Fixed Points of Contractions

by

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Abstract

An overview of fixed point theory is given, followed by a survey of topological and metric fixed point theorems; then some fixed point results involving for contractions in compact metric spaces are proved, which are then followed by some results in complete metric spaces. Over all, the presentation herein is rather elementary.

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Chapter 1

Introduction

1.1 Overview

Fixed point theory has in recent times emerged as an important field of both pure and applied mathematics due to its numerous and diverse applications to problems of science, engineering, and economics. The utility of the fixed point methods stems mainly from their applicability not only to establishing the existence of solutions to algebraic, differential, and integral equations (or systems of equations), which arise in any field in which mathematical methods could be applied, but these methods are useful also in devising various iterative, numerical procedures for computing and / or approximating these solutions. (See [19], Chapter 5.)

However, fixed point theory has that inherent quality of elegance that makes its pursuit worthwhile even for its own sake—as a branch of pure mathematics: Fixed point theory is beautiful mathematics! This field can aptly be regarded as the climax of pure mathematics, particularly analysis, topology, and functional analysis.

In this dissertation, I will not say much about applications of fixed point theory and confine myself mainly to the purer side of things.

Fixed point theory is the study of conditions that guarantee the existence of fixed points for different classes of self-maps of sets of different kinds. Since we have an inexhaustible supply of sets and an even more diverse range of self-maps that could be defined on those sets, we have obtained and will continue to obtain an abundant collection of fixed point results. Fixed point theory is thus a burgeoning field of mathematics that will continue to remain an active area of research for many more years!

We have the following basic problem:

Given an arbitrary (non-empty) set X and a (single-valued) map $f\colon X \to X$, a point $x \in X$ is said to be a *fixed point* of f iff $f(x) = x$.

We can also state this problem with a little more generality as follows: If A and B are any two (overlapping) subsets of a set X and $f\colon A \to B$, then a point $x \in A \cap B$ is a fixed point of f iff $f(x) = x$.

Many types of existence theorems in mathematics could be formulated in an equivalent form as a fixed point problem:

For example, the problem of finding the solution of the equation $p(z) = 0$, where p is a polynomial over the complex numbers, is equivalent to that of determining the fixed points of the self-map $z \to z - p(z)$ of the set $\mathbf{C}$ of complex numbers. (See [10], Chapter 0.)

Similarly, for a self-map T of a vector space, showing that the equation $T(u) = 0$ (respectively $u - \lambda T(u) = 0$, λ being a scalar) has a solution is equivalent to showing that the mapping $u \to u - T(u)$ (respectively $u \to \lambda T(u)$) admits a fixed point. (See [10], Chapter 0.)

For multi-valued maps (also called set-valued maps), however, the fixed point problem takes the following form:

Given a set X and a map $f\colon X \to P(X)$, where $P(X)$ here denotes the power set (i.e. the collection of all possible subsets) of X, a point $x \in X$ is defined to be a fixed point of f iff $x \in f(x)$: here $f(x)$ is of course a subset of X. Multi-valued maps are (for the most part) beyond the scope of the present work.

A fixed point is a particular example of a *periodic point*, which we now define.

Given a set X and a map $f\colon X \to X$, a point $x \in X$ is said to be a periodic point of f iff $f^k(x) = x$ for some positive integer k, where f^k denotes the composition of f with itself k times; such an x, which is a fixed point of f^k, is a periodic point of f with period k if k is the smallest positive integer such that $f^k(x) = x$. Every fixed point of f is also its periodic point of unit period. Not every periodic point is a fixed point, however, as is illustrated by the following example:

Let $f\colon [0,1] \to [0,1]$ be defined by

$$f(x) := 1 - x$$

for all $x \in [0,1]$. Then, for any $x \in [0,1]$, $f(x) = x$ iff $x = 1/2$. However,

$$f^2(x) = f(f(x)) = f(1-x) = 1 - (1-x) = x$$

for all $x \in [0,1]$. Thus, every point of $[0,1]$ is a periodic point of f, of period 2, but the only fixed point for f is $x = 1/2$.

A (single-valued) self-map of a set can have no fixed points, a unique fixed point, finitely many fixed points, or infinitely—countably or uncountably—many fixed points, as the following examples illustrate:

1. Let $f : \mathbf{R} \to \mathbf{R}$ be defined as $f(x) := e^x \ \ \forall x \in \mathbf{R}$. Then f has no fixed points: To see this we note that the function $g\colon \mathbf{R} \to \mathbf{R}$ defined as $g(x) := e^x - x \ \ \forall x \in \mathbf{R}$ is strictly increasing over the interval $[0, +\infty)$ because $g'(x) = e^x - 1 > 0 \ \forall x > 0$ and g is strictly decreasing on $(-\infty, 0]$ because $g'(x) < 0 \ \forall x < 0$. So g has an absolute minimum at $x = 0$, and $g(0) = 1$ is the absolute minimum value of g.
Thus $g(x) \geq 1 \ \forall x \in \mathbf{R}$, so $f(x) \geq x + 1$ or $e^x \geq x + 1 \ \forall x \in \mathbf{R}$. Hence $e^x \neq x$ for any x in $\mathbf{R}$.

2. The function $f\colon \mathbf{R} \to \mathbf{R}$ defined by $f(x) = x/2$ for all $x \in \mathbf{R}$ has a unique fixed point, namely $x = 0$.

3. The map $x \to x^2$, where $x \in \mathbf{R}$ has only two fixed points, viz. $x = 0$ and $x = 1$.

4. Given a natural number n, let $f\colon \mathbf{R} \to \mathbf{R}$ be defined as follows:

$$f(x) := \begin{cases} x & \text{for all } x \in \{1, 2, \ldots, n\}\ ; \\ x + 1 & \text{for all } x \notin \{1, 2, \ldots, n\}. \end{cases}$$

Then f has exactly n fixed points.

5. Let $f\colon \mathbf{R} \to \mathbf{R}$ be defined by

$$f(x) := \begin{cases} x & \text{for all } x \in \mathbf{Q}; \\ x + 1 & \text{for all } x \notin \mathbf{Q}. \end{cases}$$

. Then f has infinitely many—in fact countably many—fixed points.

6. Let $f\colon \mathbf{R} \to \mathbf{R}$ be given by

$$f(x) := \begin{cases} x + 1 & \text{for all } x \in \mathbf{Q}; \\ x & \text{for all } x \notin \mathbf{Q}. \end{cases}$$

Then f has uncountably many fixed points.

For any given set X, the identity function $i_X\colon X \to X$, defined by $i_X(x) := x$ for all $x \in X$, has as its fixed point every element of X.

For pairs of maps, we make the following basic definitions:

Definition 1.1.1. *Let f, $g\colon X \to X$ be two self-maps of a non-empty set X.*
A point $x \in X$ is said to be a coincidence point of f and g iff $f(x) = g(x)$.
The maps f and g are said to commute *at a point $x \in X$ iff $f(g(x)) = g(f(x))$.*
Furthermore, f and g are said to be commuting *iff they commute at each point of X,*
and f and g are weakly *compatible iff $f(g(x)) = g(f(x))$ for all points $x \in X$ for which $f(x) = g(x)$.*

For self-maps of metric spaces, we have the following definitions:

Definition 1.1.2. *Let* (X, d) *be a metric space, and let* f, g *be self-maps of* X. *Then* f *and* g *are said to be*

weakly commuting *iff* $d(f(g(x)), g(f(x))) \leq d(f(x), g(x))$ *for all* $x \in X$*;*

r-weakly commuting *if* $d(f(g(x)), g(f(x))) \leq r \cdot d(f(x), g(x))$ *for all* $x \in X$, *for a given* $r \geq 0$*; and*

compatible *iff* $\lim_{n\to\infty} d(f(g(x_n)), g(f(x_n))) = 0$ *for any sequence* (x_n) *in* X *for which* $\lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} g(x_n) = t$ *for some point* $t \in X$.

Evidently, f and g in the preceding set of definitions are weakly commuting if they are r-weakly commuting for some real number $r \in [0, 1]$; this need not hold for an $r > 1$ however: e.g., let f, g: $\mathbf{R} \to \mathbf{R}$ be defined by

$$f(x) := ax + b$$

and

$$g(x) := cx + d$$

for all $x \in \mathbf{R}$, a, b, c, d being real numbers. Then, for any real number x,

$$|f(x) - g(x)| = |(a - c)x + (b - d)|,$$

$$f(g(x)) = f(cx + d) = a(cx + d) + b = acx + ad + b,$$

and

$$g(f(x)) = g(ax + b) = c(ax + b) + d = acx + bc + d;$$

so

$$d(f(g(x)), g(f(x))) = |ad + b - bc - d| = |(a - 1)d - (c - 1)b|.$$

Now by suitably restricting the constants a, b, c, and d or the domain of the mappings we can obtain different sorts of conclusions as to the commutativity, weak commutativity, or r-weak commutativity of this pair.

Fixed point results can be categorized as *topological* or *metric*.

Topological fixed point theory is concerned with those results in which no conditions involving a metric are there. Such results usually specify what condition(s) a set (i.e. a topological space of some sort) must satisfy and what global conditions (like continuity over the entirety of some subspace or subset of the whole space) a mapping or set of mappings must obey.

On the other hand, the metric fixed point theory usually comprises those results that are about self-maps of metric spaces on which some sort of condition(s) explicitly involving the metric itself are also given.

Although our main focus here will be on the second category, in the following few pages we give a brief description of the first as well.

1.2 Topological Fixed Point Theory

The most basic (topological) fixed point result is the following:

Theorem 1.2.1. *Given any two real numbers a and b such that $a < b$, any self-map of the closed interval $[a, b]$ in $\mathbf{R}$ has at least one fixed point.*

The proof of this result is based on the intermediate-value theorem of elementary calculus.

We can thus say that a closed interval is a *fixed point space* for the set of its continuous self-maps. (See [10], Chapter 0.)

More generally, we have the *Brouwer's fixed point theorem*, which states that any continuous self-map of a compact, convex set in the real Euclidean space $\mathbf{R}^n$, where n is a positive integer, has a fixed point; thus every compact, convex set in $\mathbf{R}^n$ is a fixed point space relative to its set of continuous self-maps. (See [10], Chapter 0.)

The Brouwer's fixed point theorem in particular states that any continuous self-map of a closed ball in $\mathbf{R}^n$ has a fixed point. This is a generalization to $\mathbf{R}^n$ of the fixed point theorem for continuous self-maps of the closed intervals in $\mathbf{R}$ given above. This theorem is also considered a prototype of the topological fixed point results.

A self-map f of a metric space—or more generally a topological space—X is said to be *compact* if the image set $f(X)$ has compact closure, that is if $\mathrm{cl}(f(X))$ is a compact subset of X. (See [10], Chapter 0.)

In this connection, we have the following assertion:

Any continuous, compact self-map of $\mathbf{R}$ has a fixed point. To see how this is true, we note that, the closure of the set $f(\mathbf{R})$, being compact, is a closed and bounded subset of $\mathbf{R}$, and so is contained in some finite closed interval. Thus f maps this particular closed interval into itself; so by virtue of continuity of f we're assured of the existence of a fixed point. (See [10], Chapter 0.)

The Schauder fixed point theorem states that any compact, continuous self-map of a convex subset of a normed space has a fixed point. (See [10], Chapter 0.)

This fixed point may not be unique,however; e.g., the function $f\colon [-1, 1] \to [-1, 1]$ defined by $f(x) := x^2$ for all $x \in [-1, 1]$ has a fixed point at $x = 0$ and at $x = 1$; the same function when restricted to the open interval $(0, 1)$ does not have any fixed point, however, although it is still continuous. On the other hand, continuity of the map is also essential because the function $f\colon [-1, 1] \to [-1, 1]$ defined by

$$f(x) := \begin{cases} x^2 & \text{if } -1 \le x < 0; \\ 1 & \text{if } x = 0; \\ -x & \text{if } 0 < x \le 1 \end{cases}$$

has no fixed points.

A subset A of a metric space—or more generally a topological space— X is said to be a *retract* of X if there exists a continuous map $r\colon X \to A$, called a *retraction mapping* or simply a *retraction*, such that $r(a) = a$ for all $a \in A$. (See [10], Chapter 0.)

In this connection, we have the assertion that if a topological space X is a fixed point space, then so is a retract of X. To see this, we let A be a retract of X with r as the corresponding retraction, let $i\colon A \to X$ be the inclusion map; that is we let $i(x) := x$ for all $x \in A$. Let $f\colon A \to A$ be an arbitrary self-map of A. Then $iofor\colon X \to X$. So if, for some $x_0 \in X$, we have

$$(iofor)(x_0) = x_0,$$

then

$$x_0 = i(f(r(x_0))) = f(r(x_0)),$$

from which we have

$$r(x_0) = r(f(r(x_0))) = f(r(x_0)),$$

showing that the map f has a fixed point, namely the point $r(x_0) \in A$. Since f was arbitrary, we have our desired conclusion.

On the other hand, if X has a retract which is a fixed point space, then X need not be a fixed point space, for if $a \in X$, the subset $\{a\}$ of X is always a retract of X (under the identity map, for example). (See [10], Chapter 0.)

Let X be a normed space, and let

$$\overline{B}(\theta, \epsilon) := \{x \in X\colon ||x|| \leq \epsilon\},$$

where θ is the zero vector in our normed space, and $\epsilon > 0$ is arbitrary. Then the map $r\colon X \to \overline{B}(\theta, \epsilon)$ defined as

$$r(\mathbf{x}) := \begin{cases} \mathbf{x} & \text{if } ||\mathbf{x}|| \leq \epsilon; \\ \frac{\epsilon}{||\mathbf{x}||} \cdot \mathbf{x} & \text{if } ||\mathbf{x}|| > \epsilon \end{cases}$$

is a retraction onto the unit closed ball centered at the zero vector θ in X. This retraction is called the natural or standard retraction. (See [10], Chapter 0.)

Theorem 1.2.2. *[10] Let X be a normed space, and let $\overline{B}(\theta, \epsilon)$ be the closed unit ball of radius $\epsilon > 0$ centered at the zero vector of X. Let $f\colon \overline{B}(\theta, \epsilon) \to X$ be a continuous, compact map. Then either f has a fixed point or there exists a vector $\mathbf{x}$ such that $||\mathbf{x}|| = \epsilon$ and*

$$\mathbf{x} = \lambda \cdot f(\mathbf{x})$$

for some $\lambda \in (0, 1)$.

Proof. Let $r\colon X \to \overline{B}(\theta,\epsilon)$ be the standard retraction. Then $rof\colon \overline{B}(\theta,\epsilon) \to \overline{B}(\theta,\epsilon)$ is continuous, and $\overline{B}(\theta,\epsilon)$ is a closed, convex subset of X. So by the Schauder's theorem rof has a fixed point; that is $x = r(f(x))$ for some $x \in X$. Now if $f(x) \in \overline{B}(\theta,\epsilon)$, then $x = r(f(x)) = f(x)$ so that f has a fixed point, but if $f(x) \notin \overline{B}(\theta,\epsilon)$, then

$$x = r(f(x)) = \frac{\epsilon}{||f(x)||} \cdot f(x) = \lambda \cdot f(x),$$

where $\lambda = \frac{\epsilon}{||f(x)||} \in (0,1)$ since in this case $||f(x)|| > \epsilon$. □

If X and Y are two homeomorphic topological spaces, then every self-map of X has a fixed point if and only if the same holds for Y. To see why this is true, let $h\colon X \to Y$ be a homeomorphism. Then, for any map $g\colon Y \to Y$, we have $h^{-1}ogoh\colon X \to X$. So for any point $x_0 \in X$, we have

$$h^{-1}ogoh(x_0) = x_0$$

if and only if

$$g(h(x_0)) = h(x_0),$$

and evidently $h(x_0) \in Y$.

1.3 Metric Fixed Point Theory

This category usually comprises those results that are about self-maps of metric spaces on which some sort of condition(s) involving the metric itself are also given. The most basic of these results is the well-known Banach contraction principle, which we state now:

Theorem 1.3.1. *Let (X,d) be a complete metric space, and let $f\colon X \to X$ be a self-map of X satisfying, for a fixed $\alpha \in [0,1)$, the following inequality:*

$$d(f(x), f(y)) \leq \alpha \cdot d(x,y) \tag{1.1}$$

for all x, $y \in X$. Then f has a unique fixed point, say, p in X.

Furthermore, for any point $x_0 \in X$, the sequence (x_n) of Picard iterates *given by*

$$x_n := f^n(x_0)$$

for $n = 0, 1, 2, \ldots$ converges to p.

Also, we have the following inequalities, which serve as bounds on the error at any stage in the above iterative procedure:

$$d(x_n, p) \leq \frac{\alpha^n}{1-\alpha} \cdot d(x_0, x_1); \tag{1.2}$$

$$d(x_n, p) \leq \alpha \cdot d(x_{n-1}, p); \tag{1.3}$$

for and

$$d(x_n, p) \leq \frac{\alpha}{1-\alpha} \cdot d(x_n, x_{n-1}) \tag{1.4}$$

for $n = 1, 2, 3, \ldots$.

For a proof, see [19].

Maps that satisfy the inequality (1.1) in the above theorem with an arbitrarily chosen, non-negative α, are said to be *Lipschitzian*; if $\alpha < 1$, then the mapping is a contraction with α as a contractive constant, whereas for $\alpha = 1$, the mappings are called contractive mappings. We here use the term non-expansive . . We note that if α is a contractive constant, then so is any real number β in the interval $(\alpha, 1)$.

The above theorem admits the following "converse":

Theorem 1.3.2. *[4] If X is any arbitrary non-empty set, $\alpha \in (0, 1)$, and $f\colon X \to X$ is a self-map of X such that each iterate $f^n\colon X \to X$ has a unique fixed point, then there could be defined on X a metric which not only makes X a complete metric space but which also has the property that f is a contraction with respect to this metric with contractive constant α, as in the statement of the Banach contraction principle given above.*

The Bancah contraction principle immediately yields the following results as a corollary:

Corollary 1.3.3. *If some iterate f^m, where m is a positive integer, of a self-map f of a complete metric space X is a contraction, then f has a unique fixed point, say, $p \in X$, and, for each $x \in X$, we have*

$$\lim_{n\to\infty} f^n(x) = p.$$

For a proof, see [9].

If a self-map f of a metric space (X, d) is a contraction, with contractive constant $\alpha < 1$, then, for any positive integer n, the iterate f^n is also a contraction with α^n as the contractive constant.

So if f is a self-map of a complete metric space and f is a contraction, then each iterate f^n, where n is a positive integer, of f, itself being a contraction, has a unique fixed point; furthermore this fixed point must be the same as the fixed point of f as ensured by the Banach contraction principle, since the unique fixed point of f is automatically a fixed point for each iterate of f.

If some iterate f^n of a function f has a unique fixed point p, then f also leaves p fixed, because from the equality $f^n(p) = p$, we get

$$f^n(f(p)) = f^{n+1}(p) = f(f^n(p)) = f(p),$$

showing that $f(p)$ is also a fixed point of f^n and thus $f(p) = p$ since f^n has a unique fixed point.

However, if f has a unique fixed point, then an iterate of f can have more than one—in fact infinitely many—fixed points; e.g., the map $f\colon [0,1] \to [0,1]$ defined by $f(x) := 1 - x$ for all $x \in [0,1]$ has the unique fixed point $x = 1/2$, but its second iterate f^2, which is given by $f^2(x) = f(f(x)) = f(1-x) = 1 - (1-x) = x$ for all $x \in [0,1]$, leaves each point of the closed unite interval $[0,1]$ fixed.

Here are some more results in metric fixed point theory. For proofs, the interested reader can consult the appropriate references.

Theorem 1.3.4. *Let (X, d) be a complete metric space, and let*

$$B(x_0, r) := \{x \in X\colon d(x, x_0) < r\}$$

for a given point $x_0 \in X$ and a given real number $r > 0$. Let $f\colon B(x_0, r) \to X$ satisfy the inequality

$$d(f(x), f(y)) \leq \alpha \cdot d(x, y)$$

for all x, $y \in B(x_0, r)$, where $0 \leq \alpha < 1$, and suppose that

$$d(x_0, f(x_0)) < (1 - \alpha) r.$$

Then f has a unique fixed point in $B(x_0, r)$.

For a proof, refer to [1], Chapter 1.

Theorem 1.3.5. *[1] Let B_r be a closed ball of radius r, for a given $r > 0$, centered at the point zero in a Banach space X; let $f\colon B_r \to X$ be a contraction such that $||f(x)|| \leq r$ whenever $||x|| = r$. Then f has a unique fixed point in B_r.*

Theorem 1.3.6. *[1] Let f be a self-map of a complete metric space (X, d) such that, for any given $\epsilon > 0$, we can find a real number $\delta(\epsilon) > 0$ in a such a manner that the following condition holds:*

for any points x, $y \in X$,

$$d(f(y), x) < \epsilon$$

whenever

$$d(x, f(x)) < \delta(\epsilon)$$

and

$$d(y, x) < \epsilon.$$

Moreover, if for some point $u \in X$, we have

$$\lim_{n\to\infty} d(f^n(u), f^{n+1}(u)) = 0,$$

then the sequence $(f^n(u))$ converges to a fixed point of f.

Theorem 1.3.7. *[1] Let f be a self-map of a complete metric space (X, d) such that*

$$d(f(x), f(y)) \leq \phi(d(x, y))$$

for all x, $y \in X$, where $\phi\colon [0, +\infty) \to [0, +\infty)$ is a monotonic non-decreasing, not necessarily continuous, function for which

$$\lim_{n\to\infty} \phi^n(t) = 0$$

for any fixed $t > 0$.

Then f has a unique fixed point p in X, and for each $x \in X$, we have

$$\lim_{n\to\infty} f^n(x) = p.$$

The Banach contraction principle is a special case of the above theorem with $\phi(t) := \alpha t$, where $0 \leq \alpha < 1$.

In the last few pages, we have given a brief, non-exhaustive survey of fixed point theory. We will say no more about results in the topological domain. However discussion of metric fixed point results continues in the sequel. .

Chapter 2

Some Fixed Point Theorems in Compact Metric Spaces

In this chapter, I have presented some fixed point results for self-maps of compact metric spaces. For a brief description of compactness in metric spaces, please refer to the appendix. Of the many interesting consequences of compactness, we have made particular use of is the following: *In a compact metric space, every sequence has a convergent subsequence.*

2.1 Self-Maps of the Unit Closed Interval

Of all the compact metric spaces, the most notable are the closed, bounded intervals in $\mathbf{R}$, and since any two such intervals are homoemorphic—or topologically identical—, nothing essential is lost if we restrict ourselves to the unit closed interval $[0, 1]$.

In the present section, let I denote the unit closed interval $[0, 1]$ on the real line. Then we have the following:

Theorem 2.1.1. *[14] Let f, g be a commuting pair of continuous self-maps of the unit closed interval I. Then the set*

$$A := \{x \in I : f(x) = g(x)\}$$

is non-empty. Furthermore, if $x \in A$, then $f(x)$, $g(x) \in A$.

Proof. Let us assume that A *is* empty. Then the function $h \colon I \to I$ defined as

$$h(x) := f(x) - g(x)$$

for each $x \in I$ is continuous on I, and $h(x) \neq 0$ for any $x \in I$.

So if, for some points a and b in I, h were to satisfy the inequality $h(a) < 0 < h(b)$, then by the intermediate value theorem for continuous functions on closed intervals—applied to h on the interval $[\min(a,\ b), \max(a,\ b)]$—, there must exist some number c between a and b such that $h(c) = 0$ or $f(c) = g(c)$, which is contrary to our assumption.
Thus either $h(x) < 0$ for each $x \in I$, or $h(x) > 0$ for each $x \in I$. That is, either $f(x) < g(x)$ for each $x \in I$ or $f(x) > g(x)$ for each $x \in I$.

Case (i):

We have $f(x) < g(x)$ for each $x \in I$. Then the set

$$S := \{x \in I : x \leq f(x)\}$$

is non-empty since $0 \in S$ by the definition of f, and $S \subseteq I$ so that S is bounded.

Now we show that S is closed: Let x be any point in $\text{cl}(S)$, the closure of S. Then there exists a sequence (x_n) in S that converges to x and so by the sequential criterion for continuity of f at x, the sequence $(f(x_n))$ must converge to $f(x)$. Also since $x_n \in S$, we have $x_n \in I$ and $x_n \leq f(x_n)$ for each $n \in N$; thus it follows that $x \in I$ and $x \leq f(x)$, showing that $x \in S$ and hence that S is closed.

Thus S is a non-empty, closed, bounded subset of $\mathbf{R}$, the set of real numbers; so S has a maximum element, say, p. Thus $p \in I$ and $p \leq f(p)$.

Now if $p < f(p)$, then $p \neq 1$ because $1 \geq f(1)$, so $p < 1$.

Moreover by the continuity at point p of the function $f - i_I$, where i_I denotes the identity function on I, we see that for $\epsilon := (f(p) - p)/2 > 0$, there exists a real number $\delta > 0$ such that, whenever $x \in (p - \delta,\ p + \delta)$, then not only that $x \in I$ but also that

$$(f(p) - p) - \epsilon < f(x) - x < (f(p) - p) + \epsilon.$$

But

$$(f(p) - p) - \epsilon = (f(p) - p) - \frac{f(p) - p}{2} = \frac{f(p) - p}{2} = \epsilon > 0.$$

Thus $f(x) - x > 0$ for all $x \in (p - \delta,\ p + \delta)$. In particular, we note that

$$f(p + \delta/2) - (p + \delta/2) > 0,$$

and also that the real number $(p + \delta/2) \in I$ according to our choice of δ, showing that $(p + \delta/2) \in S$. But $(p + \delta/2) > p$, which contradicts the choice of p as the maximum element of S. Thus $f(p) = p$.

Now as f and g commute, we have

$$g(p) = g(f(p)) = f(g(p)),$$

and since $g(p) \in I$ by definition of g, we conclude that $g(p) \in S$.

Since p is the maximum element of S, we must have $g(p) \leq p = f(p)$.

But $p \in I$, so we have assumed that $f(p) < g(p)$. Therefore our assumption that A is empty leads to a contradiction. Hence the set A is non-empty.

Case (ii):

In this case we have $g(x) < f(x)$ for all $x \in I$, and we proceed as in Case (i) but interchanging the roles of f and g.

Now if $x \in A$, then $x \in I$, which implies that $f(x),\ g(x) \in I$ by the definition of f and g, and also that $f(x) = g(x)$. So by the commutativity of f and g, we obtain

$$f(f(x)) = f(g(x)) = g(f(x)),$$

showing that $f(x) \in A$. and also

$$g(g(x)) = g(f(x)) = f(g(x)),$$

showing that $g(x) \in A$, as required. □

Theorem 2.1.2. *[14] Let $f,\ g\colon I \to I$ be continuous self-maps of I such that f and g commute. If f and g have no common fixed point in I, then there exist points a, $b \in I$ such that*
(i) $f(a) = g(a) \geq b > a \geq f(b) = g(b)$, and
(ii) $f(x) \neq g(x)$ for any $x \in (a,\ b)$.

Proof. The set $A = \{x \in I\colon f(x) = g(x)\}$ of the previous theorem, being a non-empty, closed, and bounded subset of $\mathbf{R}$, has a minimum element, say c, and a maximum element, say d.

Since $c \in A$, we have $c \in I$ and $f(c) = g(c)$; also, as shown in the proof of the previous theorem, $f(c) = g(c) \in A$; and since $c = \min A$, we have $f(c) = g(c) \geq c$. But f and g have no common fixed point, so we must have

$$f(c) = g(c) > c. \tag{2.1}$$

So the set $S := \{x \in I : f(x) = g(x) \geq x\}$ is non-empty, because $c \in S$, and S is also bounded because $S \subseteq A \subseteq I$.

To show that S is a closed set, we note that if some point $x \in I$ is in $\mathrm{cl}(S)$, then there is a sequence (x_n) in S that converges to x and so by the continuity of f and g, we conclude that the sequences $(f(x_n))$ and $(g(x_n))$ converge, respectively, to $f(x)$ and $g(x)$. Moreover since (x_n) is a sequence in S, we have $x_n \in I$ and

$$f(x_n) = g(x_n) \geq x_n$$

for all $n \in \mathbf{N}$, and thus it follows that $x \in I$ and also that

$$f(x) = g(x) \geq x,$$

showing that $x \in S$; hence it follows that S is closed.

Consequently S has a maximum element, say a, so that $a \in I$ and $f(a) = g(a) \geq a$, but since f and g have no common fixed point, we must have

$$f(a) = g(a) > a. \tag{2.2}$$

Since $c \in S$, we have $c \leq a$ by the maximality of a; furthermore since $a \in S$ and $S \subseteq A$, we have $a \in A$ and so $a \leq d$ because by our choice d is the maximum element of A. Thus we have the inequality

$$c \leq a \leq d.$$

We now show that $a < d$. Since $d \in A$, $f(d) = g(d) \in A$, as established in the previous theorem; and since $d = \max A$, we have

$$f(d) = g(d) \leq d.$$

But f and g have no common fixed points, so we must have

$$f(d) = g(d) < d.. \tag{2.3}$$

Thus d is not in set S.

But $a \in S$, so $a \neq d$ and thus $a < d$. Thus we have obtained the inequality $c \leq a < d$.

Now let's consider a set T defined as follows:

$$T := \{x : x \in [a\ , d], f(x) = g(x) \leq x\}.$$

Then since

$$f(d) = g(d) < d,$$

we can conclude that $d \in T$. Thus the set T is non-empty and also bounded.

Further if $y \in \mathrm{cl}(T)$, then there is a sequence (y_n) in T that converges to y so that by virtue of continuity of f and g, the sequences $(f(y_n))$ and $(g(y_n))$ converge respectively to $f(y)$ and $g(y)$. Since (y_n) is a sequence in T, we have $y_n \in [a\ , d]$ and

$$f(y_n) = g(y_n) \leq y_n$$

for all $n \in \mathbf{N}$; thus it follows that $y \in [a\ , d]$ and $f(y) = g(y) \leq y$, showing that $y \in T$ and hence that T is a closed set.

So T has a minimum element, say b. Since $b \in T$, and since f and g have no common fixed point, we must have

$$a \leq b \leq d$$

and

$$f(b) = g(b) < b. \tag{2.4}$$

But by (2.2),

$$f(a) = g(a) > a.$$

So

$$a < b \le d. \tag{2.5}$$

Now if $a < x < b$, then $x \in I$ and since a is the maximum element of the set

$$S := \{x \in I\colon f(x) = g(x) \le x\}$$

and b is the minimum element of the set

$$T := \{x \in [a, d], f(x) = g(x) \le x\},$$

x is an element of neither S nor T.

Now since $x \in I$ but x is not in $S = \{x \in I\colon f(x) = g(x) \ge x\}$, therefore either $f(x) \ne g(x)$ or $f(x) = g(x) < x$.

And since x is not in $T = \{x \in [a, d]\colon f(x) = g(x) \le x\}$, therefore either $f(x) \ne g(x)$ or $f(x) = g(x) > x$; we note that $a < x < b \le d$ by (2.5).

So if we assume $f(x) = g(x)$, then we have the contradiction that the inequalities $f(x) = g(x) < x$ and $f(x) = g(x) > x$ hold simultaneously.

Therefore $f(x) \ne g(x)$ whenever $a < x < b$, proving part (b) of our assertion.

Thus we have shown that if $x \in (a, b)$, then $x \notin A = \{x \in I\colon f(x) = g(x)\}$. So $A \cap\ (a\ , b) = \emptyset$, the empty set.

Now since by (2.4) $f(b) = g(b) < b$, we have $b \in A$, and so by the previous theorem we conclude that $f(b) = g(b)$ is an element of set A less than b, and since A has no elements in common with the open interval $(a\ , b)$, we must have

$$f(b) = g(b) \le a.$$

Likewise since we have $f(a) = g(a) > a$ by (2.2), we infer that $a \in A$ and so again by the previous theorem $f(a) = g(a)$ is an element of A greater than a, and since A has no elements in common with $(a\ , b)$, we must have

$$f(a) = g(a) \ge b.$$

Also $a < b$ by virtue of (2.5) above.

Thus we have shown that

$$f(b) = g(b) \le a < b \le f(a) = g(a),$$

as required. □

We note that for commuting self-maps f and g of a non-empty set X, if, for some point $x \in X$, we have $g(x) = x$, then for the same x, we can conclude that

$$f(x) = f(g(x)) = g(f(x))$$

so that $f(x)$ is a fixed point of g whenever x is a fixed point of g, and analogously $g(x)$ is a fixed point f whenever x is a fixed point of f.

Theorem 2.1.3. *[14] Let f and g be continuous self-maps of I that commute. If there exists a real number $\alpha > 0$ such that*

$$|f(x) - f(y)| \leq \alpha \cdot |g(f(x)) - g(f(y))| + |x - y|$$

for x, $y \in I$, then f and g have a common fixed point.

Proof. Suppose that f and g have no common fixed point. The preceding theorem then furnishes points a, $b \in I$ such that
(a) $f(a) = g(a) \geq b > a \geq f(b) = g(b)$, and
(b) $f(x) \neq g(x)$ whenever $a < x < b$.

So if there exist points c_1, $c_2 \in (a\ ,b)$ such that

$$f(c_1) - g(c_1) < 0 < f(c_2) - g(c_2),$$

then by the intermediate-value theorem—applied to the continuous function $f - g$ on the interval $[\min(c_1,\ c_2), \max(c_1\ ,c_2)]$—we conclude that f and g must have the same value at some point between c_1 and c_2 and hence in (a, b), contrary to what (b) above states.

So either (i) $f(x) < g(x)$ for each $x \in (a,\ b)$ or (ii) $f(x) > g(x)$ for each $x \in (a,\ b)$.

Case (i):

Let's assume first that $f(x) < g(x)$ for each $x \in (a,\ b)$. Since by (a) above,

$$g(a) \geq b > a \geq g(b),$$

we have the inequalities

$$g(a) - a > 0 > g(b) - b$$

and so by the intermediate value theorem—applied to the continuous function $g - i_I$ on $[a, b]$—, we can be sure that the set

$$G := \{x \in (a, b) \colon g(x) = x\}$$

of fixed points of g in $(a\ ,b)$, which is evidently bounded, is also non-empty.

To show that G is closed, we note that if $x \in \mathrm{cl}(G)$, then there is a sequence (x_n), say, in G that converges to x and so the image sequence $(g(x_n))$ converges to $g(x)$ by the continuity of g.

Now since $x_n \in G$ for all $n \in \mathbf{N}$, we must have

$$a < x_n < b$$

and

$$g(x_n) = x_n$$

for all $n \in \mathbf{N}$; thus it follows that

$$a \leq x \leq b$$

and also

$$g(x) = x$$

by the uniqueness of the limit of a sequence.

But, as per our choice of a and b satisfying (2.2) and (2.4) in the previous theorem, we know that $g(a) \neq a$ and $g(b) \neq b$, whereas $g(x) = x$; so we must have $a < x < b$, showing that $x \in G$.

Thus for any element x of the closure of G we have obtained the relations $a < x < b$ and $g(x) = x$, showing that $x \in G$, from which it follows that G is a closed set. Therefore G has a minimum element d, say. That is, we let

$$d := \min\{x \in (a, b) \colon g(x) = x\}. \tag{2.6}$$

Thus

$$a < d < b$$

and

$$g(d) = d,$$

so by the commutativity of f and g we have

$$g(f(d)) = f(g(d)) = f(d). \tag{2.7}$$

so that $f(d)$ is also a fixed point of g, and as $d \in (a,\ b)$, we have

$$f(d) < g(d) = d$$

as per our assumption in this particular one of the two cases above that we're considering right now.

So by the minimality of d as an element of set G, we can conclude that $f(d)$ is not in G. However since

$$g(f(d)) = d$$

by (2.7), we can conclude that $f(d) \notin a, b$.

But as per our choice of a, b, and d, we must have the inequalities

$$f(d) \leq a < d = g(d) < b \leq f(a).$$

But $g(f(d)) = f(d)$, whereas $g(a) \neq a$ by (b) above; so we must have

$$f(d) < a < d = g(d) < b \leq f(a). \tag{2.8}$$

Thus

$$f(d) < a < d < f(a).$$

Now since $f(d) < a < f(a)$, the set

$$H := \{x \in (a, d)\colon f(x) = a\}$$

is non-empty by the intermediate value theorem—applied to the continuous function f on the interval $[a, d]$—, and this set is also bounded.

If y is any point in the closure of H, then there is a sequence (y_n) in H that converges to y, and by the continuity of f the image sequence $(f(y_n))$ converges to $f(y)$.

But since $y_n \in H$, we have $a < y_n < d$ and $f(y_n) = a$ for each $n \in \mathbf{N}$. So $a \leq y \leq d$ and so we have $f(y) = a$.

But by (2.8) $f(a) \neq a$ and $f(d) \neq a$, so we must have $a < y < d$. Thus $y \in H$, from which it follows that H is a closed set.

So H, being a closed and bounded subset of $\mathbf{R}$, has a maximum element, say, c. That is, we let

$$c := \max\{x \in (a, d)\colon f(x) = a\}. \tag{2.9}$$

Since $c \in H$, we have $a < c < d$ and $f(c) = a$; further by the maximality of c, $f(x) \neq a$ if $x \in (c\ , d)$.

But as $f(d) \neq a$ by virtue of (2.8), we conclude that $f(x) \neq a$ if $x \in (c\ , d]$.

Now as $f(d) < a$ again by (2.8), thus if $f(x) > a$ for some $x \in (c\ , d)$, then by the intermediate value theorem—applied to the continuous function f on the interval $[x,\ d]$—, there must be a point in $(x\ , d) \subset (c,\ d]$ at which f assumes the value a, contrary to the maximality of c. Thus

$$f(x) < a \text{ if } x \in (c\ , d] \tag{2.10}$$

Since $f(c) = a$, we must have by (b) at the beginning of this proof and by (2.8)

$$g(f(c)) = g(a) \geq b > d. \tag{2.11}$$

On the other hand, since $g(d) = d$, we obtain by virtue of (2.8) again and by commutativity of f and g in our hypothesis the following set of inequalities:

$$g(f(d)) = f(g(d)) = f(d) < a < d. \tag{2.12}$$

Thus by combining the last two sets of inequalities that we've obtained, we have

$$g(f(c)) = g(a) \geq b > d = g(d) > a > f(d) = g(f(d)). \tag{2.13}$$

Therefore,

$$g(f(c)) > d > g(f(d)).$$

So

$$g(f(z)) = d \text{ for some point } z \in (c\ , d). \tag{2.14}$$

Moreover since

$$f(c) = a < d < b \leq f(a),$$

by (2.9), (2.13), and (a) at the start of this proof, and hence

$$f(c) < d < f(a),$$

we must have

$$f(w) = d \text{ for some point } w \in (a\ , c) \tag{2.15}$$

by the intermediate value theorem—applied to the continuous function f on the interval $[a\ , c]$, whence

$$g(f(w)) = g(d) = d,$$

by (2.6).

Thus we have

$$g(f(w)) = d = g(f(z)). \tag{2.16}$$

And by (2.14) and (2.15)

$$a < w < c < z < d. \tag{2.17}$$

Since $z \in (c\ , d)$, we have must have

$$f(z) < a < d = f(w).$$

by (2.10), (2.13), and (2.15). In fact,

$$f(z) < a < w < c < z < d = f(w)$$

by (2.17). So

$$f(z) < w < z < f(w).$$

Therefore,

$$|f(w) - f(z)| = f(w) - f(z) > z - w = |w - z|.$$

Thus

$$|f(w) - f(z)| > |w - z|.$$

And as

$$|g(f(w)) - g(f(z))| = |d - d| = 0$$

by (2.16), we see that for any real number $\alpha > 0$,

$$|f(w) - f(z)| > \alpha \cdot |g(f(w)) - g(f(z))| + |w - z|.$$

Thus we have shown that whenever f and g have no common fixed point in I, then there exist points w, $z \in I$ such that

$$|f(w) - f(z)| > \alpha \cdot |g(f(w)) - g(f(z))| + |w - z|,$$

which is the contrapositive of the required assertion.

Case (ii):

We of course proceed as in case (i) but by interchanging the roles of f and g. $\square$

Theorem 2.1.4. *[14] Let f, g be a commuting pair of continuous self-maps of the unit closed interval $I := [0, 1]$. If*

$$|f(x) - f(y)| \leq |g(x) - g(y)| + |x - y|$$

for all x, $y \in I$ such that

$$(f(x) - g(x))(f(y) - g(y)) = 0,$$

then f and g have a common fixed point in I.

Proof. If f and g have no common fixed point in I, then as in (a) and (2.13) in the proof of the last theorem, we have

$$f(d) < a < d = g(d) < b \leq f(a).$$

In fact,

$$f(a) = g(a) > g(d) = d > a > f(d).$$

So

$$\begin{aligned} |g(a) - g(d)| + |a - d| &= g(a) - g(d) + d - a \\ &= g(a) - d + d - a \\ &= g(a) - a = f(a) - a \\ &< f(a) - f(d) \\ &= |f(a) - f(d)|. \end{aligned}$$

Thus

$$|f(a) - f(d)| > |g(a) - g(d)| + |a - d|,$$

and

$$f(a) - g(a) = 0$$

so that

$$(f(a) - g(a))(f(d) - g(d)) = 0,$$

which is the contrapositive of our assertion. □

Theorem 2.1.5. *[14] For a commuting pair of continuous self-maps f and g of the unit closed interval I, if there exists a positive real number α such that*

$$|x - g(x)| \leq \alpha \cdot |f(x) - g(f(x))| + |x - f(x)|$$

for all $x \in I$, then f and g have a common fixed point in I.

Proof. In the proof of Theorem 2.1.3, we note that, if f and g have no common fixed point in I, then for some points a, b, c, d, and w in I, we have the inequalities

$$f(x) < g(x)$$

for all $x \in (a, b)$ and

$$b > g(f(w)) = f(w) = d > c > w > a$$

by (a), (2.9), (2.13), and (2.15).

But $w \in (a\ , c) \subset (a\ , b)$, which implies that $f(w) < g(w)$.

Thus

$$g(w) > f(w) = g(f(w)) > w.$$

So

$$|w - g(w)| = g(w) - w > f(w) - w = |w - f(w)|.$$

In short

$$|w - g(w)| > |w - f(w)|.$$

Also

$$f(w) = d = g(f(w)).$$

So for any $\alpha > 0$, we have

$$\alpha \cdot |f(w) - g(f(w))| = 0$$

and thus

$$|w - g(w)| > \alpha \cdot |f(w) - g(f(w))| + |w - f(w)|.$$

Therefore whenever f and g have no common fixed points in I, there exists a point $w \in I$ such that for any positive real number α, we have

$$|w - g(w)| > \alpha \cdot |f(w) - g(f(w))| + |w - f(w)|,$$

which is contrary to our hypothesis. Hence f and g do have a common fixed point in I. □

2.2 Non-Expansive Mappings in Compact Metric Spaces

Now we discuss some results for non-expansive mappings in compact metric spaces. It is my conjecture that, in all the results in this section, we can replace the compactness condition by the hypothesis that there is a sequence of Picard iterates which has a convergent subsequence; however I will not attempt a proof of this here and just assume that the underlying space is compact.

Theorem 2.2.1. *Let (X, d) be a compact metric space and let $f\colon X \to X$ be a continuous self-map of X such that for all x, $y \in X$, where $x \neq y$, we have the following condition:*

$$d(f(x), f(y)) < \mathbf{max}(d(x, y), d(x, f(x)), d(y, f(y)), \frac{d(x, f(y)) + d(y, f(x))}{2}). \quad (2.18)$$

Then f has a unique fixed point u in X. Further, for any point $x_0 \in X$, the sequence $(x_n)_{n=0}^{\infty}$, where $x_n := f^n(x_0)$ for each $n = 0, 1, 2, 3, \ldots$, has a subsequence that converges to u.

Proof. If f has fixed points u and v such that $u \neq v$, then we have $f(u) = u$ and $f(v) = v$ and also

$$\begin{aligned} 0 &< d(u, v) = d(f(u), f(v)) \\ &< \mathbf{max}(d(u, v), d(u, f(u)), d(v, f(v)), \frac{d(u, f(v)) + d(v, f(u))}{2}) \\ &= d(u, v), \end{aligned}$$

a contradiction. So f can have at most one fixed point.

We note that $x_n = f(x_{n-1})$ for each $n = 1, 2, 3, \ldots$. Now if, for some positive integer n, we were to have $x_n = x_{n-1}$, then we would have $f(x_{n-1}) = x_{n-1}$ and thus x_{n-1} would be a fixed point of f. So we assume that $x_n \neq x_{n-1}$ for each $n = 1, 2, 3, \ldots$. Then

$$\begin{aligned} d(x_n, x_{n+1}) &= d(f(x_{n-1}), f(x_n)) \\ &< \mathbf{max}(d(x_{n-1}, x_n), d(x_{n-1}, f(x_{n-1})), d(x_n, f(x_n)), \\ &\qquad \frac{d(x_{n-1}, f(x_n)) + d(x_n, f(x_{n-1}))}{2}) \\ &= \mathbf{max}(d(x_{n-1}, x_n), d(x_n, x_{n+1}), \frac{d(x_{n-1}, x_{n+1})}{2}) \\ &\leq \mathbf{max}(d(x_{n-1}, x_n), d(x_n, x_{n+1}), \frac{d(x_{n-1}, x_n) + d(x_n, x_{n+1})}{2}) \\ &= \mathbf{max}(d(x_{n-1}, x_n), d(x_n, x_{n+1})), \end{aligned}$$

from which it follows that

$$d(x_n, x_{n+1}) < d(x_{n-1}, x_n).$$

for each $n = 1, 2, 3, \cdots$.

Thus the sequence $(d(x_{n-1}, x_n))_{n=1}^{\infty}$ in $\mathbf{R}$ is monotonic decreasing and bounded below by 0; so there exists a real number $\alpha \geq 0$ such that

$$\alpha = \lim_{n\to\infty} d(x_{n-1}, x_n).$$

Thus α belongs to the closure of the set

$$S := \{d(x, f(x))\colon x \in X\},$$

which is a subset of $\mathbf{R}$.

Now as f is continuous, so is the map $g\colon X \to \mathbf{R}$ defined by

$$g(x) := d(x, f(x))$$

for all $x \in X$ because the metric $d\colon X \times X \to \mathbf{R}$ is continuous.

Thus, set S is the image of the *compact* set X under the *continuous* mapping g (or S is a continuous image of a compact set); so S is a compact—and hence a closed and bounded—subset of $\mathbf{R}$.

Since α is in the closure of S and S is closed, $\alpha \in S$; that is $\alpha = d(u, f(u))$ for some point u in X. We now show that $f(u) = u$ as follows:

If $u \neq f(u)$, then by virtue of (2.18) we must have

$$\begin{aligned} d(f(u), f^2(u)) &< \mathbf{max}(d(u, f(u)), d(f(u), f^2(u)), \frac{d(u, f^2(u))}{2}) \\ &\leq \mathbf{max}(d(u, f(u)), d(f(u), f^2(u)), \frac{d(u, f(u)) + d(f(u), f^2(u))}{2}) \\ &= \mathbf{max}(d(u, f(u)), d(f(u), f^2(u))). \end{aligned}$$

from which it follows that

$$d(f(u), f^2(u)) < d(u, f(u)). \tag{2.19}$$

This inequality holds when $u \neq f(u)$.

Since (x_n) is a sequence in X, which is *compact*, there exists a sequence n_k of positive integers such that $n_k < n_{k+1}$ for each $k = 1, 2, 3, \ldots$ and such that the subsequence (x_{n_k}) of (x_n) converges to a point $a \in X$.

Since f is continuous, the sequence $(f(x_{n_k}))$ converges to $f(a)$ and therefore by the continuity of d the sequence $(d(x_{n_k}, f(x_{n_k})))$ converges to the real number $d(a, f(a))$.

But the sequence $(d(x_{n_k}, f(x_{n_k})))$ — being a subsequence of the sequence $(d(x_{n-1}, x_n))$, which is identical with the sequence $(d(x_{n-1}, f(x_{n-1})))$, must also converge to $d(u, f(u))$. So

$$d(a, f(a)) = d(u, f(u)). \tag{2.20}$$

Now if $a \neq f(a)$, then as in (2.19) we must have

$$d(f(a), f^2(a)) < d(a, f(a)). \tag{2.21}$$

However as

$$a = \lim_{k\to\infty} x_{n_k}$$

and f is continuous, we have

$$f(a) = \lim_{k\to\infty} f(x_{n_k}) = \lim_{k\to\infty} x_{1+n_k}$$

and

$$f^2(a) = \lim_{k\to\infty} f^2(x_{n_k}) = \lim_{k\to\infty} x_{2+n_k}.$$

So

$$d(f(a), f^2(a)) = \lim_{k\to\infty} d(x_{1+n_k}, x_{2+n_k}).$$

But as the sequence $(d(x_{1+n_k}, x_{2+n_k}))$ is a subsequence of the sequence $(d(x_{n-1}, x_n))$ and since the latter converges to $d(u, f(u))$, we must have

$$d(f(a), f^2(a)) = d(u, f(u)). \tag{2.22}$$

Now from (2.20) and (2.22), we arrive at the equality

$$d(f(a), f^2(a)) = d(a, f(a)),$$

which contradicts (2.21). So $f(a) = a$, and then by (2.20) we conclude that $u = f(u)$.

But f can have at most one fixed point. So we have

$$u = a = \lim_{k\to\infty} x_{n_k},$$

as required. □

In corollaries 2.2.5 and 2.2.8 as well as the next one, we have tacitly assumed that the sequence (f_n) of self-maps of X is point-wise convergent in the metric space (X, d); this assumption is implied by the definition of the map f in each of these results.

Corollary 2.2.2. *If (X, d) is a compact metric space and if, for each $n = 1, 2, 3, \ldots$, the self-map $f_n \colon X \to X$ is continuous and satisfies*

$$\begin{aligned} d(f_n(x), f_n(y)) \quad \leq \quad & a_n < b_n \leq \mathbf{max}(d(x,y), d(x, f_n(x)), d(y, f_n(y)), \\ & \frac{d(x, f_n(y)) + d(y, f_n(x))}{2}) \end{aligned} \tag{2.23}$$

for all x, $y \in X$ such that $x \neq y$ where the sequences (a_n) and (b_n) satisfy

$$\limsup_{n\to\infty} a_n < \liminf_{n\to\infty} b_n. \tag{2.24}$$

Now let $f \colon X \to X$ be given by

$$f(x) := \lim_{n\to\infty} f_n(x) \text{ for all } x \in X.$$

and let f be continuous. Then f has a unique fixed point u in X.

Proof. Taking limit superior and limit inferior in (2.23), we obtain

$$d(f(x), f(y)) \leq a_* \leq b_* \leq \mathbf{max}(d(x,y), d(x, f(x)), d(y, f(y)), \frac{d(x, f(y)) + d(y, f(x))}{2}), \tag{2.25}$$

and

$$d(f(x), f(y)) \leq a^* \leq b^* \leq \mathbf{max}(d(x,y), d(x, f(x)), d(y, f(y)), \frac{d(x, f(y)) + d(y, f(x))}{2}), \tag{2.26}$$

where $a_* := \liminf_{n\to\infty} a_n$ and $a^* := \limsup_{n\to\infty} a_n$, and similarly for b_* and b^*.

But we know that $a_* \leq a^*$ and $b_* \leq b^*$. Then putting together relations (2.24), (2.25), and (2.26), we get

$$\begin{aligned} d(f(x), f(y)) \quad & \leq \quad a_* \leq a^* \\ & < \quad b_* \leq b^* \\ & \leq \quad \mathbf{max}(d(x,y), d(x, f(x)), d(y, f(y)), \\ & \qquad \frac{d(x, f(y)) + d(y, f(x))}{2}), \end{aligned} \tag{2.27}$$

so

$$d(f(x), f(y)) < \mathbf{max}(d(x,y), d(x, f(x)), d(y, f(y)), \frac{d(x, f(y)) + d(y, f(x))}{2}),$$

for all x, $y \in X$ such that $x \neq y$, thus showing that f satisfies (2.18). Since f is assumed to be continuous, it follows from the preceding theorem that f has a unique fixed point u in X. □

In fact, if u_n is the unique fixed point of f_n — whose existence is guaranteed by Theorem 2.2.1 since f_n is continuous and satisfies the inequality (2.18) because f_n satisfies (2.23) — for each $n = 1, 2, 3, \ldots$, and if u is the unique fixed point of f as in the last corollary, then we have the following:

Corollary 2.2.3.

$$u = \lim_{n\to\infty} u_n.$$

Proof. If $u_n \neq u$ for at most finitely many n, then we have nothing to prove; so let's assume that $u_n \neq u$ for infinitely many n. Since X is compact, there is a subsequence (u_{n_k}), say, of (u_n) that converges to some point v of X and for which $u_{n_k} \neq u$ for any $k = 1, 2, 3, \ldots$. Then we have the inequality

$$\begin{aligned} d(u_{n_k}, f_{n_k}(u)) &= d(f_{n_k}(u_{n_k}), f_{n_k}(u)) \\ &\leq a_{n_k} < b_{n_k} \\ &\leq \mathbf{max}(d(u_{n_k}, u), d(u_{n_k}, f_{n_k}(u_{n_k})), d(u, f_{n_k}(u)), \\ &\qquad \frac{d(u_{n_k}, f_{n_k}(u)) + d(u, f_{n_k}(u_{n_k}))}{2}) \\ &= \mathbf{max}(d(u_{n_k}, u), d(u, f_{n_k}(u)), \frac{d(u_{n_k}, f_{n_k}(u)) + d(u, u_{n_k})}{2}). \end{aligned}$$

for each $k = 1, 2, 3, \ldots$. Taking the limit as $k \to \infty$, we obtain

$$d(u, v) \leq a_* \leq a^* < b_* \leq b^* \leq d(u, v),$$

a contradiction. So our assumption that $u_n \neq u$ for infinitely many n is wrong, and the desired conclusion follows. □

Theorem 2.2.4. *Let f be a continuous self-map of a compact metric space (X, d) such that the following condition holds:*

$$d(f(x), f(y)) < \mathbf{max}(d(x, y), \frac{d(x, f(x)) + d(y, f(y))}{2}, d(x, f(y)), d(y, f(x))), \quad (2.28)$$

for all x , $y \in X$, where $x \neq y$, and also assume that f satisfies the condition

$$d(x, f^2(x)) \leq d(x, f(x)) \quad (2.29)$$

for all $x \in X$ such that $x \neq f(x)$. Then f has a unique fixed point u. Further, for any point $x_0 \in X$, the sequence (x_n) given by

$$x_n := f^n(x_0)$$

for each $n = 1, 2, 3, \ldots$, has a subsequence that converges to u.

Proof. If f has fixed points u and v such that $u \neq v$, then by (2.28)

$$\begin{aligned} 0 < d(u,v) &= d(f(u), f(v)) \\ &< \mathbf{max}(d(u,v), \frac{d(u,f(u)) + d(v,f(v))}{2}, d(u,f(v)), d(v,f(u))) \\ &= d(u,v) \end{aligned}$$

a contradiction. So f can have at most one fixed point.

Now if, for some integer $n \geq 0$, we have $x_n = x_{n+1}$, then we have $x_n = f(x_n)$ for this same n and so x_n is a fixed point of f, and we're done. So we assume that $x_n \neq x_{n+1}$ for each $n = 0, 1, 2, 3, \ldots$. Then by (2.29) we have the inequality

$$d(x_n, x_{n+2}) = d(x_n, f^2(x_n)) \leq d(x_n, f(x_n)) = d(x_n, x_{n+1}). \tag{2.30}$$

And then by (2.28) we have

$$\begin{aligned} d(x_{n+1}, x_{n+2}) &= d(f(x_n), f(x_{n+1})) \\ &< \mathbf{max}(d(x_n, x_{n+1}), \frac{d(x_n, f(x_n)) + d(x_{n+1}, f(x_{n+1}))}{2}, \\ &\qquad d(x_n, f(x_{n+1})), d(x_{n+1}, f(x_n))) \\ &= \mathbf{max}(d(x_n, x_{n+1}), \frac{d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2})}{2}, \\ &\qquad d(x_n, x_{n+2}), d(x_{n+1}, x_{n+1})) \\ &= \mathbf{max}(d(x_n, x_{n+1}), \frac{d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2})}{2}, d(x_n, x_{n+2})) \\ &\leq \mathbf{max}(d(x_n, x_{n+1}), \frac{d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2})}{2}, \\ &\qquad d(x_n, x_{n+1})) \text{ using (2.29)} \\ &= \mathbf{max}(d(x_n, x_{n+1}), \frac{d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2})}{2}), \end{aligned}$$

from which it follows that

$$d(x_{n+1}, x_{n+2}) < d(x_n, x_{n+1}) \tag{2.31}$$

for each $n = 0, 1, 2, 3, \ldots$. Using the same argument as in the proof of our first theorem, we can conclude that there exists a point $u \in X$ such that

$$\lim_{n \to \infty} d(x_n, x_{n+1}) = d(u, f(u)).$$

If $u = f(u)$, we're done; so let's assume that $u \neq f(u)$. Then by (2.29) we have

$$d(u, f^2(u)) \leq d(u, f(u)),$$

and then by (2.28),

$$\begin{aligned} d(f(u), f^2(u)) &< \mathbf{max}(d(u, f(u)), (d(u, f(u)) + d(f(u), f^2(u)))/2, d(u, f^2(u))) \\ &\leq \mathbf{max}(d(u, f(u)), (d(u, f(u)) + d(f(u), f^2(u)))/2, d(u, f(u))) \\ &= \mathbf{max}(d(u, f(u)), (d(u, f(u)) + d(f(u), f^2(u)))/2). \end{aligned}$$

Thus it follows that

$$d(f(u), f^2(u)) < d(u, f(u)). \tag{2.32}$$

Now since X is compact, there exists a point $a \in X$ such that a subsequence (x_{n_k}) of the sequence (x_n) converges to a. If $a \neq f(a)$, then as before we must have

$$d(f(a), f^2(a)) < d(a, f(a)). \tag{2.33}$$

Moreover since f and d are continuous, it follows that

$$\lim_{k \to \infty} f(x_{n_k}) = \lim_{k \to \infty} x_{1+n_k} = f(a)$$

and

$$\lim_{k \to \infty} f^2(x_{n_k}) = \lim_{k \to \infty} x_{2+n_k} = f^2(a).$$

So

$$\lim_{k \to \infty} d(x_{n_k}, x_{1+n_k}) = d(a, f(a)),$$

and

$$\lim_{k \to \infty} d(x_{1+n_k}, x_{2+n_k}) = d(f(a), f^2(a)).$$

But both the sequences $(d(x_{n_k}, x_{1+n_k}))$ and $(d(x_{1+n_k}, x_{2+n_k}))$, being subsequences of the sequence $(d(x_n, x_{n+1}))$, converge to the real number $d(u, f(u))$. So we must have

$$d(a, f(a)) = d(u, f(u)) = d(f(a), f^2(a)), \tag{2.34}$$

which contradicts (2.33). So we must have $a = f(a)$ and hence $u = f(u)$ because of (2.34). But f can have at most one fixed point. So we must have

$$u = a = \lim_{k \to \infty} x_{n_k},$$

which is our desired conclusion. □

Corollary 2.2.5. *Let, for each $n = 1, 2, 3, \ldots$, f_n be a continuous self-map of a compact metric space (X, d) such that*

$$\begin{aligned} d(f_n(x), f_n(y)) \leq a_n \quad < \quad b_n \leq \mathbf{max}(d(x, y), \frac{d(x, f_n(x)) + d(y, f_n(y))}{2}, \\ d(x, f_n(y)), d(y, f_n(x))) \end{aligned} \tag{2.35}$$

for all x, $y \in X$ *such that* $x \neq y$, *where the sequences* (a_n) *and* (b_n) *satisfy (2.24), and let*

$$d(x, f_n^2(x)) \leq d(x, f_n(x)) \tag{2.36}$$

for all $x \in X$ *such that* $x \neq f_n(x)$.

Now let $f \colon X \to X$ *be given by*

$$f(x) := \lim_{n\to\infty} f_n(x) \text{ for all } x \in X.$$

and let f *be continuous. Then* f *has a unique fixed point in* X

Proof. Taking the limit inferior and limit superior in (2.35), we obtain

$$\begin{aligned} d(f(x), f(y)) &\leq a_* \leq b_* \\ &\leq \mathbf{max}(d(x,y), \frac{d(x,f(x)) + d(y,f(y))}{2}, d(x,f(y)), d(y,f(x))) \end{aligned}$$

and

$$\begin{aligned} d(f(x), f(y)) &\leq a^* \leq b^* \\ &\leq \mathbf{max}(d(x,y), \frac{d(x,f(x)) + d(y,f(y))}{2}, d(x,f(y)), d(y,f(x))) \end{aligned}$$

Then using (2.24), we have

$$d(f(x), f(y)) \leq a^* \quad < \quad b_* \leq \mathbf{max}(d(x,y), \frac{d(x,f(x)) + d(y,f(y))}{2}, \\ d(x,f(y)), d(y,f(x)))$$

for all x, $y \in X$ such that $x \neq y$, where a_* is the $\liminf$ of (a_n) etc as in Corollary 2.2.2.

Further taking the limit as $n \to \infty$ in (2.36) and using the continuity of the f_n for each $n = 1, 2, 3, \ldots$ and the continuity of d, we get

$$d(f(x), f^2(x)) \leq d(x, f(x))$$

for all $x \in X$ such that $x \neq f(x)$.

Since f is assumed to be continuous, f satisfies all the conditions of our Theorem 2.2.4 and therefore has a unique fixed point in X.

□

In the preceding corollary, if u_n is the unique fixed point of f_n — whose existence is ensured by our Theorem 2.2.4 because f_n satisfies the conditions of that theorem— for each $n = 1, 2, 3, \ldots$, and if u is the unique fixed point of f, then we have

Corollary 2.2.6.

$$\lim_{n\to\infty} u_n = u.$$

Proof. If $u_n \neq u$ for at most finitely many n, then there's nothing to prove; so let's assume that $u_n \neq u$ for infinitely many n, Since (u_n) is a sequence in the compact space X, there exists a subsequence (u_{n_k}) such that $u_{n_k} \neq u$ for each $k = 1, 2, 3, \cdots$ and also $\lim_{k\to\infty} u_{n_k} = v$ for some point $v \in X$. Since $u_{n_k} \neq u$ for each $k = 1, 2, 3, \ldots$, we have the following chain of inequalities:

$$\begin{aligned}
d(u_{n_k}, f_{n_k}(u)) &= d(f_{n_k}(u_{n_k}), f_{n_k}(u)) \leq a_{n_k} \\
&< b_{n_k} \leq \mathbf{max}(d(u_{n_k}, u), \frac{d(u_{n_k}, f_{n_k}(u_{n_k})) + d(u, f_{n_k}(u))}{2}, \\
&\qquad d(u_{n_k}, f_{n_k}(u)), d(u, f_{n_k}(u_{n_k}))) \\
&= \mathbf{max}(d(u_{n_k}, u), \frac{d(u, f_{n_k}(u))}{2}, d(u_{n_k}, f_{n_k}(u)), d(u, u_{n_k})) \\
&\leq \mathbf{max}(d(u_{n_k}, u), \frac{d(u, u_{n_k}) + d(u_{n_k}, f_{n_k}(u))}{2}, d(u_{n_k}, f_{n_k}(u))) \\
&= \mathbf{max}(d(u_{n_k}, u), d(u_{n_k}, f_{n_k}(u))).
\end{aligned}$$

Taking the limit superior and inferior as $k \to \infty$ and then using the relation (2.24), we get

$$d(u, v) \leq a^* < b_* \leq d(u, v),$$

a contradiction. So we msut have $u_n \neq u$ for only finitely many values of n, and the conclusion follows. □

Theorem 2.2.7. *Let f be a continuous self-map of a compact metric space (X, d) such that*

$$d(f(x), f(y)) < \mathbf{max}(d(x, y), d(x, f(x)), d(y, f(y)), d(x, f(y)), d(y, f(x))) \quad (2.37)$$

for all x, $y \in X$ such that $x \neq y$, and let f satisfy the inequality (2.29) for all $x \in X$ such that $x \neq f(x)$. Then f has a unique fixed point $u \in X$.

Furthermore, for any point $x_0 \in X$, the sequence (x_n) given by $x_n := f^n(x_0)$ for each $n = 0, 1, 2, 3, \ldots$, has a subsequence that converges in (X, d) to u.

Proof. The proof is almost identical to that of the last theorem. If u and v are fixed points of f such that $u \neq v$, then we must have the inequality

$$\begin{aligned}
0 < d(u, v) &= d(f(u), f(v)) \\
&< \mathbf{max}(d(u, v), d(u, f(u)), d(v, f(v)), d(u, f(v)), d(v, f(u))) \\
&= d(u, v),
\end{aligned}$$

a contradiction. So f can have at most one fixed point.

Now if, for some non-negative integer n, we had $x_n = x_{n+1}$, then $x_n = f(x_n)$ and we're done. So let's assume that $x_n \neq x_{n+1}$ for all n. Then we have the inequalities

$$d(x_n, x_{n+2}) = d(x_n, f^2(x_n)) \leq d(x_n, f(x_n)) = d(x_n, x_{n+1}), \text{ using (2.29)}$$

and then

$$\begin{aligned} d(x_{n+1}, x_{n+2}) &= d(f(x_n), f(x_{n+1})) \\ &< \mathbf{max}(d(x_n, x_{n+1}), d(x_n, f(x_n)), d(x_{n+1}, f(x_{n+1})), \\ &\qquad\qquad d(x_n, f(x_{n+1})), d(x_{n+1}, f(x_n))) \\ &= \mathbf{max}(d(x_n, x_{n+1}), d(x_{n+1}, x_{n+2}), d(x_n, x_{n+2}), d(x_{n+1}, x_{n+1})) \\ &\leq \mathbf{max}(d(x_n, x_{n+1}), d(x_{n+1}, x_{n+2})), \end{aligned}$$

from which it follows that

$$d(x_{n+1}, x_{n+2}) < d(x_n, x_{n+1}).$$

for each non-negative integer n. Then as in the proof of our first theorem we can assume the existence of a point $u \in X$ such that

$$d(u, f(u)) = \lim_{n\to\infty} d(x_n, x_{n+1}) = \lim_{n\to\infty} d(x_n, f(x_n)).$$

If $u = f(u)$, we're done; so let's assume that $u \neq f(u)$. Then

$$d(u, f^2(u)) \leq d(u, f(u)).$$

and

$$\begin{aligned} d(f(u), f^2(u)) &< \mathbf{max}(d(u, f(u)), d(f(u), f^2(u)), d(u, f^2(u))) \\ &\leq \mathbf{max}(d(u, f(u)), d(f(u), f^2(u))). \end{aligned}$$

Thus it follows that

$$d(f(u), f^2(u)) < d(u, f(u)).$$

Since (x_n) is a sequence in a compact space. it has a convergent subsequence (x_{n_k}), say. Let

$$a := \lim_{k\to\infty} x_{n_k}.$$

Then by the continuity of f and $d : X \times X \to \mathbf{R}$, we can conclude that

$$\lim_{k\to\infty} x_{1+n_k} = \lim_{k\to\infty} f(x_{n_k}) = f(a)$$

and

$$\lim_{k\to\infty} x_{2+n_k} = \lim_{k\to\infty} f^2(x_{n_k}) = f^2(a).$$

So

$$\lim_{k\to\infty} d(x_{n_k}, x_{1+n_k}) = \lim_{k\to\infty} d(x_{n_k}, f(x_{n_k})) = d(a, f(a))$$

and

$$\lim_{k\to\infty} d(x_{1+n_k}, x_{2+n_k}) = \lim_{k\to\infty} d(f(x_{n_k}), f^2(x_{n_k})) = d(f(a), f^2(a)).$$

However, as each of the sequences $(d(x_{n_k}, x_{1+n_k}))$ and $(d(x_{1+n_k}, x_{2+n_k}))$ is a subsequence of the sequence $(d(x_n, x_{n+1}))$, we must have

$$\lim_{k\to\infty} d(x_{n_k}, x_{1+n_k}) = \lim_{k\to\infty} d(x_{n_k}, f(x_{n_k})) = d(u, f(u))$$

and

$$\lim_{k\to\infty} d(x_{1+n_k}, x_{2+n_k}) = \lim_{k\to\infty} d(f(x_{n_k}), f^2(x_{n_k})) = d(u, f(u)).$$

Therefore,

$$d(a, f(a)) = d(f(a), f^2(a)) = d(u, f(u)).$$

If $a = f(a)$, then $u = f(u)$ and we're done. So we assume the contrary. Then as in the case of u we can prove that

$$d(a, f^2(a)) \leq d(a, f(a)),$$

which then leads to

$$d(f(a), f^2(a)) < d(a, f(a)),$$

which gives rise to a contradiction. So $a = f(a)$ and thence $u = f(u)$. But f can have at most one fixed point. So we must have

$$u = a = \lim_{k\to\infty} x_{n_k},$$

as required. □

Corollary 2.2.8. *Let, for each $n = 1, 2, 3, \ldots$, f_n be a continuous self-map of a compact metric space (X, d) that satisfies the following condition:*

$$\begin{aligned} d(f_n(x), f_n(y)) &\leq a_n \\ &< b_n \leq \mathbf{max}(d(x,y), d(x, f_n(x)), d(y, f_n(y)), \\ &\qquad\qquad d(x, f_n(y)), d(y, f_n(x))) \end{aligned} \tag{2.38}$$

for all x, $y \in X$ such that $x \neq y$, where the sequences (a_n) and (b_n) satisfy the inequality (2.24), and let f_n satisfy (2.36) for all $x \in X$ such that $x \neq f(x)$.
Let $f\colon X \to X$, defined as

$$f(x) := \lim_{n\to\infty} f_n(x) \text{ for all } x \in X,$$

be continuous. Then f has a unique fixed point in X.

Proof. As before, taking the limit inferior and limit superior in (2.38) and using (2.24), we obtain

$$\begin{aligned} d(f(x), f(y)) &\leq a_* \leq a^* \\ &< b_* \leq b^* \leq \mathbf{max}(d(x,y), d(x,f(x)), d(y,f(y)), d(x,f(y)), d(y,f(x))) \end{aligned}$$

for all x, $y \in X$ such that $x \neq y$. Now taking the limit as $n \to \infty$ in (2.36), we obtain

$$d(x, f^2(x)) \leq d(x, f(x))$$

for all $x \in X$ such that $x \neq f(x)$. Thus f satisfies all the conditions of Theorem 2.2.7 and therefore has a unique fixed point in X. □

In the last corollary, if, for each $n = 1, 2, 3, \ldots$, u_n is the unique fixed point of f_n — whose existence follows from Theorem 2.2.7 since f_n satisfies all the conditions of that theorem —, and if u is the unique fixed point of f, then

Corollary 2.2.9.

$$\lim_{n \to \infty} u_n = u.$$

Proof. If $u_n \neq u$ for only finitely many values of n, then we're done; so let's assume that $u_n \neq u$ for infinitely many n. Since (u_n) is a sequence in a compact metric space (X, d), it has a subsequence (u_{n_k}) such that $u_{n_k} \neq u$ for each $k = 1, 2, 3, \cdots$, and

$$\lim_{k \to \infty} u_{n_k} = v$$

for some point $v \in X$. Then using the inequality (2.38), we obtain

$$\begin{aligned} d(u_{n_k}, f_{n_k}(u)) &= d(f_{n_k}(u_{n_k}), f_{n_k}(u)) \\ &\leq a_{n_k} < b_{n_k} \\ &< \mathbf{max}(d(u_{n_k}, u), d(u_{n_k}, f_{n_k}(u_{n_k})), d(u, f_{n_k}(u)), \\ &\qquad d(u_{n_k}, f_{n_k}(u)), d(u, f_{n_k}(u_{n_k}))) \\ &= \mathbf{max}(d(u_{n_k}, u), d(u, f_{n_k}(u)), d(u_{n_k}, f_{n_k}(u))). \end{aligned}$$

Taking the limit (inferior and superior) as $k \to \infty$ and using (2.24), we arrive at

$$d(u, v) \leq a_* \leq a^* < b_* \leq b^* \leq d(u, v),$$

which gives rise to a contradiction.

Hence $u_n \neq u$ for only finitely many n, and the result follows. □

The contractive conditions given by (2.18) and (2.28) both imply the one given by (2.37) but not conversely. As an example, let $f\colon [0,1] \to [0,1]$ be defined by

$$f(x) := \begin{cases} 1/4 & \text{if } 0 \le x \le 1/2 \\ 1/2 & \text{if } 1/2 < x \le 1\ . \end{cases}$$

We only need to consider the case $0 \le x \le 1/2 < y \le 1$. Then $d(f(x), f(y)) = 1/4$, $d(x, f(x)) = |x - 1/4|$, $d(y, f(y)) = y - 1/2$, $d(x, f(y)) = 1/2 - x$, and $d(y, f(x)) = y - 1/4$. Since

$$\begin{aligned} d(f(x), f(y)) = 1/4 &< y - 1/4 = d(y, f(x)) \\ &< \mathbf{max}(d(x,y), (d(x, f(x)) + d(y, f(y)))/2, d(x, f(y)), d(y, f(x))) \\ &< \mathbf{max}(d(x,y), d(x, f(x)), d(y, f(y)), d(x, f(y)), d(y, f(x))), \end{aligned}$$

f satisfies the condition (2.28) and hence also (2.37). However, for $x = 2/5$, say, and $y \to 1/2 + 0$ (in fact for any y such that $1/2 < y < 13/20$), we see that

$$\begin{aligned} d(f(x), f(y)) &= 1/4 \\ &> \mathbf{max}(d(x,y), d(x, f(x)), d(y, f(y)), (d(x, f(y)) + d(y, f(x)))/2), \end{aligned}$$

showing that condition (2.18) fails.

On the other hand, let's consider the function $f\colon [0,1] \to [0,1]$ defined by

$$f(x) := \begin{cases} 1/2 & \text{if } 0 \le x \le 1/2\ ; \\ 1/4 & \text{if } 1/2 < x \le 1\ . \end{cases}$$

Then we need only consider the case $0 \le x \le 1/2 < y \le 1$. In this case $d(f(x), f(y)) = 1/4$, $d(x, f(x)) = 1/2 - x$, $d(y, f(y)) = y - 1/4$, $d(x, f(y)) = |x - 1/4|$, and $d(y, f(x)) = y - 1/2$. So

$$\begin{aligned} d(f(x), f(y)) &= 1/4 \\ &< y - 1/4 = d(y, f(y)) \\ &\le \mathbf{max}(d(x,y), d(x, f(x)), d(y, f(y)), (d(x, f(y)) + d(y, f(x)))/2) \\ &\le \mathbf{max}(d(x,y), d(x, f(x)), d(y, f(y)), d(x, f(y)), d(y, f(x))), \end{aligned}$$

showing that f satisfies condition (2.18) and hence condition (2.37). However, again for $x = 2/5$ and $y \to 1/2 + 0$ (for $1/2 < y < 13/20$ to be precise), we note that

$$\begin{aligned} d(f(x), f(y)) &= 1/4 \\ &> \mathbf{max}(d(x,y), (d(x, f(x)) + d(y, f(y)))/2, d(x, f(y)), d(y, f(x))), \end{aligned}$$

thus showing that f does not satisfy condition (2.28).

These two examples also serve to illustrate the fact that the conditions given by (2.18) and by (2.28) are *independent*.

In the hindsight, we can treat Theorem 2.2.4 as a corollary of Theorem 2.2.7 since the condition (2.28) implies condition (2.37) and (2.29) is common between (i.e. shared by) the hypotheses of both the theorems. Our first theorem in this section, however, may not follow from the third as the following example elucidates: let $f\colon [0,1] \to [0,1]$ be defined by

$$f(x) := \begin{cases} 1/5 & \text{if } 0 \le x \le 1/2 \ ; \\ 1/4 & \text{if } 1/2 < x \le 1 \ . \end{cases}$$

For $0 \le x \le 1/2 < y \le 1$, we have

$$d(f(x), f(y)) = 1/4 - 1/5 = 1/20,$$

$$d(x, f(x)) = |x - 1/5|,$$
$$d(y, f(y)) = y - 1/4,$$
$$d(x, f(y)) = |x - 1/4|,$$

and

$$d(y, f(x)) = y - 1/5.$$

Then

$$\begin{aligned} d(f(x), f(y)) = 1/20 &< y - 1/4 = d(y, f(y)) \\ &\le \mathbf{max}(d(x,y), d(x, f(x)), d(y, f(y)), \frac{d(x, f(y)) + d(y, f(x))}{2}) \\ &\le \mathbf{max}(d(x,y), d(x, f(x)), d(y, f(y)), d(x, f(y)), d(y, f(x))). \end{aligned}$$

But $f^2(x) = 1/5$ for all $x \in [0,1]$. So

$$d(x, f^2(x)) = x - 1/5 \not\le x - 1/4 = d(x, f(x)) \text{ for all } x \in (1/2, 1].$$

Thus although f satisfies (2.18) and hence (2.37), it does not satisfy (2.28) and thus we cannot apply our third theorem in this section in this particular situation. However, the first theorem is clearly applicable since $[0,1]$ is compact and f satisfies (2.18). Evidently f has a unique fixed point, namely $x = 1/5$.

It is my feeling that Condition (2.28)—or condition (2.37) for that matter—by itself is not sufficient to guarantee the existence of a fixed point, but I have not been able to come up with an illustrative example to establish this for a fact.

Rhoades [23] in 1977 asserted that there is no fixed point theorem known for maps satisfying our condition (2.37); I have not come upon any such theorem either, in the literature I've surveyed so far. If that is the actual situation even now, then this work is certainly a step forward in our quest for (conditions for) existence of unique fixed points.

Finally, our condition (2.37) follows from all the (basic) contractive conditions examined by Rhoades in [23]. So Theorem 2.2.7 above shows that a fixed point theorem follows from all the conditions of Rhoades, provided we add the auxiliary condition (2.29).

2.3 Periodic Points for Continuous Self-Maps of Compact Metric Spaces

Now we give a theorem about periodic points:

Theorem 2.3.1. *[16] A continuous, surjective self-map g of a compact metric space (X, d) has a periodic point if g satisfies the following condition: There exists a positive real number ϵ such that, whenever*

$$0 < d(g(x), g(y)) < \epsilon,$$

we have the inequality

$$d(f(z), f(y)) < d(g(x), g(y))$$

for some continuous self-map f of X such that f commutes with g and for some point $z \in X$ such that $g(z) = g(x)$.

Further, if, for some positive integer k and for some point x in X, we have the inequality

$$d(x, g^k(x)) < \epsilon,$$

then

$$g^k(a) = a$$

for some point a in X.

Proof. Since (X, d) is compact, the sequence $(g^n(x_0))$ of Picard iterates under g of any point $x_0 \in X$ has a convergent subsequence. So there exist positive integers m and k such that

$$d(g^m(x_0), g^{k+m}(x_0) < \epsilon$$

or

$$d(g^m(x_0), g^k(g^m(x_0))) < \epsilon.$$

So

$$d(t, g^k(t)) < \epsilon$$

for some $t \in X$.

Now since g is continuous, so is g^k; thus because of the continuity of the metric in both its variables the map $x \to d(x, g^k(x))$ of X into $\mathbf{R}$, the set of real numbers, is also continuous. So the set

$$\{d(x, g^k(x)) \colon x \in X\}$$

is a compact—and hence closed and bounded—subset of $\mathbf{R}$. Therefore, for some point $a \in X$, we have

$$d(a, g^k(a)) \leq d(x, g^k(x)) \tag{2.39}$$

for all $x \in X$. So

$$d(a, g^k(a)) \leq d(t, g^k(t)) < \epsilon.$$

Hence

$$d(a, g^k(a)) < \epsilon. \tag{2.40}$$

Suppose $g^k(a) \neq a$. Then since g is assumed to be surjective, we can find a point $b \in X$ such that $a = g(b)$. Thus

$$0 < d(a, g^k(a)) = d(g(b), g^k(g(b))) = d(g(b), g(g^k(b))) < \epsilon.$$

So we can find a map f which commutes with g and hence with g^k and a point $c \in X$ such that $g(c) = g(b)$ and hence $g^k(c) = g^k(b)$ such that

$$d(f(c), f(g^k(b))) < d(g(b), g(g^k(b)))$$

or

$$d(f(c), g^k(f(c))) < d(a, g(a)),$$

which is a contradiction to (2.39). Hence $a = g(a)$, as required. □

In [16], the author merely states and then make use of the following elegant result; I've filled in the details of the proof on my own.

Theorem 2.3.2. *Let g be a continuous self-map of a compact metric space (X, d). Then the set $A := \cap_{n=1}^{\infty} g^n(X)$ is compact and $g(A) = A$.*

Proof. Since g is continuous, so is g^n for each $n = 2, 3, 4, \ldots$. So due to the compactness of (X, d), each of the sets $g^n(X)$ for $n = 1, 2, 3, \ldots$, being a continuous image of a compact set, is compact.

In a metric space, compactness is equivalent to sequential compactness, and every sequentially compact set is closed (See [25], Chapter 4). So each $g^n(X)$ is a closed set. Then A, being the intersection of a family of closed sets, is a closed set. Hence A is compact as it is a closed subset of the compact set X.

Let $x \in g(A)$. Then $x = g(a)$ for some point $a \in A$. Since $a \in A$, we must have $a \in g^n(X)$ for each $n = 1, 2, 3, \ldots$. So $a = g^n(x_n)$ for some point $x_n \in X$ for each $n = 1, 2, 3, \ldots$. Thus

$$x = g(a) = g(g^n(x_n)) = g^{n+1}(x_n) = g^n(g(x_n))$$

for each $n = 1, 2, 3, \ldots$.

Now, for each $n = 1, 2, 3, \ldots$, since $x_n \in X$, and since $g \colon X \to X$ so that $g(x_n) \in X$, therefore we must have $x = g^n(g(x_n)) \in g^n(X)$ for each $n = 1, 2, 3, \ldots$. So $x \in A$, and thus it follows that $g(A) \subseteq A$.

Conversely, let $a \in A$. Then $a \in g^n(X)$ for each $n = 1, 2, 3, \ldots$; hence $a = g^n(a_n)$ for some $a_n \in X$, for each $n = 1, 2, 3, \ldots$. So

$$a = g(a_1) = g(g(a_2)) = g(g^2(a_3)) = g(g^3(a_4)) = \ldots = g(g^{n-1}(a_n)) = \ldots.$$

Thus $a = g(x_n)$ for a sequence (x_n) in X, where

$$x_n := g^{n-1}(a_n)$$

for each $n = 1, 2, 3, \ldots$.

Now as (x_n) is a sequence in a compact metric space (X, d), this sequence has a convergent subsequence (x_{k_n}), say, which converges to some point b of X. We now show that $b \in A$ as follows:

We note that, by the definition of a subsequence, (k_n) is a strictly increasing sequence of natural numbers; that is, $k_n \in \mathbf{N}$ such that $k_n < k_{n+1}$ so that $k_n \geq n$ for each $n \in \mathbf{N}$.

Now as $x_{k_n} = g^{k_n - 1}(a_{k_n})$ for each n, so for $n > 1$ we can write $x_{k_n} = g(g^{k_n - 2}(a_{k_n}))$; thus (x_{k_n}) is ultimately a sequence in $g(X)$, which is a compact and hence closed set. Therefore, $b \in g(X)$ because every convergent sequence in a closed set must converge in the set.

For $n > 2$, we can write $x_{k_n} = g^2(g^{k_n - 3}(a_{k_n}))$ so that the sequence (x_{k_n}) is ultimately a sequence in the compact—and hence closed—set $g^2(X)$. So $b \in g^2(X)$.

Similarly, for any natural number $m > 2$, we can write $x_{k_n} = g^m(g^{k_n - (m+1)}(a_{k_n}))$ for each $n > m$ so that the sequence (x_{k_n}) can be regarded as being ultimately a sequence in the set $g^m(X)$. Now since X is compact and g is continuous (and since consequently g^m is also continuous), we can conclude that $g^m(X)$ is also compact and therefore closed. Thus, (x_{k_n}), being ultimately a sequence in the closed set $g^m(X)$, must converge in $g^m(X)$. So $b \in g^m(X)$.

Thus we have shown that $b \in g^m(X)$ for each natural number m. Therefore, $b \in \bigcap_{m=1}^{\infty} g^m(X)$ or $b \in A$.

Since (x_{k_n}) converges to b and g is continuous, the sequence $(g(x_{k_n}))$ must converge to $g(b)$. But $g(x_{k_n}) = a$ for each n as per our choice of the x_n and hence the x_{k_n}. So $a = g(b)$, where $b \in A$, whence $a \in g(A)$, from which it follows that $A \subseteq g(A)$.

Hence $A = g(A)$, as required. □

We can also improved upon Corollary 2.2 in [16] as follows:

Corollary 2.3.3. *For a continuous self-map g of a compact metric space (X, d), let there exist a real number $\epsilon > 0$ such that, for any x, $y \in X$, whenever*

$$0 < d(g(x), g(y)) < \epsilon,$$

we have

$$d(f(x), f(y)) < d(g(x), g(y)).$$

for a continuous self-map f of X such that f and g commute. Then g has a periodic point. In fact, if

$$d(x, g^k(x)) < \epsilon$$

for some point $x \in A := \cap_{n=1}^{\infty} g^n(X)$ and for some positive integer k, then we have $g^k(a) = a$ for some point $a \in A$.

Proof. Since by Theorem 2.3.2 $g(A) = A$, the restriction $g|_A\colon A \to A$, of the map g to the set A, is a surjective self-map of A, which is itself a compact metric space again by Theorem 2.3.2. Further, if f and g commute on X, then $f|_A$ and $g|_A$ commute on A. So we can apply Theorem 2.3.1 to $g|_A$ to get the desired conclusion. □

Note that the author merely states that the point a is in X, but we have been able to pinpoint this point a more accurately because the set A could well be a proper subset of X.

In the next chapter, we present some fixed point results for pairs of self-maps of complete metric spaces.

Chapter 3

Some Fixed Point Theorems in Complete Metric Spaces

There is a whole host of papers featuring fixed point results in complete metric spaces. The cornerstone of all these results is the famous Banach's Contraction Principle [19]. This classical result has been extended among others by E. Rakotch [20], M. Edelstein [11], R. Kannan [17] and [18], Singh & Russell [26],, Hardy & Rogers [12], Simeon Reich [22] and [21], Tudor Zamfirescu [27], and Lj. B. Ciric [7] and [6]. Rhoades [23] and Collaco & Silva [8] have presented a detailed comparison of the various contractive conditions which have been studied by the above authors. There is also a nice expository article by Keith Conrad [9] on the Banach's contraction principle. Another frequently cited work in the literature that I have surveyed is a compendium of fixed point results by F. F. Bonsall [5].

3.1 Commuting Pairs of Continuous Self-Maps of Complete Metric Spaces

In this section, we present a couple of results about pairs of self-maps of complete metric spaces, but for self-maps of arbitrary metric spaces, we have the following theorem; note that in the direct part we don't require continuity of the map or completeness of the space.

Theorem 3.1.1. *Let f be a mapping of a metric space (X, d) into itself. Then if f has a fixed point in X, then there exists a real number $\alpha \in (0, 1)$ and a mapping $g\colon X \to X$ such that*

1. $f(g(x)) = g(f(x))$ *for all* $x \in X$;
2. $g(X) \subseteq f(X)$; *and*

3. $d(g(x), g(y)) \leq \alpha \cdot d(f(x), f(y))$ *for all* x, $y \in X$.

And the converse also holds provided that f *is continuous and* (X, d) *is complete. Indeed* f *and* g *have a unique common fixed point if these maps have a common fixed point and if (3) above holds.*

Proof. Suppose that f has a fixed point a in X so that $f(a) = a$. Define $g\colon X \to X$ as $g(x) := a$ for $x \in X$.

Then for any x in X, we have

$$f(g(x)) = f(a) = a = g(f(x)).$$

So $f(g(x)) = g(f(x))$ for all $x \in X$.

Since $g(x) = a = f(a)$ for all $x \in X$,

$$g(X) = \{a\} \subseteq f(X).$$

And for all x, y in X, we have

$$d(g(x), g(y)) = d(a, a) = 0 \leq \alpha \cdot d(f(x), f(y))$$

for any $\alpha \in (0, 1)$.

Conversely, let's suppose that (X, d) is a complete metric space and that there exists an $\alpha \in (0, 1)$ and a mapping $g\colon X \to X$ satisfying (1), (2), and (3) above. Then, for any point $x_0 \in X$,

$$g(x_0) \in g(X) \subseteq f(X),$$

which implies that $g(x_0) = f(x_1)$ for some point x_1 of X. So after x_0, x_1, x_2, ..., x_{n-1} have been defined, let $x_n \in X$ be chosen such that $f(x_n) = g(x_{n-1})$ for any natural number n. Thus we have inductively defined a sequence (x_n) in X, and corresponding to any particular choice of x_0 this sequence is not necessarily unique.

Now for any natural number n, we have

$$d(g(x_{n+1}), g(x_n)) \leq \alpha \cdot d(f(x_{n+1}), f(x_n)) = \alpha \cdot d(g(x_n), g(x_{n-1})).$$

Thus there exists a real number $\alpha \in (0, 1)$ such that for every $n \in \mathbf{N}$,

$$d(g(x_{n+1}), g(x_n)) \leq \alpha \cdot d(g(x_n), g(x_{n-1})).$$

So

$$d(g(x_{n+1}), g(x_n)) \leq \alpha \cdot d(g(x_n), g(x_{n-1})) \leq \ldots \leq \alpha^n \cdot d(g(x_1), g(x_0))$$

for any natural number n. Then for any m, n such that $m > n$, we have

$$d(g(x_n), g(x_m)) \leq d(g(x_n), g(x_{n+1})) + \ldots + d(g(x_{m-1}), g(x_m))$$

$$
\begin{aligned}
&\leq \ (\alpha^{n-1} + \alpha^n + \ldots + \alpha^{m-1}) \cdot d(x_1, x_0) \\
&\leq \ (\alpha^{n-1} + \alpha^n + \ldots) \cdot d(x_1, x_0) \\
&= \ \frac{\alpha^{n-1}}{1-\alpha} \cdot d(x_1, x_0).
\end{aligned}
$$

Since $0 \leq \alpha < 1$,

$$\lim_{n\to\infty} \alpha^n = 0,$$

and therefore it follows that the sequence $(g(x_n))$ is a Cauchy sequence, and so this sequence converges in (X, d) to some point $t \in X$ since (X, d) is a complete metric space.

Then the sequence $(f(x_n))$, which is essentially the same sequence, also converges to the same point t.

Now we show that g is also continuous. Let $s \in X$ and let (s_n) be a sequence in X that converges to the point s. Then the sequence $(f(s_n))$ must converge to $f(s)$ because of the continuity of f; so

$$\lim_{n\to\infty} d(f(s_n), f(s)) = 0.$$

But by (3) above we have the relation

$$0 \leq d(g(s_n), g(s)) \leq \alpha \cdot d(f(s_n), f(s))$$

for all n, where α is a constant. Therefore by the squeeze theorem we have

$$\lim_{n\to\infty} d(g(s_n), g(s)) = 0$$

also.

So the sequence $(g(s_n))$ converges to the point $g(s)$, showing that g is continuous at s as the sequence (s_n) was arbitrary except that it converged to s. Since $s \in X$ is arbitrary, it follows that g is continuous on X.

Now as f and g are continuous and as

$$\lim_{n\to\infty} f(x_n) = t = \lim_{n\to\infty} g(x_n),$$

we must have

$$\lim_{n\to\infty} g(f(x_n)) = g(t)$$

and

$$\lim_{n\to\infty} f(g(x_n)) = f(t)$$

and therefore

$$\lim_{n\to\infty} d(f(g(x_n)), g(f(x_n))) = d(f(t), g(t)).$$

But by virtue of (1) above,

$$f(g(x_n)) = g(f(x_n))$$

for all n, so

$$d(f(g(x_n)), g(f(x_n))) = 0$$

for all n, and then

$$d(f(t), g(t)) = \lim_{n\to\infty} d(f(g(x_n)), g(f(x_n))) = 0$$

and hence it follows that

$$f(t) = g(t).$$

Then

$$f(f(t)) = f(g(t)) = g(f(t))$$

and

$$g(g(t)) = g(f(t)) = f(g(t)).$$

Now we show that $g(t)$ is a common fixed point of f and g. Indeed,

$$d(g(t), g(g(t))) \leq \alpha \cdot d(f(t), f(g(t))) = \alpha \cdot d(g(t), g(g(t)))$$

by (3) in our hypothesis above and the equalities just established, so we have

$$d(g(t), g(g(t))) \cdot (1 - \alpha) \leq 0.$$

But as $0 < \alpha < 1$ and as $d(g(t), g(g(t))) \geq 0$, we must have

$$d(g(t), g(g(t))) \cdot (1 - \alpha) \geq 0.$$

Hence

$$d(g(t), g(g(t))) \cdot (1 - \alpha) = 0.$$

But $1 - \alpha > 0$, so $d(g(t), g(g(t))) = 0$, which implies that $g(g(t)) = g(t)$. Thus $g(t)$ is a fixed point of g.

Now as $f(t) = g(t)$ and as f and g commute, we have

$$f(g(t)) = g(f(t)) = g(g(t)) = g(t).$$

Hence $f(t) = g(t)$ is a common fixed point of f and g.

Now we show that f and g have a unique common fixed point if they have a common fixed point and if (3) above holds: If, for some x, y in X, we have $x = f(x) = g(x)$ and $y = f(y) = g(y)$, then

$$d(x, y) = d(g(x), g(y)) \leq \alpha \cdot d(f(x), f(y)) = \alpha \cdot d(x, y),$$

so

$$0 \leq d(x, y) \cdot (1 - \alpha) \leq 0.$$

But $1 - \alpha > 0$, which implies that $d(x, y) = 0$ and thus $x = y$. □

The last theorem as stated here is a slight improvement upon Theorem 2.1 in [15].

We can even include $\alpha = 0$ in the statement of the last theorem without changing anything else. The direct part holds trivially for any non-negative α.

Corollary 3.1.2. *[15] Let $f,\ g : (X, d) \to (X, d)$ be commuting mappings, where (X, d) is complete, f is continuous, and $g(X) \subseteq f(X)$. If there exists $\alpha \in (0, 1)$ and a positive integer k such that*

$$d(g^k(x), g^k(y)) \leq \alpha \cdot d(f(x), f(y))$$

for all x and y in X, then f and g have a unique common fixed point.

Proof. Since f and g commute, for any $x \in X$, we have $f(g(x)) = g(f(x))$ and then

$$f(g^2(x)) = f(g(g(x))) = g(f(g(x))) = g(g(f(x))) = g^2(f(x)),$$

so f and g^2 commute; then

$$f(g^3(x)) = f(g^2(g(x))) = g^2(f(g(x))) = g^2(g(f(x))) = g^3(f(x)),$$

so f and g^3 commute; and continuing in this way for any positive integer k.

To use induction, if f and g^k commute for any $k \in \mathbf{N}$, then for any x in X, we have

$$f(g^k(x)) = g^k(f(x)),$$

and so

$$f(g^{k+1}(x)) = f(g^k(g(x))) = g^k(f(g(x))) = g^k(g(f(x))) = g^{k+1}(f(x)),$$

so f and g^{k+1} also commute.

Hence by induction f and g^k commute for all k in $\mathbf{N}$.

For any $y \in X$, if $y \in g^k(X)$, then $y = g^k(x)$ for some $x \in X$, so

$$y = g(g^{k-1}(x))$$

and $g^{k-1}(x) \in X$, which implies that $y \in g(X)$.

Thus

$$g^k(X) \subseteq g(X) \subseteq f(X)$$

for all $k \in \mathbf{N}$.

And since for all x, y in X,

$$d(g^k(x), g^k(y)) \leq \alpha \cdot d(f(x), f(y)),$$

we conclude that f and g^k satisfy all the conditions of the preceding theorem and so there exists a unique point $a \in X$ such that

$$a = f(a) = g^k(a).$$

Then as f and g commute,

$$g(a) = g(f(a)) = f(g(a)),$$

and also

$$g(a) = g(g^k(a)) = g^k(g(a)),$$

Which shows that $g(a)$ also turns out to be a common fixed point of f and g^k whenever a is a common fixed point of f and g^k.

But a is the unique common fixed point of f and g^k. So we must have

$$g(a) = a$$

also, and thus a is a common fixed point of f and g.

Now if b is also a common fixed point of f and g, then we have

$$b = f(b) = g(b),$$

which implies that

$$g^k(b) = g^{k-1}(g(b)) = g^{k-1}(b) = g^{k-2}(g(b)) = g^{k-2}(b) = \ldots = g(b) = b$$

and thus b turns out to be a common fixed point of f and g^k, and so $b = a$ by the uniqueness of a as a common fixed point of f and g^k.

Hence a is also the unique common fixed point of f and g. □

When f is the identity map i_X defined as $i_X(x) = x$ for $x \in X$. in the converse part of the proof of Theorem 3.1.1, we obtain the Banach contraction principle; and when f is the identity map in the last corollary, we obtain the generalized form of the same fundamental result.

Corollary 3.1.3. *[15] Let n be a positive integer and let κ be a real number > 1. If g is a continuous mapping of a complete metric space (X,d) onto itself such that*

$$d(g^n(x), g^n(y)) \geq \kappa d(x, y)$$

for all x, y in X, then g has a unique fixed point.

Proof. Let $\alpha := 1/\kappa$. Then $0 < \alpha < 1$. The inequality

$$d(g^n(x), g^n(y)) \geq \kappa \cdot d(x, y)$$

for all x, y in X then takes the form

$$d(x, y) \leq \alpha \cdot d(g^n(x), g^n(y))$$

for all x, y in X.

So for any x, y in X, we have

$$d(g(x), g(y)) \leq \alpha \cdot d(g^n(g(x)), g^n(g(y))) \leq \cdot\alpha \cdot d(g^{n+1}(x), g^{n+1}(y)).$$

Now since g is continuous, so is g^{n+1}. And as $g\colon X \to X$ is onto, $g(X) = X$.

So if $x \in X = g(X)$, then there exists some point $x_1 \in X$ such that $x = g(x_1)$.

Now as $x_1 \in X = g(X)$, we have $x_1 = g(x_2)$ for some point $x_2 \in X$ and then

$$x = g(x_1) = g(g(x_2)) = g^2(x_2).$$

Continuing in this way we obtain a sequence of points x_1, x_2, ..., x_n in X such that

$$x = g(x_1) = g^2(x_2) = \ldots = g^n(x_n).$$

Finally as $x_n \in X = g(X)$, we can write $x_n = g(x_{n+1})$ for some point $x_{n+1} \in X$.

Hence we have

$$x = g(x_1) = g^2(x_2) = \ldots = g^n(x_n) = g^{n+1}(x_{n+1})$$

or

$$x = g^{n+1}(x_{n+1}).$$

Since $x \in X$ was arbitrary, it follows that

$$X \subseteq g^{n+1}(X) \subseteq X,$$

which implies that

$$g^{n+1}(X) = X;$$

so g^{n+1} is also an onto map, whence

$$g(X) \subseteq g^{n+1}(X).$$

Therefore we have two commuting mappings g and g^{n+1} of X onto itself such that (for our interest) g^{n+1} is continuous, $g(X) \subseteq g^{n+1}(X)$, and also for all x, $y \in X$, we have the inequality

$$d(g(x), g(y)) \leq \alpha \cdot d(g^{n+1}(x), g^{n+1}(y)),$$

where α is a real number in the open interval $(0, 1)$.

Thus the maps g and g^{n+1} of X onto (and hence into) itself, where (X, d) is a complete metric space, satisfy all the conditions in Theorem 3.1.1. So they have a unique common fixed point, say, $a \in X$. That is,

$$a = g(a) = g^{n+1}(a).$$

Thus a is a fixed point of g.

Now if some $b \in X$ also happens to be a fixed point of g, then we have $g(b) = b$; so

$$g^{n+1}(b) = g^n(g(b)) = g^n(b) = g^{n-1}(g(b)) = g^{n-1}(b) = \ldots = g^2(b) = g(g(b)) = g(b) = b,$$

so

$$b = g(b) = g^{n+1}(b),$$

which implies that $b = a$ owing to the uniqueness of a as a common fixed point of g and g^{n+1}.

Hence g has a unique fixed point. □

If we take $n := 1$ in the last corollary, then the mapping g is said to be *expansive*.

Example

[15] Let $X := \mathbf{R}^2$, the real Euclidean 2-space with the usual metric, denoted by d, and let f, $g \colon X \to X$ be defined as follows:

$$g((x, y)) := (7x, \frac{y}{3} + 4)$$

and

$$f((x, y)) := (11x, \frac{y}{2} + 3)$$

for each $(x, y) \in X$.

Then as the real-valued functions of the real variables x and y given by $x \to 7x$, $x \to 11x$, $y \to y/3 + 4$, and $y \to y/2 + 3$ are continuous everywhere, so the functions f and g are also continuous on all of $\mathbf{R}^2$.

For any $(x, y) \in \mathbf{R}^2$, we have

$$f(g((x, y))) = f((7x, \frac{y}{3} + 4)) = (11(7x), \frac{\frac{y}{3} + 4}{2} + 3) = (77x, \frac{y}{6} + 5)$$

and

$$g(f((x, y))) = g((11x, \frac{y}{2} + 3)) = (7(11x), \frac{\frac{y}{2} + 3}{3} + 4) = (77x, \frac{y}{6} + 5) = f(g((x, y))).$$

Thus f and g commute.

Now for any two points (a,b), $(x,y) \in \mathbf{R}^2$, we have

$$\begin{aligned}
d(g((a,b)),g((x,y))) &= d((7a,\frac{b}{3}+4),(7x,\frac{y}{3}+4)) \\
&= \sqrt{(7a-7x)^2+((\frac{b}{3}+4)-(\frac{y}{3}+4))^2} \\
&= \sqrt{49(a-x)^2+\frac{1}{9}(b-y)^2} \\
&\leq \frac{2}{3}\cdot\sqrt{121(a-x)^2+\frac{1}{4}(b-y)^2} \\
&= \frac{2}{3}\cdot\sqrt{(11a-11x)^2+((\frac{b}{2}+3)-(\frac{y}{2}+3))^2} \\
&= \frac{2}{3}\cdot d((11a,\frac{b}{2}+3),(11x,\frac{y}{2}+3)) \\
&= \frac{2}{3}\cdot d(f((a,b)),f((x,y))).
\end{aligned}$$

Thus for all (a,b), $(x,y) \in \mathbf{R}^2$, we have

$$d(g((a,b)),g((x,y))) \leq \frac{2}{3}\cdot d(f((a,b)),f((x,y))).$$

And we also note that for any point $(x,y) \in \mathbf{R}^2$,

$$(x,y) = (7\cdot\frac{x}{7},\frac{3y-12}{3}+4) = g(\frac{x}{7},3y-12)$$

and also that

$$(x,y) = (11\cdot\frac{x}{11},\frac{2y-6}{2}+3) = f((\frac{x}{11},2y-6)).$$

Thus both f and g are onto.

In short, we have seen that the mappings f and g are both continuous maps of the real Euclidean 2-space $\mathbf{R}^2$—which is a complete metric space—onto itself such that for all (a,b), $(x,y) \in \mathbf{R}^2$, we have

$$d(g((a,b)),g((x,y))) \leq \frac{2}{3}\cdot d(f((a,b)),f((x,y))).$$

So by our main theorem in this section, f and g must have a unique common fixed point.

To find the fixed points of g we set

$$g((x,y)) = (x,y).$$

Then

$$(7x, \frac{y}{3}+4) = (x, y),$$

which implies that

$$(x, y) = (0, 6),$$

and conversely.

Hence $(0, 6)$ is the only fixed point of g.

To find the fixed points of f we set

$$f((x, y)) = (x, y).$$

Then

$$(11x, \frac{y}{2}+3) = (x, y),$$

which implies that

$$(x, y) = (0, 6),$$

and conversely.

So $(0, 6)$ is the only fixed point of f also.

Hence $(0, 6)$ is the unique common fixed point of f and g.

However, neither of f and g is contractive or expansive. In fact, for any a, b, x, $y \in \mathbf{R}^2$, we find that

$$d(f((x, b)), f((x, y))) = d((11x, \frac{b}{2}+3), (11x, \frac{y}{2}+3)) = \frac{1}{2} \cdot |b-y| = \frac{1}{2} \cdot d((x, b), (x, y))$$

and

$$d(g((x, b)), g((x, y))) = d((7x, \frac{b}{3}+4), (7x, \frac{y}{3}+4)) = \frac{1}{3} \cdot |b-y| = \frac{1}{3} \cdot d((x, b), (x, y)),$$

showing that neither of f and g is expansive.

Similarly, we note that

$$d(f((a, b)), f((x, b))) = d((11a, \frac{b}{2}+3), (11x, \frac{b}{2}+3)) = 11|a-x| = 11d((a, b), (x, b))$$

and

$$d(g((a, b)), f((x, b))) = d((7a, \frac{b}{3}+4), (7x, \frac{b}{3}+4)) = 7 \cdot |b-y| = 7 \cdot d((a, b), (x, b)),$$

showing that neither of f and g is contractive.

Chapter 4

Appendix

The most fundamental notion with which we are concerned in this dissertation is that of a metric space. However, we first define the more general notion of a topological space.

Topological Spaces

Let X be a non-empty set. Then a collection Ω of subsets of X is called a *topology* on X iff the union of any subcollection of Ω is again in Ω and the intersection of any finite sub-collection of Ω is also in Ω, where for an empty collection of subsets of X, its union is to be defined to be the empty set and its intersection to be the set X itself. Sets that make up Ω are said to be *open* and the complements in X of these open sets are said to *closed*. The set X together with a topology Ω is called a *topological space* and is denoted by (X, Ω) or simply X when the topology is clear from the context. In every topological space X, both the empty set and the set X itself are closed as well as open; however there may be other subsets of X with this property.

For any non-empty set X, the collection consisting of the empty set and the set X itself is a topology on X, called the *indiscrete* topology; similarly, provided X have more than one element, the collection of all possible subset of X is another topology. on X, called the *discrete* topology. Between these two extremes may lie an abundant collection of topologies.

If (X, Ω) is a topological space and if Y is a non-empty subset of X. then we can show that the collection of subsets of X consisting of the intersection with Y of the sets that make up the collection Ω is a topology on Y, which we call the *induced* or *relative* topology and denote it by Ω_Y. The topological space (Y, Ω_Y) is said to be a topological *subspace* of X.

Now for the notion of a metric space.

Metric Spaces

A *metric* is an abstraction of the notion of distance (between points on a line, in the plane or in the three-dimensional space) in elementary geometry.

Let X be a given non-empty set. Then a function $d\colon X \times X \to [0, +\infty)$ is said to be a *metric* if it satisfies the following conditions for all x, y, z in X:
[**definiteness**] $d(x, y) = 0$ iff $x = y$
[**symmetry**] $d(x, y) = d(y, x)$
[**triangle inequality**] $d(x, y) \leq d(x, z) + d(z, y)$

A set X, together with a metric d defined on it, is called a *metric space* and is usually denoted by (X, d), or simply by X when the metric is clear from the context. Of course, we can regard every non-empty subset Y of a metric space X to be a metric space in its own right by restricting the metric in question to the elements of Y. In this case Y is said to be a metric *subspace* of X.

If in the above set of conditions on d we merely require that $d(x, x) = 0$ for all x in X in place of definiteness, then we obtain what is called a *semi-metric*. A semi-metric space (X, d) can easily be made into a metric space by first defining any two points x and y of X to be related iff $d(x, y) = 0$; then observing that this relation, being an equivalence relations, partitions the set X into disjoint subsets; and finally defining the "distance" $d\prime$ between the subset $[x]$ containing x and the subset $[y]$ containing y as $d\prime([x], [y]) := d(x, y)$. Here we need to show this "distance" is independent of the choice of the points x and y as representative elements of the particular subsets.

Of course, given any (non-empty) set X, we can define a metric on it as follows: Let r be any fixed but arbitrary positive real number, and let $d_r\colon X \times X \to [0, +\infty)$ be defined as

$$d_r(x, y) := \begin{cases} 0 & \text{if } x = y \ ; \\ r & \text{if } x \neq y \ . \end{cases}$$

That d_r as defined here is a metric can be easily verified. For $r = 1$, we get what is called the *discrete* metric. Note that in this case we obtain an infinite collection of metric spaces with the same underlying set as r varies over the open interval $(0, +\infty)$.

That it is possible for a set to be underlying several different metrics is also brought out by the following very important example: Let n be a positive integer, let $p \geq 1$, and let $d_p\colon \mathbf{R}^n \to \mathbf{R}^n$, where $\mathbf{R}$ denotes the set of real numbers, be defined as

$$d_p((x_1, \ldots, x_n), (y_1, \ldots, y_n)) := \sqrt[p]{\sum_{k=1}^{n} |x_k - y_k|^p}.$$

The metric space $(\mathbf{R}^n, d_p)$ — which encompasses all the structures dealt with in the Euclidean analytic geometry of the line, plane, and space — is in fact a particular example of the following: Let l^p denote the set of all (real or complex) sequences $(x_n)_{n=1}^{\infty}$ for which $\sum_{n=1}^{\infty} |x_n|^p < +\infty$. Then we can define

$$d_p((x_n)_{n=1}^{\infty}, (y_n)_{n=1}^{\infty}) := \sqrt[p]{\sum_{n=1}^{\infty} |x_n - y_n|^p},$$

where we also have $(y_n)_{n=1}^{\infty}$ also satisfies $\sum_{n=1}^{\infty} |y_n|^p < +\infty$. Refer to [19] and [25].

In the theory of the metric and topological spaces, amongst some particularly relevant concepts are those of the open and closed balls (also called spheres); the interior, exterior, and boundary (points) of a set; and the open, closed, and compact sets.

Let (X, d) be a metric space, let $x_0 \in X$, and let $\epsilon > 0$ be given. Then the subset $B(x_0, r)$ defined as

$$B(x_0, r) := \{x \in X \colon d(x, x_0) < \epsilon\}$$

is an *open sphere* with center x_0 and radius r; this set is also called a *neighborhood* of x_0 and is denoted by $N_\epsilon(x_0)$ or simply as $N(x_0)$ when we are not concerned with what the radius of the neighborhood is.

Let A be a set in a metric space X. A point a in X is said to be an *interior point* of A iff there exists a neighborhood $N(a)$ of a that is contained in A, and the set of all interior points of A is called the interior of A. Further, we define A to be an open set iff each point of A happens to be an interior point.

A point x in a metric is said to be an *adherent point* of a set A in the space iff every neighborhood of x intersects A and is said to be a *limit*, or *cluster*, point of A iff every neighborhood of x contains a point of A other than the point x itself. If x is a limit point of A, then it can be shown that every neighborhood of x contains infinitely many points of A. Thus no finite set in a metric space can have limit points. The set of all the limit points of A is called the *derived* set of A and the set of all the adherent points the *closure* of A. The set A is defined to be *closed* iff it contains all of its limit points, and A is called *perfect* iff each point of A is a limit point of A. Evidently A is contained in its closure, and it can be shown that A is closed if and only if the reverse inclusion also holds. In particular, in a metric space every finite set, having no limit points, is necessarily closed.

A set S in a metric space X is said to be *bounded* iff there exists a point a in x and a real number $\epsilon > 0$ such that $d(x, a) < \epsilon$ for all points x in S (i.e. S is contained in some open sphere).

In a metric space, it can be shown that the union of any collection—finite, countable, or uncountable—of open sets and the intersection of any finite collection of open sets is again open; it can be also shown that a set is open if and only if its complement is closed; and consequently the union of any finite collection or the intersection of an arbitrary collection of closed sets is also closed by virtue of the De

Morgan's laws of set theory. Furthermore, the empty set and the whole space can each be shown to be both open and closed. Thus the collection of all open sets in a metric space defines a topology called the *metric topology.*

A sequence (x_n) in a metric space is said to be a *Cauchy* sequence iff the following holds: given any positive real number ϵ, there exists a natural number N_ϵ such that $d(x_n, x_m) < \epsilon$ whenever $n > N_\epsilon$ and $m > N_{epsilon}$ (i.e. the terms of the sequence ultimately get arbitrarily close to each other). Every Cauchy sequence can be easily shown to be bounded, though every bounded sequence need not be Cauchy.

We define (x_n) to be *convergent* iff there exists a point x in X, called a limit of (x_n), for which the following holds: given any real number $\epsilon > 0$, we can find a natural number N_ϵ such that $d(x_n, x) < \epsilon$ whenever $n > N_\epsilon$ (i.e. the terms of the sequence ultimately get arbitrarily close to x). It can be shown that a sequence can have at most one limit and that every convergent sequence is necessarily Cauchy— and thus bounded— but not conversely.

A metric space in which every Cauchy sequence is convergent is said to be *complete.*For example, $\mathbf{R}^n$, n being a positive integer, is complete (See [24], Theorem 3.11 (c)), whereas $\mathbf{Q}$ is not.a complete subspace of $\mathbf{R}$. Any closed subset of a complete metric space is itself complete (as a metric subspace) [19], Section 1.3 and 1.4.

A metric space in which every Cauchy sequence is convergent is said to be *complete.*For example, $\mathbf{R}^n$, n being a positive integer, is complete (See [24], Theorem 3.11 (c)), whereas $\mathbf{Q}$ is not.a complete subspace of $\mathbf{R}$.

If A is a set in a metric space X, then a point x in X belongs to the closure of A if and only if there is a sequence (x_n) of points of A that converges to x. In particular, x is a limit point of A if and only if there exists a sequence in $A - \{x\}$ that converges to x. Thus A is closed if and only if every sequence in A converges in A. Refer to [19], Section 1.4.

For a set A of real numbers which is bounded above (respectively bounded below), an upper bound (respectively lower bound) x is the supremum (respectively infimum) of A if and only if there exists a sequence in A that converges to x; hence every closed and bounded set of real numbers has a maximum and a minimum element.

A sequence (x_n) in a metric space converges to a point x in X if and only if every subsequence of (x_n) converges to x, Indeed, the set of all subsequential limits of a sequence in a metric space is a closed set see Theorem 3.7 in [24]); therefore in the case of a real sequence this set—with the $\pm\infty$ being added if either of these happens to be a subsequential limit—is a bounded subset of the set of extended real numbers and thus has a maximum element—called the *limit superior* and denoted by $\limsup$—and a minimum element—called the *limit inferior* and denoted by $\liminf$. A real sequence thus converges in the extended reals if and only if its limit superior and limit inferior coincide. We can even conclude that if (x_n) is a real sequence and if $x > \limsup x_n$ (respectively $x < \liminf x_n$), then there exists a natural number N such that $x > x_n$ (respectively $x < x_n$) for all $n > N$; furthermore $\limsup x_n$ (respectively $\liminf x_n$) are the unique numbers in the set of subsequential limits—finite or infinite—of (x_n) that have these properties Refer to [24], Theorem 3.17). Moreover, if (x_n) and (y_n) are two real sequences for which there exists an N such that $x_n \leq y_n$ for all $n > N$, then $\liminf x_n \leq \liminf y_n$ and $\limsup x_n \leq \limsup y_n$; refer to Theorem 3.19 in [24].

Given two metric spaces (X, d_X) and (Y, d_Y), a map $f \colon X \to Y$, and a point a in X, the mapping f is said to be *continuous* at a iff the following holds: for any positive real number ϵ, we can find a positive real number δ, usually dependent upon both ϵ and the point a, such that $d(f(x), f(a)) < \epsilon$ for all points x in X for which $d(x, a) < \delta$. The mapping f is said to be continuous on a subset A of X iff f is continuous at each point of A. Furthermore, the mapping f is said to be *uniformly* continuous on A iff given $\epsilon > 0$ we can find a $\delta > 0$ such that $d(f(x), f(y)) < \epsilon$ for all points x, y in A for which $d(x, y) < \delta$. Clearly every uniformly continuous mappings is continuous, though the converse need not hold; e.g., consider the function $f \colon [0, +\infty) \to [0, +\infty)$ defined by $f(x) := x^2$ for all x in $[0, +\infty)$.

Continuity at a point can be characterized in terms of convergence of sequences as follows: a map f of a metric space X into a metric space Y is continuous at a point x in X if and only if the sequence $(f(x_n))$ converges in Y to the point $f(x)$ for every sequence (x_n) in X that converges to x; for a proof, refer to [24], Theorems 4.2 and 4.6.

Continuous mappings have the following "global" characterization: a mapping f of a metric space X into a metric space Y is open respectively closed if and only if $f^{-1}(A)$ is open (respectively closed) in X for every open (respectively closed) set A in Y; refer to Theorem 4.8 in [24].

For a fuller discussion of these concepts we refer the reader to [24] or [2]. We here summarize the concept of compactness in metric spaces.

Compactness

A set A in a topological space is said to be *compact* (respectively, *countably compact*) iff for every collection of open sets whose union contains A as a subset—called an open cover for A—, we can find a finite (respectively, countable) sub-collection whose union also contains A.

And, a set A in a metric space X is said to be *sequentially compact* iff every sequence in A has a convergent subsequence (i.e. a subsequence that converges to some point of A). Of course the set A can be the whole space X.

In a metric or topological space, every finite set is evidently compact. Further, every compact metric space is necessarily complete (Theorem 3.11 (b) in [24]), though the converse may be false, as in the case of the complete metric space $\mathbf{R}$, which by virtue of Theorem 3.38 in [2] is not compact because it fails to be bounded.

For metric spaces, compactness is equivalent to sequential compactness. For a proof, see Section 24 in [25]. Every compact subset of a metric space is necessarily closed and bounded [Refer to Theorem 3.38 in [2]) though the converse is not always true, as illustrated by Problem 3.42 in [2]; however for the Euclidean spaces $\mathbf{R}^n$, where n is a positive integer, the converse also holds (See Theorem 2.41 (a) and (b) in [24]). In particular, every closed interval $[a, b]$ in $\mathbf{R}$ is compact.

The continuous image of a compact set is compact; that is, if X and Y are two metric spaces, if $f\colon X \to Y$ is continuous, and if A is a compact subset of X, then $f(A) := \{f(x)\colon x \in A\}$ is a compact subset of Y, so $f(A)$ is closed and bounded if Y is the Euclidean space $\mathbf{R}^n$ for any positive integer n; in particular $f(A)$ has a maximum and a minimum element if Y is $\mathbf{R}$; as another particular case the function f attains a maximum and a minimum value if A coincides with X and Y is $\mathbf{R}$; refer to Theorems 4.14 through 4.16 in [24].

In the last few pages, we have given a brief description of some of the properties of metric spaces—and more generally topological spaces—that we have drawn upon freely throughout this work. However, the present discussion is by no means exhaustive! For a detailed account of these matters, we refer the interested reader to the references quoted herein.

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