Hadamard-type fractional operators and their Mellin transform analysis

by

Rashida Zafar

Regn.#203269

A dissertation submitted in partial fulfillment of the requirements for the degree of

Master of Sciences in Mathematics

Supervised by : Dr. Moniba Shams

Co-Supervised by : Prof. Faiz Ahmed

DEPARTMENT OF MATHEMATICS,

SCHOOL OF NATURAL SCIENCES,

NATIONAL UNIVERSITY OF SCIENCES AND TECHNOLOGY

ISLAMABAD, PAKISTAN

FORM TH-4

National University of Sciences & Technology

MS THESIS WORK

We hereby recommend that the dissertation prepared under our supervision by: Rashida Zafar, Regn No. 00000203269 Titled: Hadamard-type fractional operators and their Mellin transform analysis accepted in partial fulfillment of the requirements for the award of **MS** degree.

Examination Committee Members

1. Name: DR. MUJEEB UR REHMAN Signature:

2. Name: DR. UMER SAEED Signature:

External Examiner: DR. NAZIR AHMED MIR Signature:

Supervisor's Name DR. MONIBA SHAMS Signature:

Co-Supervisor's Name PROF. FAIZ AHMAD Signature: Faiz Ahmad

Head of Department

12-9-19.
Date

COUNTERSINGED

Date: 12/9/2019

Dean/Principal

THESIS ACCEPTANCE CERTIFICATE

Certified that final copy of MS thesis written by **Ms. Rashida Zafar** (Registration No. **00000203269**), of **School of Natural Sciences** has been vetted by undersigned, found complete in all respects as per NUST statutes/regulations, is free of plagiarism, errors, and mistakes and is accepted as partial fulfillment for award of MS/M.Phil degree. It is further certified that necessary amendments as pointed out by GEC members and external examiner of the scholar have also been incorporated in the said thesis.

Signature: ____________

Name of Supervisor: Dr. Moniba Shams

Date: 12-09-19

Signature (HoD): ____________

Date: 12-9-19

Signature (Dean/Principal): ____________

Date: 12-09-2019

Dedicated

to

My Beloved Parents

Acknowledgments

In the name of Almighty Allah, the most merciful and the benevolent. I am thankful and greatly obliged to Him for showering me with His countless blessings and continuous assistance in every phase of life.

I would like to express my deepest gratitude to my supervisor Dr. Moniba Shams, for her generous support, guidance and encouragement throughout research phase. Secondly, I also want to pay my regards to my co-supervisor Prof. Faiz Ahmed and my GEC members, Dr. Mujeeb-ur-Rehman (SNS) and Dr. Umer Saeed (NICE) for providing their precious time and guidance. I am very grateful to Dr. Mujeeb-ur-Rehman for his continued support, enthusiasm and dedication.

I am thankful to all the faculty members of SNS for helping, teaching and guiding me during my studies.

I would like to express my deepest gratitude to my parents and siblings for their endless prayers, love, continuous encouragement and moral support.

I am also thankful to my friends especially Salma, Tayyaba, Urooj, Maryam, Linta, Javeria, Aniqa and other fellows.

Rashida Zafar

Abstract

This thesis shows devotion towards the study of Caputo Hadamard-type fractional derivatives and their Mellin transform analysis. The generalization of the Taylor's formula in sense of Caputo Hadamard-type derivatives is introduced. Also, few related results are presented. Furthermore, Mellin transform of some generalized fractional differential operators is evaluated. An example for Mellin transform of generalized Caputo fractional derivative is also discussed.

Contents

Chapter 1

Introduction

Fractional calculus is an important developing field in both pure and applied mathematics. Its applications are found in different areas of science and engineering, for example, electrical circuits with fractance, tracer in fluid flows and model of neurons. In fractional calculus, integrals and derivatives are of arbitrary or fractional order. Calculus was discovered in 17^{th} century by Newton and Leibniz independently. At first Leibniz discovered the concept of a symbolic method, he found notation of n^{th} derivative, where n be a non-negative integer. A question was raised by L'Hospital on Leibniz discovery that what happened about "$n = \frac{1}{2}$", that question leads to the concept of fractional calculus. Few mathematicians made contributions in 18^{th} century such as Euler, who found a ratio $d^n p$ (p is a function of x and n be a positive integer) to dx^n. In 1772, Lagrange introduced the law of exponents for differential operators of integer order. In case of fractional calculus, mathematicians were eager to find such conditions for which this law holds, when m and n are arbitrary. During 19^{th} century, studies in the field were made by Laplace, Lacroix, Riemann and Liouville and by some other mathematicians. In 1812, Laplace introduced a fractional derivative by an integral. The formula for derivative of x^β where β is fractional number, was introduced by Lacroix in his book [1]. Further more, the formula for fractional order derivative of a function $k(x)$ was expanded by Liouville [2] known as Liouville's first formula. Later he introduced a second formula but that was applicable for rational functions. None of his definition was applicable for large number of classes of functions. According to his second definition, $\frac{d}{dt_1}c = 0$ where c is any constant, but according to [1], derivative of a constant is non-zero. In 1822, Fourier [3] proposed the integral representation of function and its derivatives. In 1823, Abel gave an application of fractional calculus. He solved isochrone problem by using derivatives of arbitrary order. Peacock supported Lacroix formula but several mathematicians acknowledged Liouville's definition in 1833.

Riemann-Liouville fractional (RLF) integrals and derivatives were properly introduced in Riemann and Liouville in 1847, where as in 1858, Greer [4] developed fractional derivative for hyperbolic and trigonometric functions using first definition of Liouville. In 1892, Hadamard [5] introduced new formulas for frac-integrals and derivatives by using kernel of logarithmic form and such formulas are known as Hadamard frac-integrals and derivatives. Many mathematicians worked in the development of fractional calculus during 20^{th} century, Hardy, Weyl, Little wood, Fabian and Erdelyi are few of them. In 1967, Caputo [6] introduced a new concept of frac-derivative by interchanging the order of integral and derivative of RLF-derivative, is known as Caputo frac-derivative (for more details see [7]).

In 1993, Samko [8] used Hadamard frac-operators to develop properties and showed a link between R-L and Hadamard operators by using change of variables. Some other properties of Hadamard operators were introduced in [9].

In the beginning of 21^{st} century, Hilfer [10] introduced a new definition for frac-derivatives which insets RL-derivative (Riemann-Liouville derivative) and Caputo derivative known as Hilfer derivative. A simple modification in the definition of Hadamard frac-operator leads to a new formula which is known as Hadamard-type frac-operator (HTF-operator) introduced in [11]. At the same time Kilbas [9] found some properties of Hadamard-type frac-integrals and derivatives (HTF-integrals and derivatives). In 2014, Gambo et al. [12] discussed several new results such as semi-group property about Caputo Hadamard derivatives (CH-derivatives). Further more, generalizations of R-L, Caputo and Hilfer frac-operators were introduced in [13, 14, 15].

In this thesis, our main focus is to explore properties of Caputo Hadamard-type frac-operators (CHTF-operators), Taylor's formula for CHTF-derivatives following [9, 12, 16, 17]. Furthermore, Mellin transform of generalized Caputo, Hilfer and generalized Hilfer frac-derivatives is also evaluated.

Chapter 2 consists of the fundamental concepts such as definitions, properties and examples of frac-calculus and Mellin transform. Mellin transform is an integral transform which was introduced by Mellin in [18] . Many mathematicians worked on Mellin transform in different aspects such as, Butzer [11, 19], Katugampola [13], Carlo [20] and Podlubny [21].

Chapter 3 presents the review work of Hadamard, Caputo Hadamard and Hadamard-type operators while some new results about CHT-derivatives are introduced. Taylor and integration by parts formulas are also discussed in this chapter. Chapter 4 contains work of Butzer [11, 19], Katugampola [13] and Carlo [20] about Mellin transform of frac-operators whereas some new results about generalized frac-derivatives are obtained.

Chapter 2

Preliminaries of fractional calculus and Mellin transform

In this chapter, some fundamental concepts of the frac-calculus and Mellin transform are discussed. Many mathematicians worked for the growth of frac-calculus, for example, Lacroix, Riemann, Liouville, Caputo, Hilfer, Katugampola and Oliveira, to name a few. To move further, we need to discuss gamma function and beta function.

2.1 Euler gamma function

Euler gamma function was introduced by Leonhard Euler to generalize the factorial function to non integer values. It belongs to the group of unusual transcendental functions and appears in various areas of mathematics such as integration, number theory, hyper-geometric series etc (see [22]).

The gamma function, denoted by $\Gamma(.)$ is defined as

$$\Gamma(\alpha) = \int_0^\infty t_1^{\alpha-1} \exp(-t_1) dt_1, \qquad \alpha > 0. \tag{2.1.1}$$

where x is a dummy variable. Here are some properties of Gamma function such as

$$\Gamma(1) = 1,$$
$$\Gamma(1+\alpha) = \alpha\Gamma(\alpha). \tag{2.1.2}$$

Relation between gamma function and factorial function is given as:

$$\Gamma(\alpha) = (\alpha-1)!, \quad \alpha > 0, \tag{2.1.3}$$

where ! denotes the factorial.

By using Eq. (2.1.2), gamma function for negative values of α is given by

$$\Gamma(\alpha) = \frac{\Gamma(\alpha+1)}{\alpha}, \quad \alpha > -1, \alpha \neq 0. \tag{2.1.4}$$

In general,

$$\Gamma(\alpha) = \frac{\Gamma(\alpha+n)}{\alpha}, \quad \text{for} \quad \alpha \in \mathbb{R} - \{... - 3, -2, -1, 0\} \tag{2.1.5}$$

2.2 Beta function

Now we discuss a useful function related to gamma function which occurs in computation of many definite integrals. The Beta function $B(\tau_1, \tau_2)$ is the name used by Legendre, Whittaker and Watson 1990. It is also known as the Eulerian integral of first kind, defined as

$$B(\tau_1, \tau_2) = \int_0^1 k^{\tau_1 - 1}(1-k)^{\tau_2 - 1} dk. \tag{2.2.1}$$

Relation of gamma function with Beta function for positive τ_1 and τ_2 is given as

$$B(\tau_1, \tau_2) = \frac{\Gamma(\tau_1)\Gamma(\tau_2)}{\Gamma(\tau_1 + \tau_2)}. \tag{2.2.2}$$

2.3 Fractional calculus

In this section, basic definitions of some well known frac-operators are presented. The following definition of integral and derivative in frac-calculus was defined by Riemann and Liouville known as RLF-integrals and derivatives (see [7] for more detail).

Definition 1. For $x \in [a, b]$, the left/right-sided RLF-integrals of order $\alpha > 0$ are given as

$$\mathfrak{I}^{\alpha}_{a+}k(x_1) = \frac{1}{\Gamma(\alpha)} \int_a^{x_1} (x_1 - t_1)^{\alpha-1} k(t_1) dt_1, \tag{2.3.1}$$

$$\mathfrak{I}^{\alpha}_{b-}k(x_1) = \frac{1}{\Gamma(\alpha)} \int_{x_1}^{b} (t_1 - x_1)^{\alpha-1} k(t_1) dt_1. \tag{2.3.2}$$

Definition 2. For $x \in [a, b]$, the left/right-sided RLF-derivatives of order $\alpha > 0$ are given as

$$\begin{aligned} {}^{RL}D^{\alpha}_{a+}k(x) &= \left(\frac{d}{dx}\right)^m \frac{1}{\Gamma(m-\alpha)} \int_a^x (x - t_1)^{m-\alpha-1} k(t_1) dt_1 \\ &= \left(\frac{d}{dx}\right)^m (\mathfrak{I}^{m-\alpha}_{a+}k)(x). \end{aligned} \tag{2.3.3}$$

$$\begin{aligned} {}^{RL}D^{\alpha}_{b-}k(x) &= (-1)^m \left(\frac{d}{dx}\right)^m \frac{1}{\Gamma(m-\alpha)} \int_x^b (t_1 - x)^{m-\alpha-1} k(t_1) dt_1 \\ &= \left(-\frac{d}{dx}\right)^m (\mathfrak{I}^{m-\alpha}_{b-}k)(x). \end{aligned} \tag{2.3.4}$$

such that $m = \lceil \alpha \rceil$, where $\lceil . \rceil$ denotes the ceiling function.

Hadamard frac-operators were introduced in [5]. The definition follows:

Definition 3. For $x \in [a,b]$, the left/right-sided Hadamard frac-integrals of order $\alpha > 0$ are defined as

$$\mathcal{J}^{\alpha}_{a+}k(x_1) = \frac{1}{\Gamma(\alpha)}\int_a^{x_1}\left(\ln\frac{x_1}{t_1}\right)^{\alpha-1}k(t_1)\frac{dt_1}{t_1}, \tag{2.3.5}$$

$$\mathcal{J}^{\alpha}_{b-}k(x_1) = \frac{1}{\Gamma(\alpha)}\int_{x_1}^{b}\left(\ln\frac{t_1}{x_1}\right)^{\alpha-1}k(t_1)\frac{dt_1}{t_1}. \tag{2.3.6}$$

Definition 4. For $x \in [a,b]$, the left/right-sided Hadamard frac-derivatives of order $\alpha > 0$ are defined as

$$\begin{aligned}\mathcal{D}^{\alpha}_{a+}k(x_1) &= \frac{1}{\Gamma(m-\alpha)}\left(x_1\frac{d}{dx_1}\right)^m\int_a^{x_1}\left(\ln\frac{x_1}{t_1}\right)^{m-\alpha-1}k(t_1)\frac{dt_1}{t_1},\\ &= \delta^m(\mathcal{J}^{m-\alpha}_{a+}k)(x_1).\end{aligned} \tag{2.3.7}$$

$$\begin{aligned}\mathcal{D}^{\alpha}_{b-}k(x_1) &= \frac{(-1)^m}{\Gamma(m-\alpha)}\left(x_1\frac{d}{dx_1}\right)^m\int_{x_1}^{b}\left(\ln\frac{t_1}{x_1}\right)^{m-\alpha-1}k(t_1)\frac{dt_1}{t_1},\\ &= (-\delta)^m(\mathcal{J}^{m-\alpha}_{b-}k)(x_1).\end{aligned} \tag{2.3.8}$$

where $m = \lceil\alpha\rceil$ and $\delta^m = \left(x_1\frac{d}{dx_1}\right)^m$.

Hadamard's work was different in two ways from R-L (compare Definitions 1,2 with 3,4 in this section) as he used kernel of logarithmic form $\ln(t)$ instead of t and in case of derivative, $\delta = x\frac{d}{dx}$ instead of $\frac{d}{dx}$.

There are some disadvantages of RLF-derivative (see chapter 2 and 3 of [7]), as it is not consistent with initial and boundary value problems. Also, R-L derivative of a constant is non zero. In order to overcome these shortcomings, refined concept of frac-derivative was introduced by Caputo. The formal definition of Caputo frac-derivatives is given below.

Definition 5. For $x \in [a,b]$, the left/right-sided Caputo frac-derivatives of order

$\alpha > 0$ are defined as

$$
\begin{aligned}
{}_*\mathcal{D}^{\alpha}_{a+}k(x) &= \frac{1}{\Gamma(m-\alpha)}\int_a^x (x-t_1)^{m-\alpha-1}k^{(m)}(t_1)dt_1, \\
&= \Big(\mathfrak{I}^{m-\alpha}_{a+}\Big(\frac{d}{dt_1}\Big)^m k\Big)(x). \qquad (2.3.9) \\
{}_*\mathcal{D}^{\alpha}_{b-}k(x) &= \frac{(-1)^m}{\Gamma(m-\alpha)}\int_x^b (t_1-x)^{m-\alpha-1}k^{(m)}(t_1)dt_1, \\
&= \Big(\mathfrak{I}^{m-\alpha}_{b-}\Big(-\frac{d}{dt_1}\Big)^m k\Big)(x), \qquad (2.3.10)
\end{aligned}
$$

where $m = \lceil \alpha \rceil$, $k^{(m)}(t_1) = \Big(\frac{d}{dt_1}\Big)^m k(t_1)$.

Hilfer derivative was introduced by Hilfer [10]. It is considered as an inset between R-L and Caputo derivative. The formal definition of the Hilfer derivative is given as:

Definition 6. For $x \in [a, b]$, the left/right-sided Hilfer derivatives of order $\alpha > 0$ and $\beta = [0, 1]$ are defined as

$$\mathcal{D}^{\alpha,\beta}_{a+}k(x) = (\mathfrak{I}^{\beta(1-\alpha)}_{a+}\frac{d}{dx}\mathfrak{I}^{(1-\alpha)(1-\beta)}_{a+}k)(x). \qquad (2.3.11)$$

$$\mathcal{D}^{\alpha,\beta}_{b-}k(x) = -(\mathfrak{I}^{\beta(1-\alpha)}_{b-}\frac{d}{dx}\mathfrak{I}^{(1-\alpha)(1-\beta)}_{b-}k)(x). \qquad (2.3.12)$$

In special case, for $\beta = 0$, we get RLF-derivative (defined in Eqs. (2.3.3) and (2.3.4)) and for $\beta = 1$, the Caputo frac-derivative is obtained (defined in Eqs. (2.3.9) and (2.3.10)).

Like R-L, Hadamard derivative of a constant is also non zero. To overcome this shortcoming, CHF-derivative was introduced [16], the definition of which is as follows:

Definition 7. For $x \in [a, b]$, the left/right-sided CHF-derivatives of order $\alpha > 0$ are defined as

$$
\begin{aligned}
{}_*\mathcal{D}^{\alpha}_{a+}k(x) &= \frac{1}{\Gamma(m-\alpha)}\int_a^x \Big(\ln\frac{x}{t_1}\Big)^{m-\alpha-1}\Big(t_1\frac{d}{dt_1}\Big)^m k(t_1)\frac{dt_1}{t_1}, \\
&= \Big(\mathcal{J}^{m-\alpha}_{a+}\delta^m k\Big)(x). \qquad (2.3.13) \\
{}_*\mathcal{D}^{\alpha}_{b-}k(x) &= \frac{(-1)^m}{\Gamma(m-\alpha)}\int_x^b \Big(\ln\frac{t_1}{x}\Big)^{m-\alpha-1}\Big(t_1\frac{d}{dt_1}\Big)^m k(t_1)\frac{dt_1}{t_1}, \\
&= \Big(\mathcal{J}^{m-\alpha}_{b-}(-\delta)^m k\Big)(x), \qquad (2.3.14)
\end{aligned}
$$

where $\delta^m = (x\frac{d}{dx})^m$ and $m = \lceil\alpha\rceil$.

Butzer et al. added a simple modification in the definition of Hadamard operators, to introduce a new concept known as HTF-operators ([20]), defined below.

Definition 8. For $x \in [a,b]$, left HTF-integral and derivative of order $\alpha > 0$ is defined as

$$
\begin{aligned}
\mathcal{J}^{\alpha}_{a+,c}k(x) &= \frac{1}{\Gamma(\alpha)}\int_a^x \Big(\frac{t_1}{x}\Big)^c\Big(\ln\frac{x}{t_1}\Big)^{\alpha-1}k(t_1)\frac{dt_1}{t_1} \qquad (2.3.15) \\
\mathcal{D}^{\alpha}_{a+,c}k(x) &= \frac{1}{\Gamma(m-\alpha)}x^{-c}\delta^m x^c\int_a^x \Big(\frac{t_1}{x}\Big)^c\Big(\ln\frac{x}{t_1}\Big)^{m-\alpha-1}k(t_1)\frac{dt_1}{t_1} \\
&= D^m_c(\mathcal{J}^{m-\alpha}_{a+,c}k)(x), \qquad (2.3.16)
\end{aligned}
$$

respectively. Here $D^m_c = x^{-c}\delta^m x^c$, $m = \lceil\alpha\rceil$ and $\delta^m = (x\frac{d}{dx})^m$. For $\alpha = 0$ and $c = 0$, we set $\mathcal{J}^0_{a+,0} = I$ (identity operator) and also for $\alpha = 0$ and $c = 0$, $\mathcal{D}^0_{a+,0} = I$.

Definition 9. For $x \in [a,b]$, right HTF-integral and derivative of order $\alpha > 0$ is defined as

$$
\begin{aligned}
\mathcal{J}^{\alpha}_{b-,c}k(x) &= \frac{1}{\Gamma(\alpha)}\int_x^b \Big(\frac{x}{t_1}\Big)^c\Big(\ln\frac{t_1}{x}\Big)^{\alpha-1}k(t_1)\frac{dt_1}{t_1} \qquad (2.3.17) \\
\mathcal{D}^{\alpha}_{b-,c}k(x) &= \frac{(-1)^m}{\Gamma(m-\alpha)}x^{c}\delta^m x^{-c}\int_x^b \Big(\frac{x}{t_1}\Big)^c\Big(\ln\frac{t_1}{x}\Big)^{m-\alpha-1}k(t_1)\frac{dt_1}{t_1}, \\
&= D^m_c(\mathcal{J}^{m-\alpha}_{b-,c}k)(x), \qquad (2.3.18)
\end{aligned}
$$

where $D_c^m = x^c(-\delta)^m x^{-c}$, $m = \lceil \alpha \rceil$ and $(-\delta)^m = (-x\frac{d}{dx})^m$.

2.4 Spaces

In this section, definitions of spaces of p-integrable, absolutely continuous functions and their weighted modifications are presented [8, 17].

Definition 10. Let $\mathcal{L}^p(a,b)(1 \leq p \leq \infty)$ be the set of those Lebesgue complex-valued measurable functions k on $[a,b]$ for which $\|k\|_p < \infty$, where

$$\|k\|_p = \Big(\int_a^b |k(t)|^p dt \Big)^{\frac{1}{p}}, \quad 1 \leq \ p < \infty, \tag{2.4.1}$$

and

$$\|k\|_\infty = \text{ess} \sup_{a \leq \ x < b} |k(x)|, \tag{2.4.2}$$

where $\text{ess}\sup k(x)$ is the essential supremum of $k(x)$.

Definition 11. Let $\mathcal{X}_c^p(a,b)$ $(c \in \mathbb{R}; 1 \leq p \leq \infty)$, be the set of those Lebesgue complex-valued measurable functions k on (a,b) for which $\|k\|_{X_c^p} < \infty$, with

$$\|k\|_{X_c^p} = \Big(\int_a^b |t^c k(t)|^p \frac{dt}{t} \Big)^{\frac{1}{p}}, \quad 1 \leq \ p < \infty, \tag{2.4.3}$$

and

$$\|k\|_{X_c^\infty} = \text{ess} \sup_{a \leq \ x < b} |x^c k(x)|. \tag{2.4.4}$$

In special case, when $c = \frac{1}{p}$, the space $\mathcal{X}_c^p(a,b)$ coincides with the $\mathcal{L}^p(a,b)$ space in Eq. (2.4.1).

Definition 12. Let $[a, b]$ be a finite interval and let $\mathcal{AC}[a, b]$ be the space of absolutely continuous functions.

It is known that $\mathcal{AC}[a, b]$ coincides with the space of Lebesgue summable functions:

$$k \in \mathcal{AC}[a, b] \implies k(x) = c + \int_a^x f(t)dt. \tag{2.4.5}$$

where $f \in \mathcal{L}(a, b)$, and therefore an absolutely continuous function $k(x)$ has a summable derivative $k^{'}(x) = f(x)$ almost everywhere on [a,b]. Thus

$$f(t) = k^{'}(t), \qquad c = k(a). \tag{2.4.6}$$

For $n \in \mathbb{N}$, we denote by $\mathcal{AC}^n[a, b]$ the space of complex-valued functions $k(x)$ which have continuous derivatives up to order $(n-1)$ on $[a, b]$ such that $k^{n-1}(x) \in \mathcal{AC}[a, b]$:

$$\mathcal{AC}^n[a, b] = \Big\{ k : [a, b] \to \mathbb{C} \quad \text{and} \quad (D^{n-1}k)(x) \in \mathcal{AC}[a, b] \Big\}, \tag{2.4.7}$$

$\mathbb{C}$ being the set of complex numbers. In particular, $\mathcal{AC}^1[a, b] = \mathcal{AC}[a, b]$. Such changes, which we represented by $\mathcal{AC}^n_{\delta,\mu}[a, b](n \in \mathbb{N}, \mu \in \mathbb{R})$, it includes functions h, which are the complex-valued Lebesgue measurable on (a, b) such that $x_1^{\mu} h(x_1)$ has δ-derivatives order $(n-1)$ on $[a, b]$ and $\delta^{n-1}[x_1^{\mu} h(x_1)]$ is absolutely continuous on $[a, b]$. Therefore, a generalization of space in Eq (2.4.7) is

$$\mathcal{AC}^n_{\delta,\mu}[a, b] = \Big\{ h : [a, b] \to \mathbb{C} : \delta^{n-1}[x_1^{\mu} h(x_1)] \in \mathcal{AC}[a, b], \mu \in \mathbb{R}, \delta = x_1 \frac{d}{dx_1} \Big\}. \tag{2.4.8}$$

As special case, when $\mu = 0$,

$$\mathcal{AC}^n_{\delta}[a, b] = \Big\{ h : [a, b] \to \mathbb{C} : \delta^{n-1}[h(x_1)] \in \mathcal{AC}[a, b], \delta = x_1 \frac{d}{dx_1} \Big\}. \tag{2.4.9}$$

If $\mu = 0$ and $n = 1$, the space $\mathcal{AC}^1_\delta[a,b]$ coincides with $\mathcal{AC}[a,b]$.

Definition 13. Let $-\infty \leq a < b \leq \infty$ and $m \in \mathbb{N}_0$. Then $C^m[a,b]$ is given as:

$$\|k\|_{C^m} = \sum_{q=0}^{m} \|k^{(q)}\|_C = \sum_{q=0}^{m} \max_{x\in[a,b]} |k^{(q)}(x)|. \tag{2.4.10}$$

For $m = 0$, $C^0[a,b] = C[a,b]$

2.5 Generalized fractional operators

In this section, a new approach for frac-operators is presented which was introduced by Katugampola in [23] and [24]. He has generalized R-L and Hadamard frac-operators into single form which now are the generalized RLF-integrals and derivatives, respectively. Furthermore, Oliveira [14] introduced a new operator which generalizes Caputo and Hadamard frac-derivatives into single form known as generalized Caputo frac-derivatives. where as, generalized Hilfer frac-derivatives were introduced in [15].

Definition 14. For $x \in [a,b]$, the left/right sided generalized RLF-integrals of order $\alpha > 0$, $\nu > 0$ and $k \in \mathcal{X}^p_c$ are defined as

$$ {}^\nu\mathfrak{I}^\alpha_{a+}k(x) = \frac{\nu^{1-\alpha}}{\Gamma(\alpha)} \int_a^x (x^\nu - \tau_1^\nu)^{\alpha-1}\tau_1^{\nu-1}k(\tau_1)d\tau_1. \tag{2.5.1}$$

$$ {}^\nu\mathfrak{I}^\alpha_{b-}k(x) = \frac{\nu^{1-\alpha}}{\Gamma(\alpha)} \int_x^b (\tau_1^\nu - x^\nu)^{\alpha-1}\tau_1^{\nu-1}k(\tau_1)d\tau_1. \tag{2.5.2}$$

For $x \in [a,b]$, the left/right sided generalized RLF-derivatives of order $\alpha > 0$, $\nu > 0$ and $k \in \mathcal{X}^p_c$ are defined as

$$\begin{aligned} {}^\nu\mathcal{D}^\alpha_{a+}k(x) &= \frac{\nu^{1-m+\alpha}}{\Gamma(m-\alpha)}\Big(x^{1-\nu}\frac{d}{dx}\Big)^m \int_a^x (x^\nu - \tau_1^\nu)^{m-\alpha-1}\tau_1^{\nu-1}k(\tau_1)d\tau_1, \\ &= \delta^m_\nu(\mathfrak{I}^{m-\alpha}_{a+}k)(x). \end{aligned} \tag{2.5.3}$$

$$
\begin{aligned}
{}^{\nu}\mathcal{D}^{\alpha}_{b-}k(x) &= \frac{\nu^{1-m+\alpha}}{\Gamma(m-\alpha)}\Big(-x^{1-\nu}\frac{d}{dx}\Big)^m \int_x^b (\tau_1^{\nu}-x^{\nu})^{m-\alpha-1}\tau_1^{\nu-1}k(\tau_1)d\tau_1,\\
&= (-\delta)^m_{\nu}(\mathfrak{I}^{m-\alpha}_{b-}k)(x), \qquad (2.5.4)
\end{aligned}
$$

where $\delta^m_{\nu} = \Big(x^{1-\nu}\frac{d}{dx}\Big)^m$ and $m = \lceil \alpha \rceil$.

After that Oliveira et al. [14] introduced generalization of Caputo and Hadamard derivative, defined below.

Definition 15. For $x \in [a,b]$, the left/right sided generalized Caputo frac-derivatives of order $\alpha > 0$, $\nu > 0$ and $k \in \mathcal{X}^p_c$ are defined as

$$
\begin{aligned}
{}^{\nu}_{*}\mathcal{D}^{\alpha}_{a+}k(x) &= \frac{\nu^{1-m+\alpha}}{\Gamma(m-\alpha)}\int_a^x \frac{t^{\nu-1}}{(x^{\nu}-t^{\nu})^{1-m+\alpha}}\Big(t^{1-\nu}\frac{d}{dt}\Big)^m k(t)dt,\\
&= (\mathfrak{I}^{m-\alpha}_{a+}\delta^m_{\nu}k)(t). \qquad (2.5.5)\\
{}^{\nu}_{*}\mathcal{D}^{\alpha}_{b-}k(x) &= (-1)^m\frac{\nu^{1-m+\alpha}}{\Gamma(m-\alpha)}\int_a^x \frac{t^{\nu-1}}{(t^{\nu}-x^{\nu})^{1-m+\alpha}}\Big(t^{1-\nu}\frac{d}{dt}\Big)^m k(t)dt,\\
&= (-1)^m(\mathfrak{I}^{m-\alpha}_{b-}\delta^m_{\nu}k)(t), \qquad (2.5.6)
\end{aligned}
$$

where $\delta^m_{\nu} = \Big(x^{1-\nu}\frac{d}{dx}\Big)^m$ and $m = \lceil \alpha \rceil$.

Oliveira [15] also introduced generalization of Hilfer and Hilfer-Hadamard frac-derivatives as:

Definition 16. For $x \in [a,b]$, the left & right-sided generalized Hilfer derivatives of order $m-1 < \alpha \le m$ and $\beta = [0,1]$ with $m \in \mathbb{N}$, $\nu > 0$ and $k \in \mathcal{X}^p_c$ are defined as

$$
{}^{\nu}\mathcal{D}^{\alpha,\beta}_{a+}k(t) = ({}^{\nu}\mathfrak{I}^{\beta(m-\alpha)}_{a+}\delta^m_{\nu}\mathfrak{I}^{(m-\alpha)(1-\beta)}_{a+}k)(t). \qquad (2.5.7)
$$

$$
{}^{\nu}\mathcal{D}^{\alpha,\beta}_{b-}k(t) = (-1)^m({}^{\nu}\mathfrak{I}^{\beta(m-\alpha)}_{b-}\delta^m_{\nu}\mathfrak{I}^{(m-\alpha)(1-\beta)}_{b-}k)(t). \qquad (2.5.8)
$$

where $\delta^m_{\nu} = \Big(x^{1-\nu}\frac{d}{dx}\Big)^m$ and $m = \lceil \alpha \rceil$.

In special case, when $\nu = 1$ and $\beta = 0$, we get the definition of RLF-derivatives (defined in Eqs. (2.3.3), (2.3.4)) and for $\nu = 1$ and $\beta = 1$, we get the definition of Caputo frac-derivatives (defined in Eqs. (2.3.9), (2.3.10)).

2.6 Mellin transform

Mellin transform occurs in many areas of engineering and applied mathematics. Its utilization can be found in the study of classes of functions which are defined on the positive real line. Riemann acknowledged Mellin transform in his study on prime numbers. Further more Cahen worked on it but according to Flajolet et al. the formula of Mellin transform and its inverse was derived by Mellin [18]. He derived Mellin transform ($\mathcal{M}$) and its inverse ($\mathcal{M}^{-1}$) from complex Fourier transform and its inverse (see Chapter 8 of [25]).

Definition 17. The Mellin transform of a real valued function $k(x)$ on $(0, \infty)$ is defined by

$$\mathcal{M}[k](s) = \int_0^\infty x^{s-1} k(x) dx, \tag{2.6.1}$$

where s is a complex number.

Definition 18. The inverse Mellin transform is defined by

$$\mathcal{M}^{-1}[k](x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} x^{-s} k(s) ds. \tag{2.6.2}$$

where c is a constant.

Mellin transform is closely related to the Laplace transform, Fourier transform, theory of the gamma function and allied special functions. In case of many applications Mellin transform is more convenient to operate directly rather than the Laplace and Fourier transform. The Mellin transform can simplify some of the frac-calculus operations which makes it an interesting subject for research in frac-calculus.

2.7 Some useful properties of Mellin transform

Various properties of Mellin transform are treated in different books on integral transforms like in [25]. Following are some of them.

If $\mathcal{M}[k](x) = \tilde{k}(s)$ and $\mathcal{M}[h](x) = \tilde{h}(s)$, then the following properties hold:

i. $\mathcal{M}[k(ax)] = a^{-s}\tilde{k}(s), a > 0.$

ii. $\mathcal{M}[x^a k(x)] = \tilde{k}(s+a).$

iii. $\mathcal{M}[k(x^a)] = \frac{1}{a}\tilde{k}(\frac{s}{a}).$

iv. $\mathcal{M}[xk^{'}(x)] = -s\tilde{k}(s).$

v. $\mathcal{M}[\int_0^x k(t)dt] = \frac{-1}{s}\tilde{k}(s+1), s \neq 0$

vi. $\mathcal{M}[k^{'}(x)] = -(s-1)\tilde{k}(s-1).$

vii. $\mathcal{M}[(xk^{'})^n(x)] = (-1)^n s^n \tilde{k}(s).$

viii. $\mathcal{M}[k(x) * h(x)] = \tilde{k}(s)\tilde{h}(s).$

Uniqueness theorem of Mellin transform

Theorem 2.7.1. [11] Let $k, g \in X_c$ for some $c \in \mathbb{R}$ such that $\mathcal{M}[k](c+it_1) = \mathcal{M}[h](c+it_1)$ for all $t \in \mathbb{R}$. Then $k = h$ on $\mathbb{R}+$.

Existence of Mellin transform

Theorem 2.7.2. [26] The conditions $k(t_1)_{t_1 \to 0+} = O(t_1^{u_1})$, $k(t_1)_{t_1 \to +\infty} = O(t_1^{u_2})$. when $u_1 > u_2$, guarantee that $\tilde{k}(s)$ exists in the strip $< -u_1, -u_2 >$.

Chapter 3

Hadamard-type frac-operators

In this chapter, basic properties such as semi-group property, composition of integrals and derivatives are presented. Also relevant examples about Hadamard frac-integrals and derivatives, CH-derivatives, HT-integrals and derivatives and CHT-derivatives are discussed. Furthermore, Taylor's formula and its relevant results are given for CH and CHT-derivatives. Later, the integration by parts formula for Hadamard and HT-derivatives is discussed. HTF-integrals and derivatives were introduced by Butzer et al. in 1997 by using a simple modification in the definition of Hadamard frac-operators.

3.1 Hadamard frac-integrals and derivatives

In this section, some results of Hadamard calculus are reviewed from [8], [9] and [17].

Example 3.1.1. For $\alpha > 0$ and $\beta > -1$

(a) Let $k(x) = \left(\ln \frac{x}{a}\right)^{\beta}$. Then

$$\mathcal{J}^{\alpha}_{a+} k(x) = \frac{\Gamma(\beta+1)}{\Gamma(\beta+\alpha+1)} \left(\ln \frac{x}{a}\right)^{\beta+\alpha}. \tag{3.1.1}$$

and

$$\mathcal{D}_{a+}^{\alpha} k(x) = \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)} \left(\ln \frac{x}{a}\right)^{\beta-\alpha}. \tag{3.1.2}$$

(b) Let $g(x) = \left(\ln \frac{b}{x}\right)^{\beta}$. Then

$$\mathcal{J}_{b-}^{\alpha} g(x) = \frac{\Gamma(\beta+1)}{\Gamma(\beta+\alpha+1)} \left(\ln \frac{b}{x}\right)^{\beta+\alpha}. \tag{3.1.3}$$

and

$$\mathcal{D}_{b-}^{\alpha} g(x) = \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)} \left(\ln \frac{b}{x}\right)^{\beta-\alpha}. \tag{3.1.4}$$

3.1.1 Some properties of Hadamard operators

Some results/properties of Hadamard calculus are as follows.

Semi-group property:

Theorem 3.1.2. Let $\alpha > 0$, $\beta > 0$ and $0 < a < b < \infty$. Then for $k \in \mathcal{L}^p(a,b)$, $p \in [1,\infty]$ the following equalities hold everywhere on open interval of a and b.

(a) $\mathcal{J}_{a+}^{\alpha}\ \mathcal{J}_{a+}^{\beta} k(t) = \mathcal{J}_{a+}^{\alpha+\beta} k(t)$.

(b) $\mathcal{J}_{b-}^{\alpha}\ \mathcal{J}_{b-}^{\beta} k(t) = \mathcal{J}_{b-}^{\alpha+\beta} k(t)$.

Now we discuss an interesting result.

Lemma 3.1.3. If $n \in \mathbb{N}$ and $0 < a < b < \infty$ then for every $k \in \mathcal{L}^p(a,b)$, $1 \leq p \leq \infty$, the following equalities hold almost everywhere on open interval of a and b. .

(a) $\delta^n \mathcal{J}_{a+}^{n} k(x) = k(x)$, where $\delta = x\frac{d}{dx}$.

(b) $\delta^n \mathcal{J}_{b-}^{n} k(x) = k(x)$, where $\delta = -x\frac{d}{dx}$.

Lemma 3.1.4. If $n \in \mathbb{N}$ and $0 < a < b < \infty$, then for every $k \in \mathcal{AC}_\delta^n[a,b]$ the following relations hold almost everywhere,

(a) $\mathcal{J}_{a+}^n(\delta^n k(x)) = k(x) - \sum_{q=0}^{n-1} \frac{\delta^q k(a)}{\Gamma(q+1)} \Big(\ln \frac{x}{a} \Big)^q$, where $\delta = x\frac{d}{dx}$.

(b) $\mathcal{J}_{b-}^n(\delta^n k(x)) = k(x) - \sum_{q=0}^{n-1} \frac{\delta^q k(b)}{\Gamma(q+1)} \Big(\ln \frac{b}{x} \Big)^q$, where $\delta = -x\frac{d}{dx}$.

Composition of Hadamard integrals and derivatives:

Compositions between $\mathcal{D}^\alpha$ and $\mathcal{J}^\alpha$ are given by following Theorems.

Lemma 3.1.5. Let $\alpha > 0$, $0 < a < b < \infty$ and $1 \leq p \leq \infty$, then for $k \in \mathcal{L}^p(a,b)$,

(a) $\mathcal{D}_{a+}^\alpha\big(\mathcal{J}_{a+}^\alpha k\big)(x) = k(x)$.

(b) $\mathcal{D}_{b-}^\alpha\big(\mathcal{J}_{b-}^\alpha k\big)(x) = k(x)$.

Lemma 3.1.6. If $\alpha > 0$ and $0 < a < b < \infty$. Suppose that $k \in \mathcal{L}^p(a,b)$ is such that $\mathcal{J}^{n-\alpha}k \in \mathcal{AC}_\delta^n[a,b]$. Then

(a) $\mathcal{J}_{a+}^\alpha\big(\mathcal{D}_{a+}^\alpha k\big)(x) = k(x) - \sum_{q=0}^{n-1} \frac{\delta^{n-q-1}\mathcal{J}_{a+}^{n-\alpha}k(a)}{\Gamma(\alpha-q)} \Big(\ln \frac{x}{a} \Big)^{\alpha-q-1}$.

(b) $\mathcal{J}_{b-}^\alpha\big(\mathcal{D}_{b-}^\alpha k\big)(x) = k(x) - \sum_{q=0}^{n-1} \frac{\delta^{n-q-1}\mathcal{J}_{b-}^{n-\alpha}k(b)}{\Gamma(\alpha-q)} \Big(\ln \frac{b}{x} \Big)^{\alpha-q-1}$.

In particular, for $0 < \alpha \leq 1$

$$\mathcal{J}_{a+}^\alpha \, \big(\mathcal{D}_{a+}^\alpha k\big)(x) = k(x) - \frac{\mathcal{J}_{a+}^{1-\alpha}k(a)}{\Gamma(\alpha)} \Big(\ln \frac{x}{a} \Big)^{\alpha-1}.$$

When Hadamard derivatives and integrals are of different order, then the following results hold.

Lemma 3.1.7. If $\alpha > 0$ and $\beta > 0$, where $\beta > \alpha$, $0 < a < b < \infty$ and $p \in [0,\infty]$, then for $k \in \mathcal{L}^p(a,b)$, following relations hold

(a) $\mathcal{D}_{a+}^\alpha(\mathcal{J}_{a+}^\beta k)(x) = \mathcal{J}_{a+}^{\beta-\alpha}k(x)$.

(b) $\mathcal{D}^{\alpha}_{b-}(\mathcal{J}^{\beta}_{b-}k)(x) = \mathcal{J}^{\beta-\alpha}_{b-}k(x)$.

In particular, for $\alpha = r \in \mathbb{N}$

$$\mathcal{D}^{r}_{a+}(\mathcal{J}^{\beta}_{a+}k)(x) = \mathcal{J}^{\beta-r}_{a+}k(x).$$

Theorem 3.1.8. For $\beta > \alpha > 0$, $0 < a < b < \infty$, and $1 \leq p \leq \infty$ then for $k \in \mathcal{L}^p(a,b)$,

(a) $\mathcal{J}^{\alpha}_{a+}(\mathcal{D}^{\beta}_{a+}k(x)) = \mathcal{J}^{\alpha-\beta}_{a+}k(x) - \sum\limits_{q=0}^{j-1} \frac{\delta^{j-q-1}\mathcal{J}^{j-\beta}_{a+}k(a)}{\Gamma(\alpha-q)}\left(\ln \frac{x}{a}\right)^{\alpha-q-1}$.

(b) $\mathcal{J}^{\alpha}_{b-}(\mathcal{D}^{\beta}_{b-}k(x)) = \mathcal{J}^{\alpha-\beta}_{b-}k(x) - \sum\limits_{q=0}^{j-1} \frac{\delta^{j-q-1}\mathcal{J}^{j-\beta}_{b-}k(b)}{\Gamma(\alpha-q)}\left(\ln \frac{b}{x}\right)^{\alpha-q-1}$.

Semi-group property of Hadamard frac-derivatives is discussed below.

Theorem 3.1.9. Let α, β be positive such that $m-1 < \alpha < m$, $n-1 < \beta < n$. Then

(a) $\mathcal{D}^{\alpha}_{a+}(\mathcal{D}^{\beta}_{a+}k(x)) = \mathcal{D}^{\alpha+\beta}_{a+}k(x)$.

(b) $\mathcal{D}^{\alpha}_{b-}(\mathcal{D}^{\beta}_{b-}k(x)) = \mathcal{D}^{\alpha+\beta}_{b-}k(x)$.

3.2 Caputo Hadamard frac-operators

In this section, results about CH-derivatives are reviewed from [12] and [16]. As Hadamard derivative of a constant is non zero, so to overcome this shortcoming, CH-derivatives were introduced. An example that is discussed below, plays a vital role in proving some properties of CH-operator.

Example 3.2.1. For $\alpha > 0$, $\beta > -1$ and $0 < a < b < \infty$,

(a) Let $k(x) = \left(\ln \frac{x}{a}\right)^{\beta}$, then

$${}_*\mathcal{D}^{\alpha}_{a+}k(x) = \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)}\left(\ln \frac{x}{a}\right)^{\beta-\alpha}. \tag{3.2.1}$$

(b) Let $h(x) = \left(\ln \frac{b}{x}\right)^{\beta}$, then

$$ {}_{*}\mathcal{D}^{\alpha}_{b-} h(x) = \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)} \left(\ln \frac{b}{x}\right)^{\beta-\alpha}. \tag{3.2.2} $$

In particular,

$$ {}_{*}\mathcal{D}^{\alpha}_{a+} 1 = 0 \quad \text{and} \quad {}_{*}\mathcal{D}^{\alpha}_{b-} 1 = 0. \tag{3.2.3} $$

Composition between CH-integrals and derivatives is discussed in following results.

Lemma 3.2.2. Let $\alpha > 0$ then for $k \in C[a,b]$,

(a) ${}_{*}\mathcal{D}^{\alpha}_{a+}\big(\mathcal{J}^{\alpha}_{a+} k\big)(x) = k(x)$.

(b) ${}_{*}\mathcal{D}^{\alpha}_{b-}\big(\mathcal{J}^{\alpha}_{b-} k\big)(x) = k(x)$.

Lemma 3.2.3. If $\alpha > 0$ and $k \in \mathcal{AC}^{n}_{\delta}[a,b]$ or $C^{n}_{\delta}[a,b]$ then the following relations hold:

(a) $\mathcal{J}^{\alpha}_{a+}\big({}_{*}\mathcal{D}^{\alpha}_{a+} k\big)(x) = k(x) - \sum\limits_{q=0}^{n-1} \frac{\delta^{q} k(a)}{\Gamma(q+1)} \left(\ln \frac{x}{a}\right)^{q}$.

(b) $\mathcal{J}^{\alpha}_{b-}\big({}_{*}\mathcal{D}^{\alpha}_{b-} k\big)(x) = k(x) - \sum\limits_{q=0}^{n-1} \frac{\delta^{q} k(b)}{\Gamma(q+1)} \left(\ln \frac{b}{x}\right)^{q}$.

In particular, If $0 < \alpha \leq 1$, then

$$ \mathcal{J}^{\alpha}_{a+}\big({}_{*}\mathcal{D}^{\alpha}_{a+} k\big)(x) = k(x) - k(a). $$

When CH-integrals and derivatives are of different order then following results hold.

Lemma 3.2.4. If $\alpha > 0$, $\beta > 0$ and $\beta > \alpha$ then for $k \in C^{n}_{\delta}[a,b]$ then the following relations hold:

(a) ${}_*\mathcal{D}^{\alpha}_{a+}(\mathcal{J}^{\beta}_{a+}k)(x) = \mathcal{J}^{\beta-\alpha}_{a+}k(x).$

(b) ${}_*\mathcal{D}^{\alpha}_{b-}(\mathcal{J}^{\beta}_{b-}k)(x) = \mathcal{J}^{\beta-\alpha}_{b-}k(x).$

Theorem 3.2.5. If $\alpha > 0$ and $\beta > 0$ then for $k \in C^n_\delta[a,b]$ the following relations hold:

(a) $\mathcal{J}^{\alpha}_{a+}({}_*\mathcal{D}^{\beta}_{a+}k)(x) = \mathcal{J}^{\alpha-\beta}_{a+}k(x) - \sum_{q=0}^{n-1} \frac{\delta^q k(a)}{\Gamma(q+\alpha-\beta+1)}\Big(\ln\frac{x}{a}\Big)^{q+\alpha-\beta}.$

(b) $\mathcal{J}^{\alpha}_{b-}({}_*\mathcal{D}^{\beta}_{b-}k)(x) = \mathcal{J}^{\alpha-\beta}_{b-}k(x) - \sum_{q=0}^{n-1} \frac{\delta^q k(b)}{\Gamma(q+\alpha-\beta+1)}\Big(\ln\frac{b}{x}\Big)^{q+\alpha-\beta}.$

Semi-group property for CH-derivatives is discussed below.

Theorem 3.2.6. If $k \in C^{m+n}_\delta[a,b]$. Let $\alpha \geq 0$ and $\beta \geq 0$ such that $m-1 < \alpha \leq m$ and $n-1 < \beta \leq n$. Then

(a) ${}_*\mathcal{D}^{\alpha}_{a+}({}_*\mathcal{D}^{\beta}_{a+}k)(x) = {}_*\mathcal{D}^{\alpha+\beta}_{a+}k(x).$

(b) ${}_*\mathcal{D}^{\alpha}_{b-}({}_*\mathcal{D}^{\beta}_{b-}k)(x) = {}_*\mathcal{D}^{\alpha+\beta}_{b-}k(x).$

3.3 Hadamard-type frac-integrals and derivatives

In this section, some results about HTF-operators are presented.

Following example shows that HTF-integrals and derivatives (see Definitions 8 and 9) of logarithmic functions $\Big(\ln\frac{x}{a}\Big)^{\beta}$ and $\Big(\ln\frac{b}{x}\Big)^{\beta}$ yield logarithmic function of the same form.

Example 3.3.1. For $c \in \mathbb{R}$, $\alpha > 0$ and $\beta > -1$

(a) Let $k(x) = x^{-c}\Big(\ln\frac{x}{a}\Big)^{\beta}$. Then

$$\begin{aligned}
\mathcal{J}^{\alpha}_{a+,c}k(x) &= \mathcal{J}^{\alpha}_{a+,c}x^{-c}\Big(\ln\frac{x}{a}\Big)^{\beta} \\
&= \frac{1}{\Gamma(\alpha)}\int_a^x t^{-c}\Big(\frac{t}{x}\Big)^{c}\Big(\ln(x)-\ln(t)\Big)^{\alpha-1}\Big(\ln(t)-\ln(a)\Big)^{\beta}\frac{dt}{t} \\
&= \frac{x^{-c}}{\Gamma(\alpha)}\int_a^x \Big(\ln(x)-\ln(t)-\ln(a)+\ln(a)\Big)^{\alpha-1}\Big(\ln(t)-\ln(a)\Big)^{\beta}\frac{dt}{t}
\end{aligned}$$

$$\mathcal{J}^{\alpha}_{a+,c}k(x) = \frac{x^{-c}}{\Gamma(\alpha)}\int_a^x \Big(\ln(x)-\ln(a)\Big)^{\alpha-1}\left(1-\frac{\ln(t)-\ln(a)}{\ln(x)-\ln(a)}\right)^{\alpha-1}$$
$$\times\Big(\ln(t)-\ln(a)\Big)^{\beta}\frac{dt}{t}.$$

Let $v = \frac{\ln(t)-\ln(a)}{\ln(x)-\ln(a)}$ and $dv = \frac{1}{\ln(t)-\ln(a)}\frac{dt}{t}$, then

$$\mathcal{J}^{\alpha}_{a+,c}k(x) = \frac{x^{-c}}{\Gamma(\alpha)}\Big(\ln(x)-\ln(a)\Big)^{\alpha+\beta}\int_0^1 (1-v)^{\alpha-1}v^{\beta}dv.$$

By using properties of Beta function defined in section 2.2, we get

$$\mathcal{J}^{\alpha}_{a+,c}k(x) = \frac{\Gamma(\beta+1)}{\Gamma(\beta+\alpha+1)}x^{-c}\Big(\ln\frac{x}{a}\Big)^{\beta+\alpha}. \tag{3.3.1}$$

Now to find HT-derivative of $k(x)$, first prove for $\beta = r$, where $r \in \mathbb{N}$

$$\mathcal{D}^{\alpha}_{a+,c}k(x) = \mathcal{D}^{\alpha}_{a+,c}x^{-c}\Big(\ln\frac{x}{a}\Big)^{r} = \frac{r!}{(r-\alpha)!}x^{-c}\Big(\ln\frac{x}{a}\Big)^{r-\alpha}$$

Replace r with β, we get

$$\mathcal{D}^{\alpha}_{a+,c}k(x) = \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)}x^{-c}\Big(\ln\frac{x}{a}\Big)^{\beta-\alpha}. \tag{3.3.2}$$

(b) Similarly, Let $g(x) = x^{c}\Big(\ln\frac{b}{x}\Big)^{\beta}$. Then

$$\mathcal{J}^{\alpha}_{b-,c}g(x) = x^{c}\frac{\Gamma(\beta+1)}{\Gamma(\beta+\alpha+1)}\Big(\ln\frac{b}{x}\Big)^{\beta+\alpha}. \tag{3.3.3}$$

and

$$\mathcal{D}^{\alpha}_{b-,c}g(x) = x^{c}\frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)}\Big(\ln\frac{b}{x}\Big)^{\beta-\alpha}. \tag{3.3.4}$$

Some results/properties of Hadamard calculus are formed as follows.

Semi-group property:

Theorem 3.3.2. [17] Let $\alpha > 0$, $\beta > 0$ and $0 < a < b < \infty$. Then for $k \in \mathcal{X}_c^p(a,b)$, $1 \le p \le \infty$ the following relations hold everywhere on (a,b).

(a) $\mathcal{J}^{\alpha}_{a+,c}\ \mathcal{J}^{\beta}_{a+,c}k(t) = \mathcal{J}^{\alpha+\beta}_{a+,c}k(t)$.

(b) $\mathcal{J}^{\alpha}_{b-,c}\ \mathcal{J}^{\beta}_{b-,c}k(t) = \mathcal{J}^{\alpha+\beta}_{b-,c}k(t)$.

Proof. (a) By using definition of $\mathcal{J}^{\alpha}_{a+,c}k(x)$ from Eq. (2.3.15), we have

$$\begin{aligned}
\mathcal{J}^{\alpha}_{a+,c}\ (\mathcal{J}^{\beta}_{a+,c}k)(t) &= \frac{1}{\Gamma(\alpha)}\int_a^t \Big(\frac{\tau_1}{t}\Big)^c \Big(\ln\frac{t}{\tau_1}\Big)^{\alpha-1} \mathcal{J}^{\beta}_{a+,c}k(\tau_1)\frac{d\tau_1}{\tau_1} \\
&= \frac{1}{\Gamma(\alpha)}\int_a^t \Big(\frac{\tau_1}{t}\Big)^c \Big(\ln\frac{t}{\tau_1}\Big)^{\alpha-1} \\
&\quad \times\Big[\frac{1}{\Gamma(\beta)}\int_a^{\tau_1} \Big(\frac{\xi}{\tau_1}\Big)^c \Big(\ln\frac{\tau_1}{\xi}\Big)^{\beta-1} k(\xi)\frac{d\xi}{\xi}\Big]\frac{d\tau_1}{\tau_1}.
\end{aligned}$$

By Dirichlet formula, we get

$$\begin{aligned}
&= \frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_a^t k(\xi)\Big(\frac{\xi}{t}\Big)^c\ \frac{d\xi}{\xi} \\
&\quad \times\int_\xi^t \Big(\ln(t)-\ln(\tau_1)\Big)^{\alpha-1}\Big(\ln(\tau_1)-\ln(\xi)\Big)^{\beta-1}\frac{d\tau_1}{\tau_1} \\
&= \frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_a^t k(\xi)\Big(\frac{\xi}{t}\Big)^c\ \frac{d\xi}{\xi}\int_\xi^t \Big(\ln(\tau_1)-\ln(\xi)\Big)^{\beta-1} \\
&\quad \times\Big(\ln(t)-\ln(\tau_1)-\ln(\xi)+\ln(\xi)\Big)^{\alpha-1}\frac{d\tau_1}{\tau_1} \\
&= \frac{1}{\Gamma(\alpha+\beta)}\int_a^t k(\xi)\Big(\frac{\xi}{t}\Big)^c\ \frac{d\xi}{\xi}\int_\xi^t \Big(\ln(t)-\ln(\xi)\Big)^{\alpha-1} \\
&\quad \times\left(1-\frac{\ln(\tau_1)-\ln(\xi)}{\ln(t)-\ln(\xi)}\right)^{\alpha-1}\Big(\ln(\tau_1)-\ln(\xi)\Big)^{\beta-1}\frac{d\tau_1}{\tau_1}.
\end{aligned}$$

Introduce $v = \frac{\ln(\tau_1)-\ln(\xi)}{\ln(t)-\ln(\xi)}$ and $dv = \frac{1}{\ln(t)-\ln(\xi)}\frac{d\tau_1}{\tau_1}$, then it takes the form

$$\begin{aligned}\mathcal{J}^{\alpha}_{a+,c}\,(\mathcal{J}^{\beta}_{a+,c}k)(t) &= \frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_a^t k(\xi)\Big(\frac{\xi}{t}\Big)^c\,\Big(\ln(t)-\ln(\xi)\Big)^{\alpha+\beta-1}\frac{d\xi}{\xi}\\ &\times\int_0^1 (1-v)^{\alpha-1}v^{\beta-1}dv.\\ &= \frac{1}{\Gamma(\alpha+\beta)}\int_a^t k(\xi)\Big(\frac{\xi}{t}\Big)^c\Big(\ln\frac{t}{\xi}\Big)^{\alpha+\beta-1}\frac{d\xi}{\xi}.\end{aligned}$$

By using properties of Beta function defined in section 2.2, we get the required result (a).

(b) Follows similar procedure.

□

Lemma 3.3.3. If $n \in \mathbb{N}$, $c \in \mathbb{R}$ and $0 < a < b < \infty$ then for every $k \in \mathcal{X}^p_c(a, b)$, $1 \leq p \leq \infty$, then the following relations hold:

(a) $D^n_c\big(\mathcal{J}^n_{a+,c}k(x)\big) = k(x)$.

(b) $D^n_c\big(\mathcal{J}^n_{b-,c}k(x)\big) = k(x)$.

Proof. (a) By using induction method, we have

For $n = 1$

$$D^1_c\mathcal{J}^1_{a+,c}k(x) = x^{-c}x\frac{d}{dx}x^c\mathcal{J}^1_{a+,c}k(x) = k(x). \tag{3.3.5}$$

Now we suppose that the relation holds for n and prove that it also holds for $n + 1$.

$$\begin{aligned}D^{n+1}_c\,\mathcal{J}^{n+1}_{a+,c}k(x) &= D^n_c(D^1_c\mathcal{J}^1_{a+,c})\mathcal{J}^n_{a+,c}k(x)\\ &= D^n_c\mathcal{J}^n_{a+,c}k(x) = k(x).\end{aligned}$$

using Eq. (3.3.5) and so assertion (a) of Lemma follows.

(b) Follows similar procedure.

□

Lemma 3.3.4. If $n \in \mathbb{N}$, $c \in \mathbb{R}$ and $0 < a < b < \infty$ then for every $k \in \mathcal{AC}^n_{\delta,\mu}[a,b]$, then the following relations hold:

(a) $\mathcal{J}^n_{a+,c}\big(D^n_c k\big)(x) = k(x) - a^c x^{-c} \sum\limits_{q=0}^{n-1} \frac{D^q_c k(a)}{\Gamma(q+1)} \Big(\ln \frac{x}{a} \Big)^q.$

(b) $\mathcal{J}^n_{b-,c}\big(D^n_c k\big) = k(x) - b^{-c} x^c \sum\limits_{q=0}^{n-1} \frac{D^q_c k(b)}{\Gamma(q+1)} \Big(\ln \frac{b}{x} \Big)^q.$

Proof. (a) By fundamental Theorem of calculus, we have

For $n = 1$

$$\begin{aligned} \mathcal{J}^1_{a+,c} D^1_c k(x) &= x^{-c} \int_a^x t^c D^1_c k(t) \frac{dt}{t} \\ &= x^{-c} \int_a^x \frac{d}{dt} t^c k(t) dt \\ &= x^{-c} [x^c k(x) - a^c k(a)] \\ &= k(x) - a^c x^{-c} k(a). \end{aligned} \tag{3.3.6}$$

For $n = 2$ and by Eq. (3.3.6), we have

$$\begin{aligned} \mathcal{J}^2_{a+,c} D^2_c k(x) &= \mathcal{J}^1_{a+,c} \mathcal{J}^1_{a+,c} (D^1_c D^1_c k) \\ &= \mathcal{J}^1_{a+,c} \big[D^1_c k(x) - a^c x^{-c} D^1_c k(a) \big] \\ &= \mathcal{J}^1_{a+,c} D^1_c k(x) - a^c \mathcal{J}^1_{a+,c} x^{-c} D^1_c k(a) \\ \mathcal{J}^2_{a+,c} D^2_c k(x) &= k(x) - a^c x^{-c} k(a) - a^c \mathcal{J}^1_{a+,c} x^{-c} D^1_c k(a), \end{aligned} \tag{3.3.7}$$

where,

$$\mathcal{J}^1_{a+,c} x^{-c} = x^{-c} \int_a^x \frac{dt}{t} = x^{-c} \Big(\ln \frac{x}{a} \Big). \tag{3.3.8}$$

Therefore,

$$\mathcal{J}^2_{a+,c} D^2_c k(x) = k(x) - a^c x^{-c} k(a) - x^{-c} a^c \left(\ln \frac{x}{a} \right) D^1_c k(a). \tag{3.3.9}$$

For $n = 3$ and by Eq. (3.3.9), we have

$$\begin{aligned} \mathcal{J}^3_{a+,c} D^3_c k(x) &= \mathcal{J}^1_{a+,c} \left[\mathcal{J}^2_{a+,c} D^2_c (D^1_c k) \right] \\ &= \mathcal{J}^1_{a+,c} \left[D^1_c k(x) - a^c x^{-c} D^1_c k(a) - x^{-c} a^c \left(\ln \frac{x}{a} \right) D^2_c k(a) \right] \\ &= \mathcal{J}^1_{a+,c} D^1_c k(x) - a^c x^{-c} D^1_c \mathcal{J}^1_{a+,c} k(a) \\ &\quad - x^{-c} a^c \left(\ln \frac{x}{a} \right) D^2_c \mathcal{J}^1_{a+,c} k(a). \end{aligned} \tag{3.3.10}$$

Here,

$$\mathcal{J}^1_{a+,c} \left(x^{-c} \ln \frac{x}{a} \right) = x^{-c} \int_a^x \ln \frac{t}{a} \frac{dt}{t}. \tag{3.3.11}$$

Let $\ln(\frac{t}{a}) = v$, then

$$\begin{aligned} \mathcal{J}^1_{a+,c} \left(x^{-c} \ln \frac{x}{a} \right) &= x^{-c} \int_0^{\ln(\frac{x}{a})} v dv \\ &= \frac{x^{-c} \left(\ln \frac{x}{a} \right)^2}{2}. \end{aligned} \tag{3.3.12}$$

Now using Eq. (3.3.11) and Eq. (3.3.12) in Eq. (3.3.10), we get

$$\begin{aligned} \mathcal{J}^3_{a+,c} D^3_c k(x) &= k(x) - a^c x^{-c} k(a) - \left(a^c D^1_c k(a) \right) x^{-c} \ln \left(\frac{x}{a} \right) \\ &\quad - \frac{\left(a^c D^2_c k(a) \right) x^{-c} \left(\ln \frac{x}{a} \right)^2}{2}. \end{aligned} \tag{3.3.13}$$

In general, we get the required result.

(b) Follows similar procedure. □

Composition of Hadamard-type integrals and derivatives

Lemma 3.3.5. If $\alpha > 0$, $c \in \mathbb{R}$ and $0 < a < b < \infty$ then for every $k \in \mathcal{X}_c^p(a,b)$, $1 \leq p \leq \infty$, then the following relations hold:

(a) $D_c^\alpha\big(\mathcal{J}_{a+,c}^\alpha k(x)\big) = k(x)$.

(b) $D_c^\alpha\big(\mathcal{J}_{b-,c}^\alpha k(x)\big) = k(x)$.

Lemma 3.3.6. [17] If $\alpha > 0$, $0 < a < b < \infty$ and $1 \leq p \leq \infty$, then for $k \in \mathcal{X}_c^p(a,b)$, the following relations hold

(a) $\mathcal{J}_{a+,c}^\alpha\ \big(\mathcal{D}_{a+,c}^\alpha k\big)(x) = k(x) - \sum\limits_{q=0}^{n-1} \frac{a^c x^{-c} D_c^{n-q-1} \mathcal{J}_{a+,c}^{n-\alpha} k(a)}{\Gamma(\alpha-q)} \left(\ln \frac{x}{a}\right)^{\alpha-q-1}$.

(b) $\mathcal{J}_{b-,c}^\alpha\ \big(\mathcal{D}_{b-,c}^\alpha k\big)(x) = k(x) - \sum\limits_{q=0}^{n-1} \frac{b^{-c} x^c D_c^{n-q-1} \mathcal{J}_{b-,c}^{n-\alpha} k(a)}{\Gamma(\alpha-q)} \left(\ln \frac{b}{x}\right)^{\alpha-q-1}$.

In particular, for $0 < \alpha \leq 1$

$$\mathcal{J}_{a+,c}^\alpha\ \big(\mathcal{D}_{a+,c}^\alpha k\big)(x) = k(x) - \frac{x^{-c} a^c \mathcal{J}_{a+,c}^{1-\alpha} k(a)}{\Gamma(\alpha)} \left(\ln \frac{x}{a}\right)^{\alpha-1}. \tag{3.3.14}$$

Proof. (a) Note that

$$D_c^n\ \big(\mathcal{J}_{a+,c}^{\alpha+n} k\big)(x) = D_c^n\ \mathcal{J}_{a+,c}^n\ \big(\mathcal{J}_{a+,c}^\alpha k\big) = \mathcal{J}_{a+,c}^\alpha k(x). \tag{3.3.15}$$

Using Eq. (3.3.15) in left hand side of assertion (a) of this Lemma and by definition of $\mathcal{D}_{a+,c}^\alpha k$ from Eq. (2.3.16), we get

$$\begin{aligned}\mathcal{J}_{a+,c}^\alpha\ \big(\mathcal{D}_{a+,c}^\alpha k\big)(x) &= D_c^n \mathcal{J}_{a+,c}^{\alpha+n} \mathcal{D}_{a+,c}^\alpha k(x)\\ &= D_c^n\ \mathcal{J}_{a+,c}^{\alpha+n}\ \big(D_c^n\ \mathcal{J}_{a+,c}^{n-\alpha} k\big)(x)\\ &= D_c^n\ \mathcal{J}_{a+,c}^\alpha \left[\mathcal{J}_{a+,c}^n D_c^n\ (\mathcal{J}_{a+,c}^{n-\alpha} k)(x)\right].\end{aligned}$$

By using Lemma 3.3.4(a) and Example 3.3.1(a)

$$\begin{aligned}\mathcal{J}^{\alpha}_{a+,c}\left(\mathcal{D}^{\alpha}_{a+,c}k\right)(x) &= D^n_c\ \mathcal{J}^{\alpha}_{a+,c}\Big[\mathcal{J}^{n-\alpha}_{a+,c}k(x)\\ &\quad -\sum_{q=0}^{n-1}\frac{a^c x^{-c}\ D^{n-q-1}\ \mathcal{J}^{n-\alpha}_{a+,c}k(a)}{\Gamma(n-q)}\Big(\ln\frac{x}{a}\Big)^{n-q-1}\Big]\\ &= D^n_c\Big[\mathcal{J}^{n}_{a+,c}k(x)\\ &\quad -\sum_{q=0}^{n-1}\frac{a^c x^{-c} D^{n-q-1}\ \mathcal{J}^{n-\alpha}_{a+,c}k(a)}{\Gamma(n-q+\alpha)}\Big(\ln\frac{x}{a}\Big)^{n-q-1+\alpha}\Big]\\ &= k(x)-\sum_{q=0}^{n-1}\frac{a^c x^{-c} D^{n-k-1}_c \mathcal{J}^{n-\alpha}_{a+,c}k(a)}{\Gamma(\alpha-q)}\Big(\ln\frac{x}{a}\Big)^{(\alpha-q-1)}.\end{aligned}$$

for $0<\alpha<1$ and $n=1$ we get the required result Eq. (3.3.14).

(b) Follows similar procedure.

□

Lemma 3.3.7. [9] If $\alpha>0$ and $\beta>0$ where $\beta>\alpha$, $0<a<b<\infty$ and let $c\in\mathbb{R}$, $1\le p\le\infty$ then for $k\in\mathcal{X}^p_c(a,b)$, following relations hold

(a) $\mathcal{D}^{\alpha}_{a+,c}(\mathcal{J}^{\beta}_{a+,c}k)(x)=\mathcal{J}^{\beta-\alpha}_{a+,c}k(x)$.

(b) $\mathcal{D}^{\alpha}_{b-,c}(\mathcal{J}^{\beta}_{b-,c}k)(x)=\mathcal{J}^{\beta-\alpha}_{b-,c}k(x)$.

In particular, for $\alpha=r\in\mathbb{N}$

$$\mathcal{D}^{r}_{a+,c}(\mathcal{J}^{\beta}_{a+,c}k)(x)=\mathcal{J}^{\beta-r}_{a+,c}k(x). \tag{3.3.16}$$

Proof. (a) If $\beta>\alpha\ge0$, then by using Theorem 3.3.2(a), we get

$$\begin{aligned}\mathcal{D}^{\alpha}_{a+,c}(\mathcal{J}^{\beta}_{a+,c}k)(x) &= \mathcal{D}^{\alpha}_{a+,c}\Big(\mathcal{J}^{\alpha}_{a+,c}\Big(\mathcal{J}^{\beta-\alpha}_{a+,c}k(x)\Big)\Big)\\ &= \mathcal{J}^{\beta-\alpha}_{a+,c}k(x).\end{aligned}$$

In particular, for $\alpha = r \in \mathbb{N}$, then by using Theorem 3.3.2(a), we get

$$\begin{aligned}\mathcal{D}^{r}_{a+,c}(\mathcal{J}^{\beta}_{a+,c}k)(x) &= \mathcal{D}^{r}_{a+,c}(\mathcal{J}^{r+\beta-r}_{a+,c}k)(x)\\ &= \mathcal{D}^{r}_{a+,c}\Big(\mathcal{J}^{r}_{a+,c}\Big(\mathcal{J}^{\beta-r}_{a+,c}k(x)\Big)\Big)\\ &= \mathcal{J}^{\beta-r}_{a+,c}k(x).\end{aligned}$$

(b) Follows similar procedure.

□

Lemma 3.3.8. If $\alpha > 0$ and $\beta > 0$ where $\beta > \alpha$, $0 < a < b < \infty$ and $1 \leq p \leq \infty$ then for $k \in \mathcal{X}^{p}_{c}(a,b)$, following relations hold

(a) $\mathcal{J}^{\alpha}_{a+,c}(\mathcal{D}^{\beta}_{a+,c}k)(x) = \mathcal{J}^{\alpha-\beta}_{a+,c}k(x) - a^{c}x^{-c}\sum_{q=0}^{j-1}\frac{D^{j-q-1}_{c}\mathcal{J}^{j-\beta}_{a+,c}k(a)}{\Gamma(\alpha-q)}\left(\ln\frac{x}{a}\right)^{\alpha-q-1}$.

(b) $\mathcal{J}^{\alpha}_{b-,c}(\mathcal{D}^{\beta}_{b-,c}k)(x) = \mathcal{J}^{\alpha-\beta}_{b-,c}k(x) - b^{-c}x^{c}\sum_{q=0}^{j-1}\frac{D^{j-q-1}_{c}\mathcal{J}^{j-\beta}_{b-,c}k(a)}{\Gamma(\alpha-q)}\left(\ln\frac{b}{x}\right)^{\alpha-q-1}$.

Proof. (a) By using Example 3.3.1(a), Lemma 3.3.3(a) and Lemma 3.3.6(a), we get

$$\begin{aligned}\mathcal{J}^{\alpha}_{a+,c}(\mathcal{D}^{\beta}_{a+,c}k)(x) &= \mathcal{J}^{\alpha-\beta}_{a+,c}\Big(\mathcal{J}^{\beta}_{a+,c}\mathcal{D}^{\beta}_{a+,c}k(x)\Big)\\ &= \mathcal{J}^{\alpha-\beta}_{a+,c}\Big[k(x) - \sum_{q=0}^{j-1}\frac{a^{c}x^{-c}D^{j-q-1}_{c}\mathcal{J}^{j-\beta}_{a+,c}k(a)}{\Gamma(\beta-q)}\left(\ln\frac{x}{a}\right)^{\beta-q-1}\Big]\\ &= \mathcal{J}^{\alpha-\beta}_{a+,c}k(x) - \sum_{q=0}^{j-1}\frac{a^{c}x^{-c}D^{j-k-1}_{c}\mathcal{J}^{j-\beta}_{a+,c}k(a)}{\Gamma(\alpha-q)}\left(\ln\frac{x}{a}\right)^{\alpha-k-1}.\end{aligned}$$

(b) Follows similar procedure.

□

Semi-group property of HTF-derivatives is given below.

Theorem 3.3.9. Let $\alpha > 0$, $\beta > 0$ such that $m-1 < \alpha < m$, $n-1 < \beta < n$. Then

(a) $\mathcal{D}^{\alpha}_{a+,c}(\mathcal{D}^{\beta}_{a+,c}k)(x) = \mathcal{D}^{\alpha+\beta}_{a+,c}k(x)$.

(b) $\mathcal{D}^{\alpha}_{b-,c}(\mathcal{D}^{\beta}_{b-,c}k)(x) = \mathcal{D}^{\alpha+\beta}_{b-,c}k(x)$.

Proof. By Lemma 3.3.3(a), Lemma 3.3.8(a) and Example 3.3.1, we have

$$\begin{aligned}\mathcal{D}^{\alpha}_{a+,c}(\mathcal{D}^{\beta}_{a+,c}k)(x) &= D^m_c\Big(\mathcal{J}^{m-\alpha}_{a+,c}\mathcal{D}^{\beta}_{a+,c}k(x)\Big)\\ &= D^m_c\Big[\mathcal{J}^{m-\alpha-\beta}_{a+,c}k(x) - a^c x^{-c}\sum_{q=0}^{j-1}\frac{D^{j-q-1}_c\mathcal{J}^{j-\beta}_{a+,c}k(a)}{\Gamma(m-\alpha-q)}\Big(\ln\frac{x}{a}\Big)^{m-\alpha-q-1}\Big]\\ &= \mathcal{D}^{\alpha+\beta}_{a+,c}k(x) - a^c x^{-c}\sum_{q=0}^{j-1}\frac{D^{j-q-1}_c\mathcal{J}^{j-\beta}_{a+,c}k(a)}{\Gamma(-\alpha-q)}\Big(\ln\frac{x}{a}\Big)^{-\alpha-q-1}. \quad (3.3.17)\end{aligned}$$

Interchanging α and β, we can write,

$$\begin{aligned}\mathcal{D}^{\beta}_{a+,c}(\mathcal{D}^{\alpha}_{a+,c}k)(x) &= D^n_c\Big(\mathcal{J}^{n-\beta}_{a+,c}\mathcal{D}^{\alpha}_{a+,c}k(x)\Big)\\ &= \mathcal{D}^{\alpha+\beta}_{a+,c}k(x) - a^c x^{-c}\sum_{q=0}^{j-1}\frac{D^{j-q-1}\mathcal{J}^{j-\alpha}_{a+,c}k(a)}{\Gamma(-\beta-q)}\Big(\ln\frac{x}{a}\Big)^{-\beta-q-1}. \quad (3.3.18)\end{aligned}$$

The comparison of the relationships Eq. (3.3.17) and Eq. (3.3.18) says that in general case the HTF-derivatives do not commute, with only one exception ($\alpha = \beta$), for ($\alpha \neq \beta$) we have

$$\mathcal{D}^{\alpha}_{a+,c}(\mathcal{D}^{\beta}_{a+,c}k)(x) = \mathcal{D}^{\beta}_{a+,c}(\mathcal{D}^{\alpha}_{a+,c}k)(x) = \mathcal{D}^{\alpha+\beta}_{a+,c}k(x).$$

only if both sums in the right hand sides of Eq. (3.3.17) and Eq. (3.3.18) vanish.

(b) Follows similar procedure. □

For $c = 0$, HTF-operators and their results given in this section, coincide with Hadamard frac-operators and their results given in section 3.1.

3.4 Caputo Hadamard-type frac-operators

In this section, motivated from CHF-derivative, a new differential operator known as CHTF-derivative is presented by interchanging integral and differential operators of HTF-derivatives. Some results such as semi-group property for derivatives and composition of integrals and derivatives are discussed.

Definition 19. For $x \in [a, b]$, the left and right-sided CHTF-derivatives of order $\alpha > 0$ and $m = \lceil \alpha \rceil$, are defined as

$$
\begin{aligned}
{}_*\mathcal{D}^{\alpha}_{a+,c}k(x) &= \frac{1}{\Gamma(m-\alpha)} \int_a^x \Big(\frac{t}{x}\Big)^c \Big(\ln \frac{x}{t}\Big)^{m-\alpha-1} x^{-c}\delta^m x^c k(t)\frac{dt}{t}, \\
&= \Big(\mathcal{J}^{m-\alpha}_{a+,c} D^m_c k\Big)(x),
\end{aligned} \tag{3.4.1}
$$

where $D^m_c = x^{-c}\delta^m x^c$ and $\delta^m = (x\frac{d}{dx})^m$.

$$
\begin{aligned}
{}_*\mathcal{D}^{\alpha}_{b-,c}k(x) &= \frac{(-1)^m}{\Gamma(m-\alpha)} \int_x^b \Big(\frac{x}{t}\Big)^c \Big(\ln \frac{t}{x}\Big)^{m-\alpha-1} x^{c}\delta^m x^{-c} k(t)\frac{dt}{t}, \\
&= \Big(\mathcal{J}^{m-\alpha}_{b-,c} D^m_c k\Big)(x),
\end{aligned} \tag{3.4.2}
$$

where $D^m_c = x^{c}(-\delta)^m x^{-c}$ and $(-\delta)^m = (-x\frac{d}{dx})^m$. For $c = 0$, we get the definition of CHF-derivative (Definition 7).

Now we discuss an example about CHTF- derivative.

Example 3.4.1. For $\alpha > 0$ and $\beta > -1$

(a) Let $k(x) = x^{-c}\Big(\ln \frac{x}{a}\Big)^{\beta}$, to find CHTF-derivative of $k(x)$, first prove for $\beta = r \in \mathbb{N}$

$$
{}_*\mathcal{D}^{\alpha}_{a+,c}k(x) = {}_*\mathcal{D}^{\alpha}_{a+,c}x^{-c}\Big(\ln \frac{x}{a}\Big)^{r} = \frac{\Gamma(r+1)}{\Gamma(r-\alpha+1)}x^{-c}\Big(\ln \frac{x}{a}\Big)^{r-\alpha}.
$$

Replace r with β, we get

$$ {}_*\mathcal{D}^{\alpha}_{a+,c}k(x) = \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)} x^{-c}\Big(\ln\frac{x}{a}\Big)^{\beta-\alpha}. \tag{3.4.3} $$

(b) Let $h(x) = x^{c}\Big(\ln\frac{b}{x}\Big)^{\beta}$

$$ {}_*\mathcal{D}^{\alpha}_{b-,c}h(x) = \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)} x^{c}\Big(\ln\frac{b}{x}\Big)^{\beta-\alpha}. \tag{3.4.4} $$

The following assertions show relation between operators as:

Theorem 3.4.2. Let $\alpha > 0$, $0 < a < b < \infty$ and $c \in \mathbb{R}$. Then

(a) Hadamard and HT-integrals have the following relations

$$ \mathcal{J}^{\alpha}_{a+,c}k(x) = x^{-c}\mathcal{J}^{\alpha}_{a+}x^{c}k(x), \tag{3.4.5} $$

and

$$ \mathcal{J}^{\alpha}_{b-,c}k(x) = x^{c}\mathcal{J}^{\alpha}_{b-}x^{-c}k(x). \tag{3.4.6} $$

(b) Hadamard and HT-derivatives have the following relations

$$ \mathcal{D}^{\alpha}_{a+,c}k(x) = x^{-c}\mathcal{D}^{\alpha}_{a+}x^{c}k(x), \tag{3.4.7} $$

and

$$ \mathcal{D}^{\alpha}_{b-,c}k(x) = x^{c}\mathcal{D}^{\alpha}_{b-}x^{-c}k(x). \tag{3.4.8} $$

(c) CH and CHT-derivatives have the following relations

$$ {}_*\mathcal{D}^{\alpha}_{a+,c}k(x) = x^{-c}\ {}_*\mathcal{D}^{\alpha}_{a+}x^{c}k(x), \tag{3.4.9}$$

and

$$ {}_*\mathcal{D}^{\alpha}_{b-,c}k(x) = x^{c}\ {}_*\mathcal{D}^{\alpha}_{b-}x^{-c}k(x). \tag{3.4.10}$$

Proof. (a) By Eq. (2.3.5) and Eq. (2.3.15), we have

$$\begin{aligned}
\mathcal{J}^{\alpha}_{a+,c}k(x) &= \int_a^x \left(\frac{t}{x}\right)^c \left(\ln\frac{x}{t}\right)^{\alpha-1} k(t)\frac{dt}{t} \\
&= x^{-c}\int_a^x \left(\ln\frac{x}{t}\right)^{\alpha-1} t^c k(t)\frac{dt}{t} \\
&= x^{-c}\mathcal{J}^{\alpha}_{a+}x^c k(x).
\end{aligned}$$

Now by Eq. (2.3.6) and Eq. (2.3.17) , we have

$$\begin{aligned}
\mathcal{J}^{\alpha}_{b-,c}k(x) &= \int_x^b \left(\frac{x}{t}\right)^c \left(\ln\frac{t}{x}\right)^{\alpha-1} k(t)\frac{dt}{t} \\
&= x^{c}\int_x^b \left(\ln\frac{t}{x}\right)^{\alpha-1} t^{-c} k(t)\frac{dt}{t} \\
&= x^{c}\mathcal{J}^{\alpha}_{b-}x^{-c} k(x).
\end{aligned}$$

(b) By Eq. (2.3.7), Eq. (2.3.16) and Eq. (3.4.5), we have

$$\begin{aligned}
\mathcal{D}^{\alpha}_{a+,c}k(x) &= x^{-c}\delta^m x^c \mathcal{J}^{m-\alpha}_{a+,c}k(x) \\
&= x^{-c}\delta^m \mathcal{J}^{m-\alpha}_{a+}x^c k(x) \\
&= x^{-c}\mathcal{D}^{\alpha}_{a+}x^c k(x).
\end{aligned}$$

Now by Eq. (2.3.8), Eq. (2.3.1) and Eq. (3.4.6), we have

$$\begin{aligned}\mathcal{D}^{\alpha}_{b-,c}k(x) &= (-1)^m x^c \delta^m x^{-c} \mathcal{J}^{m-\alpha}_{b-,c} k(x)\\ &= (-1)^m x^c \delta^m x^{-c}\Big(x^c \mathcal{J}^{m-\alpha}_{b-} x^{-c} k\Big)(x)\\ &= x^c \mathcal{D}^{\alpha}_{b-} x^{-c} k(x).\end{aligned}$$

(c) By Eq. (3.4.1), Eq. (2.3.13) and Eq. (3.4.5), we have

$$\begin{aligned}{}_*\mathcal{D}^{\alpha}_{a+,c}k(x) &= \mathcal{J}^{m-\alpha}_{a+,c} x^{-c} \delta^m x^c k(x)\\ &= x^{-c} \mathcal{J}^{m-\alpha}_{a+} \delta^m x^c k(x)\\ &= x^{-c}\ {}_*\mathcal{D}^{\alpha}_{a+} x^c k(x).\end{aligned}$$

Now by Eq. (3.4.2), Eq. (2.3.14) and Eq. (3.4.6), we have

$$\begin{aligned}{}_*\mathcal{D}^{\alpha}_{b-,c}k(x) &= (-1)^m \mathcal{J}^{m-\alpha}_{b-,c} x^{-c} \delta^m x^c k(x)\\ &= (-1)^m \Big(x^c \mathcal{J}^{m-\alpha}_{b-} x^{-c} x^c \delta^m x^{-c} k\Big)(x)\\ &= x^c\ {}_*\mathcal{D}^{\alpha}_{b-} x^{-c} k(x).\end{aligned}$$

□

Composition of HT-integrals and CHT-derivatives is discussed below.

Lemma 3.4.3. Let $\alpha > 0$ then for $k \in \mathcal{X}^p_c(a,b)$, $1 \leq\ p \leq \infty$,

(a) ${}_*\mathcal{D}^{\alpha}_{a+,c}\ \big(\mathcal{J}^{\alpha}_{a+,c} k\big)(x) = k(x)$.

(b) ${}_*\mathcal{D}^{\alpha}_{b-,c}\ \big(\mathcal{J}^{\alpha}_{b-,c} k\big)(x) = k(x)$.

Lemma 3.4.4. If $n-1 < \alpha \leq n$ and $k \in \mathcal{X}^p_c(a,b)$, $1 \leq\ p \leq \infty$, then the following relation holds.

(a) $\mathcal{J}^{\alpha}_{a+,c}\ \big(\ {}_*\mathcal{D}^{\alpha}_{a+,c} k\big)(x) = k(x) - a^c x^{-c} \sum\limits_{q=0}^{n-1} \frac{D^q_c k(a)}{\Gamma(q+1)} \Big(\ln \frac{x}{a}\Big)^q$.

(b) $\mathcal{J}^{\alpha}_{b-,c}\left({}_{*}\mathcal{D}^{\alpha}_{b-,c}k\right)(x) = k(x) - b^{-c}x^{c}\sum_{q=0}^{n-1}\frac{D^{q}_{c}k(b)}{\Gamma(q+1)}\left(\ln\frac{b}{x}\right)^{q}.$

In particular, If $0 < \alpha \leq 1$, then

$$\mathcal{J}^{\alpha}_{a+,c}\left({}_{*}\mathcal{D}^{\alpha}_{a+,c}k\right)(x) = k(x) - a^{c}x^{-c}k(a). \tag{3.4.11}$$

Proof. (a) By Eq. (3.4.1) and Theorem 3.3.2, we have

$$\begin{aligned}\mathcal{J}^{\alpha}_{a+,c}\left({}_{*}\mathcal{D}^{\alpha}_{a+,c}k\right)(x) &= \mathcal{J}^{\alpha}_{a+,c}(\mathcal{J}^{m-\alpha}_{a+,c}D^{m}_{c}k)(x)\\ &= \mathcal{J}^{\alpha+m-\alpha}_{a+,c}D^{m}_{c}k(x)\\ &= \mathcal{J}^{m}_{a+,c}D^{m}_{c}k(x). \end{aligned} \tag{3.4.12}$$

Using Lemma 3.3.4(a), we have

$$\mathcal{J}^{\alpha}_{a+,c}\left({}_{*}\mathcal{D}^{\alpha}_{a+,c}k\right)(x) = k(x) - a^{c}x^{-c}\sum_{q=0}^{n-1}\frac{D^{q}_{c}k(a)}{\Gamma(q+1)}\left(\ln\frac{x}{a}\right)^{q}.$$

In particular, if $n = 1$ in Eq. (3.4.12)

$$\begin{aligned}\mathcal{J}^{\alpha}_{a+,c}\left({}_{*}\mathcal{D}^{\alpha}_{a+,c}k\right)(x) &= \mathcal{J}^{1}_{a+,c}D^{1}_{c}k(x)\\ &= k(x) - a^{c}x^{-c}k(a).\end{aligned}$$

(b) Follows similar procedure.

□

When HT-integrals and CHT-derivatives are of different order then the following results hold.

Lemma 3.4.5. If $\alpha > 0$ and $\beta > 0$ where $\beta > \alpha$, $0 < a < b < \infty$ and $1 \leq p \leq \infty$ then for $k \in X^{p}_{c}(a, b)$, following relations hold

(a) ${}_{*}\mathcal{D}^{\alpha}_{a+,c}(\mathcal{J}^{\beta}_{a+,c}k)(x) = \mathcal{J}^{\beta-\alpha}_{a+,c}k(x).$

(b) ${}_*\mathcal{D}^{\alpha}_{b-,c}(\mathcal{J}^{\beta}_{b-,c}k)(x) = \mathcal{J}^{\beta-\alpha}_{b-,c}k(x)$.

Proof. (a) If $\beta > \alpha \geq 0$, then by using Theorem 3.3.2 , we get

$$\begin{aligned} {}_*\mathcal{D}^{\alpha}_{a+,c}(\mathcal{J}^{\beta}_{a+,c}k)(x) &= {}_*\mathcal{D}^{\alpha}_{a+,c}\Big(\mathcal{J}^{\alpha}_{a+,c}(\mathcal{J}^{\beta-\alpha}_{a+,c}k(x))\Big) \\ &= \mathcal{J}^{\beta-\alpha}_{a+,c}k(x). \end{aligned}$$

(b) Follows similar procedure.

□

Theorem 3.4.6. If $\alpha > 0$ and $\beta > 0$ where $\beta > \alpha$, $0 < a < b < \infty$ and $1 \leq p \leq \infty$, then for $k \in X^p_c(a, b)$, following relations hold

(a) $\mathcal{J}^{\alpha}_{a+,c}({}_*\mathcal{D}^{\beta}_{a+,c}k)(x) = \mathcal{J}^{\alpha-\beta}_{a+,c}k(x) - a^c x^{-c}\sum_{q=0}^{n-1}\frac{D^q_c k(a)}{\Gamma(q+\alpha-\beta+1)}\Big(\ln\frac{x}{a}\Big)^{q+\alpha-\beta}$.

(b) $\mathcal{J}^{\alpha}_{b-,c}({}_*\mathcal{D}^{\beta}_{b-,c}k)(x) = \mathcal{J}^{\alpha-\beta}_{a+,c}k(x) - b^{-c} x^{c}\sum_{q=0}^{n-1}\frac{D^q_c k(b)}{\Gamma(q+\alpha-\beta+1)}\Big(\ln\frac{b}{x}\Big)^{q+\alpha-\beta}$.

Proof. (a) By using Example 3.3.1, Theorems 3.3.2(a) and Lemma 3.4.4(a), we get

$$\begin{aligned} \mathcal{J}^{\alpha}_{a+,c}({}_*\mathcal{D}^{\beta}_{a+,c}k)(x) &= \mathcal{J}^{\alpha-\beta}_{a+,c}\Big(\mathcal{J}^{\beta}_{a+,c}\, {}_*\mathcal{D}^{\beta}_{a+,c}k(x)\Big) \\ &= \mathcal{J}^{\alpha-\beta}_{a+,c}\Big[k(x) - \sum_{q=0}^{n-1}\frac{a^c x^{-c} D^q_c k(a)}{\Gamma(q+1)}\Big(\ln\frac{x}{a}\Big)^{q}\Big] \\ &= \mathcal{J}^{\alpha-\beta}_{a+,c}k(x) - \sum_{q=0}^{n-1}\frac{a^c x^{-c} D^q_c k(a)}{\Gamma(q+\alpha-\beta+1)}\Big(\ln\frac{x}{a}\Big)^{q+\alpha-\beta}. \end{aligned}$$

(b) Follows similar procedure.

□

Semi-group property for CHT-derivatives is as follows.

Theorem 3.4.7. Let $\alpha \geq 0$ and $\beta \geq 0$ such that $m-1 < \alpha \leq m$ and $n-1 < \beta \leq n$ and also let $k \in C^{m+n}_{\delta,\mu}[a, b], 0 < a < b < \infty$. Then

(a) ${}_{*}\mathcal{D}^{\alpha}_{a+,c}(\,{}_{*}\mathcal{D}^{\beta}_{a+,c}k)(x) = {}_{*}\mathcal{D}^{\alpha+\beta}_{a+,c}k(x)$.

(b) ${}_{*}\mathcal{D}^{\alpha}_{b-,c}(\,{}_{*}\mathcal{D}^{\beta}_{b-,c}k)(x) = {}_{*}\mathcal{D}^{\alpha+\beta}_{b-,c}k(x)$.

Proof. (a) Without the loss of generality, let $n \geq m$. Thus $n = m + q$, $q \in \{0, 1, 2...\}$. Since $\alpha + \beta = m + n$, then by definition of ${}_{*}\mathcal{D}^{\alpha}_{a+,c}$ and Theorem 3.3.2, we have

$$\begin{aligned}
{}_{*}\mathcal{D}^{\alpha}_{a+,c}(\,{}_{*}\mathcal{D}^{\beta}_{a+,c}k)(x) &= \mathcal{J}^{m-\alpha}_{a+,c}D^{m}_{c}(\,{}_{*}\mathcal{D}^{\beta}_{a+,c}k(x)) \\
&= \mathcal{J}^{m-\alpha}_{a+,c}D^{m}_{c}(\mathcal{J}^{n-\beta}_{a+,c}D^{n}_{c}k(x)) \\
&= \mathcal{J}^{m-\alpha}_{a+,c}D^{m}_{c}(\mathcal{J}^{m+q-\beta}_{a+,c}D^{m+q}_{c}k(x)) \\
&= \mathcal{J}^{m-\alpha}_{a+,c}D^{m}_{c}\mathcal{J}^{m-\beta}_{a+,c}\mathcal{J}^{q}_{a+,c}D^{m+q}_{c}k(x) \\
&= \mathcal{J}^{m-\alpha}_{a+,c}\mathcal{D}^{\beta}_{a+,c}\mathcal{J}^{q}_{a+,c}D^{m+q}_{c}k(x) \\
&= \mathcal{J}^{m-\alpha-\beta}_{a+,c}\mathcal{J}^{\beta}_{a+,c}\mathcal{D}^{\beta}_{a+,c}\mathcal{J}^{q}_{a+,c}D^{m+q}_{c}k(x).
\end{aligned}$$

Using Lemma 3.3.6(a), we get

$$\begin{aligned}
{}_{*}\mathcal{D}^{\alpha}_{a+,c}(\,{}_{*}\mathcal{D}^{\beta}_{a+,c}k)(x) = \mathcal{J}^{m-\alpha-\beta}_{a+,c}\Big[&\mathcal{J}^{q}_{a+,c}D^{m+q}_{c}k(x) \\
&-a^{c}x^{-c}\sum_{j=0}^{m-1}\frac{D^{m-j-1}_{c}\mathcal{J}^{m-\beta}_{a+,c}\mathcal{J}^{q}_{a+,c}D^{m+q}_{c}k(a)}{\Gamma(\beta-j)}\Big(\ln\frac{x}{a}\Big)^{\beta-j-1}\Big].
\end{aligned}$$

Let $\lceil\beta\rceil = n = m+q$, since $\mathcal{J}^{m-\beta}_{a+,c}\mathcal{J}^{q}_{a+,c}D^{m+q}_{c}k(a) = \mathcal{J}^{\lceil\beta\rceil-\beta}_{a+,c}D^{\lceil\beta\rceil}_{c}k(a) = {}_{*}\mathcal{D}^{\beta}_{a+,c}k(a)$. So

$$\begin{aligned}
{}_{*}\mathcal{D}^{\alpha}_{a+,c}(\,{}_{*}\mathcal{D}^{\beta}_{a+,c}k)(x) = \mathcal{J}^{m-(\alpha+\beta)}_{a+,c}\Big[&\mathcal{J}^{q}_{a+,c}D^{m+q}_{c}k(x) \\
&-a^{c}x^{-c}\sum_{j=0}^{m-1}\frac{D^{m-j-1}_{c}\,{}_{*}\mathcal{D}^{\beta}_{a+,c}k(a)}{\Gamma(\beta-j)}\Big(\ln\frac{x}{a}\Big)^{\beta-j-1}\Big].
\end{aligned}$$

For $\beta \geq 0$ we have ${}_*\mathcal{D}^{\beta}_{a+,c}k(a) = 0$, so

$$\begin{aligned} {}_*\mathcal{D}^{\alpha}_{a+,c}({}_*\mathcal{D}^{\beta}_{a+,c}k)(x) &= \mathcal{J}^{m-(\alpha+\beta)}_{a+,c}\Big(\mathcal{J}^{q}_{a+,c}D^{m+q}_{c}k(x)\Big) \\ &= \mathcal{J}^{m+q-(\alpha+\beta)}_{a+,c}D^{m+q}_{c}k(x) \\ &= {}_*\mathcal{D}^{\alpha+\beta}_{a+,c}k(x). \end{aligned}$$

(b) Follows similar procedure.

□

For $c = 0$, CHTF-derivatives and their results given in this section, coincide with CHF-derivatives and their results given in section 3.2.

3.5 Taylor's formula

In this section, Taylor's formula of CHTF-derivatives is presented. Taylor series for frac-derivatives was introduced in 1971 by Osler in [28]. It had been developed also for RLF-derivative by Trujillo et al. in his book [29]. Further more, Usero worked on it in case of Caputo frac-derivative in 2007 [30]. Recently, Gambo et al. had found Taylor's formula for CHF-derivative in [12], reviewed below.

3.5.1 Taylor's formula for Caputo Hadamard frac-derivatives

We first give an analogous theorem to the MVT.

Theorem 3.5.1. Let $0 < \alpha \leq 1$. Then for all $x \in [a, b]$ there exists $\xi \in [a, x]$ such that

$$k(x) = k(a) + \frac{1}{\Gamma(\alpha+1)}\Big(\ln\frac{x}{a}\Big)^{\alpha}\ {}_*\mathcal{D}^{\alpha}_{a+}k(\xi). \tag{3.5.1}$$

and

$$k(x) = k(b) + \frac{1}{\Gamma(\alpha+1)}\left(\ln\frac{b}{x}\right)^{\alpha} {}_*\mathcal{D}^{\alpha}_{b-}k(\xi). \tag{3.5.2}$$

Proof. Since $0 < \alpha \leq 1$ we have by Lemma 3.2.3(a) that

$$\mathcal{J}^{\alpha}_{a+}\left({}_*\mathcal{D}^{\alpha}_{a+}k\right)(x) = k(x) - k(a).$$

Using definition of $\mathcal{J}^{\alpha}_{a+}$ and by integral MVT, we obtain

$$\begin{aligned}
\mathcal{J}^{\alpha}_{a+}({}_*\mathcal{D}^{\alpha}_{a+}k)(x) &= \frac{1}{\Gamma(\alpha)}\int_a^x \left(\ln\frac{x}{t}\right)^{\alpha-1} {}_*\mathcal{D}^{\alpha}_{a+}k(t)\frac{dt}{t} \\
&= \frac{1}{\Gamma(\alpha+1)} {}_*\mathcal{D}^{\alpha}_{a+}k(\xi)\left(\ln\frac{x}{a}\right)^{\alpha} \\
&= \frac{1}{\Gamma(\alpha+1)}\left(\ln\frac{x}{a}\right)^{\alpha} {}_*\mathcal{D}^{\alpha}_{a+}k(\xi). \qquad (3.5.3)
\end{aligned}$$

Hence, we get

$$k(x) = k(a) + \frac{1}{\Gamma(\alpha+1)}\left(\ln\frac{x}{a}\right)^{\alpha} {}_*\mathcal{D}^{\alpha}_{a+}k(\xi).$$

with $\xi = [a, x]$. □

Eq. (3.5.2) can be proved in similar way.

Theorem 3.5.2. Let $0 < \alpha \ \leq 1$ and $j \in \mathbb{N}_0$, then

$$\mathcal{J}^{j\alpha}_{a+} {}_*\mathcal{D}^{j\alpha}_{a+}k(x) - \mathcal{J}^{(j+1)\alpha}_{a+} {}_*\mathcal{D}^{(j+1)\alpha}_{a+}k(x) = \frac{1}{\Gamma(j\alpha+1)}\left(\ln\frac{x}{a}\right)^{j\alpha} {}_*\mathcal{D}^{j\alpha}_{a+}k(a). \tag{3.5.4}$$

and

$$\mathcal{J}^{j\alpha}_{b-} {}_*\mathcal{D}^{j\alpha}_{b-}k(x) - \mathcal{J}^{(j+1)\alpha}_{b-} {}_*\mathcal{D}^{(j+1)\alpha}_{b-}k(x) = \frac{1}{\Gamma(j\alpha+1)}\left(\ln\frac{b}{x}\right)^{j\alpha} {}_*\mathcal{D}^{j\alpha}_{b-}k(b). \tag{3.5.5}$$

Proof. Consider Eq. (3.5.4) and using Lemma 3.2.3, we have

$$
\begin{aligned}
\mathcal{J}^{j\alpha}_{a+}\ {}_*\mathcal{D}^{j\alpha}_{a+}k(x) - \mathcal{J}^{(j+1)\alpha}_{a+}\ {}_*\mathcal{D}^{(j+1)\alpha}_{a+}k(x) &= \mathcal{J}^{j\alpha}_{a+}\Big(\ {}_*\mathcal{D}^{j\alpha}_{a+}k - \mathcal{J}^{\alpha}_{a+}\ {}_*\mathcal{D}^{(j+1)\alpha}_{a+}k\Big)\\
&= \mathcal{J}^{j\alpha}_{a+}\Big(\ {}_*\mathcal{D}^{j\alpha}_{a+}k - \Big(\mathcal{J}^{\alpha}_{a+}\ {}_*\mathcal{D}^{\alpha}_{a+}k\Big)\ {}_*\mathcal{D}^{j\alpha}_{a+}k\Big)\\
&= \mathcal{J}^{j\alpha}_{a+}\Big(\ {}_*\mathcal{D}^{j\alpha}_{a+}k(a)\Big)\\
\mathcal{J}^{j\alpha}_{a+}\ {}_*\mathcal{D}^{j\alpha}_{a+}k(x) - \mathcal{J}^{(j+1)\alpha}_{a+}\ {}_*\mathcal{D}^{(j+1)\alpha}_{a+}k(x) &= \frac{1}{\Gamma(j\alpha+1)}\Big(\ln\frac{x}{a}\Big)^{j\alpha}\ {}_*\mathcal{D}^{j\alpha}_{a+}k(a).
\end{aligned}
$$

Eq. (3.5.5) can be proved similarly. □

Now we discuss the Taylor's formula for CHF-derivative.

Theorem 3.5.3. Let $0 < a < b < \infty$ and $0 < \alpha \leq 1$. Let m be an arbitrary non-negative integer. Then the Taylor's formula involving CHF-derivatives is given as:

$$
\begin{aligned}
k(x) = &\sum_{j=0}^{m}\frac{1}{\Gamma(j\alpha+1)}\Big(\ln\frac{x}{a}\Big)^{j\alpha}\ {}_*\mathcal{D}^{j\alpha}_{a+}k(a)\\
&+\frac{1}{\Gamma((m+1)\alpha+1)}\Big(\ln\frac{x}{a}\Big)^{(m+1)\alpha}\ {}_*\mathcal{D}^{(m+1)\alpha}_{a+}k(\xi).
\end{aligned} \tag{3.5.6}
$$

and

$$
\begin{aligned}
k(x) = &\sum_{j=0}^{m}\frac{1}{\Gamma(j\alpha+1)}\Big(\ln\frac{b}{x}\Big)^{j\alpha}\ {}_*\mathcal{D}^{j\alpha}_{b-}k(b)\\
&+\frac{1}{\Gamma((m+1)\alpha+1)}\Big(\ln\frac{b}{x}\Big)^{(m+1)\alpha}\ {}_*\mathcal{D}^{(m+1)\alpha}_{b-}k(\xi),
\end{aligned} \tag{3.5.7}
$$

where $\xi = [a, b]$.

Proof. Consider Eq. (3.5.6) and using Theorem 3.5.2, we have

$$
\sum_{j=0}^{m}\mathcal{J}^{j\alpha}_{a+}\ {}_*\mathcal{D}^{j\alpha}_{a+}k(x) - \mathcal{J}^{(j+1)\alpha}_{a+}\ {}_*\mathcal{D}^{(j+1)\alpha}_{a+}k(x) = \sum_{j=0}^{m}\frac{1}{\Gamma(j\alpha+1)}\Big(\ln\frac{x}{a}\Big)^{j\alpha}\ {}_*\mathcal{D}^{j\alpha}_{a+}k(a).
$$

It follows

$$\mathcal{J}^{0}_{a+}\ {}_{*}\mathcal{D}^{0}_{a+}k(x) - \mathcal{J}^{(m+1)\alpha}_{a+,c}\ {}_{*}\mathcal{D}^{(m+1)\alpha}_{a+}k(x) = \sum_{j=0}^{m}\frac{1}{\Gamma(j\alpha+1)}\left(\ln\frac{x}{a}\right)^{j\alpha}\ {}_{*}\mathcal{D}^{j\alpha}_{a+}k(a).$$

As we know that $\mathcal{J}^{0}_{a+}\ {}_{*}\mathcal{D}^{0}_{a+}k(x) = k(x)$,

$$k(x) = \sum_{j=0}^{m}\frac{1}{\Gamma(j\alpha+1)}\left(\ln\frac{x}{a}\right)^{j\alpha}\ {}_{*}\mathcal{D}^{j\alpha}_{a+}k(a) + \mathcal{J}^{(m+1)\alpha}_{a+}\ {}_{*}\mathcal{D}^{(m+1)\alpha}_{a+}k(x). \quad (3.5.8)$$

Replacing α by $(m+1)\alpha$ in Eq. (3.5.3), we get

$$\mathcal{J}^{(m+1)\alpha}_{a+}\ {}_{*}\mathcal{D}^{(m+1)\alpha}_{a+}k(x) = \frac{1}{\Gamma((m+1)\alpha+1)}\left(\ln\frac{x}{a}\right)^{(m+1)\alpha}\ {}_{*}\mathcal{D}^{(m+1)\alpha}_{a+}k(\xi). \quad (3.5.9)$$

So

$$\begin{aligned} k(x) = &\sum_{j=0}^{m}\frac{1}{\Gamma(j\alpha+1)}\left(\ln\frac{x}{a}\right)^{j\alpha}\ {}_{*}\mathcal{D}^{j\alpha}_{a+}k(a) \\ &+\frac{1}{\Gamma((m+1)\alpha+1)}\left(\ln\frac{x}{a}\right)^{(m+1)\alpha}\ {}_{*}\mathcal{D}^{(m+1)\alpha}_{a+}k(\xi), \end{aligned}$$

with $\xi = [a, x]$. □

3.5.2 Taylor's formula for Caputo Hadamard-type f-derivatives

Firstly, we need to discuss analogous result to MVT.

Theorem 3.5.4. Let $0 < \alpha \leq 1$ and $k \in \mathcal{AC}[a,b]$ such that ${}_{*}\mathcal{D}^{\alpha}_{a+,c}k \in C[a,b]$. Then for all $x \in [a,b]$ there exists $\xi \in [a,b]$ such that

(a) For $\xi \in [a,x]$

$$k(x) = a^{c}x^{-c}k(a) + \frac{1}{\Gamma(\alpha+1)}\left(\ln\frac{x}{a}\right)^{\alpha}\left(\frac{\xi}{x}\right)^{c}\ {}_{*}\mathcal{D}^{\alpha}_{a+,c}k(\xi). \quad (3.5.10)$$

(b) For $\xi \in [x,b]$

$$k(x) = b^{-c}x^{c}k(b) + \frac{1}{\Gamma(\alpha+1)}\Big(\ln\frac{b}{x}\Big)^{\alpha}\Big(\frac{x}{\xi}\Big)^{c}\ {}_{*}\mathcal{D}^{\alpha}_{b-,c}k(\xi). \tag{3.5.11}$$

Proof. (a) Since $0 < \alpha \leq 1$ we have by Eq. (3.3.14) that

$$\mathcal{J}^{\alpha}_{a+,c}\Big(\ {}_{*}\mathcal{D}^{\alpha}_{a+,c}k\Big)(x) = k(x) - a^{c}x^{-c}k(a).$$

Using definition of $\mathcal{J}^{\alpha}_{a+,c}$ and by integral MVT, we obtain

$$\begin{aligned}\mathcal{J}^{\alpha}_{a+,c}({}_{*}\mathcal{D}^{\alpha}_{a+,c}k)(x) &= \frac{1}{\Gamma(\alpha)}\int_{a}^{x}\Big(\frac{t}{x}\Big)^{c}\Big(\ln\frac{x}{t}\Big)^{\alpha-1}\ {}_{*}\mathcal{D}^{\alpha}_{a+,c}k(t)\frac{dt}{t}\\ &= \frac{1}{\Gamma(\alpha+1)}\Big(\frac{\xi}{x}\Big)^{c}\Big(\ln\frac{x}{a}\Big)^{\alpha}\ {}_{*}\mathcal{D}^{\alpha}_{a+,c}k(\xi). \end{aligned} \tag{3.5.12}$$

Hence, we get

$$k(x) = a^{c}x^{-c}k(a) + \frac{1}{\Gamma(\alpha+1)}\Big(\frac{\xi}{x}\Big)^{c}\Big(\ln\frac{x}{a}\Big)^{\alpha}\ {}_{*}\mathcal{D}^{\alpha}_{a+,c}k(\xi).$$

with $\xi = [a,x]$.

(b) Follows similar procedure.

□

Theorem 3.5.5. Let $0 < \alpha \leq 1$ and $j \in \mathbb{N}_0$. Let $k \in \mathcal{AC}^{n}_{\delta}[a,b]$, then

(a) $\mathcal{J}^{j\alpha}_{a+,c}\ {}_{*}\mathcal{D}^{j\alpha}_{a+,c}k(x) - \mathcal{J}^{(j+1)\alpha}_{a+,c}\ {}_{*}\mathcal{D}^{(j+1)\alpha}_{a+,c}k(x) = \frac{a^{c}x^{-c}}{\Gamma(j\alpha+1)}\Big(\frac{\xi}{x}\Big)^{c}\Big(\ln\frac{x}{a}\Big)^{j\alpha}\ {}_{*}\mathcal{D}^{j\alpha}_{a+,c}k(a).$

(b) $\mathcal{J}^{j\alpha}_{b-,c}\ {}_{*}\mathcal{D}^{j\alpha}_{b-,c}k(x) - \mathcal{J}^{(j+1)\alpha}_{b-,c}\ {}_{*}\mathcal{D}^{(j+1)\alpha}_{b-,c}k(x) = \frac{b^{-c}x^{c}}{\Gamma(j\alpha+1)}\Big(\frac{x}{\xi}\Big)^{c}\Big(\ln\frac{b}{x}\Big)^{j\alpha}\ {}_{*}\mathcal{D}^{j\alpha}_{b-,c}k(b).$

Proof. (a) Using the Theorem 3.3.2, Lemma 3.4.3 and Theorem 3.4.7, we have

$$\begin{aligned}
\mathcal{J}^{j\alpha}_{a+,c}\ {}_*\mathcal{D}^{j\alpha}_{a+,c}k(x) - \mathcal{J}^{(j+1)\alpha}_{a+,c}\ {}_*\mathcal{D}^{(j+1)\alpha}_{a+,c}k(x) &= \mathcal{J}^{j\alpha}_{a+,c}\Big(\ {}_*\mathcal{D}^{j\alpha}_{a+,c}k(x) - \mathcal{J}^{\alpha}_{a+,c}\ {}_*\mathcal{D}^{(j+1)\alpha}_{a+,c}k(x)\Big)\\
&= \mathcal{J}^{j\alpha}_{a+,c}\Big(\ {}_*\mathcal{D}^{j\alpha}_{a+,c}k(x)\\
&\quad -\Big(\mathcal{J}^{\alpha}_{a+,c}\ {}_*\mathcal{D}^{\alpha}_{a+,c}k(x)\Big)\ {}_*\mathcal{D}^{j\alpha}_{a+,c}k(x)\Big)
\end{aligned}$$

By using Lemma 3.4.4

$$\begin{aligned}
\mathcal{J}^{j\alpha}_{a+,c}\ {}_*\mathcal{D}^{j\alpha}_{a+,c}k(x) - \mathcal{J}^{(j+1)\alpha}_{a+,c}\ {}_*\mathcal{D}^{(j+1)\alpha}_{a+,c}k(x) &= \mathcal{J}^{j\alpha}_{a+,c}\Big(\ {}_*\mathcal{D}^{j\alpha}_{a+,c}k(x) - \ {}_*\mathcal{D}^{j\alpha}_{a+,c}k(x)\\
&\quad + a^c x^{-c}\ {}_*\mathcal{D}^{j\alpha}_{a+,c}k(a)\Big).\\
&= a^c x^{-c}\mathcal{J}^{j\alpha}_{a+,c}\Big(\ {}_*\mathcal{D}^{j\alpha}_{a+,c}k(a)\Big)\\
&= \frac{a^c x^{-c}}{\Gamma(j\alpha+1)}\Big(\frac{\xi}{x}\Big)^c\Big(\ln\frac{x}{a}\Big)^{j\alpha}\ {}_*\mathcal{D}^{j\alpha}_{a+,c}k(a).
\end{aligned}$$

(b) Follows similar procedure.

□

Following are the Taylor's formulas for CHTF-derivatives.

Theorem 3.5.6. Let $0 < a < b < \infty$ and $0 < \alpha \leq 1$. Let m be an arbitrary non-negative integer. Let $k \in \mathcal{AC}^n_\delta[a,b]$ and suppose ${}_*\mathcal{D}^{(j+1)\alpha}_{a,c}k \in C[a,b]$. Then the Taylor's formula involving CHTF-derivatives is given as:

(a) For $\xi = [a,x]$.

$$\begin{aligned}
k(x) &= \sum_{j=0}^{m}\frac{a^c x^{-c}}{\Gamma(j\alpha+1)}\Big(\frac{\xi}{x}\Big)^c\Big(\ln\frac{x}{a}\Big)^{j\alpha}\ {}_*\mathcal{D}^{j\alpha}_{a+,c}k(a)\\
&\quad + \frac{1}{\Gamma((m+1)\alpha+1)}\Big(\frac{\xi}{x}\Big)^c\Big(\ln\frac{x}{a}\Big)^{(m+1)\alpha}\ {}_*\mathcal{D}^{(m+1)\alpha}_{a+,c}k(\xi). \qquad (3.5.13)
\end{aligned}$$

(b) For $\xi = [x, b]$.

$$k(x) = \sum_{j=0}^{m} \frac{b^{-c}x^{c}}{\Gamma(j\alpha+1)} \left(\frac{x}{\xi}\right)^{c} \left(\ln\frac{b}{x}\right)^{j\alpha} {}_{*}\mathcal{D}_{b-,c}^{j\alpha} k(b)$$
$$+\frac{1}{\Gamma((m+1)\alpha+1)} \left(\frac{x}{\xi}\right)^{c} \left(\ln\frac{b}{x}\right)^{(m+1)\alpha} {}_{*}\mathcal{D}_{b-,c}^{(m+1)\alpha} k(\xi). \tag{3.5.14}$$

Proof. (a) Using Theorem 3.5.5, we have

$$\sum_{j=0}^{m} \mathcal{J}_{a+,c}^{j\alpha}\ {}_{*}\mathcal{D}_{a+,c}^{j\alpha} k(x) - \mathcal{J}_{a+,c}^{(j+1)\alpha}\ {}_{*}\mathcal{D}_{a+,c}^{(j+1)\alpha} k(x) = \sum_{j=0}^{m} \frac{a^{c}x^{-c}}{\Gamma(j\alpha+1)} \left(\frac{\xi}{x}\right)^{c} \times \left(\ln\frac{x}{a}\right)^{j\alpha} {}_{*}\mathcal{D}_{a+,c}^{j\alpha} k(a).$$

It follows

$$\mathcal{J}_{a+,c}^{0}\ {}_{*}\mathcal{D}_{a+,c}^{0} k(x) - \mathcal{J}_{a+,c}^{(m+1)\alpha}\ {}_{*}\mathcal{D}_{a+,c}^{(m+1)\alpha} k(x) = \sum_{j=0}^{m} \frac{a^{c}x^{-c}}{\Gamma(j\alpha+1)} \left(\frac{\xi}{x}\right)^{c} \left(\ln\frac{x}{a}\right)^{j\alpha} {}_{*}\mathcal{D}_{a+,c}^{j\alpha} k(a).$$

As $\mathcal{J}_{a+,c}^{0}\ {}_{*}\mathcal{D}_{a+,c}^{0} k(x) = k(x)$, we have

$$k(x) = \sum_{j=0}^{m} \frac{a^{c}x^{-c}}{\Gamma(j\alpha+1)} \left(\frac{\xi}{x}\right)^{c} \left(\ln\frac{x}{a}\right)^{j\alpha} {}_{*}\mathcal{D}_{a+,c}^{j\alpha} k(a) + \mathcal{J}_{a+,c}^{(m+1)\alpha}\ {}_{*}\mathcal{D}_{a+,c}^{(m+1)\alpha} k(x).$$

Replacing α by $(m+1)\alpha$ in Eq. (3.5.12), we get

$$\mathcal{J}_{a+,c}^{(m+1)\alpha}\ {}_{*}\mathcal{D}_{a+,c}^{(m+1)\alpha} k(x) = \frac{1}{\Gamma((m+1)\alpha+1)} \left(\frac{\xi}{x}\right)^{c} \left(\ln\frac{x}{a}\right)^{(m+1)\alpha} {}_{*}\mathcal{D}_{a+,c}^{(m+1)\alpha} k(\xi). \tag{3.5.15}$$

So

$$k(x) = \sum_{j=0}^{m} \frac{a^{c}x^{-c}}{\Gamma(j\alpha+1)} \left(\frac{\xi}{x}\right)^{c} \left(\ln\frac{x}{a}\right)^{j\alpha} {}_{*}\mathcal{D}_{a+,c}^{j\alpha} k(a)$$
$$+\frac{1}{\Gamma((m+1)\alpha+1)} \left(\frac{\xi}{x}\right)^{c} \left(\ln\frac{x}{a}\right)^{(m+1)\alpha} {}_{*}\mathcal{D}_{a+,c}^{(m+1)\alpha} k(\xi).$$

(b) Follows similar procedure.

□

Remark 3.5.7. Consider same assumptions of Theorem 3.5.6, the Taylor's expansion of $k(x)$ can also be written by using the definition of HHTF-integrals of order $(m+1)\alpha$ in Eq. (3.5.15).

3.6 Integration by parts formula

In this section, Integration by parts formula for HT-derivatives is discussed. It was introduced in 2009 for RLF-integrals and derivatives by Almeida in [31] also see Corollary 2 in [8] . Furthermore, it has been found for Caputo frac-derivatives in 2017 by Yufeng et al. in [32]. At first, integration by parts formula for Hadamard frac-derivatives is reviewed.

To find it for Hadamard frac-derivatives, the following result is required.

Lemma 3.6.1. For $0 < \alpha \leq 1$ and $0 < a < b < \infty$. Then

$$\mathcal{D}^{\alpha}_{a+}k(x) = \frac{1}{\Gamma(1-\alpha)}\left[\left(\ln\frac{x}{a}\right)^{-\alpha}k(a) + \int_a^x \left(\ln\frac{x}{t}\right)^{-\alpha}\delta\ k(t)\frac{dt}{t}\right]. \tag{3.6.1}$$

and

$$\mathcal{D}^{\alpha}_{b-}k(x) = \frac{1}{\Gamma(1-\alpha)}\left[\left(\ln\frac{b}{x}\right)^{-\alpha}k(b) + \int_x^b \left(\ln\frac{t}{x}\right)^{-\alpha}\delta\ k(t)\frac{dt}{t}\right]. \tag{3.6.2}$$

Proof. For $0 < \alpha \leq 1$ and by definition of $\mathcal{D}^{\alpha}_{a+}$, we have

$$\begin{aligned}\mathcal{D}^{\alpha}_{a+}k &= \delta\mathcal{J}^{1-\alpha}_{a+}k \\ &= \frac{\delta}{\Gamma(1-\alpha)}\int_a^x \left(\ln\frac{x}{t}\right)^{-\alpha}k(t)\frac{dt}{t}.\end{aligned}$$

Let $h=\int\left(\ln\frac{x}{t}\right)^{-\alpha}\frac{dt}{t}$ and $g=k(t)$ and following Appendix (A3) , we get

$$\begin{aligned}
\mathcal{D}^{\alpha}_{a+}k &= \frac{\delta}{\Gamma(2-\alpha)}\Big[\Big(\ln\frac{x}{a}\Big)^{1-\alpha}k(a)\Big]\Big|_a^x+\frac{\delta}{\Gamma(2-\alpha)}\Big[\int_a^x\Big(\ln\frac{x}{t}\Big)^{1-\alpha}\delta\ k(t)\frac{dt}{t}\Big]\\
&= \frac{\left(\ln\frac{x}{a}\right)^{-\alpha}k(a)}{\Gamma(1-\alpha)}+\frac{1}{\Gamma(1-\alpha)}\int_a^x\Big(\ln\frac{x}{t}\Big)^{-\alpha}\delta\ k(t)\frac{dt}{t}\\
\mathcal{D}^{\alpha}_{a+}k &= \frac{1}{\Gamma(1-\alpha)}\Big[\Big(\ln\frac{x}{a}\Big)^{-\alpha}k(a)+\int_a^x\Big(\ln\frac{x}{t}\Big)^{-\alpha}\delta\ k(t)\frac{dt}{t}\Big].
\end{aligned}$$

Hence the proof follows Eq. (3.6.1). Similarly Eq. (3.6.2) can be proved. □

Theorem 3.6.2. For $0<\alpha\leq 1$ and let the functions k and g have frac-derivatives, then the following integration by parts formula holds

$$\int_a^b k(x)\mathcal{D}^{\alpha}_{a+}g(x)\frac{dx}{x}=\int_a^b g(x)\mathcal{D}^{\alpha}_{b-}k(x)\frac{dx}{x}. \tag{3.6.3}$$

Proof. By using definition of $\mathcal{D}^{\alpha}_{a+}$ from Eq. (2.3.7), we have

$$\int_a^b k(x)\mathcal{D}^{\alpha}_{a+}g(x)\frac{dx}{x}=\int_a^b k(x)\delta\mathcal{J}^{1-\alpha}_{a+}g(x)\frac{dx}{x}. \tag{3.6.4}$$

Integrating by parts, we get

$$\begin{aligned}
\int_a^b k(x)\mathcal{D}^{\alpha}_{a+}g(x)\frac{dx}{x} &= \mathcal{J}^{1-\alpha}_{a+}g(x)k(x)\Big|_a^b-\int_a^b\mathcal{J}^{1-\alpha}_{a+}g(x)\delta\ k(x)\frac{dx}{x}\\
&= \frac{1}{\Gamma(1-\alpha)}\int_a^x\Big(\ln\frac{x}{t}\Big)^{-\alpha}g(t)\frac{dt}{t}k(x)\Big|_a^b-\int_a^b\mathcal{J}^{1-\alpha}_{a+}g(x)\delta\ k(x)\frac{dx}{x}\\
&= \frac{1}{\Gamma(1-\alpha)}\Bigg[\int_a^b\Big(\ln\frac{b}{t}\Big)^{-\alpha}g(t)\frac{dt}{t}k(b)\\
&\quad -\int_a^b\delta\ k(x)\int_a^x\Big(\ln\frac{x}{t}\Big)^{-\alpha}g(t)\frac{dt}{t}\frac{dx}{x}\Bigg].
\end{aligned}$$

By using Dirichlet formula on $\int_a^b \delta\ k(x) \int_a^x \left(\ln \frac{x}{t}\right)^{-\alpha} g(t) \frac{dt}{t} \frac{dx}{x}$, then we get

$$\int_a^b k(x)\mathcal{D}_{a+}^{\alpha} g(x) \frac{dx}{x} = \frac{1}{\Gamma(1-\alpha)} \left[\int_a^b \left(\ln \frac{b}{t}\right)^{-\alpha} g(t)k(b) - \int_t^b \left(\ln \frac{x}{t}\right)^{-\alpha} \delta\ k(x) \frac{dx}{x} \right] \frac{dt}{t}.$$

Using Eq. (3.6.2), we get the required result, that is

$$\int_a^b k(x)\mathcal{D}_{a+}^{\alpha} g(x) \frac{dx}{x} = \int_a^b g(x)\mathcal{D}_{b-}^{\alpha} k(x) \frac{dx}{x}.$$

□

3.6.1 Integration by parts formula for Hadamard-type frac-derivatives

Firstly, we discuss the following results.

Lemma 3.6.3. For $0 < \alpha \leq 1$, $c \in \mathbb{R}$ and $0 < a < b < \infty$. Then

(a) $\mathcal{D}_{a+,c}^{\alpha}\ k(x) = \frac{x^{-c}}{\Gamma(1-\alpha)} \Big[a^c \Big(\ln \frac{x}{a}\Big)^{-\alpha} k(a) + \int_a^x \Big(\ln \frac{x}{t}\Big)^{-\alpha} D_c k(t) \frac{dt}{t} \Big]$.

(b) $\mathcal{D}_{b-,c}^{\alpha}\ k(x) = \frac{x^{c}}{\Gamma(1-\alpha)} \Big[b^{-c} \Big(\ln \frac{b}{x}\Big)^{-\alpha} k(b) + \int_x^b \Big(\ln \frac{t}{x}\Big)^{-\alpha} D_c k(t) \frac{dt}{t} \Big]$.

Proof. (a) From Eq. (2.3.16), and following Appendix (A4), we have

$$\begin{aligned}
\mathcal{D}_{a+,c}^{\alpha} k(x) &= D_c \mathcal{J}_{a+,c}^{1-\alpha} k(t) \\
&= \frac{D_c}{\Gamma(1-\alpha)} \int_a^x \Big(\frac{t}{x}\Big)^c \Big(\ln \frac{x}{t}\Big)^{-\alpha} k(t) \frac{dt}{t} \\
&= \frac{D_c x^{-c}}{\Gamma(1-\alpha)} \int_a^x t^c \Big(\ln \frac{x}{t}\Big)^{-\alpha} k(t) \frac{dt}{t} \\
&= \frac{D_c}{\Gamma(2-\alpha)} \Big[x^{-c} a^c \Big(\ln \frac{x}{a}\Big)^{1-\alpha} k(a) \Big] \\
&\quad + \frac{D_c}{\Gamma(2-\alpha)} x^{-c} \Big[\int_a^x t^c \Big(\ln \frac{x}{t}\Big)^{1-\alpha} D_c k(t) \frac{dt}{t} \Big]
\end{aligned}$$

$$= \frac{x^{-c}a^{c}\Big(\ln\frac{x}{a}\Big)^{1-\alpha}k(a)}{\Gamma(1-\alpha)} + \frac{x^{-c}}{\Gamma(1-\alpha)}\int_a^x \Big(\ln\frac{x}{t}\Big)^{-\alpha} D_c k(t)\frac{dt}{t}$$

$$\mathcal{D}^{\alpha}_{a+,c}k(x) = \frac{x^{-c}}{\Gamma(1-\alpha)}\Big[a^{c}\Big(\ln\frac{x}{a}\Big)^{-\alpha}k(a) + \int_a^x \Big(\ln\frac{x}{t}\Big)^{-\alpha} D_c k(t)\frac{dt}{t}\Big].$$

(b) Follows similar procedure.

□

Now, we discuss the integration by parts formula for HTF-derivatives.

Theorem 3.6.4. For $0 < \alpha \leq 1$ and let the functions k and g have f-derivatives, then the following integration by parts formula holds

$$\int_a^b k(x)\mathcal{D}^{\alpha}_{a+,c}g(x)\frac{dx}{x} = \int_a^b g(x)\mathcal{D}^{\alpha}_{b-,c}k(x)\frac{dx}{x}. \tag{3.6.5}$$

Proof. By using Eq. (3.4.7), Eq. (3.4.8) and Eq. (3.6.3), we have

$$\begin{aligned}
\int_a^b k(x)\mathcal{D}^{\alpha}_{a+,c}g(x)\frac{dx}{x} &= \int_a^b k(x)x^{-c}\mathcal{D}^{\alpha}_{a+}x^{c}g(x)\frac{dx}{x}\\
&= \int_a^b x^{-c}k(x)\mathcal{D}^{\alpha}_{a+}x^{c}g(x)\frac{dx}{x}\\
&= \int_a^b x^{c}g(x)\mathcal{D}^{\alpha}_{b-}x^{-c}k(x)\frac{dx}{x}\\
&= \int_a^b g(x)x^{c}\mathcal{D}^{\alpha}_{b-}x^{-c}k(x)\frac{dx}{x}\\
&= \int_a^b g(x)\mathcal{D}^{\alpha}_{b-,c}k(x)\frac{dx}{x}.
\end{aligned}$$

□

Chapter 4

Mellin transform of frac-operators

In this chapter, Mellin transform and its various properties are applied to different types of frac-integrals and derivatives e.g., R-L, Caputo and Hilfer derivative. Also Mellin transform of Hadamard, HT-integrals and derivatives and CH, CHT-derivatives is given.

Complex Fourier transform and its inverse provides the basics for Mellin transform and its inverse and was introduced by Mellin [18]. Some properties and applications of Mellin transform are discussed in [25]. Podlubny, Katugampola, Butzer and Kilbas are few of the mathematicians who worked on Mellin transform.

4.1 Mellin transform of frac-integrals and derivatives

Following [13], the Mellin transforms of left and right sided R-L integrals and derivatives that are given as under:

Lemma 4.1.1. Let $\alpha \in \mathbb{C}$, $\text{Re}(\alpha) > 0$. Then

$$\mathcal{M}[\mathfrak{I}^{\alpha}_{a+}k](s) = \frac{\Gamma(1-s-\alpha)}{\Gamma(1-s)}\,\tilde{k}(s+\alpha), \qquad \text{Re}(s+\alpha) < 1,\ x > a, \tag{4.1.1}$$

$$\mathcal{M}[\mathfrak{I}^{\alpha}_{b-}k](s) = \frac{\Gamma(s)}{\Gamma(s+\alpha)}\,\tilde{k}(s+\alpha), \qquad \text{Re}(s) > 0,\ x < b. \tag{4.1.2}$$

for $k \in X^1_{s+\alpha}(\mathbb{R}+)$, if $\tilde{k}(s+\alpha)$ exists for $s \in \mathbb{C}$.

Theorem 4.1.2. Let $\alpha \in \mathbb{C}$, $\text{Re}(\alpha) > 0$, $s \in \mathbb{C}$ and $k \in X^1_{s-\alpha}(\mathbb{R}+)$. Then

$$\mathcal{M}[\,{}^{RL}\mathcal{D}^{\alpha}_{a+}k](s) = \frac{\Gamma(1-s+\alpha)}{\Gamma(1-s)}\,\tilde{k}(s-\alpha), \qquad \text{Re}(s) < 1,\ x > a \geq 0, \tag{4.1.3}$$

$$\mathcal{M}[\,{}^{RL}\mathcal{D}^{\alpha}_{b-}k](s) = \frac{\Gamma(s)}{\Gamma(s-\alpha)}\,\tilde{k}(s-\alpha), \qquad \text{Re}(s-\alpha) > 0,\ x < b \leq \infty. \tag{4.1.4}$$

To find the Mellin transform of Caputo frac-derivative we need to use Mellin transform of mth derivative $\left(D^m k(t) = \left(\frac{d}{dt}\right)^m k(t)\right)$, given by following lemma.

Lemma 4.1.3. [17] Let $k \in \mathbb{C}^m(\mathbb{R}+)$, $\tilde{k}(s-m)$ and $\mathcal{M}[D^m k(t)](s)$ exist, and $\lim\limits_{t\to 0}[t^{s-q-1}k^{(m-q-1)}(t)]$ and $\lim\limits_{t\to\infty}[t^{s-q-1}k^{(m-q-1)}(t)]$ are finite for $q = 0, 1, ..., (m-1), m \in \mathbb{N}$, then

$$\begin{aligned}\mathcal{M}[D^m k(t)](s) &= \frac{\Gamma(1-s+m)}{\Gamma(1-s)}\tilde{k}(s-m)\\ &\quad + \sum_{q=0}^{m-1}\frac{\Gamma(1-s+q)}{\Gamma(1-s)}\left[x^{s-q-1}k^{(m-q-1)}(x)\right]_0^{\infty}. \qquad (4.1.5)\\ &= (-1)^m\frac{\Gamma(s)}{\Gamma(s-m)}\tilde{k}(s-m)\\ &\quad + \sum_{q=0}^{m-1}(-1)^q\frac{\Gamma(s)}{\Gamma(s-q)}\left[x^{s-q-1}k^{(m-q-1)}(x)\right]_0^{\infty}. \qquad (4.1.6)\end{aligned}$$

Reader can see [17, 21] for the following lemma, which gives the Mellin transform for Caputo frac-derivative.

Theorem 4.1.4. Let $\alpha > 0$, $m-1 < \alpha \leq m$, $s \in \mathbb{C}$ and $k \in X^1_{s-\alpha}(\mathbb{R}+)$. Then,

(a) For $\mathrm{Re}(s) < 1,\ x > a \geq 0,$

$$\mathcal{M}[\,{}_*\mathcal{D}^{\alpha}_{a+}k(x)](s) = \frac{\Gamma(1-s+\alpha)}{\Gamma(1-s)}\tilde{k}(s-\alpha)+\sum_{q=0}^{m-1}\frac{\Gamma(1+q-s-m+\alpha)}{\Gamma(1-s-m+\alpha)}\Big[x^{s+m-\alpha-q-1}k^{(m-q-1)}(x)\Big]_0^{\infty}.$$

(b) For $\mathrm{Re}(s-\alpha) > 0,\ x < b \leq \infty,$

$$\mathcal{M}[\,{}_*\mathcal{D}^{\alpha}_{b-}k(x)](s) = \frac{\Gamma(s)}{\Gamma(s-\alpha)}\tilde{k}(s-\alpha)+\sum_{q=0}^{m-1}(-1)^q\frac{\Gamma(s+m-\alpha)}{\Gamma(s+m-\alpha-q)}\Big[x^{s+m-\alpha-q-1}k^{(m-q-1)}(x)\Big]_0^{\infty}.$$

Proof. (a) By definition of ${}_*\mathcal{D}^{\alpha}_{a+}$ from Eq. (2.3.9) and Eq. (2.6.1), we have

$$\begin{aligned}\mathcal{M}[\,{}_*\mathcal{D}^{\alpha}_{a+}k(x)](s) &= \mathcal{M}\Big[\frac{1}{\Gamma(m-\alpha)}\int_a^x (x-t)^{m-\alpha-1}k^{(m)}(t)dt\Big]\\ &= \mathcal{M}\Big[(\mathfrak{I}^{m-\alpha}_{a+}D^m k)(t)\Big].\end{aligned}$$

By using Eqs. (4.1.1) and (4.1.5) in the above expression, we get

$$\begin{aligned}\mathcal{M}[\,{}_*\mathcal{D}^{\alpha}_{a+}k](s) &= \frac{\Gamma(1-s-m+\alpha)}{\Gamma(1-s)}\mathcal{M}[D^m k(t)](s+m-\alpha)\\ &= \frac{\Gamma(1-s+\alpha)}{\Gamma(1-s)}\tilde{k}(s-\alpha)\\ &\quad+\sum_{q=0}^{m-1}\frac{\Gamma(1+q-s-m+\alpha)}{\Gamma(1-s-m+\alpha)}\Big[x^{s+m-\alpha-q-1}k^{(m-q-1)}(x)\Big]_0^{\infty}.\end{aligned}$$

which is the required result (a).

(b) Follows similar procedure.

□

4.2 Mellin transform of generalized R-L frac-operators

In this section, Mellin transform of generalized RL-integrals and derivatives defined by Katugampola [13] are reviewed. Afterwards, Mellin transforms of generalized Caputo and Hilfer derivatives are evaluated.

Lemma 4.2.1. Let $\alpha \in \mathbb{C}$, $\mathrm{Re}(\alpha) > 0$, $s \in \mathbb{C}$, $\nu > 0$ and $k \in X^1_{s+\alpha\nu}(\mathbb{R}+)$. Then

$$\mathcal{M}[\ ^{\nu}\mathfrak{I}^{\alpha}_{a+}k(x)](s) = \frac{\Gamma(1-\frac{s}{\nu}-\alpha)}{\nu^{\alpha}\Gamma\Big(1-\frac{s}{\nu}\Big)}\tilde{k}(s+\alpha\nu), \quad \mathrm{Re}\Big(\frac{s}{\nu}+\alpha\Big) < 1,\ x > a, \tag{4.2.1}$$

$$\mathcal{M}[\ ^{\nu}\mathfrak{I}^{\alpha}_{b-}k(x)](s) = \frac{\Gamma(\frac{s}{\nu})}{\nu^{\alpha}\Gamma(\frac{s}{\nu}+\alpha)}\tilde{k}(s+\alpha\nu), \quad \mathrm{Re}\Big(\frac{s}{\nu}\Big) > 0,\ x < b. \tag{4.2.2}$$

The next result is the Mellin transform of generalized RLF-derivatives. For simplicity we consider the case $0 < \alpha < 1$.

Theorem 4.2.2. Let $\alpha \in \mathbb{C}$, $\mathrm{Re}(\alpha) > 0$, $s \in \mathbb{C}$, $\nu > 0$ and $k \in X^1_{s+\alpha\nu}(\mathbb{R}+)$. Then,

$$\mathcal{M}[\ ^{\nu}\mathcal{D}^{\alpha}_{a+}k(x)](s) = \frac{\nu^{\alpha}\Gamma(1-\frac{s}{\nu}-\alpha)}{\Gamma(1-\frac{s}{\nu})}\tilde{k}(s-\alpha\nu), \quad \mathrm{Re}\Big(\frac{s}{\nu}\Big) < 1,\ x > a \geq\ 0, \tag{4.2.3}$$

$$\mathcal{M}[\ ^{\nu}\mathcal{D}^{\alpha}_{b-}k(x)](s) = \frac{\nu^{\alpha}\Gamma(\frac{s}{\nu})}{\Gamma(\frac{s}{\nu}-\alpha)}\tilde{k}(s-\alpha\nu), \quad \mathrm{Re}\Big(\frac{s}{\nu}-\alpha\Big) > 0,\ x < b \leq\ \infty. \tag{4.2.4}$$

4.3 Mellin transform analysis of Hadamard frac-integrals and derivatives

Mellin transform of Hadamard operators was evaluated by Kilbas et al. in [17]. The Mellin transforms of Hadamard frac-integrals defined in Eqs. (2.3.5) and (2.3.6) are given in the following result.

Lemma 4.3.1. If $\mathrm{Re}(\alpha) > 0$ and let a function $k(x)$ be such that its Mellin transform $\mathcal{M}[k](s)$ exists for $s \in \mathbb{C}$.

(a) If $\mathrm{Re}(s) < 0$, then
$$\mathcal{M}[\mathcal{J}^{\alpha}_{0+}k](s) = (-s)^{-\alpha}\mathcal{M}[k](s).$$

(b) If $\mathrm{Re}(s) > 0$, then
$$\mathcal{M}[\mathcal{J}^{\alpha}_{0-}k](s) = (s)^{-\alpha}\mathcal{M}[k](s).$$

The Mellin transforms of Hadamard frac-derivatives defined in Eqs. (2.3.7) and (2.3.8) are given by the following Theorem.

Theorem 4.3.2. If $\mathrm{Re}(\alpha) > 0$ and let a function $k(x)$ be such that its Mellin transform $\mathcal{M}[k](s)$ exists for $s \in \mathbb{C}$.

(a) If $\mathrm{Re}(s) < 0$, then

$\mathcal{M}[\mathcal{D}^{\alpha}_{0+}k](s) = (-s)^{\alpha}\mathcal{M}[k](s).$

(b) If $\mathrm{Re}(s) > 0$, then

$\mathcal{M}[\mathcal{D}^{\alpha}_{0-}k](s) = (s)^{\alpha}\mathcal{M}[k](s).$

Mellin transforms of HTF-integrals by Eqs. (2.3.15) and (2.3.17) are presented in the following lemma.

Lemma 4.3.3. If $\mathrm{Re}(\alpha) > 0$ and $c \in \mathbb{C}$. Also let a function $k(x)$ be such that its Mellin transform $\mathcal{M}[k](s)$ exists for $s \in \mathbb{C}$.

(a) If $\mathrm{Re}(c - s) > 0$, then

$\mathcal{M}[\mathcal{J}^{\alpha}_{0+,c}k](s) = (c - s)^{-\alpha}\mathcal{M}[k](s).$

(b) If $\mathrm{Re}(c + s) > 0$, then

$\mathcal{M}[\mathcal{J}^{\alpha}_{0-,c}k](s) = (c + s)^{-\alpha}\mathcal{M}[k](s).$

Mellin transforms of HTF-derivatives by Eqs. (2.3.16) and (2.3.1) is presented in the following Theorem.

Theorem 4.3.4. If $\mathrm{Re}(\alpha) > 0$ and $c \in \mathbb{C}$. Also let a function $k(x)$ be such that its Mellin transform $\mathcal{M}[k](s)$ exists for $s \in \mathbb{C}$.

(a) If $\mathrm{Re}(c - s) > 0$ and $\mathcal{M}[\mathcal{D}^{\alpha}_{0+,c}k](s)$ exists, then

$\mathcal{M}[\mathcal{D}^{\alpha}_{0+,c}k](s) = (c - s)^{\alpha}\mathcal{M}[k](s).$

(b) If $\mathrm{Re}(c + s) > 0$and $\mathcal{M}[\mathcal{D}^{\alpha}_{0-,c}k](s)$ exists, then

$\mathcal{M}[\mathcal{D}^{\alpha}_{0-,c}k](s) = (c + s)^{\alpha}\mathcal{M}[k](s).$

Following Theorem gives the Mellin transform of CH-derivative.

Theorem 4.3.5. [16] If $\text{Re}(\alpha) > 0$ and let a function $k(x)$ be such that its Mellin transform $\mathcal{M}[k](s)$ exists for $s \in \mathbb{C}$.

(a) If $\text{Re}(s) < 0$, then
$\mathcal{M}[\ {}_*\mathcal{D}^{\alpha}_{0+}k](s) = (-s)^{\alpha}\mathcal{M}[k](s).$

(b) If $\text{Re}(s) > 0$, then
$\mathcal{M}[\ {}_*\mathcal{D}^{\alpha}_{0-}k](s) = (s)^{\alpha}\mathcal{M}[k](s).$

Following definitions and results are about HT-integrals in the Mellin transform setting taken from [11] and [20]. These are required to find few results of CHT-derivatives, which come later in this section.

Definition 20. The domain of $\mathcal{J}^{\alpha}_{0+,c}$, for $\alpha > 0$ and $c \in \mathbb{R}$, is the class of all functions $k : \mathbb{R}+ \to \mathbb{C}$ such that

$$\int_0^t v^c \Big(\log \frac{t}{v}\Big)^{\alpha-1} |k(v)| \frac{dv}{v} < +\infty \tag{4.3.1}$$

for $t \in \mathbb{R}+$. The domain of $\mathcal{J}^{\alpha}_{0+,c}$ is denoted by $\text{Dom}\mathcal{J}^{\alpha}_{0+,c}$.

$X_{c,loc}$ is the space of all functions such that $k(t)t^{c-1} \in L^1(0, a)$ for every $a > 0$. For $\alpha = 1$, we have $\text{Dom}\mathcal{J}^{1}_{0+,c} = X_{c,loc}$. For $0 < \alpha < 1$, $X_{c,loc} \subset \text{Dom}\mathcal{J}^{\alpha}_{0+,c}$.

The semi group property of HT-integrals in the domain of frac-integrals is given in Theorem 2 of [20] stated as,

Theorem 4.3.6. Let α, β be positive, $c \in \mathbb{R}$ be fixed. If $k(x) \in \text{Dom}\mathcal{J}^{\alpha+\beta}_{0+,c}$. Then

i. $k(x) \in \text{Dom}\mathcal{J}^{\alpha}_{0+,c} \cap \text{Dom}\mathcal{J}^{\beta}_{0+,c}$.

ii. $\mathcal{J}^{\alpha}_{0+,c}k(x) \in \text{Dom}\mathcal{J}^{\beta}_{0+,c}$ and $\mathcal{J}^{\beta}_{0+,c}k(x) \in \text{Dom}\mathcal{J}^{\alpha}_{0+,c}$.

iii. $(\mathcal{J}^{\alpha+\beta}_{0+,c}k)(x) = (\mathcal{J}^{\alpha}_{0+,c}(\mathcal{J}^{\beta}_{0+,c}k)), x \in \mathbb{R}+.$

iv. If $\alpha < \beta$ then $\mathrm{Dom}\mathcal{J}^{\beta}_{0+,c} \subset \mathrm{Dom}\mathcal{J}^{\alpha}_{0+,c}$.

Definition 21. Let $k \in X_c$, $c \in \mathbb{R}$ then the Mellin integral $\mathcal{J}^m_c$ of order $m \in \mathbb{N}$, $t \in \mathbb{R}+$ is defined as

$$\mathcal{J}^m_c k(x) = x^{-c} \int_0^x \frac{dt_1}{t_1} \int_0^{t_1} \frac{dt_2}{t_2} ... \int_0^{t_{m-1}} t^c_m k(t_m) \frac{dt_m}{t_m}. \tag{4.3.2}$$

where we set, $\mathcal{J}_c = \mathcal{J}^1_c$ and $\mathcal{J}^m = \mathcal{J}^m_0$.

Definition 22. The Mellin translation operator τ^c_h for $k(x) : \mathbb{R}+ \to \mathbb{C}$ where $c \in \mathbb{R}$ and $h \in \mathbb{R}+$ is defined by

$$(\tau^c_h k)(t) = h^c k(ht), \qquad t \in \mathbb{R}+ \tag{4.3.3}$$

The concept of a derivative of a function say k is defined by the limit of the difference quotient involving the Mellin translation , if $k^{'}(t)$ exists, is

$$\begin{aligned} \lim_{h \to 1} \frac{\tau^c_h k(t) - k(t)}{h-1} &= \lim_{h \to 1} \left\{ h^c t \frac{k(ht) - k(t)}{ht - t} + \frac{h^c - 1}{h-1} k(t) \right\} \\ &= tk^{'}(t) + ck(t). \end{aligned}$$

This motivates the following:

Definition 23. The Mellin derivative θ_c of a function $k : \mathbb{R}+ \to \mathbb{C}$ and $c \in \mathbb{R}$ is defined as

$$\theta_c k(t) = tk^{'}(t) + c\ k(t), \tag{4.3.4}$$

where $t \in \mathbb{R}+$ and $k^{'}(t)$ exists on $\mathbb{R}+$. θ_c of order $m \in \mathbb{N}$ can be defined as $\theta^1_c = \theta_c$ and $\theta^m_c = \theta_c(\theta^{m-1}_c)$. For $m = 0$, $\theta^0_c = I$, where I is the identity and for $c = 0, \theta^m = \theta^m_0$.

The following lemma is about the relation between Mellin and Stirling numbers.

Lemma 4.3.7. The Mellin derivative θ_c of a function $k(t)$ of order $m \in \mathbb{N}$ for $c \in \mathbb{R}$ can be written as

$$\theta_c^m k(t) = \sum_{q=0}^{m} S_c(m,q) t^q k^{(q)}(t), \tag{4.3.5}$$

where $S_c(m,q)$, $0 \leq q \leq m$ are the generalized Stirling numbers of second kind [27].

An important result is given below.

Proposition 4.3.8. Let $m \in \mathbb{N}$ and t is positive, then

$$\delta^m t^c k(t) = t^c \theta_c^m k(t). \tag{4.3.6}$$

4.4 Some new results on generalized frac-derivatives

In this section, Mellin transform of the generalized Caputo and Hilfer frac-derivatives is evaluated.

4.4.1 Mellin transform of generalized Caputo frac-derivative

To evaluate the Mellin transform of right and left sided generalized Caputo frac-derivative, we need to use Mellin transform of $\left(x^{1-\nu}\frac{d}{dx}\right)^m k(t)$, given by following lemma.

Lemma 4.4.1. Let $k \in C^m(\mathbb{R}+)$, $\tilde{k}(s - m\nu)$ and $\mathcal{M}[\left(x^{1-\nu}\frac{d}{dx}\right)^m k(t)](s)$ exist, and $\lim_{t\to 0}[x^{s-m\nu+(q-1)}k^{(q-1)}(t)]$ and $\lim_{t\to\infty}[x^{s-m\nu+(q-1)}k^{(q-1)}(t)]$ are finite for $q = 0, 1, ..., (m-$

1), $m \in \mathbb{N}$, then

$$\mathcal{M}\Big[\Big(x^{1-\nu}\frac{d}{dx}\Big)^m k(x)\Big] = \frac{\Gamma(1-\frac{s}{\nu}+m)}{\Gamma(1-\frac{s}{\nu})}\tilde{k}(s-m\nu) + \sum_{q=1}^{m}\Big[x^{s-m\nu+(q-1)}k^{(q-1)}(x)\Big]_0^\infty. \tag{4.4.1}$$

$$= (-1)^m\frac{\Gamma(\frac{s}{\nu})}{\Gamma(\frac{s}{\nu}-m)}\tilde{k}(s-m\nu) + \sum_{q=1}^{m}(-1)^q\Big[x^{s-m\nu+(q-1)}k^{(q-1)}(x)\Big]_0^\infty. \tag{4.4.2}$$

Proof. From Section 2.7 property (ii) and (vi), we have

For $m = 1$

$$\mathcal{M}\Big[\Big(x^{1-\nu}\frac{d}{dx}\Big)k(x)\Big](s) = -(s-\nu)\tilde{k}(s-\nu) + \Big|t^{s-\nu}k(t)\Big|_0^\infty. \tag{4.4.3}$$

For $m = 2$

$$\mathcal{M}\Big[\Big(x^{1-\nu}\frac{d}{dx}\Big)^2 k(x)\Big] = (-1)^2(s-\nu)(s-2\nu)\tilde{k}(s-2\nu) + \Big|t^{s-2\nu}k(t)\Big|_0^\infty + \Big|t^{s-2\nu+1}\frac{d}{dt}k(t)\Big|_0^\infty. \tag{4.4.4}$$

For $m = 3$

$$\mathcal{M}\Big[\Big(x^{1-\nu}\frac{d}{dx}\Big)^3 k(x)\Big] = (-1)^3(s-\nu)(s-2\nu)(s-3\nu)\tilde{k}(s-3\nu) + \Big|t^{s-3\nu}k(t)\Big|_0^\infty + \Big|t^{s-3\nu+1}\frac{d}{dt}k(t)\Big|_0^\infty + \Big|t^{s-3\nu+2}\Big(\frac{d}{dt}\Big)^2 k(t)\Big|_0^\infty. \tag{4.4.5}$$

Following similar procedure, in general we get

$$\mathcal{M}\Big[\Big(x^{1-\nu}\frac{d}{dx}\Big)^{m}k(x)\Big] = \frac{\Gamma(1-\frac{s}{\nu}+m)}{\Gamma(1-\frac{s}{\nu})}\tilde{k}(s-m\nu) + \sum_{q=1}^{m}\Big[x^{s-m\nu+(q-1)}k^{(q-1)}(x)\Big]_{0}^{\infty}.$$

Similarly Eq. (4.4.2) can be proved. □

The following result is about the Mellin transform of generalized Caputo frac-derivatives.

Theorem 4.4.2. Let $\alpha > 0$, $m-1 < \alpha \leq m$, $\nu > 0$, $s \in \mathbb{C}$ and $k \in X^{1}_{s+\alpha\nu}(\mathbb{R}+)$. Then,

(a) For $\mathrm{Re}\Big(\frac{s}{\nu}\Big) < 1$ and $x > a \geq 0$,

$$\mathcal{M}[\,{}^{\nu}_{*}\mathcal{D}^{\alpha}_{a+}k](s) = \frac{\nu^{\alpha}\Gamma(1-\frac{s}{\nu}+\alpha)}{\Gamma(1-\frac{s}{\nu})}\tilde{k}(s-\alpha\nu)+\sum_{q=0}^{m-1}\frac{\Gamma(1+q-\frac{s}{\nu}-m+\alpha)}{\Gamma(1-\frac{s}{\nu}-m+\alpha)}\Big[x^{s+(m-\alpha)\nu-q-1}k^{(m-q-1)}(x)\Big]_{0}^{\infty}.$$

(b) For $\mathrm{Re}\Big(\frac{s}{\nu}-\alpha\Big) > 0$ and $x < b \leq \infty$,

$$\mathcal{M}[\,{}^{\nu}_{*}\mathcal{D}^{\alpha}_{b-}k](s) = \frac{\nu^{\alpha}\Gamma(\frac{s}{\nu})}{\Gamma(\frac{s}{\nu}-\alpha)}\tilde{k}(s-\alpha\nu)+\sum_{q=0}^{m-1}(-1)^{q}\frac{\Gamma(\frac{s}{\nu}+m-\alpha)}{\Gamma(\frac{s}{\nu}+m-\alpha-q)}\Big[x^{s+(m-\alpha)\nu-q-1}k^{(m-q-1)}(x)\Big]_{0}^{\infty}.$$

Proof. (a) By definition of ${}^{\nu}_{*}\mathcal{D}^{\alpha}_{a+}$ from Eq. (2.5.5) and Eq. (2.6.1), we have

$$\begin{aligned}\mathcal{M}[\,{}^{\nu}_{*}\mathcal{D}^{\alpha}_{a+}k(x)] &= \mathcal{M}\Big[\frac{\nu^{1-m+\alpha}}{\Gamma(m-\alpha)}\int_{a}^{x}(x^{\nu}-t^{\nu})^{m-\alpha-1}\Big(x^{1-\nu}\frac{d}{dx}\Big)^{m}k(t)dt\Big]\\ &= \mathcal{M}\Big[(\,{}^{\nu}\mathfrak{I}^{m-\alpha}_{a+}\delta^{m}_{\nu}k)(t)\Big].\end{aligned}$$

By using Eq. (4.2.1) and Eq. (4.4.1), we get

$$\begin{aligned}\mathcal{M}[\,{}^{\nu}_{*}\mathcal{D}^{\alpha}_{a+}k](s) &= \frac{\Gamma(1-\frac{s}{\nu}-m+\alpha)}{\nu^{m-\alpha}\Gamma(1-\frac{s}{\nu})}\mathcal{M}[\delta^{m}_{\nu}k(t)](s+(m-\alpha)\nu)\\ &= \frac{\nu^{\alpha}\Gamma(1-\frac{s}{\nu}+\alpha)}{\Gamma(1-\frac{s}{\nu})}\tilde{k}(s-\alpha\nu)\\ &\quad+\sum_{q=0}^{m-1}\frac{\Gamma(1+q-\frac{s}{\nu}-m+\alpha)}{\Gamma(1-\frac{s}{\nu}-m+\alpha)}\Big[x^{s+(m-\alpha)\nu-q-1}k^{(m-q-1)}(x)\Big]_{0}^{\infty}.\end{aligned}$$

we get the required result (a).

(b) By definition of ${}^{\nu}_{*}\mathcal{D}^{\alpha}_{b-}$ from Eq. (2.5.6) and Eq. (2.6.1) we have

$$\mathcal{M}\left[\ {}^{\nu}_{*}\mathcal{D}^{\alpha}_{b-}k(x)\right] = \mathcal{M}[(-1)^m({}^{\nu}\mathfrak{I}^{m-\alpha}_{b-}\delta^m_{\nu}k)(t)].$$

By using Eq. (4.2.2) and Eq. (4.4.2), we get

$$\begin{aligned}\mathcal{M}[\ {}^{\nu}_{*}\mathcal{D}^{\alpha}_{b-}k](s) &= (-1)^m\frac{\Gamma(\frac{s}{\nu})}{\Gamma(\frac{s}{\nu}+m-\alpha)}\mathcal{M}[\delta^m_{\nu}k(t)](s+(m-\alpha)\nu)\\ &= (-1)^{2m}\frac{\nu^{\alpha}\Gamma(\frac{s}{\nu})}{\Gamma(\frac{s}{\nu}-\alpha)}\tilde{k}(s-\alpha\nu)\\ &\quad+\sum_{q=0}^{m-1}(-1)^q\frac{\Gamma(\frac{s}{\nu}+m-\alpha)}{\Gamma(\frac{s}{\nu}+m-\alpha-q)}\left[x^{s+(m-\alpha)\nu-q-1}k^{(m-q-1)}(x)\right]_0^{\infty}.\end{aligned}$$

we get the required result (b).

□

4.4.2 Mellin transform of Hilfer and generalized Hilfer frac-derivative

In following Theorem, Mellin transform of right and left sided Hilfer derivative is defined as

Theorem 4.4.3. Let $\alpha \in \mathbb{C}$, $\mathrm{Re}(\alpha) > 0$, $s \in \mathbb{C}$ and $k \in X^1_{s-\alpha}(\mathbb{R}+)$. Then

(a) For $\mathrm{Re}(s) < 1,\ x > a \geq 0$
$\mathcal{M}[\mathcal{D}^{\alpha,\beta}_{a+}k(t)] = \frac{\Gamma(1-s+\alpha)}{\Gamma(1-s)}\tilde{k}(s-\alpha) + \left[t^{s+\beta(1-\alpha)-1}\mathfrak{I}^{(1-\alpha)(1-\beta)}_{a+}k(t)\right]_0^{\infty}.$

(b) For $\mathrm{Re}(s-\alpha) > 0,\ x < b \leq \infty$
$\mathcal{M}[\mathcal{D}^{\alpha,\beta}_{b-}k(t)] = \frac{\Gamma(s)}{\Gamma(s-\alpha)}\tilde{k}(s-\alpha) + \left[t^{s+\beta(1-\alpha)-1}\mathfrak{I}^{(1-\alpha)(1-\beta)}_{b-}k(t)\right]_0^{\infty}.$

Proof. (a) By definition of $\mathcal{D}_{a+}^{\alpha,\beta}$ from Eq. (2.3.11) and Eq. (2.6.1), we have

$$\begin{aligned}\mathcal{M}[\mathcal{D}_{a+}^{\alpha,\beta}k(t)](s) &= \mathcal{M}\Big(\mathfrak{I}_{a+}^{\beta(1-\alpha)}\frac{d}{dt}\mathfrak{I}_{a+}^{(1-\alpha)(1-\beta)}k(t)\Big)(s)\\ &= \mathcal{M}\Big(\mathfrak{I}_{a+}^{\beta(1-\alpha)}\frac{d}{dt}g(t)\Big)(s),\end{aligned}$$

where $g(t) = \mathfrak{I}_{a+}^{(1-\alpha)(1-\beta)}k(t)$.

Using Eq. (4.1.1) and Eq. (4.1.5), we get

$$\begin{aligned}\mathcal{M}[\mathcal{D}_{a+}^{\alpha,\beta}k(t)](s) &= \frac{\Gamma(1-s-\beta(1-\alpha))}{\Gamma(1-s)}\mathcal{M}\Big[\frac{d}{dt}g(t); s+\beta(1-\alpha)\Big]\\ &= \frac{\Gamma(2-s-\beta(1-\alpha))}{\Gamma(1-s)}\mathcal{M}\Big[g(t); s+\beta(1-\alpha)-1\Big]\\ &\quad + \Big[t^{s+\beta(1-\alpha)-1}g(t)\Big]_0^\infty\\ &= \frac{\Gamma(1-s-\beta(1-\alpha)+1-(1-\alpha)(1-\beta))}{\Gamma(1-s)}\\ &\quad \times\mathcal{M}\Big[k(t); s+\beta(1-\alpha)-1+(1-\alpha)(1-\beta)\Big]\\ &\quad + \Big[t^{s+\beta(1-\alpha)-1}\mathfrak{I}_{a+}^{(1-\alpha)(1-\beta)}k(t)\Big]_0^\infty.\end{aligned}$$

Therefore,

$$\mathcal{M}[\mathcal{D}_{a+}^{\alpha,\beta}k(t)](s) = \frac{\Gamma(1-s+\alpha)}{\Gamma(1-s)}\tilde{k}(s-\alpha) + \Big[t^{s+\beta(1-\alpha)-1}\mathfrak{I}_{a+}^{(1-\alpha)(1-\beta)}k(t)\Big]_0^\infty.$$

Hence, we get the required result. As special cases, for $\beta = 0$, we get Eq. (4.1.3) and for $\beta = 1$, we get assertion (a) of Theorem 4.1.4.

(b) By definition of $\mathcal{D}_{b-}^{\alpha,\beta}$ from Eq. (2.3.12) and Eq. (2.6.1), we have

$$\begin{aligned}\mathcal{M}[\mathcal{D}_{b-}^{\alpha,\beta}k(t)] &= \mathcal{M}(\mathfrak{I}_{b-}^{\beta(1-\alpha)}\frac{d}{dt}\mathfrak{I}_{b-}^{(1-\alpha)(1-\beta)}k)(t)\\ &= \mathcal{M}\Big(\mathfrak{I}_{b-}^{\beta(1-\alpha)}\frac{d}{dt}g(t)\Big)(s),\end{aligned}$$

where $g(t) = \mathfrak{I}_{b-}^{(1-\alpha)(1-\beta)}k(t)$.

Using Eq. (4.1.2) and Eq. (4.1.6), we get

$$\begin{aligned}\mathcal{M}[\mathcal{D}_{b-}^{\alpha,\beta}k(t)] &= -\frac{\Gamma(s)}{\Gamma(s+\beta(1-\alpha))}\mathcal{M}\Big[\frac{d}{dt}g(t); s+\beta(1-\alpha)\Big] \\ &= \frac{\Gamma(s)}{\Gamma(s+\beta(1-\alpha)-1)}\mathcal{M}\Big[g(t); s+\beta(1-\alpha)-1\Big] \\ &\quad+\Big[t^{s+\beta(1-\alpha)-1}g(t)\Big]_0^\infty.\end{aligned}$$

Therefore,

$$\mathcal{M}[\mathcal{D}_{b-}^{\alpha,\beta}k(t)] = \frac{\Gamma(s)}{\Gamma(s-\alpha)}\tilde{k}(s-\alpha) + \Big[t^{s+\beta(1-\alpha)-1}\mathfrak{I}_{b-}^{(1-\alpha)(1-\beta)}k(t)\Big]_0^\infty.$$

Hence, we get the required result.

As special cases, for $\beta = 0$, we get Eq. (4.1.4) and for $\beta = 1$, we get assertion (b) of Theorem 4.1.4. □

Mellin transform of generalized Hilfer derivative is presented below.

Theorem 4.4.4. Let $\alpha \in \mathbb{C}$, $\mathrm{Re}(\alpha) > 0$, $\nu > 0$, $s \in \mathbb{C}$ and $k \in X^1_{s+\alpha\nu}(\mathbb{R}+)$. Then

(a) For $\mathrm{Re}\Big(\frac{s}{\nu}\Big) < 1$ and $x > a \geq 0$.
$${}^{\nu}\mathcal{D}_{a+}^{\alpha,\beta}k(t) = \frac{\nu^{\alpha}\Gamma(1-\frac{s}{\nu}+\alpha)}{\Gamma(1-\frac{s}{\nu})}\tilde{k}(s-\alpha\nu)+\sum_{q=1}^{m}\Big[t^{s+\beta(m-\alpha)\nu-m\nu+(q-1)}\Big(\frac{d}{dt}\Big)^{q-1}\mathfrak{I}_{a+}^{(m-\alpha)(1-\beta)}k(t)\Big]_0^\infty.$$

(b) For $\mathrm{Re}\Big(\frac{s}{\nu}-\alpha\Big) > 0$ and $x < b \leq \infty$.
$$\mathcal{M}[\,{}^{\nu}\mathcal{D}_{b-}^{\alpha,\beta}k(t)] = \frac{\Gamma(\frac{s}{\nu})}{\Gamma(\frac{s}{\nu}-\alpha)}\tilde{k}(s-\alpha\nu)+\sum_{q=1}^{m}(-1)^q\Big[t^{s+\beta(m-\alpha)\nu-m\nu+(q-1)}\Big(\frac{d}{dt}\Big)^{q-1}\mathfrak{I}_{b-}^{(m-\alpha)(1-\beta)}k(t)\Big]_0^\infty.$$

Proof. (a) By definition of ${}^{\nu}\mathcal{D}_{a+}^{\alpha,\beta}$ from Eq. (2.5.7) and Eq. (2.6.1), we have

$$\begin{aligned}\mathcal{M}[\,{}^{\nu}\mathcal{D}_{a+}^{\alpha,\beta}k](s) &= \mathcal{M}\Big({}^{\nu}\mathfrak{I}_{a+}^{\beta(m-\alpha)}\delta_\nu^m\mathfrak{I}_{a+}^{(m-\alpha)(1-\beta)}k\Big)(s) \\ &= \mathcal{M}\Big({}^{\nu}\mathfrak{I}_{a+}^{\beta(m-\alpha)}\delta_\nu^m g(t)\Big)(s),\end{aligned}$$

where

$$g(t) = \mathfrak{I}_{a+}^{(m-\alpha)(1-\beta)} k(t).$$

Using Eqs. (4.2.1) and (4.4.1), we get

$$\begin{aligned}
\mathcal{M}[\,{}^{\nu}\mathcal{D}_{a+}^{\alpha,\beta} k](s) &= \frac{\Gamma(1-\frac{s}{\nu}-\beta(m-\alpha))}{\nu^{\beta(m-\alpha)}\Gamma(1-\frac{s}{\nu})}\mathcal{M}\Big[\delta_{\nu}^{m} g(t); s+\beta(m-\alpha)\nu\Big] \\
&= \frac{\Gamma(1-\frac{s}{\nu}-\beta(m-\alpha)+m)}{\nu^{\beta(m-\alpha)-m}\Gamma(1-\frac{s}{\nu})}\mathcal{M}\Big[g(t); s+\beta(m-\alpha)\nu-m\nu\Big] \\
&\quad +\sum_{q=1}^{m}\Big[t^{s+\beta(m-\alpha)\nu-m\nu+(q-1)}\Big(\frac{d}{dt}\Big)^{q-1} g(t)\Big]_0^{\infty} \\
&= \frac{\nu^{\alpha}\Gamma(1-\frac{s}{\nu}-\beta(m-\alpha)+m-(m-\alpha)(1-\beta))}{\Gamma(1-\frac{s}{\nu})} \\
&\quad \times\mathcal{M}\Big[k(t); s+\beta(m-\alpha)\nu-m\nu+(m-\alpha)(1-\beta)\nu\Big] \\
&\quad +\sum_{q=1}^{m}\Big[t^{s+\beta(m-\alpha)\nu-m\nu+(q-1)}\Big(\frac{d}{dt}\Big)^{q-1}\mathfrak{I}_{a+}^{(m-\alpha)(1-\beta)} k(t)\Big]_0^{\infty} \\
&= \frac{\nu^{\alpha}\Gamma(1-\frac{s}{\nu}+\alpha)}{\Gamma(1-\frac{s}{\nu})}\tilde{k}(s-\alpha\nu) \\
&\quad +\sum_{q=1}^{m}\Big[t^{s+\beta(m-\alpha)\nu-m\nu+(q-1)}\Big(\frac{d}{dt}\Big)^{q-1}\mathfrak{I}_{a+}^{(m-\alpha)(1-\beta)} k(t)\Big]_0^{\infty}.
\end{aligned}$$

Hence we get the required result.

As special case for $\beta = 0$, we get the Eq. (4.2.3) and for $\beta = 1$, we get assertion (a) of Theorem 4.4.2.

(b) By definition of ${}^{\nu}\mathcal{D}_{b-}^{\alpha,\beta}$ from Eq. (2.5.8) and Eq. (2.6.1), we have

$$\begin{aligned}
\mathcal{M}[\,{}^{\nu}\mathcal{D}_{b-}^{\alpha,\beta} k(t)] &= \mathcal{M}(\mathfrak{I}_{b-}^{\beta(m-\alpha)}\delta_{\nu}^{m}\mathfrak{I}_{b-}^{(m-\alpha)(1-\beta)} k)(t) \\
&= \mathcal{M}\Big(\mathfrak{I}_{b-}^{\beta(1-\alpha)}\delta_{\nu}^{m} g(t)\Big)(s),
\end{aligned}$$

where $g(t) = \mathfrak{I}_{b-}^{(1-\alpha)(1-\beta)} k(t)$.

Using Eq. (4.2.2) and Eq. (4.4.2), we get

$$\begin{aligned}
\mathcal{M}[\mathcal{D}_{b-}^{\alpha,\beta}k(t)] &= -\frac{\Gamma(\frac{s}{\nu})}{\Gamma(\frac{s}{\nu}+\beta(m-\alpha))}\mathcal{M}\Big[\delta_\nu^m g(t); s+\beta(m-\alpha)\nu\Big] \\
&= \frac{\Gamma(\frac{s}{\nu})}{\Gamma(\frac{s}{\nu}+\beta(m-\alpha)-m)}\mathcal{M}\Big[g(t); s+\beta(m-\alpha)\nu-m\nu\Big] \\
&\quad + \sum_{q=1}^{m}(-1)^q\Big[t^{s+\beta(m-\alpha)\nu-m\nu+(q-1)}\Big(\frac{d}{dt}\Big)^{q-1}g(t)\Big]_0^\infty \\
&= \frac{\Gamma(\frac{s}{\nu})}{\Gamma(\frac{s}{\nu}-\alpha)}\tilde{f}(s-\alpha\nu) \\
&\quad + \sum_{q=1}^{m}(-1)^q\Big[t^{s+\beta(m-\alpha)\nu-m\nu\ \ \nu}\Big(\frac{d}{dt}\Big)^{q-1}\mathfrak{I}_{b-}^{(m-\alpha)(1-\beta)}k(t)\Big]_0^\infty.
\end{aligned}$$

Hence we get the required result.

□

As special cases, for $\beta = 0$, we get the Eq. (4.2.4) and for $\beta = 1$, we get assertion (b) of Theorem 4.4.2.

4.4.3 Mellin transform of Caputo Hadamard-type frac-derivatives

Mellin transform of CHTF-derivative is obtained in following Theorem.

Theorem 4.4.5. If $\mathrm{Re}(\alpha) > 0$ and $c \in \mathbb{C}$. Also let a function $k(x)$ be such that its Mellin transform $\mathcal{M}[k](s)$ exists for $s \in \mathbb{C}$.

(a) If $\mathrm{Re}(c-s) > 0$ and $\mathcal{M}[\ {}_*\mathcal{D}_{0+,c}^{\alpha}k](s)$ exists, then
$\mathcal{M}[\ {}_*\mathcal{D}_{0+,c}^{\alpha}k](s) = (c-s)^\alpha\mathcal{M}[k](s)$.

(b) If $\mathrm{Re}(c+s) > 0$and $\mathcal{M}[\ {}_*\mathcal{D}_{0-,c}^{\alpha}k](s)$ exists, then
$\mathcal{M}[\ {}_*\mathcal{D}_{0-,c}^{\alpha}k](s) = (c+s)^\alpha\mathcal{M}[k](s)$.

Proof. (a) By using Property (ii) and (vii) of Section 2.7, that is

$$\mathcal{M}[\delta^m k](s) = (-s)^m \mathcal{M}[k](s)$$
$$\mathcal{M}[D_c^m k](s) = (c-s)^m \mathcal{M}[k](s). \tag{4.4.6}$$

Now we can find Mellin transform of CHTF-derivatives as

$$\begin{aligned}\mathcal{M}[\,{}_*\mathcal{D}^{\alpha}_{0+,c}k](s) &= \mathcal{M}[D_c^m \mathcal{J}^{m-\alpha}_{a+,c}k](s)\\ &= (c-s)^{m-\alpha}(c-s)^m \mathcal{M}[k](s)\\ &= (c-s)^{\alpha}\mathcal{M}[k](s).\end{aligned}$$

(b) Follows similar procedure.

□

Now we discuss some results related to Mellin derivative and CHTF-derivative.

The following Proposition shows connection between Mellin derivative and ordinary derivative.

Proposition 4.4.6. If $k \in X_{c,loc}$ then Mellin derivative $\theta_c^m (m \in \mathbb{N})$ and CHTF-derivative exists, and both are equivalent, that is

$$(\,{}_*\mathcal{D}^{m}_{0+,c}k)(t) = \theta_c^m k(t). \tag{4.4.7}$$

Proof. By induction, we have

For $m = 1$, $n = m + 1 = 2$ and $\delta = w\frac{d}{dw}$,

$$\begin{aligned}(\,{}_*\mathcal{D}^{1}_{0+,c}k)(t) &= \frac{1}{\Gamma(1)}\int_0^t \big(\frac{w}{t}\big)^c \big(\ln\frac{t}{w}\big)^{2-2} w^{-c}\delta^2 w^c k(w)\frac{dw}{w}\\ &= t^{-c}\int_0^t w^{-1}\delta\big(w\frac{d}{dw}\big)w^c k(w)dw\\ &= t^{-c}\left[c^2\int_0^t w^{c-1}k(w)dw + (2c+1)\int_0^t w^c k'(w)dw + \int_0^t w^{c+1}k''(w)dw\right]\end{aligned}$$

$$\begin{aligned}(\,{}_*\mathcal{D}^1_{0+,c}k)(t) &= tk^{'}(t) + ck(t)\\ &= \theta_c k(t). \end{aligned} \tag{4.4.8}$$

Now we suppose that the relation holds for m and prove that it holds for $m+1$ by Proposition 4.3.8 and Definitions 23 and 21, we have

$$\begin{aligned}(\,{}_*\mathcal{D}^{m+1}_{0+,c}k)(t) &= \mathcal{J}^{n-m}_{0+,c}(t^{-c}\delta^{(n)}t^c)k(t) = \mathcal{J}^{1}_{0+,c}(t^{-c}\delta^{(m+2)}t^c)k(t)\\ &= \mathcal{J}^{1}_{0+,c}t^{-c}\delta^{m+1}(\delta\ t^c k)(t)\\ &= \mathcal{J}^{1}_{0+,c}t^{-c}\delta^{m+1}(t^c\ \theta_c k(t))\\ &= \mathcal{J}^{1}_{0+,c}\ \theta_c^{m+2}k(t) = \theta_c^{m+1}k(t). \end{aligned}$$

and so the assertion follows. □

By Lemma 4.3.7, we can also write it as

$$(\,{}_*\mathcal{D}^m_{0+,c}k)(t) = \theta_c^m k(t) = \sum_{q=0}^{m} S_c(m,q)t^q k^{(q)}(t). \tag{4.4.9}$$

Fractional derivative of $k(t)$ can also be written as,

$$(\,{}_*\mathcal{D}^\alpha_{0+,c}k)(t) = \mathcal{J}^{m-\alpha}_{0+,c}(\theta_c^m k)(t), \quad \alpha > 0. \tag{4.4.10}$$

where $m = \lceil \alpha \rceil$.

The following result shows the relation that how CHTF-derivative can be written in form of Stirling numbers.

Proposition 4.4.7. Let α be positive and c be a real number. If $k \in X_{c,loc}$ such that $k^{(m)}(t) \in X_{c,loc}$, then

$$(\,{}_*\mathcal{D}^\alpha_{0+,c}k)(t) = \mathcal{J}^{m-\alpha}_{0+,c}\Big(\sum_{q=0}^{m} S_c(m,q)t^q k^{(q)}\Big)(t). \tag{4.4.11}$$

Proof. It can be directly proved by using Eqs. (4.4.9) and (4.4.10). □

Now we discuss an interesting result.

Proposition 4.4.8. Let $k \in X_{c,loc}$ such that $\theta_c^m k(t) \in X_{c,loc}$, where $m = \lceil \alpha \rceil$. Then

$$(\, _*\mathcal{D}^{\alpha}_{0+,c} k)(t) = \mathcal{J}^{m-\alpha}_{0+,c}(\theta_c^m k)(t) = \theta_c^m (\mathcal{J}^{m-\alpha}_{0+,c} k)(t). \tag{4.4.12}$$

Proof. Proposition 18 of [20] shows that

$$\theta_c^m (\mathcal{J}^{m-\alpha}_{0+,c} k)(t) = \mathcal{J}^{m-\alpha}_{0+,c}(\theta_c^m k)(t).$$

Hence we have,

$$(\, _*\mathcal{D}^{\alpha}_{0+,c} k)(t) = \mathcal{J}^{m-\alpha}_{0+,c}(\theta_c^m k)(t) = \theta_c^m (\mathcal{J}^{m-\alpha}_{0+,c} k)(t).$$

□

4.5 An application

Here is an applications of Mellin transform of frac-differential equations.

Example 4.5.1. Consider the equation

$$t^{(\alpha+1)\nu} \, {}^{\nu}_{*}\mathcal{D}^{\alpha+1} y(t) + t^{\alpha\nu} \, {}^{\nu}_{*}\mathcal{D}^{\alpha} y(t) = k(t), \quad \text{for} \quad 2 < \alpha \leq 3. \tag{4.5.1}$$

we suppose that

$$y(0) = y^{'}(0) = 0, \qquad\qquad y(\infty) = y^{'}(\infty) = 0. \tag{4.5.2}$$

Now by applying Mellin transform on both sides of Eq. (4.5.1) and using Mellin transform of generalized Caputo frac-derivative given in Theorem 4.4.2 and from

Section 2.6 (property ii), we get

$$\tilde{y}(s) = \tilde{k}(s)\frac{\Gamma(1-\frac{s}{\nu}-\alpha)}{-\nu^{\alpha}(s+\alpha\nu-1)\Gamma(1-\frac{s}{\nu})}.$$
$$\tilde{y}(s) = \tilde{k}(s)\tilde{h}(s).$$

To find solution of Eq. (4.5.1), we need to know the inverse Mellin transform $h(t)$ of the function $\tilde{h}(s)$ that is the Mellin convolution of the function $k(t)$ and $h(t)$,

$$y(t) = \int_0^{\infty} k(ts)h(s)ds.$$

Mellin transform of $h(t)$ can be written as

$$\tilde{h}(s) = \tilde{h_1}(s)\tilde{h_2}(s),$$

where

$$\tilde{h_1}(s) = \frac{-1}{(s+\alpha\nu-1)}.$$
$$\tilde{h_2}(s) = \frac{\Gamma(1-\frac{s}{\nu}-\alpha)}{\nu^{\alpha}\Gamma(1-\frac{s}{\nu})}.$$

The inverse Mellin transform of $\tilde{h_1}(s)$ and $\tilde{h_2}(s)$ can be found by using formulas 7.1(4) and 7.3(22) from tables [35]:

$$h_1(t) = \begin{cases} 0 & \text{for} \quad 0 < t < 1, \\ t^{\alpha\nu-1} & \text{for} \quad t > 1. \end{cases}$$
$$h_2(t) = \begin{cases} 0 & \text{for} \quad 0 < t < 1, \\ \frac{-(t-1)^{\alpha-1}}{\nu^{\alpha}\Gamma(\alpha)} & \text{for} \quad 1 < t < \infty. \end{cases}$$

By convolution property the inverse transform of $\tilde{g}(s)$ is,

$$h(t) = \int_0^\infty h_1\left(\frac{t}{s}\right)\, h_2(s)\frac{ds}{s}. \tag{4.5.3}$$

From Eq. (4.5.3), we know that $g(t) = 0$ for $0 < t < s < 1$, so

$$h(t) = \int_1^\infty h_1\left(\frac{t}{s}\right) h_2(s)\frac{ds}{s}.$$

Therefore,

$$y(t) = \int_1^\infty k(ts)h(s)ds,$$

where

$$h(s) = -\frac{s^{(\alpha\nu-1)}\Gamma(\alpha\nu-\alpha)}{\nu^\alpha\Gamma(\alpha\nu)}.$$

Chapter 5

Conclusions

In this thesis, a study is presented for Caputo Hadamard-type frac-derivative(fractional derivative) (by changing the order of integral and differential operator of Hadamard-type frac-derivative), motivated from Caputo Hadamard frac-derivative.

In [12, 16] semi group property and composition of operators for Caputo Hadamard frac-derivatives were discussed and in [30] Taylor's formula for Caputo Hadamard frac-derivative was presented. This work is extended in this thesis on the same lines for Caputo Hadamard-type frac-derivatives. Integration by parts formula for Hadamard-type frac-derivatives is also discussed.

In the rest of the thesis, Mellin transform is discussed and evaluated for generalized Caputo, Hilfer and generalized Hilfer frac-derivatives. In [20], Mellin transform analysis of Hadamard-type frac-derivatives was discussed. Here, a similar analysis is presented for Caputo Hadamard-type frac-derivative. An example following [21] is presented to study the Mellin transform of generalized Caputo frac-derivative used to solve a boundary value problem with a frac differential equation.

Appendix

A1

Let

$$\mathcal{J}_{a+}k(x) = \int_a^x k(t_1)\frac{dt_1}{t_1}.$$

Then

$$\mathcal{J}_{a+}^2 k(x) = \int_a^x \frac{1}{s}\int_a^s \frac{k(t_1)}{t_1}dt_1 ds.$$

By Dirichlet formula,

$$= \int_a^x \ln\left(\frac{x}{t_1}\right)\frac{dt_1}{t_1}.$$

In general,

$$\begin{aligned}\mathcal{J}_{a+}^n k(x) &= \int_a^x \frac{dt_1}{t_1}\int_a^{t_1}\frac{dt_2}{t_2}...\int_a^{t_{n-1}} k(t_n)\frac{dt_n}{t_n}\\ &= \frac{1}{(n-1)!}\int_a^x \left(\ln\frac{x}{t_1}\right)^{n-1} k(t_1)\frac{dt_1}{t_1}.\end{aligned}$$

Replace n with α

$$\mathcal{J}_{a+}^\alpha k(x) = \frac{1}{(n-1)!}\int_a^x \left(\ln\frac{x}{t_1}\right)^{\alpha-1} k(t_1)\frac{dt_1}{t_1}.$$

A2

Leibniz rule:

$$\frac{d}{dt_1}\int_{k(t_1)}^{g(t_1)} h(t_1,s)ds = h(t_1,g(t_1))g'(t_1) - h(t_1,k(t_1))f'(t_1) + \int_{k(t_1)}^{g(t_1)}\frac{\partial\ h(t_1,s)}{\partial\ t_1}ds.$$

A3

Integration by parts for Hadamard frac-operator

$$
\begin{aligned}
\delta(kh) &= k\delta h + \delta k h \\
\mathcal{J}\delta(kh) &= \mathcal{J}k\delta h + \mathcal{J}\delta k h \\
kh\Big|_a^x &= \int_a^x k\delta h \frac{dt_1}{t_1} + \int_a^x \delta k h \frac{dt_1}{t_1} \\
\int_a^x \delta\ kh \frac{dt_1}{t_1} &= kh\Big|_a^x - \int_a^x k\delta h \frac{dt_1}{t_1}.
\end{aligned}
$$

A4

Integration by parts formula for HTF- operator

$$
\begin{aligned}
D_c[t_1^{-c}k(t_1)h(t_1)] &= [D_c t_1^{-c}k(t_1)]h(t_1) + k(t_1)[D_c t_1^{-c}h(t_1)] \\
\mathcal{J}_c D_c[t_1^{-c}k(t_1)h(t_1)] &= \mathcal{J}_c[D_c t_1^{-c}k(t_1)]h(t_1) + \mathcal{J}_c k(t_1)[D_c t_1^{-c}h(t_1)] \\
\int_a^{t_1} \left(\frac{s}{t_1}\right)^c s^{-c}\delta s^c k(s)h(s)\frac{ds}{s} &= \int_a^{t_1} \left(\frac{s}{t_1}\right)^c D_c\Big(s^{-c}k(s)\Big)h(s)\frac{ds}{s} \\
&\quad + \int_a^{t_1} \left(\frac{s}{t_1}\right)^c k(s) D_c\Big(s^{-c}h(s)\Big)\frac{ds}{s} \\
\int_a^{t_1} \left(\frac{s}{t_1}\right)^c D_c\Big(s^{-c}k(s)\Big)h(s)\frac{ds}{s} &= t_1^{-c}k(s)h(s)\Big|_a^{t_1} - \int_a^{t_1} \left(\frac{s}{t_1}\right)^c k(s) D_c\Big(s^{-c}h(s)\Big)\frac{ds}{s}.
\end{aligned}
$$

Bibliography

[1] Lacroix, S. F., *Traité Du Calcul Differential at du Calcul integral*, 2nd edition, Paris: Courcier, vol 3, (1819).

[2] Liouville, J., sur le calcul des différentielles à indices quelconques, vol 13, 1-69, (1832).

[3] Fourier, J., *La Théorie Analytique de la Chaleur*, English Translation by A. Freeman, Dover Publications, vol 1955, (1822).

[4] Greer, H. R., On Fractional Differentiation, Quart. J. Math., vol 3, 327–330, (1858).

[5] Hadamard, J., *Essai sur l'étude des fonctions données par leur développement de Taylor*, Journal de Mathématiques Pures et Appliquées, vol 8, 101-186, (1892).

[6] Caputo, M., Linear models of dissipation whose Q is almost frequency independent, Annals of Geophysics, vol 19, 383-393, (1967).

[7] Diethelm, K., *The Analysis of Fractional Differential Equations*, Springer Science & Business Media, (2010).

[8] Samko, S. G., Kilbas, A. A., and Marichev, O. I., *Fractional integrals and derivatives*, Theory and Applications, Gordon and Breach, Yverdon et alibi, (1993).

[9] Kilbas, A. A., HTF- calculus, Journal of the Korean Mathematical Society, vol 38,1191-1204, (2001).

[10] Hilfer, R., *Applications of Fractional Calculus in Physics*, World Scientific Computer Sci, 10.1142/3779, (2000).

[11] Butzer, P. and Jansche, S., A direct approach to the Mellin transform, Journal of Fourier Analysis and Applications, vol 3, 325-376, (1997).

[12] Gambo, Y. Y., Jarad, F., Baleanu, D. and Abdeljawad, T., On Caputo modification of the Hadamard f-derivatives, Advances in Difference Equations, vol 2014, 1-10, (2014).

[13] Katugampola, U., Mellin Transforms of the Generalized Fractional Integrals and Derivatives, Applied Mathematics and Computation, vol 257, 566-580, (2015).

[14] Oliveira, D. and de Oliveira, E. C., On a Caputo-type fractional derivative, Advances in Pure and Applied Mathematics, vol 10, 81-91, (2018).

[15] Oliveira, D. and de Oliveira, E. C., Hilfer–Katugampola f-derivatives, Computational and Applied Mathematics, vol 37, 3672–3690, (2018).

[16] Jarad, F., Abdeljawad, T. and Baleanu, D., Caputo-type modification of the Hadamard f-derivatives, Advances in Difference Equations, vol 2012, 1-42, (2012).

[17] Kilbas, A. A., Srivastava, H. M. and Trujillo, J. J., *Theory and Applications of Fractional Differential Equations*, North-Holland Mathematics Studies, Elsevier Science Inc., New York, USA, Vol 204, (2006).

[18] Mellin, H., Über die fundamentale Wichtigkeit des Satzes von Cauchy für die Theorien der Gamma- und hypergeometrischen Funktionen, Acta Soc. Fennicae, vol 21, (1896).

[19] Butzer, P., Kilbas, A. A. and Trujillo, J., Fractional calculus in Mellin setting and HTF- integrals, Journal of Mathematical Analysis and Applications, vol 269, 1-27, (2002).

[20] Bardaro, C., Butzer, P. and Mantellini, I., The Foundations of Fractional Calculus in the Mellin Transform Setting with Applications, Journal of Fourier Analysis and Applications, vol 21, 1-57, (2015).

[21] Podlubny, I., *Fractional differential equations. Academic Press*, San Diego, (1999).

[22] A. Loverro, Fractional Calculus: History, Definitions and Applications for the Engineer, Notre dame, IN 46556, U.S.A, (2004).

[23] Katugampola, U., New approach to a generalized fractional integral, Applied Mathematics and Computation, vol 218, 860-865, (2011).

[24] Katugampola, U., A new approach to a generalized f-derivatives, Bulletin of Mathematical Analysis and Applications, vol 6, 1-15, (2014).

[25] Debnath, L., Bhatta, D., *Integral transforms and their applications*, Chapman and Hall/CRC, vol 1, (1995).

[26] Flajolet, P., Gourdon, X. and Dumas, P., Mellin Transforms and Asymptotics: Harmonic Sums, Theoretical Computer Science, vol 144, 3-58, (1995).

[27] Butzer, P., Kilbas, A., and Trujillo, J., *Stirling Functions of the Second Kind in the Setting of Difference and Fractional Calculus*, Numerical Functional Analysis and Optimization, vol 24, 673-711, (2003).

[28] Osler, T., Taylor's Series Generalized for f-derivatives and Applications, SIAM Journal on Mathematical Analysis, vol 2, 37-48, (1971).

[29] Trujillo, J., Rivero, M. and Bonilla, B.,*On a Riemann–Liouville Generalized Taylor's Formula*, Journal of Mathematical Analysis and Applications, vol 231, (1999).

[30] Usero Mainer, D., Fractional Taylor Series for Caputo f-derivatives. Construction of Numerical Schemes, (2007).

[31] Almeida, R. and Torres, D. F. M., *Calculus of variations with f-derivatives and fractional integrals*, Applied Mathematics Letters, vol 22, 1816-1820, (2009).

[32] Xu, Y., Sun, H. and Sheng, Q., On variational properties of balanced central f-derivatives, International Journal of Computer Mathematics, vol 95, 1-16, (2017).

[33] Ricardo A., Caputo-Hadamard f-derivatives of Variable Order, Numerical Functional Analysis and Optimization, Taylor and Francis, vol 38, 1-19, (2016).

[34] Butzer, P., Kilbas, A. A. and Trujillo, J., Mellin transformation and integration by parts for HTF- integrals, Journal of Mathematical Analysis and Applications, vol 270, 1-15, (2002).

[35] Bateman, H., *Tables of Integral Transforms*, Mcgraw-Hill Book Company, INC, vol 1, 341-350, (1954).