Delay Differential Equations

by

Rahila Jumani

A thesis submitted to the
School of Natural Sciences,
National University of Sciences and Technology,
Islamabad, Pakistan.
For the degree of
Master of Philosophy

Supervised by

Prof. Faiz Ahmad

October 2013

Dedicated

To My Parents

Acknowledgements

Primarily and foremost, all praise for the Almighty Allah, the benevolent and merciful, the creator of the universe, who provided me the apt ability and strength to complete the work presented here.

I express my profound gratitude to Prof. Faiz Ahmad, who supervised my MPhil research work, and without who's help it would have been impossible to complete it. I am really thankful to him and pray for his long life and happiness.

I cannot appreciate enough the whole Department of Mathematics of NUST-SNS, specially the principal of SNS, Prof. Azad Siddiqui and the Head of department Dr. Rashid Farooq for providing us the invaluable assets and enabling atmosphere that helped a lot in the completion of the degree, and also I would thank my GEC members, Dr. Mujeeb-ur-Rehman, Dr. Abid Ali Lashari, and Dr. Asim Aziz for their invaluable comments and time.

Last but not least my debt of gratitude to my parents, for their never-ending love and support in all my efforts, and for giving me the foundation to be who I am. May Allah bless them all.

Rahila Jumani

Abstract

In this thesis we discuss the theory of delay differential equations. We have given basic concepts and definition of delay differential equation. We have also pointed out some major differences between delay and ordinary differential equations. The elementary methods for solving delay differential equations i.e the method of steps and the Laplace transform method have been discussed.

The oscillatory behavior of delay differential equation has been analyzed by considering the characteristic equation associated to the first order delay equation. Necessary and sufficient conditions for the oscillation of all the solutions of the delay differential equation have also been given in detail.

A brief account of the stability of delay differential equation is given, by which stability of delay differential equation depends on the location of roots of the characteristic function and the steady state is stable if and only if all the roots have negative real parts. Then, we have also discussed the stability of delay differential equation using Sturm sequences.

Contents

Chapter 1

Introduction

In this chapter we have discussed some basic concepts and definition of delay differential equation. We have also pointed out some major differences between delay differential equations and ordinary differential equations. The elementary methods for solving delay differential equations i.e the methods of steps and the Laplace transform method has been illustrated. We have considered only first order delay differential equation.

Definition 1.0.1. Functional differential equation

A functional differential equation (FDE) is an equation for an unknown function which involves derivatives of the function and in which the function, and possibly its derivative(s), occur with various different arguments. Functional differential equations are also referred as differential equations with deviating arguments.

Example 1.0.2. The following are some examples of functional differential equations.

$$
\begin{aligned}
&y'(t) = -y(t-\pi),\\
&y'(t) = y(t) - y\left(\frac{t}{3}\right),\\
&y'(t) = t^3 y(t) - y'(t-1),\\
&y'(t) = y(t)y(t-1) + y(t+4),\\
&y'(t) = -y'(t) + \cos(y(t)) + y(t-3) + y^2(t-5),\\
&y'(t) = y^2(t) + \left(\frac{5}{\pi}\right)\int_0^\infty \mathrm{e}^{-s^2} y(t-s)ds.
\end{aligned}
$$

Definition 1.0.3. Delay differential equation

A delay differential equation (DDE) is a functional differential equation where the highest order derivative only occurs with one value of the argument, and this argument is not less than the arguments of the unknown function and its lower order derivatives appearing in the equation. Such equations are also called retarded functional differential equations (RFDE) or differential equations with retarded argument, or, in other words, an equation expressing some derivative of y at time t in terms of y and its lower-order derivatives, if any, at t and at earlier instants, is called a delay differential equation.

General form of delay differential equation is

$$y'(t) = f(t, y(t), y(t-\tau_1), y(t-\tau_2), ..., y(t-\tau_k)),$$

where τ_i are known as delays or time lags, they might be constant functions $\tau(t)$ of t called *'time-dependent delays'*, or functions $\tau(t, y(t))$ called *'state-dependent delays'*. Delay equations with delays of the derivatives are referred to as *'neutral delay differential equations'* (NDDEs). In this thesis, we have considered constant delay. Some examples of DDEs are

$$\begin{aligned}
y'(t) &= -y(t-\pi), \\
y'(t) &= -y'(t) + \cos(y(t)) + y(t-3) + y^2(t-5), \\
y'(t) &= y^2(t) + \left(\frac{5}{\pi}\right) \int_0^\infty \mathrm{e}^{-s^2} y(t-s) ds.
\end{aligned}$$

Delay differential equations basically describe a physical process with an after effect. Such equations appear when we consider a problem of force acting on a point that depends on the velocity and position of the point not only at any given moment but also at some moment proceeding the given moment. In the study of population of species, it is more realistic to assume that the rate of change of the population at time t is proportional not to the population at time t but to an earlier time $t - \tau$. For this purpose. delay models are becoming more common, and are appearing in many branches of biological modeling.

Delay differential equations have their applications not only in physics, engineering and mathematical biology but they also serve as useful tools in economics, statistics and social sciences etc. They are often used as modeling tools in many different areas of applied mathematics that include the study of age structured population growth, traffic flow, in the study of epidemics and also problems related to the engineering of high-rise buildings for earthquake protection.

The first systematic study of delay differential equations was made in Russia by Myskis [1]. An introductory account of the theory may be found in Elsgolts and

Norkin [2], and Bellman and Cooke [3]. A comprehensive account of the oscillation theory of the delay equations appears in Györi and Ladas [4].

When we discuss delay differential equations we adopt some of the terminologies from ordinary differential equations, thus, the order of a delay differential equation will mean the order of the highest derivative involved in the equation as in the case for ordinary differential equation. Therefore

$$y'(t) = my(t) + ny(t - \tau), \tag{1.0.1}$$

and

$$y'(t) = -dy(t - 1) + [1 - y(t)], \tag{1.0.2}$$

are first order delay differential equation, and

$$ly''(t) + my'(t) + ny'(t - \tau) + py(t) = k(t), \tag{1.0.3}$$

is second-order delay differential equation. Also the concept of linear and homogeneous equations is same in both ordinary and delay differential equation. Equations (1.0.1) and (1.0.3) are linear whereas equation (1.0.2) is non-linear, and equation (1.0.1) is homogeneous but equation (1.0.3) is nonhomogeneous. [5], [6].

1.1 Difference between ordinary and delay differential equation

The most obvious difference between ODE's and DDE's is the initial data, the solution of an ODE is determined by its value at the initial point. For example if we are given the slope of tangent at any point on a curve by

$$\frac{dy}{dx} = \frac{3}{4}x^2 + x - 3,$$

with initial point

$$y(0) = 2.$$

Using method of separation of variables, we have

$$y = \frac{1}{4}x^3 + \frac{x^2}{2} - 3x + 2.$$

Thus, we can obtain a solution of an ODE with an initial condition. However to solve a delay differential equation, one needs an initial history to obtain a solution. Consider a delay differential equation of the form

$$y'(t) = f(t, y(t), y(t - \tau_1), y(t - \tau_2), ..., y(t - \tau_k)),$$

for $t \in [a, b]$ with initial data $y(0) = \theta(t)$, $t \leq a$.

A term $y(t - \tau_k)$ may represent values of the solution at points prior to the initial point, for example to get a solution at $t = a$, we must have a solution at $a - \tau_j$. We can see that if, for example, T is the longest delay, the delay equation requires us to provide the solution $\theta(t)$ for $t \in [a - T, a]$. Thus for the solution of DDEs one must provide not just the initial point but also the "history" of the solution at the time previous to the initial point. For further illustration consider an equation of the form

$$y'(t) = f(t, y(t), y(t - \tau_1)).$$

For an ordinary differential system, a unique solution can be determined by an initial point in Euclidean space at an initial time t_0. But for a delay differential system, one requires information on the whole interval $[t_0 - \tau, t_0]$. To know the rate of change at t_0, one needs $y(t_0)$ and $y(t_0 - \tau)$, and for $y'(t_0 + \epsilon)$, one needs to know $y(t_0 + \epsilon)$, and $y(t_0 + \epsilon - \tau)$. So to solve a delay differential equation one needs to give an initial history, the value $y(t)$ for the interval $[-\tau, 0]$.

If we require that initial function to be continuous, then the space of solution has the same dimensionality, and we can say that the delay differential equations generates a system that is infinite dimensional. The characteristic equation of a linear DDE with discrete delays of the form

$$y'(t) = A_0 y(t) + A_1 y(t - \tau_1) + ... + A_m y(t - \tau_m),$$

is

$$det(-\lambda I + A_0 + A_1 \mathrm{e}^{-\tau_1 \lambda} + ... + A_m \mathrm{e}^{-\tau_m \lambda}) = 0.$$

Since the exponential is involved in the characteristic equation, the delay differential equation has infinite number of eigen values, therefore an infinite dimensional system is generated.

Consider an example

$$y'(t) = -y(t - 1),$$

the characteristic equation is

$$-\lambda + \mathrm{e}^{-\lambda} = 0,$$

there are infinite number of solutions to the equation for complex λ . Whereas for a linear, homogenous, constant ordinary differential equations one has the solution of the form

$$y(t) = \mathrm{e}^{\lambda t},$$

any root λ of the characteristic equation provides a solution. The above equation has a finite number of roots. Thus an ODE generates a system that is finite dimensional.

Delay differential equations also differ from ODEs in the manner that the derivative at any time depends on the solution at prior times which is not the case with ODEs. In ODEs the effect of any change in the system is instantaneous, but in DDEs the effect of any change is not instantaneous. A small delay can produce a large effect in the solution when we are solving DDEs.

A first-order, linear, homogeneous delay differential equation with real coefficients can have a non-trivial oscillatory solution whereas for first-order ordinary differential equation it may not be possible. Consider a scalar linear homogeneous delay differential equation

$$y'(t) = c(t)y(t) + d(t)y(t - \tau), \tag{1.1.1}$$

where c and d are continuous real-valued functions on $\mathbb{R}$ and $\tau > 0$ is a constant. Equation (1.1.1) can have a non-trivial solution which is oscillatory , i.e. a solution that can take both positive and negative values for arbitrarily large values of t. For example, consider the equation

$$y'(t) = -y(t - \frac{\pi}{2}), \quad \text{for } t \geq 0, \tag{1.1.2}$$

with initial function

$$y(t) = \sin(t), \quad \text{for } \frac{-\pi}{2} \leq t < 0.$$

Simply $y(t) = \sin t$ is the solution of (1.1.2). Thus (1.1.2) has an oscillatory solution, whereas for a first order initial value problem

$$y'(t) = -y(t),$$

with initial point

$$y(0) = 0,$$

we do not get an oscillatory solution. Its solution is

$$y(t) = \mathrm{e}^{-t} - 1. \tag{1.1.3}$$

Thus, the solution is obviously not oscillatory.

The solution of an initial value problem may not be unique whereas the presence of a delay in a delay differential equation causes uniqueness of the solution. For example, consider the first order differential equation

$$y'(t) = [y(t)]^{\frac{3}{4}}, \tag{1.1.4}$$

with initial point,

$$y(0) = 0.$$

Using method of separation of variables, we have

$$4[y(t)]^{\frac{1}{4}} = t.$$

or

$$y(t) = \frac{t^4}{256},$$

is the solution of (1.1.4). Also, $y(t) \equiv 0$ is another solution of (1.1.4). Thus, the solution is not unique.

Now, consider a first order delay differential equation,

$$y'(t) = [y(t-\tau)]^{\frac{3}{4}}, \tag{1.1.5}$$

with initial function

$$y(t) = 1 + t, \qquad -\tau \leq t \leq 0.$$

Integrating (1.1.5) w.r.t. t on the interval $[0, 1]$, we have

$$\int_0^t y'(t)dt = \int_0^t [y(t-\tau)]^{\frac{3}{4}} dt,$$

$$\begin{aligned} y(t) &= y(0) + \int_0^t [y(t-\tau)]^{\frac{3}{4}} dt, \\ &= 1 + \int_0^t (1 + t - \tau)^{\frac{3}{4}} dt, \\ &= 1 + \frac{4}{7}\Big[(1 + t - \tau)^{\frac{7}{4}}\Big] \Big|_0^t, \end{aligned}$$

$$y(t) = 1 + \frac{4}{7} t^{\frac{7}{4}}. \tag{1.1.6}$$

Thus, we obtain a unique solution on the interval $[0, 1]$. Similarly, on the interval $[1, 2]$ by integrating (1.1.6) w.r.t. t, we have

$$\begin{aligned} y(t) &= y(0) + \int_1^t [y(t-\tau)]^{\frac{3}{4}} dt, \\ &= 1 + \int_1^t [1 + \frac{4}{7}(t-\tau)^{\frac{7}{4}}] dt, \\ &= 1 + [\, t\,] \Big|_1^t + \frac{16}{77}\Big[(t-\tau)^{\frac{11}{4}}\Big] \Big|_1^t, \\ &= 1 + (t-1) + \frac{16}{77}[(t-\tau)^{\frac{11}{4}} - (1-\tau)^{\frac{11}{4}}], \\ y(t) &= t + \frac{16}{77}(t-1)^{\frac{11}{4}}. \end{aligned}$$

Thus, we again obtain a unique solution in the interval $[1, 2]$. Similarly for a first order delay differential equation we obtain a unique solution in each interval.

Sometimes the solution of ordinary differential equation can have unpredictable endings, for example consider an ordinary differential equation

$$y'(t) = 1 + y^2(t), \tag{1.1.7}$$

with initial point $y(0) = 1$.
Using the method of separation of variables, we obtain

$$\int \frac{dy}{1 + y^2(t)} = \int dt,$$

$$\tan^{-1} y(t) = t + c_1,$$

as $y(0) = 1$ so, $c_1 = \frac{\pi}{4}$. We have

$$\tan^{-1} y(t) = t + \frac{\pi}{4},$$

or

$$y(t) = \tan(t + \frac{\pi}{4}).$$

This solution is unique on $\frac{-3\pi}{4} < t < \frac{\pi}{4}$. Further it cannot be continued beyond $t = \frac{\pi}{4}$ and $t = \frac{-3\pi}{4}$, this was completely unpredictable from the differential equation (1.1.7). This kind of unpredictable endings can be obtained from differential equation whereas for a delay differential equation

$$y'(t) = 1 + y^2(t-\tau), \quad t \geq 0,$$

with initial function

$$y(t) = \theta(t), \quad -\tau \leq t \leq 0,$$

the solution exists for $[0, \infty)$.

1.2 Elementary methods for solving delay differential equations.

Some of the elementary methods for solving delay differential equations includes, the method of steps, and the Laplace Transform.

1.2.1 The method of steps.

The most common method used to solve delay differential equations is known as the method of steps. In this method any delay differential equation is converted to an ordinary differential equation over a given interval by using the initial data and this processes is repeated over the next interval where the previous solution serves as an initial data. We will explain this process by considering the following example. [5]

Example 1.2.1. Consider a simple delay differential equation

$$y'(t) = -y(t-1), \tag{1.2.1}$$

with initial function

$$y(t) = 1, \quad \text{for} \quad t \in [-1, 0].$$

By integrating both sides of (1.2.1) from 0 to t w.r.t. t, on the interval $[0, 1]$, we have

$$\int_0^t y'(t)dt = -\int_0^t y(t-1)dt,$$

$$y(t) - y(0) = -\int_0^t y(t-1)dt,$$

$$y(t) = y(0) - \int_0^t y(t-1)dt, y(t) = 1 - \int_0^t dt,$$

or

$$y(t) = 1 - t.$$

For the next step, $y(t) = 1 - t$ serves as initial data, for $1 \leq t \leq 2$, we have

$$
\begin{aligned}
y(t) &= -\int_1^t [1 - (t-1)]dt, \\
&= -\int_1^t (2-t)dt, \\
&= -\Big[2t - \frac{1}{2}t^2\Big] \Big|_1^t, \\
&= -\Big[2t - \frac{1}{2}t^2 - \frac{3}{2}\Big], \\
y(t) &= \frac{1}{2}t^2 - 2t + \frac{3}{2}.
\end{aligned}
$$

Next $y(t) = \frac{1}{2}t^2 - 2t + \frac{3}{2}$, serves as the initial function, so on the interval $[2, 3]$, we have

$$
\begin{aligned}
y(t) &= \Big[-2(2) + \frac{1}{2}(2)^2 + \frac{3}{2}\Big] - \int_2^t [-2(t-1) + \frac{1}{2}(t-1)^2 + \frac{3}{2}]dt, \\
&= \Big[-4 + \frac{4}{2} + \frac{3}{2}\Big] - \int_2^t \Big[-2(t-1) + \frac{1}{2}(t-1) + \frac{3}{2}\Big]dt, \\
&= -\frac{1}{2} - \Big[-\frac{2(t-1)^2}{2} + \frac{(t-1)^3}{6} + \frac{3}{2}t\Big] \Big|_2^t, \\
&= \frac{5}{3} + (t-1)^2 - \frac{1}{6}(t-1)^3 - \frac{3}{2}t, \\
y(t) &= \frac{17}{6} - 4t + \frac{3}{2}t^2 - \frac{t^3}{6}.
\end{aligned}
$$

Similarly further using mathematica, we have

$for\ \ 3 \leq t \leq 4,$

$$y(t) = \frac{149}{24} - \frac{17}{2}t + \frac{15}{4}t^2 - \frac{2}{3}t^3 + \frac{t^4}{24},$$

$for\ \ 4 \leq t \leq 5,$

$$y(t) = \frac{1769}{120} - \frac{115}{6}t + \frac{109}{12}t^2 - 2t^3 + \frac{5}{24}t^4 - \frac{t^5}{120},$$

$for\ \ 5 \leq t \leq 6,$

$$y(t) = \frac{26239}{720} - \frac{1085}{24}t + \frac{1061}{48}t^2 - \frac{197}{36}t^3 + \frac{35}{48}t^4 - \frac{t^5}{20} + \frac{t^6}{720},$$

$for\ \ 6 \leq t \leq 7,$

$$y(t) = \frac{463609}{5040} - \frac{13201}{120}t + \frac{13081}{240}t^2 - \frac{521}{36}t^3 + \frac{107}{48}t^4 - \frac{t^5}{5} + \frac{7}{720}t^6 - \frac{t^7}{5040},$$

for $7 \leq t \leq 8$,

$$y(t) = \frac{3157891}{13440} - \frac{39371}{144}t + \frac{39227}{288}t^2 - \frac{27227}{720}t^3 + \frac{3685}{576}t^4 - \frac{487}{720}t^5 + \frac{7}{160}t^6 - \frac{t^7}{630} + \frac{t^8}{40320},$$

for $8 \leq t \leq 9$,

$$y(t) = \frac{43896157}{2576} - \frac{1158379}{1680}t + \frac{1156699}{3360}t^2 - \frac{212753}{2160}t^3 + \frac{51193}{2880}t^4 - \frac{1511}{720}t^5 + \frac{701}{4320}t^6 - \frac{t^7}{126} + \frac{t^8}{4480} - \frac{t^9}{362880}.$$

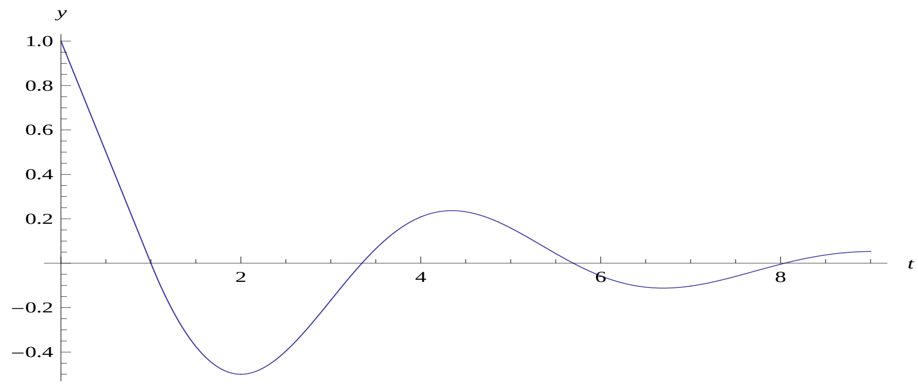

Figure 1.1: *The graph of the solution of* $y'(t) = -y(t-1)$, *on the interval* $[0, 9]$.

1.2.2 The Laplace transform.

For Laplace the method of solution is same as for the ordinary differential equation. That is, we first take the transform of the equation and also do some calculations and then take the inverse transform. [7].

Example 1.2.2. Consider a delay differential equation

$$y'(t) = -y(t-1), \tag{1.2.2}$$

with initial data

$$y(t) = 1, \quad \text{for} \quad t \in [-1, 0].$$

Multiplying both sides of (1.2.2) with e^{-st} and integrating from 0 to ∞ with respect to t, we get

$$\int_0^\infty \mathrm{e}^{-st} y'(t) dt = -\int_0^\infty \mathrm{e}^{-st} y(t-1) dt,$$

$$\int_0^\infty e^{-st}y'(t)dt + \int_0^\infty e^{-st}y(t-1)dt = 0. \tag{1.2.3}$$

As for a function $y(x)$ defined for $t \geq 0$ the laplace transform is given by

$$\mathcal{L}\{y(x)\} = \int_0^\infty e^{-sx}y(x)dx = Y(s).$$

Also, the Laplace transform of a function $y'(t)$ for $t \geq 0$ is comes out as

$$\mathcal{L}\{y(t)\} = \int_0^\infty e^{-st}y'(t)dt = sY(s) - y_0.$$

So, the Laplace transform of the first integral in equation (1.2.3) would be

$$\int_0^\infty e^{-st}y'(t)dt = sY(s) - 1.$$

Now, consider the second integral in equation (1.2.3)

$$\int_0^\infty e^{-st}y(t-1)dt.$$

Let us substitute $x = t-1$, i.e $t = x+1$, then $dt = dx$ when $t \to 0, \quad x \to -1$ and when $t \to \infty, \quad x \to \infty$. Therefore, we have

$$\begin{aligned}
\int_{-1}^\infty e^{-s(x+1)}y(x)dx &= \int_{-1}^0 e^{-s(x+1)}y(x)dx + \int_0^\infty e^{-s(x+1)}y(x)dx, \\
&= e^{-s}\int_{-1}^0 e^{-sx}y(x)dx + e^{-s}\int_0^\infty e^{-sx}y(x)dx, \\
&= e^{-s}\int_{-1}^0 e^{-sx}dx + e^{-s}\int_0^\infty e^{-sx}y(x)dx, \\
&= e^{-s}\left[\frac{e^{-sx}}{-s}\right]\Big|_{-1}^0 + e^{-s}Y(s), \\
&= e^{-s}\left[\frac{1-e^{-s}}{-s}\right] + e^{-s}Y(s), \\
&= \left[\frac{e^{-s}-1}{-s}\right] + e^{-s}Y(s).
\end{aligned}$$

Therefore, we have

$$\int_0^\infty e^{-st}y'(t)dt + \int_0^\infty e^{-st}y(t-1)dt = sY(s) - 1 + \left[\frac{e^{-s}-1}{-s}\right] + e^{-s}Y(s) = 0.$$

Solving it for Y(s)

$$\begin{aligned}
sY(s) - 1 + \left[\frac{e^{-s}-1}{-s}\right] + e^{-s}Y(s) &= 0, \\
(s+e^{-s})Y(s) - \left[\frac{s+e^{-s}-1}{s}\right] &= 0, \\
(s+e^{-s})Y(s) &= \frac{s+e^{-s}-1}{s},
\end{aligned}$$

or

$$\begin{aligned} Y(s) &= \frac{s + \mathrm{e}^{-s} - 1}{s(s + \mathrm{e}^{-s})}, \\ &= \frac{s + \mathrm{e}^{-s}}{s(s + \mathrm{e}^{-s})} - \frac{1}{s(s + \mathrm{e}^{-s})}, \\ Y(s) &= \frac{1}{s} - \frac{1}{s(s + \mathrm{e}^{-s})}. \end{aligned}$$

This $Y(s)$ can be expanded as

$$Y(s) = \frac{1}{s} - \sum_{k=0}^{\infty} (-1)^k \, \mathrm{e}^{-ks} s^{-k-2},$$

then taking inverse Laplace transform we get

$$y(t) = \sum_{k=0}^{\infty} (-1)^k \, \frac{(t - k + 1)^k}{k!}.$$

Numerical methods are also used to solve delay differential equations. Mostly the methods used for ordinary differential equations are extended to solve delay differential equations. The most common method is "Explicit Runge Kutta method", and it uses the same idea as for the methods of steps, it reduces the DDE into an ODE and then it can be easily solved. Matlab code "dde23" also uses the same technique, it extends the Matlab ODE solver "ode223" to determine the solution of the DDE.

Note: The work presented in this thesis is a review of some of the material provided in Ordinary and Delay Differential Equations by R. D. Driver [5], Oscillation Theory of Delay Differential Equations by I. Gyori and G. Ladas [10], and Delay Differential Equation Models in Mathematical Biology by Jonathan Erwin Forde [13].

Chapter 2

Oscillation behavior of delay differential equations

In this chapter, we will discuss the periodic and bounded solution of delay differential equation and also give the necessary and sufficient conditions for oscillation of all solutions of delay differential equations. Firstly, we will look at the characteristic equation for the delay differential equation.

Definition 2.0.3. (Characteristics equation) Consider a first order delay differential equation

$$y'(t) + \sum_{i=1}^{n} m_i y(t - \tau_i) = 0, \tag{2.0.1}$$

where m_i and τ_i are constants.
Let

$$y(t) = \mathrm{e}^{\lambda t},$$

be a solution of (2.0.1). When we substitute $y(t) = \mathrm{e}^{\lambda t}$ in (2.0.1) we get

$$\lambda \mathrm{e}^{\lambda t} + \sum_{i=1}^{n} m_i \mathrm{e}^{\lambda(t-\tau_i)} = 0,$$

$$\mathrm{e}^{\lambda t}\Big(\lambda + \sum_{i=1}^{n} m_i \mathrm{e}^{-\lambda\tau_i}\Big) = 0.$$

As $\mathrm{e}^{\lambda t} \neq 0$ so

$$\lambda + \sum_{i=1}^{n} m_i \mathrm{e}^{-\lambda\tau_i} = 0, \tag{2.0.2}$$

which is said to be the characteristics equation associated with the delay differential equation (2.0.1). The nature of solutions of (2.0.1) depends on the roots of the characteristics equation (2.0.2).

2.1 Periodic and bounded solutions of a delay differential equation.

Consider a delay differential equation

$$y'(t) + my(t-\tau) = 0, \tag{2.1.1}$$

where $m, \tau \in \mathbb{R}^+$. We will obtain conditions for the existence of a periodic solution and a condition for all solutions to be bounded. The characteristic equation corresponding to (2.1.1) is

$$\lambda + m\mathrm{e}^{-\lambda\tau} = 0. \tag{2.1.2}$$

Let $\lambda = \alpha + i\beta$ be a root, then from above equation, we have

$$\alpha + i\beta + m\mathrm{e}^{-(\alpha+i\beta)\tau} = 0.$$

Equating the real and imaginary parts to zero gives

$$\alpha + m\mathrm{e}^{-\alpha\tau}\cos\beta\tau = 0,$$
$$\beta - m\mathrm{e}^{-\alpha\tau}\sin\beta\tau = 0,$$

or

$$\alpha = -m\mathrm{e}^{-\alpha\tau}\cos\beta\tau, \tag{2.1.3}$$

$$\beta = m\mathrm{e}^{-\alpha\tau}\sin\beta\tau. \tag{2.1.4}$$

Dividing (2.1.4) by (2.1.3), we get

$$\frac{\beta}{\alpha} = -\tan\beta\tau,$$

or

$$\alpha = -\beta\cot\beta\tau. \tag{2.1.5}$$

Substituting (2.1.5) in (2.1.4), we get

$$\beta = m\mathrm{e}^{\beta\cot\beta\tau}\sin\beta\tau.$$

Let $b = \beta\tau$ then above equation becomes

$$\frac{b}{\tau} = m\mathrm{e}^{b\cot b}\sin b,$$

or

$$m\tau = \frac{b}{\sin b}\mathrm{e}^{-b\cot b}. \tag{2.1.6}$$

A periodic solution corresponds to $\alpha = 0$. If $\lambda = i\beta$ in (2.1.2), we get

$$i\beta + me^{i\beta\tau} = 0,$$

or

$$i\beta = -me^{i\beta\tau}.$$

On taking the absolute value, we get

$$|\beta| = m. \tag{2.1.7}$$

We consider only positive β because the roots occur in conjugate pairs $\alpha \pm i\beta$. Also $m > 0$ and $\tau > 0$. Therefore, for a periodic solution

$$\beta = m. \tag{2.1.8}$$

If $\alpha = 0$ in (2.1.3), then it becomes

$$\cos \beta\tau = 0,$$

which gives

$$\beta\tau = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \cdots .$$

From (2.1.4), we see that

$$m\tau = \frac{3\pi}{2}, \frac{7\pi}{2}, \cdots .$$

Will give $\beta = -m$, which is a contradiction to (2.1.8). Hence, such values are excluded and

$$\beta\tau = \frac{\pi}{2}, \frac{5\pi}{2}, \frac{9\pi}{2}, \cdots ,$$

or

$$\beta\tau = (4n+1)\frac{\pi}{2}, \quad n = 0, 1, 2, \cdots .$$

Thus, a periodic solution of (2.1.1) will exist if and only if

$$\tau = (4n+1)\frac{\pi}{2\beta} = (4n+1)\frac{\pi}{2m}. \tag{2.1.9}$$

This means the equation

$$y'(t) + my\Big(t - \frac{(4n+1)\pi}{2m}\Big) = 0, \tag{2.1.10}$$

has a periodic solution $\cos(mt)$ or $\sin(mt)$. Thus, we have found that the condition for the existence of a periodic solution of (2.1.1) is that (2.1.9) holds i.e.

$$\tau m = \frac{(4n+1)\pi}{2}.$$

Now, we will find a condition for the solution of (2.1.1) to be bounded. For this, let

$$f(b) = \frac{b}{\sin b} \mathrm{e}^{-b \cot b}.$$

We see that

$$f(0^+) = \frac{1}{\mathrm{e}}.$$

Also, when $b > 0$, $f(b)$ increases such that $f(\frac{\pi}{2}) = (\frac{\pi}{2})$ and $f(\pi^-) \to \infty$. There are infinite branches of the function

$$f(b) = \frac{b\mathrm{e}^{-b \cot b}}{\sin b}.$$

Let $c = m\tau$, (2.1.6) becomes

$$c = f(b). \tag{2.1.11}$$

If $m\tau < \frac{1}{\mathrm{e}}$, then the line $y = m\tau$ will not intersect the first positive branch but it will intersect all other positive branches. We see that there are always infinite roots of the equation $c = f(b)$ because, no matter what c is, the line $y = c$ will intersect infinite number of branches. This shows that the characteristic equation has infinitely many complex roots.

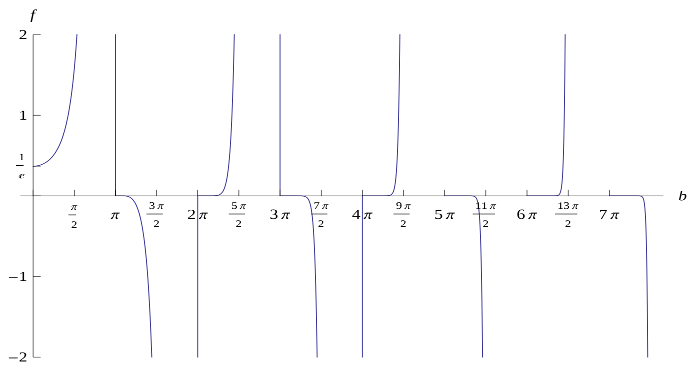

Figure 2.1: *Graph of* $f(b) = \frac{be^{-bcotb}}{sinb}$

Let $m\tau < \frac{\pi}{2}$. The line $y = m\tau$ will intersect the branches in such a way that for the first root $0 < b_1 < \frac{\pi}{2}$, for the second root $2\pi < b_2 < \frac{5\pi}{2}$, for the third root $4\pi < b_3 < \frac{9\pi}{2}$ and so on. From (2.1.3), we see that for all these values of $b = \beta\tau$ we get a negative α. Thus we find that all roots of the characteristic equation will have negative real part if and only if $m\tau < \frac{\pi}{2}$. This means all solutions of (2.1.1) are bounded if and only if $m\tau < \frac{\pi}{2}$. Let $m\tau = \frac{\pi}{2}$, then a periodic solution exists. Let $\frac{\pi}{2} < m\tau < \frac{5\pi}{2}$, label roots of (2.1.11) as $b_1, b_2, \cdots$. Then, $\frac{\pi}{2} < b_1 < \pi$, $2\pi < b_2 < \frac{5\pi}{2}$,

$4\pi < b_3 < \frac{9\pi}{2}$ and so on.

The real part of the first root will be positive and real part of all other roots will be negative. The solution corresponding to the first root will be unbounded while other solutions will be bounded. Since, a solution to a problem, where an arbitrary initial function is specified will in general be a series containing all the solutions, including the first one mentioned above, such a solution will be unbounded on $[0, \infty)$. Thus, we can say that all solutions of (2.1.1) are bounded if and only if $m\tau \leq \frac{\pi}{2}$. [8]

Definition 2.1.1. (Oscillatory solution) A nontrivial solution y is said to be oscillatory if it has arbitrarily large zeros for $t \geq t_0$, that is there exists a sequence of zeros $\{t_n\}$ (that is $y(t_n) = 0$) of y such that $\lim_{n\to\infty} t_n = \infty$. Otherwise, y is said to be non oscillatory.

Definition 2.1.2. (Autonomous equation) Consider a scalar delay differential equation

$$y'(t) + \sum_{i=1}^{n} m_i y(t - \tau_i) = 0, \tag{2.1.12}$$

where m_i and τ_i are positive constants for $i = 1, 2, \cdots, n$. The equation in which the delays τ_i and the coefficients m_i are constants is called autonomous equation as in (2.1.12). Otherwise, it is called non-autonomous equations.

Definition 2.1.3. (Laplace transform) Let $y : [0, \infty) \to \mathbb{R}$ be a real-valued function. The Laplace transform of $y(t)$ denoted by $Y(s)$ is given by the improper integral

$$Y(s) = \int_0^\infty \mathrm{e}^{-st} y(t)dt. \tag{2.1.13}$$

The Laplace transform of a function $y'(t)$ for $t \geq 0$ is

$$\mathcal{L}\{y(t)\} = \int_0^\infty \mathrm{e}^{-st} y'(t)dt = sY(s) - y_0. \tag{2.1.14}$$

Similarly, the Laplace transform of $y(t - \tau)$ is given by

$$\mathcal{L}\{y(t-\tau)\} = \int_0^\infty \mathrm{e}^{-st} y(t-\tau)dt = \mathrm{e}^{-st} Y(s) + \mathrm{e}^{-st} \int_{-\tau}^0 \mathrm{e}^{-st} y(t)dt. \tag{2.1.15}$$

Definition 2.1.4. (Abscissa of convergence) For given function y(t), the integral in (2.1.13) can behave in one of the following three ways,

(a) it converges for all complex numbers s;

(b) it diverges for all complex numbers s;

(c) there exists a real number such that the integral (2.1.13) converges for all s with

$Re\ s > \xi_0$ and diverges for all s with $Re\ s \le \xi_0$.
When (c) holds, the number 'ξ_0' is known as the abscissa of convergence of $Y(s)$. When (a) holds, the abscissa of convergence of $Y(s)$ is $\xi_0 = -\infty$, and if (b) holds then the abscissa of convergence of $Y(s)$ is $\xi_0 = +\infty$.

For example, the abscissa of convergence of Laplace transform of the functions e^{-t^2}, e^{3t} and e^{t^2} are $-\infty$, 3 and $+\infty$ respectively.

Definition 2.1.5. (Entire function) If a function $g(z)$ has a representation of the form

$$g(z) = \sum_{k=0}^{\infty} a_k z^k, \quad \text{valid for } |z| < \infty.$$

Then $g(z)$ is called an entire function. An entire function $g(z)$ is analytic on the whole complex plane.

2.2 Necessary and sufficient conditions for oscillation.

Our first aim is to establish a fundamental result concerning oscillation of all solutions of the delay differential equation (2.0.1), for which we first state some basic lemmas and theorems which will be useful to prove the fundamental result.

Lemma 2.2.1. *Let $y \in \boldsymbol{C}[[0,\infty),\mathbb{R}]$ and suppose that there exist positive constant N and μ such that*

$$|y(t)| \le N e^{\mu t}, \quad \textit{for } t \ge 0.$$

Then the abscissa of convergence ξ_0 of the Laplace transform $Y(s)$ of $y(t)$ satisfies, $\xi_0 \le \mu$.
Furthermore, $Y(s)$ exists and it is an analytic function of s for $Re\ s > \xi_0$.

Proof. For proof see [9]. □

Theorem 2.2.2. *Let $y \in \boldsymbol{C}[[0,\infty),\mathbb{R}^+]$, and let us assume that abscissa of convergence ξ_0 of the Laplace transform $Y(s)$ of $y(t)$ is finite. Then, $Y(s)$ has a singularity at the point $s = \xi_0$.*

Proof. If ξ_0 is the abscissa of convergence , then there must be a singularity on the boundary of the region of convergence $Re\ s < \xi_0$. Thus, there is a singularity on the line $Re\ s = \xi_0$.

Suppose the singularity nearest to the origin is not $(\xi_0, 0)$ but some other point (ξ_0, η_0), $\eta_0 \neq 0$, then $Y(s)$ will be analytic in the disk $| z | < \sqrt{\xi_0^2 + \eta_0^2}$ which implies its analyticity at points to the left of the abscissa of convergence which is false. Hence $Y(s)$ must have a singularity at the point $s = \xi_0$. □

Now, we will establish the fundamental result concerning oscillation of all solutions of the delay differential equation (2.0.1).

Theorem 2.2.3. *Consider a linear autonomous delay differential equation*

$$y'(t) + \sum_{i=1}^{n} m_i y(t - \tau_i) = 0, \tag{2.2.1}$$

where the coefficients m_i are real numbers and τ_i are non-negative real numbers, for $i = 1, 2, 3, ..., n$.
The characteristic equation corresponding to (2.2.1) *is*

$$\lambda + \sum_{i=1}^{n} m_i \mathrm{e}^{-\lambda \tau_i} = 0. \tag{2.2.2}$$

Then the following statements are equivalent,
(a) Every solution of (2.2.1) *oscillates.*
(b) Equation (2.2.2) *has no real root.*

Proof. (a) ⇒ (b) If the characteristic equation (2.2.2) has a real root λ_0 then $e^{\lambda_0 t}$ is non-oscillatory solution of (2.2.1).
(b) ⇒ (a) Let us assume a contradiction, that (b) holds and equation (2.2.1) has an eventually positive solution $y(t)$. As equation (2.2.1) is autonomous, we may assume that,

$$y(t) > 0, \text{ for } t \geq -\tau, \text{ where } \tau = \max_{1 \leq i \leq n} \tau_i.$$

Clearly $\tau > 0$, or otherwise equation (2.2.2) has a real root. We know by Lemma 2.2.1 that there exist constants N and μ such that

$$|y(t)| \leq N e^{\mu t}, \quad t \geq -\tau.$$

Thus, the Laplace transform

$$Y(s) = \int_0^\infty \mathrm{e}^{-st} y(t) dt, \tag{2.2.3}$$

exists for $Re\ s > \mu$. Let ξ_0 be the abscissa of convergence of $Y(s)$, $\xi_0 = \inf \xi \in \mathbb{R} : Y(\xi)$ exists. Then, for any $i = 1, 2, ..., n$, the Laplace transform of $y(t-\tau)$ exists and has abscissa of convergence ξ_0. As we know

$$\int_0^\infty \mathrm{e}^{-st} y'(t)dt = sY(s) - y(0), \quad Re\ s > \xi_0,$$

and for $i = 1, 2, ..., n$.

$$\int_0^\infty \mathrm{e}^{-st} y(t-\tau_i)dt = \mathrm{e}^{-s\tau_i} Y(s) + \mathrm{e}^{-s\tau_i} \int_{-\tau_i}^0 \mathrm{e}^{-st} y(t)dt, \quad Re\ s > \xi_0.$$

Therefore by taking Laplace transform of both sides of equation (2.2.1), we have

$$y'(t) + \sum_{i=1}^n m_i y(t-\tau_i) = 0,$$

$$\int_0^\infty \mathrm{e}^{-st} y'(t)dt + \sum_{i=1}^n m_i \int_0^\infty \mathrm{e}^{-st} y(t-\tau_i)dt = 0,$$

$$sY(s) - y(0) + \sum_{i=1}^n m_i \Big[\mathrm{e}^{-s\tau_i} Y(s) + \mathrm{e}^{-s\tau_i} \int_{-\tau_i}^0 \mathrm{e}^{-st} y(t)dt\Big] = 0,$$

$$Y(s)\Big[s + \sum_{i=1}^n m_i \mathrm{e}^{-s\tau_i}\Big] = y(0) - \sum_{i=1}^n m_i \mathrm{e}^{-s\tau_i} \int_{-\tau_i}^0 \mathrm{e}^{-st} y(t)dt. \tag{2.2.4}$$

Let

$$s + \sum_{i=1}^n m_i \mathrm{e}^{-s\tau_i} = G(s). \tag{2.2.5}$$

$$y(0) - \sum_{i=1}^n m_i \mathrm{e}^{-s\tau_i} \int_{-\tau_i}^0 \mathrm{e}^{-st} y(t)dt = H(s). \tag{2.2.6}$$

where $G(s)$ and $H(s)$ are clearly entire functions. Therefore equation (2.2.4) becomes

$$G(s)Y(s) = H(s), \quad Re\ s > \xi_0, \tag{2.2.7}$$

also by (b), $G(s) \neq 0$ for all real s. It follows from equation (2.2.7) that

$$Y(s) = \frac{H(s)}{G(s)}, \quad Re\ s > \xi_0. \tag{2.2.8}$$

We now claim that $\xi_0 = -\infty$, otherwise $\xi_0 > -\infty$ and by Theorem 2.2.2 the point $s = \xi_0$ must be a singularity of the quotient $\frac{H(s)}{G(s)}$. But this quotient has no singularity on the real axis, since the numerator and denominator are entire functions and by hypothesis the denominator has no real zeros. Thus $\xi_0 = -\infty$ and equation (2.2.8) becomes

$$Y(s) = \frac{H(s)}{G(s)}, \text{ for all } s \in \mathbb{R}. \tag{2.2.9}$$

We now see that as $s \to -\infty$ through real values, equation (2.2.9) leads to a contradiction because $Y(s)$ and $G(s)$ are always positive while $H(s)$ becomes eventually negative. The positivity of $Y(s)$ follows from equation (2.2.3) and the fact that $y(t) > 0$ for $t > 0$. The positivity of $G(s)$ follows from equation (2.2.5) and the fact that $G(\infty) = \infty$ and the characteristic equation has no real roots. Without loss of generality, we suppose that the delays in equation (2.2.1) are distinct and the coefficients m_i are not zero. Let τ_{i_0} be the maximum delay in equation (2.2.1), then the corresponding coefficients $m_{i_0} > 0$, for otherwise

$$\lim_{s \to -\infty} G(s) = -\infty,$$

and the dominant term in equation (2.2.6) as $s \to -\infty$, is $m_{i_0} e^{-s\tau_{i_0}}$. We apply the mean value theorem for integrals to

$$\int_{-\tau_{i_0}}^{0} \mathrm{e}^{-st} y(t)dt.$$

Here $\mathrm{e}^{-st} > 0$ on $[-\tau_i, 0]$ and $y(t) \in \mathbf{C}[-\tau_i, 0]$. Therefore, for some $c \in [-\tau_i, 0)$

$$\begin{aligned} \int_{-\tau_{i_0}}^{0} \mathrm{e}^{-st} y(t)dt &= y(c) \int_{-\tau_{i_0}}^{0} \mathrm{e}^{-st} dt, \\ &= -\frac{1}{s} y(c)[1 - \mathrm{e}^{s\tau_i}]. \end{aligned}$$

Now, equation (2.2.6) becomes

$$H(s) = y(0) - \frac{y(c)}{s} \sum_{i=1}^{n} m_i + y(c) \sum_{i=1}^{n} \frac{m_i s^{-s\tau i}}{s}.$$

Clearly

$$\lim_{s \to -\infty} H(s) = -\infty.$$

This completes the proof. [4] □

Theorem 2.2.4. ***(A sufficient condition for oscillation)***

Let

$$m_i,\ \tau_i \geq 0, \quad for \ \ i = 1, 2, ..., n.$$

Then

$$\sum_{i=1}^{n} m_i \tau_i > \frac{1}{\mathrm{e}}, \tag{2.2.10}$$

is sufficient for the oscillation of all the solutions of the delay equation

$$y'(t) + \sum_{i=1}^{n} m_i y(t - \tau_i) = 0. \tag{2.2.11}$$

Proof. The characteristic equation corresponding to (2.2.11) is

$$\lambda + \sum_{i=1}^{n} m_i \mathrm{e}^{-\lambda\tau_i} = 0. \tag{2.2.12}$$

Consider the inequality,

$$\mathrm{e}^y \geq \mathrm{e}y, \quad \text{for } y \geq 0.$$

We will make use of above to prove the required result. We see that for $\lambda < 0$

$$\lambda + \sum_{i=1}^{n} m_i \mathrm{e}^{-\lambda\tau_i} \geq \lambda + \sum_{i=1}^{n} m_i(-\lambda\tau_i)\mathrm{e} = -\lambda\mathrm{e}\Big(-\frac{1}{\mathrm{e}} + \sum_{i=1}^{n} m_i\tau_i \Big) > 0$$

which shows that (2.2.12) has no negative roots. As equation (2.2.12) has no roots in $\mathbb{R}^+$ either, by Theorem 2.2.3 all solutions of equation (2.2.11) oscillates. This completes the proof. [10]. □

Theorem 2.2.5. ***(A necessary condition for oscillation)***
Consider a delay differential equation

$$y'(t) + my(t-\tau) - ny(t-\sigma) = 0, \tag{2.2.13}$$

where $m, n, \tau, \sigma \in \mathbb{R}^+$, $\tau \geq \sigma$ and $p > q$. Then all solutions of (2.2.13) *are oscillating if*

$$m\tau - n\sigma > \frac{1}{\mathrm{e}}.$$

Proof. The characteristic equation corresponding to (2.2.13) is

$$y + m\mathrm{e}^{-\tau y} - n\mathrm{e}^{-\sigma y} = 0.$$

Let

$$f(y) = y + m\mathrm{e}^{-\tau y} - n\mathrm{e}^{-\sigma y} = 0,$$

and assume that $m\tau - n\sigma = \frac{1}{\mathrm{e}}$. Since

$$f(0) = 0 + m\mathrm{e}^0 - n\mathrm{e}^0,$$

or

$$f(0) = m - n > 0,$$

because $m > n$, and

$$\begin{aligned} f(-\frac{1}{\tau}) &= -\frac{1}{\tau} + m\mathrm{e}^{-\tau(-\frac{1}{\tau})} - n\mathrm{e}^{-\sigma(-\frac{1}{\tau})}, \\ &= -\frac{1}{\tau} + m\mathrm{e} - n\mathrm{e}^{\frac{\sigma}{\tau}}, \end{aligned}$$

$$f(-\frac{1}{\tau}) = \frac{-1 + me\tau}{\tau} - ne^{\frac{\sigma}{\tau}}. \tag{2.2.14}$$

As we assumed that

$$m\tau - n\sigma = \frac{1}{\mathrm{e}},$$

or

$$m\tau\mathrm{e} - n\sigma\mathrm{e} = 1,$$

or

$$m\tau\mathrm{e} = 1 - n\sigma\mathrm{e}.$$

Therefore (2.2.14) becomes

$$\begin{aligned} f(-\frac{1}{\tau}) &= \frac{-1 + me\tau}{\tau} - ne^{\frac{\sigma}{\tau}}, \\ &= \frac{-1 + 1 + ne\sigma}{\tau} - ne^{\frac{\sigma}{\tau}}, \\ f(-\frac{1}{\tau}) &= n[\frac{\mathrm{e}\sigma}{\tau} - \mathrm{e}^{\frac{\sigma}{\tau}}] \leq 0, \end{aligned}$$

because $m\tau - n\sigma = \frac{1}{\mathrm{e}}$ and $\tau \geq \sigma$. Hence, f has a zero in $[-\frac{1}{\tau}, 0)$. Now assume that $m\tau - n\sigma < \frac{1}{\mathrm{e}}$. If $n\sigma = 0$, then, $n = 0$ is trivial, since it is covered by Theorem 2.2.4. If $\sigma = 0$,

$$\begin{aligned} f(0) &= m - n > 0, \\ f(-\frac{1}{\tau}) &= \frac{-1 + me\tau}{\tau} - ne^{\frac{\sigma}{\tau}}, \\ &< \frac{-1 + 1}{\tau} - n, \\ &= -n, \end{aligned}$$

because $n\sigma = 0$. Thus

$$f(-\frac{1}{\tau}) < 0.$$

Now, let $n\sigma > 0$ and let us define $k = n\sigma\mathrm{e}$ then there exists an $\varepsilon > 0$ such that

$$m\tau - n\sigma = \frac{1 - k\varepsilon}{\mathrm{e}} < \frac{1 - \varepsilon}{\mathrm{e}},$$

because of the assumption $m\tau - n\sigma < \frac{1}{\mathrm{e}}$ without the loss of generality, we let $\varepsilon < k$. Thus

$$\begin{aligned} m\tau - n\sigma &< \frac{1 - \varepsilon}{\mathrm{e}}, \\ m\tau &< n\sigma + \frac{1 - \varepsilon}{\mathrm{e}}, \\ m\tau &< \frac{k}{\mathrm{e}} + \frac{1 - \varepsilon}{\mathrm{e}}, \\ &= \frac{k + 1 - \varepsilon}{\mathrm{e}} \quad (\text{as } n\sigma\mathrm{e} = k). \end{aligned}$$

Therefore

$$m\tau < n\sigma + \frac{1-\varepsilon}{\mathrm{e}} = \frac{k+1-\varepsilon}{\mathrm{e}}. \tag{2.2.15}$$

Let us write

$$f(y) = y + m\mathrm{e}^{-\tau y} - n\mathrm{e}^{-\sigma y}.$$

As

$$f(y) = f_1(y) + f_2(y),$$

where

$$f_1(y) = (k+1-\varepsilon)y + m\mathrm{e}^{-\tau y},$$

and

$$f_2(y) = -(k-\varepsilon)y - n\mathrm{e}^{-\sigma y}.$$

Since an equation of the form $y + c\mathrm{e}^{-dy} = 0$ has a real root if and only if $cd < \frac{1}{\mathrm{e}}$, it follows from (2.2.15) that $f_1(y)$ has a real root , while

$$n\sigma = \frac{k}{\mathrm{e}} > \frac{k-\varepsilon}{\mathrm{e}},$$

shows that $f_2(y) = 0$ does not have any real root. Since

$$f_2(0) = -(k-\varepsilon)(0) - n\mathrm{e}^0,$$

or

$$f_2(0) = -n < 0 \quad (\text{as } n \in \mathbb{R}^+),$$

it follows that $f_2(y_0) < 0$. Now

$$f(y_0) = f_1(y_0) + f_2(y_0) < 0,$$

and also $f(0) = m - n > 0$. Hence, the characteristics equation possesses a root in $(0, y_0)$. This proves the necessity of the condition $m\tau - n\sigma > \frac{1}{\mathrm{e}}$ for the oscillation of all solutions of (2.2.13). This completes the proof of the theorem. [11] □

Example 2.2.6. Consider a simple delay differential equation

$$y'(t) + y(t-\tau) = 0. \tag{2.2.16}$$

If $\tau = 1.5$, then (2.2.16) becomes

$$y'(t) + y(t-1.5) = 0, \tag{2.2.17}$$

then the necessary condition for oscillation holds true

$$m\tau = 1.5 > \frac{1}{\mathrm{e}}.$$

Therefor,e all solutions of (2.2.17) are oscillatory.

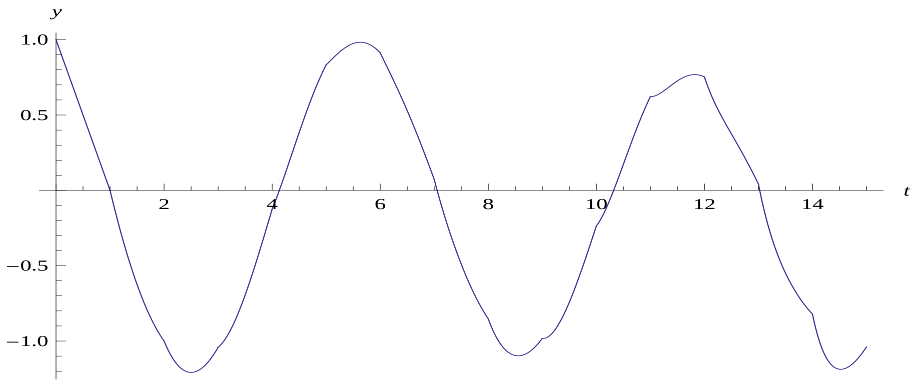

Figure 2.2: *The solutions of* (2.2.16) *with* $\tau = 1.5$

Example 2.2.7. Consider a delay differential equation

$$y'(t) + 2y(t - \frac{9}{4}\pi) = 0. \tag{2.2.18}$$

Here $m = 2$ and $\tau = \frac{9}{4}\pi$. So

$$m\tau = 14.13 > \frac{1}{\mathrm{e}},$$

thus the necessary condition for oscillation is satisfied, therefore the (2.2.18) has an oscillatory solution of the form $y(t) = \cos(2t)$.

Example 2.2.8. The delay differential equation

$$y'(t) + y(t - \frac{\pi}{2}) = 0, \tag{2.2.19}$$

has an oscillatory solution of the form $y(t) = \sin(2t)$, as the necessary condition for oscillation holds true, i.e.

$$m\tau = \frac{\pi}{2} > \frac{1}{\mathrm{e}}.$$

Example 2.2.9. Consider a delay differential equation

$$y'(t) + 2y(t - \frac{5}{4}\pi) - y(t - \frac{3}{4}\pi) = 0. \tag{2.2.20}$$

Here $m = 2$, $n = 1$,$\tau = \frac{5}{4}\pi$ and $\sigma = \frac{3}{4}\pi$. Also

$$m > n, \quad \tau \geq \sigma.$$

Thus the necessary condition for oscillation

$$m\tau - n\sigma > \frac{1}{\mathrm{e}},$$

is satisfied, therefore the (2.2.20) has an oscillatory solution of the form $y(t) = \cos(t)$.

Chapter 3

Stability analysis of delay differential equation

Definition 3.0.10. (Equilibrium point) For a system of differential equation,

$$y'(t) = f(t, y),$$

the point $\tilde{y} \in \mathbb{R}^n$ is known as the equilibrium point if $f(t,\tilde{y}) = 0 \; \forall \, t$.

Definition 3.0.11. (Bifurcation) The solution of an equation is said to undergo a bifurcation if, at some critical value of a parameter, the number of solutions to the equation changes. For instance, in a quadratic equation with real coefficients, as the constant term changes the number of real solutions can change from 0 to 2.

Definition 3.0.12. (Descartes rule of sign) Let P be a polynomial function with real coefficients and with the terms arranged in order of decreasing powers of x.
The number of positive real zeros of P is equal to the number of variations in sign of $P(x)$ or to the number decreased by an even integer.
The number of negative real zeros of P is equal to the number of variations in sign of $P(-x)$ or to the number decreased by an even integer.

Definition 3.0.13. (Quasi-polynomials) Let λ be a real number. A quasi-polynomial with exponent λ is a product $e^{\lambda x} p(x)$, where p is a polynomial. The degree of the polynomial p is called the degree of the quasi-polynomial.

3.1 Stability of delay differential equation.

For ordinary differential equations, the stability of a steady state depends on the location of roots of the characteristic function, which is polynomial in form.

The steady state is stable if and only if all the roots have negative real parts. For delay differential equations, stability is also determined by the same idea but in this case the stability is determined by the form of quasi-polynomial, which usually have infinitely many roots.

In a state space, equilibrium point is a point for which $y^* = y(t)$ is the solution for all t. Therefore for a delay differential equation of the form

$$y'(t) = f(y(t), y(t-\tau_1), y(t-\tau_2), ..., y(t-\tau_k)), \tag{3.1.1}$$

equilibrium point must satisfy

$$f(y^*, y^*, y^*, ..., y^*) = 0.$$

When we discuss the stability for the point of equilibrium of an ordinary differential equation, we consider that the system has gone under small displacement in phase space from the equilibrium. The phase space for ordinary differential equations is a finite dimensional coordinate space whereas, when we look for the stability of the point of equilibrium of delay differential equations the concept is same except the phase space, in this case, is infinite-dimensional function space. So we consider the displacement from the equilibrium in the infinite-dimensional function space, i.e the displacements in this case are time-dependent functions say $\delta y(t)$, persisting over the interval of at least of the longest delay say τ_{max}.

Let the equilibrium point of the delay equation (3.1.1) be y^* and we assume that the system has been disturbed from the state of equilibrium by a small perturbation which lasts from t to t_0, where $t = t_0 - \tau_{max}$. Let us assume that $\delta y(t)$ is the displacement from the state of equilibrium which we have assumed to be small, at any time in the open interval $[t_0 - \tau_{max}, \infty)$. Then accordingly

$$y = y^* + \delta y(t),$$

and

$$y'(t) = \delta' y(t) = f(y^*(t) + \delta y^*(t), y^*(t) + \delta y(t-\tau_1), y^*(t) + \delta y(t-\tau_2), ..., y^*(t) + \delta y(t-\tau_k)).$$

Since $\delta y(t), \delta y(t-\tau_1), \delta y(t-\tau_2), \delta y(t-\tau_3), \delta y(t-\tau_k)$ are small quantities, we can use the Taylor series method to linearize the differential equation about the equilibrium point

$$\delta' y(t) \approx J_0 \delta y(t) + J(t-\tau_1)\delta y(t-\tau_1) + J(t-\tau_2)\delta y(t-\tau_2) + ... + J(t-\tau_k)\delta y(t-\tau_k), \tag{3.1.2}$$

where the quantity J_0 is the Jacobian w.r.t. y which we have evaluated at the point of equilibrium, and the matrices $J(t-\tau_i)$ are the Jacobian w.r.t. $y(t-\tau_i)$ again evaluated at $y = y(t-\tau_1) = y(t-\tau_2) = ... = y(t-\tau_k) = y^*$.

In the linear ordinary differential equations, exponential function of time are the solutions, in which the exponents are given by the eigenvalues of Jacobian matrix. Now, let us the assume that the linear delay differential equation given in (3.1.2) also has the exponential solutions, then we write

$$\delta y(t) = A\mathrm{e}^{\lambda t}. \tag{3.1.3}$$

Substituting (3.1.3) in equation (3.1.2), we have

$$\lambda A\mathrm{e}^{\lambda t} = (J_0 A\mathrm{e}^{\lambda t} + J(t-\tau_1)A\mathrm{e}^{\lambda(t-\tau_1)} + J(t-\tau_2)A\mathrm{e}^{\lambda(t-\tau_2)} + ... + J(t-\tau_k)A\mathrm{e}^{\lambda(t-\tau_k)}),$$

$$\lambda A\mathrm{e}^{\lambda t} = (J_0 + J(t-\tau_1)\mathrm{e}^{-\lambda\tau_1} + J(t-\tau_2)\mathrm{e}^{-\lambda\tau_2} + ... + J(t-\tau_k)\mathrm{e}^{-\lambda\tau_k})A\mathrm{e}^{\lambda t},$$

$$\lambda AA^{-1} = (J_0 + J(t-\tau_1)\mathrm{e}^{-\lambda\tau_1} + J(t-\tau_2)\mathrm{e}^{-\lambda\tau_2} + ... + J(t-\tau_k)\mathrm{e}^{-\lambda\tau_k}),$$

or

$$\lambda I = (J_0 + J(t-\tau_1)\mathrm{e}^{-\lambda\tau_1} + J(t-\tau_2)\mathrm{e}^{-\lambda\tau_2} + ... + J(t-\tau_k)\mathrm{e}^{-\lambda\tau_k}),$$

where I represents the identity matrix.

We know that from the linear algebra the above equation can be satisfied only with non-zero displacement amplitudes A if

$$|J_0 + J(t-\tau_1)\mathrm{e}^{-\lambda\tau_1} + J(t-\tau_2)\mathrm{e}^{-\lambda\tau_2} + ... + J(t-\tau_k)\mathrm{e}^{-\lambda\tau_k} - \lambda I| = 0. \tag{3.1.4}$$

Equation (3.1.4) is known as the characteristic equation of the point of equilibrium, equation (3.1.4) looks like the ordinary eigenvalue problem, except the exponential terms appearance. If we try to expand out the equation (3.1.4) we will have parts which are polynomials but it will also include some exponential terms $e^{\lambda\tau_i}$ which are known as the quasi-polynomials. We basically have to concentrate on the fact that if any solution of the equation (3.1.4) has positive real parts then the point of equilibrium is unstable and if they have negative real parts , the point of equilibrium is stable. If values of the equation (3.1.4) are zero, then further checking is required about the stability of the equation.

When considering polynomials of degree n, we know that they have exactly n complex roots, therefore we can easily investigate all roots to determine its stability at the equilibrium state. But when we look at the quasi-polynomials, we cannot find all the roots , so therefore it is not always that simple to work out the stability of the equilibrium points of the delay differential equation. [12].

Example 3.1.1. Let us discuss the stability of the following delay differential equation

$$y'(t) = -y(t-\tau). \tag{3.1.5}$$

It would be really simple, as it is a single equation and the matrices become scalars. In equation (3.1.5) the delay τ is 1, $y^* = 0$ is clearly the steady state and, $J_0 = 0$ and $J_1 = -1$ are the Jacobian.

As we know that the determinant of a single number is the number itself therefore the characteristic equation (3.1.4) becomes

$$Y(\lambda) = \mathrm{e}^{-\lambda} + \lambda = 0. \tag{3.1.6}$$

The absolute minimum of $Y(\lambda)$ is 1 at λ=0. So there is no real root solutions for the characteristic equation.

Now to find the complex solutions, let us write

$$\lambda = \alpha + i\beta, \tag{3.1.7}$$

where α and β are real and imaginary parts of λ. We now substitute (3.1.7) in the characteristic equation (3.1.6)

$$(\alpha + i\beta) + \mathrm{e}^{-(\alpha+i\beta)} = 0,$$

or

$$\alpha + i\beta + \mathrm{e}^{-\alpha}\mathrm{e}^{-i\beta} = 0,$$

or

$$\alpha + i\beta + \mathrm{e}^{-\alpha}(\cos\beta - i\sin\beta) = 0, \tag{3.1.8}$$

$$\alpha + \mathrm{e}^{-\alpha}\cos\beta + i(\beta - \mathrm{e}^{-\alpha}\sin\beta) = 0.$$

Comparing real and imaginary parts, we have

$$\mathrm{e}^{-\alpha}\cos\beta = -\alpha, \tag{3.1.9}$$

and

$$\mathrm{e}^{-\alpha}\sin\beta = \beta. \tag{3.1.10}$$

Here we want to see whether equations (3.1.9) and (3.1.10) can have solutions with positive values of the real part α. As we can note that, we have the complex conjugate parts in the characteristic value, we can say that if (α, β) is the solution of the equation (3.1.8), then $(\alpha, -\beta)$ is also a solution of equation (3.1.8).Here we restrict ourself to positive values. Now let us assume that there are solutions with positive α. From the equation (3.1.9), we must have $\cos\beta < 0$. This means that we must have $\beta > \frac{\pi}{2}$. Since $\cos\beta$ will be positive for any other smaller , positive values of β, on the other hand, $e^{-\alpha} < 1$ if $\alpha > 0$, and $|\sin\alpha| < 1$, so that equation (3.1.10) gives $|\beta| < 1$, which is a contradiction. Since β cannot be smaller in magnitude than 1 and be larger than $\frac{\pi}{2}$ at the same time. Therefore we conclude that the point of equilibrium is stable, as the real part of the characteristic value cannot be positive.

3.2 Stability analysis of delay differential equation using Sturm sequences.

The steady state can become unstable if by increasing the delay, a root of the delay differential equation changes from having a negative real part to a positive real part, and this occur only when the root of the characteristic equation transverses the imaginary axis. Here we are interested in the existence of any such critical delay at which the root of the equation changes from having a negative real part to a positive real part.

Example 3.2.1. Consider a delay differential equation

$$y'(t) + ay(t-\tau) = 0, \;\; a > 0. \tag{3.2.1}$$

The steady state is stable if $\tau = 0$. The characteristic equation corresponding to the equation (3.2.1) is

$$\lambda + a\mathrm{e}^{-\lambda\tau} = 0,$$

Assume that it has a pure imaginary root $\lambda = i\sigma$, then

$$i\sigma + a\mathrm{e}^{-i\sigma\tau} = 0,$$

gives

$$i\sigma + a(\cos(\tau\sigma) - i\sin(\tau\sigma)) = 0,$$

or

$$a\cos(\tau\sigma) = 0, \quad a\sin(\tau\sigma) = \sigma,$$

From above

$$\tau\sigma = \frac{\pi}{2},$$

and

$$\sigma = a,$$

which together gives

$$\tau = \frac{\pi}{2a}.$$

Hence the steady state $y = 0$ will be stable if $\tau < \frac{\pi}{2a}$ and will become unstable if $\tau > \frac{\pi}{2a}$. This is the same result which we have found in chapter 2.

Example 3.2.2. Consider a delay differential equation

$$y''(t) + ay'(t) + by(t-\tau) = 0, \quad a > 0, b > 0. \tag{3.2.2}$$

The steady state is stable if $\tau = 0$. The characteristic equation corresponding to the equation (3.2.2) is

$$\lambda^2 + a\lambda + b\mathrm{e}^{-\lambda\tau} = 0,$$

Assume that it has a pure imaginary root $\lambda = i\sigma$, then

$$-\sigma^2 + ai\sigma + b\mathrm{e}^{-i\sigma\tau} = 0,$$

gives

$$-\sigma^2 + ai\sigma + b(\cos(\tau\sigma) - i\sin(\tau\sigma)) = 0,$$

or

$$-\sigma^2 + b\cos(\tau\sigma) = 0,$$

$$a\sigma - b\sin(\tau\sigma) = 0,$$

or

$$b\cos(\tau\sigma) = \sigma^2, \tag{3.2.3}$$

$$b\sin(\tau\sigma) = a\sigma. \tag{3.2.4}$$

Squaring both (3.2.3) and (3.2.4), then adding yields

$$b^2 = a^2\sigma^2 + \sigma^4$$

or

$$a^2\sigma^2 + \sigma^4 - b^2 = 0$$

then

$$\sigma^2 = \frac{-a^2 \pm \sqrt{a^4 + 4b^2}}{2}$$

A positive root exists,

$$\sigma = \sqrt{\frac{-a^2 + \sqrt{a^4 + 4b^2}}{2}} \tag{3.2.5}$$

Also

$$\tan(\tau\sigma) = \frac{a}{\sigma}. \tag{3.2.6}$$

Both equations (3.2.5) and (3.2.6) will be satisfied simultaneously, for some discrete values of τ. For these critical values the steady state solution will be unstable, otherwise it will continue to be stable.

3.2.1 Critical delay existence

The characteristic equation of a delay differential equation at the steady state has the form

$$L(\lambda, \tau) \equiv L_1(\lambda) + \mathrm{e}^{-\lambda\tau} L_2(\lambda) = 0, \tag{3.2.7}$$

where τ is the delay and L_1 and L_2 are polynomials. We can write (3.2.7) as

$$\sum_{k=0}^{n} a_k \lambda^k + \mathrm{e}^{-\lambda\tau} \sum_{k=0}^{m} b_k \lambda^k = 0.$$

Let us assume that in the absence of delay the steady state is stable. Then for $\tau = 0$, all the roots of the polynomials will have negative real parts. If τ varies, these roots will change. Here we are looking for any critical value of τ at which the roots of this equation transitions from having negative real part to positive real part. At this critical value of τ, we will have a purely imaginary root for the characteristic equation. We will determine that whether this critical value of τ exists or not, by reducing (3.2.7) to a polynomial problem and then finding the particular types of roots.

We start by considering purely imaginary root, $i\eta$, where $\eta \in \mathbb{R}$, of equation (3.2.7).

$$L_1(i\eta) + \mathrm{e}^{-i\eta\tau} L_2(i\eta) = 0$$

We break up the above polynomial into real and imaginary parts

$$N_1(\eta) + iM_1(\eta) + (N_2(\eta) + iM_2(\eta))(\cos(\eta\tau) - i\sin(\eta\tau)) = 0. \tag{3.2.8}$$

We can write $N_1(\eta)$,$M_1(\eta)$, $N_2(\eta)$ and $M_2(\eta)$ as,

$$\begin{aligned}
N_1(\eta) &= \sum_k (-1)^{k+1} a_{2k} \eta^{2k}, \\
M_1(\eta) &= \sum_k (-1)^{k} a_{2k+1} \eta^{2k+1}, \\
N_2(\eta) &= \sum_k (-1)^{k+1} b_{2k} \eta^{2k}, \\
M_2(\eta) &= \sum_k (-1)^{k} b_{2k+1} \eta^{2k+1}.
\end{aligned}$$

Since $i\eta$ is purely imaginary, so N_1, N_2 are even polynomials and M_1, M_2 are odd polynomials. In order for equation (3.2.8) to be true, both the imaginary and real parts must be equal to zero.

$$\begin{aligned}
N_1(\eta) + N_2(\eta)\cos(\eta\tau) + M_2(\eta)\sin(\eta\tau) &= 0, \\
M_1(\eta) - N_2(\eta)\sin(\eta\tau) + M_2(\eta)\cos(\eta\tau) &= 0.
\end{aligned}$$

which can be written as

$$N_2(\eta)\cos(\eta\tau) + M_2(\eta)\sin(\eta\tau) = -N_1(\eta), \tag{3.2.9}$$

and

$$N_2(\eta)\sin(\eta\tau) - M_2(\eta)\cos(\eta\tau) = M_1(\eta). \tag{3.2.10}$$

Squaring (3.2.9) and (3.2.10), and then adding both equations gives

$$N_1(\eta)^2 + M_1(\eta)^2 = N_2(\eta)^2 + M_2(\eta)^2, \tag{3.2.11}$$

or

$$N_1(\eta)^2 + M_1(\eta)^2 - N_2(\eta)^2 - M_2(\eta)^2 = 0. \tag{3.2.12}$$

In the above equation we can see that the delay has been eliminated and trigonometric terms have also disappeared, and it is now a polynomial equation. Equation (3.2.12) is an even polynomial as we know that squaring an odd or even function gives an even function, i.e., $g(-y)^2 = (\pm g(y))^2 = g(y)^2$.

Let us define $\xi = \eta^2 \in \mathbb{R}$. Then the equation (3.2.12) can be written in terms of ξ as

$$P(\xi) = 0, \tag{3.2.13}$$

where P is a polynomial. Here we are concerned in $\eta \in \mathbb{R}$, and if all real roots of P are negative, then there can be no solution η^* simultaneously of equation (3.2.9) and (3.2.10). Conversely, if P has a positive real root ξ^*, there is a delay τ, which is corresponding to the positive real root $\eta^* = \pm\sqrt{\xi^*}$ which solves both of the equations (3.2.9) and (3.2.10). For this, suppose that there is a η^* such that

$$N_1(\eta^*)^2 + M_1(\eta^*)^2 = N_2(\eta^*)^2 + M_2(\eta^*)^2 = 0.$$

Let

$$B = \sqrt{N_2(\eta^*)^2 + M_2(\eta^*)^2}.$$

From the above equation we can say that the point $(-N_1(\eta^*), M_1(\eta^*))$ lies on the circle of radius B. Now equation (3.2.9) and (3.2.10) can be written as:

$$B(\frac{N_2(\eta^*)}{B}\cos(\eta^*\tau) + \frac{M_2(\eta^*)}{B}\sin(\eta^*\tau)) = -N_1(\eta^*),$$

and

$$B(\frac{N_2(\eta^*)}{B}\sin(\eta^*\tau) - \frac{M_2(\eta^*)}{B}\cos(\eta^*\tau)) = M_1(\eta^*).$$

We write

$$\frac{N_2(\eta^*)}{B} = \cos\alpha,$$

and

$$\frac{M_2(\eta^*)}{B} = \sin\alpha,$$

then

$$B\cos(\eta^*\tau - \alpha) = -N_1(\eta^*),$$

and

$$B\sin(\eta^*\tau - \alpha) = M_1(\eta^*).$$

There must be a positive value of $\tau = \tau^*$ which satisfies both the above equations simultaneously, as $(-N_1(\eta^*), M_1(\eta^*))$ lies on the circle of radius B.

We now find the roots for the characteristic equation (3.2.7). Let $\lambda = i\eta$, we can write (3.2.7) as

$$e^{-i\eta\tau} = -\frac{L_1(i\eta)}{L_2(i\eta)}. \tag{3.2.14}$$

As the delay τ varies, plotting the left hand side of (3.2.14) in the complex plane yields a rational curve, and the right hand side is a unit circle. The intersections of these two curves represents the critical delays which we are looking for. Thus we need to find the value of η for which the right hand side of (3.2.14) has 1 modulus. This reproduces the equation (3.2.11), and thus for any τ the equation (3.2.7) which is the original characteristic polynomial is satisfied for some τ^*. [13].

Sturm sequences and positive real roots

Once we obtain the polynomial (3.2.13), we have to determine if it has any positive real roots. There are a lot of different approaches we might take to determine the root, for example for a polynomial of degree two, one can use quadratic formula. Similarly there are other ways of finding the root of a polynomial of degree three and four. To determine the existence of a real root, a simple approach is "Descartes rule of signs". In this the number of sign in the coefficients is equal to the number of positive real roots, modulo 2. There is a solution if the number of sign changes is odd, and if it is even then this rule is not any help.

Sturm sequences are used to determine the existence of positive real root for polynomial. Suppose that $g(y)$ is a polynomial with no repeated roots. Then $g(y)$ and $g'(y)$ have two common factors or they are relatively prime. Let $g(y) = g_0(y)$ and $g'(y) = g_1(y)$. By the use of division algorithm we can have the following sequence of equations.

$$
\begin{aligned}
g_0(y) &= r_0 g_1(y) - g_2(y),\\
g_1(y) &= r_1 g_2(y) - g_3(y),\\
&\vdots\\
g_{s-2}(y) &= r_{s-2}\, g_{s-1}(y) - C.
\end{aligned}
$$

where C is a constant. The sequence $g_0(y), g_1(y), g_2(y), ..., g_{s-1}(y), g_s(y)(= C)$ of Sturm functions is known as the Sturm chain. In any interval , we can determine the number of positive roots of polynomial $g(y)$ by putting each of the endpoint of the interval and get the sequence of signs. The number of real root in the interval is the difference between the number of sign changes in the sequence at each endpoint.

Example 3.2.3. Let $g(y) = y^2 - 1$. Then $g'(y) = 2y$, so by the division algorithm we have

$$y^2 - 1 = \frac{y}{2}(2y) - 1,$$

thus simply the Sturm chain is $y^2 - 1, (2y), 1$. Now let us consider the interval $[0, \infty)$, then the sequence of signs are

$$
\begin{aligned}
at\; 0 &: -, 0, + \;,\\
at\; \infty &: +, +, +.
\end{aligned}
$$

Since there is only one sign change in the first sequence and zero in the last, so we conclude that $g(y)$ has only one positive real root. Similarly if we consider the interval $[-4, 4]$, then the chain of signs are

$$at\ -4: +, -, +\ ,$$
$$at\ +4: +, +, +.$$

In the first sequence there are two sign changes, and in the second sequence there is no sign change thus this shows that $g(y)$ has two positive real roots in $[-4, 4]$.

If we have any specified parameter set, this method gives us a simple, applicable algorithm by which we can determine whether the bifurcation occurs. [13]

3.2.2 Applications

Perelson and Nelson studied the Mathematical analysis of delay differential equation models of HIV-1 infection, in [14] they had encountered the following characteristic equation

$$\lambda^2 + (\alpha + d)\lambda + \alpha d - \beta \mathrm{e}^{-\lambda\tau}) = 0,$$

where α, β, d are positive constants. So we have

$$L_1(\lambda) = \lambda^2 + (\alpha + d)\lambda + \alpha d,$$

and

$$L_2(\lambda) = -\beta.$$

Therefore

$$N_1(\eta) = -\eta^2 + \alpha d,$$
$$M_1(\eta) = -(\alpha + d)\eta,$$
$$N_2(\eta) = -\beta,$$
$$M_2(\eta) = 0.$$

Using (3.2.12) we have

$$\begin{aligned}
N_1^2(\eta) + M_1^2(\eta) - N_2^2(\eta) - M_2^2(\eta) &= 0,\\
(-\eta^2 + \alpha d)^2 + ((\alpha + d)\eta)^2 - (-\beta)^2 &= 0,\\
(-\eta^2 + \alpha d)^2 + (\alpha + d)^2\eta^2 + \beta^2 &= 0,\\
\beta^2 &= (\eta^2 - \alpha d)^2 + (\alpha + d)^2\eta^2,\\
\beta^2 = \eta^4 - 2\eta^2\alpha d + \alpha^2 d^2 + (\alpha^2 + 2\alpha^2 d + d^2)\eta^2&,\\
\beta^2 = \eta^4 - 2\eta^2\alpha d + \alpha^2 d^2 + \alpha^2\eta^2 + 2\alpha^2 d\eta^2 + d^2\eta^2&,\\
\beta^2 &= \eta^4 + (\alpha^2 + d^2)\eta^2 + \alpha^2 d^2,
\end{aligned}$$

$$\eta^4 + (\alpha^2 + d^2)\eta^2 + \alpha^2 d^2 - \beta^2 = 0. \tag{3.2.15}$$

At $\xi = \eta^2$, the above becomes

$$P(\xi) \equiv \xi^2 + (\alpha^2 + d^2)\xi + \alpha^2 d^2 - \beta^2 = 0,$$

where P is a polynomial. Since the P has positive linear coefficients, so by the Descartes rule of signs, the positive real root will occur if and only if the constant coefficient is negative. So there is a change in the stability if and only if

$$0 > (\alpha^2 d^2 - \beta^2) = (\alpha d - \beta)(\alpha d + \beta),$$

that is, if $\alpha d < \beta$. At $\eta = \eta^*$ the equation (3.2.15) becomes

$$(\eta^*)^4 + (\alpha^2 + d^2)(\eta^*)^2 + \alpha^2 d^2 - \beta^2 = 0.$$

Taking the derivative of above w.r.t. η^* to check the non-degeneracy

$$\begin{aligned}
4(\eta^*)^3 + 2\eta^*(\alpha^2 + d^2) &= 0,\\
2(\eta^*)^3 + \eta^*(\alpha^2 + d^2) &= 0.
\end{aligned}$$

Again differentiating w.r.t. η^* yields

$$(\eta^*)^2 + (\alpha^2 + d^2) = 0,$$

thus it does not have any root, which shows that a non-degenerate bifurcation occurs for $\alpha d < \beta$, which reproduces the result shown by Perelson and Nelson.

Perelson, and Nelson studied the models of HIV-1 infection [15] in which they showed that intracellular delays are more accurate representations of the biology, they also changed the estimated values of kinetic parameters and compared them to models without delays.

$$\lambda^3 + R\lambda^2 + (Q - \alpha d\mathrm{e}^{-\lambda\tau})\lambda + \alpha d\delta - \alpha d(\delta - \phi')\mathrm{e}^{-\lambda\tau} = 0,$$

where

$$R \equiv \alpha + d + \delta,$$
$$Q \equiv \alpha d + (\alpha + d)\delta,$$

and

$$\phi' \equiv \delta - h_T > 0.$$

In [15], it is shown that there is no change in the stability for $\tau > 1$ and $\tau < 1$. We will extend it for all $\tau > 0$. Therefore using the same method we have,

$$\begin{aligned} N_1(\eta) &= -R\eta^2 + \alpha d\delta, \\ M_1(\eta) &= -\eta^3 + Q\eta, \\ N_2(\eta) &= -\alpha d h_T, \\ M_2(\eta) &= -\alpha d\eta. \end{aligned}$$

Using (3.2.12) we have

$$N_1^2(\eta) + M_1^2(\eta) - N_2^2(\eta) - M_2^2(\eta) = 0,$$

$$\begin{aligned} (-R\eta^2 + \alpha d\delta)^2 + (-\eta^3 + Q\eta)^2 - (-\alpha d h_T)^2 - (-\alpha d\eta)^2 &= 0, \\ R^2\eta^4 + (\alpha d)^2\delta^2 - 2R\eta^2\alpha d\delta + Q^2\eta^2 + \eta^6 - 2\eta^4 Q - (\alpha d)^2 h_T^2 - (\alpha d)^2\eta^2 &= 0, \\ \eta^6 + (R^2 - 2Q)\eta^4 + (Q^2 - (\alpha d)^2 - 2R\alpha d\delta)\eta^2 - (h_T^2 - \delta^2)(\alpha d)^2 &= 0, \\ \eta^6 + (R^2 - 2Q)\eta^4 + (Q^2 - (\alpha d)^2 - 2R\alpha d\delta)\eta^2 - (\phi'^2 - 2\delta\phi')(\alpha d)^2 &= 0. \end{aligned}$$

At $\xi = \eta^2$,

$$\xi^3 + (R^2 - 2Q)\xi^2 + (Q^2 - (\alpha d)^2 - 2R\alpha d\delta)\xi - (\phi'^2 - 2\delta\phi')(\alpha d)^2 = 0. \tag{3.2.16}$$

Comparing coefficients of ξ^2,

$$\begin{aligned} R^2 - 2Q &= (\alpha + d + \delta)^2 - 2(\alpha d + (\alpha + d)\delta), \\ &= \alpha^2 + d^2 + \delta^2 + 2\alpha d + 2\delta d + 2\alpha\delta - 2\alpha d - 2\delta d - 2\alpha\delta, \\ &= \alpha^2 + d^2 + \delta^2. \end{aligned}$$

Comparing coefficients of ξ,

$$
\begin{aligned}
Q^2 - (\alpha d)^2 - 2R\alpha d\delta &= (\alpha d + (\alpha + d)\delta)^2 - (\alpha d)^2 - 2\alpha d\delta(\alpha + d + \delta), \\
&= ((\alpha d)^2 + (\alpha\delta)^2 + (d\delta)^2 + 2\alpha^2 d\delta + 2\alpha d^2\delta + 2\alpha d\delta^2) \\
&- (\alpha d)^2 - 2\alpha d\delta(\alpha + d + \delta), \\
&= (\alpha d)^2 + (\alpha\delta)^2 + (d\delta)^2 + 2\alpha^2 d\delta + 2\alpha d^2\delta + 2\alpha d\delta^2 \\
&- (\alpha d)^2 - 2\alpha^2 d\delta - 2\alpha d^2\delta - 2\alpha d\delta^2, \\
&= (\alpha\delta)^2 + (d\delta)^2.
\end{aligned}
$$

Now we have for constant term,

$$
\begin{aligned}
\phi'^2 - 2\delta\phi' &= (\delta - h_T)^2 - 2\delta(\delta - h_T), \\
&= \delta^2 - 2\delta h_T + h_T^2 - 2\delta^2 + 2\delta h_T, \\
&= -\delta^2 + h_T^2, \\
&= (h_T - \delta)(h_T + \delta), \\
&= -\phi'(h_T + \delta).
\end{aligned}
$$

So (3.2.16) can be written as:

$$
P(\xi) \equiv \xi^3 + (\alpha^2 + d^2 + \delta^2)\xi^2 + ((\alpha\delta)^2 + (d\delta)^2)\xi + \phi'(h_T + \delta)(\alpha d)^2,
$$

where P is a polynomial with positive coefficients, and thus cannot have any positive real roots, therefore we can say that an introduction of a delay into the Perelson's and Nelson's model presented in the paper [15] cannot lead to bifurcation.

Bibliography

[1] A. D. Myskis, General theory of differential equations with a retarded arguments, *Differential Equations, Translations Series 1, Vol.4* **53** (1951) 207-262.

[2] L. E. El'sgol'ts, S. B. Norkin, An Introduction to the Theory and Application of Differential Equations with Deviating Arguments, *Academic Press* New York, 1973.

[3] R. Bellman and K. L. Cooke, Differential-Difference Equations, *Academic Press* New York, 1963.

[4] I. Gyori and G. Ladas, Oscillation Theory of Delay Differential Equations *Clarendon Press Oxford University Press* New York, 1991.

[5] R. D. Driver, Ordinary and Delay Differential Equations, *Springer-Verlog* NewYork, 1977

[6] V. I. Arnol'd, Ordinary Differential Equations, *Springer-Verlog* New York 1984.

[7] D. Zwillinger, Handbook of Differential Equations, *Academic Press* New York, 1977.

[8] G. Ladas, Y. G. Sficas and I. P. Stavroulakis, Necessary and sufficient conditions for oscillations, *American Mathematical Monthly* **90** (1983) 637-640.

[9] D.V. Widder, An Introduction to transform theory, *Academic Press* New York, 1971.

[10] O. Arino and I. Gyori, Necessary and sufficient condition for oscillation of a neutral differential system with several delays, *Journal of Differential Equations* **81** (1989) 98-105.

[11] F. Ahmad, Linear delay differential equations with positive and a negative term, *Electronic Journal of Differential Equations* **92** (2003) 1-6.

[12] M. Roussel, Delay Differential Equations, *http://people.uleth.ca/∼roussel/nld/delay.pdf* (2005).

[13] J. E. Forde, Delay Differential Equation Models in Mathematical Biology, *University of Michigan* 2005.

[14] P. W. Nelson and A .S. Perelson, Mathematical analysis of delay differential equation models of HIV-1 infection. *Mathematical Biosciences* **179** (2002) 73-94.

[15] P. W. Nelson, J. D. Murray and A .S. Perelson, A model of HIV-1 pathogenesis that include an intracellular delay, *Mathematical Biosciences* **163** (2000) 201-215.