\Big(\frac{1}{2}\Big) = \frac{\sqrt{\pi}}{2}.$$

In the same way we can get the values of $\Gamma\Big(\frac{5}{2}\Big),\ \Gamma\Big(\frac{7}{2}\Big), \cdots,\ \Gamma\Big(\frac{2n+1}{2}\Big)$.

(v) To extend the domain of the gamma function to negative values of z, we rewrite (1.2.2) for $n = 1, 2, 3 \ldots n-1$, as

$$\Gamma(z) = \frac{\Gamma(z+n)}{z(z+1)(z+2)\cdots(z+n-1)},\ \text{ for } z \neq 0, -1, -2 \ldots.$$

For some values of z

$$\Gamma\Big(-\frac{1}{2}\Big) = \frac{\Gamma\Big(\frac{1}{2}\Big)}{-\frac{1}{2}} = -2\Gamma\Big(\frac{1}{2}\Big) = -2\sqrt{\pi}.$$

1.3 Mittag-Leffler Function

Definition 1.3.1. The function $E_\alpha(z)$ in infinite series representation, defined by

$$E_\alpha(z) = \sum_{k=0}^{\infty} \frac{(z)^k}{\Gamma(\alpha k + 1)}, \quad \alpha > 0,\ z \in \mathbf{C}, \tag{1.3.1}$$

is called the one-parameter Mittag-Leffler function of order α.

This function is convergent in the entire complex plane. The parameter α may be a complex number, provided $\mathrm{Re}(\alpha) > 0$ for the convergence of the function (1.3.1). The two-parameter Mittag-Leffler function for $\alpha > 0,\ \beta > 0$ is defined by

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{(z)^k}{\Gamma(\alpha k + \beta)}, \quad z \in \mathbb{C}. \tag{1.3.2}$$

Remark 1.3.1. It is obvious that for $\beta = 1$, the two-parameter Mittag-Leffler function equals the one-parameter Mittag-Leffler function via the relation

$$E_{\alpha,1}(z) = \sum_{k=0}^{\infty} \frac{(z)^k}{\Gamma(\alpha k + 1)} = E_\alpha(z). \tag{1.3.3}$$

1.4 Fractional Integrals and Derivatives

Here in this section we study the Riemann-Liouville fractional integral and derivative operators with their basic properties. We also, discuss the Caputo approach by studying the Caputo derivative operator with its properties. A theorem based on the relationship between the Riemann-Liouville fractional derivative and the Caputo fractional derivative is also discussed here in this section.

1.4.1 Riemann-Liouville Fractional Integral

Let us consider a function $f(t) \in L_1[a,b]$ of real variable t, namely complex or real valued functions, that are vanishing for $t < a$. The notion of Riemann-Liouville fractional integral of order α $(\alpha > 0)$ for $f(t)$, is a natural analogue of the renowned formula (usually credited to Cauchy).

The Cauchy's formula for $t > a$ is defined as;

$$ {}_aI_t^n f(t) = f_n(t) = \frac{1}{(n-1)!}\int_a^t (t-\tau)^{n-1} f(\tau)d\tau, \quad n \in \mathbb{N}, $$

here $\mathbb{N}$ is the set of natural numbers. The integral vanishes at $t = a$.

In a natural way we can extend this formula from positive integer values to any positive real values by introducing the gamma function. Now, introducing the arbitrary positive real number α and using $(n-1)! = \Gamma(n)$, we define for $t > a$, the Riemann-Liouville fractional integral as follows;

Definition 1.4.1. For $\alpha \in \mathbb{R}^+$, the operator ${}_aI_t^\alpha$ defined on $L_1[a,b]$ by

$$ {}_aI_t^\alpha f(t) = \frac{1}{\Gamma(\alpha)}\int_a^t (t-\tau)^{\alpha-1} f(\tau)d\tau, \quad a \le t \le b, $$

is called the Riemann-Liouville fractional integral of the function $f \in L_1[a,b]$ of order $\alpha > 0$.

Properties:

Some well-known properties of the Riemann-Liouville fractional integral operator are as follows;

(**i**) For $\alpha = 0$, we define ${}_aI_t^0 := I$ as an identity operator.

(ii) These fractional integral operators satisfy the semigroup property as;

$$ {}_aI_t^{\alpha}\, {}_aI_t^{\beta} f(t) = {}_aI_t^{\alpha+\beta} f(t), \quad \alpha,\ \beta \geq 0, $$

which implies ${}_aI_t^{\alpha}\, {}_aI_t^{\beta} = {}_aI_t^{\beta}\, {}_aI_t^{\alpha}$ (a commutative property).

(iii) The Riemann-Liouville fractional integral operator is linear as;

$$ {}_aI_t^{\alpha}(af(t) + bg(t)) = a({}_aI_t^{\alpha} f(t)) + b({}_aI_t^{\alpha} g(t)). $$

(iv) The Riemann-Liouville fractional integral operator of $f(t) = (t-a)^{\gamma}$ (a power function) gives the following formula

$$ {}_aI_t^{\alpha}(t-a)^{\gamma} = \frac{\Gamma(\gamma+1)}{\Gamma(\gamma+1+\alpha)}(t-a)^{\gamma}, \quad \alpha \geq 0,\ \gamma > -1, \quad t > a. $$

1.4.2 Riemann-Liouville Fractional Derivative

Definition 1.4.2. The Riemann-Liouville fractional derivative of order $\alpha > 0$ of f is defined as

$$ {}_a^{RL}\mathcal{D}_t^{\alpha} f(t) = \mathcal{D}_t^{m} \circ {}_aI_t^{m-\alpha} f(t), \quad m-1 < \alpha \leq m, \quad m \in \mathbb{N}. \tag{1.4.1} $$

More explicitly as

$$ {}_a^{RL}\mathcal{D}_t^{\alpha} f(t) = \begin{cases} \dfrac{1}{\Gamma(m-\alpha)} \dfrac{d^m}{dt^m} \displaystyle\int_a^t \frac{f(\tau)d\tau}{(t-\tau)^{\alpha+1-m}}, & m-1 < \alpha < m, \\ \dfrac{d^m}{dt^m} f(t), & \alpha = m. \end{cases} \tag{1.4.2} $$

Properties

(i) For $\alpha = 0$, we obtain the identity operator

$$ {}_a^{RL}\mathcal{D}_t^{0} = \mathcal{D}^m\, {}_aI_t^{m-0} = \mathcal{D}^m\, {}_aI_t^{m} = I $$

(ii). Let $m,\ n \in \mathbb{N}$, $m-1 \leq \alpha \leq m$, $n-1 \leq \beta \leq n$, we write relations [23]

$$ {}_a^{RL}\mathcal{D}_t^{\alpha}\left({}_a^{RL}\mathcal{D}_t^{\beta} f(t)\right) = {}_a^{RL}\mathcal{D}_t^{\alpha+\beta} f(t) - \sum_{j=1}^{n} \left[{}_a^{RL}\mathcal{D}_t^{\beta-j} f(a)\right] \frac{(t-a)^{-\alpha-j}}{\Gamma(1-\alpha-j)}. \tag{1.4.3} $$

And

$$ {}_{a}^{RL}\mathcal{D}_t^{\beta}\left({}_{a}^{RL}\mathcal{D}_t^{\alpha} f(t)\right) = {}_{a}^{RL}\mathcal{D}_t^{\alpha+\beta} f(t) - \sum_{j=1}^{n}\left[{}_{a}^{RL}\mathcal{D}_t^{\alpha-j} f(a)\right]\frac{(t-a)^{-\beta-j}}{\Gamma(1-\beta-j)}. \quad (1.4.4)$$

In general the Riemann-Liouville fractional derivative operators ${}_{a}^{RL}\mathcal{D}_t^{\alpha}$ and ${}_{a}^{RL}\mathcal{D}_t^{\beta}$ do not commute except for $\alpha = \beta$. For $\alpha \neq \beta$, the Riemann-Liouville fractional derivative operators commute only if both sums in the right-hand sides of (1.4.3) and (1.4.4) vanish and it possible only if

$$ {}_{a}^{RL}\mathcal{D}_t^{\beta-j} f(a) = 0 \quad \text{for } j = 1, 2, \ldots, m.$$

$$ {}_{a}^{RL}\mathcal{D}_t^{\alpha-j} f(a) = 0 \quad \text{for } j = 1, 2, \ldots, n.$$

(iii) Let $a_1,\ a_2 \in \mathbb{R}$, then the Riemann-Liouville fractional derivative is a linear operator as

$$ {}_{a}^{RL}\mathcal{D}_t^{\alpha}(a_1 f_1 + a_2 f_2) = a_1 {}_{a}^{RL}\mathcal{D}_t^{\alpha} f_1 + a_2 {}_{a}^{RL}\mathcal{D}_t^{\alpha} f_2.$$

(iv) The Riemann-Liouville fractional derivative of $f(t) = (t-a)^{\gamma}$ (a power function) is

$$ {}_{a}^{RL}\mathcal{D}_t^{\alpha}(t-a)^{\gamma} = \frac{\Gamma(\gamma+1)}{\Gamma(\gamma+1-\alpha)}(t-a)^{\gamma}, \quad \alpha \geq 0,\ \gamma > -1, \quad t > a.$$

(v) For constant function $f(t) \equiv 1$ and when ($\alpha \notin \mathbb{N}$), then

$$ {}_{a}^{RL}\mathcal{D}_t^{\alpha} 1 = \frac{(t-a)^{-\alpha}}{\Gamma(1-\alpha)}, \quad \alpha \geq 0, \quad \gamma = 0, \quad t > a. \quad (1.4.5)$$

1.4.3 Caputo Fractional Derivative

By interchanging in (1.4.1) the processes for differentiation and integration, the Caputo fractional derivative of order α is obtained as follows

$$ {}_{a}^{C}\mathcal{D}_t^{\alpha} f(t) = {}_{a}I_t^{m-\alpha}\mathcal{D}_t^{m} f(t), \quad m-1 < \alpha \leq m, \quad m \in \mathbb{N}. \quad (1.4.6)$$

More explicitly as

$$ {}_{a}^{C}\mathcal{D}_t^{\alpha} f(t) = \begin{cases} \dfrac{1}{\Gamma(m-\alpha)}\displaystyle\int_a^t \frac{f^{m}(\tau)d\tau}{(t-\tau)^{\alpha+1-m}}, & m-1 < \alpha < m, \\ \dfrac{d^m}{dt^m} f(t), & \alpha = m. \end{cases} \quad (1.4.7)$$

Properties

Some basic properties of the Caputo fractional derivative operators are as follows;

(i) The Caputo fractional operator is a linear operator as

$$ {}_a^C\mathcal{D}_t^\alpha(\lambda f(t) + \mu g(t)) = \lambda {}_a^C\mathcal{D}_t^\alpha f(t) + \mu {}_a^C\mathcal{D}_t^\alpha g(t). $$

(ii) Commutavity property does not satisfy by the Caputo fractional operators

$$ {}_a^C\mathcal{D}_t^\alpha {}_a^C D_t^\beta = {}_a^C\mathcal{D}_t^{\alpha+\beta} \neq {}_a^C\mathcal{D}_t^\beta {}_a^C\mathcal{D}_t^\alpha. $$

(iii) The Caputo fractional derivative of the power function satisfies

$$ {}_a^C\mathcal{D}_t^\alpha t^\gamma = \begin{cases} \dfrac{\Gamma(\gamma+1)}{\Gamma(\gamma-\alpha+1)} t^{\gamma-\alpha}, & m-1 < \alpha < m, \gamma > m-1, \gamma \in \mathbb{R}, \\ 0, & m-1 < \alpha < m, \gamma \leq m-1, \gamma \in \mathbb{N}. \end{cases} \tag{1.4.8} $$

(iv) Let $\alpha \in \mathbb{R},\ m-1 < \alpha < m,\ m \in \mathbb{N},\ \lambda \in \mathbb{C}$. Then the Caputo fractional derivative of the exponential function has the form

$$ {}_a^C\mathcal{D}_t^\alpha e^{\lambda t} = \sum_{k=0}^{\infty} \frac{\lambda^{k+m} t^{k+m-\alpha}}{\Gamma(k+1+m-\alpha)} = \lambda^m t^{m-\alpha} E_{1,m-\alpha+1}(\lambda t). \tag{1.4.9} $$

1.4.4 Relationship Between the Riemann-Liouville Fractional Derivative and the Caputo Fractional Derivative

The following theorem relates the Riemann-Liouville fractional derivative and the Caputo fractional derivative as follows;

Theorem 1.4.1. [14]. *Let $t > 0,\ \alpha \in \mathbb{R},\ m-1 < \alpha < m \in \mathbb{N}$. Then the following relation between the Riemann-Liouville and the Caputo operators holds*

$$ {}_a^C\mathcal{D}_t^\alpha f(t) = {}_a^{RL}\mathcal{D}_t^\alpha f(t) - \sum_{k=0}^{m-1} \frac{t^{k-\alpha}}{\Gamma(k+1-\alpha)} f^{(k)}(0). \tag{1.4.10} $$

Proof. The expansion of Taylor series about the point **0** expresses

$$f(t) = f(0) + tf'(0) + \frac{t^2}{2!}f''(0) + \frac{t^3}{3!}f'''(0) + \cdots + \frac{t^{m-1}}{(m-1)!}f^{m-1}(0) + R_{m-1}$$
$$= \sum_{k=0}^{m-1} \frac{t^k}{\Gamma(k+1)} f^{(k)}(0) + R_{m-1}.$$

Now considering the Cauchy's formula for repeated integration [23] as

$$R_{m-1} = \int_a^t \frac{f^m(\tau)(t-\tau)^{(m-1)}}{(m-1)!} d\tau = \frac{1}{\Gamma(m)} \int_a^t f^m(\tau)(t-\tau)^{(m-1)} d\tau = {}_tI_a^m f^m(t) \tag{1.4.11}$$

$$\begin{aligned}
{}_a^{RL}\mathcal{D}_t^\alpha f(t) &= {}_a^{RL}\mathcal{D}_t^\alpha \left(\sum_{k=0}^{m-1} \frac{t^k}{\Gamma(k+1)} f^{(k)}(0) + R_{m-1} \right) \\
&= \sum_{k=0}^{m-1} \frac{{}_a^{RL}\mathcal{D}_t^\alpha t^k}{\Gamma(k+1)} f^{(k)}(0) + {}_a^{RL}\mathcal{D}_t^\alpha R_{m-1} \\
&\qquad \text{(using property } \textbf{(ii)} \text{ of the Riemann-Liouville fractional derivative)} \\
&= \sum_{k=0}^{m-1} \frac{t^{k-\alpha}}{\Gamma(k-\alpha+1)} f^{(k)}(0) + {}_a^{RL}\mathcal{D}_t^\alpha {}_{m-a}I_t^m f^m(t) \\
&\qquad \text{(using property } \textbf{(iii)} \text{ of the Riemann-Liouville derivative and (1.4.11))} \\
&= \sum_{k=0}^{m-1} \frac{t^{k-\alpha}}{\Gamma(k-\alpha+1)} f^{(k)}(0) + {}_a^{C}\mathcal{D}_t^\alpha f^m(t).
\end{aligned}$$

This completes the proof. □

Lemma 1.4.2 (Gronwall Inequality). *Suppose that $g(t)$ and $\varphi(t)$ are continuous on $[t_0, t]$, $g(t) \geq 0$, $\lambda > 0$ and $r \geq 0$ are constants. If*

$$\varphi(t) \leq \lambda + \int_{t_0}^t [g(\tau)\varphi(\tau) + r] d\tau, \tag{1.4.12}$$

then

$$\varphi(t) \leq (\lambda + r(t_1 - t_0)) \exp\left(\int_{t_0}^t g(\tau) d\tau \right), \quad t_0 \leq t \leq t_1. \tag{1.4.13}$$

Proof. Since $\lambda > 0$, therefore

$$\lambda + \int_{t_0}^t (g(\tau)\varphi(\tau) + r) d\tau \neq 0. \tag{1.4.14}$$

So, then from (1.4.12) we get the following inequality

$$\frac{\varphi(t)g(t)}{\lambda + \int_{t_0}^{t}(g(\tau)\varphi(\tau) + r)d\tau} \leq g(t),$$

or

$$\frac{\varphi(t)g(t)}{(\lambda t - t_0)r + \int_{t_0}^{t} g(\tau)\varphi(\tau)d\tau} \leq g(t).$$

Now integrating from t_0 to t, we get

$$\begin{aligned}
\int_{t_0}^{t}\left\{\frac{\varphi(s)g(s)}{\lambda + (t-t_0)r + \int_{t_0}^{\tau} g(s)\varphi(s)d\tau}\right\}ds &\leq \int_{t_0}^{t} g(\tau)d\tau,\\
\ln\left[\lambda + (t-t_0)r + \int_{t_0}^{\tau} g(s)\varphi(s)d\tau\right]_{t_0}^{t} &\leq \int_{t_0}^{t} g(\tau)d\tau,\\
\ln\left[\lambda + (t-t_0)r + \int_{t_0}^{t} g(\tau)\varphi(\tau)d\tau\right] - \ln\left[\lambda + (t-t_0)r\right] &\leq \int_{t_0}^{t} g(\tau)d\tau,\\
\ln\left[\frac{\lambda + (t-t_0)r + \int_{t_0}^{t} g(\tau)\varphi(\tau)d\tau}{\lambda + (t-t_0)r}\right] &\leq \int_{t_0}^{t} g(\tau)d\tau,\\
\frac{\lambda + (t-t_0)r + \int_{t_0}^{t} g(\tau)\varphi(\tau)d\tau}{\lambda + (t-t_0)r} &\leq e^{\int_{t_0}^{t} g(\tau)d\tau},
\end{aligned}$$

or

$$\lambda + (t-t_0)r + \int_{t_0}^{t} g(\tau)\varphi(\tau)d\tau \ \leq \ (\lambda + r(t-t_0))e^{\int_{t_0}^{t} g(\tau)d\tau}.$$

So,

$$\varphi(t) \ \leq \ \lambda + (t-t_0)r + \int_{t_0}^{t} g(\tau)\varphi(\tau)d\tau \ \leq \ (\lambda + r(t-t_0))e^{\int_{t_0}^{t} g(\tau)d\tau}.$$

This gives $\varphi(t) \ \leq \ (\lambda + r(t-t_0))e^{\int_{t_0}^{t} g(\tau)d\tau}$.
Hence, we have proved the lemma. □

1.5 Initial and Final Value Theorems

We obtain the time representation of a Laplace function by taking the inverse Laplace transform. Sometimes we are interested to find the value of the given function at its very star that is $y(t=0)$, or its very end that is as $y(t\to\infty)$. Two useful theorems of Laplace transform that can provide us this information are the initial value theorem and the final value theorem. These two theorems help us to find the initial and the final values of the function without taking the inverse Laplace transform. The initial value theorem associate the 'initial value' $y(0+)$ of a function $y(t)$ to the behaviour of the Laplace transform $Y(s)$ for $s\to\infty$. Similarly, the final value theorem associate the 'final value' $\lim_{t\to\infty} y(t)$ of a function $y(t)$ to the behaviour of the Laplace transform $Y(s)$ for $s\to 0$. Here we will only prove the final value theorem.

1.5.1 Final Value Theorem

If $Y(s)$ is the Laplace transform of $y(t)$, then the final value theorem states that

$$\lim_{t\to\infty} y(t) = \lim_{s\to 0} sY(s).$$

The roots of the denominators polynomial $Y(s)$ that is the poles of $Y(s)$ must have negative or zero real parts. which is the only restriction of the final value theorem.

Proof. By the definition of the Laplace transform of the time derivative of $y(t)$ as

$$\mathcal{L}\frac{dy(t)}{dt} = \int_{0^-}^{\infty} \frac{dy(t)}{dt} e^{-st} dt = sY(s) - y(0^-). \tag{1.5.1}$$

Consider that $y(t)$ is continuous at $t=0$, by taking limit as $s\to 0$ we get

$$\lim_{s\to 0}\int_{0^-}^{\infty} \frac{dy(t)}{dt} e^{-st} dt = \int_{0^-}^{\infty} \frac{dy(t)}{dt} (\lim_{s\to 0} e^{-st}) dt = \lim_{s\to 0} sY(s) - y(0^-),$$

or

$$\int_{0^-}^{\infty} \frac{dy(t)}{dt} dt = \lim_{s\to 0} sY(s) - y(0^-),$$

$$y(\infty) - y(0^-) = \lim_{s\to 0} sY(s) - y(0^-),$$

or

$$y(\infty) = \lim_{s\to 0} sY(s),$$

or

$$\lim_{s\to\infty} y(t) = \lim_{s\to 0} sY(s).$$

Hence we have proved the final value theorem. □

1.6 Some Results from Analysis

Some basic definitions and familier results are to be given here in this section, which later on will be used to establish the existence and uniqueness of solutions for a coupled system of fractional impulsive differential boundary value problem involving the Caputo fractional derivative.

Definition 1.6.1. A set $A \subset C([a, b])$ is said to be equicontinuous if for given $\epsilon > 0$, there exists a $\delta > 0$ such that
$|T_n(y_1) - T_n(y_2)| < \epsilon$, for all $y_1, y_2 \in [a, b]$ whenever $|y_1 - y_2| < \delta$ for all $T_n \in A,\ n = 1, 2, \dots$.

Lemma 1.6.1 (Leray-Schauder alternative). [12] *Let $T : E \to E$ be a completely continuous operator (i.e., a map that restricted to any bounded set in E is compact). Let*

$$\mathcal{E}(T) = \{y \in E : y = \lambda T(y) \text{ for some } 0 < \lambda < 1\}.$$

Then either the set $\mathcal{E}(T)$ is unbounded, or T has at least one fixed point.

Definition 1.6.2. Let $T : Y \to Y$ be a mapping, then $y \in Y$ is **fixed point** of T iff

$$T(y) = y.$$

In other words, the image Ty coincides with y.

Definition 1.6.3. Let Y be a Banach Space. A mapping $T : Y \to Y$ is called a **contraction**, if there exists a nonnegative real number $k < 1$ such that for any $y_1,\ y_2 \in Y$,

$$||T(y_1) - T(y_2)|| \leq k||y_1 - y_2||.$$

Theorem 1.6.2 (Banach Fixed Point Theorem). *Let Y be a non-empty Banach space. Let $T : Y \to Y$ be a contraction mapping on Y. Then T has exactly one fixed point.*

Chapter 2

Existence and Uniqueness of Solutions for Impulsive Fractional Differential Equations

In recent years, existence theory for boundary value problems involving fractional impulsive differential equations has been analyzed extensively by many researchers. Among the previous research in the field of fractional calculus, little is concerned with impulsive fractional differential equations. Very recently, many researchers show their great interest in the field of impulsive problems for fractional differential equations. Recently, Benchohra [1, 4] proved sufficient conditions for the existence of solutions for impulsive fractional differential equations with initial condition involving the Caputo fractional derivative of order $\alpha \in (0, 1]$ and $\alpha \in (1, 2]$. Xiao-Bao Shu [28] analyzed the existence of mild solutions for impulsive fractional differential equations. Balachandran [2] proved the existence of solutions of nonlinear fractional integro-differential equations with impulsive conditions by using the fixed point principle.

Here in this chapter we will establish the existence and uniqueness of solutions for coupled system of impulsive fractional differential boundary value problem involving the Caputo fractional derivative by applying the Leray-Schauder alternative and the Banach contraction principle. Before doing this we give the brief background of impulsive equations (impulsive ordinary and fractional differential equations).

2.1 Impulsive Equations

Dynamics of some evolutionary processes from many fields such as physics, biology, population dynamics, control theory, and medicine go under abrupt changes at some certain moments of time like harvesting, shock, earthquake and so forth. These perturbations could be well estimated as instantaneous change of states or impulses. All these processes are modeled via *impulsive differential equations*. In 1960, these impulsive differential equations were first introduced by Milman and Myshkis in their paper [20]. Some authors have considered 'impulsive differential equations' as ordinary differential equations coupled with impulsive effects. On the other hand, also the impulsive fractional differential equations give a real framework for mathematical modeling towards the real world problems.

In general, the impulsive equations have two parts. A differential equation part, that describes the continuous part of the solution. It could be ordinary differential equations, partial differential equations, functional differential equations, integro-differential equations, etc. And an impulsive part, that describes the instantaneous changes and the discontinuity of the solution. This part is called a jump condition. The points, at which the impulses take place, are known as moments of impulses and the functions, that give the amount of impulses, are named as impulsive functions.

Here the basic two types of impulsive differential equations are defined according to the type of the moments of impulses as follows [13]:

- impulsive equations with fixed moments of impulses (i.e. the impulses occur at initially given fixed points),
- impulsive equations with variable moments of impulses (i.e. the impulses occur on initially given sets, i.e. the impulse occurs when the integral curve of the solution hits a given set).

Here we describe only the first type of impulsive equations in detail. For detailed description of the second type of the impulsive differential equations site [13].

Type 1. *Impulsive differential equations with fixed moments of impulses.*
Let $t_k \in \mathbb{R}$ be the fixed points such that $t_{k+1} > t_k$, $k = 0, 1, 2\dots$ and also $\lim_{k\to\infty} t_k = \infty$. Now consider the impulsive differential equation as:
differential equation (continuous part)

$$\dot{y}(t) = f(t, y(t)) \quad \text{for} \;\; t \geq t_0 \;\; t \neq t_k, \tag{2.1.1}$$

impulsive part (jump condition)

$$y(t_k^+) - y(t_k^-) = I_k((t_k^-)), \quad \text{for} \;\; k = 0, 1, 2\dots. \tag{2.1.2}$$

where $y \in \mathbb{R}^n$, $f : \mathbb{R} \times \mathbb{R}^n$ is the given function, $I_k : \mathbb{R}^n \to \mathbb{R}^n$ is the jump of state at each t_k and $t_0 < t_1 < \cdots < t_k < t_{k+1} < \cdots$, $k = 1, 2, 3\dots$.
The equations (2.1.1) and (2.1.2) and the initial condition

$$y(t_0) = y_0, \tag{2.1.3}$$

define the initial value problem for the system of impulsive differential equations. The solution of the system of equations (2.1.1), (2.1.2), (2.1.3) is denoted by $y(t; t_0, y_0)$. The point (t, y) of the integral curve of the solution $y(t; t_0, y_0)$ starts its motion from (t_0, y_0) of the set $D \subset \mathbb{R} \times \mathbb{R}$. It continues its motion along the integral curve $(t, y(t))$ of the solution of the ordinary differential equation (2.1.1) with initial condition (2.1.3) up to moment $t_1 > t_0$. At this moment the point instantaneously moves from (t_1, y_1) to (t_1, y_1^+), where $y_1 = y_1(t_1)$, $y_1^+ = y_1 + I_1(y_1)$. Then the point keeps its motion along the integral curve of the solution of ordinary differential equation (2.1.1) with initial condition $y(t_1) = y_1^+$ until moment $t_2 > t_1$ at which it jumps. By the equality (2.1.2) the amount of jump is determined and so on.

The impulsive differential equations with non-integer(fractional) order are called the "***fractional impulsive differential equations***".

Example 2.1.1. Consider the following fractional impulsive differential equation

$$\mathcal{D}^\alpha y(t) = 1, \quad t \neq k \text{ for } k = 1, 2, 3\dots \;\text{ and }\; \alpha \in (0, 1], \tag{2.1.4}$$

$$y(t_k^+) - y(t_k^-) = b, \tag{2.1.5}$$

$$y(0) = y_0. \tag{2.1.6}$$

The solution of the above ordinary differential equation (2.1.4) with initial condition (2.1.6) is

$$\begin{aligned} I^\alpha \mathcal{D}^\alpha y(t) &= I^\alpha 1, \\ y(t) + c &= \frac{1}{\Gamma(\alpha+1)} \int_0^t (t-s)^{\alpha-1} 1 ds \\ y(t) + c &= \frac{t^\alpha}{\Gamma(\alpha+1)}. \end{aligned}$$

Using $y(0) = y_0$ to find the value of the constant c and then we get

$$y(t) = y_0 + \frac{t^\alpha}{\Gamma(\alpha+1)},$$

for $t \geq 0$. Now the solution of the impulsive system (2.1.4), (2.1.5) with initial condition (2.1.6) is:

$$y(t) = y_0 + \frac{t^\alpha}{\Gamma(\alpha+1)} + kb,$$

for $t \in (k, k+1]$, $k = 1, 2, \ldots$. It is increasing for $b > 0$, decreasing for $b < 0$, and for $b = 0$ the solution becomes constant and coincides with the solution of the corresponding differential equation (2.1.4) with initial condition (2.1.6).

2.2 Coupled System of Impulsive Fractional Differential Equations with Integral Boundary Conditions

The study of a coupled system of fractional differential equations is also significant because this system can often occur in various applications. Recently, Ntouyas and Obaid [22] discussed boundary value problem for a coupled system of fractional differential equations involving the Caputo fractional derivative. In their work they proved two results for the existence and uniqueness of solutions by applying the Leray-Schauder alternative and the Banach contraction principle. Motivated by the work of Ntouyas and Obaid, here we are to prove the same two results for the existence and uniqueness of solutions for a coupled system of impulsive fractional differential equations with integral boundary conditions by imposing some restrictions with non-linear functions.

Consider a non-local boundary value problem for a coupled system of impulsive fractional differential equations as follows

$$\begin{cases} {}^C\mathcal{D}^{\alpha}_{0+}y_1(t) = f(t, y_1(t), y_2(t)), \quad t \neq t_k,\ t \in [0,1], \\ {}^C\mathcal{D}^{\beta}_{0+}y_2(t) = g(t, y_1(t), y_2(t)), \quad t \neq t_k,\ t \in [0,1], \\ y_1(t_k^+) - y_1(t_k^-) = I_k(y_1(t_k)),\ k = 1, 2, \cdots m, \\ y_2(t_k^+) - y_2(t_k^-) = J_k(y_2(t_k)),\ k = 1, 2, \cdots m, \\ y_1(0) = \int_0^1 \phi(s)y_1(s)ds, \quad y_2(0) = \int_0^1 \psi(s)y_2(s)ds, \end{cases} \tag{2.2.1}$$

where ${}^C\mathcal{D}^{\alpha}_{0+},\ {}^C\mathcal{D}^{\beta}_{0+}$ denote the Caputo fractional derivatives , $0 < \alpha, \beta \leq 1,\ f, g \in C([0,1] \times \mathbb{R}^2, \mathbb{R})$ and $I_k, J_k : \mathbb{R} \to \mathbb{R}$ for $k = 1, 2, \cdots m$.

Let the space $Y_1 = \{y_1(t)|y_1(t) \in C^1([0,1])\}$ with the norm $||Y_1|| = \max\{|y_1(t)|, t \in [0,1]\}$. $(Y_1, ||.||)$ is a Banach space. Also, $(Y_2, ||.||)$ is a Banach space defined by the space $Y_2 = \{y_2(t)|y_2(t) \in C^1([0,1])\}$ with the norm $||Y_2|| = \max\{|y_2(t)|, t \in [0,1]\}$. The product space $(Y_1 \times Y_2, ||(y_1, y_2)||)$ with the norm $||(y_1, y_2)|| = ||y_1|| + ||y_2||$ is also a Banach space.

Lemma 2.2.1. [25]. *Assume that $h : \mathbb{R} \to \mathbb{R}$ and $I_k : \mathbb{R} \to \mathbb{R}$ are continuous and $0 < \alpha < 1$. A function y is solution of impulsive problem*

$$\begin{cases} {}^C\mathcal{D}^{\alpha}_{0+}y_1(t) = h(t), \quad t \neq t_k,\ t \in [0,t], \\ y(t_k^-) - y(t_k^+) = I_k(y(t_k)),\ k = 1, 2, \cdots m, \\ y(0) = \int_0^1 \phi(s)y(s)ds, \end{cases} \tag{2.2.2}$$

if and only if y is solution of the integral equation

$$y(t) = \int_0^1 \phi(s)y(s)ds + \int_0^t \frac{(t-s)^{(\alpha-1)}}{\Gamma(\alpha)}h(s)ds + \sum_{0<t_k<t} I_k(y(t_k)). \tag{2.2.3}$$

For simplicity we define $m_1 = \dfrac{1}{\Gamma(\alpha+1)}$ and $m_2 = \dfrac{1}{\Gamma(\beta+1)}$.

and define the operator $T : Y_1 \times Y_2 \to Y_1 \times Y_2$ by

$$
\begin{aligned}
T(y_1, y_2) &= \begin{pmatrix} T_1(y_1, y_2) \\ T_2(y_1, y_2) \end{pmatrix} \\
&= \begin{pmatrix} \int_0^1 \phi(s) y_1(s) ds + \displaystyle\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f(s, y_1, y_2) ds + \sum_{0<t_k<t} I_k(y_1(t_k)) \\ \int_0^1 \phi(s) y_2(s) ds + \displaystyle\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\beta)} g(s, y_1, y_2) ds + \sum_{0<t_k<t} J_k(y_2(t_k)) \end{pmatrix}.
\end{aligned}
$$

The prove of the first result related to the existence of solutions for the system (2.2.1) is mainly based on the Leray-Schauder alternative.

Theorem 2.2.2. *Assume that there exist real constants* $\gamma_i, \zeta_i, \eta_i, \xi_i \geq 0$ $(i = 1, 2)$ *and* $\gamma_0, \zeta_0, \eta_0, \xi_0 > 0$ *such that* $\forall y_i \in \mathbb{R}$ $(i = 1, 2)$ *and the following conditions are satisfied:*

$$
\begin{aligned}
|f(t, y_1, y_2)| &\leq \gamma_0 + \gamma_1 |y_1| + \gamma_2 |y_2|, \\
|g(t, y_1, y_2)| &\leq \zeta_0 + \zeta_1 |y_1| + \zeta_2 |y_2|, \\
|I_k(y_1(t_k))| &\leq \eta_0 + \eta_1 |y_1|, \\
|J_k(y_2(t_k))| &\leq \xi_0 + \xi_1 |y_2|.
\end{aligned}
$$

In addition, it is assumed that

$$m_1\gamma_1 + m_2\zeta_1 + m\eta_1 < 1 \quad \textit{and} \quad m_1\gamma_2 + m_2\zeta_2 + m\xi_1 < 1.$$

Then the boundary value impulsive problem (2.2.1) *has at least one solution.*

Proof. First, we show that $T : Y_1 \times Y_2 \to Y_1 \times Y_2$ is a completely continuous operator. Since f, g, I_k, J_k are continuous functions, therefore the operator T is continuous.

Let $\Omega \subset Y_1 \times Y_2$ be bounded. Then for some positive constants $M_1,\ M_2,\ L_1,\ L_2$ and $\forall\ (y_1, y_2) \in \Omega$, we have

$$|f(t, y_1(t), y_2(t))| \leq M_1, \quad |g(t, y_1(t), y_2(t))| \leq M_2, \quad |I_k(y_1)| \leq L_1 \ \text{ and } \ |J_k(y_2)| \leq L_2.$$

Then for any $(y_1, y_2) \in \Omega$, we have

$$\begin{aligned}|T_1(y_1,y_2)(t)| &\le |y_1(0)| + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}|f(s,y_1,y_2)|ds + \sum_{0<t_k<t}|I_k(y_1(t_k))| \\ &\le |y_1(0)| + \frac{t^\alpha M_1}{\Gamma(\alpha+1)} + mL_1 \\ &\le |y_1(0)| + m_1M_1 + mL_1.\end{aligned}$$

Similarly

$$||T_1(y_1,y_2)(t)|| \le |y_2(0)| + m_2M_2 + mL_2.$$

Consequently

$$||T(y_1,y_2)(t)|| \le |y_1(0) + |y_2(0)|| + m_1M_1 + m_2M_2 + m(L_1+L_2) := R.$$

So, it follows that the operator T is uniformly bounded.

Now we show that T is equicontinuous. Let $0 \le t_1 \le t_2 \le 1$. Then

$$\begin{aligned}&|T_1(y_1(t_2),y_2(t_2)) - T_1(y_1(t_1),y_2(t_1))| \\ &\quad = \Bigg| \int_0^{t_2} \frac{(t_2-s)^{\alpha-1}}{\Gamma(\alpha)} f(s,y_1(s),y_2(s))ds - \int_0^{t_1} \frac{(t_1-s)^{\alpha-1}}{\Gamma(\alpha)} f(s,y_1(s),y_2(s))ds \\ &\qquad + \sum_{0<t_k<t_2} I_k(y_1(t_k)) - \sum_{0<t_k<t_1} I_k(y_1(t_k))\Bigg| \\ &\quad \le \Bigg| \int_0^{t_1} \frac{(t_2-s)^{\alpha-1}}{\Gamma(\alpha)} f(s,y_1(s),y_2(s))ds + \int_{t_1}^{t_2} \frac{(t_2-s)^{\alpha-1}}{\Gamma(\alpha)} f(s,y_1(s),y_2(s))ds \\ &\qquad - \int_0^{t_1} \frac{(t_1-s)^{\alpha-1}}{\Gamma(\alpha)} f(s,y_1(s),y_2(s))\Bigg| \\ &\quad \le \frac{M_1}{\Gamma(\alpha)} \Bigg| \int_0^{t_1} \big[(t_2-s)^{\alpha-1} - (t_1-s)^{\alpha-1}\big]ds + \int_{t_1}^{t_2} (t_2-s)^{\alpha-1}ds \Bigg| \\ &\quad \le \frac{M_1}{\Gamma(\alpha)}(t_2^\alpha - t_1^\alpha).\end{aligned}$$

So $|T_1(y_1(t_2),y_2(t_2)) - T_1(y_1(t_1),y_2(t_1))| \to 0$ as $t_2 \to t_1$.

Similarly

$$|T_2(y_1(t_2),y_2(t_2)) - T_2(y_1(t_1),y_2(t_1))| \le \frac{M_2}{\Gamma(\beta)}(t_2^\beta - t_1^\beta)$$

Thus $|T_2(y_1(t_2), y_2(t_2)) - T_2(y_1(t_1), y_2(t_1))| \to 0$ as $t_2 \to t_1$.

Consequently the operator T is equicontinuous, and hence T is completely continuous.

Finally it will be shown that $\mathcal{E} = \big\{(y_1, y_2) \in Y_1 \times Y_2 | (y_1, y_2) = \lambda T(y_1, y_2)\big\}, 0 \leq \lambda \leq 1$ is a bounded set. Let $(y_1, y_2) \in \mathcal{E}$, then $(y_1, y_2) = \lambda T(y_1, y_2)$. So for any $t \in [0, 1]$
$y_1(t) = \lambda T_1(y_1, y_2)(t), \quad y_2 = \lambda T_2(y_1, y_2)(t)$.
Now

$$\begin{aligned}|y_1(t)| &\leq \theta + \frac{t^\alpha}{\Gamma(\alpha+1)}\Big(\gamma_0 + \gamma_1|y_1| + \gamma_2|y_2|\Big) + m(\eta_0 + \eta_1|y_1|) \\ &\leq \frac{1}{\Gamma(\alpha+1)}\Big(\gamma_0 + \gamma_1|y_1| + \gamma_2|y_2|\Big) + m(\eta_0 + \eta_1|y_1|).\end{aligned}$$

Hence

$$||y_1(t)|| \leq m_1(\gamma_0 + \gamma_1||y_1|| + \gamma_2||y_2||) + m(\eta_0 + \eta_1||y_1||).$$

Similarly

$$||y_2(t)|| \leq m_2(\zeta_0 + \zeta_1||y_1|| + \zeta_2||y_2||) + m(\xi_0 + \xi_1||y_2||),$$

Now

$$\begin{aligned}||y_1(t)|| + ||y_2(t)|| = m_1\gamma_0 + m_2\zeta_0 + m(\eta_0 + \xi_0) + (m_1\gamma_1 + m_2\zeta_1 + m\eta_1)||y_1|| \\ + (m_1\gamma_2 + m_2\zeta_2 + m\xi_1)||y_2||.\end{aligned}$$

For simplicity, we define $K_1 = m_1\gamma_1 + m_2\zeta_1 + m\eta_1 \ \ and \ \ K_2 = m_1\gamma_2 + m_2\zeta_2 + m\xi_1$.
Obviously
$||y_1(t)|| + ||y_2(t)|| = m_1\gamma_0 + m_2\zeta_0 + K_1||y_1|| + K_2||y_2||$.

Here two cases are to be discussed for $K_1, \ K_2$ as follows:

Case 1. If $K_1 \leq K_2$, then

$||y_1(t)|| + ||y_2(t)|| \leq m_1\gamma_0 + m_2\zeta_0 + K_2||y_1|| + K_2||y_2||$,

or

$(1-K_2)(||y_1(t)||+||y_2(t)||) \leq m_1\gamma_0 + m_2\zeta_0.$

This implies

$||(y_1, y_2)|| \leq \dfrac{m_1\gamma_0 + m_2\zeta_0}{(1-K_2)}.$

Case 2. If $K_2 \leq K_1$, then

$$||y_1(t)||+||y_2(t)|| \leq m_1\gamma_0 + m_2\zeta_0 + K_1||y_1|| + K_1||y_2||,$$

or

$(1-K_1)(||y_1(t)||+||y_2(t)||) \leq m_1\gamma_0 + m_2\zeta_0.$

This implies

$||(y_1, y_2)|| \leq \dfrac{m_1\gamma_0 + m_2\zeta_0}{(1-K_1)}.$

Both cases are discussed for any $t \in [0,1]$, which shows that the set $\mathcal{E}$ is bounded. By Lemma (1.6.1), the operator T has at least one fixed point. Hence, the given impulsive problem (2.2.1) has at least one solution. □

With the help of the Banach fixed-point theorem we prove the uniqueness of the solution for the system (2.2.1).

Theorem 2.2.3. *Let there exist constants* $\mu_i, \nu_i, \mu, \nu, i = 1,2$ *such that for all* $t \in [0,1]$ *and* $y_1^*, y_2^*, y_1^{**}, y_2^{**} \in \mathbb{R}$ *and the following conditions are satisfied:*

$$\begin{aligned}
|f(t, y_1^*, y_1^{**}) - f(t, y_2^*, y_2^{**})| &\leq \mu_1|y_1^* - y_1^{**}| + \mu_2|y_2^* - y_2^{**}|,\\
|g(t, y_1^*, y_1^{**}) - g(t, y_2^*, y_2^{**})| &\leq \nu_1|y_1^* - y_1^{**}| + \nu_2|y_2^* - y_2^{**}|,\\
|I_k(y_1^{**}(t_k)) - I_k(y_1^*(t_k))| &\leq \mu|y_1^* - y_1^{**}|,\\
|J_k(y_2^{**}(t_k)) - J_k(y_2^*(t_k))| &\leq \nu|y_2^* - y_2^{**}|.
\end{aligned}$$

In addition, assume that

$$m_1(\mu_1+\mu_2) + m_2(\nu_1+\nu_2) + m(\mu+\nu) < 1.$$

Then the impulsive problem (2.2.1) *has unique solution.*

Proof. Since I_k, J_k are continuous functions, so let $\sup_{t_k\in[0,1]} |I_k| = \theta_1$, $\sup_{t_k\in[0,1]} |J_k| = \theta_2$. Also we define $\sup_{t\in[0,1]} |f(t,0,0)| = M_1$, $\sup_{t\in[0,1]} |g(t,0,0)| = M_2$ and set $|y_1(0)| = \theta$, $|y_2(0)| = \theta^*$. Then

$$r \geq \frac{m_1 M_1 + m_2 M_2 + m(\theta_1+\theta_2)}{1 - m_1(\mu_1+\mu_2) - m_2(\nu_1+\nu_2)}.$$

We show that $T(B_r) \subset B_r$, where $B_r = (y_1, y_2) \in Y_1 \times Y_2 : \| (y_1, y_2) \| \leq r$.
Now for $(y_1, y_2) \in B_r$, we have
$|T_1(y_1, y_2)(t)|$

$$\begin{aligned}
&\leq |y_0| + |I^\alpha f(t,y_1,y_2)| + |\sum_{0<t_k<t} I_k(y_1(t_k))| \\
&\leq |y_0| + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} |f(s,y_1,y_2)| ds + \sum_{0<t_k<t} |I_k(y_1(t_k))| \\
&\leq |y_0| + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} (|f(s,y_1,y_2) - f(s,0,0)| + |f(s,0,0)|) ds + \sum_{0<t_k<t} |I_k y_1(t_k)| \\
&\leq |y_0| + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} (\mu_1|y_1| + \mu_2|y_2|) ds + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} |f(s,0,0)| ds \\
&\quad + \sum_{0<t_k<t} |I_k y_1(t_k)| \\
&\leq \theta + (\mu_1|y_1| + \mu_2|y_2|) \frac{t^\alpha}{\Gamma(\alpha+1)} + M_1 \frac{t^\alpha}{\Gamma(\alpha+1)} + m\theta_1 \\
&\leq \theta + (\mu_1|y_1| + \mu_2|y_2|) \frac{t^\alpha}{\Gamma(\alpha+1)} + M_1 \frac{t^\alpha}{\Gamma(\alpha+1)} + m\theta_1 \\
&\leq m\theta_1 + m_1((\mu_1+\mu_2)r + M_1).
\end{aligned}$$

Hence

$$||T_1(y_1, y_2)(t)|| \leq m\theta_1 + m_1((\mu_1+\mu_2)r + M_1).$$

Similarly

$$||T_2(y_1, y_2)(t)|| \leq m\theta_2 + m_2((\nu_1+\nu_2)r + M_2).$$

Consequently

$$
\begin{aligned}
||T(y_1,y_2)(t)|| &= ||T_1(y_1,y_2)(t)|| + ||T_2(y_1,y_2)(t)|| \\
&\le m\theta_1 + m_1((\mu_1+\mu_2)r + M_1) + m\theta_2 + m_2((\nu_1+\nu_2)r + M_2) \\
&\le m(\theta_1+\theta_2) + (m_1(\mu_1+\mu_2) + m_2(\nu_1+\nu_2))r + m_1M_1 + m_2M_2 \\
||T(y_1,y_2)(t)|| &\le r.
\end{aligned}
$$

Hence $T(B_r) \subset B_r$.

Now for $(y_1^*, y_2^*), (y_1^{**}, y_2^{**}) \in Y_1 \times Y_2$, and for any $t \in [0,1]$, we obtain

$$
\begin{aligned}
&|T_1(y_1^{**}, y_2^{**})(t) - T_1(y_1^*, y_2^*)(t)| \\
&\quad \le \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} |f(s, y_1^{**}, y_2^{**}) - f(s, y_1^*, y_2^*)| ds + \sum_{0<t_k<t} |I_k(y_1^{**}(t_k)) - I_k(y_1^*(t_k))| \\
&\quad \le \frac{t^\alpha}{\Gamma(\alpha+1)} |f(s, y_1^{**}, y_2^{**}) - f(s, y_1^*, y_2^*)| + \sum_{k=1}^m |I_k(y_1^{**}(t_k)) - I_k(y_1^*(t_k))| \\
&\quad \le \frac{t^\alpha}{\Gamma(\alpha+1)} \Big(\mu_1 |y_1^{**} - y_1^*| + \mu_2 |y_2^{**} - y_2^*| \Big) + m\mu |y_1^{**} - y_1^*| \\
&\quad \le (m_1\mu_1 + m\mu) |y_1^{**} - y_1^*| + m_1\mu_2 |y_2^{**} - y_2^*| \\
&\quad \le (m_1\mu_1 + m\mu + m_1\mu_2)(|y_1^{**} - y_1^*| + |y_2^{**} - y_2^*|)
\end{aligned}
$$

Hence

$$||T_1(y_1^{**}, y_2^{**})(t) - T_1(y_1^*, y_2^*)(t)|| \le (m_1(\mu_1+\mu_2)) + m\mu)(||y_1^{**} - y_1^*|| + ||y_2^{**} - y_2^*||),$$

Similarly

$$||T_2(y_1^{**}, y_2^{**})(t) - T_2(y_1^*, y_2^*)(t)|| \le (m_2(\nu_1+\nu_2) + m\nu)(||y_1^{**} - y_1^*|| + ||y_2^{**} - y_2^*||).$$

Consequently

$$
\begin{aligned}
&||T(y_1^{**}, y_2^{**})(t) - T(y_1^*, y_2^*)(t)|| \\
&\qquad \le (m_1(\mu_1+\mu_2) + m_2(\nu_1+\nu_2) + m(\mu+\nu))(||y_1^{**} - y_1^*|| + ||y_2^{**} - y_2^*||).
\end{aligned}
$$

Since $m_1(\mu_1+\mu_2) + m_2(\nu_1+\nu_2) + m(\mu+\nu) < 1$, therefore the operator T is a contraction operator. Thus, by Banach's fixed-point theorem, T has a unique fixed-point, which we call the solution of the impulsive problem (2.2.1). Hence proved. □

Example 2.2.1. Consider the following coupled system of fractional impulsive boundary value problem:

$$\begin{cases} {}^C\mathcal{D}^{1/2}_{0+}y_1(t) = \dfrac{1}{4}\sin y_1(t) + \dfrac{|y_2(t)|}{5(t+1)(1+|y_2(t)|)} = f(t,y_1,y_2), \quad t \neq t_k,\ t \in [0,1], \\ {}^C\mathcal{D}^{1/2}_{0+}y_2(t) = \dfrac{|y_1(t)|}{9(1+|y_1(t)|)} + \dfrac{1}{2e^t}\sin^2 y_2(t) = g(t,y_1,y_2), \quad t \neq t_k,\ t \in [0,1], \\ I_k(y_1(t_k))|_{t=\frac{1}{2}^-} = \dfrac{|y_1(\frac{1}{2}^-)|}{3+|y_1(\frac{1}{2}^-)|}, \\ J_k(y_2(t_k))|_{t=\frac{1}{2}^-} = \dfrac{|y_2(\frac{1}{2}^-)|}{3+|y_2(\frac{1}{2}^-)|}, \\ y_1(0) = \int_0^1 \phi(s)y_1(s)ds, \quad y_2(0) = \int_0^1 \psi(s)y_2(s)ds. \end{cases} \tag{2.2.4}$$

Here $\alpha = 1/2 = \beta$,
$|f(t,y_1,y_2)| \leq \gamma_0 + \dfrac{1}{4}|y_1(t)| + \dfrac{1}{5}|y_2(t)|$, $|g(t,y_1,y_2)| \leq \zeta_0 + \dfrac{1}{9}|y_1(t)| + \dfrac{1}{2}|y_2(t)|$,
$|I_k(y_1(t_k))| \leq \eta_0 + \dfrac{1}{3}|y_1(t)|$, $|J_k(y_1(t_k))| \leq \xi_0 + \dfrac{1}{3}|y_2(t)|$, where $\gamma_0,\ \zeta_0,\ \eta_0,\ \xi_0 > 0$ are some real constants. Also, $\gamma_1 = \dfrac{1}{4},\ \gamma_2 = \dfrac{1}{5},\ \zeta_1 = \dfrac{1}{9},\ \zeta_2 = \dfrac{1}{2}\ \eta_1 = \dfrac{1}{3} = \xi_1$,
$m_1 = \dfrac{1}{\Gamma(\alpha+1)} = 1.571,\ m_2 = \dfrac{1}{\Gamma(\beta+1)} = 1.571.$

Now $m_1\gamma_1 + m_2\zeta_1 + m\eta_1 = 1.571(0.25) + 1.571(0.111) + 0.333 \approx 0.9002 < 1$
and
$m_1\gamma_2 + m_2\zeta_2 + m\xi_1 = 1.571(0.2) + 1.571(0.111) + 0.333 \approx 0.8216 < 1.$
Thus, all the conditions of Theorem (2.2.2) are satisfied. So, the boundary value problem (2.2.4) has at least one solution.

Example 2.2.2. Consider the following coupled system of fractional impulsive boundary value problem:

$$\begin{cases} {}^C\mathcal{D}^{1/2}_{0+}y_1(t) = \dfrac{e^{-t}|y_1(t)|}{(9+e^t)(1+|y_1(t)|)} + \dfrac{|y_2(t)|}{10e^t(1+|y_2(t)|)} = f(t,y_1,y_2), \quad t \neq t_k,\ t \in [0,1], \\ {}^C\mathcal{D}^{1/2}_{0+}y_2(t) = \dfrac{|y_1(t)|}{4(t+2)^2(1+|y_1(t)|)} + \dfrac{|y_2|}{16(1+|y_2(t)|)} = g(t,y_1,y_2), \quad t \neq t_k,\ t \in [0,1], \end{cases}$$

$$
\begin{cases}
I_k(y_1(t_k))\big|_{t=\frac{1}{2}^-} = \dfrac{|y_1(\frac{1}{2}^-)|}{9+|y_1(\frac{1}{2}^-)|}, \\
J_k(y_2(t_k))\big|_{t=\frac{1}{2}^-} = \dfrac{|y_2(\frac{1}{2}^-)|}{9+|y_2(\frac{1}{2}^-)|}, \\
y_1(0) = \int_0^1 \phi(s)y_1(s)ds, \quad y_2(0) = \int_0^1 \psi(s)y_2(s)ds.
\end{cases} \tag{2.2.5}
$$

Here $\alpha = 1/2 = \beta$,
$|f(t, y_1^*, y_1^{**}) - f(t, y_2^*, y_2^{**})| \leq \dfrac{1}{10}|y_1^* - y_2^*| + \dfrac{1}{10}|y_1^{**} - y_2^{**}|$,
$|g(t, y_1^*, y_1^{**}) - g(t, y_2^*, y_2^{**})| \leq \dfrac{1}{16}|y_1^* - y_2^*| + \dfrac{1}{16}|y_1^{**} - y_2^{**}|$,
$|I_k(y_1^*(t_k)) - I_k(y_1^{**}(t_k))| \leq \dfrac{1}{9}|y^* - y_1^{**}|, \quad |J_k(y_1^*(t_k)) - J_k(y_1^{**}(t_k))| \leq \dfrac{1}{9}|y_1^* - y_1^{**}|$.
After doing some calculations, we have
$\mu_1 = \dfrac{1}{10} = \mu_2$, $\nu_1 = \dfrac{1}{16} = \nu_2$, $\mu = \dfrac{1}{9} = \nu$, $m_1 = \dfrac{1}{\Gamma(\alpha+1)} = 1.571 = m_2 = \dfrac{1}{\Gamma(\beta+1)}$.
Also

$$
\begin{aligned}
m_1(\mu_1 + \mu_2) + m_2(\nu_1 + \nu_2) + m(\mu + \nu) &= 1.571(0.2) + 1.571(0.125) + 1.571(0.222) \\
&\approx 0.732575 < 1.
\end{aligned}
$$

Thus, all the conditions of Theorem (2.2.3) are satisfied. So, the boundary value problem (2.2.5) has a unique solution.

Chapter 3

The Stability of Ordinary Differential Equations

Theory of differential equations has been of great interest for many years. It plays a prominent role in many other disciplines including engineering, physics, economics and biology. In this chapter we will focus on the stability theory of the systems of the ordinary differential equations, concentrating in particular on systems of first-order linear and non-linear ordinary differential equations. For this we will review some basic definitions, theorems and methods for the suitable stability concepts of the equilibrium points of the systems. Also an alternative method for studying stability, called Liapunove method, is explained here in this chapter.

The general form of the first order n-dimensional systems of differential equations in n unknowns is

$$\dot{y} = f(t, y(t)), \tag{3.0.1}$$

with $f : [0, \infty) \times D \to \mathbb{R}^n$ is piecewise continuous in t and locally Lipschitz in y, D is a domain containing the origin. Here

$$f = \begin{pmatrix} f_1 \\ f_2 \\ \vdots \\ f_n \end{pmatrix}, \text{ and the } n\text{-vector function } y = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix}.$$

The system (3.0.1) is called the **non-autonomous system**.
If t, the time variable, does not appear explicitly in the right hand side of (3.0.1), then the system is called **autonomous system** having the general n-dimensional form as

$$\dot{y}(t) = f(y). \tag{3.0.2}$$

Definition 3.0.1. The point $\bar{y} \in \mathbb{R}^n$ is an **equilibrium point** for the autonomous system (3.0.2), if $f(\bar{y}) = 0$ for all t. Similarly, if $f(t, \bar{y}) = 0$, then the point $\bar{y} \in \mathbb{R}^n$ becomes the equilibrium point for the non-autonomous system (3.0.1) for all t.

Definition 3.0.2. An equilibrium point $\bar{y}(t)$ of the system (3.0.2) is **stable** (or **Liapunov stable**) if, given $\epsilon > 0$, $\exists\ \delta = \delta(\epsilon) > 0$, such that for any other solution, $y(t)$, of (3.0.2) satisfying $|\bar{y}(t_0) - y(t_o)| < \delta$ (where $|\ .\ |$ is a norm on $\mathbb{R}^n$) implies $|\bar{y}(t) - y(t)| < \epsilon$ for all $t > t_0$ and $t_0 \in \mathbb{R}$.

Definition 3.0.3. $\bar{y}(t)$ is said to be **asymptotically stable** if it is Liapunov stable and for any other solution $y(t)$ of (3.0.2), there exist a constant $b > 0$ such that, if $|\bar{y}(t_0) - y(t_0)| < b$ then $\lim_{t\to\infty} |\bar{y}(t) - y(t)| = 0$.

The geometrical interpretation of the above two definitions is given in Figure 3.1 ([29]).

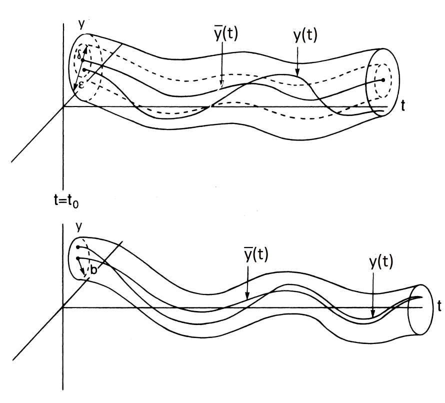
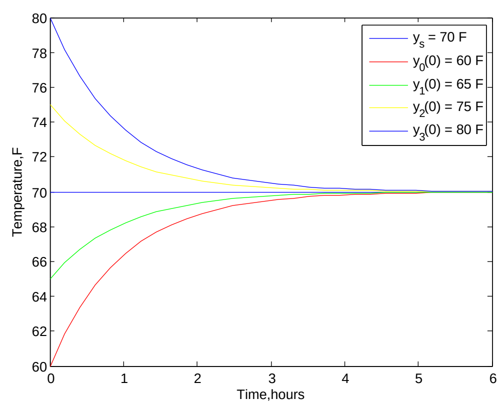


Figure 3.1: Liapunov stability and Asymptotic stability.

Example 3.0.3. Consider Newton's law of cooling

$$\frac{dy}{dt} = -k(y - y_s), \qquad y(0) = y_0 \tag{3.0.3}$$

where y is temperature of the object at time t, y_s is temperature of surroundings and k is positive constant.
The equation (3.0.3) has $y = y_s$ as the only equilibrium point. $y = y_s + (y_0 - y_s)e^{-kt}$ represents the equation of Newton's law of cooling. For $k > 0$, $\lim_{t\to\infty} e^{-kt} = 0$, so $y = y_s$. Thus our analysis suggest that the equilibrium $y = y_s$ is stable.



Definition 3.0.4. Let $\bar{y}(t, t_0, \bar{y}_0)$ denote the solution of the initial value problem

$$(\dot{y} = f(t, y),\ y(t_0) = y_0), \tag{3.0.4}$$

indicating its dependence on t and also the initial point t_0 and initial value y_0. This solution is said to be stable if for a given $\epsilon > 0$, $\exists\ \delta > 0$ such that $||\Delta y_0|| < \delta$ implies $||\bar{y}(t, t_0, \bar{y}_0 + \Delta y_0)|| < \epsilon$.

3.1 Stability of Autonomous Non-Linear Systems of Ordinary Differential Equations

Consider a two-dimensional system in the following form

$$\begin{cases} \dfrac{dy_1}{dt} = f(y_1, y_2), \\ \dfrac{dy_2}{dt} = g(y_1, y_2). \end{cases} \tag{3.1.1}$$

Suppose that $(\bar{y_1}, \bar{y_2})$ is the equilibrium point, so that $f(\bar{y_1}, \bar{y_2}) = 0 = g(\bar{y_1}, \bar{y_2})$. Now a question arises that either the equilibrium point is stable or not? From the equilibrium point we consider a small perturbation by letting

$$y_1 = \bar{y_1} + u,$$

$$y_2 = \bar{y_2} + v.$$

Here it is understood that u and v both are to be small $(u, v << 1)$. Either u and v will grow or decay? u or v is growing if y_1 and y_2 both are moving away from the equilibrium point and the equilibrium point is unstable. The equilibrium point is stable, if y_1 and y_2 both move towards it (u and v are decaying).
Now we want to derive the differential equations for u and v, from which we will see whether the perturbation grows or decays, as follows

$$\begin{aligned}
\frac{du}{dt} &= \frac{dy_1}{dt} \\
&= f(y_1, y_2) \\
&= f(\bar{y_1} + u, \bar{y_2} + v) \\
&= f(\bar{y_1}, \bar{y_2}) + \frac{\partial f}{\partial y_1}(\bar{y_1}, \bar{y_2})u + \frac{\partial f}{\partial y}(\bar{y_1}, \bar{y_2})v + h.o.t. \quad \text{(by expansion of Taylor series)} \\
&= \frac{\partial f}{\partial y_1}(\bar{y_1}, \bar{y_2})u + \frac{\partial f}{\partial y_2}(\bar{y_1}, \bar{y_2})v + h.o.t.
\end{aligned}$$

Similarly,

$$\frac{dv}{dt} = \frac{\partial g}{\partial y_1}(\bar{y_1}, \bar{y_2})u + \frac{\partial g}{\partial y_2}(\bar{y_1}, \bar{y_2})v + h.o.t. \text{ (since } g(\bar{y_1}, \bar{y_2}) = 0)$$

h.o.t represent higher order terms, as $O(u^2, v^2, uv)$. Since we have assumed that u and v both are small, so these h.o.t become extremely small.

So by neglecting the higher order terms, the following linear system of equations is obtained, which governs the progression of the perturbations of both u and v :

$$\begin{aligned}
\begin{pmatrix} \frac{du}{dt} \\ \frac{dv}{dt} \end{pmatrix} &= \begin{pmatrix} \frac{\partial f}{\partial y_1}(\bar{y_1}, \bar{y_2}) & \frac{\partial f}{\partial y_2}(\bar{y_1}, \bar{y_2}) \\ \frac{\partial g}{\partial y_1}(\bar{y_1}, \bar{y_2}) & \frac{\partial g}{\partial y_2}(\bar{y_1}, \bar{y_2}) \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} \\
&= J_e \begin{pmatrix} u \\ v \end{pmatrix},
\end{aligned}$$

where J_e is the Jacobian matrix evaluated at the equilibrium point $(\bar{y_1}, \bar{y_2})$. As J_e is a constant matrix, so the above differential equation is linear. Then this linear system has the trivial equilibrium point $(u, v) = (0, 0)$, and by the eigenvalues of the J_e we determine the stability of $(u, v) = (0, 0)$, as by the following theorem:

Theorem 3.1.1. *Assume that*

(**1**) *The equilibrium point is stable, if all the eigenvalues of J_e have the real negative parts.*

(**2**) *The equilibrium point is unstable, if J_e has at least one eigenvalue which has real positive part.*

(**3**) *No conclusion is obtained about the equilibrium point, if J_e has at least one eigenvalue which has a zero real part.*

Sometimes the negative real parts of eigenvalues do not give stability of the systems. For a general time dependent solution $\bar{y}(t)$ it may be tempting to deduce the stability properties of this solution from the eigenvalues of the Jacobian J_e. This can lead to wrong answers, as shown by the following example from Hale (1980) [29]. In the example we assumed a linear vector field having time periodic coefficients as follows

$$\begin{pmatrix} \dot{y_1} \\ \dot{y_1} \end{pmatrix} = A(t) \begin{pmatrix} y_1 \\ y_1 \end{pmatrix},$$

where

$$A(t) = \begin{pmatrix} -1 + \frac{3}{2}\cos^2 t & -1 - \frac{3}{2}\cos t \sin t \\ -1 - \frac{3}{2}\cos t \sin t & -1 + \frac{3}{2}\sin^2 t \end{pmatrix}. \qquad (3.1.2)$$

The eigenvalues of $A(t)$ determined to be independent of t are given by

$$\lambda_1 = \frac{-1 + i\sqrt{7}}{4}, \quad \lambda_2 = \frac{-1 - i\sqrt{7}}{4}.$$

In particular, these eigenvalues have real negative parts for all values of t. We have determined the following two linearly independent solutions of the above systems as follows

$$v_1(t) = \begin{pmatrix} -\cos t \\ \sin t \end{pmatrix} e^{\frac{t}{2}} \text{ and } v_2(t) = \begin{pmatrix} \sin t \\ \cos t \end{pmatrix} e^{-t}.$$

Hence the solutions are unstable. A conclusion that does not obtained from the eigenvalues of $A(t)$.

3.1.1 Classification of Equilibrium Points

On the basis of signs of the real parts of the eigenvalues of the J_e, we can classify the equilibrium points of system (3.1.1) as follows:

1. The equilibrium point $(\bar{y_1}, \bar{y_2})$ is a stable node (or a sink), if $\lambda_{1,2} < 0$ for $\lambda_i \in \mathbb{R}$ $(i = 1, 2)$.

2. The equilibrium point $(\bar{y_1}, \bar{y_2})$ is an unstable node (or a source), if $\lambda_{1,2} > 0$ for $\lambda_i \in \mathbb{R}$ $(i = 1, 2)$.

3. The equilibrium point $(\bar{y_1}, \bar{y_2})$ is a saddle point (or unstable), if $\lambda_1 < 0 < \lambda_2$ for $\lambda_i \in \mathbb{R}$ $(i = 1, 2)$.

4. The equilibrium point $(\bar{y_1}, \bar{y_2})$ is astable spiral (or stable focus), if $\lambda_i = \alpha \pm i\beta$ for $\alpha < 0$ and $\lambda_i \in \mathbb{C}$ $(i = 1, 2)$.

5. The equilibrium point $(\bar{y_1}, \bar{y_2})$ is an unstable spiral (or unstable focus), if $\lambda_i = \alpha \pm i\beta$ for $\alpha > 0$ and $\lambda_i \in \mathbb{C}$ $(i = 1, 2)$.

6. The equilibrium point $(\bar{y_1}, \bar{y_2})$ is a center (or neutrally stable), if $\lambda_i = \alpha \pm i\beta$ for $\alpha = 0$ and $\lambda_i \in \mathbb{C}$ $(i = 1, 2)$.

3.2 Stability of Linear Autonomous Systems

The following is the matrix-vector form of the n linear first order differential equations in n unknowns

$$\dot{y} = Ay + b.$$

Here the matrix function $A = [a_{ij}],\ i = 1, 2, \dots n$ is the coefficient matrix and $b = [b_i(t)],\ i = 1, 2, \dots n$ is the vector function.
If b is the zero vector then it is a homogeneous system and has the form

$$\dot{y} = Ay.$$

Theorem 3.2.1. [6] *Let $\Phi(t)$ be a fundamental matrix solution of $\dot{y} = A(t)y$. Then the system is stable for any $t_0 \in \mathbb{R}$ if and only if there is a positive constant $K = K(t_0)$ such that $|\Phi(t)| \leq K$ for all $t_0 \geq t$.*

Theorem 3.2.2. [6] *Let $\Phi(t)$ be a fundamental matrix solution of $\dot{y} = A(t)y$. Then the system is asymptotically stable for any $t_0 \in \mathbb{R}$ if and only if $|\Phi(t)| \to 0$ as $t \to \infty$.*

3.3 Stability of Linear Non-Autonomous Systems

The system

$$\dot{y} = A(t)y,$$

can be written as

$$\dot{y} = \{B + C(t)\}y,$$

where B is an $n \times n$ constant matrix.

Theorem 3.3.1. [27]. *Assume that*

i. *B is an $n \times n$ matrix and the eigenvalues of B have negative real parts;*

ii. *C(t) is continuous for $t \geq t_0$ and*

$$\int_{t_0}^{t} ||C(t)||dt,$$

is bounded for $t_0 \leq t \leq \infty$.

Then all solutions of the system $\dot{y} = \{B + C(t)\}y$ are asymptotically stable.

Proof. $\dot{y} = \{A + B(t)\}y = Ay + B(t)y$, where $h(t) = B(t)y$, is an inhomogeneous term. Let $\phi(t)$ be the solution of ODE with $\phi(t_0) = y_0$. Then by using the variation of constants formula:

$$\phi(t) = e^{A(t-t_0)}y_0 + \int_{t_0}^{t} e^{A(t-s)}B(s)\phi(s)ds$$
$$|\phi(t)| \leq |e^{A(t-t_0)}||y_0| + \int_{t_0}^{t} |e^{A(t-s)}||\phi(s)||B(s)|ds$$

$Re(\lambda_i) < 0 \Rightarrow \exists K, \rho > 0$, such that

$$|A(t-t_0)| \leq Ke^{-\rho(t-t_0)}, \quad t_0 \leq t < \infty$$

$$|A(t-s)| \leq Ke^{-\rho(t-s)}, \quad t_0 \leq s < \infty,$$

$$|\phi(t)| \leq Ke^{-\rho(t-t_0)}|y_0| + K\int_{t_0}^{t} e^{-\rho(t-s)}|\phi(s)||B(s)|ds,$$

$$|\phi(t)| \leq Ke^{-\rho t}e^{\rho t_0}|y_0| + Ke^{-\rho t}\int_{t_0}^{t} e^{\rho s}|\phi(s)||B(s)|ds,$$

$$e^{\rho t}|\phi(t)| \leq Ke^{\rho t_0}|y_0| + K\int_{t_0}^{t} e^{\rho s}|\phi(s)||B(s)|ds,$$

this inequality is a Gronwall Inequality of the form

$$u(t) = c_1 + c_2\int_{t_0}^{t} u(s)v(s)ds.$$

So by this Gronwall Inequality

$$e^{\rho t}|\phi(t)| \leq Ke^{\rho t_0}|y_0|e^{K\int_{t_0}^{t} e^{\rho s}|B(s)|ds},$$

$$|\phi(t)| \leq Ke^{-\rho(t-t_0)}|y_0|e^{K\int_{t_0}^{t}|B(s)|ds},$$

since $K\displaystyle\int_{t_0}^{t}|B(s)|ds < M_0 < \infty \Rightarrow e^{K\int_{t_0}^{t}|B(s)|ds} < e^{M_0} = M,$

$$|\phi(t)| \leq KMe^{-\rho(t-t_0)}|y_0| \to 0 \text{ as } t \to \infty.$$

Hence the **0**-solution of the system $\dot{y} = \{B + C(t)\}y$ is asymptotically stable. □

Corollary 3.3.2. [27] *If the solutions of $\dot{y} = A(t)y$ are only bounded and $C(t)$ satisfies the conditions of Theorem* (4.1.2)*, then all the solutions of $\dot{y} = \{B+C(t)\}y$ are bounded, hence stable.*

Example 3.3.1. Consider the linear non-autonomous sytem

$$\dot{y_1} = -2t^{-2}\ y_1 + y_2 + t^2,$$

$$\dot{y_2} = -y_1 - 2\ t^{-2}\ y_2 + t.$$

In the matrix form the system is

$$\dot{y} = \{B + C(t)\}y,$$

where,

$$B = \begin{bmatrix} -2 & 1 \\ -1 & -2 \end{bmatrix}, \quad C(t) = \begin{bmatrix} t^{-2} & 0 \\ 0 & t^{-2} \end{bmatrix}, \quad f(t) = \begin{bmatrix} t^2 \\ t \end{bmatrix}.$$

As we know that the system $\dot{y} = \{B + C(t)\}y$ has the same stability properties as a homogeneous equation $\dot{y} = A(t)y$, we have

$$|B - \lambda I| = \begin{vmatrix} -2-\lambda & 1 \\ -1 & -2-\lambda \end{vmatrix} = (-2-\lambda)^2 + 1 = 0.$$

By solving, we find eigenvalues as

$$\lambda_1 = -2 + i,$$
$$\lambda_2 = -2 - i.$$

So we conclude that both the eigenvalues have negative real parts. On the other hand, $C(t)$ is continuous for $t > 0$ and

$$\begin{aligned}\int_{t_0}^{\infty} ||C(s)||ds &= \lim_{t\to\infty} \int_{t_0}^{t} |s^{-2}|ds \\ &= \lim_{t\to\infty} 2|-s^{-1}||_1^2 \\ &= \lim_{t\to\infty} 2\left| -\left(\frac{1}{t} - \frac{1}{t_0}\right)\right| \\ &= \frac{2}{t_0} < \infty, \quad t_0 > 0.\end{aligned}$$

According to the Theorem (4.1.2), all solutions of $\dot{y} = \{B+C(t)\}y$ are asymptotically stable. Hence, the solutions of $\dot{y} = \{B+C(t)\}y+f(t)$ are also asymptotically stable.

3.4 Stability of Linear Systems with Constant Coefficients

Assume an $n \times n$ constant matrix A in equation and consider the following linear autonomous homogeneous system

$$\dot{y} = Ay, \tag{3.4.1}$$

$\bar{y} = 0$ is the only equilibrium solution of $\dot{y} = Ay$, and from the eigenvalues of A we can find the stability of the given system.

Definition 3.4.1 (**Liapunov Function**)**.** A continuously differentiable function V defined on an open set $U \in \mathbb{R}^n$ with $\bar{y} \in U(\bar{y}$ is an equilibrium of $\dot{y} = f(y)$ is called Liapunov function for $\dot{y} = f(y)$ provided,

(i) $V(\bar{y}) = 0$

(ii) $V(y) > 0$ for $y \neq \bar{y},\; y \in U$

(iii) $\frac{dV}{dt} \leq 0$

Definition 3.4.2. [27]. Let $f(y)$ be a scalar function such that $f(0) = 0$. If, $y \neq 0$,

(i) $f(y) > 0$ (or < 0), then it is called positive (or negative) definite.

(ii) $f(y) \geq 0$ (or ≤ 0), then it is called positive (or negative) semidefinite.

Theorem 3.4.1 (Liapunov stability for autonomous systems)**.** *Suppose that there is a continuously differentiable positive definite function $V(y) : D \in \mathbb{R}^n \to \mathbb{R}$, where D is an open set containing the origin, and $V(y)$ is negative semi definite for $y \in D$. Then the **0**-solution of $\dot{y} = f(y)$ is stable.*

Theorem 3.4.2 (Liapunov asymptotic stability for autonomous systems)**.** *Suppose that there is a continuously differentiable positive definite function $V(y) : D \in \mathbb{R}^n \to \mathbb{R}$, with D an open set containing the origin, and let $V(y)$ be negative definite for $y \in D$. Then the **0**-solution $\dot{y} = f(y)$ is asymptotically stable.*

Example 3.4.1. For the autonomous non-linear system
$\dot{y_1} = -y_1 + y_2(1 - y_1^2 - y_2^2)$
$\dot{y_2} = -y_2 - y_1(1 - y_1^2 - y_2^2)$.
Consider the Liapunov function
$V(y_1, y_2) = y_1^2 + y_2^2$.

Now taking the time derivative of $V(y_1, y_2)$ as follows

$$\begin{aligned}\dot{V}(y_1, y_2) &= 2y_1\dot{y}_1 + 2y_2\dot{y}_2 \\ &= 2y_1(-y_1 + y_2(1 - y_1^2 - y_2^2)) + 2y_2(-y_2 - y_1(1 - y_1^2 - y_2^2)) \\ &= -2y_1^2 - 2y_2^2 \\ &= -2(y_1^2 + y_2^2) < 0.\end{aligned}$$

Since $V(y_1, y_2)$ is positive definite and $\dot{V}(y_1, y_2)$ is negative everywhere except at the origin, so the 0-solution of the system is asymptotically stable.

Theorem 3.4.3 (Liapunov stability for non autonomous systems)**.** *Suppose that there is a continuously differentiable positive definite function $V(t, y) : [0, \infty) \times D \to \mathbb{R}$ such that $\dot{V}(t, y) \leq 0$. Then the **0**-solution of $\dot{y} = f(t, y)$ is stable.*

Example 3.4.2. For the non autonomous system
$\dot{y}_1 = y_2,$
$\dot{y}_2 = -y_2 - e^{-t},$
consider the Liapunov function
$V(t, y_1, y_2) = y_1 + e^t y_2$
Now taking the time derivative of the above Liapunov function as follows

$$\begin{aligned}\dot{V}(t, y_1, y_2) &= 2y_1\dot{y}_1 + e^t y_2^2 + 2y_2\dot{y}_2 e^t \\ &= 2y_1 y_2 + e^t y_2^2 + 2y_2 e^t(-y_2 - e^{-t}y_1) = -\dot{y}_2^2 e^- t.\end{aligned}$$

Since $V(t, y_1, y_2)$ is positive definite and also $\dot{V}(t,\ y_1,\ y_2) \leq 0$, so by the above theorem the **0**-solution of the non-autonomous system is stable.

3.5 Stability of Periodic Solutions: Floquet Theory

Differential equations comprising periodic functions play a significant role in various applications. Let's consider the n-dimensional first-order linear system

$$\dot{y} = A(t)y, \tag{3.5.1}$$

where $A(t)$ is a continuous, periodic $n \times n$ matrix function with minimum positive period ω; that is

$$A(t+\omega) = A(t), \quad -\infty < t < \infty.$$

($A(t)$, also has periods $2\omega, 3\omega, \dots$). Such type of systems are called Floquet systems and its study is called Floquet theory.

Definition 3.5.1. (**Fundamental Matrix**). Let $\phi_1(t), \phi_2(t), \dots, \phi_n(t)$ be n solutions of vector differential equation $\dot{y} = A(t)y$. Then an $n \times n$ matrix function $\Phi(t)$ is a fundamental matrix for $\dot{y} = A(t)y$, where

$$\Phi(t) = [\phi_1(t), \phi_2(t), \dots, \phi_n(t)],$$

with columns $\phi_1(t), \phi_2(t), \dots, \phi_n(t)$, provided that $\Phi(t)$ is an $n \times n$ matrix solution of the matrix equation $\dot{Y} = A(t)Y$ on I and $\det \Phi(t) \neq 0$ on I.

Definition 3.5.2. (**Floquet Multipliers**). Let $\Phi(t)$ be a fundamental matrix for the Floquet system (3.5.1), then the eigenvalues λ of

$$B := \Phi^{-1}(0)\Phi(\omega),$$

are called the Floquet multipliers of the Floquet system (2.6.1).

Theorem 3.5.1. [16]. *Let $\lambda_1, \lambda_2, \dots, \lambda n$ be the Floquet multipliers of the Floquet system* (2.6.1). *Then the trivial solution is*

(i) *globally asymptotically stable on* $[0, \infty)$ *iff* $|\lambda_i| < 1,\ 1 \leq i \leq n$;

(ii) *stable on* $[0, \infty)$, *provided* $|\lambda_i| \leq 1,\ 1 \leq i \leq n$ *and whenever* $|\lambda_i| = 1,\ \lambda_i$ *is a simple eigenvalue;*

(iii) *unstable on* $[0, \infty)$, *provided there is an* $i_0,\ 1 \leq i_0 \leq n$, *such that* $|\lambda_{i_0}| > 1$.

Example 3.5.1. Consider the Floquet system

$$y' = \begin{pmatrix} -\sin(2t) & \cos(2t) - 1 \\ \cos(2t) + 1 & \sin(2t) \end{pmatrix} y, \tag{3.5.2}$$

and

$$\Phi(t) = \begin{pmatrix} e^t(\cos t - \sin t) & e^{-t}(\cos t + \sin t) \\ e^t(\cos t + \sin t) & e^{-t}(-\cos t + \sin t) \end{pmatrix}, \tag{3.5.3}$$

be a fundamental matrix for the Floquet system (3.5.2). The Floquet multipliers are $\lambda_1 = -e^{\frac{\pi}{2}},\ \lambda_2 = -e^{-\pi}$. So by the Theorem 3.5.1 the trivial solution is unstable on $[0, \infty)$.

3.6 The Routh-Hurwitz Stability Criterion

The Routh-Hurwitz stability criterion was independently published by A. Hurwitz(1895) in Germany and E.J.Routh(1892) in United States. The Routh-Hurwitz stability criterion provides a necessary and sufficient condition to accertain the stability of a linear control systems. Without solving for the poles of the closed loop system the stability of a closed loop system can be judged by this criterion.

Consider a transfer function of a single-input, a single-output closed loop system is given by

$$F(s) = \frac{b_0 s^m + b_1 s^{m-1} + \cdots + b_{m-1} s^m + b_m}{a_0 s^n + a_1 s^{n-1} + \cdots + a_{n-1} s^n + a_n} = \frac{p(s)}{q(s)}. \tag{3.6.1}$$

The method was originally introduced in terms of determinants, however here we use the more suitable array formulation. The characteristic polynomial in (3.6.1) is given by

$$q(s) = a_0 s^n + a_1 s^{n-1} + \cdots + a_{n-1} s^n + a_n. \tag{3.6.2}$$

This criterion is based on arranging the coefficients of the characteristic polynomial in (3.6.2) into a schedule or an array[8]:

$$\begin{array}{c|ccccc} s^n & a_n & a_{n-2} & a_{n-4} & \cdots \\ s^{n-1} & a_{n-1} & a_{n-3} & a_{n-5} & \cdots \\ s^{n-2} & b_{n-1} & b_{n-3} & b_{n-5} & \cdots \\ s^{n-3} & c_{n-1} & c_{n-3} & c_{n-5} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ s^0 & h_{n-1} & h_{n-3} & h_{n-5} & \cdots \end{array}$$

where

$$b_{n-1} = \frac{(a_{n-1})(a_{n-2}) - a_n(a_{n-3})}{a_{n-1}},$$
$$b_{n-3} = \frac{(a_{n-1})(a_{n-4}) - a_n(a_{n-5})}{a_{n-1}},$$
$$\vdots$$
$$c_{n-1} = \frac{(b_{n-1})(a_{n-3}) - (a_{n-1})(b_{n-3})}{b_{n-1}}$$

and so on.

The Routh-Hurwitz criterion states that the number of roots of characteristic polynomial $q(s)$ with positive real parts is equal to the number of changes of sign in the first column of the array. So, the system is stable if and only if there are no changes in sign in the first column of the Routh array.

Example 3.6.1. Let

$q(s) = s^4 + 4s^3 + 16s^2 + 32s + 40.$

The Routh table is obtained as follows

s^4	1	16	40
s^3	4	32	
s^2	8	40	
s^1	12		
s^0	40		

Here we observe that in the first column no sign changes occur which indicates that no root lies in the right half of the s-plane and so, the system is stable.

3.6.1 Special Cases

Case 1. No element(entry) in the first column in Routh table is zero.

Example 3.6.2. Consider the characteristic equation
$q(s) = a_3 s^3 + a_2 s^2 + a_1 s + a_0$.
The Routh table will be as follows

$$\begin{array}{c|cc} s^3 & a_3 & a_1 \\ s^2 & a_2 & a_0 \\ s^1 & b_1 & 0 \\ s^0 & c_1 & 0 \end{array}$$

For stable third-order system, the necessary and sufficient conditon is that the coefficients of the characteristic polynomial must be positive and $a_2 a_1 - a_0 a_3 > 0$. The condition $a_2 a_1 = a_0 a_3$ gives rise to a marginal stability case, and a pair of roots of characteristic polynomial lies on the imaginary axis in the s-plane. Our case 3 is based on this marginal stability because for $a_2 a_1 = a_0 a_3$ there is a zero in the first column. Later on we will discuss it under case 3.

Case 2. If an element of the first column in the Routh table is zero, it might be replaced by a small positive number,ϵ, to complete the table. The sign of the elements in the first column is observerd as the ϵ approaches to zero.

Example 3.6.3. Let the characteristic equation be

$$q(s) = s^4 + s^3 + s^2 + s + K.$$

Here the gain K is to be determined, that results in marginal stability. The Routh table is as follows

$$\begin{array}{c|ccc} s^4 & 1 & 1 & K \\ s^3 & 1 & 1 & 0 \\ s^2 & \epsilon & K & 0 \\ s^1 & c_1 & 0 & 0 \\ s^0 & K & 0 & 0 \end{array}$$

where
$c_1 = \dfrac{\epsilon - K}{\epsilon} = \dfrac{-K}{\epsilon}$ *as* $\epsilon \to 0$.
So for $K > 0$ ($\epsilon > 0$), the system is unstable. As the last element in the first column is K, so for $k < 0$ the system is unstable. Thus for all values of gain K the system

is unstable.

Case 3. When all the elements of a row in the Routh table is zero or when a row comprises of a single zero element. This indicates the existence of some roots of the characteristic polynomial that are located symmetrically about the origin in the s-plane. In this case we should utilize an auxiliary polynomial from the row that precedes the zero row of the Routh table. This auxiliary polynomial is always with even order and shows the number of symmetrical root pairs.

Example 3.6.4.

$$q(s) = s^3 + 2s^2 + 4s + K,$$

$$\begin{array}{c|cc} s^3 & 1 & 4 \\ s^2 & 2 & K \\ s^1 & \frac{8-K}{2} & 0 \\ s^0 & K & 0 \end{array}$$

For $0 < K < 8$, the system is stable.

When $K = 8$, we find that two roots are on the imaginary axis and gives a marginal stability case. And also for $K = 8$, case 3 arises that is a row of zero elements is obtained. Therefore an auxiliary polynomial with even order is formed from the s^2 row which preceds the row of zeros as follows

$Q(s) = 2s^2 + Ks^0 = 2s^2 + 8 = 2(s^2 + 4) = 2(s + j2)(s - j2)$. Now

$\Delta(s) = \dfrac{s^3 + 2s^2 + 4s + K}{2s^2 + 8}$

Chapter 4

Stability of Fractional Differential Equations

Stability analysis is a an essential task in the theory of fractional differential systems. Many researchers contributed in the stability theory for fractional differential equations through their research papers. For instance, Matignon studied the stability of linear fractional order systems with commensurate order [18] and with incommensurate order [19]. Here in this chapter we will deal with some stability properties of linear fractional differential equations and also we will review some important theorems related to stability of linear fractional differential equations and perturbed fractional differential equations from the paper of Qian, Li, Agarwal and Wong [24].

The system of fractional differential equations with the Caputo derivative or the Riemann-Liouville derivative have the following general form

$$ {}_{t_0}\mathcal{D}_t^{\bar{\alpha}} y(t) = f(t, y), \tag{4.0.1}$$

with appropriate initial values $y_k = [y_{k1}, y_{k2}, \ldots, y_{kn}]^T \in \mathbb{R}^n$ $(k = 1, 2, \ldots, m-1)$, where $y(t) = [y_1(t), y_2(t), \ldots, y_n(t)]^T \in \mathbb{R}^n$, $\bar{\alpha} = [\alpha_1, \alpha_2, \ldots, \alpha_n]^T$, $m - 1 < \alpha_i < m \in \mathbb{Z}_+$ $(i = 1, 2, \ldots, n)$, ${}_{t_0}\mathcal{D}_t^{\bar{\alpha}} y(t) = [{}_{t_0}\mathcal{D}_t^{\alpha_1} y_1(t),\ {}_{t_0}\mathcal{D}_t^{\alpha_2} y_2(t), \ldots, {}_{t_0}\mathcal{D}_t^{\alpha_n} y_n(t)]^T$, $f : [t_0, \infty) \times \mathbb{R}^n \to \mathbb{R}^n$, ${}_{t_0}\mathcal{D}_t^{\bar{\alpha}}$ represents either ${}_{t_0}^{C}\mathcal{D}_t^{\bar{\alpha}}$ or ${}_{t_0}^{RL}\mathcal{D}_t^{\bar{\alpha}}$.
Particularly, if $\alpha_1 = \alpha_2 = \cdots = \alpha_n = \alpha$, then equation (4.0.1) can be written as

$$ {}_{t_0}\mathcal{D}_t^{\alpha} y(t) = f(t, y). \tag{4.0.2}$$

Here we say that equation (4.0.2) is the same order fractional differential system, while equation (4.0.1) is the multi-order fractional differential system.

Now we give some basic definitions which are mainly related to the stability problems of the fractional differential systems.

Definition 4.0.1. The constant vector $\bar{y}$ is an **equilibrium point** of fractional differential system (4.0.1), if and only if $f(t, \bar{y}) = {}_{t_0}\mathcal{D}_t^{\bar{\alpha}} y(t)|_{y(t)=\bar{y}}$ for all $t > t_0$.

Without loss of generality, let the equilibrium point of system (4.0.1) be $\bar{y} = \mathbf{0}$, give the following definition

Definition 4.0.2. . The **0**-solution of fractional differential system (4.0.1) is said to be **stable** if, for any initial values $y_k = [y_{k1}, y_{k2}, \ldots, y_{kn}]^T \in \mathbb{R}^n$ $(k = 0, 1, \ldots, m-1)$, there exists $\epsilon > 0$ such that any solution $y(t)$ of (4.0.1) satisfies $||y(t)|| < \epsilon$ for all $t > t_0$. The **0**-solution is said to be **asymptotically stable** if, addition to being stable, $||y(t)|| \to 0$ as $t \to +\infty$.

Lemma 4.0.1. [23]. *Let $0 < \alpha < 2$, β be an arbitrary complex number and μ be an arbitrary real number such that $\frac{\pi\alpha}{2} < \mu < \min\{\pi, \pi\alpha\}$. Then, for $p \geq 1$, an arbitrary integer, we have an expansion as follows:*
$E_{\alpha,\beta}(z) = \frac{1}{\alpha} z^{(1-\beta)/\alpha} \exp(z^{1/\alpha}) - \sum_{k=1}^{p} \frac{z^{-k}}{\Gamma(\beta - \alpha k)} + O(|z|^{-1-p})$. *when* $|\arg(z)| \leq \mu$ *and* $|z| \to \infty$; $E_{\alpha,\beta}(z) = -\sum_{1}^{p} \frac{z^{-k}}{\Gamma(\beta - \alpha k)} + O(|z|^{-1-p})$, *when* $\mu \leq |\arg(z)| \leq \pi$ *and* $|z| \to \infty$.

Remark 4.0.1. [23] If $\beta = \alpha$ in Lemma (4.0.1), then

(i)

$$E_{\alpha,\alpha}(z) = \frac{1}{\alpha} z^{(1-\alpha)/\alpha} \exp(z^{1/\alpha}) - \sum_{k=2}^{p} \frac{z^{-k}}{\Gamma(\alpha - \alpha k)} + O(|z|^{-1-p}). \qquad (4.0.3)$$

when $|\arg(z)| \leq \mu$ and $|z| \to \infty$;

(ii)

$$E_{\alpha,\alpha}(z) = -\sum_{k=2}^{p} \frac{z^{-k}}{\Gamma(\alpha - \alpha k)} + O(|z|^{-1-p}), \qquad (4.0.4)$$

when $\mu \leq |\arg(z)| \leq \pi$ and $|z| \to \infty$.

Lemma 4.0.2 (Jordan Decomposition [30])**.** *Let A be a square complex matrix, then there exists an invertible matrix such that*

$$Q^{-1}AQ = J_1 \oplus J_2 \oplus \cdots \oplus J_r, \tag{4.0.5}$$

where J_i $i = 1, 2 \cdots r$ are the Jordan blocks of A with the eigenvalues of A on the diagonal. The Jordan blocks are uniquely determined by A.

Here some results are to be given which are mainly used in the stability analysis of the fractional differential equations. These results based on the Laplace transform of the fractional derivatives and the Mittag-Leffler function [23].

• The Laplace transform of the Riemann-Liouville fractional derivative ${}^{RL}_{a}\mathcal{D}^{\alpha}_{t}y(t)$ is

$$\int_0^\infty e^{-st}\,{}^{RL}_{a}\mathcal{D}^{\alpha}_{t}y(t)dt = s^\alpha Y(s) - \sum_{k=0}^{n-1} s^\alpha[\mathcal{D}^{\alpha-k-1}y(t)]_{t=a}, \quad (n-1 \leq \alpha < n). \tag{4.0.6}$$

• The Laplace transform of the Caputo fractional derivative ${}^{C}_{a}\mathcal{D}^{\alpha}_{t}y(t)$ is

$$\int_0^\infty e^{-st}\,{}^{C}_{a}\mathcal{D}^{\alpha}_{t}y(t)dt = s^\alpha Y(s) - \sum_{k=0}^{n-1} s^{\alpha-k-1}y(a)^{(k)}, \quad (n-1 \leq \alpha < n). \tag{4.0.7}$$

• The Laplace transform of the Mittag-Leffler function is to be found as

$$\int_0^\infty e^{-st}t^{\alpha k-\beta-1}E^{(k)}_{\alpha,\beta}(\pm at^\alpha)dt = \frac{k!s^{\alpha-\beta}}{(s^\alpha \mp a)^{k+1}}, \quad (Re(s) > |a|^{\frac{1}{\alpha}}) \tag{4.0.8}$$

4.1 Stability of Linear Fractional Differential Equations

Here in this section we will analyze some important theorems and results on the stability of linear fractional differential equations and will give some related examples.

Consider the linear fractional differential equation of the form

$${}_{t_0}\mathcal{D}^{\bar{\alpha}}_{t}y(t) = Ay(t), \tag{4.1.1}$$

where matrix $A \in \mathbb{R}^{n\times n}$, $y(t) = [y_1(t), y_2(t), \ldots, y_n(t)]^T \in \mathbb{R}^n$, $\bar{\alpha} = [\alpha_1, \alpha_2, \ldots, \alpha_n]^T$, ${}_{t_0}\mathcal{D}^{\bar{\alpha}}_{t_0,t}y(t) = [{}_{t_0}\mathcal{D}^{\alpha_1}_t y_1(t),\ {}_{t_0}\mathcal{D}^{\alpha_2}_t y_2(t), \ldots, {}_{t_0}\mathcal{D}^{\alpha_n}_t y_n(t)]^T$ and ${}_{t_0}\mathcal{D}^{\alpha_i}_t$ is the Caputo derivative or Riemann-Liouville derivative of order α_i, $0 < \alpha_i \le 2$, for $i = 1, 2, \ldots, n$. Particularly if $\alpha_1 = \alpha_2 = \cdots = \alpha_n = \alpha$, then the system (4.1.1) can be the same order linear system

$$ {}_{t_0}\mathcal{D}^{\alpha}_t y(t) = Ay(t). \tag{4.1.2}$$

Theorem 4.1.1. *The autonomous system* (4.1.2) *with Caputo derivative and initial value* $y_0 = y(0)$, $\alpha \in (0, 1]$, *is*

(i) *asymptotically stable iff* $|\arg(\lambda(A))| > \dfrac{\alpha\pi}{2}$. *In this case the components of the state decay towards 0 like* $t^{-\alpha}$, *(Here* $\arg(\lambda(A))$ *denotes the arguments of the eigenvalues of the square matrix* A*.)*

(ii) *stable iff either it is asymptotically stable or those critical eigenvalues which satisfy* $|\arg(\lambda(A))| = \dfrac{\alpha\pi}{2}$ *have geometric multiplicity one.*

Proof. **(i)** Taking the Laplace transform of (4.1.2) and $0 < \alpha < 1$

$$s^{\alpha}Y(s) - \sum_{k=0}^{n-1} s^{\alpha-k-1}y^k(0) = AY(s), \quad (n-1 \le \alpha < n).$$

Using the initial condition $y^k(0) = y_0$ $(k = 0)$ we get

$$\begin{aligned} s^{\alpha}Y(s) - s^{\alpha-1}y_0 &= AY(s), \\ (s^{\alpha}I - A)Y(s) &= s^{\alpha-1}y_0, \\ Y(s) &= s^{\alpha-1}[s^{\alpha}I - A]^{-1}y_0. \end{aligned}$$

Now, taking inverse Laplace transform of the above equation we get the solution of the given system by (4.0.8) as follows

$$y(t) = y_0E_{\alpha,1}(At^{\alpha}). \tag{4.1.3}$$

Suppose that the matrix A is similar to a Jordan canonical form, i.e., there is an invertible matrix Q such that

$$A = Q^{-1}JQ = \mathrm{diag}(J_1, J_2, \ldots, J_r). \tag{4.1.4}$$

Here $J_i,\ 1 \le i \le r$ is the jordan block (by Lemma(4.0.2))

$$J_i = \begin{pmatrix} \lambda_i & 1 & & \\ & \lambda i & \ddots & \\ & & \ddots & 1 \\ & & & \lambda_i \end{pmatrix}_{n_i \times n_i}$$

and $\sum_{i=1}^{r} n_i = n$. Now

$E_{\alpha,1}(At^\alpha) = Q\text{diag}[E_{\alpha,1}(J_1 t^\alpha), E_{\alpha,1}(J_2 t^\alpha), \ldots, E_{\alpha,1}(J_n t^\alpha)]Q^{-1}$.

Where for $1 \le i \le r$,

$$E_{\alpha,1}(J_i t^\alpha) = \sum_{k=0}^{\infty} \frac{(J_i t^\alpha)^k}{\Gamma(k\alpha+1)} = \sum_{k=0}^{\infty} \frac{(t^\alpha)^k}{\Gamma(k\alpha+1)} J_i^k$$

$$= \sum_{k=0}^{\infty} \frac{(t^\alpha)^k}{\Gamma(k\alpha+1)} \begin{pmatrix} \lambda_i & 1 & & \\ & \lambda_i & \ddots & \\ & & \ddots & 1 \\ & & & \lambda_i \end{pmatrix}^k$$

$$= \sum_{k=0}^{\infty} \frac{(t^\alpha)^k}{\Gamma(k\alpha+1)} \begin{pmatrix} \lambda_i^k & C_1^k \lambda_i^{k-1} & \ldots & C_{n_i-1}^k \lambda_i^{k-n_i+1} \\ & \lambda_i^k & \ddots & \vdots \\ & & \ddots & C_1^k \lambda_i^{k-1} \\ & & & \lambda_i^k \end{pmatrix}$$

$$= \begin{pmatrix} \sum_{k=0}^{\infty} \frac{(\lambda_i t^\alpha)^k}{\Gamma(k\alpha+1)} & \sum_{k=0}^{\infty} \frac{(t^\alpha)^k}{\Gamma(k\alpha+1)} C_1^k \lambda_i^{k-1} & \cdots & \sum_{k=0}^{\infty} \frac{(t^\alpha)^k}{\Gamma(k\alpha+1)} C_{n_i-1}^k \lambda_i^{k-n_i+1} \\ & \sum_{k=0}^{\infty} \frac{(\lambda_i t^\alpha)^k}{\Gamma(k\alpha+1)} & \ddots & \vdots \\ & & \ddots & \sum_{k=0}^{\infty} \frac{(t^\alpha)^k}{\Gamma(k\alpha+1)} C_1^k \lambda_i^{k-1} \\ & & & \sum_{k=0}^{\infty} \frac{(\lambda_i t^\alpha)^k}{\Gamma(k\alpha+1)} \end{pmatrix}$$

(where $C_k^j = \frac{k!}{j!(k-j)!},\ 1 \le j \le n_i - 1$ are the binomial coefficients.)

$$\begin{aligned}
\sum_{k=0}^{\infty} \frac{(t^{\alpha})^k}{\Gamma(k\alpha+1)} C_1^k \lambda_i^{k-1} &= \sum_{k=0}^{\infty} \frac{(t^{\alpha})^k}{\Gamma(k\alpha+1)} \frac{k!}{1!(k-1)!} \lambda_i^{k-1} \\
&= \sum_{k=0}^{\infty} \frac{(t^{\alpha})^k}{\Gamma(k\alpha+1)} \frac{k(k-1)!}{1!(k-1)!} \lambda_i^{k-1} \\
&= \sum_{k=0}^{\infty} \frac{(t^{\alpha})^k}{\Gamma(k\alpha+1)} \frac{k}{1!} \lambda_i^{k-1} \\
&= \sum_{k=0}^{\infty} \frac{(t^{\alpha})^k}{\Gamma(k\alpha+1)} \frac{1}{1!} \left(\frac{\partial}{\partial \lambda_i}\right) \lambda_i^{k} \\
&= \frac{1}{1!} \left(\frac{\partial}{\partial \lambda_i}\right) \sum_{k=0}^{\infty} \frac{(t^{\alpha}\lambda_i)^k}{\Gamma(k\alpha+1)} \\
&= \frac{1}{1!} \frac{\partial}{\partial \lambda_i} E_{\alpha,1}(\lambda_i t^{\alpha})
\end{aligned}$$

$$\begin{aligned}
\sum_{k=0}^{\infty} \frac{(t^{\alpha})^k}{\Gamma(k\alpha+1)} C_{n_i-1}^k \lambda_i^{k-n_i+1} &= \sum_{k=0}^{\infty} \frac{(t^{\alpha})^k}{\Gamma(k\alpha+1)} \frac{k!}{(n_i-1)!(k-n_i+1)!} \lambda_i^{k-n_i+1} \\
&= \sum_{k=0}^{\infty} \frac{(t^{\alpha})^k}{\Gamma(k\alpha+1)} \\
&\quad \frac{k(k-1)\cdots(k-n_i+2)(k-n_i+1)\cdots 3.2.1}{(n_i-1)!(k-n_i+1)!} \lambda_i^{k-n_i+1} \\
&= \sum_{k=0}^{\infty} \frac{(t^{\alpha})^k}{\Gamma(k\alpha+1)} \\
&\quad \frac{k(k-1)\cdots(k-n_i+2)(k-n_i+1)!}{(n_i-1)!(k-n_i+1)!} \lambda_i^{k-n_i+1} \\
&= \sum_{k=0}^{\infty} \frac{(t^{\alpha})^k}{\Gamma(k\alpha+1)} \frac{1}{(n_i-1)!} \left(\frac{\partial}{\partial \lambda_i}\right)^{n_i-1} \lambda_i^{k} \\
&= \frac{1}{(n_i-1)!} \left(\frac{\partial}{\partial \lambda_i}\right)^{n_i-1} \sum_{k=0}^{\infty} \frac{(t^{\alpha}\lambda_i)^k}{\Gamma(k\alpha+1)}
\end{aligned}$$

$$= \begin{pmatrix} E_{\alpha,1}(\lambda_i t^\alpha) & \frac{1}{1!}\frac{\partial}{\partial \lambda_i}E_{\alpha,1}(\lambda_i t^\alpha) & \dots & \frac{1}{(n_i-1)!}\left(\frac{\partial}{\partial \lambda_i}\right)^{n_i-1}E_{\alpha,1}(\lambda_i t^\alpha) \\ & E_{\alpha,1}(\lambda_i t^\alpha) & \ddots & \vdots \\ & & \ddots & \frac{1}{1!}\frac{\partial}{\partial \lambda_i}E_{\alpha,1}(\lambda_i t^\alpha) \\ & & & E_{\alpha,1}(\lambda_i t^\alpha) \end{pmatrix}.$$

By Remark (4.0.1), if $|\arg(\lambda_i(A))| > \frac{\alpha\pi}{2},\ 1 \le i \le r$ and $t \to \infty$, then $|E_{\alpha,1}(\lambda_i t^\alpha)| \to 0$ and so $\left|\frac{1}{j_i}\left(\frac{\partial}{\partial \lambda_i}\right)^j E_{\alpha,1}(\lambda_i t^\alpha)\right| \to 0,\quad 0 \le j \le n_i - 1,\ 1 \le i \le r.$ Since, $E_{\alpha,1}(\lambda_i t^\alpha) = -\sum_{k=2}^{p}\frac{(\lambda_i t^\alpha)^{-k}}{\Gamma(1-k\alpha)} + O(|(\lambda_i t^\alpha)^{-1-p}|)$, which implies that $|E_{\alpha,1}(\lambda_i t^\alpha)| \to 0$ as $t \to \infty$; and

$$\begin{aligned}
\frac{1}{j!}\left(\frac{\partial}{\partial \lambda_i}\right)^j E_{\alpha,1}(\lambda_i t^\alpha) &= \frac{1}{j!}\left(\frac{\partial}{\partial \lambda_i}\right)^j \left\{-\sum_{k=2}^{p}\frac{(\lambda_i t^\alpha)^{-k}}{\Gamma(1-\alpha k)} \ + \ O(|\lambda_i t^\alpha|^{-1-p})\right\} \\
&= \frac{1}{j!}\left\{-\sum_{k=2}^{p}\frac{(t^\alpha)^{-k}}{\Gamma(1-\alpha k)}\left(\frac{\partial}{\partial \lambda_i}\right)^j \lambda_i^{-k} \ + \ O\left(\frac{\partial}{\partial \lambda_i}\right)^j (|\lambda_i t^\alpha|^{-1-p})\right\} \\
&= -\sum_{k=2}^{p}\frac{(-k-(j+1))\cdots(-k-1)(-k)\lambda_i^{-k-j}t^{-\alpha k}}{j!\Gamma(1-\alpha k)} \ + \\
&\quad O(|\lambda_i|^{-1-p-j}|t^\alpha|^{-1-p}) \\
&= -\sum_{k=2}^{p}\frac{(-1)^j(k+j-1)\cdots(k+1)k\lambda_i^{-k-j}t^{-\alpha k}}{j!\Gamma(1-\alpha k)} \ + \\
&\quad O(|\lambda_i|^{-1-p-j}|t^\alpha|^{-1-p}).
\end{aligned}$$

Multiply and divide the first term on the right hand side of the above equation with $(k-1)!$, we get

$$\begin{aligned}
&= -\sum_{k=2}^{p}\frac{(-1)^j(k+j-1)\cdots(k+1)k(k-1)!\lambda_i^{-k-j}t^{-\alpha k}}{j!\Gamma(1-\alpha k)(k-1)!} \\
&\quad + \ O(|\lambda_i|^{-1-p-j}|t^\alpha|^{-1-p}) \\
&= -\sum_{k=2}^{p}\frac{(-1)^j(k+j-1)!\lambda_i^{-k-j}t^{-\alpha k}}{j!(k-1)!\Gamma(1-\alpha k)} \ + \ O(|\lambda_i|^{-1-p-j}|t^\alpha)|^{-1-p}).
\end{aligned}$$

As $t \to \infty$, $\left|\frac{1}{j!}(\frac{\partial}{\partial \lambda_i})^j E_{\alpha,1}(\lambda_i t^\alpha)\right| \to 0$ for $1 \le j \le n_i - 1$. It follows that
$||y(t)|| = ||y_0 E_{\alpha,\alpha}(At^\alpha)|| \to 0$ as $t \to \infty$,
for any non-zero initial value y_0. Hence proof **(i)** of Theorem (4.1.1) is complete.
(ii) Suppose λ_i be a critical eigenvalue satisfying $|\arg(\lambda_i)| = \dfrac{\alpha\pi}{2}$ with same algebraic and geometric multiplicity equal to one. Now, from (4.1.3) the solution of the given system is

$$\begin{aligned} y(t) &= y_0 E_{\alpha,1}(At^\alpha) \\ &= y_0 Q \text{diag}[E_{\alpha,1}(J_1 t^\alpha), E_{\alpha,1}(J_2 t^\alpha), \dots, E_{\alpha,1}(J_{i-1} t^\alpha), E_{\alpha,1}(J_i t^\alpha), E_{\alpha,1}(J_{i+1} t^\alpha), \\ &\qquad \dots, E_{\alpha,1}(J_n t^\alpha)]Q^{-1}, \end{aligned}$$

where $J_k{}'s$ represent Jordan block matrices of order k, $|\arg(\lambda_k(A))| > \frac{\alpha\pi}{2}$, and $\sum_{k=1}^{i-1} n_k + \sum_{k=i+1}^{r} n_k n_k + 1 = n,\ k = 1, \dots, i-1, i+1, \dots, r.$

By using (4.0.3), we have
$$E_{\alpha,1}(\lambda_i t^\alpha) = \frac{1}{\alpha}(\lambda_i t^\alpha)^{(1-1)/\alpha} \exp((\lambda_i t^\alpha)^{1/\alpha}) - \sum_{k=2}^{p} \frac{(\lambda_i t^\alpha)^{-k}}{\Gamma(1-\alpha k)} + O(|(\lambda_i t^\alpha)|^{-1-p}).$$

Suppose $\lambda_i = r\left(\cos\frac{\alpha\pi}{2} + j\sin\frac{\alpha\pi}{2}\right)$, where r represents the modulus of λ_i, and $j^2 = -1$. Then,

$$\begin{aligned} E_{\alpha,1}(\lambda_i t^\alpha) &= \frac{1}{\alpha}\exp\left\{\left(rt^\alpha\left(\cos\frac{\alpha\pi}{2} + j\sin\frac{\alpha\pi}{2}\right)\right)^{1/\alpha}\right\} - \sum_{k=2}^{p} \frac{(rt^\alpha(\cos\frac{\alpha\pi}{2} + j\sin\frac{\alpha\pi}{2}))^{-k}}{\Gamma(1-\alpha k)} \\ &\quad + O\left(\left|rt^\alpha\left(\cos\frac{\alpha\pi}{2} + j\sin\frac{\alpha\pi}{2}\right)\right|^{-1-p}\right) \\ &= \frac{1}{\alpha}\exp\left\{r^{1/\alpha} t\left(\cos\frac{\pi}{2} + j\sin\frac{\pi}{2}\right)\right\} \\ &\quad - \sum_{k=2}^{p} \frac{(r^{-k} t^{-\alpha k}(\cos\frac{-\alpha k\pi}{2} + j\sin\frac{-\alpha k\pi}{2}))}{\Gamma(1-\alpha k)} + O\left((rt)^{-1-p}\right) \\ &= \frac{1}{\alpha}\exp\left\{jr^{1/\alpha} t\right\} - \sum_{k=2}^{p} \frac{(r^{-k} t^{-\alpha k}(\cos\frac{\alpha k\pi}{2} - j\sin\frac{\alpha k\pi}{2}))}{\Gamma(1-\alpha k)} + O\left((rt)^{-1-p}\right), \end{aligned}$$

which leads to $\left|E_{\alpha,1}(\lambda_i t^\alpha)\right| \to \dfrac{1}{\alpha}$ as $t \to \infty$, and also from the proof of the Theorem (4.1.1)**(i)**, $E_{\alpha,\alpha}(J_k t^\alpha) \to 0$ as $t \to +\infty$ for $k = 1, \dots, i-1, i+1, \dots, r$, we find that the **0**-solution of the given system is stable but not asymptotically stable. □

Remark 4.1.1. In Theorem (4.1.1)**(ii)**, if the critical eigenvalues have their algebraic multiplicities greater then their geometric multiplicities, and the other conditions remain unchanged then its **0**-solution is unstable.

Theorem 4.1.2. *The* 0*-solution of the autonomous same order system* (4.1.2) *with Riemann-Liouville derivative and initial value* $y_0 = {}_0^{RL}\mathcal{D}_t^{\alpha-1}y(t)|_{t=0}$, *where* $0 < \alpha < 1$ *and* $t_0 = 0$, *is*

(i) *asymptotically stable iff all the non-zero eigenvalues of* A *satisfy* $|\arg(\lambda(A))| > \frac{\alpha\pi}{2}$.

(ii) *is stable but not asymptotically stable, if all the eigenvalues of* A *satisfying* $|\arg(\lambda(A))| \geq \frac{\alpha\pi}{2}$ *and the critical eigenvalues satisfying* $|\arg(\lambda(A))| = \frac{\alpha\pi}{2}$ *have the same algebraic and geometric multiplicities.*

Proof. Taking the Laplace transform of (4.1.2) with Riemann-Liouville derivative and $0 < \alpha < 1$

$$s^\alpha Y(s) - \sum_{k=0}^{n-1} s^k[\mathcal{D}^{\alpha-k-1}y(t)]_{t=0} = AY(s), \quad (n-1 \leq \alpha < n).$$

Using the initial condition $y_0 = {}_0^{RL}\mathcal{D}_t^{\alpha-1}y(t)|_{t=0}$, we get

$$\begin{aligned} s^\alpha Y(s) - y_0 &= AY(s), \\ (s^\alpha I - A)Y(s) &= y_0, \\ Y(s) &= [s^\alpha I - A]^{-1}y_0. \end{aligned}$$

Now, taking inverse Laplace transform of the above equation we get the solution of the given system by using (4.0.8) as follows

$$y(t) = y_0 t^{\alpha-1}E_{\alpha,\alpha}(At^\alpha). \tag{4.1.5}$$

First, suppose that A is diagonalizable, i.e., there exists an invertible matrix Q such that

$$\Lambda = Q^{-1}AQ = \mathrm{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n).$$

Then,

$E_{\alpha,\alpha}(At^\alpha) = QE_{\alpha,\alpha}(\Lambda t^\alpha)Q^{-1} = Q\mathrm{diag}[E_{\alpha,\alpha}(\lambda_1 t^\alpha), E_{\alpha,\alpha}(\lambda_2 t^\alpha), \ldots, E_{\alpha,\alpha}(\lambda_n t^\alpha)]Q^{-1}$.

Applying
$E_{\alpha,\alpha}(\lambda_i t^\alpha) = -\sum_{k=2}^{p} \frac{(\lambda_i t^\alpha)^{-k}}{\Gamma(\alpha - k\alpha)} + O(|(\lambda t^\alpha)^{-1-p}|) \to 0$ as $t \to +\infty,\ 1 \le i \le n.$

Thus,

$$\| E_{\alpha,\alpha}(\Lambda t^\alpha) \| = \| \operatorname{diag}[E_{\alpha,\alpha}(\lambda_1 t^\alpha), E_{\alpha,\alpha}(\lambda_2 t^\alpha), \ldots, E_{\alpha,\alpha}(\lambda_n t^\alpha)] \| \to 0.$$

Hence, the conclusion holds.
Next, suppose that A is similar to a Jordan canonical form, i.e., there exists an invertible matrix Q such that

$$J = Q^{-1}AQ = \operatorname{diag}(J_1, J_2, \ldots, J_r).$$

Where $J_i,\ 1 \le i \le r$ is in the following form

$$\begin{pmatrix} \lambda & 1 & & \\ & \lambda & \ddots & \\ & & \ddots & 1 \\ & & & \lambda_i \end{pmatrix}_{n_i \times n_i}$$

and $\sum_{i=1}^{r} n_i = n$.
Obviously, $E_{\alpha,\alpha}(At^\alpha) = Q\operatorname{diag}[E_{\alpha,\alpha}(J_1 t^\alpha), E_{\alpha,\alpha}(J_2 t^\alpha), \ldots, E_{\alpha,\alpha}(J_n t^\alpha)]Q^{-1}$.
Where for $1 \le i \le r$,
$E_{\alpha,\alpha}(J_i t^\alpha) = \sum_{k=0}^{\infty} \frac{(J_i t^\alpha)^k}{\Gamma(k\alpha + \alpha)} = \sum_{k=0}^{\infty} \frac{(t^\alpha)^k}{\Gamma(k\alpha + \alpha)} J_i^k$

$$= \begin{pmatrix} E_{\alpha,\alpha}(\lambda_i t^\alpha) & \frac{1}{1!}\frac{\partial}{\partial\lambda_i}E_{\alpha,\alpha}(\lambda_i t^\alpha) & \ldots & \frac{1}{(n_i-1)!}\left(\frac{\partial}{\partial\lambda_i}\right)^{n_i-1} E_{\alpha,\alpha}(\lambda_i t^\alpha) \\ & E_{\alpha,\alpha}(\lambda_i t^\alpha) & \ddots & \vdots \\ & & \ddots & \frac{1}{1!}\frac{\partial}{\partial\lambda_i}E_{\alpha,\alpha}(\lambda_i t^\alpha) \\ & & & E_{\alpha,\alpha}(\lambda_i t^\alpha) \end{pmatrix}.$$

By Remark (4.0.1), if $|\arg(\lambda_i(A))| > \frac{\alpha\pi}{2},\ 1 \le i \le r$ and $t \to \infty$, then $|E_{\alpha,\alpha}(\lambda_i t^\alpha)| \to 0$ and so $\left|\frac{1}{j_i}\left(\frac{\partial}{\partial\lambda_i}\right)^{j} E_{\alpha,\alpha}(\lambda_i t^\alpha)\right| \to 0,\quad 0 \le j \le n_i - 1,\ 1 \le i \le r.$

Since, $E_{\alpha,\alpha}(\lambda_i t^\alpha) = -\sum_{k=2}^{p} \frac{(\lambda_i t^\alpha)^{-k}}{\Gamma(\alpha - k\alpha)} + O(|(\lambda_i t^\alpha)^{-1-p}|)$, this implies that $|E_{\alpha,\alpha}(\lambda_i t^\alpha)| \to 0$ as $t \to \infty$.

Now by some calculations as we have done in Theorem (4.1.1), we obtain the following

$$\frac{1}{j!}\left(\frac{\partial}{\partial \lambda_i}\right)^j E_{\alpha,\alpha}(\lambda_i t^\alpha) = -\sum_{k=2}^{p} \frac{(-1)^j (k+j-1)! \lambda_i^{-k-j} t^{-\alpha k}}{j!(k-1)!\Gamma(\alpha - \alpha k)} \;+\; O(|\lambda_i|^{-1-p-j}|t^\alpha)|^{-1-p}).$$

As $t \to \infty$, $\left|\frac{1}{j!}\left(\frac{\partial}{\partial \lambda_i}\right)^j E_{\alpha,\alpha}(\lambda_i t^\alpha)\right| \to 0$ for $1 \le j \le n_i - 1$. It follows that $||y(t)|| = ||y_0 t^{\alpha-1} E_{\alpha,\alpha}(At^\alpha)|| \to 0$ as $t \to \infty$, for any non-zero initial value y_0. Hence proof **(i)** is complete.

(ii) Suppose λ_i be a critical eigenvalue, satisfying $|\arg(\lambda_i)| = \dfrac{\alpha\pi}{2}$ with same algebraic and geometric multiplicity equal to one. Now, from (4.1.5) the solution of the given system is

$$\begin{aligned} y(t) &= y_0 t^{\alpha-1} E_{\alpha,\alpha}(At^\alpha) \\ &= y_0 t^{\alpha-1} Q \text{diag}[E_{\alpha,\alpha}(J_1 t^\alpha), E_{\alpha,\alpha}(J_2 t^\alpha), \ldots, E_{\alpha,\alpha}(J_{i-1} t^\alpha), E_{\alpha,\alpha}(J_i t^\alpha), E_{\alpha,\alpha}(J_{i+1} t^\alpha), \\ &\qquad \ldots, E_{\alpha,\alpha}(J_n t^\alpha)] Q^{-1}, \end{aligned}$$

where $J_k's$ represent Jordan block matrices of order k, $|\arg(\lambda_k(A))| > \frac{\alpha\pi}{2}$ and $\sum_{k=1}^{i-1} n_k + \sum_{k=i+1}^{r} n_k n_k + 1 = n,\ k = 1, \ldots, i-1, i+1, \ldots, r$. By using (4.0.3), we have

$$E_{\alpha,\alpha}(\lambda_i t^\alpha) = \tfrac{1}{\alpha}(\lambda_i t^\alpha)^{(1-\alpha)/\alpha} \exp((\lambda_i t^\alpha)^{1/\alpha}) - \sum_{k=2}^{p} \frac{(\lambda_i t^\alpha)^{-k}}{\Gamma(\alpha - \alpha k)} + O(|(\lambda_i t^\alpha)|^{-1-p}).$$

Suppose $\lambda_i = r\left(\cos\frac{\alpha\pi}{2} + j \sin\frac{\alpha\pi}{2}\right)$, where r represents the modulus of λ_i, and $j^2 = -1$. Now by some tedious calculations as we have done in Theorem(4.1.1), we get

$$\begin{aligned} E_{\alpha,\alpha}(\lambda_i t^\alpha) = \frac{1}{\alpha}&\left(r^{(1-\alpha)/\alpha} t^{(1-\alpha)} \left(\sin\frac{\alpha\pi}{2} + j\cos\frac{\alpha\pi}{2}\right)\right) \exp\left\{j r^{1/\alpha} t\right\} \\ &- \sum_{k=2}^{p} \frac{(r^{-k} t^{-\alpha k}(\cos\frac{\alpha k\pi}{2} - j\sin\frac{\alpha k\pi}{2}))}{\Gamma(\alpha - \alpha k)} + O\left((rt)^{-1-p}\right). \end{aligned}$$

Consequently

$$
\begin{aligned}
t^{\alpha-1}E_{\alpha,\alpha}(\lambda_i t^{\alpha}) &= t^{\alpha-1}\frac{1}{\alpha}\left(r^{(1-\alpha)/\alpha}t^{(1-\alpha)}\left(\sin\frac{\alpha\pi}{2}+j\cos\frac{\alpha\pi}{2}\right)\right)\exp\left\{jr^{1/\alpha}t\right\} \\
&\quad - t^{\alpha-1}\sum_{k=2}^{p}\frac{(r^{-k}t^{-\alpha k}(\cos\frac{\alpha k\pi}{2}-j\sin\frac{\alpha k\pi}{2}))}{\Gamma(\alpha-\alpha k)}+t^{\alpha-1}O\left((rt^{\alpha})^{-1-p}\right) \\
&= \frac{1}{\alpha}\left(r^{(1-\alpha)/\alpha}\left(\sin\frac{\alpha\pi}{2}+j\cos\frac{\alpha\pi}{2}\right)\right)\exp\left\{jr^{1/\alpha}t\right\} - \\
&\quad t^{\alpha-1}\sum_{k=2}^{p}\frac{(r^{-k}t^{-\alpha k}(\cos\frac{\alpha k\pi}{2}-j\sin\frac{\alpha k\pi}{2}))}{\Gamma(\alpha-\alpha k)}+O\left(t^{-p\alpha-1}\right),
\end{aligned}
$$

which leads to $\left|t^{\alpha-1}E_{\alpha,\alpha}(\lambda_i t^{\alpha})\right| \to \frac{1}{\alpha}r^{(1-\alpha)/\alpha}, 1 \geq j_i \geq n_i - 1$ as $t \to \infty$. And also from the proof of Theorem(4.1.2)**(i)**, $E_{\alpha,\alpha}(J_k t^{\alpha}) \to 0$ as $t \to +\infty$ for $k = 1, \ldots, i-1, i+1, \ldots, r$, we find that the **0**-solution of the given system is stable but not asymptotically stable. □

Example 4.1.1.

$$
{}_0\mathcal{D}_t^{\frac{1}{4}}\begin{pmatrix} y_1(t) \\ y_2(t) \\ y_3(t) \end{pmatrix} = \begin{pmatrix} -8 & 2 & -3 \\ -1 & -2 & 0.5 \\ 0.2 & -1 & -2 \end{pmatrix}\begin{pmatrix} y_1(t) \\ y_2(t) \\ y_3(t) \end{pmatrix}. \tag{4.1.6}
$$

Here $y \in \mathbb{R}^3$. The eigenvalues of matrix A are $\lambda_{1,2} = -2.18378 \pm 0.99799j$, $\lambda_3 = -7.63244$. As it has found that $|\arg(\lambda_{1,2})| = 2.71293$ and $|\arg(\lambda_3)| = \pi$ such that $|\arg(eig(A))| > \frac{1}{4}.\frac{\pi}{2}$, which satisfy the stability conditions given in Theorem 4.1.2. Therefore the given system is stable.

Remark 4.1.2. The conclusion does not hold if the critical eigenvalues in Theorem (4.1.2)**(ii)** are such that their algebraic multiplicities are greater then their geometric multiplicities and the other conditions remain same.

Theorem 4.1.3. *Suppose all the non-zero eigenvalues of A satisfy $|\arg(A(\lambda))| \geq \frac{\alpha\pi}{2}$, the critical eigenvalues having the same algebraic and geometric multiplicities, satisfy $|\arg(A(\lambda))| =$*
*displaystyle$\frac{\alpha\pi}{2}$, and A has k-multiple zero eigenvalues corresponding to a Jordan block matrix $diag(J_1, J_2, \ldots, J_{n_i})$, where J_l is a Jordan canonical form with order n_l, $\sum_{l=1}^{i} n_l = k$ and $n_l\alpha \leq 1, 1 \leq l \leq i$. Then, the **0**-solution of the system* (4.1.2) *with Riemann-Liouville derivative is stable but not asymptotically stable.*

Proof. If A has k-multiple simple zero eigenvalues, then

$$y(t) = y_0 t^{\alpha-1} Q \text{diag}\left[T. \underbrace{\frac{1}{\Gamma(\alpha)}, \dots, \frac{1}{\Gamma(\alpha)}}_{k} \right] Q^{-1},$$

where T represents all the non-zero eigenvalues of matrix A and the expression $t_{\alpha-1}T$ is bounded as in part **(ii)** of (4.1.2) we have proved this. Thus in this case the **0**-solution is stable but not asymptotically stable.

Now we examine the Jordan canonical form relating to the k-multiple of zero eigenvalues

$$\begin{aligned}
\frac{1}{j!}\left(\frac{\partial}{\partial\lambda}\right)^j E_{\alpha,\alpha}(\lambda t^\alpha)|_{\lambda=0} &= \frac{1}{j!}\left(\frac{\partial}{\partial\lambda}\right)^j \sum_{k=j}^{\infty} \frac{(\lambda t^\alpha)^k}{\Gamma(\alpha k+\alpha)}\Bigg|_{\lambda=0} \\
&= \frac{1}{j!}\sum_{k=j}^{\infty} \frac{t^{\alpha k}}{\Gamma(\alpha k+\alpha)}\left(\frac{\partial}{\partial\lambda}\right)^j \lambda^k\Bigg|_{\lambda=0}
\end{aligned}$$

$$\begin{aligned}
&= \sum_{k=j}^{\infty} \frac{k(k-1)\cdots(k-j+1)t^{\alpha k}\lambda^{k-j}}{j!\Gamma(\alpha k+\alpha)}\Bigg|_{\lambda=0} \\
&= \frac{j(j-1)\cdots(j-j+2)(j-j+1)t^{\alpha j}\lambda^{j-j}}{j!\Gamma(\alpha j+\alpha)}\Bigg|_{\lambda=0} + \\
&\quad \sum_{k=j+1}^{\infty} \frac{k(k-1)\cdots(k-j+1)t^{\alpha k}\lambda^{k-j}}{j!\Gamma(\alpha k+\alpha)}\Bigg|_{\lambda=0} \\
&= \frac{j(j-1)\cdots 2.1 t^{\alpha j}\lambda^0}{j!\Gamma(\alpha j+\alpha)}\Bigg|_{\lambda=0} + \sum_{k=j+1}^{\infty} \frac{k(k-1)\cdots(k-j+1)t^{\alpha k}0^{k-j}}{j!\Gamma(\alpha k+\alpha)} \\
&= \frac{j!t^{\alpha j}}{j!\Gamma(\alpha j+\alpha)} = \frac{t^{\alpha j}}{\Gamma(\alpha j+\alpha)}.
\end{aligned}$$

The eigenvalue having multiplicity n_l relates to the Jordan canonical block of order $n_l \times n_l,\ 1 \le l \le i$, in the following form

$$J_{n_l} = \begin{pmatrix} \frac{1}{\Gamma(\alpha)} & \frac{t^\alpha}{\Gamma(2\alpha)} & \cdots & \frac{t^{(n_l-1)}}{\Gamma(n_l\alpha)} \\ & \frac{1}{\Gamma(\alpha)} & \ddots & \vdots \\ & & \ddots & \frac{t^\alpha}{\Gamma(2\alpha)} \\ & \frac{1}{\Gamma(\alpha)} & & \end{pmatrix}_{n_l \times n_l}.$$

Here $t^{(\alpha-1)}J_{n_l}$ for $1 \le l \le i$, are bounded for $t > t_0 > 0$ under the condition $n_l\alpha \le 1$. So the **0**-solution is stables but not asymptotically stable. Hence the proof of the theorem is complete. □

4.2 Stability Analysis of Perturbed Fractional Differential System

The perturbed system of (4.1.2) with Riemann-Liouville derivative is given by

$$ {}_0^{RL}\mathcal{D}_t^{\alpha} y(t) = Ay(t) + B(t)y(t), \quad (0 < \alpha < 1). \tag{4.2.1}$$

Here $y(t)$ and A are the same as in system (4.1.2), the $n \times n$ matrix $B(t)$ depends on time t and has the initial condition as follows

$$ {}_0^{RL}\mathcal{D}_t^{\alpha-1} y(t)|_{t=0} = y_0. \tag{4.2.2}$$

Theorem 4.2.1. (**a**) *Suppose* $||B(t)||$ *is bounded, i.e.,* $||B(t)|| \le M$ for some $M > 0$*, and all the eigenvalues of matrix* A *satisfy the inequality*

$$|\arg(\lambda(A))| > \frac{\alpha\pi}{2}. \tag{4.2.3}$$

*Then, the **0**-solution of system* (4.2.1) *is asymptotically stable.*

(**b**) *Suppose matrix* $B(t)$ *is bounded in* $[0, \delta)$ *for any small* $\delta > 0$*, also* $B(t) \in L[0, +\infty)^n$*, all the eigenvalues of matrix* A *satisfy inequality*

$$|\arg(\lambda(A))| \ge \frac{\alpha\pi}{2} \tag{4.2.4}$$

and the critical eigenvalues having the same algebraic and geometric multiplicities, satisfy $|\arg(\lambda(A))| = 0$*. Then the **0**-solution of* (4.2.1) *is stable.*

(**c**) *Suppose all the non-zero eigenvalues of matrix* A *satisfy* (4.2.3)*, the critical eigenvalues satisfying* $|\arg(\lambda(A))| = 0$*, have the same algebraic and geometric multiplicities, corresponding to a Jordan block matrix diag*$(J_1, J_2, \ldots, J_{n_i})$ *matrix* A *has k-multiple zero eigenvalues, where* J_l *is a Jordan canonical form* *with order n_l, $\sum_{l=1}^{i} n_l = k$ and $n_l\alpha \le 1,\ 1 \le l \le i$, $B(t)$ is bounded in $[0,\delta)$ for any small $\delta > 0$, and*

$||B(t)|| \le t^{\omega}$ *as* $t \to +\infty,\ \omega \le -\min_{1\le l\le n_l} n_l\alpha, \quad \omega \ne -1,-2,\dots,-m,\dots$

*Then the **0**-solution of (4.2.1) is stable.*

Proof. Taking Laplace transform of (4.2.1) as follows

$$s^{\alpha}Y(s) - \sum_{k=0}^{n-1} s^k[\mathcal{D}^{\alpha-k-1}y(t)]_{t=t_0} = AY(s) + B(s)Y(s), \quad (n-1 \le \alpha < n).$$

Using initial condition, we get

$$\begin{aligned} s^{\alpha}Y(s) - y_0 &= AY(s) + B(s)Y(s) \\ (s^{\alpha} - A)Y(s) &= y_0 + B(s)Y(s) \\ Y(s) &= (s^{\alpha} - A)^{-1}y_0 + (s^{\alpha} - A)^{-1}B(s)Y(s) \end{aligned}$$

Now, for solution of system taking inverse Laplace transform of the above equation as follows

$$y(t) = y_0t^{\alpha-1}E_{\alpha,\alpha}(At^{\alpha}) + \int_0^t (t-\theta)^{\alpha-1}E_{\alpha,\alpha}(A(t-\theta)^{\alpha})B(\theta)y(\theta)d\theta,$$

$$||y(t) = y_0t^{\alpha-1}E_{\alpha,\alpha}(At^{\alpha}) + \int_0^t (t-\theta)^{\alpha-1}E_{\alpha,\alpha}(A(t-\theta)^{\alpha})B(\theta)y(\theta)d\theta||,$$

$$||y(t)|| \le ||y_0t^{\alpha-1}E_{\alpha,\alpha}(At^{\alpha})|| + \int_0^t (t-\theta)^{\alpha-1}||E_{\alpha,\alpha}(A(t-\theta)^{\alpha})|| \cdot ||B(\theta)|| \cdot ||y(\theta)||d\theta.$$

(a) Since all the eigenvalues of A satisfy inequality (4.2.4). By using Gronwall inequality we get

$$\begin{aligned} ||y(t)|| &\le ||y_0t^{\alpha-1}E_{\alpha,\alpha}(At^{\alpha})|| \exp\left\{\int_0^t ||(t-\theta)^{\alpha-1}E_{\alpha,\alpha}(A(t-\theta)^{\alpha})|| \cdot ||B(\theta)||d\theta\right\} \\ &= ||y_0t^{\alpha-1}E_{\alpha,\alpha}(At^{\alpha})|| \exp\left\{\int_0^t ||\theta^{\alpha-1}E_{\alpha,\alpha}(A\theta^{\alpha})|| \cdot ||B(t-\theta)||d\theta\right\} \\ &\le ||y_0t^{\alpha-1}E_{\alpha,\alpha}(At^{\alpha})|| \exp\left\{\int_0^t ||\theta^{\alpha-1}E_{\alpha,\alpha}(A\theta^{\alpha})M||d\theta\right\} \\ &\le ||y_0t^{\alpha-1}E_{\alpha,\alpha}(At^{\alpha})|| \exp\left\{M\int_0^t ||\theta^{\alpha-1}E_{\alpha,\alpha}(A\theta^{\alpha})||d\theta\right\} \end{aligned}$$

(since $||B(t)|| \le M$, so $||B(t-\theta)|| \le M$.)

We first suppose that A is similar to Q which is a diagonal matrix, then

$$\int_0^t ||\theta^{\alpha-1}E_{\alpha,\alpha}(A\theta^\alpha)||d\theta$$
$$= \int_0^t ||Q\text{diag}(\theta^{\alpha-1}E_{\alpha,\alpha}(\lambda_1\theta^\alpha), \theta^{\alpha-1}E_{\alpha,\alpha}(\lambda_2\theta^\alpha), \ldots, \theta^{\alpha-1}E_{\alpha,\alpha}(\lambda_n\theta^\alpha))Q^{-1}||d\theta.$$

Now we shall show that $\exists$ a positive constant N such that

$$\int_0^t |\theta^{\alpha-1}E_{\alpha,\alpha}(\lambda_i\theta^\alpha)|d\theta \leq N, \quad 1 \leq i \leq n.$$

For $t > t_0 > 0$

$$\int_0^t |\theta^{\alpha-1}E_{\alpha,\alpha}(\lambda_i\theta^\alpha)|d\theta$$
$$= \int_0^{t_0} |\theta^{\alpha-1}E_{\alpha,\alpha}(\lambda_i\theta^\alpha)|d\theta + \int_{t_0}^t |\theta^{\alpha-1}E_{\alpha,\alpha}(\lambda_i\theta^\alpha)|d\theta$$
$$= \int_0^{t_0} |\theta^{\alpha-1}E_{\alpha,\alpha}(\lambda_i\theta^\alpha)|d\theta + \int_{t_0}^t \left|\theta^{\alpha-1}\Big(- \sum_{k=2}^{p} \frac{(\lambda_i\theta^\alpha)^{-k}}{\Gamma(\alpha-\alpha k)} + O(|(\lambda_i\theta^\alpha)|^{-1-p})\Big)\right|d\theta,$$
$$\Big(\text{using } (4.0.4)\Big)$$
$$= \int_0^{t_0} |\theta^{\alpha-1}E_{\alpha,\alpha}(\lambda_i\theta^\alpha)|d\theta + \int_{t_0}^t \left| - \sum_{k=2}^{p} \frac{\lambda_i^{-k}\theta^{-k\alpha+\alpha-1}}{\Gamma(\alpha-\alpha k)} + O(|(\lambda_i|^{-1-p}\theta^{-\alpha-\alpha p+\alpha-1})\right|d\theta$$
$$\leq \int_0^{t_0} |\theta^{\alpha-1}||E_{\alpha,\alpha}(\lambda_i\theta^\alpha)|d\theta + \int_{t_0}^t \left\{\left| - \sum_{k=2}^{p} \frac{\lambda_i^{-k}\theta^{-k\alpha+\alpha-1}}{\Gamma(\alpha-\alpha k)}\right| + |O(|\lambda_i|^{-1-p}\theta^{-\alpha p-1})|\right\} d\theta$$
$$= \int_0^{t_0} |\theta^{\alpha-1}|E_{\alpha,\alpha}(|\lambda_i|\theta^\alpha)d\theta + \int_{t_0}^t \sum_{k=2}^{p} \frac{|\lambda_i|^{-k}\theta^{-k\alpha+\alpha-1}}{|\Gamma(\alpha-\alpha k)|}d\theta + \int_{t_0}^t O(|\lambda_i|^{-1-p}\theta^{-\alpha p-1})d\theta$$
$$= \sum_{k=0}^{\infty} \frac{|\lambda_i|^k}{\Gamma(\alpha k+\alpha)} \int_0^{t_0} \theta^{\alpha k+\alpha-1}d\theta + \sum_{k=2}^{p} \frac{|\lambda_i|^{-k}}{|\Gamma(\alpha-\alpha k)|} \int_{t_0}^t \theta^{-k\alpha+\alpha-1}\text{d}\theta + O(|\lambda_i|^{-1-p}t^{-\alpha p})$$
$$= \sum_{k=0}^{\infty} \frac{|\lambda_i|^k t_0^{\alpha k+\alpha}}{(\alpha k+\alpha)\Gamma(\alpha k+\alpha)} + \sum_{k=2}^{p} \frac{|\lambda_i|^{-k}t^{-k\alpha+\alpha}}{(-\alpha k+\alpha)|\Gamma(\alpha-\alpha k)|} - \sum_{k=2}^{p} \frac{|\lambda_i|^{-k}t_0^{-k\alpha+\alpha}}{(-\alpha k+\alpha)|\Gamma(\alpha-\alpha k)|}$$
$$+ O(|\lambda_i|^{-1-p}t^{-\alpha p})$$
$$= \sum_{k=0}^{\infty} \frac{|\lambda_i|^k t_0^{\alpha k+\alpha}}{\Gamma(\alpha k+\alpha+1)} + \sum_{k=2}^{p} \frac{|\lambda_i|^{-k}t^{-k\alpha+\alpha}}{|\Gamma(\alpha-\alpha k+1)|} - \sum_{k=2}^{p} \frac{|\lambda_i|^{-k}t_0^{-k\alpha+\alpha}}{|\Gamma(\alpha-\alpha k+1)|} + O(|\lambda_i|^{-1-p}t^{-\alpha p})$$

$$\sum_{k=2}^{p} \frac{|\lambda_i|^{-k}t^{-k\alpha+\alpha}}{|\Gamma(\alpha-\alpha k+1)|} \to 0 \quad \text{and also} \quad O(|\lambda_i|^{-1-p}t^{-\alpha p}) \to 0 \quad \text{as} \quad t \to \infty.$$

Therefore

$$= \sum_{k=0}^{\infty} \frac{|\lambda_i|^k t_0^{\alpha k+\alpha}}{\Gamma(\alpha k+\alpha+1)} - \sum_{k=2}^{p} \frac{|\lambda_i|^{-k} t_0^{-k\alpha+\alpha}}{|\Gamma(\alpha-\alpha k+1)|} \leq N \quad \text{as} \quad t \to +\infty.$$

It directly follows that $\int_0^t ||\theta^{\alpha-1} E_{\alpha,\alpha}(A\theta^\alpha)||d\theta \leq C_1,$ for any $t \geq 0$.

Next, we suppose that A is similar to Jordan form as in the proof of Theorem (4.1.2)**(i)**.

For $t > t_0 > 0$, we have

$$\int_0^t \left|\theta^{\alpha-1}\frac{1}{j!}\left(\frac{\partial}{\partial\lambda_i}\right)^j E_{\alpha,\alpha}(\lambda_i\theta^\alpha)\right|d\theta$$

$$\begin{aligned}
&= \int_0^{t_0} \left|\theta^{\alpha-1}\frac{1}{j!}\left(\frac{\partial}{\partial\lambda_i}\right)^j E_{\alpha,\alpha}(\lambda_i\theta^\alpha)\right|d\theta + \int_{t_0}^t \left|\theta^{\alpha-1}\frac{1}{j!}\left(\frac{\partial}{\partial\lambda_i}\right)^j E_{\alpha,\alpha}(\lambda_i\theta^\alpha)\right|d\theta \\
&\leq \int_0^{t_0} \left|\theta^{\alpha-1}\frac{1}{j!}\left(\frac{\partial}{\partial\lambda_i}\right)^j \sum_{k=0}^{\infty}\frac{(\lambda_i\theta^\alpha)^k}{\Gamma(\alpha k+\alpha)}\right|d\theta \\
&\quad + \int_{t_0}^t \left|\theta^{\alpha-1}\frac{1}{j!}\left(\frac{\partial}{\partial\lambda_i}\right)^j \left\{-\sum_{k=2}^{p}\frac{(\lambda_i\theta^\alpha)^{-k}}{\Gamma(\alpha-\alpha k)} + O(|(\lambda_i\theta^\alpha)|^{-1-p})\right\}\right|d\theta \\
&= \int_0^{t_0} \sum_{k=0}^{\infty} \frac{k(k-1)\cdots(k-j+1)|\lambda_i|^{k-j}\theta^{\alpha k+\alpha-1}}{j!\Gamma(\alpha k+\alpha)}d\theta \\
&\quad + \int_{t_0}^t \left|\theta^{\alpha-1}\left\{-\sum_{k=2}^{p}\frac{(-k)(-k-1)\cdots(-k-j+1)\lambda_i^{-k-j}\theta^{-\alpha k}}{j!\Gamma(\alpha-\alpha k)} + \right.\right. \\
&\quad \left.\left. O(|\lambda_i|^{-1-p-j}\theta^{-\alpha p-\alpha})\right\}\right|d\theta \\
&\leq \sum_{k=0}^{\infty} \frac{k(k-1)\cdots(k-j+1)|\lambda_i|^{k-j}}{j!\Gamma(\alpha k+\alpha)} \int_0^{t_0} \theta^{\alpha k+\alpha-1}d\theta \\
&\quad + \int_{t_0}^t \theta^{\alpha-1}\left\{\sum_{k=2}^{p}\frac{|(-1)^j|(k+j-1)\cdots(k)(k+1)(k-1)!|\lambda_i|^{-k-j}\theta^{-\alpha k}}{j!(k-1)!|\Gamma(\alpha-\alpha k)|}\right. \\
&\quad \left. + O(|\lambda_i|^{-1-p-j}\theta^{-\alpha p-\alpha})\right\}d\theta
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{k=0}^{\infty} \frac{k(k-1)\cdots(k-j+1)|\lambda_i|^{k-j}}{j!(\alpha k+\alpha)\Gamma(\alpha k+\alpha)} t_0^{\alpha k+\alpha} \\
&\quad + \sum_{k=2}^{p} \frac{(k+j-1)!|\lambda_i|^{-k-j}}{j!(k-1)!|\Gamma(\alpha-\alpha k)|} \int_{t_0}^{t} \theta^{-\alpha k+\alpha-1} d\theta + \int_{t_0}^{t} O(|\lambda_i|^{-1-p-j}\theta^{-\alpha p-1}) d\theta \\
&= \sum_{k=0}^{\infty} \frac{k(k-1)\cdots(k-j+1)|\lambda_i|^{k-j} t_0^{\alpha k+\alpha}}{j!\Gamma(\alpha k+\alpha+1)} \\
&\quad + \sum_{k=2}^{p} \frac{(k+j-1)!|\lambda_i|^{-k-j}}{j!(k-1)!(-\alpha k+\alpha)|\Gamma(\alpha-\alpha k)|} \big(t^{-\alpha k+\alpha} - t_0^{-\alpha k+\alpha}\big) + O(|\lambda_i|^{-1-p-j}\theta^{-\alpha p}) d\theta \\
&= t_0^{\alpha} \frac{1}{j!} \left(\frac{\partial}{\partial|\lambda_i|}\right)^j E_{\alpha,\alpha+1}(|\lambda_i| t_0^{\alpha}) \\
&\quad + \sum_{k=2}^{p} \frac{(k+j-1)!|\lambda_i|^{-k-j} t^{-\alpha k+\alpha}}{j!(k-1)!|\Gamma(\alpha-\alpha k+1)|} - \sum_{k=2}^{p} \frac{(k+j-1)!|\lambda_i|^{-k-j} t_0^{-\alpha k+\alpha}}{j!(k-1)!|\Gamma(\alpha-\alpha k+1)|} \\
&\quad + O(|\lambda_i|^{-1-p-j}\theta^{-\alpha p}) d\theta, \\
&= t_0^{\alpha} \frac{1}{j!} \left(\frac{\partial}{\partial|\lambda_i|}\right)^j E_{\alpha,\alpha+1}(|\lambda_i| t_0^{\alpha}) - \sum_{k=2}^{p} \frac{(k+j-1)!|\lambda_i|^{-k-j} t_0^{-\alpha k+\alpha}}{j!(k-1)!|\Gamma(\alpha-\alpha k+1)|} \le C_2 \text{ as } t \to +\infty,
\end{aligned}
$$

where $1 \le j \le n_i - 1$. So $\exp\left\{M \int_0^t ||\theta^{\alpha-1} E_{\alpha,\alpha}(A\theta^{\alpha})|| d\theta\right\}$ is bounded.
Also, we find that $||y_0 t^{\alpha-1} E_{\alpha,\alpha}(At^{\alpha})|| \to 0$ as $t \to +\infty$. Thus, we have $\lim\limits_{t\to+\infty} = 0$. Thus part **(a)** of the Theorem (4.2.1) is completed.

(b) Without loss of generality we can assume the case when there exist one critical eigenvalue, say λ_i, satisfying $|\arg(\lambda_i)| = \frac{\alpha\pi}{2}$ with same algebraic and geometric multiplicity both equal to one. We can see from the proof (ii) of Theorem(4.1.2), that

$$
|E_{\alpha,\alpha}(\lambda_i t^{\alpha})| \le \frac{1}{\alpha} |\lambda_i|^{(1-\alpha)/\alpha} t^{1-\alpha} + O(t^{-2\alpha}) \text{ as } t \to +\infty,
$$

which implies that

$$
|t^{\alpha-1} E_{\alpha,\alpha}(\lambda_i t^{\alpha})| \le \frac{1}{\alpha} |\lambda_i|^{(1-\alpha)/\alpha} + O(t^{-1-\alpha}) \text{ as } t \to +\infty.
$$

So, it immediately follows that

$\int_0^t |\theta^{\alpha-1}E_{\alpha,\alpha}(\lambda_i\theta^\alpha)|\cdot||B(t-\theta)||d\theta$

$$= \int_0^t |\theta^{\alpha-1}E_{\alpha,\alpha}(\lambda_i\theta^\alpha)|\cdot||B(t-\theta)||d\theta + \int_{t_0}^t |\theta^{\alpha-1}E_{\alpha,\alpha}(\lambda_i\theta^\alpha)|\cdot||B(t-\theta)||d\theta$$
$$\le \int_0^t |\theta^{\alpha-1}E_{\alpha,\alpha}(\lambda_i\theta^\alpha)|\cdot||B(t-\theta)||d\theta + \int_{t_0}^t \frac{1}{\alpha}|\lambda_i|^{(\alpha-1)/\alpha}\cdot||B(\theta)||d\theta$$
$$\le \int_0^t |\theta^{\alpha-1}E_{\alpha,\alpha}(\lambda_i\theta^\alpha)|\cdot||B(t-\theta)||d\theta + \frac{1}{\alpha}|(\lambda_i|^{(\alpha-1)/\alpha}\int_{t_0}^t ||B(\theta)||d\theta$$

So by the assumption on $B(t)$, we get
$\int_0^t |\theta^{\alpha-1}E_{\alpha,\alpha}(\lambda_i\theta^\alpha)|\cdot||B(t-\theta)||\mathrm{d}\theta \le C_3$. By using Theorem (4.2.1)(a) and the proof of Theorem (4.1.2)(ii), we obtain
$||y_t|| \le C_4||y_0||$.
The proof of Theorem (4.2.1)(b) is complete.

(c) We shall discuss the case of only zero-eigenvalue, in view of **(a)** and **(b)** of Theorem (4.2.1). Suppose that $\exists$ n_l-multiple zero eigenvalues $\lambda_l = 0$. We find that
$\int_0^t |\theta^{\alpha-1}E_{\alpha,\alpha}(\lambda_i\theta^\alpha)|_{\lambda_l=0}\cdot||B(t-\theta)||d\theta$

$$= \int_0^{t_0} |\theta^{\alpha-1}E_{\alpha,\alpha}(\lambda_i\theta^\alpha)|_{\lambda_l=0}||B(t-\theta)||d\theta + \int_{t_0}^t |\theta^{\alpha-1}E_{\alpha,\alpha}(\lambda_i\theta^\alpha)|_{\lambda_l=0}||B(t-\theta)||d\theta$$
$$= \int_0^{t_0} \left|\theta^{\alpha-1}\sum_{k=0}^\infty \frac{(\lambda_l\theta^\alpha)^k}{\Gamma(\alpha k+\alpha)}\right|_{\lambda_l=0}||B(t-\theta)||d\theta + \int_{t_0}^t \left|\theta^{\alpha-1}\sum_{k=0}^\infty \frac{(\lambda_l\theta^\alpha)^k}{\Gamma(\alpha k+\alpha)}\right|_{\lambda_l=0}||B(t-\theta)||d\theta$$
$$= \int_0^{t_0} \theta^{\alpha-1}\sum_{k=0}^\infty \frac{|\lambda_l|^k\theta^{\alpha k}}{\Gamma(\alpha k+\alpha)}\bigg|_{\lambda_l=0}||B(t-\theta)||d\theta + \int_{t_0}^t \theta^{\alpha-1}\sum_{k=0}^\infty \frac{|\lambda_l|^k\theta^{\alpha k}}{\Gamma(\alpha k+\alpha)}\bigg|_{\lambda_l=0}||B(t-\theta)||d\theta$$
$$= \int_0^{t_0} \theta^{\alpha-1}\left\{\frac{1}{\Gamma(\alpha)} + \sum_{k=1}^\infty \frac{|\lambda_l|^k\theta^{\alpha k}}{\Gamma(\alpha k+\alpha)}\bigg|_{\lambda_l=0}\right\}\cdot||B(t-\theta)||d\theta$$
$$+ \int_{t_0}^t \theta^{\alpha-1}\left\{\frac{1}{\Gamma(\alpha)} + \sum_{k=1}^\infty \frac{|\lambda_l|^k\theta^{\alpha k}}{\Gamma(\alpha k+\alpha)}\right\}\bigg|_{\lambda_l=0}\cdot||B(t-\theta)||d\theta$$
$$= \int_0^{t_0} \frac{1}{\Gamma(\alpha)}\theta^{\alpha-1}||B(t-\theta)||d\theta + \int_{t_0}^t \frac{1}{\Gamma(\alpha)}\theta^{\alpha-1}||B(t-\theta)||d\theta$$
$$= \frac{1}{\Gamma(\alpha)}\left(\int_0^{t_0} \theta^{\alpha-1}||B(t-\theta)||d\theta + \int_{t_0}^t \theta^{\alpha-1}||B(t-\theta)||\right)d\theta$$

is bounded due to the assumptions on $B(t)$. Also,

$$\int_0^t \left|\frac{\theta^{\alpha-1}}{j!}\left(\frac{\partial}{\partial\lambda_l}\right)^j E_{\alpha,\alpha}(\lambda_l\theta^\alpha)\right|_{\lambda_l=0} ||B(t-\theta)||d\theta$$

$$= \int_0^t \left|\frac{\theta^{\alpha-1}}{j!}\left(\frac{\partial}{\partial\lambda_l}\right)^j \sum_{k=0}^{\infty}\frac{\lambda_l^k\theta^{\alpha k}}{\Gamma(\alpha k+\alpha)}\right|_{\lambda_l=0} ||B(t-\theta)||d\theta$$

$$= \int_0^t \frac{\theta^{\alpha-1}}{j!}\sum_{k=0}^{\infty}\left(\frac{\partial}{\partial|\lambda_l|}\right)^j \frac{|\lambda_l|^k\theta^{\alpha k}}{\Gamma(\alpha k+\alpha)}\bigg|_{\lambda_l=0} ||B(t-\theta)||d\theta$$

$$= \int_0^t \frac{\theta^{\alpha-1}}{j!}\sum_{k=0}^{\infty}\frac{k(k-1)\cdots(k-j+1)|\lambda_l|^{k-j}\theta^{\alpha k}}{\Gamma(\alpha k+\alpha)}\bigg|_{\lambda_l=0} ||B(t-\theta)||d\theta$$

$$= \int_0^t \frac{\theta^{\alpha-1}}{j!}\sum_{k=0}^{\infty}\frac{k!|\lambda_l|^{k-j}\theta^{\alpha k}}{(k-j)!\Gamma(\alpha k+\alpha)}\bigg|_{\lambda_l=0} ||B(t-\theta)||d\theta$$

$$= \int_0^t \frac{\theta^{\alpha-1}}{j!}\Bigg(\sum_{k=0}^{j-1}\frac{k!|\lambda_l|^{k-j}\theta^{\alpha k}}{(k-j)!\Gamma(\alpha k+\alpha)} + \frac{j!\theta^{\alpha j}}{\Gamma(\alpha j+\alpha)}$$

$$+ \sum_{k=j-1}^{\infty}\frac{k!|\lambda_l|^{k-j}\theta^{\alpha k}}{(k-j)!\Gamma(\alpha k+\alpha)}\Bigg)\bigg|_{\lambda_l=0} \cdot ||B(t-\theta)||d\theta.$$

For $t > t_0 > 0$,

$$\int_0^t \left|\frac{\theta^{\alpha-1}}{j!}\left(\frac{\partial}{\partial\lambda_l}\right)^j E_{\alpha,\alpha}(\lambda_l\theta^\alpha)\right|_{\lambda_l=0} ||B(t-\theta)||d\theta$$

$$= \frac{1}{\Gamma(\alpha j+\alpha)}\left[\int_0^{t_0}\theta^{\alpha j+\alpha-1}||B(t-\theta)||d\theta + \int_{t_0}^t \theta^{\alpha k+\alpha-1}||B(t-\theta)||d\theta\right]$$

is also bounded due to the assumptions on $B(t)$. Combining all these with the proof of Theorem (4.1.3) and Theorem(4.2.1)(a)(b), we find that the zero solution of (4.2.1) is stable. Thus, the proof is complete. □

Chapter 5

Stability of Ordinary and Fractional Delay Differential Equations

The earliest systematic study of delay differential equations was made by Myskis [21] in Russia. An introductory concepts of the theory can be found in Elsgolts and Norkin [9], and Bellman and Cooke [3]. Delay differential equations have their applications in various fields of science such as physics, engineering economics, mathematical biology, statistics and social sciences. They also serve as modeling tools in several different areas of applied mathematics that include the analyzes of age structured population growth, in the study of epidemics, traffic flow and also problems based on the engineering of high-rise buildings for earthquake protection.

Here In this chapter we will discuss some basic concepts and definitions of delay differential equations (ordinary and fractional). Also we will study the stability behaviour of ordinary and fractional delay differential equations.

Definition 5.0.1. Functional differential equation
An equation is called a functional differential equation for an unknown function y, if it involves the derivatives of the function y and also the function y, and possibly its derivative(s) with various different arguments. Functional differential equations are also known as differential equations with deviating arguments.

A general first-order functional differential equation is as follows

$$\dot{y}(t) = f(t, y(t), y(u(t))). \tag{5.0.1}$$

Delay differential equations, known as difference differential equations, are an important class of differential equations called functional differential equations. In the 18th century the delay differential equations were firstly introduced by Laplace and Condorcet [11].

A general form of a delay differential equation for $y(t) \in \mathbb{R}$ is as follows

$$\dot{y}(t) = f(t, y(t), y(t - \tau_1), y(t - \tau_2), \ldots, y(t - \tau_k)); \quad t \geq t_0, \tag{5.0.2}$$

where the quantities $\tau_i,\ i = 1, 2, \ldots, k$, are known as delays(or time lags), they may be constants, functions of time t i.e., $\tau_i(t)$, called 'time-dependent delays', or functions that may depend on solution $y(t)$ i.e., $\tau_i(t, y(t))$, called 'state-dependent delay.' The delay equations with delays of the derivatives are known as 'neutral delay differential equations' (NDDEs).

5.1 Linearized Stability Analysis

An equilibrium point in the state space is a point for which $y(t) = \bar{y}$ is a solution for all t. Therefore for a DDE (5.0.2), the equilibrium point satisfy

$$f(\bar{y}, \bar{y}, \bar{y}, \ldots, \bar{y}) = 0.$$

When we analyze the stability of the equilibrium point of ODDEs, we consider that the system has been displaced through a small distance from the equilibrium. For ODEs, the phase space is a finite dimensional coordinate space, so the concept for the stability of the delay differential equations is same except the phase space, in this case, is an infinite-dimensional function space. Therefore we assume the displacements from equilibrium in infinite-dimensional function space such that our displacements are time-dependent functions $\delta y(t)$ continuing over an interval of at least of the longest delay τ_{max}.

Let $\bar{y}$ be the equilibrium point of the delay equation (5.0.2), and consider that the

system has been disturbed from equilibrium by a small perturbation which lasts from $t = t_0 - \tau_{max}$ *to* t_0.
Let the displacement from equilibrium be $\delta y(t)$, which we have assumed to be small, taken at any time in open interval $[t, \tau_{max})$. So then

$$y = \bar{y} + \delta y,$$

and

$$\dot{y} = \delta\dot{y} = f(\bar{y} + \delta y, \bar{y} + \delta y(t - \tau_1), \bar{y} + \delta y(t - \tau_2), \ldots, , \bar{y} + \delta y(t - \tau_k)).$$

Since the quantities $\delta y, \delta y(t - \tau_1), \delta y(t - \tau_2, \ldots, \delta y(t - \tau_k)$ are small, so we can, by using Taylor series method linearize the differential equation about the equilibrium point as follows

$$\delta\dot{y} \approx J_0\delta y(t) + J(t-\tau_1)\delta y(t-\tau_1) + J(t-\tau_2)\delta y(t-\tau_2) + \cdots + J(t-\tau_k)\delta y(t-\tau_k), \quad (5.1.1)$$

where J_0 is the Jacobian with respect to y calculated at the equilibrium point, while the quantities $J(t-\tau_i)$ are the Jacobian matrices with respect to $y(t-\tau_i)$ calculated at the equilibrium points $y = y(t - \tau_1) = y(t - \tau_2) = \cdots = y(t - \tau_k) = \bar{y}$.

For linear ordinary differential equations the exponential functions of time, in which the exponents contain the eigenvalues of the Jacobian matrices, are the solutions. Assume that the linear delay differential equation (5.1.1) also has solutions as exponential functions, then we can write

$$\delta y(t) = Ae^{\lambda t}.$$

Using the above equation in (5.1.1), we get

$$\begin{aligned}
\lambda Ae^{\lambda t} &= J_0Ae^{\lambda t} + J(t - \tau_1)Ae^{\lambda(t-\tau_1)} + J(t - \tau_2)Ae^{\lambda(t-\tau_2)} + \cdots + J(t - \tau_k)Ae^{\lambda(t-\tau_k)} \\
&= (J_0 + J(t - \tau_1)e^{-\lambda\tau_1} + J(t - \tau_2)e^{-\lambda\tau_2} + \cdots + J(t - \tau_k)e^{-\lambda\tau_k})Ae^{\lambda t} \\
\lambda AA^{-1} &= J_0 + J(t - \tau_1)e^{-\lambda\tau_1} + J(t - \tau_2)e^{-\lambda\tau_2} + \cdots + J(t - \tau_k)e^{-\lambda\tau_k},
\end{aligned}$$

or

$$\lambda I = J_0 + J(t - \tau_1)e^{-\lambda\tau_1} + J(t - \tau_2)e^{-\lambda\tau_2} + \cdots + J(t - \tau_k)e^{-\lambda\tau_k}.$$

We know form linear algebra theory that this equation can only be satisfied by non-zero amplitude A if,

$$|J_0 + J(t-\tau_1)e^{-\lambda\tau_1} + J(t-\tau_2)e^{-\lambda\tau_2} + \cdots + J(t-\tau_k)e^{-\lambda\tau_k} - \lambda I| = 0, \qquad (5.1.2)$$

which is known as the characteristic equation of the equilibrium point. By expanding the determinant, we will find the equations with polynomial parts which have some terms in $e^{\lambda\tau_i}$. These polynomials are called quasi-polynomials.

(**a**) The equilibrium point is stable, if all solutions of the characteristic equation (5.1.2) have negative real parts.

(**b**) The equilibrium point is unstable, if any solution of the characteristic equation (5.1.2) have positive real parts.

(**c**) No conclusion is obtained about the equilibrium point, if the leading values are zero.

Example 5.1.1. Consider a delay differential equation as follows

$$\dot{y} = y(t-1). \qquad (5.1.3)$$

As it is a single equation and also the matrices become scaler, so it would be fairly simple. Here the delay $\tau = 1$, the equilibrium point is clearly $\bar{y} = 0$ and, $J_0 = 0$ and $J_1 = -1$ are the Jacobians.

As the determinant of a single number is that number itself, therefore we get the characteristic equation (5.1.2) as

$$Y(\lambda) = e^{-\lambda} + \lambda = 0. \qquad (5.1.4)$$

At $\lambda = 0$, $Y(\lambda)$ has absolute minimum of 1, so no real solutions exist for the characteristic equation (5.1.4).

Next to find the complex solutions, we write

$$\lambda = \alpha + i\beta, \qquad (5.1.5)$$

where the real and imaginary parts of λ are, respectively, α and β. Substituting (5.1.5) into (5.1.4), we get

$$e^{-(\alpha+i\beta)} + (\alpha + i\beta) = 0,$$

or

$$e^{-\alpha}e^{-i\beta} + \alpha + i\beta = 0,$$

or

$$e^{-\alpha}(\cos\beta - i\sin\beta) + \alpha + i\beta = 0, \tag{5.1.6}$$

$$\alpha + e^{-\alpha}\cos\beta + i(\beta + \sin\beta) = 0.$$

Then

$$e^{-\alpha}\cos\beta = -\alpha, \tag{5.1.7}$$

and

$$\sin\beta = -\beta. \tag{5.1.8}$$

Now we want to know that whether the equations (5.1.7) and (5.1.8) can have solutions having positive values of the real part α. It is noted that the characteristic values arise in complex-conjugate pairs. Therefore, if (α, β) is the solution of (5.1.6), then so is $(\alpha, -\beta)$. Therefore we can restrict ourself only to positive values. Now assume that there exist solutions with positive α. Then in (5.1.7) we must have $\cos\beta < 0$. So this means that $\beta > \frac{\pi}{2}$, since $\cos\beta$ is positive for any other smaller, positive values of β. On the other hand, if $\alpha > 0$ then $e^{-\alpha} < 1$, and $|\sin\beta| < 1$, which implies that $|\beta| < 1$ from equation (5.1.8). This gives a contradiction, since β cannot simultaneously be smaller in magnitude than 1 and larger than $\frac{\pi}{2}$. Therefore the equilibrium point is stable, as the characteristic value cannot have real part with positive value.

5.2 Stability Analysis of Delay Differential Equation Using Sturm Sequences.

A stable equilibrium point can become unstable if, via increasing the length of delay, the eigenvalues of the delay differential equation transform from having negative real parts to having positive real parts, and this happens only when they traverse the imaginary axis.

5.2.1 Existence of Critical Delay

The characteristic equation of a delayed differential equation, at the equilibrium point determine for $\tau = 0$, has the following form

$$P(\lambda, \tau) \equiv P_1(\lambda) + P_2(\lambda)e^{-\lambda\tau} = 0, \tag{5.2.1}$$

where τ is the time delay and P_1 and P_2 are the two polynomials in λ. Rewriting (5.2.1) as

$$\sum_{k=0}^{N} a^k \lambda^k + e^{-\lambda\tau} \sum_{k=0}^{M} b^k \lambda^k = 0.$$

And assume that in the absence of delay the equilibrium point is stable. Then for $\tau = 0$, all of the roots of the polynomial will have negative real part. These roots will change if, τ varies. Here we are observing for any critical value of τ for which a root of this equation changes from having negative real parts to having positive real parts. If this is to happen, then at this critical value of τ, the characteristic equation will have a purely imaginary root[9]. Now we are interested in determining whether or not such a critical value of τ exist, by reducing the equation (5.2.1) to a polynomial problem and then seeking for the particular types of roots.

We begin by considering a purely imaginary root $i\beta$, $\beta \in \mathbb{R}$, of equation (5.2.1),

$$P_1(i\beta) + e^{-i\beta\tau} P_2(i\beta) = 0.$$

We can break the polynomial into real and imaginary parts and also write the exponential in the form of trigonometric functions as follows

$$R_1(\beta) + iQ_1(\beta) + (R_2(\beta) + iQ_2(\beta))(\cos(\beta) - i\sin(\beta)) = 0. \tag{5.2.2}$$

Since $i\beta$ is purely imaginary, so R_1, R_2 are even polynomials and Q_1, Q_2 are odd polynomials. In order for equation (5.2.2) to hold, both the parts(real and imaginary) must be zero such that

$$R_1(\beta) + R_2(\beta)\cos(\beta) + Q_2(\beta)\sin(\beta) = 0,$$

or

$$R_2(\beta)\cos(\beta) + Q_2(\beta)\sin(\beta) = -R_1(\beta). \tag{5.2.3}$$

And

$$Q_1(\beta) - R_2(\beta)\sin(\beta) + Q_2(\beta)\cos(\beta) = 0,$$

or

$$R_2(\beta)\sin(\beta) - Q_2(\beta)\cos(\beta) = Q_1(\beta). \tag{5.2.4}$$

Squaring equations (5.2.3) and (5.2.4) and adding the results yields

$$R_1(\beta)^2 + Q_1(\beta)^2 = R_2(\beta)^2 + Q_2(\beta)^2, \tag{5.2.5}$$

or

$$R_1(\beta)^2 + Q_1(\beta)^2 - R_2(\beta)^2 - Q_2(\beta)^2 = 0. \tag{5.2.6}$$

This is a polynomial as there is no delay τ and no trigonometric terms. Also it is an even polynomial because squaring an odd or an even function gives an even function, i.e., $f(-y)^2 = (\pm f(y))^2 = f(y)^2$.

A new variable is defined as $\gamma = \beta^2 \in \mathbb{R}$. Then equation (5.2.6) can becomes in terms of γ as

$$S(\gamma) = 0, \tag{5.2.7}$$

where S is a polynomial. Here we are interested in $\beta \in \mathbb{R}$, and thus S has all of the roots negative, we will have revealed that no simultaneous solution β^* of equations. (5.2.3) and (5.2.4) can be exist. Conversely, if polynomial S has a positive real root γ^*, then there is a time delay τ corresponding to $\beta^* = \pm\sqrt{\gamma^*}$, which solve both the equations (5.2.3) and (5.2.4).

Now we find the roots of the equation. (5.2.6). Taking $\lambda = i\beta$, then we rewrite (5.2.1) as

$$-\frac{P_1(i\beta)}{P_2(i\beta)} = e^{-i\beta\tau}. \tag{5.2.8}$$

As β varies, in the complex plane plotting the right-hand side traces out a unit circle and the left-hand side yields a rational curve. By the intersection of these two curves we obtain the critical delays for which we are looking. Thus finding the values of β, for which the left-hand side of equation.(5.2.8) has modulus 1. This reproduces the equation. (5.2.5) and the freely choose of τ ensures that for some τ^* the original characteristic polynomial (5.2.1) is satisfied(see [10]).

5.3 Positive Real Roots and Sturm Sequences

Once the polynomial (5.2.7) is obtain, we must determine whether any positive real roots exist for this polynomial. To determine the root we might take many different approaches, e.g., for a characteristic polynomial of degree two, a quadratic formula is always used. There are also some explicit algorithms, for third and fourth degree characteristic polynomials (see [15] or [17]). To determining the existence of a positive real root, a simple method is "Descartes rule of signs is used. this method gives "the number of sign changes in the coefficients is equal to the number of positive real roots, modulo 2." If odd time a sign changes, then the existence of a solution is guaranteed. If a sign changes even time, then this rule is not helpful.

To this problem a more general approach is Sturm sequences. Suppose that p is a polynomial with no repeated roots. Then the polynomial p and its derivative $\dot{P}$ are relatively prime. Let $p = p_0$ and $p = p_1$, then by the division algorithm a sequence of equations are obtained as follows

$$\begin{aligned}
p_0(y) &= q_0 p_1(y) - p_2(y) \\
p_1(y) &= q_1 p_2(y) - p_3(y) \\
&\vdots \\
p_{s-2}(y) &= q_{s-2} p_{s-1}(y) - K.
\end{aligned}$$

Where K is some constants. The sequence $p_0, p_1, p_2, \ldots, p_{s-1}, p_s (= K)$ of *sturm functions* is called a *sturm chain*. In any interval, the number of real roots of $p(y)$ can be determined by putting each endpoint of that interval and in this way a sequence of signs is obtained. In the sequence the difference between the number of sign changes at each end point gives the number of real roots in the interval.

Example 5.3.1. Let $p(y) = y^2 - 2$, then $\dot{P} = 2y$. By division algorithm

$$y^2 - 2 = \frac{y}{2}.2y - 2.$$

So $y^2 - 2, 2y, 2$ is the sturm chain. We consider the interval $[0, \infty)$, then the sign sequences will be

$$\text{at } 0 : -, 0, +, \ \textit{and}$$

$$\text{at } \infty : +, +, +.$$

Since in the first sequence only one sign change occurs, while in the second sequence there is no sign change, so the polynomial $p(y)$ has one positive real root in $[0, \infty)$. Similarly, if we take the interval $[-3, 3]$, then the sign sequences will be

$$\text{at } -3 : +, -, +, \; and$$

$$\text{at } 3 : +, +, +.$$

In the first sequence two sign changes and in the second no sign change are observed, so we conclude that $p(y)$ has two positive real roots in this interval.

5.4 Stability Analysis of Fractional Delay Differential Equations

The stability analysis of the time-delayed fractional differential is as important as of fractional differential equations. Recently, Chen and Moore [7] analyzed the stability of 1-dimensional fractional systems with retard time. Here in this section we will review a result based on the stability of the following time-delayed fractional differential system form the paper of Qian, Li, Agarwal and Wong [24].

Consider the following time-delayed fractional differential system involving multiple Riemann-Liouville derivative as follows

$$\begin{cases} {}_0^{RL}\mathcal{D}_t^{\alpha_1} y_1(t) = a_{11}y_1(t-\tau_{11}) + a_{12}y_2(t-\tau_{12}) + \cdots + a_{1n}y_n(t-\tau_{1n}), \\ {}_0^{RL}\mathcal{D}_t^{\alpha_2} y_2(t) = a_{21}y_1(t-\tau_{21}) + a_{22}y_2(t-\tau_{22}) + \cdots + a_{2n}y_n(t-\tau_{2n}), \\ \cdots \\ {}_0^{RL}\mathcal{D}_t^{\alpha_n} y_n(t) = a_{n1}y_1(t-\tau_{n1}) + a_{n2}y_2(t-\tau_{n2}) + \cdots + a_{nn}y_n(t-\tau_{nn}), \end{cases} \tag{5.4.1}$$

where $\alpha_i \in (0,1),\; i = 1, 2, \cdots n$. The multiple fractional order of system (5.4.1) is $\alpha = (\alpha_1, \alpha_2, \cdots \alpha_n)$. The initial condition is given by
${}_0^{Rl}\mathcal{D}_t^{\alpha_i - 1} y_i(t) = \phi_i(t) \in C^0[-\tau_{max}, 0], \quad 1 \le i \le n,$ where $\tau_{max} = \max\limits_{1 \le ij \le n} \tau_{ij}$.
But here we shall set

$${}_{t_0}^{RL}\mathcal{D}_t^{\alpha-1} y(t)|_{t=t_0} = y_{i0}, \quad 1 \le i \le n.$$

Laplace Transform of Time-Delayed Fractional Differential Equation

The Laplace transform of a time-delayed fractional differential equation is to be taken in the same way as for the ordinary differential equation.

Consider the following equation

$$\dot{y}(t) = y(t-\tau), \quad t \geq 0, \tag{5.4.2}$$

with initial condition

$$y(t) = y_0, \quad t \in [-\tau, 0]. \tag{5.4.3}$$

Now taking the Laplace transform

$$\int_0^{\infty} e^{-st}\dot{y}(t)dt = \int_0^{\infty} e^{-st}y(t-\tau)dt, \tag{5.4.4}$$

substituting $t-\tau = u$ i.e., $t = u+\tau$, then $dt = du$, also when $t \to 0,\ y \to -\tau$ and when $t \to \infty,\ y \to \infty$ in the R.H.S of the above equation. Then

$$\begin{aligned} sY(s) - y_0 &= \int_{-\tau}^{\infty} e^{-s(u+\tau)}y(u)du \\ &= e^{-s\tau}\Big[\int_{-\tau}^{0} e^{-su}y(u)du + \int_0^{\infty} e^{-su}y(u)du\Big] \\ &= e^{-s\tau}\Big[y_0 \frac{e^{-su}}{-s}\Big|_{-\tau}^{0} + Y(s)\Big] \\ &= e^{-s\tau}\Big[\frac{y_0}{-s}(1-e^{s\tau}) + Y(s)\Big] \\ &= y_0\Big[\frac{1-e^{-s\tau}}{s}\Big] + s^{-s\tau}Y(s), \end{aligned}$$

or

$$(s - s^{-s\tau})Y(s) - \Big(\frac{1+s-e^{-s\tau}}{s}\Big) = 0.$$

Theorem 5.4.1. *If all the roots of the characteristic equation* $\det(\Delta(s)) = 0$ *have negative real parts, then the* ***0****-solution of the above system* (5.4.1) *is asymptotically stable.*

Proof. Taking Laplace transform of (5.4.1) and using initial condition, we get

$$
\begin{aligned}
s^{\alpha_1}Y_1(s)+y_{10} &= a_{11}e^{-s\tau_{11}}\Big(Y_1(s)+\int_{-\tau_{11}}^{0}e^{-st}y_1(t)dt\Big)+a_{12}e^{-s\tau_{12}}\Big(Y_2(s)+\\
&\qquad \int_{-\tau_{12}}^{0}e^{-st}y_2(t)dt\Big)+\cdots+a_{1n}e^{-s\tau_{1n}}\Big(Y_n(s)+\int_{-\tau_{1n}}^{0}e^{-st}y_n(t)dt\Big),\\
s^{\alpha_2}Y_2(s)+y_{20} &= a_{21}e^{-s\tau_{21}}\Big(Y_1(s)+\int_{-\tau_{21}}^{0}e^{-st}y_1(t)dt\Big)+a_{22}e^{-s\tau_{12}}\Big(Y_2(s)+\\
&\qquad \int_{-\tau_{22}}^{0}e^{-st}y_2(t)dt\Big)+\cdots+a_{2n}e^{-s\tau_{2n}}\Big(Y_n(s)+\int_{-\tau_{2n}}^{0}e^{-st}y_n(t)dt\Big),\\
&\cdots\\
s^{\alpha_n}Y_n(s)+y_{n0} &= a_{n1}e^{-s\tau_{1n}}\Big(Y_1(s)+\int_{-\tau_{1n}}^{0}e^{-st}y_2(t)dt\Big)+a_{2n}e^{-s\tau_{2n}}\Big(Y_2(s)\\
&\qquad \int_{-\tau_{2n}}^{0}e^{-st}y_2(t)dt\Big)+\cdots+a_{nn}e^{-s\tau_{nn}}\Big(Y_n(s)+\int_{-\tau_{nn}}^{0}e^{-st}y_n(t)dt\Big),
\end{aligned}
$$

writing the above equations in the following form

$$
\Delta(s)\begin{pmatrix}Y_1(s)\\ Y_2(s)\\ \vdots\\ Y_n(s)\end{pmatrix}=\begin{pmatrix}b_1(s)\\ b_2(s)\\ \vdots\\ b_n(s)\end{pmatrix}, \tag{5.4.5}
$$

$$
\text{where}\ \ \Delta(s)=\begin{pmatrix}
s^{\alpha_1}-a_{11}e^{-s\tau 11} & -a_{12}e^{-s\tau 12} & \dots & -a_{1n}e^{-s\tau 1n}\\
-a_{21}e^{-s\tau 21} & s^{\alpha_2}-a_{22}e^{-s\tau 22} & \dots & -a_{2n}e^{-s\tau 2n}\\
\vdots & \vdots & \ddots & \vdots\\
-a_{n1}e^{-s\tau n1} & -a_{n2}e^{-s\tau n2} & \dots & s^{\alpha_n}-a_{nn}e^{-s\tau nn}
\end{pmatrix},
$$

and

$$
\begin{aligned}
b_1(s) = a_{11}e^{-s\tau_{11}}\int_{-\tau_{11}}^{0}e^{-st}y_1(t)dt+a_{12}e^{-s\tau_{12}}\int_{-\tau_{12}}^{0}e^{-st}y_2(t)dt\\
+\cdots+a_{1n}e^{-s\tau_{1n}}\int_{-\tau_{1n}}^{0}e^{-st}y_n(t)dt+y_{10}
\end{aligned}
$$

$$b_2(s) = a_{21}e^{-s\tau_{21}} \int_{-\tau_{21}}^{0} e^{-st}y_1(t)dt + a_{22}e^{-s\tau_{22}} \int_{-\tau_{22}}^{0} e^{-st}y_2(t)dt$$
$$+ \cdots + a_{2n}e^{-s\tau_{2n}} \int_{-\tau_{2n}}^{0} e^{-st}y_n(t)dt + y_{20}$$
$$\vdots$$
$$b_n(s) = a_{n1}e^{-s\tau_{n1}} \int_{-\tau_{n1}}^{0} e^{-st}y_1(t)dt + a_{n2}e^{-s\tau_{n2}} \int_{-\tau_{n2}}^{0} e^{-st}y_2(t)dt$$
$$+ \cdots + a_{nn}e^{-s\tau_{nn}} \int_{-\tau_{nn}}^{0} e^{-st}y_n(t)dt + y_{n0}.$$

Here $\Delta(s)$ is a characteristic matrix and $\det(\Delta(s))$ is a characteristic polynomial of system (5.4.1). From the distribution of the eigenvalues of $\det(\Delta(s))$, we determine the stability of system (5.4.1). Multiplying both sides of (5.4.5) with ' s ' gives

$$\Delta(s) \begin{pmatrix} sY_1(s) \\ sY_2(s) \\ \vdots \\ sY_n(s) \end{pmatrix} = \begin{pmatrix} sb_1(s) \\ sb_2(s) \\ \vdots \\ sb_n(s) \end{pmatrix}. \tag{5.4.6}$$

We consider (5.4.6) in $\mathrm{Re}(s) \geq 0$, if the transcendental equation $\det(\Delta(s)) = 0$ has all roots lying in left half open complex plane, i.e, $\mathrm{Re}(s) < 0$. In this limited area, there exist a unique solution $(sY_1(s), sY_2(s), \ldots, sY_n(s))$ of (5.4.6). So, we then have

$$\lim_{s\to 0, \mathrm{Re}\geq 0} sY_i(s) = 0, \ i = 1, 2, \ldots, n.$$

From the supposition of all roots of $\det(\Delta(s)) = 0$ and the final-value theorem of Laplace transform, we find $\lim_{s\to\infty} y_i(t) = \lim_{s\to 0, \mathrm{Re}\geq 0} sY_i(s) = 0, \ i = 1, 2, \ldots, n$. This completes the proof of the theorem. □

Remark 5.4.1. Suppose the Caputo derivative in place of the Riemann-Liouville derivative in system (5.4.1) and all other assumed conditions remain the same, then the conclusion of Theorem (5.4.1) still holds.

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