Periodic Solutions of Non-Linear Ordinary Differential Equations

by

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To My Wonderful Parents

Mr. & Mrs. Waheed Zaffar

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Abstract

The existence of periodic orbits in the solution of non-linear ordinary differential equations has always been of special interest in the theory of non-linear ordinary differential equation. If a system of non-linear ordinary differential equation contains periodic orbits then a lot can be predicted about the nature of the solutions of that system. Such systems often appear in different branches of science. Hence it is in our interest to know if in a given system periodic orbits exist or not and if they do how many are there?

The aim of our work involves the discussion of non-linear systems which exhibit periodic orbits. However we shall only be concerned with second-order autonomous ordinary differential equations. We start by defining plane autonomous systems and then obtain their representation on phase plane to observe the solutions graphically. Then a result, Bendixson's negative criterion is used to check the non-existence of periodic orbits of simply connected regions in plane. Poincaré Bendixson theorem is used for closed and bounded regions to establish existence of periodic orbits along with some other results that hold for particular systems. Finally, some approximate methods are presented related to finding the number of periodic solutions of Liénard equation along with their comparison on the accuracy and fast convergence to the results.

Keywords: Periodic solutions, Limit cycles, Non-linear ordinary differential equation.

Contents

Chapter 1

Introduction

A differential equation is linear if the unknown function and its derivatives appear to the power one. An nth order linear ordinary differential equation, in the independent variable x and the dependent variable y, is an equation that can be expressed in the form:

$$a_0(x)y^n(x) + a_1(x)y^{n-1}(x) + ... + a_{n-1}(x)\acute{y}(x) + a_n(x)y(x) = b(x).$$

Equations that are not linear are called non-linear. Non-linear differential equations are accurate models of most of the real life physical problems so these equations are of special interest. However, due to their non-linearity, these equations are very complex to analyze. Instead of actually solving them, it is easier to analyze their solutions graphically. In the theory of non-linear differential equations the existence of periodic solutions of a system plays a very prominent role. It is a key to determine the behavior of the solutions of the system.

Consider a plane autonomous system defined by two first order differential equations of the form:

$$\dot{x} = X(x, y); \qquad \dot{y} = Y(x, y). \tag{1.0.1}$$

Here the overdots denote derivative with respect to time. This system has been extensively studied and often appears in problems related to many branches of science, from technology to biology [1, 2].

In the study of the system (1.0.1), an important question about the existence and the number of limit cycles arises. However, it is a delicate matter to determine the number of limit cycles of a certain class of systems even under certain conditions. In 1928 Liénard developed a criterion for the existence and uniqueness of periodic solutions for a general class of system, now referred to as generalized Liénard equation

$$\frac{d^2x}{dt^2} + f(x)\frac{dx}{dt} + g(x) = 0. \tag{1.0.2}$$

Liénard transformed this equation to a first order system by placing

$$\frac{dx}{dt} = h(y) - F(x), \qquad \frac{dy}{dt} = -g(x), \tag{1.0.3}$$

where $F(x) = \int_0^x f(u)du$. The system (1.0.3) is called Liénard system and is a particular case of (1.0.1). The Liénard equation has been of interest of research for many scientists but most of the research done so far is related to the uniqueness and existence of limit cycles of this system and less on their exact location in the phase plane. Huang and Sun [3] proposed a theorem that guarantees the uniqueness of limit cycles for the generalized Liénard system (1.0.3) under certain conditions. But Kooji and Jianhua [4] further improved their result by showing with a counter example that the theorem proposed by Huang and Sun was false and required extra conditions on $F(x)$ to guarantee uniqueness of limit cycle of (1.0.3). Leopold Herrmann [5] also studied Liénard equation of the type

$$\ddot{x} + f(x, \dot{x})\dot{x} + g(x) = 0,$$

for existence and uniqueness of solutions of the corresponding initial-value problem and studied their oscillatory properties. Then conditions on the functions $f(x, \dot{x})$ and $g(x)$ were imposed under which the solution is oscillatory. Recently, Liu and Huang [6] discussed the existence and uniqueness of periodic solutions for a Liénard equation with delay, of form

$$\ddot{x} + f(x)\dot{x} + g(t, x(t - \tau(t))) = p(t), \tag{1.0.4}$$

where τ and p are periodic functions. Liu, Tang and Martin [7] improved their result concerning the existence and uniqueness of periodic solutions for a Liénard equations with delay. Gutiérrez and Torres [8] studied the existence and stability of periodic solutions of a second order differential equation of Liénard type

$$\ddot{x} + f(x)\dot{x} + g(x) = p(t),$$

where p is a periodic function under condition that $f(x)$ is positive and $g(x)$ has one or two weak singularities.

Apart from existence and uniqueness of periodic solutions of Liénard equation, research has been done on the number of periodic solutions of Liénard equation. Giacomini and Neukrich [9] presented an approximate method for determining the number of limit cycles of Liénard system of the form $\dot{x} = y - F(x)$, $\dot{y} = -x$ for an odd polynomial $F(x)$. The proposed method gave a sequence of polynomial that approximated to the equation of each limit cycle of the system hence giving an idea about their number and location in phase plane. An improved nonperturbative method was devised by Delamotte [10] for solving second order differential equation for finding limit cycles and is illustrated on anharmonic oscillator and Van der Pol equation. It gives amplitude, period and equation of limit cycle. The most interesting feature of this method is it's simplicity and accuracy. Maesschalck and Dumortier [11] studied the Liénard equation to generelize the number of limit cycles. They conjectured that the Liénard equation of degree $n \geq 6$ can have at least $\frac{n-1}{2} + 2$ limit cycles. In 1976 A. Lins, W. de Melo and C. Pugh [12] conjectured that the maximum number of limit cycles for a classical Liénard equation of degree n would be equal to $\frac{n-1}{2}$. In [12] the existence of at least 4 limit cycles of classical Liénard equations of degree 7 has also been proved. Maesschalck and Dumortier [11] contradicted the conjecture from Lins, de Melo and Pugh and also improved the counterexample from Dumortier, Panazzolo and Roussarie [13] by supplying one additional limit cycle from degree 7 on, and by finding a counterexample of degree 6.

This dissertation is divided into four chapters. In chapter 2 we have discussed one of the useful technique in finding the solution to non-linear differential equations through a graphical method i.e through phase diagram. We start by devising a procedure for the representation of second-order (or plane) autonomous ordinary differential equations on phase plane to study their solutions. The plane autonomous ordinary differential equations can not directly be represented in phase plane, for that we make linear approximation to those equations. The corresponding linear equations are solved and a general solutions to those equations is provided. The solutions obtained, called the phase paths, are then generally characterized. Later in this chapter an important geometrical aspect of phase diagram, known as index, is discussed to better understand the phase paths obtained. Certain results related to finding the index of a plane autonomous system are also presented in this chapter.

Periodic solutions are one of the prominent features of a non-linear plane autonomous ordinary differential equations. The aim of our work is the study of existence and non-existence of periodic solutions in phase plane of a non-linear plane autonomous ordinary differential equations. In chapter 3 we start by establishing criterions for non-existence of periodic solutions of which Bendixson's negative criterion holds importance. Another result called the critical point criterion has been discussed for the same reason where both these results hold only for a simply connected region in plane. Then we establish the existence of such paths for a closed and bounded region in Poincaré Bendixson theorem. Some other prominent results on the existence of periodic solutions covering certain types of differential equations under certain conditions have also been presented in this chapter.

After dealing with the existence of limit cycles, question about the exact number of such periodic orbits comes to mind and it has always intrigued the researchers to find out the number of periodic orbits of a differential system. In the final chapter we deal with this questions by presenting two methods to find the number of limit cycles. But the results are not general. The methods are approximate and cover only special type of differential equations called the Liénard equation. The comparison of the two methods based on accuracy and fast convergence to the results has also been presented in this chapter.

Chapter 2

Preliminaries

In this chapter we introduce a qualitative technique for studying a differential equation: phase diagram. Then we'll further look how to obtain general solution of plane autonomous system and observe the nature of solutions obtained by representing them in the phase plane. Lastly we study a certain geometrical aspect, called the index, of that system which is useful for establishing the structure of phase diagram.

2.1 Phase Plane

One of the useful techniques in analyzing a differential equation is through its qualitative study. The qualitative study of the differential equations involves studying the characteristics of solution without solving it. The geometrical procedure introduced helps in studying many features of the non-linear system, whose solutions are represented on the phase plane. Generally mechanical systems are initial value problems governed by ordinary differential equations or partial differential equations or difference equations. In this study we consider the non-linear systems that arise from ordinary differential equation. The geometrical procedure introduced helps in studying many features of the non-linear system, whose solutions are represented on the phase plane.

2.1.1 Introduction

Firstly the plane autonomous differential equation considered i.e.,

$$\ddot{x} = f(x, \dot{x}), \tag{2.1.1}$$

will be converted to its equivalent first-order system

$$\dot{x} = X(x, y); \qquad \dot{y} = Y(x, y), \tag{2.1.2}$$

to acquire a representation on the phase plane. This approach is useful in the mechanical interpretation of the original equation. Frequently, the fitting formulation of mechanical, biological, and geometrical problems is not through a second-order equation, but directly as a more general type of first-order system of the form (2.1.2)[14]. We construct its phase plane which represents the solution curves. Obtain the equilibrium points, make a linear approximation near them, solve the simpler equations obtained, and so determine the local structure of the system.

2.1.2 Plane Autonomous System

A general form of system of first order ordinary differential equations is:

$$\dot{x_1} = X_1(x_1, ...; x_n; t);$$
$$\vdots$$
$$\dot{x_n} = X_n(x_1, ...; x_n; t).$$

If $n = 2$ then we have a **plane system**. It will be further called **autonomous** if Xi's are independent of t. Hence we get a **plane autonomous system**, written as:

$$\dot{x_1} = X_1(x_1, x_2); \qquad \dot{x_2} = X_2(x_1, x_2). \tag{2.1.3}$$

2.1.3 Plane Autonomous Systems in Phase Plane

Assume that we are given second order ordinary differential equation in two variables (2.1.1) and write its equivalent autonomous system (2.1.2). Let $(x(t), y(t))$ be the solution of system (2.1.2), which, when traced on the plane, gives a curve in $\mathbb{R}^2$ and that is the solution curve of the system (2.1.2). The system (2.1.2) gives the tangent vector to the solution curve $(X(x(t), y(t)), Y(x(t), y(t)))$. Then as t increases the solution $(x(t), y(t))$ traces out a directed curve in that plane called **phase path**. The plane $\mathbb{R}^2$ together with the solution curves is called **phase plane** of the differential system and the diagram depicting the phase paths in the phase plane is called **phase diagram**. A typical point (x, y) is called the **state** of the system. The state of the system at a particular time t_0 consists of the pair of values $(x(t_0), y(t_0))$, which can be regarded as a pair of *initial conditions* for the original differential equation (2.1.2). To obtain a relation between x and y that defines the phase paths, we eliminate the parameter t between (2.1.2) by using the identity

$$\frac{dy}{dx} = \frac{Y(x, y)}{X(x, y)}. \tag{2.1.4}$$

A point (x_0, y_0) is called a **critical point** of (2.1.2) if $X(x_0, y_0) = Y(x_0, y_0) = 0$. The **direction** to be assigned to a phase path is obtained from the relation (2.1.2). The signs of X and Y at any particular point determine the direction through the point, and generally the directions of all other paths follows from the requirement of *continuity of direction of adjacent paths.* For example when $X > 0$, then $\dot{x} > 0$, therefore x is increasing with time, and when $X < 0$, x is decreasing with time. Therefore the direction of the solution curve is from *left to right in the upper half-plane* and from *right to left in the lower half-plane.*

2.1.4 Linear Approximation at Equilibrium Points

Qualitative behavior of nonlinear systems is obtained locally by linearization around the equilibrium points. Linearization is an important tool in phase plane analysis. We linearize the non-linear system near it's equilibrium point to classify the equilibrium points after solving these equations. This does not change the nature of local structure of the system.

Consider the system (2.1.2)

We assume that the equilibrium point to be studied has been shifted to the origin, by translation of axes if necessary, so that

$$X(0,0) = Y(0,0) = 0. \tag{2.1.5}$$

Therefore by Taylor expansion,

$$X(x,y) = ax + by + P(x,y), \quad Y(x,y) = cx + dy + Q(x,y),$$

where

$$a = \frac{\partial X}{\partial x}(0,0), \qquad b = \frac{\partial X}{\partial y}(0,0), \qquad c = \frac{\partial Y}{\partial x}(0,0), \qquad d = \frac{\partial Y}{\partial y}(0,0),$$

and $P(x,y)$, $Q(x,y)$ are of lower order of magnitude than the linear terms as (x,y) approaches the origin $(0,0)$. The linear approximation to (2.1.2) in the neighborhood of the origin is defined as the system

$$\dot{x} = ax + by, \qquad \dot{y} = cx + dy. \tag{2.1.6}$$

We expect that the solutions of (2.1.6) will be geometrically similar to those of (2.1.2) near the origin, as the main idea was to preserve the local structure of the system. The purpose is to see how the simple relations between coefficients a, b, c, d enable the classification of equilibrium points of system (2.1.6) which helps in understanding local character of the system (2.1.2), which we will see in the next section.

2.1.5 General Solution of Linear Plane Autonomous Systems

For solving the system of linear differential equations with constant coefficients (2.1.6) for $x(t)$ and $y(t)$, the general linear autonomous case is more manageable (especially for higher order systems) when the system is expressed in matrix form.

Following the procedure as in Non-linear differential equation by Jordan Smith[14]. The system (2.1.6) can be written as

$$\dot{\mathbf{x}} = \mathbf{A}\ \mathbf{x}, \quad where \quad \mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}. \tag{2.1.7}$$

Define the column vectors

$$x(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}, \quad \dot{x}(t) = \begin{pmatrix} \dot{x}(t) \\ \dot{y}(t) \end{pmatrix}.$$

We shall only consider the case where there is a single equilibrium point, at the origin, the condition for this being

$$det\mathbf{A} = ad - bc \neq 0. \tag{2.1.8}$$

As for $detA = 0$ any non-trivial solution will be another equilibrium point.

We search for a fundamental solution consisting of two linearly independent solutions of (2.1.7), having the form:

$$\mathbf{x_1}(t) = \mathbf{v_1} e^{\lambda_1 t}, \quad \mathbf{x_2}(t) = \mathbf{v_2} e^{\lambda_2 t}, \tag{2.1.9}$$

where λ_1, λ_2 are constants, and $\mathbf{v_1}, \mathbf{v_2}$ are constant vectors. The general solution is given by:

$$\mathbf{x}(t) = C\mathbf{x}_1(t) + D\mathbf{x}_2(t), \tag{2.1.10}$$

where C and D are arbitrary constants. To determine λ_1, λ_2, $\mathbf{v_1}, \mathbf{v_2}$ in (2.1.9) substitute

$$\mathbf{x}(t) = \mathbf{v}e^{\lambda}t \tag{2.1.11}$$

into the system (2.1.7). After canceling the common factor $e^{\lambda}t$, we obtain:

$$(\mathbf{A} - \lambda\mathbf{I})\mathbf{v} = 0, \tag{2.1.12}$$

where $\mathbf{I}$ is the identity matrix. Now we put

$$\mathbf{v} = \begin{pmatrix} r \\ s \end{pmatrix}, \tag{2.1.13}$$

in (2.1.12), where r and s are constants, it represents the pair of scalar equations

$$(a - \lambda)r + bs = 0, \qquad cr + (d - \lambda)s = 0, \tag{2.1.14}$$

for λ, r, s. We know from algebraic theory that eq. (2.1.12) has nonzero solutions for $\mathbf{v}$ only if the determinant of the matrix of the coefficients in eq's. (2.1.14) is zero. Therefore

$$det \begin{pmatrix} a-\lambda & b \\ c & d-\lambda \end{pmatrix} = 0, \tag{2.1.15}$$

or

$$\lambda^2 - (a+d)\lambda + (ad - bc) = 0. \tag{2.1.16}$$

This is called the **characteristic equation**, and its solutions, λ_1 and λ_2 , the **eigenvalues** of the matrix **A**, are the **characteristic exponents** for the problem. For classifying the solutions of the characteristic equation (2.1.16) we use the following notations for the purpose of our convenience:

$$\lambda^2 - p\lambda + q = 0,$$

where,

$$p = a + d, \qquad q = ad - bc. \tag{2.1.17}$$

Also put

$$\Delta = p^2 - 4q. \tag{2.1.18}$$

The eigenvalues $\lambda = \lambda_1$ and $\lambda = \lambda_2$ are given by

$$\lambda_1, \lambda_2 = \frac{1}{2}(p \pm \sqrt{\Delta}). \tag{2.1.19}$$

These are to be substituted successively into (2.1.14) to obtain corresponding values for the constants r and s.

Now we discuss two main cases classified on the basis of the sign of the discriminant Δ. These are (we shall not consider the special case when $\Delta = 0$):

Time solutions when $\Delta > 0$, $q \neq 0$;

In this case λ_1 and λ_2 are real and distinct. When $\lambda = \lambda_1$ eq. (2.1.14) for r and s becomes

$$(a - \lambda_1)r + bs = 0, \qquad cr + (d - \lambda_1)s = 0. \tag{2.1.20}$$

Since the determinant (2.1.15) is zero, its rows are linearly dependent. Therefore one of these eqs. (2.1.20) is simply a multiple of the other; effectively we have only one equation connecting r and s. Let $r = r_1$, $s = s_1$ be any (nonzero) solution of (2.1.20), and put:

$$\mathbf{v}_1 = \begin{pmatrix} r_1 \\ s_1 \end{pmatrix}.$$

This is called an **eigenvector** of **A** corresponding to the eigenvalue λ_1 . We have now obtained one of the two basic time solutions having form (2.1.9). This process is repeated for $\lambda = \lambda_2$, giving rise to another eigenvector

$$\mathbf{v}_2 = \begin{pmatrix} r_2 \\ s_2 \end{pmatrix}.$$

The general solution is then given by (2.1.10):

$$\mathbf{x}(t) = C\mathbf{v}_1 e^{\lambda_1 t} + D\mathbf{v}_2 e^{\lambda_2 t}, \tag{2.1.21}$$

in vector form, where C and D are arbitrary constants.

Time solutions when $\Delta < 0$, $q \neq 0$;

In this case λ_1 and λ_2 , obtained from (2.1.19), are complex, given by

$$\begin{aligned} \lambda_1 &= \tfrac{1}{2}p + i(\sqrt{(-\Delta)}) = \alpha + i\beta, \\ \lambda_2 &= \tfrac{1}{2}p - i(\sqrt{(-\Delta)}) = \alpha - i\beta, \end{aligned} \tag{2.1.22}$$

where $\alpha = \frac{1}{2}p$ and $\beta = \frac{1}{2}(\sqrt{(-\Delta)})$ are real numbers. Therefore λ_1 and λ_2 , are complex conjugates.

Obtain an eigenvector corresponding to λ_1 from (2.1.14),

$$\mathbf{v} = \mathbf{v}_1 = \begin{pmatrix} r_1 \\ s_1 \end{pmatrix}, \tag{2.1.23}$$

exactly as before, where r_1 and s_1 are now complex. Since a, b, c, d are all real numbers, a suitable eigenvector corresponding to $\lambda_2 (= (\overline{\lambda_1}))$ is given by taking $r_2 = (\overline{r_1}), s_2 = (\overline{s_1})$ as solutions of (2.1.14):

$$\mathbf{v}_2 = \overline{\mathbf{v}_1} = \begin{pmatrix} \overline{r_1} \\ \overline{s_1} \end{pmatrix},$$

Therefore, two basic complex time solutions following (2.1.9) are

$$\mathbf{v}e^{(\alpha+i\beta)t}, \qquad \overline{\mathbf{v}}e^{(\alpha-i\beta)t},$$

where $\mathbf{v}$ is given by (2.1.23). The general complex solution of (2.1.7) is

$$\mathbf{x}(t) = C\mathbf{v}e^{(\alpha+i\beta)t} + D\overline{\mathbf{v}}e^{(\alpha-i\beta)t}, \tag{2.1.24}$$

in vector form, where C and D are arbitrary constants which are, in general, complex.

Since we are interested only in real solutions, the expression (2.1.24) is real if and only if

$$D = \overline{C},$$

in which case the second term is the conjugate of the first term, and we obtain

$$\mathbf{x}(t) = 2Re\{C\mathbf{v}e^{(\alpha+i\beta)t}\},$$

or

$$\mathbf{x}(t) = Re\{C_1\mathbf{v}e^{(\alpha+i\beta)t}\}, \tag{2.1.25}$$

where C_1 is an arbitrary complex constant.

2.1.6 Phase Paths of Linear Plane Autonomous Systems

We now obtain the general character of the phase path of the time solutions obtained in (2.1.21) and (2.1.25) for the system

$$\dot{x} = ax + by, \qquad \dot{y} = cx + dy. \tag{2.1.26}$$

which is a linear approximation near the origin. As we said earlier the phase diagram of this linearization will generally approximate to the phase diagrams of the non-linear system.

The main purpose of this section is to make complete classification of the equilibrium points. We classify the phase diagrams into three classes depending upon their eigenvalues which are the solution to the characteristic equation. (2.1.16).

The three classes are defined as:

- λ_1, λ_2 are real, distinct having same signs;
- λ_1, λ_2 are real, distinct having opposite signs;
- λ_1, λ_2 are complex conjugates.

- **The eigenvalues are real, distinct and have same sign:**

Node: Let λ_1 be the larger eigenvalue than λ_2 i.e., $\lambda_1 > \lambda_2$ The general solution (2.1.21) in it's component form is

$$x(t) = Cr_1 e^{\lambda_1 t} + Dr_2 e^{\lambda_2 t}, \qquad y(t) = Cs_1 e^{\lambda_1 t} + Ds_2 e^{\lambda_2 t}, \tag{2.1.27}$$

where r and s are determined by solving (2.1.20) by putting in values of $\lambda' s$. From (2.1.27) we obtain

$$\frac{dy}{dx} = \frac{C\lambda_1 s_1 e^{\lambda_1 t} + D\lambda_2 s_2 e^{\lambda_2 t}}{C\lambda_1 r_1 e^{\lambda_1 t} + D\lambda_2 r_2 e^{\lambda_2 t}}. \tag{2.1.28}$$

Now suppose that $0 > \lambda_1 > \lambda_2$. From this and (2.1.27) we have

$$\begin{cases} x \quad and \quad y \to origin \quad as \quad t \to \infty & ; \\ x \quad and \quad y \to \infty \quad as \quad t \to -\infty & . \end{cases} \tag{2.1.29}$$

Also there are four radial phase paths, which lie along a pair of straight lines as

$$\begin{cases} if \quad D = 0, \quad \frac{y}{x} = \frac{s_1}{r_1} & ; \\ if \quad C = 0, \quad \frac{y}{x} = \frac{s_2}{r_2}, & . \end{cases} \tag{2.1.30}$$

Therefore from (2.1.28) we obtain

$$\begin{cases} \frac{dy}{dx} \to \frac{s_1}{r_1} \quad as \quad t \to \infty & ; \\ \frac{dy}{dx} \to \frac{s_2}{r_2} \quad as \quad t \to -\infty & . \end{cases} \tag{2.1.31}$$

(2.1.29) along with (2.1.30) shows that every phase path is tangential to $y = (\frac{s_1}{r_1})x$ at the origin, and approaches the direction of $y = (\frac{s_2}{r_2})x$ at infinity. The radial solutions (2.1.30) are called asymptotes of the family of phase paths.

Now if the eigenvalue are both positive i.e., $\lambda_1 > \lambda_2 > 0$ then the phase diagram has similar characteristics but now all the phase paths are directed outward, moving away from the origin to infinity. These phase paths have a special name called **node**. The cases when all the phase paths are directed inward to the origin i.e. when both eigenvalues are negative then we have a **stable node** (see Fig. 2.1) and vice versa for **unstable node** (see Fig. 2.2). The condition on the coefficient from (2.1.17) for these cases is given as:

$$\begin{cases} \textbf{stable} \quad \textbf{node} \quad \Delta = p^2 - 4q > 0, \quad q > 0, \quad p < 0 & ; \\ \textbf{unstable} \quad \textbf{node} \quad \Delta = p^2 - 4q > 0, \quad q > 0, \quad p > 0 & . \end{cases} \tag{2.1.32}$$

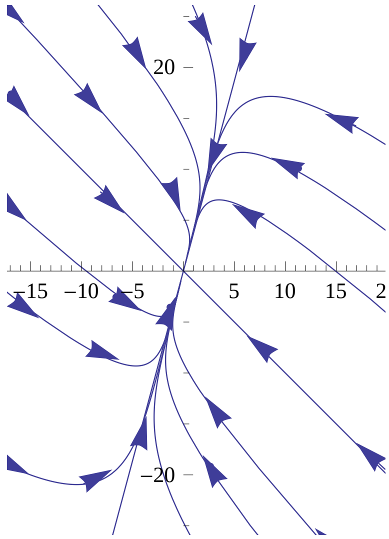

Figure 2.1: Stable Node

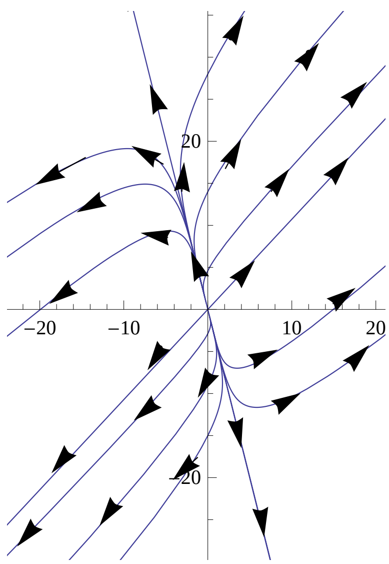

Figure 2.2: Unstable Node

- **The eigenvalues are real, distinct and have opposite sign:**

Saddle: In this case we have $\lambda_1 > 0 > \lambda_2$, the procedure and formula will be the same as in last case i.e. (2.1.27) and (2.1.28) also apply here. Also we get four radial paths, two of them along each of the lines

$$\frac{y}{x} = \frac{s_1}{r_1}, \quad and \quad \frac{y}{x} = \frac{s_2}{r_2}.$$

However there are only two phase paths that approach the origin and those are which lie along the line $\frac{y}{x} = \frac{s_2}{r_2}$. The rest of the paths go to infinity as $t \to \infty$. The fig.(2.3) is like a family of hyperbolas together with their asymptotes. In this case the equilibrium point is called **saddle**. From (2.1.17) the conditions on the coefficient becomes

$$\textbf{saddle} \qquad \Delta = p^2 - 4q > 0, \quad q < 0 \tag{2.1.33}$$

A saddle is always *unstable*.

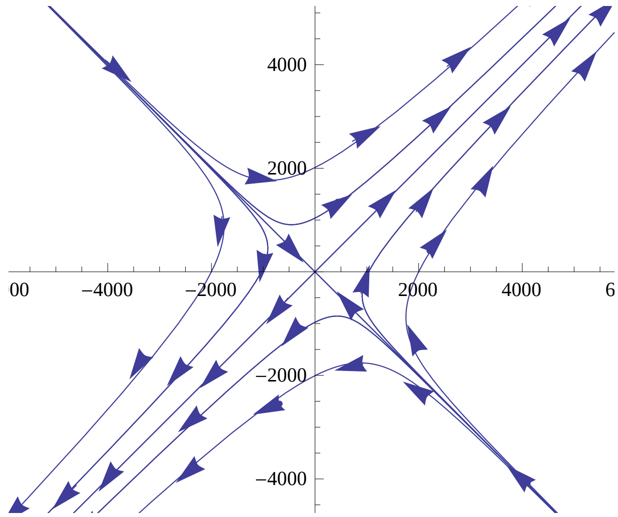

Figure 2.3: Saddle

- **The eigenvalues are complex:**

If a real matrix has complex eigenvalues then they always occur as complex conjugate pairs, so we put

$$\lambda_1 = \alpha + i\beta, \quad \lambda_2 = \alpha - i\beta, \quad (\alpha, \beta \quad \text{are} \quad \text{real}) \tag{2.1.34}$$

After separating real parts in (2.1.25), we get

$$x(t) = e^{\alpha t} Re\{C_1 r_1 e^{i\beta t}\}, \quad y(t) = e^{\alpha t} Re\{C_1 s_1 e^{i\beta t}\}, \tag{2.1.35}$$

where C_1, r_1, s_1 are in general complex.

Center: Let $\alpha = 0$. Put C_1, r_1, s_1 in polar form

$$C_1 = |C_1| e^{i\gamma}, \quad r_1 = |r_1| e^{i\sigma}, \quad s_1 = |s_1| e^{i\rho},$$

where γ, σ and ρ are real.
Then (2.1.35) implies

$$x(t) = |C_1||r_1|\cos(\beta t + \gamma + \rho), \quad y(t) = |C_1||s_1|\cos(\beta t + \gamma + \sigma). \tag{2.1.36}$$

The motion of the representative point $(x(t), y(t))$ in the phase plane consists of two simple harmonic components of equal circular frequency β, in the x and y directions, but they have different phase and amplitude. The phase paths therefore form a family of geometrically similar ellipses. The geometrical figure corresponding to this case is called **center** (see Fig. 2.4). The corresponding conditions on coefficients (2.1.17) becomes

$$\textbf{center} \qquad p = 0, \quad q > 0. \tag{2.1.37}$$

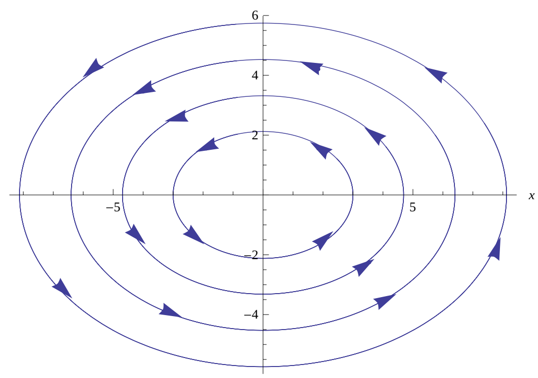

Figure 2.4: Center

Spirals: Now suppose that $\alpha \neq 0$. As t increases in eqns. (2.1.35), the elliptical paths above are modified by the factor $e^{\alpha t}$. This prevents them from closing, and each ellipse turns into a **spiral**; a **contracting spiral** if $\alpha < 0$, and an **expanding spiral** if $\alpha > 0$ (see Fig. 2.5 & Fig.2.6 respectively). The equilibrium point is then called a **spiral** or **focus**, stable if $\alpha < 0$, unstable if $\alpha > 0$. The directions may be clockwise or counterclockwise.

The conditions on coefficient (2.1.17) are

$$\begin{cases} \textbf{stable} \quad \textbf{spiral} \quad \Delta = p^2 - 4q < 0, \quad q > 0, \quad p < 0, \quad ; \\ \textbf{unstable} \quad \textbf{spiral} \quad \Delta = p^2 - 4q < 0, \quad q > 0, \quad p > 0, \quad . \end{cases} \tag{2.1.38}$$

If there exists a neighborhood of an equilibrium point such that every phase path starting in the neighborhood ultimately approaches the equilibrium point, the point is known as an **attractor**. (The term is used both for linear and nonlinear systems.) The stable node and stable spiral are attractors. An attractor with all path directions reversed is a **repeller**. Unstable nodes and unstable spirals are repellers, but a saddle point is not.

If the eigenvalues of the linearized equation have nonzero real parts then the equilibrium point is said to be **hyperbolic**. At hyperbolic points the phase diagrams of the nonlinear equations and the

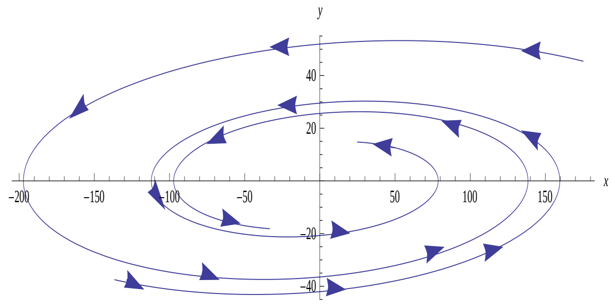

Figure 2.5: Stable Spiral

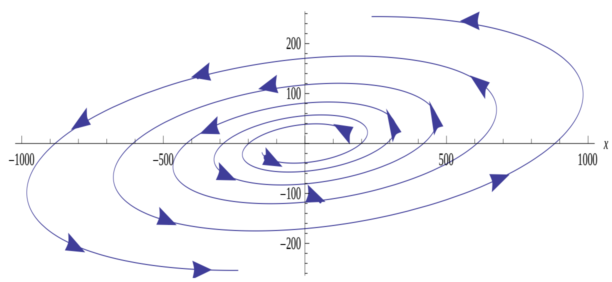

Figure 2.6: Unstable Spiral

linearized equations are, locally, qualitatively the same. Spirals, nodes, and saddles are hyperbolic but the center is not. We summarize all the cases in the following table:

	$\Delta = p^2 - 4q$	$p = a + d$	$q = ad - bc$
Saddle	$\Delta > 0$	-	$q < 0$
Stable node	$\Delta > 0$	$p < 0$	$q > 0$
Stable spiral	$\Delta < 0$	$p < 0$	$q > 0$
Unstable node	$\Delta > 0$	$p > 0$	$q > 0$
Unstable spiral	$\Delta < 0$	$p > 0$	$q > 0$
Center	$\Delta < 0$	$p = 0$	$q > 0$

Example 2.1.1. Obtain the time solutions and sketch the phase diagram of the system

$$\dot{x} = x + 2y, \qquad \dot{y} = 2x + y.$$

Solution: The matrix of coefficient of this system is

$$A = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}.$$

To locate the eigenvalues put

$$|A - \lambda I| = 0.$$

i.e.,

$$det \begin{pmatrix} 1-\lambda & 2 \\ 2 & 1-\lambda \end{pmatrix} = 0.$$

Solving the characteristic equation obtained for the values of λ, we get

$$\lambda_1 = -1 \qquad \& \quad \lambda_2 = 3.$$

We observe that the values of λ have opposite signs which satisfies for the condition of a saddle. Hence origin is a saddle for this system. Now to search the time solution we locate the eigenvectors from (2.1.12). We first solve for $\lambda = \lambda_1 = -1$ and get

$$(1-\lambda)r + 2s = 0, \qquad 2r + (1-\lambda)s = 0.$$

Solving this eq. for $r = r_1$ & $s = s_1$, we get only one eigenvector

$$2r_1 + 2s_1 = 0.$$

or

$$s_1 = -r_1.$$

Similarly solving for $\lambda = \lambda_2 = 3$, we get another eigenvector

$$r_2 = s_2.$$

Hence we arrive at two time solutions, from (2.1.27) we get

$$x(t) = Cr_1e^{-t} + Dr_2e^{3t}, \qquad y(t) = Cs_1e^{-t} + Ds_2e^{3t}.$$

Our general solution will be like

$$\mathbf{x}(t) = Ce^{-t}\begin{pmatrix} -1 \\ 1 \end{pmatrix} + De^{3t}\begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

From this we get

$$\frac{dy}{dx} = \frac{-Cs_1e^{-t} + 3Ds_2e^{3t}}{-Cr_1e^{-t} + 3Dr_2e^{3t}}.$$

Now when

$$C = 0, \qquad \frac{dy}{dx} = \frac{s_2}{r_2} = -1,$$

and

$$D = 0, \qquad \frac{dy}{dx} = \frac{s_1}{r_1} = 1.$$

We will start by sketching lines that follow the direction of the two eigenvectors. If we have $D = 0$

then the solution is an exponential times a vector. In other words, the trajectory in this case will be a straight line that is parallel to the vector $\frac{s_1}{r_1}$. Also notice that as t increases the exponential will get smaller and smaller and hence the trajectory will be moving in towards the origin. If $C > 0$ the trajectory will be in Quadrant II and if $C < 0$ the trajectory will be in Quadrant IV.

So in the Fig. (2.7) the trajectory that approach the origin as t increases is a sketch of the trajectory corresponding to $D = 0$ .If we now look at the solution corresponding to $C = 0$ we will have a trajectory that is parallel to $\frac{s_2}{r_2}$. Also, since the exponential will increase as t increases and so in this case the trajectory will now move away from the origin as t increases.

Eigenvalues that are negative will correspond to solutions that will move towards the origin as t increases in a direction that is parallel to its eigenvector. Likewise, eigenvalues that are positive move away from the origin as t increases in a direction that will be parallel to its eigenvector.

For large negative t the solution will be dominated by the portion that has the negative eigenvalue since in these cases the exponent will be large and positive. Trajectories for large negative t will be parallel to $\frac{s_1}{r_1}$ moving in the same direction.

Solutions for large positive t will be dominated by the portion with the positive eigenvalue. Trajectories in this case will be parallel to $\frac{s_2}{r_2}$ and moving in the same direction.

In general, the trajectories will start near $\frac{s_1}{r_1}$, move in towards the origin and as they approach the origin they will start moving towards $\frac{s_2}{r_2}$ and then continue up along this vector. Sketching some of these in will give the phase portrait of the system. Hence our phase diagram would be like Fig. (2.7).

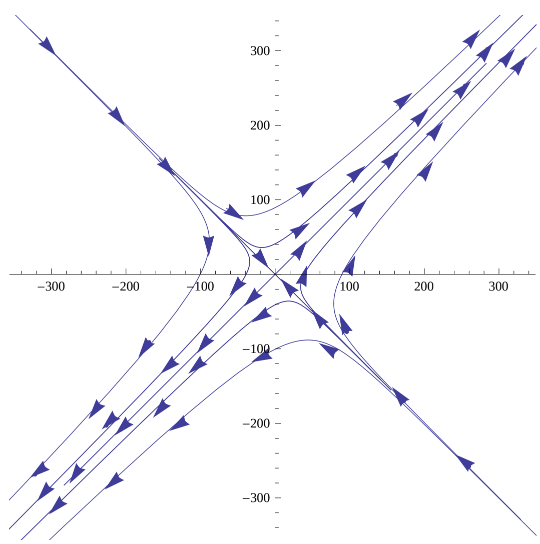

Figure 2.7: Saddle

Example 2.1.2. Classify the equilibrium points of the system

$$\dot{x} = -6x + 5y, \qquad \dot{y} = -5x + 2y.$$

Solution: We don't have to write the time solutions for this system so we will only check the conditions on the coefficients of this system.

Here we have $a = -6, \quad b = 5, \quad c = -5, \quad d = 2$. Therefore

$$p = a + d = -6 + 2 = 4 < 0,$$

$$q = ad - bc = -12 + 25 = 13 > 0,$$

so that

$$\Delta = p^2 - 4q = 16 - 52 = -36 < 0,$$

which are the conditions for a stable spiral (2.1.38). Hence origin is the stable spiral for this system.

2.2 Index in a Phase Diagram

In this section we describe a very important geometrical aspect of plane autonomous system : index of a point. It is helpful in establishing the phase diagram of autonomous systems. The index of an equilibrium point provides information about that system on the nature and behavior of the phase paths of the system which in turn helps in establishing the structure of the solution.

2.2.1 Index of a Point

For a given system (2.1.2), let Γ be any smooth, closed curve which does not pass through any equilibrium points of the system. Then at each point on the loop, say P, there is only one phase path through that point. The phase paths belong to the family described by the equation (2.1.4). The vector tangent to the phase path is $S = (X, Y)$ through P and points in the direction of increasing t and it's inclination can be measured by the angle ϕ measured from the x-axis in counter clockwise direction to S by

$$\tan\phi = \frac{Y}{X}.$$

Now as you move around Γ in the counter-clockwise sense (call this the positive direction), the vectors on Γ rotate, and when you get back to the point at which you started, they will have rotated through an angle $2\pi k_\Gamma$, where k_Γ is some integer. This integer, k_Γ, is called the **index** of Γ. The index of a closed curve containing no equilibrium points can be calculated by integrating $d\phi$ i.e change in angle of vectors at each point on Γ around Γ. So

$$\begin{aligned} k_\Gamma &= \frac{1}{2\pi}\oint_\Gamma d\phi \\ &= \frac{1}{2\pi}\oint_\Gamma d\Big(\tan^{-1}\frac{Y}{X}\Big) \\ &= \frac{1}{2\pi}\oint_\Gamma \frac{XY' - YX'}{X^2 + Y^2} \end{aligned} \tag{2.2.1}$$

where X' denotes the derivative of X w.r.t some parameter s. The index of an equilibrium point is defined to be the index of a closed curve which contains only this one equilibrium point, and where no equilibrium points are on the closed curve.

Example 2.2.1. Find the index for the system $\dot{x} = 2xy, \quad \dot{y} = x^2 - y^2$ where Γ is a unit circle centered at the origin.

Solution: Here we have $X(x, y) = 2xy$ & $Y(x, y) = x^2 - y^2$. We define Γ by $x = \cos\theta$ & $y = \sin\theta$, so $(X, Y) = (\sin 2\theta, \cos 2\theta)$,

$$k_\Gamma = \frac{1}{2\pi}\int_0^{2\pi} \frac{XY' - YX'}{X^2 + Y^2} d\theta,$$

$$k_\Gamma = \frac{1}{2\pi}\int_0^{2\pi} \frac{-(\sin^2 2\theta + \cos^2 2\theta)}{\sin^2 2\theta + \cos^2 2\theta} d\theta,$$

$$k_\Gamma = \frac{1}{2\pi}\int_0^{2\pi} -1 \; d\theta,$$

$$k_\Gamma = -1.$$

So the index of the given system is -1 of the origin.

We now present some useful theorems for computing index.

Theorem 2.2.2. *Suppose that Γ lies in a simply connected region on which X, Y and their first derivatives are continuous and X and Y are not simultaneously zero. In other words there is no equilibrium point there. Then k_Γ is zero.*

Proof. Green's theorem in the plane states that if Γ is a closed, non-self-intersecting curve, lying in a simply connected region on which the functions $f(x, y)$ and $g(x, y)$ have continuous first partial derivatives, then

$$\oint_\Gamma (f dx + g dy) = \int\int_{D_\Gamma} \Big(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\Big) dx \; dy,$$

where D_Γ is the region interior to Γ. (The first integral is a line integral round Γ, the second a double integral taken over its interior). In (2.2.1) replace

$$X' = X_x\frac{dx}{ds} + X_y\frac{dy}{ds}, \quad Y' = Y_x\frac{dx}{ds} + Y_y\frac{dy}{ds},$$

where $X_x = \frac{\partial X}{\partial x}$ and so on. Then (2.2.1) implies

$$k_\Gamma = \frac{1}{2\pi}\oint_\Gamma \Big(\frac{XY_x - YX_x}{X^2 + Y^2} dx + \frac{XY_y - YX_y}{X^2 + Y^2} dy\Big).$$

The functions $f = \frac{XY_x - YX_x}{X^2+Y^2}$ & $g = \frac{XY_y - YX_y}{X^2+Y^2}$ satisfy the condition for Greens' theorem since $X^2 + Y^2 \neq 0$ on Γ and it's interior. So

$$k_\Gamma = \frac{1}{2\pi}\int\int_{D_\Gamma} \Big[\frac{\partial}{\partial x}\Big(\frac{XY_y - YX_y}{X^2 + Y^2}\Big) - \frac{\partial}{\partial y}\Big(\frac{XY_x - YX_x}{X^2 + Y^2}\Big)\Big] dx \; dy.$$

After solving the partial derivatives we get $k_\Gamma = 0$. □

The immediate consequence of this theorem is the following corollary.

Corollary 2.2.3. *Let Γ be a simple closed curve, and Γ' a simple closed curve inside Γ. Then if, on Γ, Γ' and the region between them there is no equilibrium point, and if X, Y and their first derivatives are continuous there, then $k_{\Gamma} = k_{\Gamma'}$.*

This theorem shows that the index of a surface is independent of its vector field and that any loop Γ around that point has the same index.

Theorem 2.2.4. *If Γ surrounds n equilibrium points $P_1, P_2, ..., P_n$ then*

$$k_{\Gamma} = \sum_{i=1}^{n} k_{i}$$

where k_{i} is the index of P_i.

Another simple way to calculate the index is through the following result.

Theorem 2.2.5. *Let p be the number of times $Y(x, y) / X(x, y)$ changes from $+\infty$ to $-\infty$, and q the number of times it changes from $-\infty$ to $+\infty$, on Γ. Then $k_{\Gamma} = \frac{1}{2}(p - q)$.*

If we know the nature of the equilibrium point beforehand then the index is found by simply drawing a figure and following the angle round. The indices of the types met in previous section are:

- A saddle point: The change in ϕ in a single circuit of the curve Γ surrounding the saddle point is -2π, and the index is therefore -1.
- A center: Γ can be chosen to be a phase path, so that $k_{\Gamma} = +1$ irrespective of the direction of the paths.
- A spiral (stable or unstable): The index is $+1$.
- A node (stable or unstable): The index is $+1$.

Chapter 3

Periodic Solutions in Phase Plane

We have so far concentrated on finding the critical points for investigating non-linear systems in the xy-plane, and studied how the trajectories of the system look in the neighborhood of each critical point. This gives us some feeling for how the other trajectories can behave, at least those which pass near enough to critical points. But this much is true in study of local behavior of such systems, for global behavior of these systems periodic solutions, sometimes also called periodic orbits, are of special interest. Periodic orbits are the only type of orbits which can provide us with the understanding of the solution through its period. For this reason it is very tempting and advantageous to understand the features of non-linear systems in terms of periodic orbits. In this chapter we will discuss a special type of periodic orbit in phase plane i.e. 'Limit cycle'. Secondly an important question arises about the existence and non-existence of such orbits, so some results regarding this question will be presented in separate sections.

3.1 Limit Cycles

Periodic solutions are an important aspect of differential equations, since many physical phenomena occur roughly periodically. In phase portrait a periodic solution is represented by a closed path. In last chapter we have meet such paths in the case of pure imaginary eigenvalues. If a closed path exists for a system then it gives a good analysis of the system since the nearby paths behave in the same manner. For a given system on the plane if we know the number of critical points, the structures near these points and further know the number of closed orbits with their direction then, generally speaking, we have a good understanding of global behavior of the system. Here we are interested in a more interesting and special type of a close path which holds two properties

1. *closed*, indicating periodicity.
2. *isolated*, indicating that nearby trajectories either converge to or diverge from it.

This special type of closed path is called **"Limit cycle"**. The word 'isolated' means there are no other such paths in its immediate neighborhood. Limit cycle is a trajectory for which energy of the system would be constant over a cycle i.e. on average there is no loss or gain of energy. Limit cycle is an outcome of delicate energy balance due to presence of nonlinear term in the equation of motion [15]. Many problems in different branches of science, such as aerodynamics, biology, mechanics, chemistry etc. are modeled by systems that produce limit cycles [16–20]. The neighboring trajectories of the limit cycle either approach it from the either side or move away from it from any side. The first refers to *stable* limit cycle and the later refers to *unstable* limit cycle, out of which the most important one is the stable limit cycle. Periodic processes in nature can often be modeled as stable limit cycles eg. breathing process. When we increase our breathing rate artificially eventually we revert back to our original breathing rate similarly is the case when we hold our breath. We see that when the system is disturbed from the original path the resulting new paths are attracted back to the original path from the either side. Here our original breathing rate is the original path, our faster breathing rate constitutes the path that approach that limit cycle from outside and our slower breathing rate constitutes the path that approach the limit cycle from the inside. Further examples include blinking of the eyes, movement of pendulum clock, pumping of heart etc. Hence greater interest lies in finding these trajectories if they exist. Therefore we are interested in two aspects in the study of limit cycles: existence and non existence. We will present some of the famous results in this regard.

3.2 Non-existence of Periodic Solutions

We will first deal with the non-existence theory of limit cycles. One can not actually solve the non-linear system which contains the limit cycle, it is important to establish, by some results, if a limit cycle is there in the system or not. In this section we give negative criterions on the non-existence of such paths. However, the condition is valid for all closed paths whether isolated (limit cycle) or not.

In the following section we introduce a result due to Bendixson and is called Bendixson's negative criterion.

3.2.1 Bendixson's Negative Criterion

Theorem 3.2.1. *(Bendixson's Negative Criterion): Suppose* Ω *is simply connected region of the phase plane and in this region we define* $\nabla \boldsymbol{f} = \dfrac{\partial X}{\partial x} + \dfrac{\partial Y}{\partial y}$. *If* $\nabla \boldsymbol{f}$ *is not identically zero and does not changes sign in* Ω *, then* Ω *contains no close path for the nonlinear system (2.1.2), where* $(x, y) \in \mathbb{R}^2$ *and* X *&* Y *are at least* $\mathcal{C}^1$*.*

Proof. This is a simple result of Green's theorem on the plane[14].

On the contrary we suppose that there exist a close path D in the region Ω, where $\nabla \mathbf{f}$ is of one sign. Then using divergence theorem

$$\int\int_R \Big(\frac{\partial X}{\partial x} + \frac{\partial Y}{\partial y}\Big)\, dxdy = \int_D \mathbf{f}.\mathbf{n}\; ds = 0,$$

where R is interior of D, $\mathbf{n}$ is the unit outward normal and ds is the line element of D. Since on the surface D, $\mathbf{f}$ is perpendicular to $\mathbf{n}$ so their dot product is zero so the right hand side of the integral turns zero. But the integral on the left is zero only if $\nabla\mathbf{f} = 0$ in R or if it changes sign in R, which is contrary to the assumption that $\nabla\mathbf{f}$ is of one sign. Hence D can not be a close path implying there are no close paths in Ω □

Example 3.2.2. Find out if the system $\dot{x} = x^3 + y^3, \quad \dot{y} = 3x + y^3 + 2y$ has any limit cycles.

Solution: We'll first calculate the divergence of this system.

$$\nabla\mathbf{f} = \frac{\partial X}{\partial x} + \frac{\partial Y}{\partial y} = 3x^2 + 3y^2 + 2,$$

where

$$X = x^3 + y^3, \quad Y = 3x + y^3 + 2y.$$

We observe that the divergence of this system is always positive in the xy-plane. Hence, using Bendixon's criterion, this system can not have any close path implying that there can't be any limit cycles either.

An extension of this theorem is called 'Dulac's test' which states:

Theorem 3.2.3. *(Dulac's Test): For the system $\dot{x} = X(x,y), \dot{y} = Y(x,y)$, there are no closed paths in a simply connected region in which $\frac{\partial(\rho X)}{\partial x} + \frac{\partial(\rho Y)}{\partial y}$ is of one sign, where $\rho(x,y)$ is any function having continuous first partial derivatives.*

Proof. The proof is on the same lines as Bendixon's negative crieterion. The results obtained for $\nabla\mathbf{f}$ will be the same for $\nabla(\rho.\mathbf{f})$, since $\rho(x,y)$ has continuous first partial derivatives. □

Now we observe an interesting fact through the following example.

Example 3.2.4. Check if the system $\dot{x} = x^2 + y^2 + 1, \quad \dot{y} = x^2 - y^2$ has any limit cycles.

Solution: Using Bendixon's criterion we'll first calculate the divergence of this system.

$$\nabla\mathbf{f} = \frac{\partial X}{\partial x} + \frac{\partial Y}{\partial y} = 2x - 2y,$$

where

$$X = x^2 + y^2 + 1, \quad Y = x^2 - y^2.$$

We observe that the divergence of this system is zero along the line $x = y$ in the xy-plane. Which doesn't satisfy the Bendixon's criterion, so we can conclude that is there can be close paths in xy-plane.

But if a close curve is to contain a portion of the line $x = y$ drawn in such a way that it is in the first quadrant not containing the origin, then the divergence won't be zero there nor it will change signs. So it doesn't violates the Bendixon's criterion implying there can be no close paths. We could not have a limit cycle there. But in the same region we just concluded there is no assurity that there are no close paths. So we don't arrive at any definite conclusion. The reason is that we are using the wrong criterion. For this reason we introduce a new criterion, called the "critical point criterion".

3.2.2 Critical Point Criterion

Stating it first in a positive sense, we say that a simply connected region D containing a close path C of a system contains a critical point of that system. Since while studying index of close path we inferred that a close path must have a critical point some where and also that the close path (center) has index 1. So by using theorem (2.2.4) of the same section we conclude that if C is a limit cycle then the sum of the indices of the equilibrium points enclosed by C is 1. To summarize it in a contrapositive way we say:

Theorem 3.2.5. *(Critical Point Criterion): If a simply connected region has no critical point then it has no close path.*

This result implies for any close path and it provides the **negative criterion** where such a path can not exist and is the necessary condition. For example, a closed path cannot surround a region containing no equilibrium points, nor one containing only a saddle (since it has an index of -1). If the sum of the indices of a group of equilibrium points does equal unity, the result does not allow us to infer the existence of a close path surrounding them.
So now solving the Example (3.2.4) using this criterion we see that the system has no equilibrium point so there can't be any close path surrounding it.

We now turn to the opposite task : finding methods to establish the existence of periodic orbits of a certain system.

3.3 Existence of Periodic Solutions

Up till now we have been concerned with the cases showing negative results about the existence and non-existence of limit cycles but we have no tests giving sufficient conditions for their existence. In particular we lack theorems and methods for proving positively the existence or non-existence of limit cycles. So in this section we provide some famous results proving the existence of such paths for certain types of equation.

3.3.1 Poincar$\acute{e}$ Bendixson Theorem

The main tool which is used to show if a plane autonomous system (2.1.2) has a limit cycle is the *Poinca're Bendixson theorem*, which describes the ultimate behavior on $t \to \infty$ of a phase path which enters and remains in a closed bounded region[14].

Theorem 3.3.1. *(Poincaré Bendixson theorem): Let D be a closed, bounded region consisting of non-singular points of a plane system (2.1.2) such that some positive half-path H of the system lies entirely within D. Then either:*

1. *H is itself a closed path on D,*

2. *H approaches a closed path on D,*

3. *H approaches an equilibrium point on D.*

In particular, we consider two simple closed curves c_1 & c_2 and R is the region between these curves where c_2 is interior to c_1. If the solution which enters R and can never leave R, the only thing it can do, as $t \to \infty$, is either approach a critical point, which are none by hypothesis, or spiral in towards a closed trajectory. Thus there is a closed trajectory inside R. Also the closed path must enclose the inner curve c_2, for the index of a closed path is 1, and since R contains no equilibrium points so there has to be some in the interior of c_2. Hence the theorem implies that if D contains no equilibrium points, and some half-path H remains in D, then D must contain at least one periodic solution.

Example 3.3.2. Show that the system

$$\dot{x} = 2x + 2y - x\,(2x^2 + y^2), \qquad \dot{y} = y - 2x - y\,(2x^2 + y^2) \tag{3.3.1}$$

has a periodic solution.

Solution: The origin is the equilibrium point for this system. Using the Poincaré Bendixson theorem we'll search for two circles centered at the origin satisfying the required properties.
Consider a circle of radius r centered at origin. Let P be any point on the circle and $\mathbf{n} = (x, y)$ is a normal, pointing outward at P from the circle, and $\mathbf{X} = (X, Y)$ is in the direction of the path through P. We will calculate $\mathbf{n}.\mathbf{X}$. Also $cos\phi = \mathbf{n}.\mathbf{X}/|\mathbf{n}||\mathbf{X}|$, therefore $\mathbf{n}.\mathbf{X}$ is positive or negative according to whether X is pointing away from, or towards, the interior of the circle. So when $\mathbf{n}.\mathbf{X}$ is positive it will indicate the paths are moving away from the circle and vice versa where $\mathbf{n}.\mathbf{X}$ is negative. We have

$$\begin{aligned}
\mathbf{n}.\mathbf{X} &= xX + yY \\
&= 2x^2 + y^2 - (x^2 + y^2)(2x^2 + y^2). \\
&= (2x^2 + y^2)(1 - (x^2 + y^2)). \\
&= r^2(2\cos^2\theta + \sin^2\theta)(1 - r^2). \\
&= \frac{r^2}{2}(\cos 2\theta + 3)(1 - r^2). \\
&= \frac{r^2}{2}(\cos 2\theta + 3) - \frac{r^4}{2}(\cos 2\theta + 3).
\end{aligned}$$

We observe that when $r = 1/2$ then $\mathbf{n}.\mathbf{X}$ is positive, so all the paths on this circle are directed outwards. Similarly when we take $r = 2$ then $\mathbf{n}.\mathbf{X}$ is negative, meaning all the paths on this circle are directed inwards. So these are the two curves in which a periodic solution is said to exist containing no equilibrium point according to Poinca're Bendixson theorem (see Fig. 3.1).

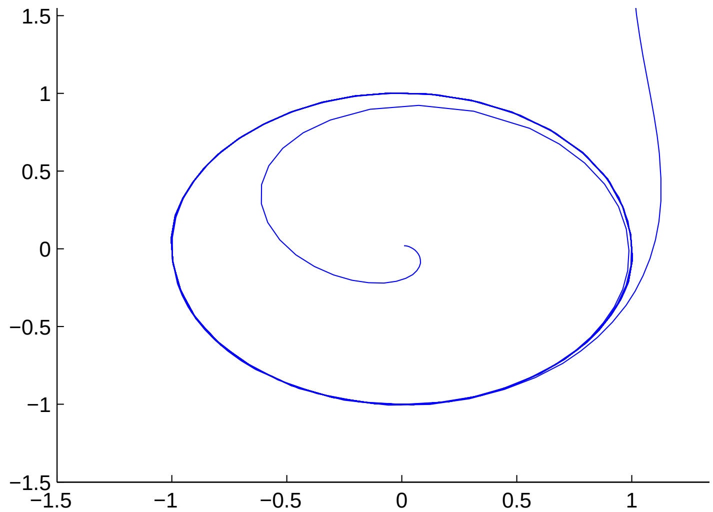

Figure 3.1: Limit Cycle for system (3.3.1)

3.3.2 Theorems on Existence of Limit Cycles

Now we introduce another useful theorem on the existence of closed paths which cover certain types of differential equation i.e., *Liénard equation.*

Theorem 3.3.3. *The differential equation*

$$\ddot{x} + f(x, \dot{x})\dot{x} + g(x) = 0, \tag{3.3.2}$$

the Liénard equation, or the equivalent system

$$\dot{x} = y, \qquad \dot{y} = -f(x, y)\, y - g(x),$$

has a unique periodic solution if f *and* g *are continuous, and:*

1. $\exists \quad r > 0$ *such that* $f(x, y) > 0$ *whenever* $x^2 + y^2 > r^2$*;*

2. $f(0, 0) < 0$ *(hence* $f(x, y) < 0$ *in a neighborhood of the origin);*

3. $g(0) = 0$*,* $g(x) > 0$ *when* $x > 0$*, and* $g(x) < 0$ *when* $x < 0$*;*

4. $G(x) = \int_0^x g(u)du \to \infty \quad as \quad x \to \infty.$

Proof. The idea to prove this theorem is to consider some closed contours in the region of our interest and study the behavior of the paths on and between them and in the end use Poincaré Bendixson theorem to deduce that there exists a periodic solution in the region bounded by these contours.

Consider the function

$$\varepsilon(x, y) = \frac{1}{2}\, y^2 + G(x) \tag{3.3.3}$$

Note that (by *4.*) $G(0) = 0$, $G(x) > 0$ when $x \neq 0$ and $G(x)$ is continuous and monotonically increasing to infinity. So $\varepsilon(0,0) = 0$, and $\varepsilon(x,y) > 0$ for $x \neq 0$ and $y \neq 0$ which implies $\varepsilon(x,y)$ is positive definite. Our function $\varepsilon(x,y)$ is continuous and is increasing (monotonically) from the origin in every direction. Therefore $\varepsilon(x,y)$ constitutes a family of contours increasing in every direction from the origin. Let

$$\varepsilon(x, y) = c \tag{3.3.4}$$

where $c > 0$ is a parameter and represents a family of simple closed curves encircling the origin. So (by *4.*) as $c \to 0$, these contours approach the origin and as $c \to \infty$, these contours also approach infinity.
Now we choose $c = c_1$ where c_1 lies entirely in the neighborhood of the origin where condition *2* of the theorem holds i.e $f(x,y) < 0$. We now examine a half-path $\mathcal{D}$ that starts at some point on c_1. Consider $\dot{\varepsilon}(x,y)$ on $\mathcal{D}$:

$$\begin{aligned} \dot{\varepsilon}(x, y) &= y\, \dot{y} + g(x)\, \dot{x}, \\ &= g(x)\, y + y\, (-f(x,y)\, y - g(x)), \\ &= -y^2\, f(x,y). \end{aligned} \tag{3.3.5}$$

Since (by *2.*) $f(x,y) < 0$ in this region so $\dot{\varepsilon}$ is positive, except for the point $y = 0$, which means all the half-paths on c_1, will leave c_1 in outward direction. We choose $\mathcal{D}$ to start on any point other than $y = 0$ on c_1. So $\mathcal{D}$ has to move in outward drection since if doesn't it will cross some contours inside c_1, which is not possible, as by (3.3.5) $\dot{\varepsilon}$ is positive.

Consider now $c = c_2$, where c_2 is large enough so that by *1.* $f(x,y) > 0$ on c_2. By (3.3.5) for $f(x,y) > 0$ we have $\dot{\varepsilon} < 0$, except for the point $y = 0$, so all half-paths on c_2 will move in the inward direction of c_2. So no half-path, once inside c_2, can escape. Hence $\mathcal{D}$ remains trapped in the region between c_1 and c_2. Therefore by theorem (3.3.1) there exists a periodic solution in the region bounded by c_1 and c_2. □

Example 3.3.4. Show that the equation $\ddot{x} + \beta(x^2 + \dot{x}^2 - 1)\dot{x} + x^3 = 0$, $(\beta > 0)$ has at least one periodic solution.

Solution: The given equation is the special case of the Liénard equation so the theorem (3.3.3) can be applied to check if it has a periodic solution or not. The equivalent system for this equation is

$$\dot{x} = y, \quad \dot{y} = \beta\, (x^2 + y^2 - 1)\, y - x^3.$$

So

$$f(x,y) = \beta\,(x^2 + y^2 - 1), \qquad g(x) = x^3,$$

which are continuous. Now checking the condition for the theorem (3.3.3):

1. for r = 1, $x^2 + y^2 > 1$ then $f(x,y) > 0$ since given $\beta > 0$.

2. for $f(x,y) = \beta\,(x^2 + y^2 - 1)$, $f(0,0) = -1 < 0$.

3. for $g(x) = x^3$, g(0)=0,$g(x) > 0$ for $x > 0$ & $g(x) < 0$ for $x < 0$.

4. $G(x) = \int_0^x g(u)\, du = \frac{x^3}{3}$ which $\to \infty$ as $x \to \infty$.

Hence all conditions of theorem (3.3.3) are satisfied so the given system has at least one periodic solution (see Fig. 3.2).

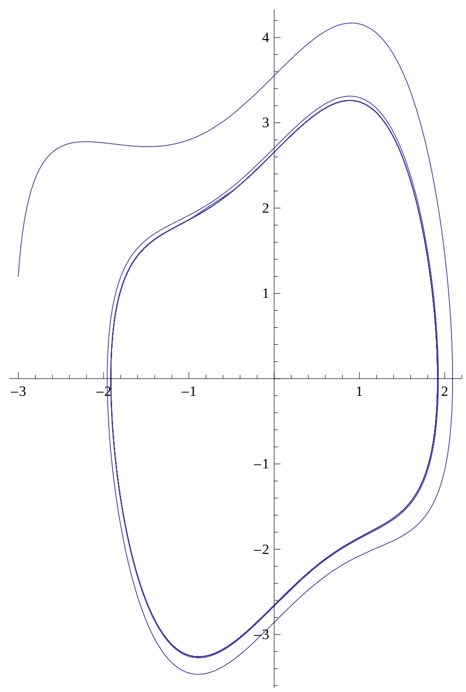

Figure 3.2: Limit Cycle for the system (3.3.8)

We have another such theorem which provides us with the indication of presence of limit cycles for the system

$$\ddot{x} + f(x)\dot{x} + g(x) = 0, \tag{3.3.6}$$

which is carried out on a different phase plane than the last one:

$$\dot{x} = y - F(x), \quad \dot{y} = -g(x), \tag{3.3.7}$$

called the *Liénard plane*, where

$$F(x) = \int_0^x f(u)\, du.$$

Now all the conditions rest on F rather than f.

Theorem 3.3.5. *The equation (3.3.6) or it's equivalent system (3.3.7) has a unique periodic solution if f and g are continuous, and*

1. *$F(x)$ is an odd function;*
2. *$F(x)$ is zero only at $x = 0$, $x = r$, $x = -r$, for some $r > 0$;*
3. *$F(x) \to \infty$ as $x \to \infty$ monotonically for $x > r$;*
4. *$g(x)$ is an odd function, and $g(x) > 0$ for $x > 0$.*

Example 3.3.6. Show that

$$\ddot{x} + \beta(x^2 - 1)\dot{x} + x^3 = 0 \tag{3.3.8}$$

has exactly one periodic solution.

Solution: Here we observe that the theorem (3.3.5) can be used to check if the given equation has a periodic solution or not. The equivalent system for the given equation is

$$\dot{x} = y - \beta\,(\frac{x^3}{3} - x), \quad \dot{y} = -x^3.$$

So $f(x) = \beta(x^2 - 1)$ and $g(x) = x^3$ are continuous. Also

$$F(x) = \int_0^x \beta(u^2 - 1)\,du$$

$$= \beta\,(\frac{x^3}{3} - x).$$

Now checking the condition for the theorem (3.3.5):

1. $F(-x) = -F(x)$ for $F(x) = \beta\,(\frac{x^3}{3} - x)$. Hence $F(x)$ is odd.
2. F(0) = 0. Also for $r = \sqrt{3}$, $F(\sqrt{3}) = 0 = F(-\sqrt{3})$.
3. For $r = \sqrt{3}$ & $x > \sqrt{3}$ $\quad F(x) \to \infty$ as $x \to \infty$.
4. $g(x) = x^3$ is odd & $g(x) > 0$ for $x > 0$.

Since all the conditions of the theorem (3.3.5) are satisfied hence the given equation has a periodic solution.

Chapter 4

Approximate Methods for Finding Limit Cycles

Here in this chapter we will present and discuss two results related to finding limit cycles of the Liénard equation. One result is due to Hector Giacomini & Sébastien Neukirch [9] and other is due to B. Delamotte [10]. We will first separately discuss these methods and then we will compare the results obtained to see which method gives better approximation and fast convergence to the results.

4.1 Number of Limit Cycles of Liénard Equation

In this section we will study a Liénard system of the form $\dot{x} = y - F(x), \dot{y} = -x$, where F(x) is an odd polynomial. Hector Giacomini & Sébastien Neukirch introduced a method that gives a sequence of algebraic approximations to the equation of each limit cycle of the system [9]. This sequence approaches the exact equation of each limit cycle and also, a sequence of polynomials $R_n(x)$ is obtained whose roots of odd multiplicity are related to the number and location of the limit cycles of the system.

We are interested in finding number of limit cycles of a two-dimensional autonomous system of the form:

$$\dot{x} = X(x, y), \qquad \dot{y} = Y(x, y). \tag{4.1.1}$$

In 1928 Liénard presented a criterion for the uniqueness of periodic solutions for an equation [9]:

$$\ddot{x} + f(x)\dot{x} + x = 0. \tag{4.1.2}$$

He transformed this equation into a first-order system by replacing $\dot{x} = z$ to get,

$$\dot{x} = z, \qquad \dot{z} = -x - f(x)\, z. \tag{4.1.3}$$

In his proof he used an equivalent form to (4.1.3) by changing variable as $z = y - F(x)$, where $F(x) = \int_0^x f(u)du$:

$$\dot{x} = y - F(x), \qquad \dot{y} = -x, \tag{4.1.4}$$

which is a particular case of (4.1.1).(4.1.2) is called Liénard equation and both system (4.1.3) and (4.1.4) are called Liénard systems.

We will deal with the question of the number of limit cycles for the system (4.1.4) and their location in phase space for a given arbitrary odd polynomial $F(x)$ of degree m. There are no general results about the number of limit cycles of (4.1.4) however, for $m = 3$, i.e., for $F(x) = a_1x + a_3x^3$, it has been shown in [12] that the system has a unique limit cycle if $a_1a_3 < 0$ and no limit cycle if $a_1a_3 > 0$. For $m = 5$ i.e., for $F(x) = a_1x + a_3x^3 + a_5x^5$, it has been shown in [21] that the maximum number of limit cycles is 2.

We now explain the method through the case where,

$$F(x) = \epsilon\,(\frac{x^3}{3} - x) \tag{4.1.5}$$

also known as the van der Pol equation.
We consider a function $h_2(x, y) = y^2 + g_{1,2}(x)y + g_{0,2}(x)$, which is second degree polynomial in y. Here $g_{1,2}(x)$ and $g_{0,2}(x)$ are arbitrary functions of x. Here, the second subindex of h_2 refers to the degree of the polynomial. Then we calculate $\dot{h}_2$ i.e.,

$$\dot{h}_2 = \dot{x}\frac{\partial h_2}{\partial x} + \dot{y}\frac{\partial h_2}{\partial y}.$$

Putting the values from eq.(4.1.4) in $\dot{h}_2$, we get:

$$\dot{h}_2 = \Big(y - F(x)\Big)\frac{\partial h_2}{\partial x} - x\frac{\partial h_2}{\partial y}.$$

$$\dot{h}_2 = \Big(y - F(x)\Big)\Big(y\frac{\partial g_{1,2}}{\partial x}(x) + \frac{\partial g_{0,2}}{\partial x}(x) - x(2y + g_{1,2}(x))\Big),$$

which is a 2nd degree polynomial in y. By arranging $\dot{h}_2$, we get

$$\dot{h}_2 = y^2\frac{\partial g_{1,2}}{\partial x}(x) + \Big(- F(x)\frac{\partial g_{1,2}}{\partial x}(x) + \frac{\partial g_{0,2}}{\partial x}(x) - 2x\Big)y \\ + \Big(- xg_{1,2} - F(x)\frac{\partial g_{0,2}}{\partial x}(x)\Big).$$

Now we are to choose $g_{1,2}(x)$ & $g_{0,2}(x)$ in such a way that the coefficients of y^2 and y in $\dot{h}_2$ are zero and we are only left with a function of x. From this we get $g_{1,2}(x) = k_1$ and $g_{0,2}(x) = x^2 + k_0$, where k_0 and k_1 are arbitrary constants. As F(x) is an odd polynomial & if (x, y) is a point of the limit cycle of (4.1.4) then so will $(-x, -y)$ be. We want that the function $h_2(x, y)$ should have this symmetry as well. For this reason we put $k_1 = 0$. We then have $\dot{h}_2 = R_2(x) = -2xF(x) = -2x\epsilon\,(\frac{x^3}{3} - x)$. The polynomial $R_2(x)$ is even and it has exactly one positive root of odd multiplicity, i.e. $x = \sqrt{3}$. If we integrate the function $\dot{h}_2$ along the limit cycle, we have : $\int_0^T \dot{h}_2(x(t), y(t))dt = \int_0^T R_2(x(t))dt$, where

T is the period; but $\int_0^T \dot{h}_2(x(t), y(t))dt = h_2(x(T), y(T)) - h_2(x(0), y(0)) = 0$ since the integral is around a closed curve. Consequently, we find : $\int_0^T R_2(x(t))dt = 0$. This last equality tells us that there cannot be any limit cycle in a region of the phase plane where $R_2(x)$ is of one sign. For the van der Pol equation, $R_2(x)$ has a root of odd multiplicity at $x = \sqrt{3}$, hence the maximum value of x for the limit cycle must be greater than $\sqrt{3}$. The curves defined by $h_2(x, y) = x_2 + y_2 + k_0 = 0$ are closed for $k_0 < 0$. We will only consider the cases for polynomial of even degree. Since the polynomials $h_n(x, y)$ with n odd do not give useful information since the level curves $h_n(x, y))$ are open and the polynomials $R_n(x)$ have always a single root of odd multiplicity at $x = 0$

Next we consider a 4th degree polynomial in y for the function $h_4(x, y)$, i.e. $h_4(x, y) = y^4+g_{3,4}(x)y^3+ g_{2,4}(x)y^2 + g_{1,4}(x)y + g_{0,4}(x)$. By imposing the condition on $\dot{h}_4$ as we did for $\dot{h}_3$ to be a function of only x, we find

$$\begin{aligned}\dot{h}_4 &= y^4\frac{\partial g_{3,4}}{\partial x} + y^3\Big(-F(x)\frac{\partial g_{3,4}}{\partial x} + \frac{\partial g_{2,4}}{\partial x} - 4x\Big)\\ &+ y^2\Big(\frac{\partial g_{1,4}}{\partial x} - F(x)\frac{\partial g_{2,4}}{\partial x} - 3xg_{3,4}\Big)\\ &+ y\Big(\frac{\partial g_{0,4}}{\partial x} - F(x)\frac{\partial g_{1,4}}{\partial x} - 2x\Big) + \Big(-F(x)\frac{\partial g_{0,4}}{\partial x} - xg_{1,4}\Big).\end{aligned}$$

Now we find all $g's$ in such a way that the coefficients of y^4, y^3, y^2 & y in $\dot{h}_2$ are zero and get $R_4(x) = -\frac{4}{27}x^{10} - \frac{4}{9}x^8 - \frac{24}{15}x^6 + \frac{14}{3}x^4 + 2x^2$. After neglecting the odd powers of x wee see $R_4(x)$ is an even polynomial of tenth degree. We will take $\epsilon = 1$ for this case. $R_4(x)$ has only one positive root of odd multiplicity, given by $x \simeq 1.824$,. This root is greater than the root of $R_2(x)$. Once again, by the same reasoning as before, the maximum value of x for the limit cycle must be greater than this value. The limit cycle for this system is given in (Fig. 4.1).

In this way a new lower bound for the maximum value of x on the limit cycle is obtained. It is also observed that the number of positive roots of odd multiplicity is equal to the number of limit cycles of the given system. Hence continuing in this way for larger even values of n the roots obtained converge to the number of the limit cycles of the system. In all cases, the polynomials $R_n(x)$ have only one positive root of odd multiplicity implying the uniqueness of limit cycle for this system. The level curves $h_n(x, y) = K$ are all closed for positive values of K. These $K's$ represents an algebraic approximation to the limit cycle. The numerical value of the maximum of x on the limit cycle, with $F(x)$ defined by (4.1.5), is $x_{max} \simeq 2.01(\epsilon = 1)$. It is clear that the roots of $R_n(x)$ seem to converge to x_{max} and the curves $f_n(x, y)$ seem to approximate the limit cycle.

We have also studied another case where:

$$F(x) = 0.8x - \frac{4}{3}x^3 + 0.32x^5. \tag{4.1.6}$$

This system has two limit cycles[21]. We have calculated the polynomials $h_n(x, y)$ and $R_n(x)$ up to $n = 4$ (see table 4.1). The polynomials $R_n(x)$ have exactly two positive roots of odd multiplicity for both cases. For each n, we determine values of K, which will be two for this case. These closed

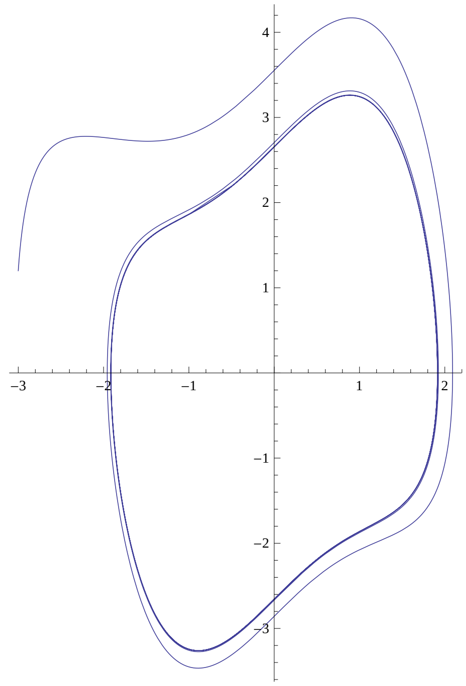

Figure 4.1: Limit Cycle for the system (4.1.5) for $\epsilon = 1$

curves $h_n(x, y) = K$ provide algebraic approximations to each cycle for each value of n even and approach each one of the limit cycles of the system. The numerical values of the maximum of x on each limit cycle are $x_{max,1} \simeq 1.0034$ and $x_{max,2} \simeq 1.9992$ respectively. The limit cycle for this system is given in (Fig. 4.2).

n	Root 1	Root 2	Exact Root 1	Exact Root 2
2	0.852	1.854	1.003	1.9992
4	0.905	1.885	-	-

Table 4.1: *For each value of* n, *we give the two roots of* $Rn(x)$ *for* $F(x)$ *defined by (4.1.6)*

Hence it has been shown that the polynomials $h_n(x, y) = y^n + g_{n-1,n}(x)y^{n-1} + g_{n-2,n}(x)y^{n-2} + ... + g_{1,n}(x)y + g_{0,n}(x)$ give a lot of information about the number and location of the limit cycles of system of type (4.1.4), in the case where $F(x)$ is an odd polynomial (for the case where $F(x)$ is not an odd polynomial, the limit cycles are not invariant under the transformation $(x, y) \to (-x, -y)$. The curves $h_n(x, y) = K_n$ give algebraic approximations to each limit cycle. The number of positive roots of odd multiplicity of the polynomials $R_n(x) = \dot{h}_n(x, y)$ give information about the number of limit cycles of (4.1.4) and they approximate each limit cycle. Hence all the relevant information about the limit cycles of (4.1.4) seems to be contained in the proposed polynomials $h_n(x, y)$.

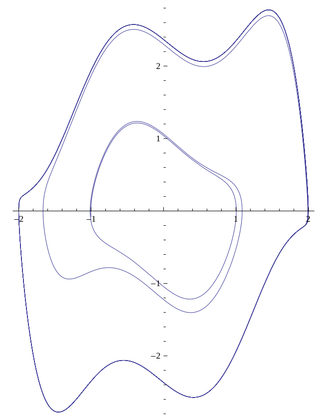

Figure 4.2: Limit Cycle for the system (4.1.6)

4.2 Non-Perturbative Method for Finding Limit Cycles of Liénard Equation

In this section we deal with same problem i.e. solving the differential equation for finding the limit cycles through another method. In a study B. Delamotte used a non-perturbative approach for solving the second order differential equation for finding the limit cycles [10]. We explain the method and then compare it with the method proposed in previous section for accuracy and speed of convergence to the solution.

We consider a particular class of differential equation through which we will explain the method and then apply it on the same equations (4.1.5) & (4.1.6).
Consider a 2nd order differential equation:

$$f(x(t), x'(t), x''(t)) = 0, \tag{4.2.1}$$

with initial conditions

$$x(t_0) = x_0, \quad x'(t_0) = x'_0, \tag{4.2.2}$$

where $x(t)$ is it's solution.
We are to replace eq.'s (4.2.1) & (4.2.2) by a linear differential equation with an explicit time-dependent right-hand side i.e.,

$$x''(t) + \omega^2 x = F(t). \tag{4.2.3}$$

Eq.'s (4.2.1) & (4.2.2) are completely equivalent to (4.2.2) & (4.2.3). F will depend on the differential equation we are studying and on the initial conditions (4.2.2). Now we are to make an initial guess more likely called ansatz i.e. F_{ans} for F and find such F_{ans} that best approximates the actual F.
Let us denote the solution of (4.2.3) by $x_{ans}(t)$ with F_{ans},

$$x''_{ans}(t) + \omega^2 x_{ans} = F_{ans}(t). \tag{4.2.4}$$

Let δx be the difference between $x(t)$ & $x_{ans}(t)$ then:

$$x(t) = x_{ans}(t) + \delta x(t). \tag{4.2.5}$$

Since $x(t)$ is the solution to eq. (4.2.1):

$$f(x_{ans}(t) + \delta x(t), x'_{ans}(t) + \delta x'(t), x''_{ans}(t) + \delta x''(t)) = 0. \tag{4.2.6}$$

Also, if for any t, $x(t)$ is very close to $x_{ans}(t)$ then:

$$f(x_{ans}(t), x'_{ans}(t), x''_{ans}(t)) \sim O(\delta x). \tag{4.2.7}$$

We will expand $x_{ans}(t)$ as a Fourier sum which is equivalent to choosing F_{ans}:

$$x_{ans}(t) = \sum_{k=1}^{N} \Big(x_k \sin k\omega t + y_k \cos k\omega t \Big). \tag{4.2.8}$$

We now impose a condition that δx contains harmonics higher than N then all harmonics less than or equal to N are contained in x_{ans}. Then eq. (4.2.8) determines the approximate solution $x_{ans}(t)$.

Let us now apply this technique to a specific example. We consider the van der Pol equation:

$$x'' + g(x^2 - 1)x' + x = 0, \tag{4.2.9}$$

with initial conditions defined as

$$x_{ans}(0) = x_0, \qquad x'_{ans}(0) = 0. \tag{4.2.10}$$

This equations has stable limit cycle in phase space. Here we replace (4.2.9) by (4.2.4). The functions $x_{ans}(t)$ & δx are expanded as Fourier sums:

$$x_{ans}(t) = \sum_{k=0}^{N} \Big[x_{(2k+1)} \sin(2k+1)\omega t + y_{(2k+1)} \cos(2k+1)\omega t \Big]. \tag{4.2.11}$$

The set of $x_{(2k+1)}$ & ω are free parameters. δx is given by (4.2.5). Replacing x from (4.2.5) then (4.2.9) becomes:

$$\Big(x_{ans}(t) + \delta x(t)\Big)'' + g\Big[\Big((x_{ans}(t) + \delta x(t))^2 - 1\Big)\Big(x_{ans}(t) + \delta x(t)\Big)'\Big] + \Big(x_{ans}(t) + \delta x(t)\Big) = 0.$$

Further replace x''_{ans} from (4.2.4) in (4.2.9) then δx obeys the equation:

$$\delta x'' = -g\Big((x_{ans} + \delta x)^2 - 1\Big)(x'_{ans} + \delta x') - (x_{ans} + \delta x) - [F_{ans} - x_{ans}\omega^2]. \tag{4.2.12}$$

The even harmonics are small enough to be ignored hence they are not considered. For our first approximation we consider $k = 0$ in (4.2.11) to get

$$x_{ans} = x_1 \sin \omega t + y_1 \cos \omega t. \tag{4.2.13}$$

Then our initial conditions from (4.2.2) become:

$$x_{ans}(0) = x_0 = y_1, \qquad x'_{ans}(0) = 0 = x_1\omega. \tag{4.2.14}$$

Here we impose the condition that δx contains harmonics higher than $2N+1$, so leaving out those harmonics in (4.2.12) we get:

$$-g(x_{ans}^2 - 1)x'_{ans} + x_{ans}(\omega^2 - 1) - F_{ans} \sim 0. \tag{4.2.15}$$

Placing (4.2.13) in (4.2.15) and then equating the coefficients of $\sin\omega t$ & $\cos\omega t$ we get two equations. These equations along with initial conditions (4.2.14) will determine x, y & ω and therefore the equation of limit cycle and it's amplitude (x_{max}) and period T. We present the value of *ansatz* for $g = 1$ in Table 4.2 and compare with the exact obtained after numerical integration of the equation [10]. The results obtained after only two iterations are excellent and show the accuracy of this method.

k	T	x_{max}
$k = 0$	6.28	2
$k = 1$	6.68	1.925
Exact	6.66	2.01

Table 4.2: *Period and amplitude of the limit cycle of the Van der Pol equation with* $g = 1$

We now compare these results with one's obtained by the method presented in previous section (call it method 1) in following table and observe that the results are strikingly better for this method. The results obtained in previous method after 6 iterations this method gives in 2nd iteration. The convergence to the result is also fast than the last method which shows this method is more accurate than the last method.

Iteration	x_{max}, Method 1	x_{max}, Method 2
1	1.732	2
2	1.824	1.925

Table 4.3: *Comparison of results for Van der Pol equation (4.1.5) and (4.2.9), where Exact* $= 2.01$

We have also studied this method for (4.1.6) with initial conditions given by (4.2.10) and this method also showed good accuracy for this case. This system has two limit cycles [21]. Following the same procedure as in last example the x_{ans} is given as (4.2.11). Solving this equation we arrive at the equation

$$(1 - \omega^2)x_{ans} + F_{ans} + g(0.8 - 4x_{ans}^2 + 1.6x_{ans}^4)x'_{ans} \sim 0. \tag{4.2.16}$$

This equation along with the initial conditions obtained after iterating (4.2.11) help us get x, y and period of the differential equation after equating the coefficients of $\sin\omega t$ & $\cos\omega t$ obtained from (4.2.16) for $g = 1$. The results for two iterations are presented in following table:

k	T	$x_{max,1}$	$x_{max,2}$
$k = 0$	6.28	0	1
$k = 1$	6.8	0.921	1.928
Exact	6.66	1.0034	1.9992

Table 4.4: *Two roots of eq.(4.2.16)*

Comparing these results with one obtained by the method presented in previous section for (4.1.6) in following table and observe that the results are also more accurate for this method in this case. The results obtained for previous method in 4th iteration this method gives in 2nd iteration.

Iteration	x_{max}, Method 1	x_{max}, Method 2
1	$x_{max,1} = 0.852$,$x_{max,2} = 1.854$	$x_{max,1} = 0$,$x_{max,2} = 1$
2	$x_{max,1} = 0.905$,$x_{max,2} = 1.885$	$x_{max,1} = 0.921$,$x_{max,2} = 1.928$

Table 4.5: *Comparison of results for (4.2.16)and (4.1.6) where exact values are:*
$x_{max,1} = 1.0034$,$x_{max,2} = 1.9992$.

Hence we conclude that the method proposed by B.Delamotte[10] gives better approximation and fast convergence to the actual results than the method proposed by H.Giacomini and S.Neukirch [9]. This method is more simple and less time consuming and it's accuracy is higher than the other.

Chapter 5

Summary

In this study we discussed representation of plane autonomous ordinary differential equations in the phase plane and how their solutions are obtained analytically. By doing so we learned about the nature of different types solutions and their behavior in phase plane. Out of those solutions periodic orbits were of special interest to us since our aim was to look for such solutions of a problem. Then we discussed in detail the periodic orbits and the question of their existence and non-existence in a particular system. Different results pertaining to existence and non-existence of such orbits were also presented. Lastly the question about the number of periodic orbits present in a particular system was dealt with by presenting some results containing approximate techniques to find the number of periodic orbits of Liénard equation along with their comparison on the accuracy and fast convergence to the results. The comparison leads us to find the limit cycles of the Liénard equation in a way that help us arrive at the solution efficiently and effectively.

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