Some Advancement in Metric Dimension and Exchange Property for Resolving Sets

by

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Dedicated to

My Loving Parents

&

Brothers

Abstract

Metric dimension is a parameter that has appeared in various applications of graph theory as diverse as, pharmaceutical chemistry, robot navigation, combinatorial optimization, master mind games and sonar and coast guard Loran, to name a few. The problem of finding the metric dimension of a graph is NP-complete.

We study the metric dimension of antiweb-wheels. We determine the exact value of metric dimension for antiweb-wheels and prove that they have unbounded metric dimension. We also study the metric dimension of quasi flower snarks, generalized antiprism and cartesian product of square cycle and path. We prove that these classes of graphs have constant or bounded metric dimension. Furthermore, we study the metric dimension of the subdivision of Möbius ladders and use this construction to study the metric dimension of generalized Petersen multigraphs $P(2n, n)$. It is natural to ask for characterization of graphs classes with respect to the nature of their metric dimension. It is also shown that the exchange property of the bases in a vector space does not hold for minimal resolving sets of antiweb-wheels, quasi flower snarks, generalized prism, generalized antiprism, barycentric subdivision of Möbius ladders and generalized Petersen multigraphs.

Acknowledgement

In the name of Almighty Allah, Who bestowed on me His blessings and gave me courage and vision to accomplish this work successfully. I invoke peace for Holy Prophet Hazrat Muhammad (PBUH) Who is forever a symbol of guidance for humanity. I would like to thank my supervisor Dr. Muhammad Imran for providing his precious time and his guidance to accomplish this work is note worthy. He is a brilliant supervisor. I would like to thank all faculty members and staff of CAMP for providing a peaceful working environment. I would like to thank my parents and friends Sobia Ashraf and Ayesha Mahmood, who provide me a full opportunity of devoting myself for M.Phil studies and whose prayers is the main cause of my success.

Rishi Naeem

Introduction

It is well known that the problem of determining whether $\beta(G) < k$ is an NP-complete problem. The aim of this thesis is to classify the several graph classes with respect to the nature of their metric dimension. We also study the exchange property for resolving sets of different families of graph. The exchange property for resolving set of graphs is relatively a new notion in metric graph theory and it helps us to determine that which methods (algorithmic or analytical) are feasible for finding the metric dimension of graph. We are able to give some advancement in metric dimension theory and exchange property for resolving sets in this thesis.

The first two chapters are devoted to some basic definitions and terminologies. In the first chapter, we give a brief introduction of graph models and basic concepts of graphs. The second chapter discusses about distance related parameters and their association. It also contains the concept of metric dimension, a brief history and known results regarding resolving sets.

In the third chapter, we study the metric dimension of different families of graphs. The metric dimension of wheels W_n has been determined in [5] showing that wheels have unbounded metric dimension. In this chapter, we extend this study to antiweb-wheels. We determine the exact value of metric dimension for antiweb-wheels and prove that they constitute a class of graphs with unbounded metric dimension. We also study the metric dimension of some graphs that are rotationally-symmetric, namely quasi flower snarks, generalized antiprism and cartesian product of square cycle and path. We prove that these classes of graphs have constant or bounded metric dimension. The metric dimension of generalized Petersen graphs for different values of $m\,(\geq 1)$ has been determined in [6, 16, 19, 20, 21]. We study the metric dimension of the subdivision of Möbius ladders and use this construction to study the metric dimension of generalized Petersen graphs $P(2n, n)$. We prove that the generalized Petersen graphs $P(2n, n)$, which are multigraphs have metric dimension equal to 3 when n is even and $n \equiv 0, 2 \pmod 4$ and equal to 4 otherwise. It is natural to ask for characterization of graphs classes with respect to the nature of their metric dimension.

In the fourth chapter, we study the exchange property for minimal resolving sets of antiweb-wheels, quasi flower snarks, generalized prism and generalized antiprism. It is shown that the exchange property of the bases in a vector space does not hold for minimal resolving sets of antiweb-wheels, quasi flower snarks, generalized prism and generalized antiprism. We also study the exchange property for resolving sets of barycentric subdivision of Möbius ladders and generalized Petersen multigraphs $P(2n, n)$ and prove that the exchange property of the bases in a vector space does not hold for minimal resolving sets of barycentric subdivision of Möbius ladders and also does not hold for minimal resolving sets of generalized Petersen multigraphs $P(2n, n)$ when n is even and $n \equiv 0, 2 \pmod 4$. Some open problems are raised in chapter 5 arising from this thesis.

Contents

Chapter 1

Preliminaries and basic concepts

This chapter is devoted to a brief introduction of basic concepts of graphs. It contains different graph theoretical terms and their illustration with examples. We also discuss connectivity, planarity, some common graph classes and few examples that explain these concepts.

1.1 Graphs

Many structures that involve physical situations can easily be depicted on paper by the use of a diagram. A diagram consists of a set of points or dots and lines that join all or some specific pairs of the given set of points. For instance, the points represent people at a party, and a line joining two points if they shake hands. In many physical problems like telephone networks, we are interested in finding the interruptions to the regular flow of networks. For this, we require to identify those particular lines

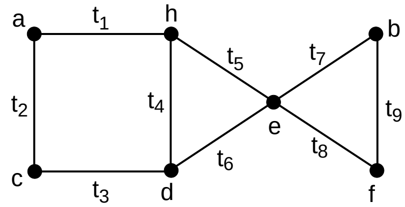

Figure 1.1: A network of telephone lines

and poles that should remain in the network service to avoid disconnecting it. The nodes in the network shown in figure 1.1 indicate poles. The removal of any single line will keep network connected. However, removing the two lines namely t_5 and t_6 will disconnect the network. The removal of the pole e also disconnects it. The set $\{t_1, t_3, t_5, t_6, t_7, t_8\}$ is the required minimum number of lines to avoid disconnecting the network.

Another problem is to find a feasible way of allocating jobs to qualified persons. Suppose that we have six people A, B, C, D, E and F and six jobs j_1, j_2, j_3, j_4, j_5 and j_6. We represent this physical situation by a diagram having dots or points for each

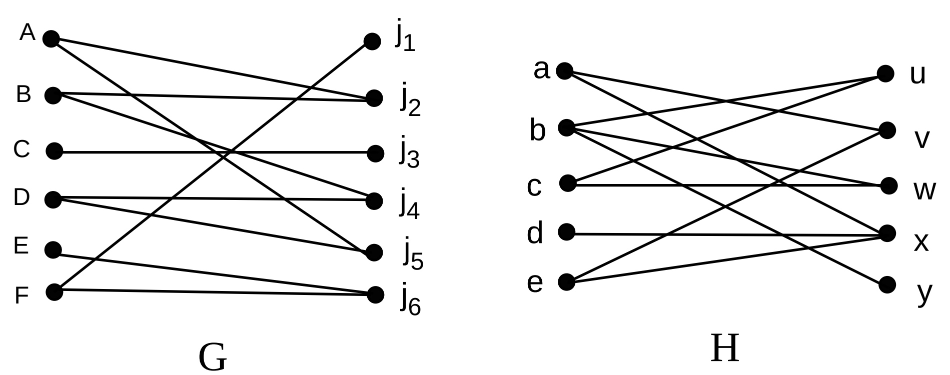

Figure 1.2: A job applications graph

person and for each job. We join people to jobs for which they are qualified with the help of lines as shown in figure 1.2. The diagram G shows that there is a feasible way for allocating one job to each people but in diagram H it is not possible. The reason can be found by considering people a, d and e. These three people as a set are collectively qualified only for two jobs v and x.

The job of a salesman requires to visit several cities each day and certain pairs of these cities are connected with highways. What should be the visit schedule so that he enables to visit each city exactly once so that the traveling time can be minimized? The transportation system of the salesman can be represented by a diagram shown in figure 1.3 whose nodes indicate cities and a line joins two cities if and only if a highway connects them. The salesman can visit each city starting

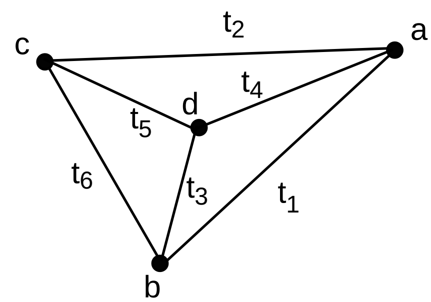

Figure 1.3: Traveling salesman's territory

from b and arrives back at b by taking the lines t_3, t_4, t_2 and t_6.

Suppose there are three houses. Each house have to be supplied with three utilities, namely electricity, water and gas. We can represent the three utilities problem by a diagram shown in figure 1.4, where the nodes h_1, h_2 and h_3 indicate houses and e, w and g represent utilities. Two vertices are joined by a line if one node

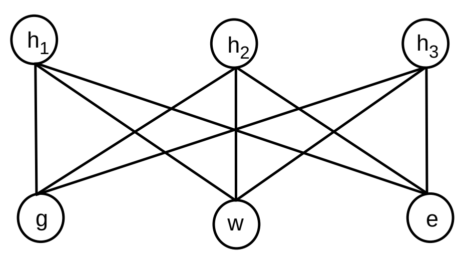

Figure 1.4: The three utilities graph

represents a house and other one is a utility. Is it possible to make such connection without intersection of lines? The diagram in figure 1.4 can not be drawn without line crossing, so the answer to this problem is no.

Seven radio broadcasting companies $R_1, \cdots, R_7$ have applied for frequency channels. To avoid interference, two companies can not be assigned the same channel if their transmitters are within three hundred kilometers. The main task is to assign a small number of different frequencies as possible. We illustrate this problem with the help of a diagram shown in figure 1.5, where $R_1, \cdots, R_7$ represent the nodes and two nodes are joined by a line if their transmitters are less than three hundred kilometers apart. Our task is to assign minimum number of different colors to the nodes such that two nodes have different color if they are joined by a line. Then the total number of colors represent the total number of frequencies. We can assign one color to R_1, R_3 and R_5. Nodes R_2, R_6 and R_7 are represented by the same color and we can assign a third color to R_4, so there are three frequencies that can be assigned to these broadcasting companies.

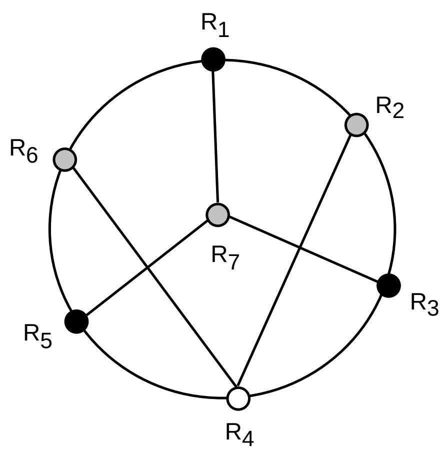

Figure 1.5: Radio transmitters and their interference graph

We list few other problems here: How can we assign colors to different region of a map by the use of four colors so that adjacent sections receive different colors? What should be the route plan in order to get cheapest fares? How can we create a fastest route structure from national to state capitals? All of the above mentioned and several other real world problems involve graph theory.

Formally, a *graph* G comprises of a non empty *vertex set* $V(G)$, and a collection of unordered pair of vertices (not necessarily distinct) called the *edge set* $E(G)$. Symbolically, a graph is represented as $G = (V(G), E(G))$, where we write $e = uv$ for an edge with *end points* u and v. A graph is usually represented by a diagram where the vertices are just points and edges are the lines or curves that serve as a link between two distinct or possibly same points.

Now we represent some basic definitions and terminologies of graph theory.

Definition 1.1.1. The number of vertices in a graph G is called *order of G* while the number of edges in a graph G is called *size of* G. These two parameters are denoted by n and m, respectively.

Definition 1.1.2. The end points of an edge are said to be *incident* with the edge. If distinct edges are incident with a common vertex then they are called *adjacent edges*. If $e = uv$ is an edge of a graph G then u and v are said to be *adjacent* in G and we also say that u and v are *joined* by the edge e. The set of all vertices that are adjacent to v in G is called the *neighborhood* of v and denoted as $N_G(v)$.

Definition 1.1.3. If two or more edges have same pair of end points, then these edges are referred as *parallel* or *multiple edges*. If an edge e is permitted to join a vertex to itself then e is called a *loop*. A graph G with no loop and multiple edges is said to be a *simple graph*.

The number of edges incident with a vertex, say v, is called the *degree* of the vertex v, denoted as $d_G(v)$. An *isolated vertex* and an *end vertex* (or a *leaf*) in a graph G are the vertices of degree 0 and 1, respectively. Each loop counts to two edges. The maximum and minimum degree is denoted by $\Delta(G)$ and $\delta(G)$, respectively and is defined as:

$$\Delta(G) = \max\{d_G(v) : v \in V(G)\};$$

$$\delta(G) = \min\{d_G(v) : v \in V(G)\}.$$

Next, we present a well known *handshaking lemma* that provides a fundamental relationship between the degree sum and size of a graph.

Theorem 1.1.1 ([10])**.** (Handshaking Lemma) *If G is a graph, then*

$$\sum_{v \in V(G)} d_G(v) = 2m,$$

where m is the size of G.

Definition 1.1.4. A graph having exactly one vertex is said to be a *trivial graph*. All other graphs are *nontrivial*.

Definition 1.1.5. If the vertex and edge set of a graph G are finite then G is called a *finite graph*, otherwise G will be an *infinite graph*.

Example 1.1.1. Consider a non trivial finite graph G shown in figure 1.6 having $V(G) = \{v_1, v_2, v_3, v_4, v_5\}$ and $E(G) = \{e_1, e_2, e_3, e_4, e_5, e_6, e_7, e_8, e_9\}$.

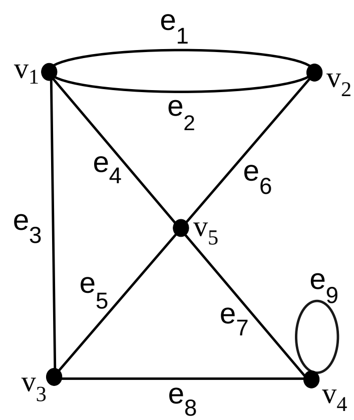

Figure 1.6: A multigraph

Here $\Delta(G) = 4$ and $\delta(G) = 3$. Since the initial and terminal vertex of the edge e_9 is same, hence it is a loop. The edges e_1 and e_2 are parallel with end points v_1 and v_2. The vertices v_1, v_2, v_3 and v_4 are joined to v_5 through edges e_4, e_6, e_5 and e_7, respectively. Thus we have $N_G\{v_5\} = \{v_1, v_2, v_3, v_4\}$. The vertices v_4, v_1 and v_4, v_2 are not joined by an edge that is why they are non adjacent vertices. Moreover, since G is having loop and parallel edges therefore G is a multigraph.

Definition 1.1.6. A graph F is a *subgraph* of G, denoted by $F \subseteq G$, if $V(F) \subseteq V(G)$ and $E(F) \subseteq E(G)$. If $V(F) = V(G)$ then F is called a *spanning subgraph* of G.

Definition 1.1.7. A subgraph F is an *induced subgraph* of G if whenever $v, u \in V(F)$ and $e = uv$ is an edge of G, then e is an edge of F as well. If $X \subseteq V(G)$, then the *subgraph of G induced by* set X is the induced subgraph with vertex set X. This induced subgraph is denoted as $G[X]$.

Example 1.1.2. A graph G and its subgraph H, spanning subgraph F and the subgraph S induced by $\{v_9, v_{10}, v_{11}, v_{12}, v_{13}\}$ are depicted in figure 1.7.

If $e \in E(G)$, then we write $G \setminus \{e\}$ for the subgraph of G where $E(G \setminus \{e\}) = E(G) \setminus \{e\}$. More generally, if $S \subseteq E(G)$, then $G \setminus S$ is the subgraph of G with $E(G \setminus$

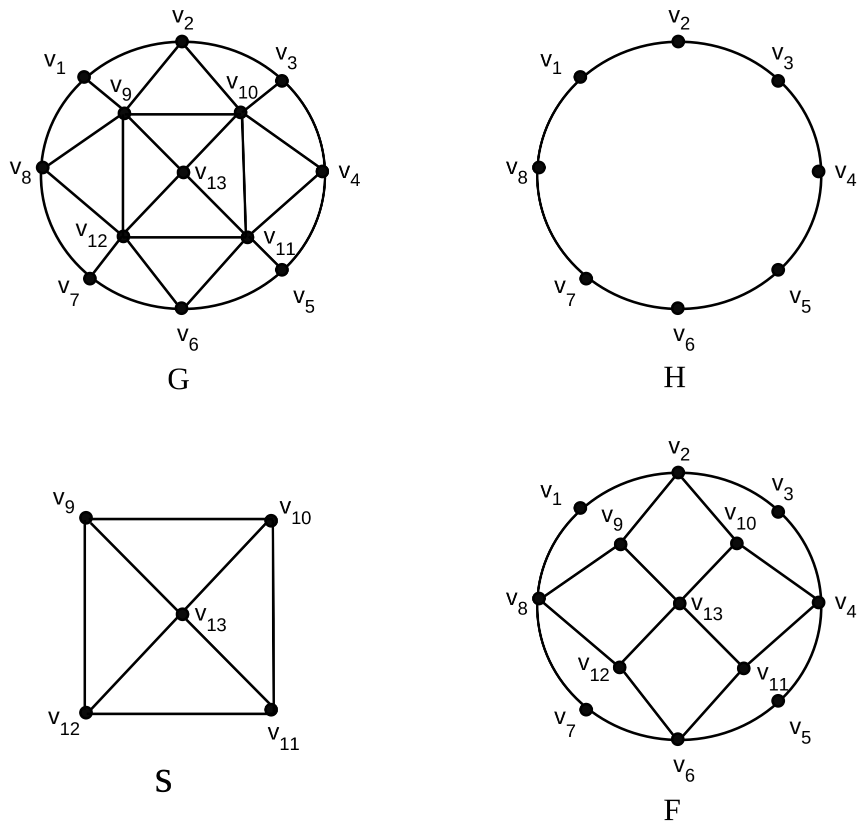

Figure 1.7: A graph G and its subgraph, spanning subgraph and induced subgraph

$S) = E(G) \setminus S$. If v is a vertex of a non trivial graph G, then the subgraph $G \setminus \{v\}$ has vertex set $V(G) \setminus \{v\}$ and $E(G \setminus \{v\}) = \{e \in E(G) : e \text{ is not incident with } v\}$. More generally, if $W \subset V(G)$, the subgraph $G \setminus W$ consists of all vertices of G that are not in W and its edge set consists of all edges of G joining two vertices in $V(G) \setminus W$.

Definition 1.1.8. A *walk* from vertex u to vertex v in a graph G is a finite alternating sequence of vertices and edges

$$W : u = v_0, e_1, v_1, e_2, \cdots, v_{l-1}, e_l, v_l = v,$$

where $e_i = v_{i-1}v_i$ such that $1 \leq i \leq l$, where u and v are the initial and terminal vertices of W, respectively. If $u = v$, then W is *closed* otherwise it is *open*. The number of edges in a walk is called its *length*.

Definition 1.1.9. A *path* P is a simple graph with $|E(P)| = |V(P)| - 1$ whose vertices can be arranged in a linear sequence in such a way that two vertices are adjacent if they occur consecutively in the sequence. A *cycle* of length $n \geq 3$ denoted by C_n is a simple graph with $|V(C_n)| = |E(C_n)|$ whose vertices can be arranged in a cyclic sequence. The number of edges encountered in a path or cycle is called its *length*. A cycle of odd length is recognized as an *odd cycle* and a cycle of even length is called an *even cycle*. A graph G is *acyclic* if it contains no cycle.

If the edges of a walk W are distinct, then W is called a *trail*. It is observed that if there is no repetition of the vertices in a walk then that walk is called a path. Thus,

$$\text{walk} \quad \xrightarrow{\text{distinct edges}} \quad \text{trail} \quad \xrightarrow{\text{distinct vertices}} \quad \text{path.}$$

So we can say that every path is a walk and every path is a trail as well. It is also noted that every cycle is a closed trail but a closed trail is a cycle if all its vertices are distinct except the initial and terminal vertices.

Definition 1.1.10. A graph G is said to be a *regular graph* if $\delta(G) = \Delta(G)$, i.e., all the vertices have same degree in G. A graph G is *k-regular* if $d_G(v) = k$ for all $v \in V(G)$ where $0 \leq k \leq n - 1$. A 3-regular graph is called a *cubic graph*.

Example 1.1.3. The graph H, in figure 1.7, is an even cycle of length 8 and $H \setminus \{v_1v_8\}$ is a path of length 7. Here in graph H, the degree of each vertex is 2, therefore it is 2-regular. A 4-regular and some cubic graphs are depicted in figure 1.8.

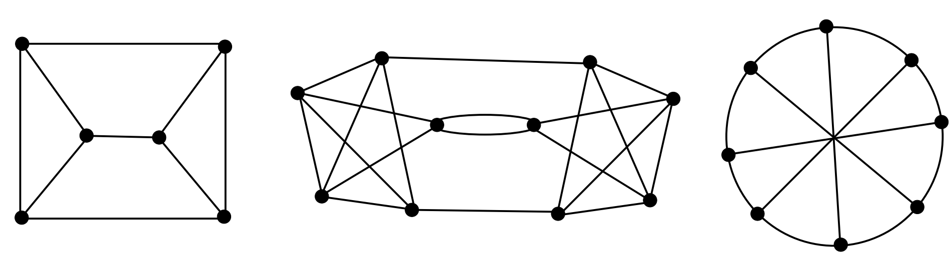

Figure 1.8: Some regular graphs

Definition 1.1.11. A cycle (path) that contains all vertices of a graph G is called a *spanning cycle (path)*.

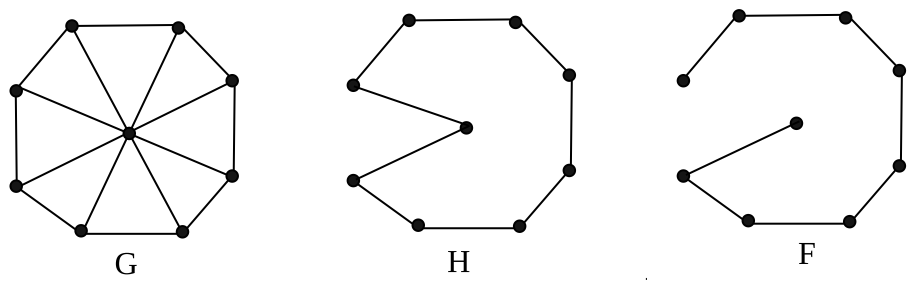

Figure 1.9: Graph G that contains spanning cycle and path

Definition 1.1.12. Let G and H be two graphs such that $V(G) = V(H)$ and $E(G) = E(H)$ then G and H are said to be *equal*. If G and H are the two graphs having same structure, then G and H are *isomorphic* and written as $G \cong H$. Formally, G and H are isomorphic if there is a mapping $\psi : V(G) \to V(H)$ such that for each edge $uv \in E(G)$ if and only if $\psi(u)\psi(v) \in E(H)$. In this case, ψ is called an isomorphism from G to H.

Example 1.1.4. The graphs G, H and G_1, H_1 in figure 1.10 are isomorphic.

Definition 1.1.13. A graph G is *connected* if any two vertices of G are connected by a path, otherwise G is *disconnected*.

A *component* of G is its connected subgraph such that it is not a proper subgraph of any other connected subgraph of G. The number of components in a graph G is denoted by $C(G)$. So, we can say a graph is connected if and only if $C(G) = 1$.

Definition 1.1.14. Let e be an edge of a connected graph G such that $G \setminus \{e\}$ is disconnected, then e is called a *bridge*. In other words, an edge e is called a bridge if $C(G \setminus \{e\}) > C(G)$.

Theorem 1.1.2 ([10]). *An edge e of a graph G is a bridge if and only if e belongs to no cycle of G.*

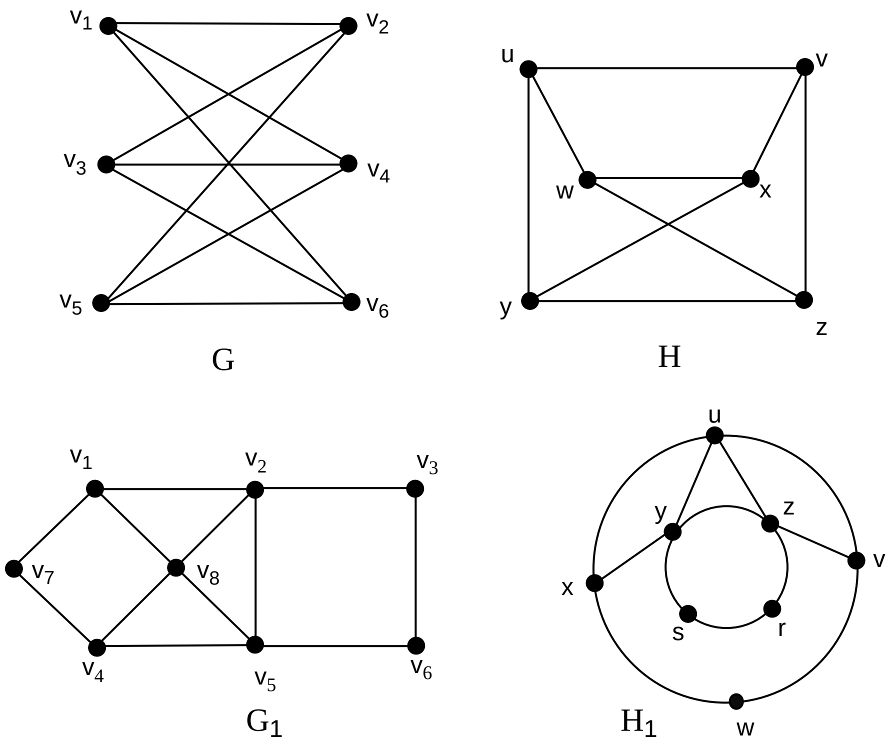

Figure 1.10: Isomorphic graphs

Definition 1.1.15. Let G be a connected graph and $v \in V(G)$, then v is referred as a *cut vertex* of G if $G \setminus \{v\}$ is disconnected.

Example 1.1.5. A graph G and the resulting graphs after deletion of the edge e and vertex v are shown in figure 1.11. Here v is a cut vertex and e is a bridge.

The following theorems give a relation between a bridge and a cut vertex of a connected graph.

Theorem 1.1.3 ([10])**.** *Let G be a connected graph and $v \in V(G)$ is incident with a bridge, then v is a cut vertex of G if and only if $d_G(v) \geq 2$.*

Theorem 1.1.4 ([10])**.** *If a connected graph G of order three or more contains a bridge then G contains a cut vertex.*

Definition 1.1.16. For two disjoint graphs G and H, $G \cup H$ is a disconnected graph

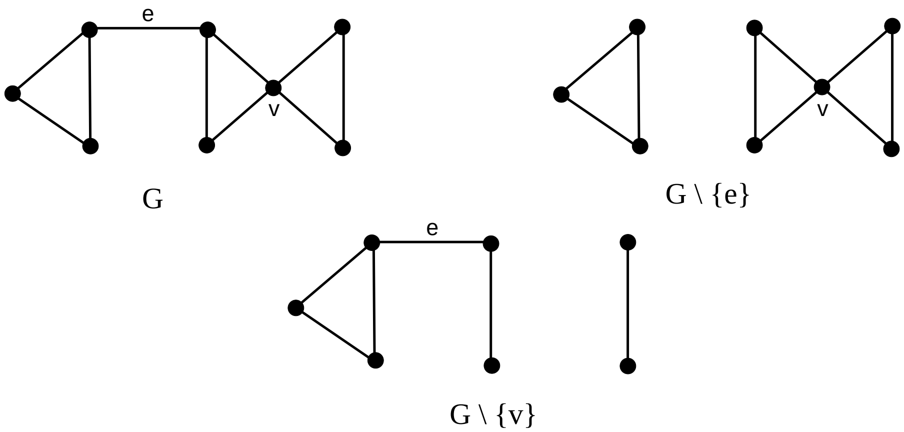

Figure 1.11: A graph G and its vertex and edge deleted subgraphs

with vertex set $V(G) \cup V(H)$ and edge set $E(G) \cup E(H)$ and *join* $G + H$ consists of $G \cup H$ and all edges joining each vertex of H with each vertex of G.

Example 1.1.6. Join of K_3 and P_3 is shown in the following figure 1.12.

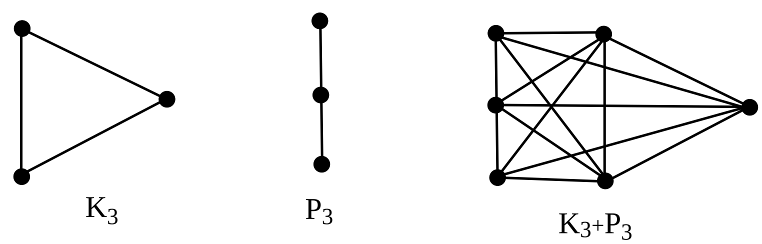

Figure 1.12: Join of two graphs K_3 and P_3

Definition 1.1.17. The *cartesian product* of two graphs G and H denoted by $G \square H$ whose vertex set is $V(G \times H) = V(G) \times V(H)$, that is $V(G \square H) = \{(g, h) | g \in V(G), h \in V(H)\}$ and the edge set $E(G \square H) = \{(g, h)(g', h') : h = h', gg' \in E(G)$ or $g = g', hh' \in E(H)\}$.

Example 1.1.7. The cartesian product of P_4 with P_3 and K_3 with P_3 are shown in figure 1.13.

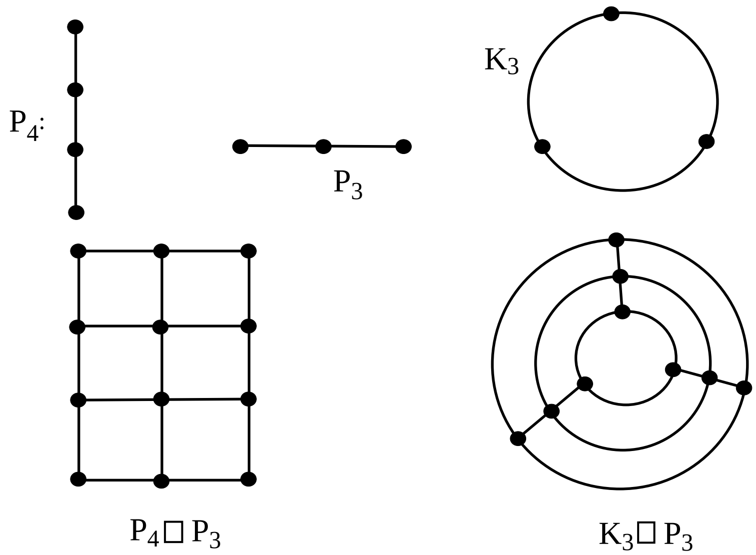

Figure 1.13: The cartesian product of two graphs

1.2 Some common graph classes

This section contains some common graph classes, namely complete graphs, bipartite graphs and trees.

Definition 1.2.1. A *complete graph* denoted by K_n, where n represents its order, is a simple graph such that every two distinct vertices are adjacent.

Example 1.2.1. Complete graphs K_8, K_6 and K_4 are shown in figure 1.14.

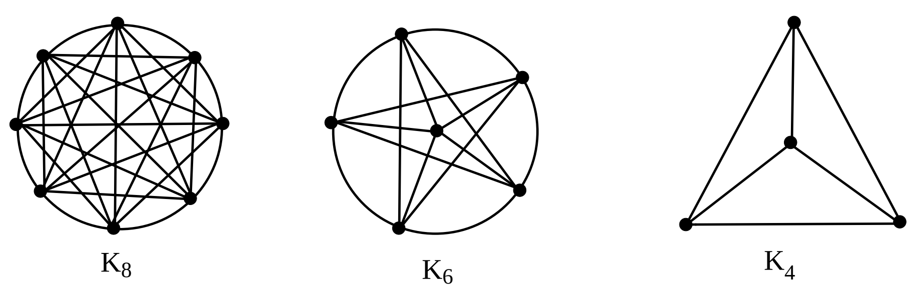

Figure 1.14: Complete graphs

Definition 1.2.2. A graph G is *bipartite* if its vertex set can be partitioned into two subsets X and Y recognized as *partite sets*, such that each edge has one end in X and other in Y. A *complete bipartite graph* is a simple graph in which every vertex of X is joined to every vertex in Y. It is denoted by $K_{r,s}$ where $r = |X|$ and $s = |Y|$.

It can be seen that a *star* is a complete bipartite graph with $|X| = 1$ or $|Y| = 1$. Now, we present a characterization of bipartite graphs by using the concept of a cycle.

Theorem 1.2.1 ([10]). *A nontrivial graph G is bipartite if and only if it has no odd cycle.*

Example 1.2.2. The graphs S and H in figure 1.15 are complete bipartite graphs $K_{1,8}$ and $K_{2,2}$, respectively. The graph G shown in figure 1.10 is a complete bipartite graph $K_{3,3}$.

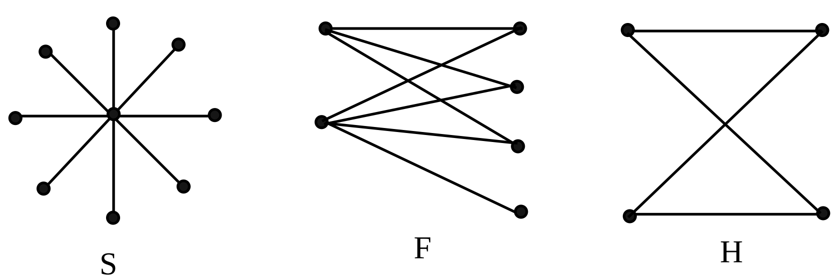

Figure 1.15: Bipartite graphs

Definition 1.2.3. An acyclic connected graph is called a *tree*. An acyclic graph is called a *forest*.

Trees have many possible characterizations and each contributes to the structural understanding of graphs in a different way. The following theorems establish some of the useful characterizations.

Theorem 1.2.2 ([10]). *A graph G is a tree if and only if any two vertices of G are connected by a unique path.*

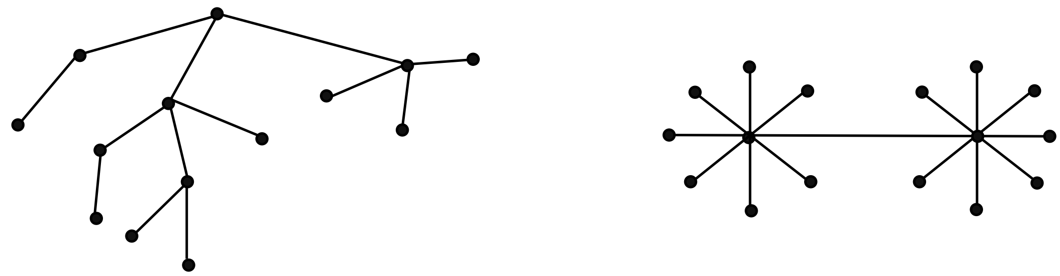

Figure 1.16: Trees

Theorem 1.2.3 ([10]). *If T is a tree then $m = n - 1$, where n is the order and m is the size of T.*

Theorem 1.2.4 ([10]). *If a graph G satisfies any two of the following properties:*

(i) *G is connected.*

(ii) *G is acyclic.*

(ii) *$m = n - 1$.*

Then G is a tree.

Definition 1.2.4. A *spanning tree* of a connected graph G is a spanning subgraph H of G such that H is a tree.

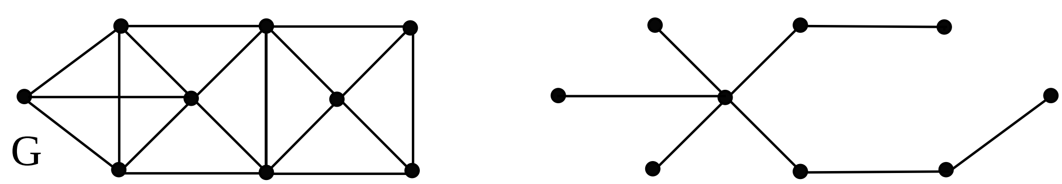

Figure 1.17: A graph G and its spanning tree

An important property related to spanning trees is established in the following theorem.

Theorem 1.2.5 ([10]). *A graph G is connected if and only if it has a spanning tree.*

Definition 1.2.5. A vertex v of a graph G is called pendant if $|N_G(v)| = 1$. If an end vertex of an edge e is pendant then e is said to be a pendant edge.

1.3 Connectivity and planarity

In this section we present vertex and edge connectivity that are useful in measuring connectedness of graphs. It also contains some portion related to planarity.

Definition 1.3.1. Let G be a connected graph and $X \subseteq V(G)$, if $G \setminus X$ is disconnected then X is referred to as a *vertex cut* of G.

Definition 1.3.2. A vertex cut of G with minimum cardinality is called a *minimum vertex cut* and the cardinality of that minimum vertex cut is recognized as *connectivity number*, denoted by $\kappa(G)$.

For a complete graph G, connectivity number $\kappa(G)$ is defined to be $n-1$. Thus, $\kappa(G) = 0$ if G is either trivial or disconnected. In a graph G, if $\kappa(G) \geq k$ then G is said to be *k-connected*.

Example 1.3.1. A cycle C_n, where $n \geq 4$ is 2-connected and a path of length two or more is 1-connected. A 3-connected graph G and 1-connected graph H are shown in figure 1.18.

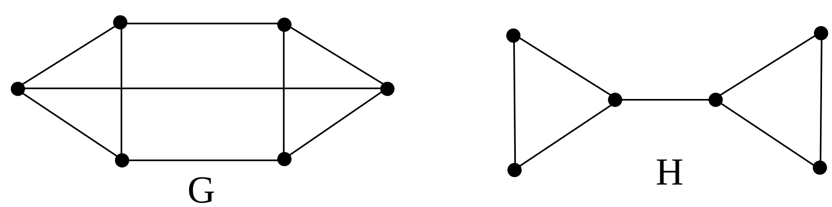

Figure 1.18: Graph

Definition 1.3.3. Let G be a connected graph and $S \subseteq E(G)$, if $G \setminus S$ is disconnected then S is referred as an *edge cut* of G.

Definition 1.3.4. An edge cut of minimum cardinality in G is called a *minimum edge cut* and the cardinality of that minimum edge cut is referred as *edge connectivity number*, denoted by $\kappa'(G)$.

Let G be a non trivial graph then $\kappa'(G) = 0$ if and only if G is disconnected. The edge connectivity number of a trivial graph is also zero. For a connected graph G, $\kappa'(G) = 1$ if and only if G contains a bridge. Furthermore, if $G \cong K_n$ then $\kappa'(G) = n - 1$. A graph G is said to be *k-edge-connected* if $\kappa'(G) \geq k$.

Example 1.3.2. Consider the graph G in figure 1.17, then we have $\kappa(G) = 2$ and $\kappa'(G) = 3$. While the graph G of figure 1.18 has $\kappa(G) = \kappa'(G) = 3$.

The next theorem gives an upper bound for connectivity and edge connectivity as well. In particular it also indicates that a high connectivity needs a large minimum degree.

Theorem 1.3.1 ([10]). *For a non trivial graph G, $\kappa(G) \leq \kappa'(G) \leq \delta(G)$.*

Theorem 1.3.2 ([10]). *If G is a cubic graph, then $\kappa(G) = \kappa'(G)$.*

To state *Menger's theorem*, first we define *seperating set.* A set $S \subseteq V(G)$ is said to separate two vertices x and y of G if $G \setminus S$ is disconnected and x and y belong to different components of $G \setminus S$. Such a set S is called an x-y separating set and an x-y separating set of minimum cardinality is said to be a *minimum x-y separating set.* Moreover, a collection $\{P_1, P_2, \ldots, P_l\}$ of x-y paths is said to be *internally disjoint* if no vertex other than x and y is common in any two paths P_i and P_j from the collection.

Theorem 1.3.3 ([10]). (Menger's Theorem) *Let x and y be two non adjacent vertices in a connected graph G. Then the maximum number of internally disjoint x-y paths in G is equal to the minimum number of vertices in an x-y separating set.*

Placing one or more vertices into the interior of an edge is known as *subdivision.* Next we present the definition of subdivision of an edge, its inverse operation and moreover introduce the concept of *barycentric subdivision.*

Definition 1.3.5. Let $x, y \in V(G)$ and $e = xy$ be an edge of G. *Subdividing the edge* e means that a new vertex v is added to $V(G)$, and that edge $e = xy$ is replaced in $E(G)$ by an edge $e' = xv$ and an edge $e'' = vy$. Geometrically, subdividing an edge is an operation that inserts a new vertex into the interior that results in splitting that edge into two edges. By an inverse operation we mean, replacement of two edges that meet at a vertex of degree two by a single edge that join their end points and this inverse operation is called *smoothing away a vertex*.

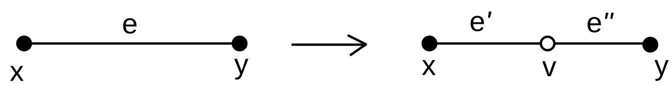

Figure 1.19: Subdividing an edge

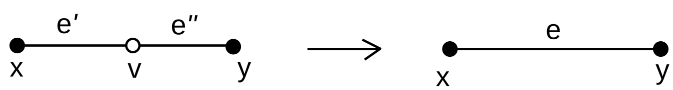

Figure 1.20: Smoothing away a vertex

Definition 1.3.6. *Subdividing a graph G* means performing a sequence of edge subdivision operations. The resulting graph is known as *subdivision of the graph G*.

We can convert a general graph into a simple graph by the use of subdivision operation.

Definition 1.3.7. The *barycentric subdivision* of a graph G is the subdivision in which a new vertex of degree two is added in the interior of each edge.

It can be noted that the resulting graph after applying barycentric subdivision is loopless and moreover a loopless graph can be converted into a simple graph by performing barycentric subdivision operation.

Example 1.3.3. Consider the graphs of figure 1.21, where G is a graph having loop and parallel edges but barycentric subdivision yields a loopless graph and further becomes a simple graph when we again apply barycentric subdivision. H is a cycle of length 5 but its barycentric subdivision yields a cycle of length 10.

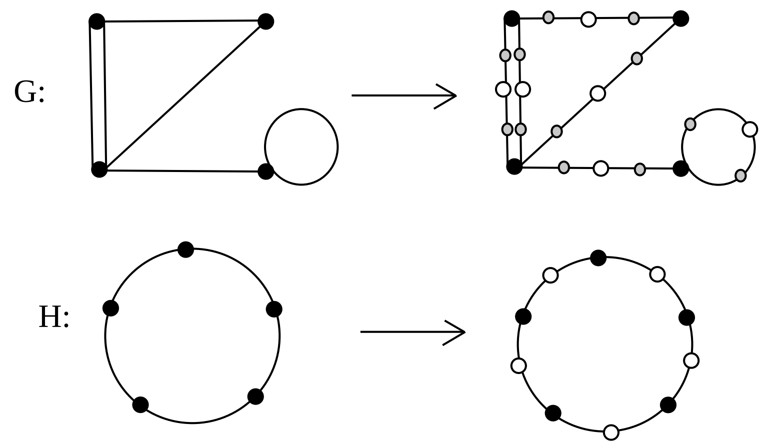

Figure 1.21: Barycentric subdivision of some graphs

Definition 1.3.8. A graph which can be drawn in a plane without edge crossing called a *planar graph*. In other words we can say a planar graph is a graph which is isomorphic to a plane graph.

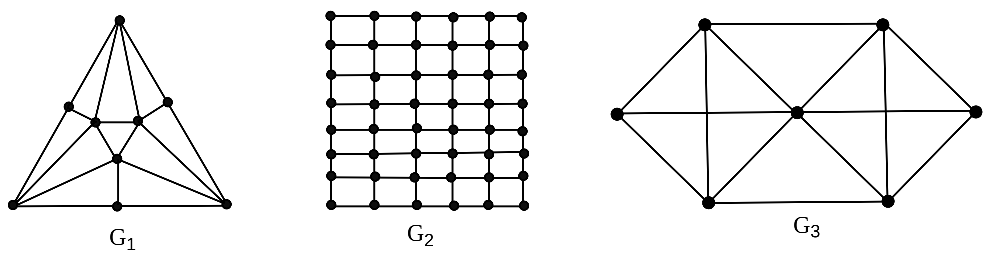

Figure 1.22: Some planar graphs

The following theorem establishes an association between the order and size of a planar graph.

Theorem 1.3.4 ([10])**.** *Let G be a planar graph of order three or more, then*

$$m \leq 3n - 6,$$

where m is the size of the graph.

Corollary 1.3.5 ([10])**.** *The complete graph K_5 is non planar.*

Theorem 1.3.6 ([10]). *The bipartite graph $K_{3,3}$ is non planar.*

A plane graph partitions the plane into connected pieces called the *regions*. In every plane graph, there is always one unbounded region called the *exterior region*. The subgraph of a plane graph whose vertices and edges are incident with a region of G is called the *boundary* of that region. It is also observed that if G is a connected graph with edges three or more, then the boundary of every region of G has more than two edges. For example, consider the graph G_1 of figure 1.22, there are 10 regions and one exterior region. Here $n = 9, m = 18$ and total number of regions are 11, we have $n - m + r = 2$ where r is the number of regions. Leonhard Euler observed that this observation is always true and is referred as the *Euler identity*.

Theorem 1.3.7 ([10]). (The Euler identity) *Let G be a connected plane graph of order n, size m, and containing r regions, then*

$$r + n - m = 2.$$

One of the milestone of graph theory is Kuratowski's characterization of planarity in terms of two forbidden subgraphs, K_5 and $K_{3,3}$.

Theorem 1.3.8 ([10]). (Kuratowski's Theorem) *A graph G is planar if and only if G does not contains $K_{3,3}, K_5$ or a subdivision of K_5 or $K_{3,3}$ as a subgraph.*

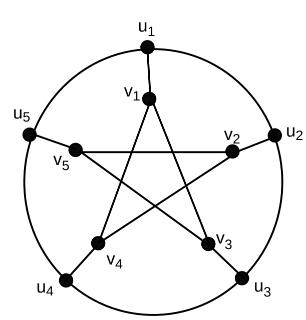

Figure 1.23: The Petersen graph

It is noted that the Petersen graph shown in figure 1.23 is non planar because it has subdivision of $K_{3,3}$.

Chapter 2

Resolvability in graphs

This chapter includes the discussion about distance related parameters and their properties. It also describes that the vertices of a graph constitute a metric space with the metric d on the vertices of G. Moreover, the concept of metric dimension and a brief history regarding resolving sets is also discussed.

2.1 Distance related parameters in graphs

Radius, diameter, eccentricity, periphery to name a few parameter that are to be discussed in this section.

Let G be a connected graph and $x, y \in V(G)$, the *distance* $d(x, y)$ from x to y is the length of a shortest x-y path in G and such an x-y path is called a *geodesic*.

Example 2.1.1. Consider the graph G in figure 2.1. Here $d(u, v) = 3$ and the geodesic between u to v is shown by dotted line.

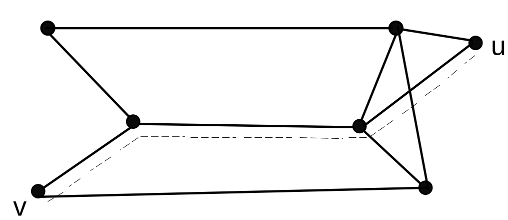

Figure 2.1: Graph

Definition 2.1.1. Let G be a connected graph. The distance function d is a *metric* on vertices of G, if it satisfies the following properties.

- $d(x, y) \geq 0$ for all $x, y \in V(G)$.
- $d(x, y) = 0$ if and only if $x = y$.
- $d(x, y) = d(y, x)$ for all $x, y \in V(G)$. *(the symmetric property)*
- $d(x, z) \leq d(x, y) + d(y, z)$ for all $x, y, z \in V(G)$. *(the triangle inequality)*

The pair $(V(G), d)$ is then called a *metric space*.

Theorem 2.1.1 ([10])**.** *For every pair of adjacent vertices x and y in a connected graph G,*

$$|d(x, v) - d(y, v)| \leq 1 \qquad \forall\, v \in V(G).$$

Definition 2.1.2. Let G be a connected graph and $v \in V(G)$. The *eccentricity* of v in G, denoted by $e(v)$, is defined as

$$e(v) = \max\{d(v, x) : x \in V(G)\}.$$

In other words, the eccentricity of v is the distance from v to a vertex farthest from v in G.

Definition 2.1.3. A vertex x in a connected graph G is referred as an *eccentric vertex of* v if $d(x, v) = e(v)$.

Example 2.1.2. Consider the graph G of figure 2.2. Here $d(v, x) = 4$, i.e., x is the vertex that lies at maximum distance from v. So $e(v) = 4$ and x is the eccentric vertex of v.

Definition 2.1.4. If a vertex x is an eccentric vertex of some vertex of G, then x is called an *eccentric vertex of* G and a subgraph of G induced by eccentric vertices of G is recognized as *eccentric subgraph* of G denoted by $\mathrm{Ecc}(G)$.

Example 2.1.3. The eccentric vertices of G shown in figure 2.2 are v and x and the subgraph induced by $\{x, v\}$ is a disconnected graph.

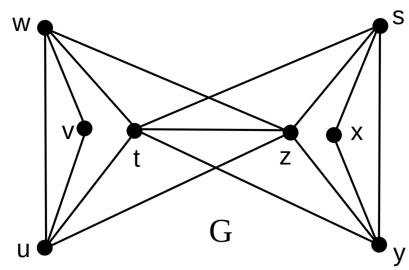

Figure 2.2: Graph

Definition 2.1.5. The *radius* denoted by $\operatorname{rad}(G)$ of a connected graph G is defined as $\operatorname{rad}(G) = \min\{e(v) : v \in V(G)\}$ (the minimum eccentricity among all vertices of G).

Definition 2.1.6. Let G be a connected graph, the *diameter* $\operatorname{diam}(G)$ of G is defined to be $\operatorname{diam}(G) = \max\{e(v) : v \in V(G)\}$ (the maximum eccentricity among all vertices of G).

Example 2.1.4. Consider the graph G of figure 2.2. Here $e(x) = e(v) = 4, e(w) = e(u) = e(s) = e(y) = 3$ and $e(t) = e(z) = 2$, so we have $\operatorname{rad}(G) = 2$ and $\operatorname{diam}(G) = 4$. Furthermore, $\operatorname{diam}(P_n) = n - 1$, $\operatorname{diam}(K_{r,s}) = 2$ where $1 \leq r \leq s\,(s > 1)$ and for every positive integer $n \geq 3$,

$$\operatorname{diam}(C_n) = \begin{cases} \frac{n}{2}, & \text{if } n \text{ is even;} \\ \frac{n-1}{2}, & \text{otherwise.} \end{cases}$$

Definition 2.1.7. A vertex v in a connected graph G is referred to as a *central vertex* if $e(v) = \operatorname{rad}(G)$ and the *center* $\operatorname{Cen}(G)$ of G is the subgraph of G induced by the set of central vertices of G. If $\operatorname{Cen}(G) = G$, then G is called *self centered.*

Example 2.1.5. The central vertices of G in figure 2.2 are t and z and the subgraph induced by $\{t, z\}$ is K_2. For some positive integer n, if $G \cong K_n$ and $G \cong C_n$ then G is self centered. Now, consider the graph of figure 2.3 where the central vertices are s and t and the subgraph induced by these vertices is a disconnected graph.

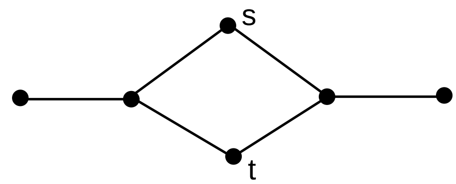

Figure 2.3: A connected graph whose center is disconnected

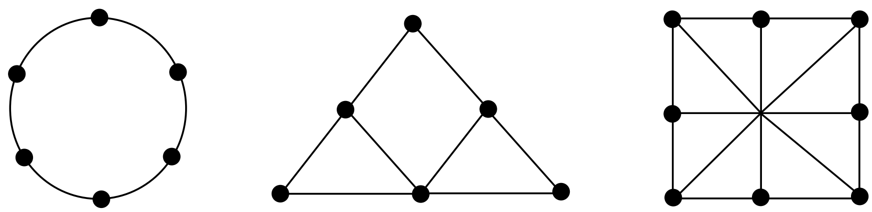

Figure 2.4: Self centered graphs

The center of a graph plays a key role in applications that involve emergency facilities where responding time (distance) to every single location (vertex) within a region (graph) is crucial. Rather than if we want to locate service facilities like general post office, commercial center, bank or power plant. In this case, we have to minimize the average distance such that a person serviced by these places must travel and it is equivalent to minimize the total distance traveled by all people in a territory. The concept of median is defined in those situations.

Definition 2.1.8. Let G be a connected graph. The *status* of a vertex v, denoted by $s(v)$, is the sum of the distances from v to every other vertex in G and *median* $M(G)$ is the set of vertices having minimum status. The minimum status of a graph G is the value of the minimum status denoted by $ms(G)$ and the total status $ts(G)$ of a graph G is the sum of all the status values. Harary introduced the concept of status.

Example 2.1.6. Consider the graph G of figure 2.5. Here $s(u) = 15, s(v) = s(w) = 12, s(x) = 9, s(y) = 10$ and $s(z) = s(r) = 14$. So median of this graph is a singleton set containing vertex x, i.e., $M(G) = \{x\}, ms(G) = 9$ and $ts(G) = 86$.

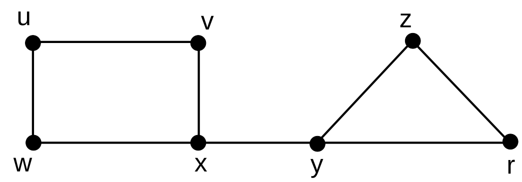

Figure 2.5: Graph

The following theorem gives a relation between the radius and diameter of a graph.

Theorem 2.1.2 ([10])**.** *Let G be a non trivial connected graph. Then*

$$\text{rad}(G) \leq \text{diam}(G) \leq 2\text{rad}(G).$$

Example 2.1.7. Consider the graph G of figure 2.6, here $e(u) = e(w) = e(r) = e(t) = 4$, $e(v) = e(x) = e(z) = e(s) = 3$ and $e(y) = 2$. So $2\text{rad}(G) = \text{diam}(G)$. In graph H, the eccentricity of each vertex is three. This implies that $\text{rad}(G) = \text{diam}(G)$.

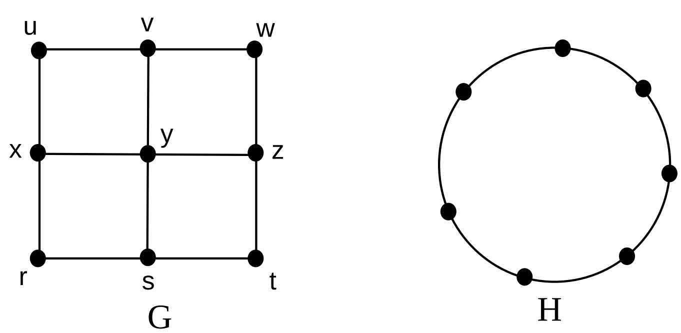

Figure 2.6: Graph

Theorem 2.1.3 ([10])**.** *Let G be a connected graph, then for every pair of adjacent vertices x and y*

$$|e(x) - e(y)| \leq 1.$$

Definition 2.1.9. Let G be a connected graph and $u \in V(G)$, the vertex u is said to be a *peripheral vertex* if $e(u) = \text{diam}(G)$ and the *periphery* $\text{Per}(G)$ of G is the subgraph of G induced by the set of peripheral vertices of G.

Example 2.1.8. For any positive integer n, if $G \cong C_n$ then $\text{Per}(C_n) = C_n$. Now consider the graph G of figure 2.7. Since $e(v_2) = 1 = \text{rad}(G)$, therefore $\text{Cen}(G)$ is a trivial graph while $e(v_1) = e(v_3) = e(v_4) = e(v_5) = 2 = \text{diam}(G)$. So $\{v_1, v_3, v_4, v_5\}$ are the peripheral vertices and $\text{Per}(G)$ is a disconnected graph having two components depicted in figure 2.7.

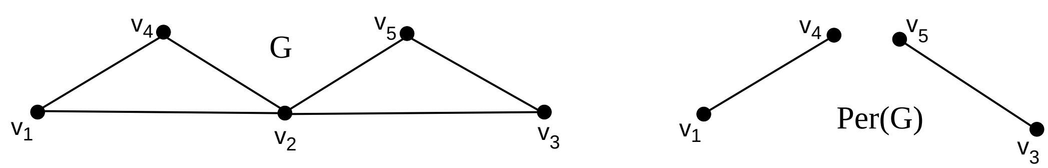

Figure 2.7: A graph G and its periphery graph $\text{Per}(G)$

Definition 2.1.10. Let G be a connected graph and $x, y \in V(G)$. The vertex x is a *boundary vertex of* y if

$$d(y,u) \leq d(x,y) \qquad \forall\, u \in N_G(x).$$

Example 2.1.9. Consider the graph G of figure 2.8. Here $d(x,y) = 3$ and $d(u,y) = d(v,y) = d(t,y) = 2$ which implies that x is a boundary vertex of y.

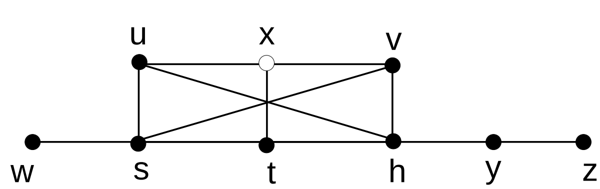

Figure 2.8: A graph and its boundary vertex

Definition 2.1.11. If a vertex x is a boundary vertex of some vertex of a graph G, then x is called a *boundary vertex of* G.

In a connected graph, there are some vertices that can not be boundary vertices.

Theorem 2.1.4 ([10]). *If u is a cut vertex of a connected graph G, then u is not a boundary vertex of G.*

Remark 2.1.1. It is noted that

Peripheral vertex of $G \Rightarrow$ eccentric vertex of $G \Rightarrow$ boundary vertex of G

Suppose, v is a peripheral vertex of G, then by definition $e(v) = d(u, v) = \text{diam}(G)$ where u is supposed to be an eccentric vertex of v. It also implies that $e(u) = d(v, u) = \text{diam}(G)$, because $d(u, y) \leq \text{diam}(G)$ for all $y \in V(G)$, which follows v is the eccentric vertex of G. Thus every peripheral vertex of G is an eccentric vertex but converse is not true. Now if we take v as an eccentric vertex of u, then by definition $e(u) = \max\{d(u, y) : y \in V(G)\} = d(v, u)$. Equivalently, v is a boundary vertex of u if $d(u, v) = \max\{d(u, y) : y \in N_G(v) \cup \{v\}\}$. So it is obvious that every eccentric vertex is a boundary vertex of G but converse is not true.

Every vertex of G is an eccentric vertex if all vertices of G have same eccentricity. However it can be happen, every vertex is an eccentric vertex without all eccentricities being same.

If u is a cut vertex, then u can not be a boundary vertex, eccentric vertex and nor a peripheral vertex.

Example 2.1.10. Consider the graph G of figure 2.8, here x is a boundary vertex of y but it is not an eccentric vertex. In the graph G shown in figure 2.9, x is an

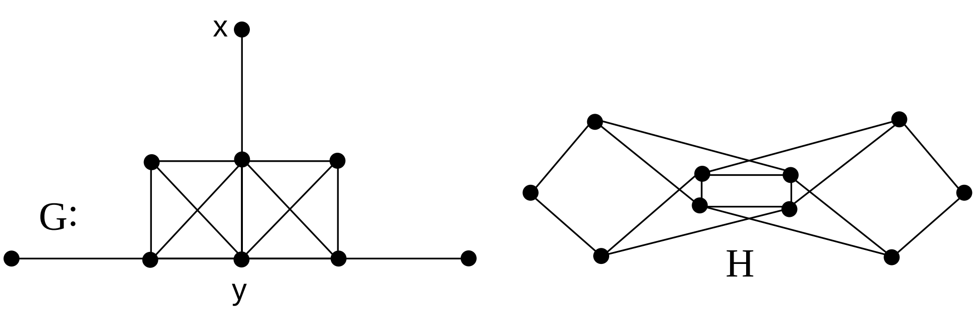

Figure 2.9: Graph

eccentric vertex of y. Since $e(x) = 3$ and $\text{diam}(G) = 4$, thus x is not a peripheral vertex. The graph H of figure 2.9 shows that every vertex is an eccentric vertex without all eccentricities being same.

2.2 Resolving sets

Navigation problems can be examined in a graph structured framework where the navigation agent considered as a point robot moves from node (vertex) to node (vertex) of a graph space. With the help of distinctly labeled landmark nodes in the graph space, the robot can locate itself. Visual detection of a distinctive landmark in a Euclidean space gives information about the direction to the landmark that helps robot to determine its location by triangulation. On the other hand, there is neither the concept of visibility nor that of direction on a graph. Here, we consider that a robot navigating can sense the distances to a set of *landmarks*. The location of a robot on a graph is uniquely determined if it knows its distances to a sufficiently large set of landmarks. This leads to the following problem: for a given graph, what should be the minimum number of landmarks and their location, so that position of the robot on the graph is uniquely determined by the distances to the landmarks? This is a classical problem about metric spaces. A set of landmarks with minimum cardinality that uniquely determines the position of a robot is said to be a *metric basis*, and its cardinality is referred as the *metric dimension* of a graph. Slater introduced the concept of metric dimension in [26] after getting motivation from the problem of uniquely determining the position of an intruder in a network. This concept was further studied independently by Harary and Melter in [12]. Slater represented the metric dimension of a graph as its *location number*. He used this concept to the placement of a smallest number of loran or sonar detecting devices in a network in order to uniquely determine the location of each vertex in the network in terms of its distances to the devices in the set.

Definition 2.2.1. Consider a connected graph G. Let $W = \{w_1, w_2, \ldots, w_k\}$ where $1 \leq k \leq n$ be an ordered set of vertices of G. The representation of a vertex v with respect to W is the k tuple

$$r(v|W) = (d(v, w_i) : 1 \leq i \leq k).$$

W is called a *resolving set or locating set* if distinct vertices of G have distinct representations with respect to the set W.

Example 2.2.1. Consider the graph G depicted in figure 2.10. If we take $W = \{v_1, v_2\}$, then $r(v_8|W) = r(v_{12}|W) = (1, 2)$ hence W is not a resolving set. Now if we consider $W' = \{v_1, v_2, v_{12}\}$ then the representation of vertices of G with respect to W' are

$$r(v_3|W') = (2, 1, 2), r(v_4|W') = (3, 2, 3), r(v_5|W') = (4, 3, 3),$$
$$r(v_6|W') = (3, 4, 2), r(v_7|W') = (2, 3, 2), r(v_8|W') = (1, 2, 1),$$
$$r(v_9|W') = (2, 1, 1), r(v_{10}|W') = (3, 2, 2), r(v_{11}|W') = (2, 3, 1),$$
$$r(v_1|W') = (0, 1, 1), r(v_2|W') = (1, 0, 2), r(v_{12}|W') = (1, 2, 0).$$

Since no two vertices of G have same representation, this mean W' is a resolving set for the vertices of G.

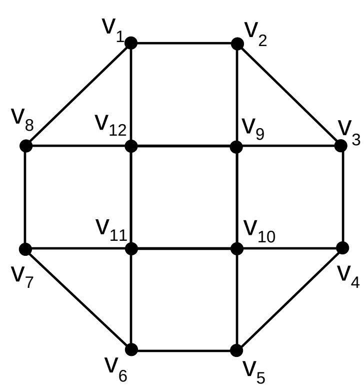

Figure 2.10: Graph

Definition 2.2.2. A resolving set with minimum cardinality is called *metric basis* for G and the cardinality of that set is referred as *location number* or *metric dimension* denoted by $\beta(G)$ or $\dim(G)$.

Example 2.2.2. Consider the graph G shown in figure 2.10, the resolving set W' in example 2.2.1 in not minimum. If we choose $W^* = \{v_1, v_3\}$, then the representation of vertices of G with respect to W^* are

$$r(v_2|W^*) = (1,1), r(v_4|W^*) = (3,1), r(v_5|W^*) = (4,2), r(v_6|W^*) = (3,3)$$
$$r(v_7|W^*) = (2,4), r(v_8|W^*) = (1,3), r(v_9|W^*) = (2,1), r(v_{10}|W^*) = (3,2)$$
$$r(v_{11}|W^*) = (2,3), r(v_{12}|W^*) = (1,2), r(v_1|W^*) = (0,2), r(v_3|W^*) = (2,0).$$

Since there is no resolving set consisting of a single vertex. So W^* is a minimum resolving set and we have $\beta(G) = 2$.

Let G be a connected graph of order two or more, we can think a resolving set of G as a set $W \subseteq V(G)$ so that each vertex in G is uniquely determined by its distances to the vertices of W. For every ordered set $W \subseteq V(G)$, the only vertex of G whose representation with respect to W is 0 in its *ith* coordinate is w_i. So the vertices of W necessarily have distinct representations, therefore we need to examine the vertices of $V(G) \setminus W$. This implies that the metric dimension of G is at most $n-1$. It follows,

$$1 \leq \beta(G) \leq n - 1.$$

The following theorems give a complete characterization of graphs having metric dimension 1 and $n-1$, respectively.

Theorem 2.2.1 ([10]). *Let G be a connected graph of order n, $\beta(G) = 1$ if and only if $G \cong P_n$.*

Theorem 2.2.2 ([10]). *For a connected graph G of order two or more, $\beta(G) = n-1$ if and only if $G \cong K_n$.*

Furthermore we also know the diameter and maximum degree of G, thus bounds for the metric dimension can be improved.

Theorem 2.2.3 ([10]). *Let G be a nontrivial connected graph of order two or more, then*

$$\lceil \log_3(\Delta + 1) \rceil \leq \beta(G) \leq n - \operatorname{diam}(G).$$

Chapter 3

Metric dimension of graphs

In this chapter, we study the metric dimension of antiweb-wheels, barycentric subdivision of Möbius ladders, generalized Petersen multigraphs $P(2n, n)$ and some graphs that are rotationally-symmetric, namely quasi flower snarks, generalized antiprism and cartesian product of square cycles and paths.

Let $\mathcal{F}$ be a family of connected graphs $G_n : \mathcal{F} = (G_n)_{n\geq 1}$ depending on n as follows: the order $|V(G)| = \varphi(n)$ and $\lim\limits_{n\to\infty} \varphi(n) = \infty$. If there exists a constant $C > 0$ such that $\beta(G_n) \leq C$ for every $n \geq 1$, then we shall say that $\mathcal{F}$ has *bounded metric dimension*; otherwise $\mathcal{F}$ has *unbounded metric dimension*. If all graphs in $\mathcal{F}$ have the same metric dimension (which does not depend on n), $\mathcal{F}$ is called a family with *constant metric dimension* [20].

3.1 Metric dimension of antiweb-wheels

If we make a small change in the graph, then how the value of its parameter is affected? This is a fundamental question that arises in graph theory. If G' is a graph obtained by adding a pendant edge to a nontrivial connected graph G, then it is easy to verify that

$$\beta(G) \leq \beta(G') \leq \beta(G) + 1.$$

However, if we add a vertex v to a connected graph G such that two or more edges are incident with v, then the metric dimension of the resulting graph remain same, increase or decrease significantly.

Denoting by $G + H$, the join of two graphs G and H, a wheel is defined as $W_n \cong C_n + K_1$. The metric dimension of wheels W_n has been determined by Buczkowski *et al.* [5] showing that wheels have unbounded metric dimension. In [20] Javaid *et al.* determined the metric dimension of square cycle C_n^2 and proved that,

$$\beta(C_n^2)=\begin{cases} 3, & \text{when } n \equiv 0, 2, 3 \pmod 4 \text{ ;} \\ \leq 4, & \text{otherwise.} \end{cases}$$

Hence, square cycles constitute a family of graphs with bounded metric dimension. However, if a new vertex v is added to the square cycle C_n^2, the dimension of resulting graph becomes unbounded as we prove it in this section.

An *antiweb-wheel* denoted by AWW_n can be defined as $AWW_n \cong C_n^2+K_1$, where we have $V(AWW_n) = V(W_n)$ and $E(AWW_n) = E(W_n) \cup \{v_iv_{i+2} : 1 \leq i \leq n\}$, where the indices are taken modulo n. Since $|E(W_n)| = 2n$, we get $|V(AWW_n)| = |V(W_n)| = n + 1$ and $|E(AWW_n)| = 3n$. The antiweb-wheels AWW_8 and AWW_9 are depicted in figure 3.1. We denote the *central vertex* of AWW_n by v and is

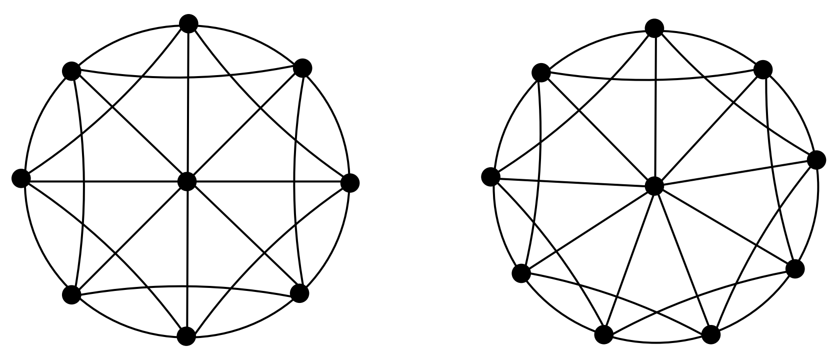

Figure 3.1: Antiweb-wheels AWW_8 and AWW_9

also called the *hub vertex* and all other vertices that induce cycle(s) are called the *rim vertices*, denoted as $v_1, v_2, \ldots, v_n$. Next, we prove that antiweb-wheels have unbounded metric dimension.

Consider the dimension of the antiweb-wheel $AWW_n \cong C_n^2 + K_1$ for $n \geq 3$. We have $\beta(AWW_3) = 3$, $\beta(AWW_4) = 4$, $\beta(AWW_5) = 5$ and $\beta(AWW_6) = \beta(AWW_7) =$

3. However, for $n \geq 8$, the dimension of AWW_n increases with number of vertices n as we prove now in the sequel.

Suppose that $AWW_n \cong C_n^2 + K_1$ for $n \geq 8$, then the central vertex v, does not belong to any metric basis. Since $\text{diam}(AWW_n) = 2$, so if the central vertex v belongs to any of the basis, say B, then there must exist two distinct vertices v_i and v_j such that $r(v_i|B) = r(v_j|B)$. Consequently, the basis belong to the only rim vertices of the antiweb-wheels AWW_n.

Let C_n be a cycle with n vertices and $V(C_n) = \{v_1, v_2, ..., v_n\}$. Let k, t be positive integers, $1 \leq k < t \leq n$. Then $\{v_{k+1}, v_{k+2}, ..., v_{t-1}\} \subset V(C_n)$ are the vertices in the *gap* denoted by G_α determined by the vertices v_k and v_t and the size of G_α is $k-t-1$. We will say that the gaps G_r and G_s are *neighboring gaps* when $|r - s| = 1$ [27]. Now, let B be a basis of antiweb-wheels AWW_n. We make the following claims.

Claim 1. *Every gap of B contains at most five vertices.* Otherwise, there is a gap containing six consecutive vertices say $v_j, v_{j+1}, v_{j+2}, v_{j+3}, v_{j+4}$ and v_{j+5}, where $1 \leq j \leq n$. But then we have $r(v_{j+2}|B) = r(v_{j+3}|B) = (2, 2, \ldots, 2)$, a contradiction.

Claim 2. *There is at most one gap of either five or four vertices.* We have two subcases here.

subcase(i): Suppose on contrary that there exist two gaps containing four and five vertices, respectively, say $\{v_p, v_{p+1}, v_{p+2}, v_{p+3}\}$ and $\{v_q, v_{q+1}, v_{q+2}, v_{q+3}, v_{q+4}\}$ where $1 \leq p \neq q \leq n$. But in this case, we get $r(v_{p+3}|B) = r(v_q|B) = (2, 1, 1, 2, \ldots, 2)$, a contradiction.

subcase(ii): Without loss of generality, suppose that there exists two distinct gaps containing four vertices each, say $\{v_p, v_{p+1}, v_{p+2}, v_{p+3}\}$ and $\{v_q, v_{q+1}, v_{q+2}, v_{q+3}\}$ where $1 \leq p \neq q \leq n$. However, then $r(v_{p+3}|B) = r(v_q|B) = (2, 1, 1, 2, \ldots, 2)$, a contradiction.

Claim 3. *If a gap contains either four or five vertices, then its both neighboring gaps are empty.* Otherwise, suppose that one neighboring gap of $N = \{v_p, v_{p+1}, v_{p+2}, v_{p+3}\}$ or $N^* = \{v_p, v_{p+1}, v_{p+2}, v_{p+3}, v_{p+4}\}$ is empty and other one contains only one vertex say v_q where $1 \leq p \neq q \leq n$. But then we receive $r(v_{p+2}|B) = r(v_{p+3}|B) = (2, 1, 2, \ldots, 2)$ being neighboring gap of N or we have $r(v_{p+3}|B) = r(v_{p+4}|B) = (2, 1, 2, \ldots, 2)$ being neighboring gap of N^*, a contradiction.

Claim 4. *If a gap contains three vertices, then its neighboring gaps may contain at most two vertices.* Suppose, there is a gap having three vertices, say $\{v_j, v_{j+1}, v_{j+2}\}$ and contrarily suppose that its one neighboring gap has three vertices, say $\{v_p, v_{p+1}, v_{p+2}\}$ where $1 \leq j \neq p \leq n$. But then $r(v_{j+2}|B) = r(v_p|B) = (2, 1, 2, \ldots, 2)$, a contradiction.

Claim 5. *If a gap of B contains two vertices, then one of its neighboring gap contains at most three vertices and other one is empty.* Suppose, there is a gap having two vertices whose one neighboring gap contains at most three vertices and suppose contrarily, its other neighboring gap has one vertex. Then there exist five consecutive vertices $v_j, v_{j+1}, v_{j+2}, v_{j+3}$ and v_{j+4} such that v_j and v_{j+3} are the basis vertices. However, then we have $r(v_{j+1}|B) = r(v_{j+2}|B) = (2, 1, 1, 2, \ldots, 2)$, a contradiction.

Claim 6. *If a gap contains exactly one vertex, then its neighboring gaps contain at most three vertices.* Suppose on contrary that its both neighboring gaps contain more than three vertices, then by claim 3 we get a contradiction.

Claim 7. *If one of the neighboring gap of an empty gap contains at most five vertices then the other gap contains at most two vertices.* Suppose that one of the neighboring gap contains at most five vertices $v_j, v_{j+1}, v_{j+2}, v_{j+3}$ and v_{j+4} where $1 \leq j \leq n$. On contrarily, suppose that the other neighboring gap contains at least three vertices, without loss of generality we suppose it has three vertices v_p, v_{p+1} and v_{p+2} where $1 \leq p \neq j \leq n$. But then we get $r(v_{j+4}|B) = r(v_p|B) = (2, 1, 1, 2, \ldots, 2)$, a contradiction.

Claim 8. *There is at least one empty gap for $n = 2k+1$, where $k \geq 4$.* Suppose on contrary that there is no empty gap. But then there exists a gap $\{v_p, v_{p+1}\}$ where $1 \leq p \leq n$ whose both neighboring gaps are non empty. By claim 5, we get a contradiction.

Now, we suppose that B is any resolving set consisting of rim vertices that satisfies claims $(1) - (8)$ and let $u \in V(AWW_n) \setminus B$. There are following six possibilities.

- *u belongs to a gap of size 5 of B.* Let $v_j, v_{j+1}, v_{j+2}, v_{j+3}, v_{j+4}, v_{j+5}, v_{j+6}$, v_{j+7} and v_{j+8} belong to the rim vertices, where v_j, v_{j+1}, v_{j+7} and $v_{j+8} \in B$. We have three subcases here.

subcase(i): If $u = v_{j+2}$, then it has distance 1 from v_j and v_{j+1} and has distance 2 from all other vertices of B. By claims (3) and (7) no other vertices of AWW_n has this property.

subcase(ii): If $u = v_{j+3}$, then it is adjacent to only one vertex v_{j+1} of B and has distance 2 from all vertices of B. By claim (3) only v_{j+3} has this representation.

subcase(iii): If $u = v_{j+4}$, then $r(v_{j+4}|B) = (2, 2, \ldots, 2)$ and by claim (3) no other vertex has this representation.

- *u belongs to a gap of size 4 of B.* Then we may assume that v_j, v_{j+1}, v_{j+2}, v_{j+3}, v_{j+4}, v_{j+5}, v_{j+6} and v_{j+7} are the rim vertices, where v_j, v_{j+1}, v_{j+6} and $v_{j+7} \in B$. Assume first that $u = v_{j+2}$, then it is adjacent to v_j and v_{j+1} and at distance 2 from all other vertices that belong to B. By claims (3) and (7) no other vertex of AWW_n has this representation. Next, we assume that $u = v_{j+3}$, then it has distance 1 from v_{j+1} and has distance 2 from all other vertices of B. By claim (3) only u has this property.
- *u belongs to a gap of size 3 of B.* Then there exist rim vertices v_j, v_{j+1}, v_{j+2}, v_{j+3}, v_{j+4}, v_{j+5} and v_{j+6}, where v_{j+1} and $v_{j+5} \in B$ and it may also contain either v_j or v_{j+6} or both. Now, Assume first that $u = v_{j+2}$, then it has distance 1 from v_j and v_{j+1} and at distance 2 from all other vertices of B. By claim (4) and (7), there exists no other vertex in AWW_n that satisfies this property. Next, assume that $u = v_{j+3}$, then it is adjacent to v_{j+1} and v_{j+5} and at distance 2 from all other vertices that contained in B. By claim (4) only u has this representation.
- *u belongs to a gap of size 2 of B.* Now we may consider $v_j, v_{j+1}, u = v_{j+2}, v_{j+3}$, v_{j+4} and v_{j+5} that belong to rim vertices, where v_j or v_{j+5} or both, v_{j+1} and $v_{j+4} \in B$. Then u has distance 1 from v_j, v_{j+1} and v_{j+4} and at distance 2 from all other vertices of B. By claim (5) and (7), there exist no other vertex that belongs to AWW_n with this property.
- *u belongs to a gap of size 1 of B.* Let v_j and $v_{j+1} \in B$ that determine this gap. Then u is adjacent to v_j and v_{j+1} and at distance 2 from all other vertices of B. Since there is no other rim vertex which has at distance 1 from both vertices v_j and v_{j+1}, so only u has this property.
- $u = v$, then $r(u|B) = (1, 1, \ldots, 1)$. There is no other vertex in AWW_n that has this representation.

So from above discussion, we have $r(u|B) \neq (u'|B)$ for $u \neq u'$, where $u, u' \in$

$V(AWW_n) \setminus B$.

Consequently, any set B having above properties is a resolving set of antiweb-wheels AWW_n. In the next theorem, we present a precise formula for the metric dimension of antiweb-wheels AWW_n.

Theorem 3.1.1. *Let $AWW_n \cong C_n^2 + K_1$ denotes the antiweb-wheels on $n+1$ vertices, then we have*

$$\beta(AWW_n) = \begin{cases} \lceil \frac{n+1}{3} \rceil, & \textit{if } n \textit{ is odd;} \\ \lceil \frac{n}{3} \rceil, & \textit{otherwise,} \end{cases}$$

for every integer $n \geq 8$.

Proof. We prove this theorem by double inequalities.

Let $AWW_n \cong C_n^2 + K_1$ denotes the antiweb-wheels on $n+1$ vertices and v is the central vertex of AWW_n. First we show that

$$\beta(AWW_n) \leq \begin{cases} \lceil \frac{n+1}{3} \rceil, & \text{if } n \text{ is odd;} \\ \lceil \frac{n}{3} \rceil, & \text{otherwise.} \end{cases}$$

For this, we construct a resolving set with $\lceil \frac{n}{3} \rceil$ vertices for n even and having $\lceil \frac{n+1}{3} \rceil$ vertices for n odd. We consider the following six cases according to the residue class modulo 6 to which rim vertices belong.

Case (i). When $n \equiv 0 \pmod 6$, then we can write $n = 6k; k \geq 2$, and $\lceil \frac{n}{3} \rceil = 2k$. Since $B = \{v_{6i+1}, v_{6i+3} : 0 \leq i \leq k-1\}$ contains $2k$ vertices and satisfies the claims $(1)-(8)$, therefore it is a resolving set for AWW_n in this case.

Case (ii). When $n \equiv 1 \pmod 6$, then we can write $n = 6k+1; k \geq 2$ and $\lceil \frac{n+1}{3} \rceil = 2k+1$. Since $B = \{v_1\} \cup \{v_{6i+3}, v_{6i+7} : 0 \leq i \leq k-2\} \cup \{v_{6k-2}, v_{6k-1}\}$ contains $2k+1$ vertices and satisfies $(1)-(8)$, so it is a resolving set in this case.

Case (iii). When $n \equiv 2 \pmod 6$, we can write $n = 6k+2; k \geq 1$ and $\lceil \frac{n}{3} \rceil = 2k+1$. Since $B = \{v_1\} \cup \{v_{6i+3}, v_{6i+5} : 0 \leq i \leq k-1\}$ contains $2k+1$ vertices and satisfies $(1)-(8)$, it is a resolving set.

Case (iv). When $n \equiv 3 \pmod 6$, we write $n = 6k+3; k \geq 1$, and $\lceil \frac{n+1}{3} \rceil = 2k+2$. Since $B = \{v_1, v_3\} \cup \{v_{6i+5}, v_{6i+9} : 0 \leq i \leq k-2\} \cup \{v_{6k}, v_{6k+1}\}$ contains $2k+2$ vertices and satisfies $(1)-(8)$, it is a resolving set.

Case (v). When $n \equiv 4 \pmod 6$ we can write as $n = 6k+4; k \geq 1$ and $\lceil \frac{n}{3} \rceil = 2k+2$. Since $B = \{v_1, v_3\} \cup \{v_{6i+5}, v_{6i+7} : 0 \leq i \leq k-1\}$ contains $2k+2$ vertices and satisfies $(1)-(8)$, it is a resolving set.

Case (vi). When $n \equiv 5 \pmod 6$, we write $n = 6k+5; k \geq 1$ and $\lceil \frac{n+1}{3} \rceil = 2k+2$. Since $B = \{v_{6i+1}, v_{6i+5} : 0 \leq i \leq k-1\} \cup \{v_{6k+2}, v_{6k+3}\}$ contains $2k+2$ vertices and satisfies $(1)-(8)$, it is a resolving set.

Next we have to show that

$$\beta(AWW_n) \geq \begin{cases} \lceil \frac{n+1}{3} \rceil, & \text{if } n \text{ is odd;} \\ \lceil \frac{n}{3} \rceil, & \text{otherwise.} \end{cases}$$

For this, suppose B be a metric basis of AWW_n. We consider three cases.

- *If* $|B| \equiv 0 \pmod 3$, then $|B| = 3l$ where $l \geq 1$. From claims $(1)-(8)$, at most one gap of B contains either four or five vertices. At most $l+1$ gaps of B contain three and at most $l-4$ gaps of B will contain two vertices. At most $l+2$ gaps contain at most one vertex out of which exactly two gaps are empty. Hence the number of vertices belonging to the gaps of B is at most $6l$ or $6l-1$. Therefore, we should have $n-3l \leq 6l$, which implies that $|B| = 3l \geq \lceil \frac{n}{3} \rceil$, or $n-3l \leq 6l-1$ showing that $|B| = 3l \geq \lceil \frac{n+1}{3} \rceil$ in this case.
- *If* $|B| \equiv 1 \pmod 3$, then $|B| = 3l+1$, where $l \geq 1$. From the claims $(1)-(8)$, at most one gap of B contains either four or five vertices. At most $l+1$ gaps of B contain three and at most $l-3$ gaps of B will contain two vertices. At most $l+2$ gaps contain at most one vertex out of which exactly two gaps are empty. So the number of vertices belonging to the gaps of B is at most $6l+2$ or $6l+1$. Hence we have $n-3l-1 \leq 6l+2$, which implies that $|B| = 3l+1 \geq \lceil \frac{n}{3} \rceil$ or $n-3l-1 \leq 6l+1$ implying that $|B| = 3l+1 \geq \lceil \frac{n+1}{3} \rceil$.
- *If* $|B| \equiv 2 \pmod 3$, then $|B| = 3l+2$ where $l \geq 1$. From calims $(1)-(8)$, at most one gap of B contains either four or five vertices. At most $l+2$ gaps of B contain three and at most $l-4$ gaps of B will contain two vertices. At most $l+3$ gaps contain at most one vertex out of which exactly two gaps are empty. So the number of vertices belonging to the gaps of B is at most $6l+4$ or $6l+3$. Hence $n-3l-2 \leq 6l+4$ which implies that $|B| = 3l+2 \geq \lceil \frac{n}{3} \rceil$ or $n-3l-2 \leq 6l+3$ showing that $|B| = 3l+2 \geq \lceil \frac{n+1}{3} \rceil$. □

3.2 Metric dimension of rotationally-symmetric graphs

This section includes the discussion about quasi flower snarks, generalized antiprism and cartesian product of square cycle and path. We prove that these classes of graphs have constant or bounded metric dimension.

3.2.1 Metric dimension of quasi flower snarks

The *quasi flower snark* denoted by G_n is a nontrivial simple connected cubic graph, where $V(G_n) = \{a_i, b_i, c_i, d_i : 0 \le i \le n-1\}$ and $E(G_n) = \{a_i a_{i+1}, b_i b_{i+1}, c_i c_{i+1}, a_i d_i, b_i d_i, c_i d_i : 0 \le i \le n-1\}$, the indices are taken modulo n. The quasi flower snarks G_{10} and G_9 are depicted in figure 3.2. Imran *et al.* [13] studied the metric dimension

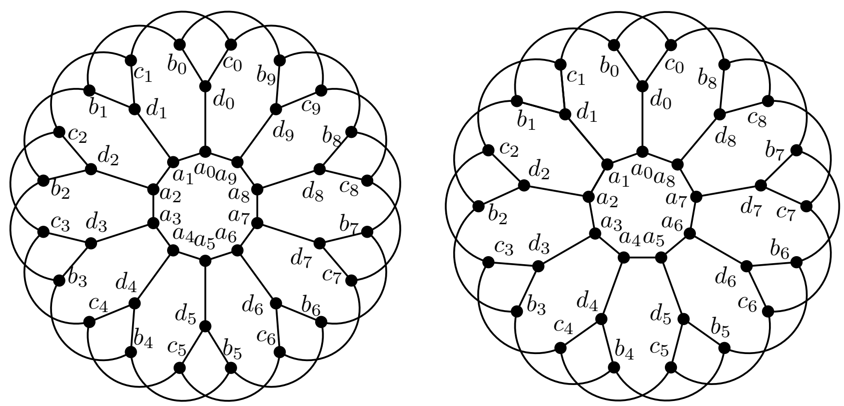

Figure 3.2: Quasi flower snark G_{10} and G_9

of flower snarks and proved that flower snakrs constitute a family of cubic graphs with constant metric dimension 3. In the next theorem, we extend this study to the metric dimension of quasi flower snarks.

Theorem 3.2.1. *Let G_n be the quasi flower snark, then for every positive integer $n \ge 4$ we have*

$$\beta(G_n) = \begin{cases} 3, & \text{if } n \text{ is odd;} \\ \le 4, & \text{otherwise.} \end{cases}$$

Proof. We consider the following cases.

Case (1). When $n \equiv 0 \pmod 2$. Then we can write as $n = 2k$, where $k \geq 2$. Suppose $W = \{b_0, c_0, d_{n-1}\}$ is a resolving set. For this, we give the representations of $V(G_n) \setminus W$.

$$r(a_i|W)=\begin{cases} (i+2, i+2, i+2), & 0 \leq i \leq k-1; \\ (2k-i+2, 2k-i+2, 2k-i), & k \leq i \leq 2k-1. \end{cases}$$

$$r(b_i|W)=\begin{cases} (i, i+2, i+2), & 0 \leq i \leq k-1; \\ (2k-i, 2k-i+2, 2k-i), & k \leq i \leq 2k-1. \end{cases}$$

$$r(c_i|W)=\begin{cases} (i+2, i, i+2), & 0 \leq i \leq k-1; \\ (2k-i+2, 2k-i, 2k-i), & k \leq i \leq 2k-1, \end{cases}$$

and

$$r(d_i|W)=\begin{cases} (i+1, i+1, i+3), & 0 \leq i \leq k-1; \\ (2k-i+1, 2k-i+1, 2k-i+1), & k \leq i \leq 2k-2. \end{cases}$$

It can be seen that for $1 \leq i \leq k-1$, $r(a_i|W)$=$r(d_{2k-i-1}|W) = (i+2, i+2, i+2)$. In order to have distinct representations, we add the vertex a_0 to W. Then for $1 \leq i \leq k-1$, we have $d(a_0, a_i) = i$ and $d(a_0, d_{2k-i-1}) = i+2$. Then we have $W' = W \cup \{a_0\}$, that resolves vertices of G_n when $n \equiv 0 \pmod 2$. Thus $\beta(G_n) \leq 4$ when $n \equiv 0 \pmod 2$.

Case (2). When $n \equiv 1 \pmod 2$. Then we have $n = 2k+1$; $k \geq 2$. We will prove this case by double inequality. First, we show that $W = \{b_0, c_0, c_k\}$ is a resolving set for G_n. For this, first we give representations of $V(G_n) \setminus W$.

$$r(a_i|W)=\begin{cases} (i+2, i+2, k-i+2), & 0 \leq i \leq k; \\ (k+2, k+2, 3), & i = k+1; \\ (2k-i+3, 2k-i+3, i-k+2), & k+2 \leq i \leq 2k. \end{cases}$$

$$r(b_i|W)=\begin{cases} (i, i+2, k-i+2), & 0 \le i \le k; \\ (k, k+2, 3), & i = k+1; \\ (2k-i+1, 2k-i+3, i-k+2), & k+2 \le i \le 2k. \end{cases}$$

$$r(c_i|W)=\begin{cases} (i+2, i, k-i), & 0 \le i \le k; \\ (k+2, k, 1), & i = k+1; \\ (2k-i+3, 2k-i+1, i-k), & k+2 \le i \le 2k, \end{cases}$$

and

$$r(d_i|W)=\begin{cases} (i+1, i+1, k-i+1), & 0 \le i \le k; \\ (k+1, k+1, 2), & i = k+1; \\ (2k-i+2, 2k-i+2, i-k+1), & k+2 \le i \le 2k. \end{cases}$$

It can be seen that all vertices in G_n have distinct representations implying that $\beta(G_n) \le 3$ when $n \equiv 1 \pmod 4$.

On the other hand, we show that $\beta(G_n) \ge 3$. Contrarily, suppose that $\beta(G_n) = 2$, then there are following possibilities to be discussed.

• *If both vertices belong to the set* $\{a_0, a_1, \ldots, a_{n-1}\}$, then we choose the resolving set is $W = \{a_p, a_q\}$, where $0 \le p < q \le 2k$. However, then we get

$$r(b_p|W) = r(c_p|W)=\begin{cases} (2, q-p+2), & 1 \le q-p \le k; \\ (2, 2k-q+p+3), & k+1 \le q-p \le 2k, \end{cases}$$

a contradiction.

• *If both vertices belong to the set* $\{d_0, d_1, \ldots, d_{n-1}\}$. We suppose that resolving set is $W = \{d_p, d_q\}$, where $0 \le p < q \le 2k$. However, then we have

$$r(b_p|W) = r(c_p|W)=\begin{cases} (1, q-p+1), & 1 \le q-p \le k; \\ (1, 2k-q+p+2), & k+1 \le q-p \le 2k, \end{cases}$$

a contradiction.

• *If both vertices belong to either* $\{b_0, b_1, \ldots, b_{n-1}\}$ or $\{c_0, c_1, \ldots, c_{n-1}\}$. Without loss of generality, we may assume that both vertices belong to the set $\{b_0, b_1, \ldots, b_{n-1}\}$. Then we may choose the resolving set $W = \{b_p, b_q\}$, where $0 \le p < q \le 2k$. Then

$$r(a_p|W) = r(c_p|W) = \begin{cases} (2, q-p+2), & 1 \le q-p \le k; \\ (2, 2k-q+p+3), & k+1 \le q-p \le 2k, \end{cases}$$

a contradiction.

• *If one vertex belongs to the set* $\{b_0, b_1, \ldots, b_{n-1}\}$ *and other one is in the set* $\{c_0, c_1, \ldots, c_{n-1}\}$. Without loss of generality we suppose, resolving set is $W = \{b_p, c_q\}$, where $0 \le p \le q \le 2k$. However, then

$r(a_{p+1}|W) = r(a_{p-1}|W) = (3, 3)$ when $p = q$;

$r(a_p|W) = r(d_{p-1}|W) = (2, q-p+2)$ when $1 \le q-p \le k-1$;

$r(c_{p+1}|W) = r(d_{p+2}|W) = (3, q-p-1)$ when $q-p = k, k+1$;

$r(a_p|W) = r(d_{p+1}|W) = (2, 2k-q+p+3)$ when $k+2 \le q-p \le 2k$.

The indices are taken modulo $2k+1$. We get a contradiction in each subcase.

• *If one vertex belongs to the set* $\{a_0, a_1, \ldots, a_{n-1}\}$ *and the other vertex belongs to the set* $\{d_0, d_1, \ldots, d_{n-1}\}$. Without loss of generality we can take $W = \{d_p, a_q\}$, where $0 \le p \le q \le 2k$. However, we have

$$r(b_p|W) = r(c_p|W) = \begin{cases} (1, q-p+2), & 0 \le q-p \le k; \\ (1, 2k-q+p+3), & k+1 \le q-p \le 2k, \end{cases}$$

a contradiction.

• *If one vertex belongs to the set* $\{a_0, a_1, \ldots, a_{n-1}\}$ *and other vertex belongs to either* $\{b_0, b_1, \ldots, b_{n-1}\}$ *or* $\{c_0, c_1, \ldots, c_{n-1}\}$. Without loss of generality we suppose that the second vertex belongs to the set $\{b_0, b_1, \ldots, b_{n-1}\}$. Then we can choose $W = \{a_p, b_q\}$, where $0 \le p \le q \le 2k$. But then we get

$r(a_{p-1}|W) = r(a_{p+1}|W) = (1, 3)$ when $p = q$;

$r(a_{p+1}|W) = r(d_p|W) = (1, q-p+1)$ when $0 \le q-p \le k$;

$r(a_{p-1}|W) = r(d_p|W) = (1, 2k-q+p+2)$ when $k+1 \le q-p \le 2k$.

The indices are taken modulo $2k+1$, a contradiction.

• *If one vertex belongs to the set* $\{d_0, d_1, \ldots, d_{n-1}\}$ *and other vertex belongs to either* $\{b_0, b_1, \ldots, b_{n-1}\}$ *or* $\{c_0, c_1, \ldots, c_{n-1}\}$. Without loss of generality we suppose the second vertex belongs to the set $\{b_0, b_1, \ldots, b_{n-1}\}$. We may suppose that the resolving set is $W = \{b_p, d_q\}$, where $0 \le p \le q \le 2k$. But then

$$r(a_p|W) = r(c_p|W) = \begin{cases} (2, q-p+1), & 0 \leq q-p \leq k; \\ (2, 2k-q+p+2), & k+1 \leq q-p \leq 2k, \end{cases}$$

we get a contradiction.

This yield that $\beta(G_n) \geq 3$. So from above, we conclude that there is no resolving set with two vertices of G_n. Hence $\beta(G_n) = 3$ when $n \equiv 1 \pmod 2$. □

3.2.2 Metric dimension of cartesian product of square cycle and path

The metric dimension of cartesian product of cycle and path has been investigated by Caceres *et al.* [6] and proved that

$$\beta(P_m \square C_n) = \begin{cases} 2, & \text{if } n \text{ is odd;} \\ 3, & \text{otherwise.} \end{cases}$$

In this section, we extend this study to the cartesian product of square cycle and path and prove that the cartesian product of square cycle and path have metric dimension equal to 3 when $n \equiv 0, 2, 3 \pmod 4$ and at most 4 otherwise. For $t \geq 2$, we have $V(C_n^2 \square P_t) = \{u_s^i : 1 \leq s \leq t \text{ and } 1 \leq i \leq n\} = \{u_1^i, u_2^i, \ldots, u_t^i : 1 \leq i \leq n\} = \{u_1^1, u_1^2, \ldots, u_t^1, u_t^2 \ldots, u_t^n\}$. In the next theorem, we determine the metric dimension of square cycle and path. Note that the choice of appropriate basis vertices (also called landmarks) is core of the problem.

Theorem 3.2.2. *For every positive integer $n \geq 5$,*

$$\beta(C_n^2 \square P_t) = \begin{cases} 3, & \text{when } n \equiv 0, 2, 3 \pmod 4; \\ \leq 4, & \text{otherwise.} \end{cases}$$

Proof. We prove this theorem by double inequality.

First we show that $\beta(C_n^2 \square P_t) \leq 3$ by showing that $W = \{u_1^1, u_1^3, u_t^2\}$ resolves all vertices of $C_n^2 \square P_t$ when $n \equiv 0, 2, 3 \pmod 4$. For this, we give representations of $V(C_n^2 \square P_t) \setminus W$ in each case.

Case(1). When $n \equiv 0 \pmod 4$. Then we can write as $n = 4k$, where $k \geq 2$. For $s = 1$,

$$r(u_s^{2i}|W)=\begin{cases}(1,1,t-1), & i=1;\\(i,i-1,t+i-2), & 2\le i\le k;\\(k,k,t+k-1), & i=k+1;\\(2k-i+1,2k-i+2,2k+t-i), & k+2\le i\le 2k.\end{cases}$$

$$r(u_s^{2i+1}|W)=\begin{cases}(i,i-1,t+i-1), & 2\le i\le k;\\(2k-i,2k-i+1,2k+t-i), & k+1\le i\le 2k-1,\end{cases}$$

for $s=t$,

$$r(u_s^{2i}|W)=\begin{cases}(t+i-1,t+i-2,i-1), & 2\le i\le k;\\(k+t-1,k+t-1,k), & i=k+1;\\(2k+t-i,2k+t-i+1,2k-i+1), & k+2\le i\le 2k.\end{cases}$$

$$r(u_s^{2i+1}|W)=\begin{cases}(t-1,t,1), & i=0;\\(t+i-1,t+i-2,i), & 1\le i\le k;\\(2k+t-i-1,2k+t-i,2k-i+1), & k+1\le i\le 2k-1,\end{cases}$$

and for $2\le s\le t-1$, we have

$$r(u_s^{2i}|W)=\begin{cases}(s,s,t-s), & i=1;\\(s+i-1,s+i-2,t-s+i-1), & 2\le i\le k;\\(k+s-1,k+s-1,k+t-s), & i=k+1;\\(2k+s-i,2k+s-i+1,2k+t-s-i+1), & k+2\le i\le 2k.\end{cases}$$

$$r(u_s^{2i+1}|W)=\begin{cases}(s-1,s,t-s+1), & i=0;\\(s+i-1,s+i-2,t-s+i), & 1\le i\le k;\\(2k+s-i-1,2k+s-i,2k+t-s-i+1), & k+1\le i\le 2k-1.\end{cases}$$

Since all the vertices have distinct representations with respect to W. This yields $\beta(C_n^2\square P_t)\le 3$ when $n\equiv 0 \pmod 4$.

Case(2). When $n\equiv 2 \pmod 4$. Then we can write as $n=4k+2$, where $k\ge 1$. For $s=1$,

$$r(u_s^{2i}|W)=\begin{cases} (1,1,t-1), & i=1; \\ (i,i-1,t+i-2), & 2\leq i\leq k+1; \\ (2k-i+2,2k-i+3,2k+t-i+1), & k+2\leq i\leq 2k+1. \end{cases}$$

$$r(u_s^{2i+1}|W)=\begin{cases} (i,i-1,t+i-1), & 2\leq i\leq k; \\ (k,k,t+k), & i=k+1; \\ (2k-i+1,2k-i+2,2k+t-i+1), & k+2\leq i\leq 2k, \end{cases}$$

for $s=t$,

$$r(u_s^{2i}|W)=\begin{cases} (t+i-1,t+i-2,i-1), & 2\leq i\leq k+1; \\ (2k+t-i+1,2k+t-i+2,2k-i+2), & k+2\leq i\leq 2k+1. \end{cases}$$

$$r(u_s^{2i+1}|W)=\begin{cases} (t-1,t,1), & i=0; \\ (t+i-1,t+i-2,i), & 1\leq i\leq k; \\ (k+t-1,k+t-1,k+1), & i=k+1; \\ (2k+t-i,2k+t-i+1,2k-i+2), & k+2\leq i\leq 2k, \end{cases}$$

and for $2\leq s\leq t-1$, we have

$$r(u_s^{2i}|W)=\begin{cases} (s,s,t-s), & i=1; \\ (s+i-1,s+i-2,t-s+i-1), & 2\leq i\leq k+1; \\ (2k+s-i+1,2k+s-i+2,2k+t-s-i+2), & k+2\leq i\leq 2k+1. \end{cases}$$

$$r(u_s^{2i+1}|W)=\begin{cases} (s-1,s,t-s+1), & i=0; \\ (s+i-1,s+i-2,t-s+i), & 1\leq i\leq k; \\ (k+s-1,k+s-1,k+t-s+1), & i=k+1; \\ (2k+s-i,2k+s-i+1,2k+t-s-i+2), & k+2\leq i\leq 2k. \end{cases}$$

It can be verified that all vertices have distinct representations with respect to W. This yields $\beta(C_n^2\Box P_t)\leq 3$ when $n\equiv 2 \pmod 4$.

Case(3). When $n\equiv 3 \pmod 4$. Then we can write as $n=4k+3$, where $k\geq 1$. For $s=1$,

$$r(u_s^{2i}|W)=\begin{cases} (1,1,t-1), & i=1; \\ (i,i-1,t+i-2), & 2\leq i\leq k+1; \\ (2k-i+2,2k-i+3,2k+t-i+2), & k+2\leq i\leq 2k+1. \end{cases}$$

$$r(u_s^{2i+1}|W)=\begin{cases} (i,i-1,t+i-1), & 2\leq i\leq k+1; \\ (2k-i+2,2k-i+3,2k+t-i+1), & k+2\leq i\leq 2k+1, \end{cases}$$

for $s=t$,

$$r(u_s^{2i}|W)=\begin{cases} (t+i-1,t+i-2,i-1), & 2\leq i\leq k+1; \\ (2k+t-i+1,2k+t-i+2,2k-i+3), & k+2\leq i\leq 2k+1. \end{cases}$$

$$r(u_s^{2i+1}|W)=\begin{cases} (t-1,t,1), & i=0; \\ (t+i-1,t+i-2,i), & 1\leq i\leq k; \\ (k+t,k+t-1,k+1), & i=k+1; \\ (2k+t-i+1,2k+t-i+2,2k-i+2), & k+2\leq i\leq 2k+1, \end{cases}$$

and for $2\leq s\leq t-1$, we have

$$r(u_s^{2i}|W)=\begin{cases} (s,s,t-s), & i=1; \\ (s+i-1,s+i-2,t-s+i-1), & 2\leq i\leq k+1; \\ (2k+s-i+1,2k+s-i+2,2k+t-s-i+3), & k+2\leq i\leq 2k+1. \end{cases}$$

$$r(u_s^{2i+1}|W)=\begin{cases} (s-1,s,t-s+1), & i=0; \\ (s+i-1,s+i-2,t-s+i), & 1\leq i\leq k; \\ (k+s,k+s-1,k+t-s+1), & i=k+1; \\ (2k+s-i+1,2k+s-i+2,2k+t-s-i+2), & k+2\leq i\leq 2k+1. \end{cases}$$

Again, in this case all the vertices have distinct representations with respect to W. This yields $\beta(C_n^2\square P_t)\leq 3$ when $n\equiv 3 \pmod 4$.

Conversely, we show that $\beta(C_n^2\square P_t)\geq 3$ when $n\equiv 0,2,3 \pmod 4$. Suppose on contrarily, $\beta(C_n^2\square P_t)=2$. Then by [22], we get a contradiction. Thus $\beta(C_n^2\square P_t)\geq 3$ when $n\equiv 0,2,3 \pmod 4$.

Case(4). When $n \equiv 1 \pmod 4$. Then we can write as $n = 4k+1$, where $k \geq 1$. First, we will give the representations of $V(C_n^2 \square P_t) \backslash W$ with respect to W. For $s=1$,

$$r(u_s^{2i}|W)=\begin{cases} (1,1,t-1), & i=1; \\ (i,i-1,t+i-2), & 2 \leq i \leq k; \\ (k,k,k+t-1), & i=k+1; \\ (2k-i+1,2k-i+2,2k+t-i+1), & k+2 \leq i \leq 2k. \end{cases}$$

$$r(u_s^{2i+1}|W)=\begin{cases} (i,i-1,t+i-1), & 2 \leq i \leq k; \\ (k,k,k+t-1), & i=k+1; \\ (2k-i+1,2k-i+2,2k+t-i), & k+2 \leq i \leq 2k, \end{cases}$$

for $s=t$,

$$r(u_s^{2i}|W)=\begin{cases} (t+i-1,t+i-2,i-1), & 2 \leq i \leq k; \\ (k+t-1,k+t-1,k), & i=k+1; \\ (2k+t-i,2k+t-i+1,2k-i+2), & k+2 \leq i \leq 2k. \end{cases}$$

$$r(u_s^{2i+1}|W)=\begin{cases} (t-1,t,1), & i=0; \\ (t+i-1,t+i-2,i), & 1 \leq i \leq k; \\ (k+t-1,k+t-1,k), & i=k+1; \\ (2k+t-i,2k+t-i+1,2k-i+1), & k+2 \leq i \leq 2k, \end{cases}$$

and for $2 \leq s \leq t-1$, we have

$$r(u_s^{2i}|W)=\begin{cases} (s,s,t-s), & i=1; \\ (s+i-1,s+i-2,t-s+i-1), & 2 \leq i \leq k; \\ (k+s-1,k+s-1,k+t-s), & i=k+1; \\ (2k+s-i,2k+s-i+1,2k+t-s-i+2), & k+2 \leq i \leq 2k. \end{cases}$$

$$r(u_s^{2i+1}|W)=\begin{cases} (s-1, s, t-s+1), & i=0; \\ (s+i-1, s+i-2, t-s+i), & 1 \leq i \leq k; \\ (k+s-1, k+s-1, k+t-s), & i=k+1; \\ (2k+s-i, 2k+s-i+1, 2k+t-s-i+1), & k+2 \leq i \leq 2k. \end{cases}$$

It can be seen that for $1 \leq s \leq t$, we have $r(u_s^{2k+2}|W) = r(u_s^{2k+3}|W) = (k+s-1, k+s-1, k+t-s)$. If we add the vertex u_t^4 to W. Then $d(u_t^4, u_s^{2k+2}) = k+t-s-1$ and $d(u_t^4, u_s^{2k+3}) = k+t-s$, where $1 \leq s \leq t$. Thus $W' = W \cup \{u_t^4\}$ resolves vertices of $C_n^2 \square P_t$. This yields $\beta(C_n^2 \square P_t) \leq 4$ when $n \equiv 1 \pmod 4$, which complete the proof. $\square$

3.2.3 Metric dimension of generalized antiprism

A generalized antiprism A_n^m can be obtained by completing the generlized prism $C_m \square P_n$ by edges $\{v_{i,j+1}v_{i+1,j} : 1 \leq i \leq m-1, 1 \leq j \leq n-1\} \cup \{v_{m,j+1}v_{1,j} : 1 \leq j \leq n-1\}$. Let $V(A_n^m) = V(C_m \square P_n)$ and $E(A_n^m) = E(C_m \square P_n) \cup \{v_{i,j+1}v_{i+1,j} : 1 \leq i \leq m-1, 1 \leq j \leq n-1\} \cup \{v_{m,j+1}v_{1,j} : 1 \leq j \leq n-1\}$ be the edge set of A_n^m, where i is taken modulo m. The metric dimension of antiprism denoted by A_n^2 has been determined by Javaid *et al.* [20].

In the next theorem, we determine the metric dimension of generalized antiprism.

Theorem 3.2.3. *Let A_n^m be the generalized antiprism. Then for every positive integer $n \geq 6$, we have*

$$\beta(A_n^m)=\begin{cases} 3, & \text{if } 2 \leq m \leq 5; \\ \geq 4, & \text{if } m \geq 6. \end{cases}$$

Proof. We denote $V(A_n^m) = \{v_{0,1}, v_{1,1}, \ldots, v_{n,1}, v_{0,2}, v_{1,2}, \ldots, v_{n,l}\}$, where $n \in \mathbb{Z}^+ \cup \{0\}$ and $1 \leq l \leq m$.

When $m = 2$, $A_n^2 \cong A_n$ (antiprism), and it was proved in [20] that $\beta(A_n) = 3$. Now, we prove that $\beta(A_n^m) = 3$ for $3 \leq m \leq 5$ and $n \geq 6$ by double inequality. We show that $W = \{v_{2,1}, v_{0,\lfloor \frac{m-1}{2} \rfloor}, v_{0,m}\}$ resolves all vertices of A_n^m by giving the representations of $V(A_n^m) \setminus W$.

Case (i). When $n = 2k$; $k \geq 3$. First, we will give the representations of all vertices of $V(A_n^m) \setminus W$ when $3 \leq m \leq 4$. For $l = 1$, we have $r(v_{1,l}|W) = (1, 1, m-1)$, and

$$r(v_{i,l}|W)=\begin{cases} (i-2,i,i), & 3\le i\le k;\\ (k-1,k-1,k+1), & i=k+1;\\ (2k-i+2,2k-i,2k+m-i-1), & k+2\le i\le 2k-1, \end{cases}$$

for $l=m$, $r(v_{1,l}|W)=(l-1,l,1)$, $r(v_{2k-2,l}|W)=(4,l-1,2)$, $r(v_{2k-1,l}\mid W)=(3,l-1,1)$ and

$$r(v_{i,l}|W)=\begin{cases} (i+l-3,i+l-1,i), & 2\le i\le k-2;\\ (i+l-3,2k-i,i), & i=k-1,k;\\ (2k-i+2,2k-i,2k-i), & k+1\le i\le 2k-3, \end{cases}$$

and for $2\le l\le m-1$, we have $r(v_{0,l}|W)=(2,l-1,m-l)$, $r(v_{1,l}|W)=(l-1,l,m-l)$, $r(v_{2k-1,l}|W)=(3,l-1,m-l+1)$ and

$$r(v_{i,l}|W)=\begin{cases} (i+l-3,i+l-1,i), & 2\le i\le k-1;\\ (k+l-3,k,k), & i=k;\\ (k+l-2,k-1,k+m-l-1), & i=k+1;\\ (2k-i+2,2k-i,2k+m-i-l), & k+2\le i\le 2k-2. \end{cases}$$

Now we will give the representations when $m=5$. For $l=1$, $r(v_{0,l}|W)=(2,1,4)$, $r(v_{1,l}|W)=(1,1,4)$, $r(v_{3,l}\mid W)=(1,3,4)$ and

$$r(v_{i,l}|W)=\begin{cases} (i-2,i,i), & 4\le i\le k;\\ (k-1,k,k+1), & i=k+1;\\ (2k-i+2,2k-i+1,2k-i+4), & k+2\le i\le 2k-1, \end{cases}$$

for $l=2$, $r(v_{1,l}|W)=(1,1,3)$, $r(v_{2,l}|W)=(1,2,3)$ and

$$r(v_{i,l}|W)=\begin{cases} (i-1,i,i), & 3\le i\le k;\\ (k,k-1,k+1), & i=k+1;\\ (2k-i+2,2k-i,2k-i+3), & k+2\le i\le 2k-1, \end{cases}$$

for $l=m$, $r(v_{1,l}|W)=(4,4,1)$, $r(v_{2k-2,l}|W)=(4,3,2)$, $r(v_{2k-1,l}|W)=(4,3,1)$ and

$$r(v_{i,l}|W)=\begin{cases} (i+2,i+3,i), & 2\le i\le k-2;\\ (k+1,k+1,k-1), & i=k-1;\\ (2k-i+2,2k-i,2k-i), & k\le i\le 2k-3,\end{cases}$$

and for $3 \le l \le m-1$, we have $r(v_{0,l}|W) = (l-1, l-2, m-l)$, $r(v_{1,l}|W) = (l-1, l-1, m-l)$, $r(v_{2k-1,l}|W) = (3, l-2, m-l+1)$ and

$$r(v_{i,l}|W)=\begin{cases} (i+l-3,i+l-2,i), & 2\le i\le k-1;\\ (k+l-3,k,k), & i=k;\\ (2k-i+2,2k-i,2k+m-i-l), & k+1\le i\le 2k-2.\end{cases}$$

Case (ii). When $n = 2k+1$; $k \ge 3$. For this, first we give the representations when $3 \le m \le 4$. For $l = 1$, $r(v_{1,l}|W) = (1, 1, m-l)$, and

$$r(v_{i,l}|W)=\begin{cases} (i-2,i,i), & 3\le i\le k;\\ (k-1,k,k+1), & i=k+1;\\ (k,k-1,k+m-2), & i=k+2;\\ (2k-i+3,2k-i+1,2k+m-i), & k+3\le i\le 2k,\end{cases}$$

for $l = m$, $r(v_{1,l}|W) = (l-1, l, 1)$, $r(v_{2k-1,l}|W) = (4, l-1, 2)$, $r(v_{2k,l}|W) = (3, m-1, 1)$ and

$$r(v_{i,l}|W)=\begin{cases} (i+l-3,i+l-1,i), & 2\le i\le k-1;\\ (k+l-3,k+1,k), & i=k;\\ (k+l-2,k,k), & i=k+1;\\ (2k-i+3,2k-i+1,2k-i+1), & k+2\le i\le 2k-2,\end{cases}$$

for $2 \le l \le m-1$, $r(v_{0,l}|W) = (2, l-1, m-l)$, $r(v_{1,l}|W) = (l-1, l, m-l)$, $r(v_{2k,l}|W) = (3, l-1, m-l+1)$ and

$$r(v_{i,l}|W)=\begin{cases} (i+l-3,i+l-1,i), & 2\le i\le k-1;\\ (i+l-3,2k-i+1,i), & i=k,k+1;\\ (2k-i+3,2k-i+1,2k+m-i-l+1), & k+2\le i\le 2k-1.\end{cases}$$

Now we will give the representations when $m = 5$. For $l = 1$, $r(v_{0,l}|W) = (2, 1, 4)$, $r(v_{1,l}|W) = (1, 1, 4)$, $r(v_{3,l}|W) = (1, 3, 4)$ and

$$r(v_{i,l}|W)=\begin{cases}(i-2,i,i), & 4\leq i\leq k+1;\\ (k,k,k+2), & i=k+2;\\ (2k-i+3,2k-i+2,2k-i+5), & k+3\leq i\leq 2k,\end{cases}$$

for $l=2$, $r(v_{1,l}|W)=(1,1,3)$, $r(v_{2,l}|W)=(1,2,3)$ and

$$r(v_{i,l}|W)=\begin{cases}(i-1,i,i), & 3\leq i\leq k;\\ (k,k,k+1), & i=k+1;\\ (2k-i+3,2k-i+1,2k-i+4), & k+2\leq i\leq 2k,\end{cases}$$

for $l=m$, $r(v_{1,l}|W)=(4,4,1)$, $r(v_{2k-1,l}|W)=(4,3,2)$, $r(v_{2k,l}|W)=(4,3,1)$ and

$$r(v_{i,l}|W)=\begin{cases}(i+2,i+3,i), & 2\leq i\leq k-1;\\ (k+2,k+1,k), & i=k;\\ (2k-i+3,2k-i+1,2k-i+1), & k+1\leq i\leq 2k-2,\end{cases}$$

and for $3\leq l\leq m-1$, we have $r(v_{0,l}|W)=(l-1,l-2,m-l)$, $r(v_{1,l}|W)=(l-1,l-1,m-l)$, $r(v_{2k,l}|W)=(3,l-2,m-l+1)$ and

$$r(v_{i,l}|W)=\begin{cases}(i+l-3,i+l-2,i), & 2\leq i\leq k-1;\\ (i+l-3,2k-i+1,i), & i=k,k+1;\\ (2k-i+3,2k-i+1,2k+m-i-l+1), & k+2\leq i\leq 2k-1.\end{cases}$$

It can be seen that all vertices of A_n^m have distinct representations with respect to W. This shows that $\beta(A_n^m)\leq 3$ for $3\leq m\leq 5$ and $n\geq 6$.

Conversely, suppose that $\beta(A_n^m)\geq 3$, where $3\leq m\leq 5$ and $n\geq 6$. Suppose on contrary that $\beta(A_n^m)=2$, but then by [22], we get a cotradiction. Hence $\beta(A_n^m)=3$, when $2\leq m\leq 5$ and $n\geq 6$. $\square$

3.3 Metric dimension of generalized Petersen multigraphs

In this section, we study the metric dimension of the subdivision of Möbius ladders and use this construction to study the metric dimension of generalized Petersen multigraphs $P(2n, n)$.

3.3.1 Metric dimension of barycentric subdivision of Möbius ladders

The *barycentric subdivision of Möbius ladders* denoted by SM_n is obtained by subdividing the edges of Möbius ladders by putting a vertex of degree two on each edge. It has $\frac{5n}{2}$ vertices and $3n$ edges and has exactly $\frac{n}{2}$ 8-cycles. There are n vertices of degree 3 and $\frac{3n}{2}$ vertices are of degree 2. Two different views of SM_{12} are shown in figure 3.3. For our convenience, we view the SM_n as a barycentric subdivision of prism D_n (the cartesain product of path on two vertices P_2 with a cylce C_n on n vertices) with one twisted edge. We denote vertices $\{v_0, v_1, \ldots, v_{2n-1}\} \subset V(SM_n)$ that are num-

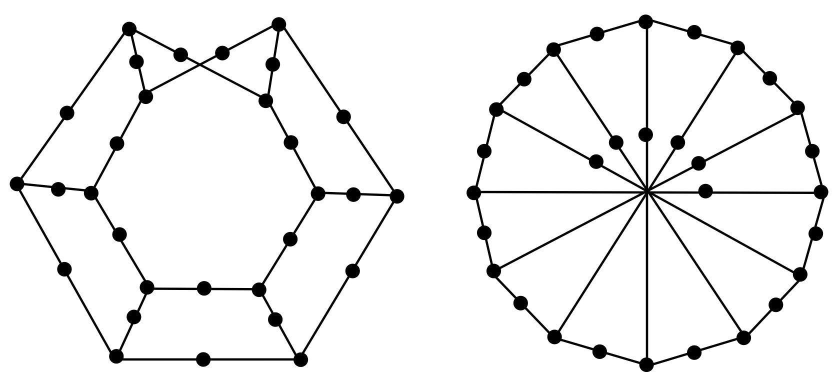

Figure 3.3: Two views of SM_{12}

bered clockwise and induce the cycle of length $2n$. Moreover $\{v_{2i+1} : 0 \leq i \leq n-1\}$ and $\{v_{2i} : 0 \leq i \leq n-1\}$ are the vertices of degree 2 and 3, respectively. We call the set of vertices $\{u_i : 0 \leq i \leq \frac{n}{2} - 1\} \subset V(SM_n)$ the internal vertices of $v_{2i} - v_{2i+n}$ paths, where $0 \leq i \leq \frac{n}{2} - 1$ and each u_i is a vertex of degree 2. So we have $V(SM_n) = \{v_i : 0 \leq i \leq 2n-1\} \cup \{u_i : 0 \leq i \leq \frac{n}{2}-1\}$. The metric dimension of Möbius ladders has been studied by Imran *et al.* [1] and proved that Möbius ladders constitute a family of cubic graphs with constant metric dimension 3 except when $n \equiv 2 \pmod 8$. In the next theorem, we extend this study to the metric dimension of barycentric subdivision of Möbius ladders denoted by SM_n. Note that the choice of appropriate basis vertices is the core of the problem.

Theorem 3.3.1. *Let SM_n denotes the barycentric subdivision of Möbius ladder, then $\beta(SM_n) = 3$ for every positive even integer $n \geq 8$.*

We prove this theorem by proving the following five lemmas. In proofs of Lemmas 3.3.2 to 3.3.5 each entry in codes tables is the distance between the vertices of column 1 and the vertices of row 1. Each row represents the code of a vertex, with respect to row 1, lying in column 1 of that row.

Lemma 3.3.2. *Let SM_n be the barycentric subdivision of Möbius ladder, then $\beta(SM_n) \leq 3$ when $n \equiv 0 \pmod 8$.*

Proof. When $n \equiv 0 \pmod 8$, we can write as $n = 8k$ where $k \geq 1$. We will prove that for a chosen index i such that $0 \leq i \leq 2n-1$, the set $W = \{v_i, v_{i+n-1}, u_{i+2k-1}\}$ is a resolving set for SM_n where $k = \frac{n}{8}$. The codes of the vertices in $V(SM_n) \setminus W$ with respect to W are the followings: $c_W(v_{i+n+4k-1}) = (4k+1, 4k, 2), c_W(v_{i+2n-1}) = (1, 4, 4k)$ and in tables 3.1 and 3.2.

$d(.,.)$	v_i	v_{i+n-1}	u_{i+2k-1}
$v_{i+j+1} : 0 \leq j \leq 4k-3$	$j+1$	$j+4$	$4k-j-2$
$v_{i+4k+j-1} : 0 \leq j \leq 2$	$4k+j-1$	$4k-j$	$j+2$
$v_{i+n-j-2} : 0 \leq j \leq 4k-4$	$j+4$	$j+1$	$4k-j+1$
$v_{i+n+j} : 0 \leq j \leq 4k-2$	$j+2$	$j+1$	$4k-j-1$
$v_{i+2n-j-2} : 0 \leq j \leq 4k-2$	$j+2$	$j+3$	$4k-j+1$

Table 3.1: Codes for the outer vertices of SM_n

$d(.,.)$	v_i	v_{i+n-1}	u_{i+2k-1}
$u_{i+j} : 0 \leq j \leq 2k-2$	$2j+1$	$2j+2$	$4k-2j$
$u_{i+2k+j} : 0 \leq j \leq 2k-1$	$4k-2j+1$	$4k-2j$	$2j+4$

Table 3.2: Codes for the inner vertices of SM_n

Since all the vertices lying in the first column of above two tables have distinct codes with respect to W, it implies that $\beta(SM_n) \leq 3$ when $n \equiv 0 \pmod 8$. ☐

Lemma 3.3.3. *Let SM_n be the barycentric subdivision of Möbius ladder, then $\beta(SM_n) \leq 3$ when $n \equiv 2 \pmod 8$.*

Proof. When $n \equiv 2 \pmod 8$, we can write $n = 8k + 2$ where $k \geq 1$. We will prove that for a chosen index i such that $0 \leq i \leq 2n - 1$, $W = \{v_i, v_{i+n-1}, u_{i+2k+1}\}$ is a resolving set for $V(SM_n)$ where $k = \frac{n-2}{8}$. The codes of the vertices in $V(SM_n) \setminus W$ with respect to W are: $c_W(v_{i+n}) = (2, 1, 4k + 1), c_W(v_{i+n+4k+1}) = (4k + 1, 4k + 2, 2), c_W(v_{i+2n-1}) = (1, 4, 4k), c_W(u_i) = (1, 2, 4k + 2)$ and in tables 3.3 and 3.4.

$d(.,.)$	v_i	v_{i+n-1}	u_{i+2k+1}
$v_{i+j+1} : 0 \leq j \leq 4k - 2$	$j + 1$	$j + 4$	$4k - j + 2$
$v_{i+4k+j} : 0 \leq j \leq 2$	$4k + j$	$4k - j + 1$	$3 - j$
$v_{i+n-j-2} : 0 \leq j \leq 4k - 3$	$j + 4$	$j + 1$	$4k - j - 1$
$v_{i+n+j+1} : 0 \leq j \leq 4k - 1$	$j + 3$	$j + 2$	$4k - j + 2$
$v_{i+2n-j-2} : 0 \leq j \leq 4k - 2$	$j + 2$	$j + 3$	$4k - j - 1$

Table 3.3: Codes for the outer vertices of SM_n

$d(.,.)$	v_i	v_{i+n-1}	u_{i+2k+1}
$u_{i+j+1} : 0 \leq j \leq 2k - 1$	$2j + 3$	$2j + 4$	$4k - 2j + 2$
$u_{i+2k+j+2} : 0 \leq j \leq 2k - 2$	$4k - 2j - 1$	$4k - 2j - 2$	$2j + 4$

Table 3.4: Codes for the inner vertices of SM_n

It can be seen that all the vertices lying in the first column of tables 3.3 and 3.4 have distinct codes with respect to W implying that $\beta(SM_n) \leq 3$ when $n \equiv 2 \pmod 8$. ☐

Lemma 3.3.4. *Let SM_n be the barycentric subdivision of Möbius ladder, then $\beta(SM_n) \leq 3$ when $n \equiv 4 \pmod 8$.*

Proof. When $n \equiv 4 \pmod 8$, we can write $n = 8k + 4$ where $k \geq 1$. We will show that for a chosen index i such that $0 \leq i \leq 2n - 1$, $W = \{v_i, v_{i+n-1}, u_{i+2k}\}$ is a resolving set for SM_n, where $k = \frac{n-4}{8}$. The codes of the vertices in $V(SM_n) \setminus W$ with respect to W are: $c_W(v_{i+n+4k+1}) = (4k + 3, 4k + 2, 2), c_W(v_{i+2n-1}) = (1, 4, 4k + 2)$

and in tables 3.5 and 3.6.

$d(.,.)$	v_i	v_{i+n-1}	u_{i+2k}
$v_{i+j+1} : 0 \leq j \leq 4k-1$	$j+1$	$j+4$	$4k-j$
$v_{i+4k+j+1} : 0 \leq j \leq 2$	$4k+j+1$	$4k-j+2$	$j+2$
$v_{i+n-j-2} : 0 \leq j \leq 4k-2$	$j+4$	$j+1$	$4k-j+3$
$v_{i+n+j} : 0 \leq j \leq 4k$	$j+2$	$j+1$	$4k-j+1$
$v_{i+2n-j-2} : 0 \leq j \leq 4k$	$j+2$	$j+3$	$4k-j+3$

Table 3.5: Codes for the outer vertices of SM_n

$d(.,.)$	v_i	v_{i+n-1}	u_{i+2k}
$u_{i+j} : 0 \leq j \leq 2k-1$	$2j+1$	$2j+2$	$4k-2j+2$
$u_{i+2k+j+1} : 0 \leq j \leq 2k$	$4k-2j+3$	$4k-2j+2$	$2j+4$

Table 3.6: Codes for the inner vertices of SM_n

It can be seen that no two vertices of SM_n lying in column 1 of tables 3.5 and 3.6 have the same code with respect to W, this yield that W is a resolving set for $V(SM_n)$. Hence $\beta(SM_n) \leq 3$ when $n \equiv 4 \pmod 8$. □

Lemma 3.3.5. *Let SM_n be the barycentric subdivision of Möbius ladder, then $\beta(SM_n) \leq 3$ when $n \equiv 6 \pmod 8$.*

Proof. When $n \equiv 6 \pmod 8$, we can write $n = 8k+6$ where $k \geq 1$. We will show that for a chosen index i such that $0 \leq i \leq 2n-1$, $W = \{v_i, v_{i+n-1}, u_{i+2k+2}\}$ is a resolving set for $V(SM_n)$, where $k = \frac{n-6}{8}$. The codes of the vertices in $V(SM_n) \setminus W$ with respect to W are: $c_W(v_{i+n}) = (2, 1, 4k+3), c_W(v_{i+n+4k+3}) = (4k+3, 4k+4, 2), c_W(v_{i+2n-1}) = (1, 4, 4k+2), c_W(u_i) = (1, 2, 4k+4)$ and in tables 3.7 and 3.8.

$d(.,.)$	v_i	v_{i+n-1}	u_{i+2k+2}
$v_{i+j+1} : 0 \leq j \leq 4k$	$j+1$	$j+4$	$4k-j+4$
$v_{i+4k+j+2} : 0 \leq j \leq 2$	$4k+j+2$	$4k-j+3$	$3-j$
$v_{i+n-j-2} : 0 \leq j \leq 4k-1$	$j+4$	$j+1$	$4k-j+1$
$v_{i+n+j+1} : 0 \leq j \leq 4k+1$	$j+3$	$j+2$	$4k-j+4$
$v_{i+2n-j-2} : 0 \leq j \leq 4k$	$j+2$	$j+3$	$4k-j+1$

Table 3.7: Codes for the outer vertices of SM_n

$d(.,.)$	v_i	v_{i+n-1}	u_{i+2k+2}
$u_{i+j+1} : 0 \leq j \leq 2k$	$2j+3$	$2j+4$	$4k-2j+4$
$u_{i+2k+j+3} : 0 \leq j \leq 2k-1$	$4k-2j+1$	$4k-2j$	$2j+4$

Table 3.8: Codes for the inner vertices of SM_n

It can be seen that all the vertices lying in the first column of tables 3.7 and 3.8 have distinct codes with respect to W implying that $\beta(SM_n) \leq 3$ when $n \equiv 6 \pmod 8$. □

Lemma 3.3.6. *Let SM_n be the barycentric subdivision of Möbius ladder, then $\beta(SM_n) \geq 3$ when $n \equiv 0, 2, 4, 6 \pmod 8$.*

Proof. Conversely, we show that $\beta(SM_n) \geq 3$ if $n \equiv 0, 2, 4, 6 \pmod 8$. Suppose on contrary that $\beta(SM_n) = 2$, then the following three possibilities arise.

(1). If both vertices belong to the set $\{v_i : 0 \leq i \leq 2n-1\}$. Without loss of generality, we can suppose that $W = \{v_i, v_{i+j}\}$ is a resolving set where $1 \leq j \leq 2n-1$. But then we get

- *If* $1 \leq j \leq \frac{n}{2}$, then $r(u_i|W) = r(v_{i+2n-1}|W) = (1, j+1)$.
- *If* $\frac{n}{2}+1 \leq j \leq n-2$, then $r(u_i|W) = r(v_{i+2n-1}|W) = (1, n-j+1)$.
- *If* $j = n-1$, then $r(v_{i+1}|W) = r(v_{i+2n-1}|W) = (1, 4)$.
- *If* $n \leq j \leq n+1$, then $r(v_{i+1}|W) = r(v_{i+2n-1}|W) = (1, j-n+3)$.
- *If* $n+2 \leq j \leq \frac{3n}{2}$, then $r(v_{i+1}|W) = r(u_i|W) = (1, j-n+1)$.
- *If* $\frac{3n}{2}+1 \leq j \leq 2n-1$, then $r(v_{i+1}|W) = r(u_i|W) = (1, 2n-j+1)$,

a contradiction.

(2). When both vertices belong to the set $\{u_i : 0 \leq i \leq \frac{n}{2}-1\}$. Without loss of generality, we can suppose that resolving set is $W = \{u_i, u_{i+j}\}$ where $1 \leq j \leq \frac{n}{2}-1$. However, we have in this case:

- *If* $1 \leq j \leq \frac{n}{4}$ *when* $n \equiv 0, 4 \pmod 8$ *and* $1 \leq j \leq \frac{n-2}{4}$ *when* $n \equiv 2, 6 \pmod 8$, then $r(v_i|W) = r(v_{i+n}|W) = (1, 2j+1)$.

- *If* $\frac{n}{4}+1 \leq j \leq \frac{n}{2}-1$ *when* $n \equiv 0,4 \pmod 8$ *and* $\frac{n+2}{4} \leq j \leq \frac{n}{2}-1$ *when* $n \equiv 2,6 \pmod 8$, then $r(v_i|W) = r(v_{i+n}|W) = (1, n-2j+1)$,

a contradiction.

(3). When one vertex belongs to the set $\{v_i : 0 \leq i \leq 2n-1\}$ and another belongs to the set $\{u_i : 0 \leq i \leq \frac{n}{2}-1\}$. Without loss of generality, we can choose resolving set as $W = \{v_i, u_{i+j}\}$ where $1 \leq j \leq \frac{n}{2}-1$. But then, we receive:

- *If* $j = 0$, then $r(v_{i+1}|W) = r(v_{i+2n-1}|W) = (1, 2)$.
- *If* $1 \leq j \leq \frac{n}{4}$ *when* $n \equiv 0,4 \pmod 8$ *and* $1 \leq j \leq \frac{n-2}{4}$ *when* $n \equiv 2,6 \pmod 8$, then $r(v_{i+2j+1}|W) = r(v_{i+n+2j-1}|W) = (2j+1, 2)$.
- *If* $\frac{n}{4}+1 \leq j \leq \frac{n}{2}-1$ *when* $n \equiv 0,4 \pmod 8$ *and* $\frac{n+2}{4} \leq j \leq \frac{n}{2}-1$ *when* $n \equiv 2,6 \pmod 8$, then $r(v_{i+2j+1}|W) = r(v_{i+n+2j-1}|W) = (n-2j+1, 2)$,

a contradiction again.

Hence in all possibilities we have $\beta(SM_n) = 3$ when $n \equiv 0,2,4,6 \pmod 8$. □

3.3.2 Metric dimension of generalized Petersen multigraphs $P(2n, n)$

The *generalized Petersen graphs* $P(n, m)$ form an important class of 3-regular graphs with $2n$ vertices and $3n$ edges having vertex set

$$V(P(n, m)) = \{u_1, u_2, ..., u_n, v_1, v_2, ..., v_n\}$$

and edge set

$$E(P(n, m)) = \{u_i u_{i+1}, u_i v_i, v_i v_{i+m} : 1 \leq i \leq n\}.$$

For $m = 1$, the generalized Petersen graph $P(n, 1)$ is called prism, denoted by D_n. In [6], Caceres *et al.* shown that

$$\beta(D_n) = \begin{cases} 2, & \text{if } n \text{ is odd;} \\ 3, & \text{otherwise.} \end{cases}$$

So prisms constitute a family of cubic graphs with constant metric dimension. In [20], Javaid *et al.* proved that the generalized Petersen graphs $P(n,2)$ is a family of graphs with constant metric dimension 3 for every positive integer $n \geq 5$.

Imran *et al.* [16] considered the generalized Petersen graphs $P(n,3)$ and deduced the following results:

Theorem 3.3.7. [16] For generalized Petersen graphs $P(n,3)$, we have

(a) $\beta(P(n,3)) = 4$ for $n \equiv 0 \pmod 6$ and $n \geq 24$.

(b) $\beta(P(n,3)) = 3$ for $n \equiv 1 \pmod 6$ and $n \geq 25$.

(c) $\beta(P(n,3)) \leq 5$ for $n \equiv 2 \pmod 6$ and $n \geq 8$.

(d) $\beta(P(n,3)) \leq 4$ for $n \equiv 3,4,5 \pmod 6$ and $n \geq 17$.

Javaid *et al.* [19] proved that

$$\beta(P(2n+1,n))=\begin{cases} 2, & \text{if } n=1; \\ 3, & \text{otherwise.} \end{cases}$$

It was proved by Javaid *et al.* [21], for all $m \geq 2$, the generalized Petersen graphs $P(2n,n-1)$ constitute a family of graphs with constant metric dimension 3 when $n(\geq 3)$ is odd, and metric dimension is 4 when $n(\geq 4)$ is even.

The generalized Petersen graphs $P(2n,n)$ are in fact the multigraphs. Here, we study their metric dimension and prove that the metric dimension is 3 when n is even and 4 otherwise.

The generalized Petersen multigraphs $P(2n,n)$ have vertex set $V(P(2n,n)) = \{v_i, u_i : 0 \leq i \leq 2n-1\}$ and the edge set $E(P(2n,n)) = \{v_iv_{i+1}, v_iu_i, u_iu_{i+n} : 0 \leq i \leq 2n-1\}$, where indices are taken modulo $2n$. Here for our convenience, we call the vertices $v_0, \ldots, v_{2n-1}$, outer vertices that numbered clockwise and $u_0, \ldots, u_{2n-1}$, the inner vertices. Note that in the generalized Petersen graphs $P(2n,n)$, the vertices u_i and u_{i+n} are joined by parallel edges, but since we are interested in finding the metric dimension of $P(2n,n)$ so these parallel edges have no role in calculating the metric dimension. So for this reason, we can view the vertices of the set $\{u_i : 0 \leq i \leq 2n-1\}$ as a vertices of degree two. Now after this observation, we can obtain the simple generalized Petersen graph $P(2n, n)$ from the graph of Möbius ladder M_{2n} by subdividing twice of its those edges that join the vertices v_i and v_{i+n}, where $0 \leq i \leq 2n-1$ and indices are taken modulo $2n$. Two different views of $P(12, 6)$ are shown in figure 3.4. For our convenience, we view the $P(12, 6)$ as

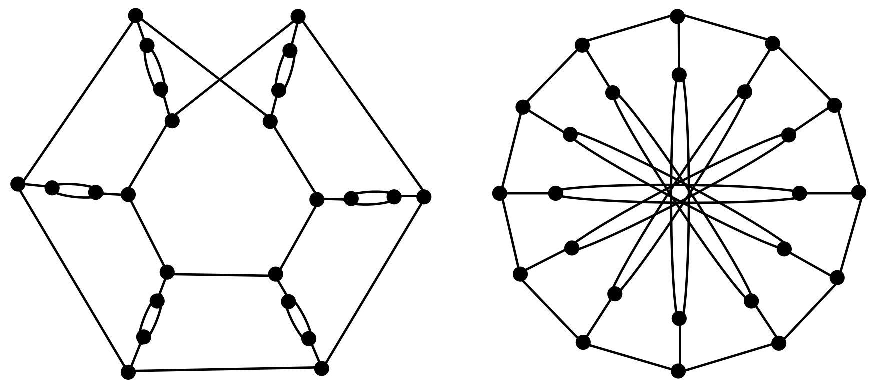

Figure 3.4: Two views of $P(12, 6)$

particular subdivision (as mentioned above) of prism with one twisted edge. In the next theorem, we extend the study to the metric dimension of generalized Petersen graphs $P(2n, n)$. Note that the choice of appropriate basis vertices is the core of the problem.

In proofs of Lemmas 3.3.8 to 3.3.11 each entry in codes tables is the distance between the vertices of column 1 and the vertices of row 1. Each row represents the code of a vertex, with respect to row 1, lying in column 1 of that row.

Lemma 3.3.8. *Let $P(2n, n)$ be the generalized Petersen multigraph, then we have $\beta(P(2n, n)) \leq 3$ when $n \equiv 0 \pmod 4$.*

Proof. When $n \equiv 0 \pmod 4$, then we can write $n = 4k$, where $k \geq 1$. We will show that for a chosen index i such that $0 \leq i \leq 2n-1$, $W = \{v_i, v_{i+n-1}, u_{i+n+2k}\}$ is a resolving set for $P(2n, n)$, where $k = \frac{n}{4}$.

For $n = 4$, the codes of the vertices in $P(2n, n) \setminus W$ with respect to $W = \{v_0, v_3, u_6\}$ are in tables 3.9 and 3.10.

$d(.,.)$	v_0	v_3	u_6
v_1	1	2	3
v_2	2	1	2
v_4	3	1	3
v_5	3	2	2
v_6	2	3	1
v_7	1	3	2

Table 3.9: Codes for the outer vertices of $P(8,4)$

$d(.,.)$	v_0	v_3	u_6
u_0	1	3	4
u_1	2	3	4
u_2	3	2	1
u_3	3	1	4
u_4	2	2	4
u_5	3	3	3
u_7	2	2	3

Table 3.10: Codes for the inner vertices of $P(8,4)$

It can be seen that all the vertices in $V(P(8,4))\setminus W$ have distinct codes with respect to W.

Now the codes of the vertices of $V(P(2n,n))\setminus W$ when $n>4$ are: $c_W(v_{i+2k+1}) = (2k+1, 2k-2, 3), c_W(u_i) = (1,3,2k+2), c_W(u_{i+2k-1}) = (2k, 2k+1, 4), c_W(u_{i+2k}) = (2k+1, 2k, 1)$ and in tables 3.11 and 3.12.

$d(.,.)$	v_i	v_{i+n-1}	u_{i+n+2k}
$v_{i+j+1} : 0 \le j \le 2k-4$	$j+1$	$j+5$	$2k-j+1$
$v_{i+2k+j-2} : 0 \le j \le 2$	$2k+j-2$	$2k-j+1$	$4-j$
$v_{i+n-j-2} : 0 \le j \le 2k-4$	$j+5$	$j+1$	$2k-j$
$v_{i+n+j} : 0 \le j \le 2k-2$	$j+3$	$j+1$	$2k-j+1$
$v_{i+n+2k+j-1} : 0 \le j \le 1$	$2k-j+1$	$2k+j$	$2-j$
$v_{i+2n-j-1} : 0 \le j \le 2k-2$	$j+1$	$j+3$	$2k-j$

Table 3.11: Codes for the outer vertices of $P(2n,n)$

$d(.,.)$	v_i	v_{i+n-1}	u_{i+n+2k}
$u_{i+j+1} : 0 \leq j \leq 2k-3$	$j+2$	$j+4$	$2k-j+2$
$u_{i+n-j-1} : 0 \leq j \leq 2k-2$	$j+3$	$j+1$	$2k-j+2$
$u_{i+n+j} : 0 \leq j \leq 2k-1$	$j+2$	$j+2$	$2k-j+2$
$u_{i+2n-j-1} : 0 \leq j \leq 2k-2$	$j+2$	$j+2$	$2k-j+1$

Table 3.12: Codes for the inner vertices of $P(2n, n)$

It can be seen that no two vertices of $P(2n, n)$ lying in column 1 of tables 3.11 and 3.12 have the same code with respect to W, this yields W is a resolving set for $P(2n, n)$. Hence $\beta(P(2n, n)) \leq 3$ when $n \equiv 0 \pmod 4$. □

Lemma 3.3.9. *Let $P(2n, n)$ be the generalized Petersen multigraph, then we have $\beta(P(2n, n)) \leq 3$ when $n \equiv 2 \pmod 4$.*

Proof. When $n \equiv 2 \pmod 4$, we can write $n = 4k + 2$, where $k \geq 1$. We will show that for a chosen index i such that $0 \leq i \leq 2n-1$, $W = \{v_i, v_{i+n-1}, u_{i+n+2k}\}$ is a resolving set for $P(2n, n)$, where $k = \frac{n-2}{4}$. The codes of the vertices in $V(P(2n, n)) \backslash W$ with respect to W are: $c_W(u_{i+2k}) = (2k+1, 2k+2, 1), c_W(u_{i+2k+1}) = (2k+2, 2k+1, 4), c_W(u_{i+n-1}) = (3, 1, 2k+3)$ and in tables 3.13 and 3.14.

$d(.,.)$	v_i	v_{i+n-1}	u_{i+n+2k}
$v_{i+j+1} : 0 \leq j \leq 2k-3$	$j+1$	$j+5$	$2k-j+1$
$v_{i+2k+j-1} : 0 \leq j \leq 1$	$2k+j-1$	$2k-j+2$	$3-j$
$v_{i+2k+j+1} : 0 \leq j \leq 1$	$2k+j+1$	$2k-j$	$j+3$
$v_{i+n-j-2} : 0 \leq j \leq 2k-3$	$j+5$	$j+1$	$2k-j+2$
$v_{i+n+j} : 0 \leq j \leq 2k-1$	$j+3$	$j+1$	$2k-j+1$
$v_{i+n+2k+j} : 0 \leq j \leq 1$	$2k-j+2$	$2k+j+1$	$j+1$
$v_{i+2n-j-1} : 0 \leq j \leq 2k-1$	$j+1$	$j+3$	$2k-j+2$

Table 3.13: Codes for the outer vertices of $P(2n, n)$

$d(.,.)$	v_i	v_{i+n-1}	u_{i+n+2k}
$u_{i+j} : 0 \leq j \leq 2k-1$	$j+1$	$j+3$	$2k-j+3$
$u_{i+n-j-2} : 0 \leq j \leq 2k-2$	$j+4$	$j+2$	$2k-j+3$
$u_{i+n+j} : 0 \leq j \leq 2k-1$	$j+2$	$j+2$	$2k-j+2$
$u_{i+2n-j-1} : 0 \leq j \leq 2k$	$j+2$	$j+2$	$2k-j+3$

Table 3.14: Codes for the inner vertices of $P(2n, n)$

Since all the vertices lying in the first column of above two tables have distinct codes with respect to W implying that $\beta(P(2n,n)) \leq 3$ when $n \equiv 2 \pmod 4$. $\square$

Lemma 3.3.10. *Let $P(2n,n)$ denotes the generalized Petersen multigraph for $n \equiv 1 \pmod 4$, then we have $\beta(P(2n,n)) \leq 4$.*

Proof. When $n \equiv 1 \pmod 4$, we can write as $n = 4k+1$, where $k \geq 1$. For a chosen index i such that $0 \leq i \leq 2n-1$, we show that $W = \{v_i, v_{i+n}, u_{i+n+2k-1}, u_{i+n+2k+1}\}$ is a resolving set for $P(2n,n)$, where $k = \frac{n-1}{4}$.

The codes of the vertices in $V(P(2n,n))\backslash W$ with respect to W are: $c_W(v_{i+n-1}) = (4, 1, 2k+1, 2k+1), c_W(u_i) = (1, 2, 2k+2, 2k+2), c_W(u_{i+2k-1}) = (2k, 2k+1, 1, 5)$, $c_W(u_{i+2k}) = (2k+1, 2k+2, 4, 4), c_W(u_{i+2k+1}) = (2k+2, 2k+1, 5, 1)$, $c_W(u_{i+n+2k}) = (2k+2, 2k+1, 3, 3)$ and in tables 3.15 and 3.16.

$d(.,.)$	v_i	v_{i+n}	$u_{i+n+2k-1}$	$u_{i+n+2k+1}$
$v_{i+j+1} : 0 \leq j \leq 2k-2$	$j+1$	$j+4$	$2k-j$	$2k-j+2$
$v_{i+2k+j} : 0 \leq j \leq 1$	$2k+j$	$2k-j+1$	$j+3$	$3-j$
$v_{i+n-j-2} : 0 \leq j \leq 2k-3$	$j+5$	$j+2$	$2k-j+2$	$2k-j$
$v_{i+n+j+1} : 0 \leq j \leq 2k-2$	$j+4$	$j+1$	$2k-j-1$	$2k-j+1$
$v_{i+n+2k+j} : 0 \leq j \leq 1$	$2k-j+1$	$2k+j$	$j+2$	$2-j$
$v_{i+2n-j-1} : 0 \leq j \leq 2k-2$	$j+1$	$j+4$	$2k-j+2$	$2k-j$

Table 3.15: Codes for the outer vertices of $P(2n,n)$

$d(.,.)$	v_i	v_{i+n}	$u_{i+n+2k-1}$	$u_{i+n+2k+1}$
$u_{i+j+1} : 0 \leq j \leq 2k-3$	$j+2$	$j+3$	$2k-j+1$	$2k-j+3$
$u_{i+n-j-2} : 0 \leq j \leq 2k-3$	$j+4$	$j+3$	$2k-j+3$	$2k-j+1$
$u_{i+n+j-1} : 0 \leq j \leq 1$	$3-j$	$2-j$	$2k-j+2$	$2k+j+2$
$u_{i+n+j+1} : 0 \leq j \leq 2k-3$	$j+3$	$j+2$	$2k-j$	$2k-j+2$
$u_{i+2n-j-1} : 0 \leq j \leq 2k-2$	$j+2$	$j+3$	$2k-j+3$	$2k-j+1$

Table 3.16: Codes for the inner vertices of $P(2n,n)$

It can be verified that all the vertices of $V(P(2n,n)) \setminus W$ that are lying in the first column of tables mentioned above have distinct codes with respect to W. This yield that $\beta(P(2n,n)) \leq 3$ for $n \equiv 1 \pmod 4$. $\square$

Lemma 3.3.11. *Let $P(2n, n)$ be the generalized Petersen multigraph for $n \equiv 3 \pmod 4$, then $\beta(P(2n, n)) \leq 4$.*

Proof. When $n \equiv 3 \pmod 4$, we write as $n = 4k + 3$ where $k \geq 1$. For a chosen index i such that $0 \leq i \leq 2n - 1$, we show that $W = \{v_i, v_{i+n}, u_{i+n+2k}, u_{i+n+2k+2}\}$ is a resolving set for $P(2n, n)$ where $k = \frac{n-3}{4}$.

The codes of the vertices in $V(P(2n, n))\backslash W$ with respect to W are: $c_W(v_{i+n-1}) = (4, 1, 2k + 2, 2k + 2)$, $c_W(u_i) = (1, 2, 2k + 3, 2k + 3)$, $c_W(u_{i+2k}) = (2k + 1, 2k + 2, 1, 5)$, $c_W(u_{i+2k+1}) = (2k+2, 2k+3, 4, 4)$, $c_W(u_{i+2k+2}) = (2k+3, 2k+2, 5, 1)$, $c_W(u_{i+n+2k+1})$ $= (2k + 3, 2k + 2, 3, 3)$ and in tables 3.17 and 3.18.

$d(.,.)$	v_i	v_{i+n}	u_{i+n+2k}	$u_{i+n+2k+2}$
$v_{i+j+1} : 0 \leq j \leq 2k - 1$	$j + 1$	$j + 4$	$2k - j + 1$	$2k - j + 3$
$v_{i+2k+j+1} : 0 \leq j \leq 1$	$2k + j + 1$	$2k - j + 2$	$j + 3$	$3 - j$
$v_{i+n-j-2} : 0 \leq j \leq 2k - 2$	$j + 5$	$j + 2$	$2k - j + 3$	$2k - j + 1$
$v_{i+n+j+1} : 0 \leq j \leq 2k - 1$	$j + 4$	$j + 1$	$2k - j$	$2k - j + 2$
$v_{i+n+2k+j+1} : 0 \leq j \leq 1$	$2k - j + 2$	$2k + j + 1$	$j + 2$	$2 - j$
$v_{i+2n-j-1} : 0 \leq j \leq 2k - 1$	$j + 1$	$j + 4$	$2k - j + 3$	$2k - j + 1$

Table 3.17: Codes for the outer vertices of $P(2n, n)$

$d(.,.)$	v_i	v_{i+n}	u_{i+n+2k}	$u_{i+n+2k+2}$
$u_{i+j+1} : 0 \leq j \leq 2k - 2$	$j + 2$	$j + 3$	$2k - j + 2$	$2k - j + 4$
$u_{i+n-j-2} : 0 \leq j \leq 2k - 2$	$j + 4$	$j + 3$	$2k - j + 4$	$2k - j + 2$
$u_{i+n+j-1} : 0 \leq j \leq 1$	$3 - j$	$2 - j$	$2k - j + 3$	$2k + j + 3$
$u_{i+n+j+1} : 0 \leq j \leq 2k - 2$	$j + 3$	$j + 2$	$2k - j + 1$	$2k - j + 3$
$u_{i+2n-j-1} : 0 \leq j \leq 2k - 1$	$j + 2$	$j + 3$	$2k - j + 4$	$2k - j + 2$

Table 3.18: Codes for the inner vertices of $P(2n, n)$

Since no two distinct vertices of $V(P(2n, n)) \setminus W$ lying in table 3.17 and table 3.18 have the same code. Thus we get $\beta(P(2n, n)) \leq 4$ when $n \equiv 3 \pmod 4$. □

Lemma 3.3.12. *Let $P(2n, n)$ be the generalized Petersen multigraph, then $\beta(P(2n, n))$ ≥ 3 when n is even and $n \equiv 0, 2 \pmod 4$.*

Proof. Suppose on contrary that $\beta(P(2n, n)) = 2$, then following three possibilities arise.

Case 1. When both vertices belong to the set $\{v_i : 0 \leq i \leq 2n-1\} \subset V(P(2n,n))$. For fixed i, suppose that resolving set is $W = \{v_i, v_{i+j}\}$. However then

- *If* $1 \leq j \leq \frac{n}{2}$, then $c_W(v_{i+2n-1}) = c_W(u_i) = (1, j+1)$.
- *If* $\frac{n}{2} + 1 \leq j \leq n-1$, then $c_W(v_{i+2n-1}) = c_W(u_i) = (1, n-j+2)$.
- *If* $j = n$, then $c_W(v_{i+1}) = c_W(v_{i+2n-1}) = (1, 4)$. And for $k = 1$ when $n \equiv 0 \pmod 4$, we have $c_W(v_{i+1}) = c_W(v_{i+7}) = (1, 3)$.
- *If* $n+1 \leq j \leq \frac{3n}{2} - 1$, then $c_W(v_{i+1}) = c_W(u_i) = (1, j-n+2)$.
- *If* $\frac{3n}{2} \leq j \leq 2n-1$, then $c_W(v_{i+1}) = c_W(u_i) = (1, 2n-j+1)$.

Case 2. When both vertices belong to the set $\{u_i : 0 \leq i \leq 2n-1\} \subset V(P(2n,n))$. For fixed i, suppose the resolving set is $W = \{u_i, u_{i+j}\}$. However, then

- *If* $1 \leq j \leq \frac{n}{2} - 1$, then $c_W(v_{i+n}) = c_W(v_{i+2n-1}) = (2, j+2)$.
- *If* $\frac{n}{2} \leq j \leq n-1$, then $c_W(v_{i+n}) = c_W(v_{i+2n-1}) = (2, n-j+1)$.
- *If* $j = n$, then $c_W(v_{i+1}) = c_W(v_{i+2n-1}) = (2, 3)$.
- *If* $n+1 \leq j \leq \frac{3n}{2}$, then $c_W(v_{i+1}) = c_W(v_{i+n}) = (2, j-n+1)$.
- *If* $\frac{3n}{2} + 1 \leq j \leq 2n-1$, then $c_W(v_{i+1}) = c_W(v_{i+n}) = (2, 2n-j+2)$.

Case 3. When one vertex belongs to the set $\{v_i : 0 \leq i \leq 2n-1\} \subset V(P(2n,n))$ and another belongs to $\{u_i : 0 \leq i \leq 2n-1\} \subset V(P(2n,n))$, then two subcases arise:

Subcase(i). For fixed i, suppose $W = \{v_i, u_{i+j}\}$ is a resolving set. However, we have

- *If* $j = 0$, then $c_W(v_{i+1}) = c_W(v_{i+2n-1}) = (1, 2)$.
- *If* $1 \leq j \leq \frac{n}{2} - 1$, then $c_W(v_{i+2n-1}) = c_W(u_i) = (1, j+2)$.

- *If* $\frac{n}{2} \leq j \leq n-2$, then $c_W(u_{i+n}) = c_W(u_{i+2n-1}) = (2, n-j+2)$.
- *If* $j = n-1$, then $c_W(v_{i+1}) = c_W(u_i) = (1, 4)$. And for $k = 1$ when $n \equiv 0 \pmod 4$, we have $c_W(v_{i+6}) = c_W(u_{i+4}) = (2, 3)$.
- *If* $j = n$, then $c_W(v_{i+1}) = c_W(v_{i+2n-1}) = (1, 3)$.
- *If* $n+1 \leq j \leq \frac{3n}{2} - 2$, then $c_W(v_{i+2n-1}) = c_W(u_i) = (1, j-n+3)$.
- *If* $\frac{3n}{2} - 1 \leq j \leq \frac{3n}{2}$, then $c_W(u_{i+1}) = c_W(u_{i+n}) = (2, j-n+2)$. For $k = 1$ when $n \equiv 0 \pmod 4$, then for $j = \frac{3n}{2} - 1$ we have $c_W(v_{i+2}) = c_W(u_{i+4}) = (2, 3)$ and when $j = \frac{3n}{2}$, we get $c_W(u_{i+1}) = c_W(u_{i+4}) = (2, 4)$.
- *If* $\frac{3n}{2} + 1 \leq j \leq 2n-1$, then $c_W(u_{i+1}) = c_W(u_{i+n}) = (2, 2n-j+3)$.

Subcase(ii). For fixed i, suppose $W = \{u_i, v_{i+j}\}$ is a resolving set. But then we receive

- *If* $0 \leq j \leq \frac{n}{2} - 2$, then $c_W(v_{i+j+2}) = c_W(u_{i+j+1}) = (j+3, 2)$.
- *If* $j = \frac{n}{2} - 1$, then $c_W(u_{i+\frac{n}{2}}) = c_W(u_{i+\frac{3n}{2}-1}) = (\frac{n}{2} + 2, 2)$.
- *If* $\frac{n}{2} \leq j \leq n-2$, then $c_W(u_{i+j+1}) = c_W(u_{i+j+n}) = (n-j+2, 2)$.
- *If* $j = n-1$, then $c_W(v_{i+n+1}) = c_W(u_{i+2n-1}) = (3, 2)$.
- *If* $n \leq j \leq \frac{3n}{2} - 3$, then $c_W(v_{i+j+2}) = c_W(u_{i+j+1}) = (j-n+4, 2)$.
- *If* $\frac{3n}{2} - 2 \leq j \leq \frac{3n}{2}$, then $c_W(u_{i+j-1}) = c_W(u_{i+j-n}) = (j-n+2, 2)$. For $k = 1$ when $n \equiv 0 \pmod 4$, then for $j = \frac{3n}{2} - 2$ we have $c_W(u_{i+3}) = c_W(u_{i+5}) = (4, 2)$. If $j = \frac{3n}{2} - 1$, then $c_W(v_{i+3}) = c_W(u_{i+1}) = (3, 2)$ and when $j = \frac{3n}{2}$, we have $c_W(u_{i+2}) = c_W(u_{i+5}) = (4, 2)$.
- *If* $\frac{3n}{2} + 1 \leq j \leq 2n-1$, then $c_W(u_{i+j-1}) = c_W(u_{i+j-n}) = (2n-j+3, 2)$.

We get a contradiction in all above cases, which implies that no two vertices for $V(P(2n, n))$ serve as basis vertices. Hence $\beta(P(2n, n)) \geq 3$ when $n \equiv 0, 2 \pmod 4$. □

Now we prove that $\beta(P(2n,n)) \geq 4$ when n is odd and $n \equiv 1,3 \pmod 4$. For this purpose, we need some more notions and definitions. Without loss of generality, we can suppose that the vertices $v_0, v_1, \ldots, v_{2n-1}$ of the outer cycle in the clockwise direction. For any two vertices v_i and v_j $(i \neq j)$, the clockwise distance, $d^*(v_i, v_j)$, is the distance measured in clockwise direction from v_i to v_j in the subgraph induced by the outer cycle. For example, $d^*(v_0, v_{2n-1}) = 2n-1$ and $d^*(v_{2n-1}, v_0) = 1$. This definition can be extended to any two vertices of $P(2n,n)$. The indices will be taken as modulo $2n$.

Consider a vertex on the outer cycle, say v_0. A vertex u_i is called a *good vertex* for v_0 if $d(v_0, u_i) = d$, where $d \in \{d(v_0, u_{i+n-1}), d(v_0, u_{i-n+1})\}$; otherwise, u_i is called a *bad vertex* for v_0. And v_i is called a good vertex for v_0 if $d(v_0, v_i) = d$, where $d \in \{d(v_0, v_{i+n-3}), d(v_0, v_{i-n+3})\}$. This definition can be extended to any two inner vertices belong to set $\{u_0, \ldots, u_{2n-1}\}$. The vertex v_i is a good vertex for u_0 if $d(u_0, v_i) = d$, where $d \in \{d(u_0, v_{i+n-1}), d(u_0, v_{i-n+1})\}$; u_i is a good vertex for u_l, say $l = 0$, if $d(u_0, u_i) = d$, where $d \in \{d(u_0, u_{i+n-1}), d(u_0, u_{i-n+1}), d(u_0, u_{i+2n-2}), d(u_0, u_{i-2n+2}), d(u_l, v_l)\}$ and bad otherwise.

It is important to note that the set of good vertices for u_0 can be obtained from the set of good vertices for v_0 by adding vertices v_1, v_2, v_{2n-2} and v_{2n-1}. Similarly a vertex u_j is good for the pair $\{v_0, v_i\}$ if it satisfies the above definition for a vertex u_j to be good for the outer vertices. If v_l is good for the pairs $\{v_0, v_i\}$ and $\{v_0, v_j\}$ then v_l is good for the triplet $\{v_0, v_i, v_j\}$. Due to rotational symmetry of the graph $P(2n,n)$ we deduce the following result:

Lemma 3.3.13. *For any two vertices x_i and y_j of $P(2n,n)$ such that $x_i \neq x_j$, we have $d(x_i, y_j) = d(x_{i+r}, y_{i+r})$ for any $1 \leq r \leq 2n-1$.*

In order to find for pairs of good vertices belonging to the outer cycle, the following lemmas will be useful.

Lemma 3.3.14. *Let $0 \leq j \leq 2n-3$. If u_i is good for v_0 and u_{i-j-1} is also good for v_0, then u_i is also good for the pair $\{v_0, v_{j+1}\}$.*

Proof. By definition, $d(v_0, u_i) = d(v_0, u_{i+n-1})$ or $d(v_0, u_i) = d(v_0, u_{i-n+1})$ and $d(v_0, u_{i-j-1}) = d(v_0, u_{i-j+n-2})$ or $d(v_0, u_{i-j-1}) = d(v_0, u_{i-j-n})$. By Lemma 3.3.13, the last two equalities imply that $d(v_{1+j}, u_i) = d(v_{1+j}, u_{i+n-1})$ or $d(v_{1+j}, u_i) = d(v_{1+j}, u_{i-n+1})$. □

Lemma 3.3.15. *Let $0 \leq j \leq 2n-7$. If v_i is good for v_0 and v_{i-j-1} is also good for v_0, then v_i is also good for the pair $\{v_0, v_{j+1}\}$.*

Clearly $\beta(P(2n, n)) > 1$ because paths are the only graph with metric dimension 1 [22].

Lemma 3.3.16. *If n is odd and $n \equiv 1, 3 \pmod 4$, then we have $\beta(P(2n, n)) \geq 3$ for every positive integer $n \geq 7$.*

Proof. We show that there is no resolving set of $V(P(2n, n))$ consisting of two vertices A and B. If both A and B belong to the outer cycle, we can suppose that $A = v_0$. Let $d^*(v_0, B) = j+1$. Since the vertices $u_1, u_2, \ldots, u_{2n-1}$ and $v_3, v_4, \ldots, v_{2n-3}$ are good vertices for v_0. By using Lemma 3.3.14, we find that u_{2n-1} is a good vertex for all pairs $\{v_0, B\}$, where $B \in \{v_{j+1} : 0 \leq j \leq 2n-3\}$ and if $B = v_{2n-1}$, then u_1 is a good vertex for the pair $\{v_0, B\}$. Similarly, by Lemma 3.3.15 we can find that v_{2n-3} is a good vertex for every pair $\{v_0, B\}$ such that $B \in \{v_{j+1} : 0 \leq j \leq 2n-7\}$ and if $B \in \{v_{2n-5}, v_{2n-4}, \ldots, v_{2n-1}\}$, then v_{2n-8} is good for all pairs $\{v_0, B\}$.

If $A, B \in \{u_i : 0 \leq i \leq 2n-1\}$, we can consider $A = u_0$ and $B = u_i$. This case can be reduced to the case when $A = v_0$ and $B = v_i$ because the set of good vertices for u_l also includes the set of good vertices for v_l for any $0 \leq l \leq 2n-1$. If $A = v_i$ and $B = u_i$, then any good vertex for v_i is also a good vertex for u_i, hence for the pair $\{A, B\}$. The remaining case when $A = v_i$ and $B = u_j$ $(i \neq j)$ can also be reduced to the case when $A = v_i$ and $B = v_j$. It follows that there is no resolving set containing two vertices in this case, which completes the proof. □

Lemma 3.3.17. *If n is odd and $n \equiv 1, 3 \pmod 4$, then we have $\beta(P(2n, n)) \geq 4$ for every positive integer $n \geq 7$.*

Proof. Clearly, $\beta(P(2n, n)) \geq 3$, by Lemma 3.3.16. Now we have to show that there is no resolving set of $V(P(2n, n))$ consisting of three vertices A, B and C when $n \equiv 1, 3 \pmod 4$. By the same reasoning as in Lemma 3.3.16, it is enough to consider only the case when A, B and C belong to the outer cycle. Since the set of good vertices for u_0 can be obtained from the set of good vertices for v_0 by adding vertices v_0, v_2, v_{2n-2} and v_{2n-1}. Without loss of generality we suppose that $d^*(A,B) < d^*(A,C)$ and let $A = v_0$. Consider $(d^*(v_0,B), d^*(v_0,C)) = (l+1, j+1)$ such that $B = v_{l+1}$ and $C = v_{j+1}$. Since $u_1, \ldots, u_{2n-1}$ and $v_3, \ldots, v_{2n-3}$ are good vertices for v_0. By applying Lemma 3.3.14, we find that u_{2n-1} is a good vertex for all the pairs $\{v_0, B\}$ and $\{v_0, C\}$, where $l, j = 0, 1, \ldots, 2n-3$ and hence for all the triplets $\{v_0, B, C\}$. And if $B = v_{2n-2}$ and $C = v_{2n-1}$, then u_1 is a good vertex for the pairs $\{v_0, B\}$ and $\{v_0, C\}$ and hence for the triplet $\{v_0, B, C\}$. Now by Lemma 3.3.15, we find v_{2n-3} is a good vertex for all the pairs $\{v_0, B\}$ and $\{v_0, C\}$, where $l, j = 0, 1, \ldots, 2n-7$ and hence for all the triplets $\{v_0, B, C\}$. When $B, C \in \{v_{2n-5}, \ldots, v_{2n-1}\}$, then v_{2n-8} is a good vertex for all pairs $\{v_0, B\}$ and $\{v_0, C\}$ and hence for all triplets $\{v_0, B, C\}$. It follows that there is no resolving set with three vertices in this case, which completes the proof. □

Theorem 3.3.18. *Let $P(2n, n)$ deontes the generalized Petersen multigraph, then for every positive integer $n \geq 2$ we have*

$$\beta(P(2n,n)) = \begin{cases} 3, & \text{if } n \text{ is even;} \\ 4, & \text{otherwise.} \end{cases}$$

Proof. **Case 1**. When n is even and $n \equiv 0, 2 \pmod 4$. By Lemmas 3.3.8, 3.3.9 and 3.3.12, $\beta(P(2n,n)) = 3$.

Case 2. When n is odd and $n \equiv 1, 3 \pmod 4$. By Lemmas 3.3.10, 3.3.11, 3.3.16 and 3.3.17, $\beta(P(2n,n)) = 4$. □

Chapter 4

Exchange property for resolving sets in graphs

We have seen that if each vertex in G is uniquely determined by its distances to the vertices of $W \subseteq V(G)$, then W is called a resolving set. Since every vertex in the graph can be uniquely identified with respect to the vertices of resolving sets, so they act like bases in a *vector space*. However, resolving sets do not always have the exchange property from linear algebra because they share some of the properties of bases in a vector space. Resolving sets are said to have the *exchange property* in G if whenever R and T are minimal resolving sets for G and $t \in T$, then there exists $r \in R$ so that $(R \setminus \{r\}) \cup \{t\}$ is a minimal resolving set [4].

If the exchange property holds for a graph G, then the size of every minimal resolving set for G is same and algorithmic methods for finding the metric dimension of G are more feasible. So, if we have to show that in a given graph the exchange property does not hold, it suffices to prove that there are two minimal resolving sets having different size. The converse is not true. Thus, if we know that the exchange property does not hold, then the existence of two distinct resolving sets having different size is not guaranteed.

The following results concerning exchange property for resolving sets were deduced by Boutin [4].

Theorem 4.0.19. [4] *The exchange property holds for resolving sets in trees.*

Theorem 4.0.20. [4] *Resolving sets do not have the exchange property in wheels W_n for every positive integer $n \geq 8$.*

In this chapter, we study the exchange property for resolving sets in antiweb-wheels, some rotationally-symmetric graphs, barycentric subdivision of Möbius ladders and generalized Petersen multigraphs $P(2n, n)$.

4.1 Exchange property for resolving sets in antiweb-wheels

The next theorem shows that the exchange property does not hold for resolving sets of antiweb-wheels for $n \geq 10$.

Theorem 4.1.1. *For every positive integer $n \geq 10$, resolving sets do not have the exchange property in antiweb-wheels AWW_n.*

Proof. We consider the six cases according to the residue class modulo six. We show that in each case, there are minimal resolving sets having different size.

(1). If $n \equiv 0 \pmod 6$ then we write it as $n = 6k$, where $k \geq 2$. Then $B = \{v_{6i+1}, v_{6i+3} : 0 \leq i \leq k-1\}$ is a metric basis [see Theorem 3.1.1] and hence a minimal resolving set.

It can be seen that $B' = \{v_1\} \cup \{v_{6i+5}, v_{6i+7} : 0 \leq i \leq k-2\} \cup \{v_{6k-3}, v_{6k-2}\}$ is also a minimal resolving set. There is no $b \in B'$ such that $B' \backslash \{b\}$ is still a resolving set. If $b = v_1$, then removal of v_1 would yield a gap having six vertices. If $b \in \{v_{6i+5}, v_{6i+7} : 0 \leq i \leq k-2\} \setminus \{v_{6k-5}\}$, then removal of b would yield neighboring gaps, one containing five vertices and other one will have at least one vertex. If $b \in \{v_{6k-5}, v_{6k-2}\}$, then there will be neighboring gaps containing three vertices each. Also if $b = v_{6k-3}$, then a gap containing two vertices will have both neighboring gaps non empty, a contradiction in all cases. Therefore we have $|B| = 2k$ and $|B'| = 2k + 1$.

(2). If $n \equiv 1 \pmod 6$, then we may write $n = 6k + 1$, where $k \geq 2$. Then $B = \{v_1\} \cup \{v_{6i+3}, v_{6i+7} : 0 \leq i \leq k-2\} \cup \{v_{6k-2}, v_{6k-1}\}$ is a metric basis [see Theorem 3.1.1] and hence a minimal resolving set.

Also $B' = \{v_1, v_5\} \cup \{v_{6i+6}, v_{6i+8} : 0 \leq i \leq k-2\} \cup \{v_{6k-2}, v_{6k-1}\}$ is a minimal resolving set. There is no $b \in B'$ such that $B' \setminus \{b\}$ is a resolving set. If $b = v_1$, then removal of v_1 would yield a gap having six vertices. If $b = v_5$, then one of the neighboring gap will contain four vertices and other will be non empty. If $b \in \{v_6, v_{6k-2}\}$, then the gap containing two vertices will have both neighboring gaps non empty. If $b \in \{v_{6i+6}, v_{6i+8} : 0 \leq i \leq k-2\} \setminus \{v_6, v_{6k-4}\}$, then removal of b would leave neighboring gaps, one having five vertices and other one will contain at least one vertex. If $b = v_{6k-4}$ for $k = 2$, then both neighboring gaps of an empty gap will contain three vertices. If $b = v_{6k-1}$ for $k = 2$ and $b \in \{v_{6k-4}, v_{6k-1}\}$ where $k \geq 3$, then there will be neighboring gaps having three vertices each. Therefore we have $|B| = 2k+1$ and $|B'| = 2k+2$.

(3). If $n \equiv 2 \pmod 6$, then $n = 6k+2$, where $k \geq 2$. Then $B = \{v_1\} \cup \{v_{6i+3}, v_{6i+5} : 0 \leq i \leq k-1\}$ is a metric basis [see Theorem 3.1.1] and hence a minimal resolving set.

Moreover $B' = \{v_1, v_5\} \cup \{v_{6i+6}, v_{6i+8} : 0 \leq i \leq k-2\} \cup \{v_{6k-1}, v_{6k}\}$ is another minimal resolving set. There is no $b \in B'$ such that $B' \setminus \{b\}$ is a resolving set. If $b = v_1$, then removal of v_1 would yield a gap of six vertices. If $b = v_5$ for $k = 2$ and $b \in \{v_5, v_{6k-4}\}$ for $k \geq 3$, then one of the neighboring gap will contain four vertices and other gap will contain at least one vertex. If $b = v_{6k-4}$ for $k = 2$, then removal of b would leave one of the neighboring gap of an empty gap containing three vertices and other one will contain four vertices. If $b \in \{v_{6i+6}, v_{6i+8} : 0 \leq i \leq k-2\} \setminus \{v_6, v_{6k-4}\}$, then removal of b would yield neighboring gaps, one having five vertices and other one will contain at least one vertex. If $b \in \{v_6, v_{6k-1}\}$, then the gap containing two vertices will have both neighboring gaps non empty. If $b = v_{6k}$, then removal of b would leave neighboring gaps having three vertices each. Therefore we have $|B| = 2k+1$ and $|B'| = 2k+2$.

(4). If $n \equiv 3 \pmod 6$, then we can write $n = 6k+3$, where $k \geq 2$. Then $B = \{v_1, v_3\} \cup \{v_{6i+5}, v_{6i+9} : 0 \leq i \leq k-2\} \cup \{v_{6k}, v_{6k+1}\}$ is a metric basis [see Theorem 3.1.1] and hence a minimal resolving set.

Also $B' = \{v_1, v_6, v_7\} \cup \{v_{6i+9}, v_{6i+11} : 0 \leq i \leq k-2\} \cup \{v_{6k}, v_{6k+3}\}$ is a minimal resolving set. There is no $b \in B'$ such that $B' \setminus \{b\}$ is a resolving set. If $b \in \{v_1, v_6\} \cup \{v_{6i+9}, v_{6i+11} : 0 \leq i \leq k-2\} \setminus \{v_9, v_{6k-1}\}$, then removal of b would leave neighboring gaps, one having five vertices and other one will contain at least one vertex. If $b \in \{v_7, v_{6k-1}\}$, then the gap containing two vertices will have both neighboring gaps non empty. If $b \in \{v_9, v_{6k}\}$, then removal of b would yield one neighboring gap of an empty gap containing three vertices and other one will contain more than three vertices. If $b = v_{6k+3}$, then it would leave neighboring gaps one having four vertices and other one will contain at least one vertex. Therefore we have $|B| = 2k + 2$ and $|B'| = 2k + 3$.

(5). If $n \equiv 4 \pmod 6$, then we may write $n = 6k + 4$, where $k \geq 1$. Then $B = \{v_1, v_3\} \cup \{v_{6i+5}, v_{6i+7} : 0 \leq i \leq k-1\}$ is a metric basis [see Theorem 3.1.1] and hence a minimal resolving set.

Moreover $B' = \{v_1, v_5\} \cup \{v_{6i+6}, v_{6i+8} : 0 \leq i \leq k-1\} \cup \{v_{6k+4}\}$ is a minimal resolving set. There is no $b \in B'$ such that $B' \backslash \{b\}$ is a resolving set. If $b \in \{v_1, v_5\}$, then removal of b would leave neighboring gaps, one containing four vertices and other one will have at least one vertex. If $b \in \{v_6, v_{6k+4}\}$, then the gap containing two vertices will have both neighboring gaps non empty. If $b \in \{v_{6i+6}, v_{6i+8} : 0 \leq i \leq k-1\} \setminus \{v_6, v_{6k+2}\}$, then removal of b would yield neighboring gaps, one containing five vertices and other one will contain at least one vertex. If $b = v_{6k+2}$, then it would leave both neighboring gaps of an empty gap containing three vertices. Therefore $|B| = 2k + 2$ and $|B'| = 2k + 3$.

(6). If $n \equiv 5 \pmod 6$, then we write $n = 6k + 5$, where $k \geq 1$. Then $B = \{v_{6i+1}, v_{6i+5} : 0 \leq i \leq k-1\} \cup \{v_{6k+2}, v_{6k+3}\}$ is a metric basis [see Theorem 3.1.1] and hence a minimal resolving set.

Also $B' = \{v_1, v_5\} \cup \{v_{6i+6}, v_{6i+8} : 0 \leq i \leq k-1\} \cup \{v_{6k+5}\}$ is a minimal resolving set too. There is no $b \in B'$ such that $B' \backslash \{b\}$ is a resolving set. If $b \in \{v_1, v_5\}$ for $k = 1$ and $b \in \{v_1, v_5, v_{6k+2}\}$, where $k \geq 2$, then removal of b would yield neighboring gaps, one having four vertices and other one will contain at least one vertex. If $b = v_{6k+2}$ for $k = 1$, then it would yield one neighboring gap of an empty gap containing three vertices and other one will have four vertices. If $b = v_6$, then the gap containing two vertices will have both neighboring gaps non empty. If $b \in \{v_{6i+6}, v_{6i+8} : 0 \leq i \leq k-1\} \setminus \{v_6, v_{6k+2}\}$, then removal of b would leave neighboring gaps, one having five vertices and other one will contain at least one vertex. If $b = v_{6k+5}$, then it would leave both neighboring gaps containing three vertices each. Therefore $|B| = 2k + 2$ and $|B'| = 2k + 3$. □

4.2 Exchange property for resolving sets in rotationally-symmetric graphs

In this section, we show that exchange property does not hold for resolving sets of quasi flower snarks, generalized antiprism when $3 \leq m \leq 5$ and generalized prism.

Theorem 4.2.1. *The exchange property for minimal resolving sets does not hold in quasi flower snarks for $n \geq 4$.*

Proof. We consider the following two cases.

Case (i). For $n = 2k+1$, where $k \geq 2$. Since $W = \{b_0, c_0, d_{n-1}\}$ is a metric basis [see Theorem 3.2.1] and hence a minimal resolving set. Also $W^* = \{a_0, b_0, c_0, a_1\}$ is a minimal resolving set. There is no $w \in W^*$ such that $S = W^* \setminus \{w\}$ is still a resolving set.

If $w = a_0$, then $r(a_{2k}|S) = r(d_2|S) = (3, 3, 2)$. When $w = b_0$, then $r(d_2|S) = r(b_1|S) = (3, 3, 2)$. If $w = c_0$, then $r(d_2|S) = r(c_1|S) = (3, 3, 2)$ and when $w = a_1$, then $r(d_k|S) = r(d_{k+1}|S) = (k + 1, k + 1, k + 1)$. Therefore, we get $|W| = 3$ and $|W^*| = 4$.

Case (ii). For $n = 2k$, where $k \geq 2$. Since $W' = \{a_0, b_0, c_0, d_{n-1}\}$ is a metric basis [see Theorem 3.2.1] and hence a minimal resolving set. Also $W^* = \{a_0, d_0, a_1, d_k, c_k\}$ is a minimal resolving set. There is no $w \in W^*$ such that $S = W^* \setminus \{w\}$ is still a resolving set.

If $w = a_0$, then $r(a_{2k-1}|S) = r(b_1|S) = (2, 2, k, k + 1)$. When $w = d_0$, then we get $r(d_{2k-1}|S) = r(c_0|S) = (2, 3, k + 1, k)$. If $w = a_1$, then $r(d_1|S) = r(d_{2k-1}|S) = (2, 3, k + 1, k)$. If $w = d_k$, then $r(a_2|S) = r(d_1|S) = (2, 3, 1, k)$ and when $w = c_k$, then $r(b_0|S) = r(c_0|S) = (2, 1, 3, k + 1)$. Therefore, $|W'| = 4$ and $|W^*| = 5$.

There are minimal resolving sets of different size in both cases. Hence exchange property for minimal resolving sets does not hold in quasi flower snarks for $n \geq 4$. □

Caceres *et al.* [6] determined the metric dimension of cartesian product of cycles and paths by using the idea of doubly resolving sets. In the next theorem, first we find the metric basis of cartesian product of cycles and paths by using the appropriate choice of basis vertices and then we use this result to explore the exchange property for resolving sets of cartesian product of cycles and paths.

Theorem 4.2.2. *For every positive integer $n \geq 3$,*

$$\beta(C_n \Box P_t) = \begin{cases} 2, & \text{if } n \text{ is odd;} \\ 3, & \text{otherwise.} \end{cases}$$

Proof. We denote $V(C_n \Box P_t) = \{v_{0,1}, v_{1,1}, \cdots, v_{n,1}, v_{0,2}, v_{1,2}, \cdots, v_{n,l}\}$, where $n \in \mathbb{Z}^+ \cup \{0\}$ and $1 \leq l \leq t$.

Case (i). When $n = 2k+1$, where $k \geq 1$. We will show that $W = \{v_{0,1}, v_{k,1}\}$ resolves all vertices of $C_n \Box P_t$ by giving the representations of $V(C_n \Box P_t) \backslash W$. For $l = 1$

$$r(v_{i,l}|W) = \begin{cases} (i, k-i), & 1 \leq i \leq k-1; \\ (2k-i+1, i-k), & k+1 \leq i \leq 2k, \end{cases}$$

and for $l \geq 2$, we have

$$r(v_{i,l}|W) = \begin{cases} (i+l-1, k+l-i-1), & 0 \leq i \leq k; \\ (2k+l-i, i+l-k-1), & k+1 \leq i \leq 2k. \end{cases}$$

It can be seen that all vertices have distinct representations. It shows that $\beta(C_n \Box P_t) \leq 2$ when $n = 2k+1$, where $k \geq 1$.

Conversely, we show that $\beta(C_n \Box P_t) \geq 2$. Suppose contrarily that $\beta(C_n \Box P_t) = 1$, but then from [22], a contradiction. Hence showed $\beta(C_n \Box P_t) = 2$ when $n = 2k+1$, where $k \geq 1$.

Case (ii). When $n = 2k$, where $k \geq 2$. First, we will give the representations of $V(C_n \Box P_t) \setminus W$ with respect to W. For $l = 1$

$$r(v_{i,l}|W) = \begin{cases} (i, k-i), & 1 \leq i \leq k-1; \\ (2k-i, i-k), & k+1 \leq i \leq 2k-1, \end{cases}$$

and for $l \geq 2$, we have

$$r(v_{i,l}|W) = \begin{cases} (i+l-1, k+l-i-1), & 0 \leq i \leq k; \\ (2k+l-i-1, i+l-k-1), & k+1 \leq i \leq 2k-1. \end{cases}$$

It can be seen that for $1 \leq i \leq k-1$, $r(v_{i,l}|W) = r(v_{2k-i,l}|W) = (i+l-1, k+l-i-1)$. In order to have distinct representations we add the vertex $v_{1,1}$ to W. Then $d(v_{1,1}, v_{i,l}) = i + l - 2$ and $d(v_{1,1}, v_{2k-i,l}) = i + l$, where $1 \leq i \leq k - 1$. Thus we have $W' = W \cup \{v_{1,1}\}$ which resolves $V(C_n \Box P_t)$. This mean that $\beta(C_n \Box P_t) \leq 3$ for $n = 2k$, where $k \geq 2$.

Conversely, we show that $\beta(C_n \Box P_t) \geq 3$ for $n = 2k$, where $k \geq 2$. Suppose that the sets of vertices $\{v_{0,1}, v_{1,1}, \cdots, v_{n,1}\}$ and $\{v_{0,t}, v_{1,t}, \cdots, v_{n,t}\}$, where $n \in \mathbb{Z}^+ \cup \{0\}$, induce the inner and outer cycles respectively. Suppose on contrary that $\beta(C_n \Box P_t) = 2$, then the following possibilities arise.

(1). If we take any of the basis vertex from the set $\{v_{0,2}, v_{1,2}, \cdots, v_{0,3}, v_{1,3}, \cdots, v_{n,t-1}\}$, where $n \in \mathbb{Z}^+ \cup \{0\}$. Then by [22], we get a contradiction.

(2). Without loss of generality, suppose that both basis vertices belong to the inner cycle. Then we may fix $v_{0,1}$ as a basis vertex. If we choose $v_{i,1}$, where $1 \leq i \leq 2k - 1$, as a second basis vertex, then we get

- *If* $i = 1$, then $r(v_{0,2}|\{v_{0,1}, v_{i,1}\}) = r(v_{2k-1,1}|\{v_{0,1}, v_{i,1}\}) = (1, 2)$
- *If* $2 \leq i \leq k - 1$, then $r(v_{i+1,1}|\{v_{0,1}, v_{i,1}\}) = r(v_{i,2}|\{v_{0,1}, v_{i,1}\}) = (i + 1, 1)$
- *If* $i = k$, then $r(v_{1,1}|\{v_{0,1}, v_{i,1}\}) = r(v_{2k-1,1}|\{v_{0,1}, v_{i,1}\}) = (1, k - 1)$
- *If* $k + 1 \leq i \leq 2k - 2$, then $r(v_{i-1,1}|\{v_{0,1}, v_{i,1}\}) = r(v_{i,2}|\{v_{0,1}, v_{i,1}\}) = (2k - i + 1, 1)$
- *If* $i = 2k - 1$, then $r(v_{2k-2,1}|\{v_{0,1}, v_{i,1}\}) = r(v_{2k-1,2}|\{v_{0,1}, v_{i,1}\}) = (2, 1)$

(3). When one vertex is from inner cycle and other vertex is from outer cycle. Now we may fix $v_{0,1}$ from inner cycle, if the other vertex is $v_{i,t}$ where $0 \leq i \leq 2k - 1$.

- *If* $i = 0$, then $r(v_{1,1}|\{v_{0,1}, v_{i,t}\}) = r(v_{2k-1,1}|\{v_{0,1}, v_{i,t}\}) = (1, t)$
- *If* $1 \leq i \leq k$, then $r(v_{i,t-1}|\{v_{0,1}, v_{i,t}\}) = r(v_{i-1,t}|\{v_{0,1}, v_{i,t}\}) = (i + t - 2, 1)$
- *If* $k + 1 \leq i \leq 2k - 1$, then $r(v_{i,t-1}|\{v_{0,1}, v_{i,t}\}) = r(v_{i+1,t}|\{v_{0,1}, v_{i,t}\}) = (2k + t - i - 2, 1)$, where the indices are taken modulo n.

A contradiction in all cases. Hence we have $\beta(C_n \Box P_t) = 3$ for $n = 2k + 1$, where $k \geq 2$. $\square$

In the next theorem, we study the exchange property for resolving sets in cartesian product of cycles and path (generalized prism).

Theorem 4.2.3. *For every positive integer $n \geq 4$, resolving sets do not have exchange property in $C_n \Box P_t$ for $n = 2k + 1$ when $t \geq 2$ and for $n = 2k$ when $t \geq 3$.*

Proof. **Case (i)**. For $n = 2k + 1$, where $k \geq 2$ and $t \geq 2$. Since $W = \{v_{0,1}, v_{k,1}\}$ is a metric basis [see Theorem 4.2.2] and hence a minimal resolving set. Also $W^* = \{v_{0,1}, v_{1,1}, v_{0,t}\}$ is a minimal resolving set. There is no $w \in W^*$ such that $S = W^* \setminus \{w\}$ is still a resolving set.

If $w = v_{0,1}$, then $r(v_{0,1}|S) = r(v_{1,2}|S) = (1, t - 1)$. When $w = v_{1,1}$, we get $r(v_{1,1}|S) = r(v_{2k,1}|S) = (1, t)$ and if $w = v_{0,t}$, then $r(v_{0,t}|S) = r(v_{2k,t-1}|S) = (t-1, t)$. Therefore, $|W| = 2$ and $|W^*| = 3$.

Case (ii). For $n = 2k$, where $k \geq 2$ and $t \geq 3$. Since $W' = \{v_{0,1}, v_{k,1}, v_{1,1}\}$ is a metric basis [see Theorem 4.2.2] and hence a minimal resolving set. Also $W^* = \{v_{0,1}, v_{1,2}, v_{k,2}, v_{2k-1,2}\}$ is a minimal resolving set. There is no $w \in W^*$ such that $S = W^* \setminus \{w\}$ is still a resolving set.

If $w = v_{0,1}$, then $r(v_{1,1}|S) = r(v_{1,3}|S) = (1, k, 3)$. When $w = v_{1,2}$, then we get $r(v_{0,2}|S) = r(v_{2k-1,1}|S) = (1, k, 1)$. If $w = v_{k,2}$, then for $n = 4$ we have $r(v_{2,1}|S) = r(v_{0,3}|S) = (2, 2, 2)$ and for $n \geq 6$ we have $r(v_{2,2}|S) = r(v_{1,3}|S) = (3, 1, 3)$. And when $w = v_{2k-1,2}$, we get $r(v_{1,1}|S) = r(v_{0,2}|S) = (1, 1, k)$. Therefore, $|W'| = 3$ and $|W^*| = 4$.

In each case, there are minimal resolving sets of different size. Hence exchange property does not hold in $C_n \Box P_t$ for $n = 2k + 1$ when $t \geq 2$ and for $n = 2k$ when $t \geq 3$. $\square$

The exchange property for resolving sets of generalized antiprism A_n^m, where $3 \leq m \leq 5$ has been discussed in the next theorem.

Theorem 4.2.4. *For every positive integer $n \geq 6$, resolving sets do not have exchange property in A_n^m when $3 \leq m \leq 5$.*

Proof. **Case (i)**. For $n = 2k+1$, where $k \geq 3$. Since $W = \{v_{2,1}, v_{0,\lfloor \frac{m-1}{2} \rfloor}, v_{0,m}\}$ is a metric basis [see Theorem 3.2.3] and hence a minimal resolving set. Also $W^* = \{v_{0,2}, v_{1,2}, v_{0,m}, v_{k,2}\}$ is a minimal resolving set. There is no $w \in W^*$ such that $S = W^* \setminus \{w\}$ is still a resolving set.

If $w = v_{0,2}$, then $r(v_{k+1,1}|S) = r(v_{k+1,2}|S) = (k, k+1, 1)$. When $w = v_{1,2}$, we get $r(v_{1,1}|S) = r(v_{2k,2}|S) = (1, m-1, k)$. If $w = v_{0,m}$, then $r(v_{1,1}|S) = r(v_{0,3}|S) = (1, 1, k)$ and when $w = v_{k,2}$, then $r(v_{0,1}|S) = r(v_{2k,2}|S) = (1, 2, m-1)$. Therefore, we get $|W| = 3$ and $|W^*| = 4$.

Case (ii). For $n = 2k$, where $k \geq 3$. Since $W = \{v_{2,1}, v_{0,\lfloor \frac{m-1}{2} \rfloor}, v_{0,m}\}$ is a metric basis [see Theorem 3.2.3] and hence a minimal resolving set. Also $W^* = \{v_{0,1}, v_{1,2}, v_{k-1,2}, v_{2k-1,2}\}$ is a minimal resolving set. There is no $w \in W^*$ such that $S = W^* \setminus \{w\}$ is still a resolving set.

If $w = v_{0,1}$, then $r(v_{1,1}|S) = r(v_{0,3}|S) = (1, k-1, 2)$. When $w = v_{1,2}$, then any pair of vertices from $\{v_{2k-1,m}, v_{2k-2,m}, v_{2k-3,m}\}$ have the same representation ($m-1$, k or $k+1$, $m-2$) with respect to S. If $w = v_{k-1,2}$, then $r(v_{k+1,2}|S) = r(v_{k+1,3}|S) = (k-1, k, k-2)$ and when $w = v_{2k-1,2}$, then we have $r(v_{1,1}|S) = r(v_{0,2}|S) = (1, 1, k-1)$. Therefore, $|W| = 3$ and $|W^*| = 4$.

Since there are minimal resolving sets having different size, hence exchange property does not hold in A_n^m for $n \geq 6$ and when $3 \leq m \leq 5$. □

4.3 Exchange property for resolving sets in barycentric subdivision of Möbius ladders

In this section, we show that exchange property does not hold for resolving sets of Möbius ladders M_n when $n \equiv 6 \pmod 8$ and also does not hold for barycentric subdivision of Möbius ladders.

Theorem 4.3.1. *The exchange property for minimal resolving sets does not hold in Möbius ladders M_n when $n \equiv 6 \pmod 8$, where $n \geq 14$.*

Proof. When $n \equiv 6 \pmod 8$. Then we can write as $n = 8k+6$, where $k \geq 1$. Since $W = \{v_1, v_2, v_{4k+3}\}$ is a metric basis (see [1]) and hence a minimal resolving set.

Also $W^* = \{v_1, v_{2k+2}, v_{4k+3}, v_{4k+4}\}$ is a minimal resolving set. There is no $w \in W^*$ such that $S = W^* \setminus \{w\}$ is still a resolving set.

If $w = v_1$, then $r(v_1|S) = r(v_{4k+5}|S) = (2k+1, 2, 1)$. When $w = v_{2k+2}$, we get $r(v_{2k+2}|S) = r(v_{6k+6}|S) = (2k+1, 2k+1, 2k+2)$. If $w = v_{4k+3}$, then $r(v_{4k+3}|S) = r(v_{4k+5}|S) = (2, 2k+1, 1)$ and when $w = v_{4k+4}$, we get $r(v_{4k+4}|S) = r(v_{8k+6}|S) = (1, 2k+2, 1)$. Therefore, $|W| = 3$ and $|W^*| = 4$.

There are minimal resolving sets of different size. Hence exchange property does not hold for resolving sets in M_n when $n \equiv 6 \pmod 8$. □

In the next theorem, we show that exchange property does not hold for resolving sets of barycentric subdivision of Möbius ladders denoted by SM_n for every positive even integer $n \geq 8$.

Theorem 4.3.2. *For every positive even integer $n \geq 8$, resolving sets do not have exchange property in barycentric subdivision of Möbius ladders denoted by SM_n.*

Proof. **Case (i)**. When $n \equiv 0 \pmod 8$, we write as $n = 8k$, where $k \geq 1$. Without loss of generality we can choose $i = 0$, $W = \{v_0, v_{n-1}, u_{2k-1}\}$ is a metric basis [see Lemma 3.3.2] and hence a minimal resolving set. Also $W^* = \{v_0, v_1, u_0, u_{2k-1}\}$ is a minimal resolving set. There is no $w \in W^*$ such that $S = W^* \setminus \{w\}$ is still a resolving set.

If $w = v_0$, then $r(v_{4k+1}|S) = r(v_{12k+1}|S) = (4k, 4k, 4)$. When $w = v_1$, we get $r(v_{4k}|S) = r(v_{12k}|S) = (4k, 4k+1, 3)$. If $w = u_0$, then $r(v_{4k+1}|S) = r(u_{2k}|S) = (4k+1, 4k, 4)$ and when $w = u_{2k-1}$, we get $r(v_{8k-1}|S) = r(v_{8k+1}|S) = (3, 4, 2)$. Therefore, $|W| = 3$ and $|W^*| = 4$.

Case (ii). When $n \equiv 2 \pmod 8$, we can write as $n = 8k+2$, where $k \geq 1$. Without loss of generality we can choose $i = 0$, $W = \{v_0, v_{n-1}, u_{2k+1}\}$ is a metric basis [see Lemma 3.3.3] and hence a minimal resolving set. Also $W^* = \{v_0, v_1, u_0, u_{2k+1}\}$ is a minimal resolving set. There is no $w \in W^*$ such that $S = W^* \setminus \{w\}$ is still a resolving set.

If $w = v_0$, then $r(v_{8k+5}|S) = r(u_{4k}|S) = (4, 4, 4k)$. When $w = v_1$, then we get $r(v_3|S) = r(u_{4k}|S) = (3, 4, 4k)$. If $w = u_0$, then $r(v_{8k+5}|S) = r(u_2|S) = (5, 4, 4k)$ and when $w = u_{2k+1}$, then $r(v_{8k+1}|S) = r(v_{8k+3}|S) = (3, 4, 2)$. Therefore, $|W| = 3$ and $|W^*| = 4$.

Case (iii). When $n \equiv 4 \pmod 8$, we can write as $n = 8k + 4$, where $k \geq 1$. Without loss of generality we choose $i = 0$, $W = \{v_0, v_{n-1}, u_{2k}\}$ is a metric basis [see Lemma 3.3.4] and hence a minimal resolving set. Also $W^* = \{v_0, v_1, u_0, u_{2k}\}$ is a minimal resolving set. There is no $w \in W^*$ such that $S = W^* \setminus \{w\}$ is still a resolving set.

If $w = v_0$, then $r(v_{4k+3}|S) = r(v_{12k+7}|S) = (4k+2, 4k+2, 4)$. When $w = v_1$, then we get $r(v_{4k+2}|S) = r(v_{12k+6}|S) = (4k+2, 4k+3, 3)$. If $w = u_0$, then $r(v_{8k+1}|S) = r(u_{4k}|S) = (5, 6, 4k+2)$ and when $w = u_{2k}$, we get $r(v_{8k+3}|S) = r(v_{8k+5}|S) = (3, 4, 2)$. Therefore, $|W| = 3$ and $|W^*| = 4$.

Case (iv). When $n \equiv 6 \pmod 8$. Then we can write as $n = 8k+6$ where $k \geq 1$. Without loss of generality we choose $i = 0$, $W = \{v_0, v_{n-1}, u_{2k+2}\}$ is a metric basis [see Lemma 3.3.5] and hence a minimal resolving set. Also $W^* = \{v_0, v_1, u_0, u_{2k+2}\}$ is a minimal resolving set. There is no $w \in W^*$ such that $S = W^* \setminus \{w\}$ is still a resolving set.

If $w = v_0$, then we have $r(v_{8k+11}|S) = r(u_{4k+1}|S) = (6, 6, 4k)$. When $w = v_1$, then $r(v_3|S) = r(u_{4k+2}|S) = (3, 4, 4k+2)$. If $w = u_0$, then $r(v_{8k+5}|S) = r(u_{4k+2}|S) = (3, 4, 4k+2)$ and when $w = u_{2k+2}$, then $r(v_{8k+5}|S) = r(v_{8k+7}|S) = (3, 4, 2)$. Therefore, we have $|W| = 3$ and $|W^*| = 4$.

In each case, there are minimal resolving sets of different size. Hence exchange property does not hold in SM_n for every positive even integer $n \geq 8$. □

4.4 Exchange property for resolving sets in generalized Petersen multigraphs $P(2n, n)$

The following theorem shows that exchange property does not hold for resolving sets of generalized Petersen multigraphs $P(2n, n)$ for every positive even integer $n \geq 4$.

Theorem 4.4.1. *For every positive even integer $n \geq 4$ and $n \equiv 0, 2 \pmod 4$, resolving sets do not have exchange property in generalized Petersen multigraphs $P(2n, n)$.*

Proof. When $n \equiv 0, 2 \pmod 4$, we can write $n = 4k, 4k+2$ where $k \geq 1$. Without loss of generality we choose $i = 0$, $W = \{v_0, v_{n-1}, u_{n+2k}\}$ is a metric basis [see

Lemma 3.3.8, 3.3.9] and hence a minimal resolving set. Also $W^* = \{v_0, v_1, u_0, v_{n-1}\}$ is a minimal resolving set. There is no $w \in W^*$ such that $S = W^* \setminus \{w\}$ is still a resolving set.

If $w = v_0$, then $r(v_{n+1}|S) = r(u_{2n-1}|S) = (3, 3, 2)$. When $w = v_1$, then $r(u_{n+1}|S) = r(u_{2n-2}|S) = (3, 4, 3)$. If $w = u_0$, then $r(u_n|S) = r(u_{2n-1}|S) = (2, 3, 2)$ and when $w = v_{n-1}$, then $r(v_2|S) = r(u_1|S) = (2, 1, 3)$. Therefore, we get $|W| = 3$ and $|W^*| = 4$.

There are minimal resolving sets of different size. Hence exchange property does not hold in $P(2n, n)$ for every positive even integer $n \geq 4$. □

Chapter 5

Conclusion and open problems

In this thesis, we have studied the metric dimension of antiweb-wheels. We have determined a precise formula for the metric dimension of antiweb-wheels and proved that the antiweb-wheel constitute a family of graphs with unbounded metric dimension. We also studied the metric dimension of several classes of rotationally-symmetric graphs namely quasi flower snarks, generalized antiprism and cartesian product of square cycle and path. It can be seen that these graphs have bounded or constant metric dimension, that is their metric dimensions do not depend upon the number of vertices they have. For generalized antiprism we can find the exact value of its metric dimension when $2 \leq m \leq 5$. Moreover, we have studied the metric dimension of the subdivision of Möbius ladders M_n and generalized Petersen multigraphs $P(2n, n)$. We proved that only three vertices are suffice to resolve all the vertices of SM_n. For the generalized Petersen graphs $P(2n, n)$ which are multigraphs, we proved that their metric dimension is 3 when n is even and $n \equiv 0, 2 \pmod 4$ and 4 otherwise. It has been shown that exchange property does not hold for minimal resolving sets of antiweb-wheels, generalized antiprism when $3 \leq m \leq 5$, generalized prism and quasi flower snarks. We have also shown that the exchange property of the bases in a vector space does not hold for minimal resolving sets of barycentric subdivision of Möbius ladders and also does not hold for minimal resolving sets of generalized Petersen multigraphs $P(2n, n)$ when n is even and $n \equiv 0, 2 \pmod 4$. We close the discussion by raising questions that naturally arise from the text.

Open Problem 1: *Determine the exact value for the metric dimension of generalized antiprism* A_n^m *when* $m \geq 6$.

Open Problem 2: *Determine the exact value for the metric dimension of quasi flower snarks* G_n *for* $n \equiv 0 \pmod 2$.

Open Problem 3: *Determine the exact value for the metric dimension of* $C_n^2 \square P_t$ *when* $n \equiv 1 \pmod 4$.

Open Problem 4: *Let* G *be a non trivial connected graph and* $S(G)$ *denotes its barycentric subdivision. Whenever* G *and* $S(G)$ *will have the same metric dimension?*

Open Problem 5: *Determine a precise formula for the metric dimension of generalized Petersen graph or some good bounds for metric dimension of* $P(n, m)$ *in terms of other graphical parameters.*

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