Rayleigh Wave Propagation in Anisotropic Elastic Materials

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I dedicate this work to my Beloved Mother and Father

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Abstract

In this dissertation, a review is presented on the propagation of Rayleigh waves in anisotropic elastic materials. The main focus is to discuss the propagation of **Rayleigh waves** in anisotropic materials. In crystals, Rayleigh wave propagation is carried out in the cases when Christoffel equations split into two parts resulting Rayleigh wave to polarize in x_1x_2-plane and a shear horizontal wave in x_3 direction. The boundary conditions are also simplified when some elastic constants vanish by applying conditions and symmetries. The conditions and equations are satisfied by different structures of crystals. Secular equation for surface waves propagating on a **monoclinic** half-space is derived using the method of first integrals. By using Stroh formalism, a system of two second order differential equations for traction components is developed. Secular equation for monoclinic material is re-derived as quartic for squared wave speed. The secular equation for surface waves propagating on **orthotropic** incompressible half-space is also re-derived in a direct manner, again by using the method of first integrals.

Contents

Chapter 1

Introduction

Waves are the disturbance in space which bring energy in space. In today's world, it is well known and much explored that waves are the main source of communication whether it is sound, radar system, electronic system or signal processing. Different types of waves exist in space and they are being used according to their properties in the field of science and technology. Mainly there are two types of waves. Waves that need medium like air for their propagation and can not transmit energy in vacuum, are named as mechanical waves. The other type of waves do not need medium for propagation, called the electromagnetic waves. As we know the role of waves is important in the atmosphere for different purposes. Likewise waves have much more significance inside the layers of earth. Further in the deep down study of mechanical waves, we come across too many types of waves which were discovered by different physicist with the passage of time. For example, a physicist named Lord Rayleigh [1] discovered the waves that travels over the surface especially associated with earthquake. This type of wave travels and vanishes when the distance from surface increases. The particle movement of the wave was also discovered that was in elliptic form. Such waves are named after him and are called the Rayleigh waves. After the Rayleigh waves discovery, Stoneley analyzed the wave propagation theoretically.

The focus in this dissertation is reviewing the work of [2], [3] and [4] for finding secular equations using different methods.

The study of [2] investigates that how the Rayleigh waves are propagated. On one hand, the Christoffel equations split in two parts giving a Rayleigh wave polarized in x_1x_2- plane and a shear horizontal wave polarized along x_3. On the other hand, we attain simplified boundary conditions when some elastic constants become zero. The study is on four materials and their configurations of crystals. Rayleigh wave velocity and displacement is calculated in this problem. After that, some suitable configurations are discussed for monoclinic, orthorhombic, tetragonal, hexagonal and cubic crystals which are later justified mathematically.

Destrade [3] worked on finding the expression of secular equation for Rayleigh waves in a half space that is an elastic monoclinic material. The author worked on the system of equations with traction components as variables and followed the method of [5].

Destrade [4] worked on the surface waves which are propagating on the surface of incompressible orthotropic half space. In this paper, the author evaluated the secular equation of surface waves by using the method of first integral followed by Mozhaev [5]. The method is applied on the system of two second order differential equations for traction components on surfaces which are parallel to free surface. The summary of dissertation is given as:

Chapter 2 comprises of the basic definitions and concepts of theory of elasticity. Some derivations of basic equations are also discussed. Apart from definitions, classification of crystals and their types are also defined by showing their matrices. Different types of waves are introduced and it is discussed how they differ from one another. Chapter 3 is the review of [2]. Velocity equation and displacement components are calculated. Some configurations for monoclinic, orthorhombic, tetragonal,hexagonal and cubic are discussed and justified.

Chapter 4 is the review work of [3] in which the purpose is to find the secular equation for surface waves in monoclinic materials by using method of first integral. Later, the equation is reduced for orthotropic material by using certain conditions.

Chapter 5 is about calculating the secular equation in an incompressible half space made of orthotropic material by using the method suggested in [5].

Chapter 6 is the conclusion of this dissertation.

Chapter 2

Basics of Elasticity Theory

This chapter is about the basic concepts and definitions in elasticity theory like stress, strain and the relation between them. Other than this, waves and their types, crystal classes are mentioned which are extensively used in this dissertation.

2.1 Elasticity Theory

The study is about the effect of deformation on a body. Deformation is caused due to the applied stress resulting in the change of shape of a body. The branch of mechanics which deals with the kinematics and mechanical properties of material considered as continuous and homogeneous rather than discrete particles. This branch of mechanics is called Continuum mechanics.

2.2 Tensors

Tensors are the mathematical geometric objects that are used to determine the physical properties of materials. We define tensor as a real valued function.

Formally the components of rank n tensor $T'_{ijkl...}$ transform as

$$T'_{ijkl...} = ...q_i^l q_j^m q_k^n q_l^o ... T_{...lmno}, \quad \text{where} \quad i, j, k, l... = \{1, 2, 3, ..., n\}. \qquad (2.2.1)$$

where q_i^l denotes the $n \times n$ transformation matrix from primed to the unprimed coordinates. Tensors can be generalized as both scalar and vector quantities as zero, first, second, third and the fourth order tensor. A scalar quantity is a zero order tensor. When the direction is involved along with magnitude we call it a vector and a vector is a first order tensor. To represent a physical quantity in a plane, we use a second order tensor. Tensors with higher order can be written as a combination of lower order tensors.

2.3 Stress-Strain Relationship

Here, we define stress and strain so that we are able to understand the relationship between them.

2.3.1 Stress

Stress is defined as a physical quantity which in simple words, is the internal force that is exerted on and by the neighboring particles and molecules that in turn tends to disturb the shape of body. Let $\Delta\mathbf{F}$ be the force that is applied and $\Delta\mathbf{s_k}$ be the surface where the force is applied then the components of the stress tensor T_{ik} can be denoted as

$$T_{ik} = lim_{\Delta\mathbf{s_k}\to 0}\frac{\Delta F_i}{\Delta\mathbf{s_k}}. \qquad k = \{1, 2, 3\} \qquad (2.3.1)$$

In Figure 2.1, T_{ik} denotes the stress tensor components. First index in T_{ik} establishes the direction where the stress is applied and second index tells us about the direction of normal to which the stress is applied and ΔF_i is the i th component of force $\Delta\mathbf{F}$

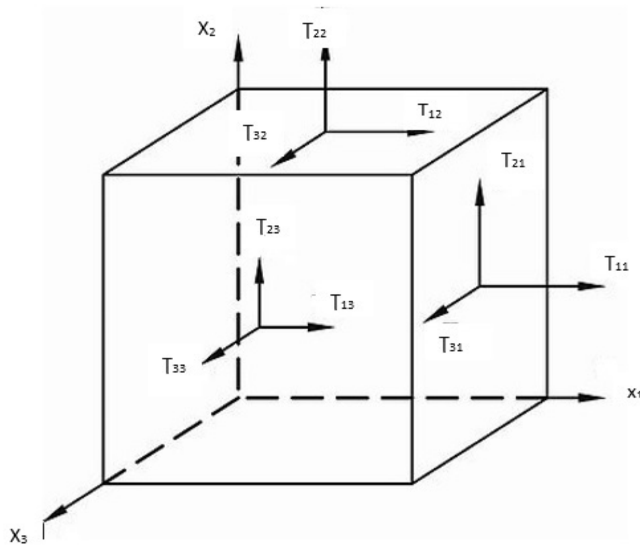

Figure 2.1: Stress Tensor Components

2.3.2 Strain

In continuum mechanics, strain is defined as the mathematical approach to measure the deformation produced in solid bodies due to applied force. This change may be in length, volume or in a shape. Strain has no unit because it is ratio between same quantities. Strain can be longitudinal, shear or volumetric. Symbolically, strain is written as **S**and mathematically strain component is defined as

$$S_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right), \qquad i, j = \{1, 2, 3\}, \tag{2.3.2}$$

where S_{ij} represents the components of linear elastic strain tensor and u_i are the displacement components. Note that strain tensor is symmetric, that is, $S_{ij}= S_{ji}$.

2.3.3 Relation between stress and strain

When stress is applied on a body, strain is produced which means change can occur in the body. For example, when a rubber band is stretched by applying stress, strain is produced and is shown in the extended rubber band. When the stress is removed, the rubber band tends to come to its original shape. This is the relation between stress and strain which is generally called Hook's Law named after Robert Hooks(1660). The law was generalized by Cauchy for materials that are linearly elastic. He described that the stress components are linear functions of strain components. Stress and strain are second rank tensors. It is formulated as

$$T_{ij} = C_{ijkl} S_{kl}, \tag{2.3.3}$$

where $i, j, k, l = \{1, 2, 3\}$. Here, C_{ijkl} are components of elastic stiffness matrix tensor $\mathbf{C}$ which contains 81 independent components in a 3 dimensional space. The entries of C_{ijkl} are called elastic constants which reduce in number when material symmetries are applied. The stiffness matrix tensor has the same SI unit as that of stress that is Pascal P where $P = N/m^2$ where N represents Newton and m is for meter.

2.4 Symmetries of elasticity tensor components

Symmetries in structure of materials exist through which we can determine material anisotropy of several types. In general, a fourth order tenor consists of 81 independent components. However, since T_{ij} is symmetric, that is $T_{ij} = T_{ji}$ so elastic stiffness tensor has symmetry in its first two indices.

$$C_{ijkl} = C_{jikl} \tag{2.4.1}$$

This symmetry reduces the components from 81 to 54. This is called left minor symmetry. The symmetry in strain tensor S_{kl} shows symmetry for the last two indices as $S_{kl} = S_{lk}$ which shows C_{ijkl} symmetric as

$$C_{ijkl} = C_{ijlk}. \tag{2.4.2}$$

This symmetry which is right minor symmetry reduces the components from 54 to 36. Symmetries in Eq. (2.4.1) and Eq. (2.4.2) hold simultaneously. Further reduction is done by strain energy function that is defined as the work that is done internally in deforming the shape of the body by activity of external forces. The expression for strain energy function is

$$W = \frac{1}{2} C_{ijkl} S_{ij} S_{kl} = \frac{1}{2} C_{klij} S_{kl} S_{ij} = \frac{1}{2} C_{klij} S_{ij} S_{kl}. \tag{2.4.3}$$

The right side of Eq. (2.4.3) involves those combinations of elasticity tensor that are pairwise symmetric thats is $(ij) \leftrightarrow (kl)$. This shows elastic tensor as

$$\begin{aligned} &\frac{1}{2}[C_{klij} - C_{ijkl}] S_{ij} S_{kl} = 0, \\ &C_{klij} = C_{ijkl}. \end{aligned} \tag{2.4.4}$$

2.5 Tensors in Voigt notation

Here, we use the well known Voigt notation for the components of the elasticity tensor. In this notation, a pair of indices is represented by a single number varying from 1 to 6 and the notations are fixed for each pair. This classical notation is given by

$$\begin{aligned} &(11) \longrightarrow (1), \quad (22) \longrightarrow (2), \quad (33) \longrightarrow (3), \\ &(23) \longrightarrow (4), \quad (13) \longrightarrow (5), \quad (12) \longrightarrow (6). \end{aligned} \tag{2.5.1}$$

More precisely, let $\alpha = [ij]$ and $\beta = [kl]$, we have the matrix $C_{\alpha\beta}$ as

$$[C_{\alpha\beta}] = \begin{bmatrix} C_{11} & C_{12} & C_{13} & C_{14} & C_{15} & C_{16} \\ * & C_{22} & C_{23} & C_{24} & C_{25} & C_{26} \\ * & * & C_{33} & C_{34} & C_{35} & C_{36} \\ * & * & * & C_{44} & C_{45} & C_{46} \\ * & * & * & * & C_{55} & C_{56} \\ * & * & * & * & * & C_{66} \end{bmatrix}, \qquad \alpha, \beta = \{1, 2, ..., 6.\} \tag{2.5.2}$$

Here in this matrix $*$ represents linearly dependent elastic tensor components $C_{\alpha\beta}$.

2.6 Crystal structures and their symmetries

A crystal structure is defined as the pattern in which atoms, ions and molecules are arranged and their repetition form crystals. The basic structure that repeats to form the crystal is called a unit cell of the crystal. Crystal structures are divided into several systems that are triclinic, monoclinic, orthorhombic, cubic, tetragonal, trigonal and hexagonal, differentiated with respect to the crystal symmetry. One needs to know about the rotation and the axis of rotation for better understanding of crystal classes. An axis of rotation is the axis or we can say a line about which rotation occurs. The angle of rotation is $2\pi/n$ where n shows the n-folds. An n-fold A_n is the proper rotation which represents counter-clockwise rotation of $(360/n)^o$ around an axis. If an n-fold rotation operation is repeated n times, then the object returns to its original position. We also define isotropic and anisotropic materials in this section. Isotropic are those materials which are not direction dependent. Their properties do not change with the change in direction or we can say orientation. On the other hand, anisotropic materials differ with respect to the change in orientation that a material posses.

2.6.1 Monoclinic crystal system

The monoclinic crystals is expressed by three sides where all three are of unequal lengths and two among three are perpendicular and the third side has acute angle with other two sides. See Figure 2.2.
where a,b,c are lengths of monoclinic crystal unit and α, β, γ are the angles between each of the two sides such that

$$a \neq b \neq c \quad \text{and} \quad \alpha = \gamma = 90^o \neq \beta. \tag{2.6.1}$$

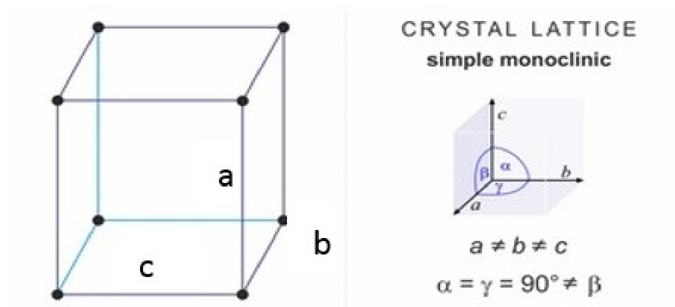

Figure 2.2: Monoclinic crystal unit

As monoclinic crystals has one dyad axis that rotates the object by 180^o. The general matrix of rotation, for example for x_3- axis as dyad axis is

$$Q = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{2.6.2}$$

A fourth order elastic tensor transforms through this rotation as follows:

$$C'_{ijkl} = Q_{pi}Q_{qj}Q_{rk}Q_{sl}C_{pqrs}, \tag{2.6.3}$$

Using Eq. (2.6.2) in Eq. (2.6.3), we can evaluate the independent components for a monoclinic material, given by

$$[C_{\alpha\beta}] = \begin{bmatrix} C_{11} & C_{12} & C_{13} & 0 & 0 & C_{16} \\ C_{12} & C_{22} & C_{23} & 0 & 0 & C_{26} \\ C_{13} & C_{23} & C_{33} & 0 & 0 & C_{36} \\ 0 & 0 & 0 & C_{44} & C_{45} & 0 \\ 0 & 0 & 0 & C_{45} & C_{55} & 0 \\ C_{16} & C_{26} & C_{36} & 0 & 0 & C_{66} \end{bmatrix} \tag{2.6.4}$$

2.6.2 Orthotropic crystal system

It is also one of the seven crystal systems having mutually perpendicular axis that are two fold which means after rotation of 90^o the material remains invariant. Orthorhombic crystal has three unequal sides but are perpendicular to each other that is given in Eq.(2.6.5). See Figure 2.3.

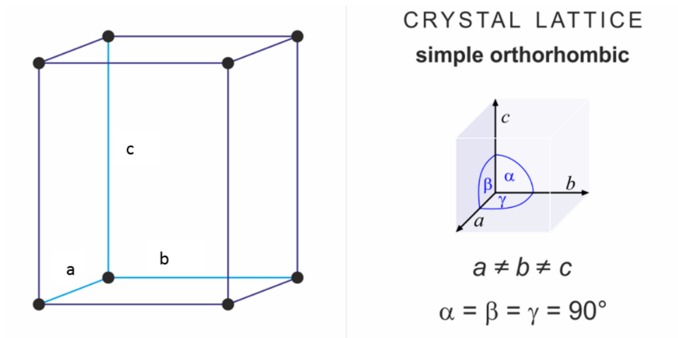

Figure 2.3: Orthorhombic crystal unit

where a,b,c are the lengths of the sides of orthotropic crystals unit and α, β, γ are the angles between each of the two sides such that

$$a \neq b \neq c, \quad \text{and} \quad \alpha = \beta = \gamma = 90^o. \tag{2.6.5}$$

There is more symmetry in orthotropic crystal than the monoclinic crystal has. When we calculate all the transformed stiffness constants, we see that we get twelve non zero components out of which nine are independent and 3 are dependent. Stiffness component matrix for an orthrotropic material is

$$[C_{\alpha\beta}] = \begin{bmatrix} C_{11} & C_{12} & C_{13} & 0 & 0 & 0 \\ C_{12} & C_{22} & C_{23} & 0 & 0 & 0 \\ C_{13} & C_{23} & C_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & C_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & C_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & C_{66} \end{bmatrix}. \tag{2.6.6}$$

2.6.3 Tetragonal crystal system

The tetragonal crystals are represented by three lengths of the sides such that $a = b \neq c$ and each side is perpendicular to each other as shown in Figure 2.4

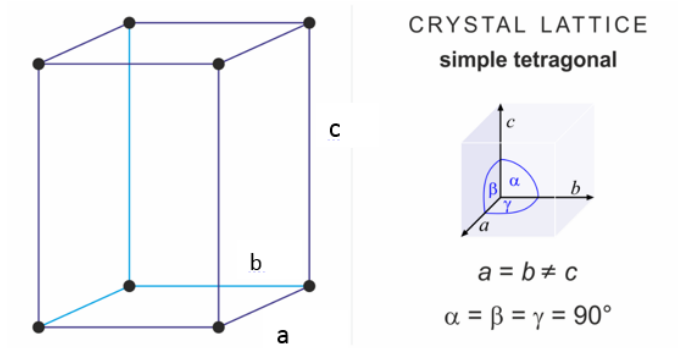

Figure 2.4: Tetragonal crystal unit

where the sides and angles between them are clearly shown. The stiffness matrix for tetragonal crystal system is more simplified than that of orthorhombic which is

$$C_{\alpha\beta} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & 0 & 0 & C_{16} \\ C_{21} & C_{11} & C_{13} & 0 & 0 & -C_{16} \\ C_{13} & C_{13} & C_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & C_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & C_{44} & 0 \\ C_{16} & -C_{16} & 0 & 0 & 0 & C_{66} \end{bmatrix} \tag{2.6.7}$$

2.6.4 Hexagonal crystal system

The hexagonal crystals are represented by three lengths of the sides such that $a = b \neq c$ as that of tetragonal crystal but it differs in angles. See Figure 2.5.

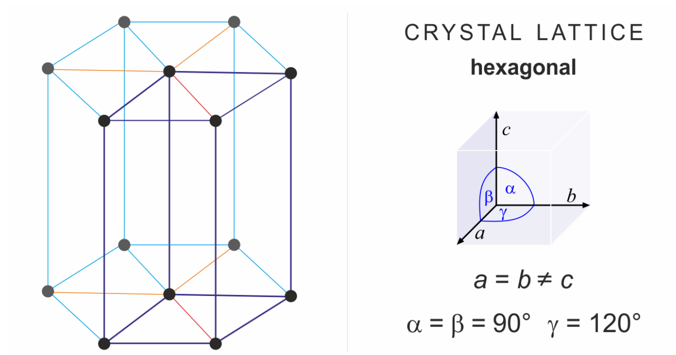

Figure 2.5: Hexagonal crystal unit

As hexagonal crystals posses more symmetries than the orthorhombic crystals which are discussed, the stiffness matrix is also more simplified given by Eq. (2.6.8).

$$\begin{bmatrix} C_{11} & C_{12} & C_{13} & 0 & 0 & 0 \\ C_{12} & C_{11} & C_{13} & 0 & 0 & 0 \\ C_{13} & C_{13} & C_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & C_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & C_{44} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{C_{11}-C_{22}}{2} \end{bmatrix} \tag{2.6.8}$$

2.6.5 Cubic crystal system

The crystal with simplest shape and with a lot of symmetries is cubic. In cubic crystal, all three sides are of equal length that is $a = b = c$ and all the three angles are equal and perpendicular to each other. Figure 2.6 is shown below.

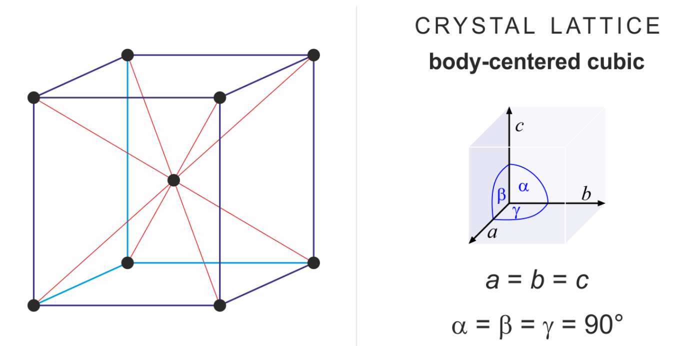

Figure 2.6: Cubic crystal unit

The stiffness matrix for cubic is

$$\begin{bmatrix} C_{11} & C_{12} & C_{12} & 0 & 0 & 0 \\ C_{12} & C_{11} & C_{12} & 0 & 0 & 0 \\ C_{12} & C_{12} & C_{11} & 0 & 0 & 0 \\ 0 & 0 & 0 & C_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & C_{44} & 0 \\ 0 & 0 & 0 & 0 & 0 & C_{44} \end{bmatrix} \tag{2.6.9}$$

in which only three independents components exist.

2.7 Wave motion and their types

Wave is defined as disturbance that transfers energy from one place to another without transferring of matter. Different types of waves exist and they are being used according to their properties in the field of science and technology. For example, waves are used for oil exploration in bore hole engineering. Velocity with which waves reflect, refract help us predict what kind of minerals are there inside the earth. Seismologist who study earthquakes and volcanic eruption also study the wave motion for understanding the geological changes occurring inside earth.

2.7.1 Types of wave

Mainly there are two types of waves.

Mechanical waves

Waves that need medium like air for their propagation and can not transmit energy are named as mechanical waves.

Electromagnetic waves

Waves which do not need medium for their propagation and can travel in vacuum, are called electromagnetic waves.

Body waves

In the thorough study of mechanical waves, scientist came across too many types of interior waves which were discovered by different physicist and seismologist with the passage of time. These waves can be classified by the type of motion in particles namely longitudinal waves and transverse waves.

Longitudinal Waves

A longitudinal wave is a wave where the particle displacement direction is parallel to direction of propagation of wave. Sound wave is an example of longitudinal wave.

Transverse Waves

A transverse wave is a type of wave in which the direction of motion is perpendicular to the direction of wave propagation. Such waves classified into shear horizontal and shear vertical waves depending on the direction of motion of the particle.

Surface Waves

Surface waves are waves that travel along the boundary surface of a material body. Surface waves travel slower than the body waves and have comparatively lower frequency than body waves. Though they are of low frequency but they are highly responsible for the the destruction during an earthquake. There are two types of surface waves, namely, Rayleigh and Love waves.

Rayleigh waves

In this thesis, the main emphasis is on Rayleigh waves in different materials. Rayleigh waves are the waves which travel along the surface of material mainly solids. Wave particles have elliptic motion and the amplitude decreases exponentially as their distance decrease from the surface of earth. Rayleigh waves were discovered by Lord Rayleigh [1] in 1885. A schematic diagram in Figure 2.7 shows the particle motion and Rayleigh surface wave propagation in an elastic material.

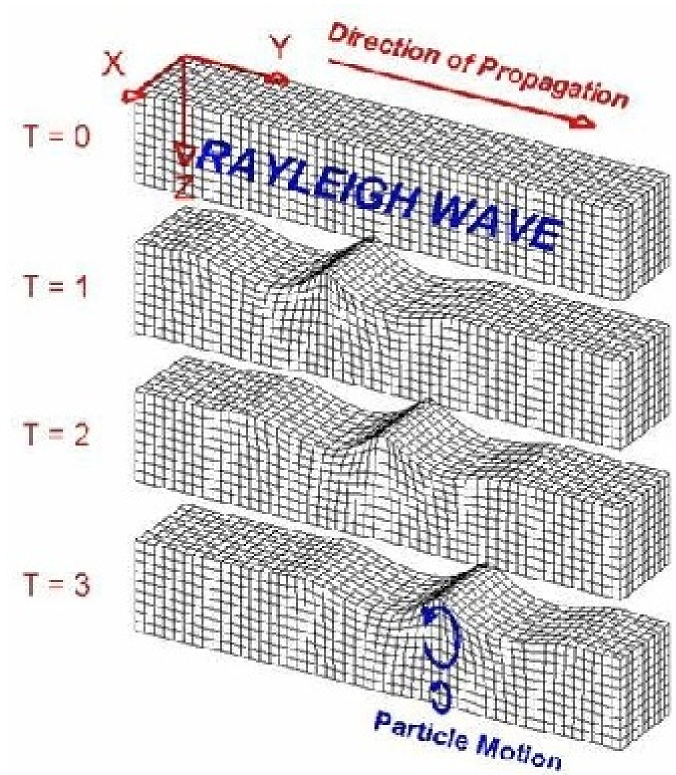

Figure 2.7: The figure showing Rayleigh wave direction and motion is taken from the web page of University of California, Berkeley, (1996).

2.7.2 Wave equation

If a disturbance travels through a medium, there will be local movements. In this case the equation of motion can be obtained using Newton's second law. As recorded before, the net of all forces acting on the volume V by using Green's Theorem can be put into the form

$$F_i = \int_s T_{ik} L_k ds + \int_V B_i dV = \int_V \left(\frac{\partial}{\partial x_k} T_{ik} + B_i \right) dV, \tag{2.7.1}$$

where s stands for surface, B_i are any other body forces here and V is the volume. Using second law of motion

$$F = ma, \tag{2.7.2}$$

where F is force, m is mass of a body and a is the acceleration. Since $m = \rho v$, where ρ denotes the density and m is the mass of the body, we get

$$F = \rho v a. \tag{2.7.3}$$

Putting Eq. (2.7.3) in Eq. (2.7.1), we get the form

$$\frac{\partial}{\partial x_k} T_{ik} + B_i = \rho \frac{\partial^2 u_i}{\partial t^2} \tag{2.7.4}$$

where B_i are the body forces. From Hook's Law in Eq. (2.3.3), we have

$$T_{ij} = C_{ijkl} \frac{\partial u_k}{\partial x_l}. \tag{2.7.5}$$

Ignoring the body forces in Eq. (2.7.4) and substituting Eq. (2.7.5) in (2.7.4) we get

$$\rho \frac{\partial^2 u_i}{\partial t^2} = C_{ijkl} \frac{\partial^2 u_k}{\partial x_j \partial x_l}, \qquad i, j, k = \{1, 2, 3.\} \tag{2.7.6}$$

These three equations are the partial differential equations of second order representing the wave motion.

2.7.3 Inner product

Inner product is defined as

$$(f, \phi) = \int (f \overline{\phi} + \overline{f} \phi) dx_2, \tag{2.7.7}$$

where f and ϕ are two second order tensor. and bar shows the conjugate of the tensor.

Chapter 3

Rayleigh wave propagation in monoclinic, orthorhombic, tetragonal and cubic materials

In this chapter, analysis on propagation of Rayleigh is presented. In this paper[2], velocity of Rayleigh wave in different materials is determined. The cases are analyzed for different elastic materials for propagation of Rayleigh wave. On one side, Christoffel equations split into two parts resulting Rayleigh wave to polarize in the x_1x_2-plane and a shear horizontal wave in the direction of x_3. On other side, the boundary conditions are also simplified, when some of the elastic constants vanish by applying conditions. The conditions and equations are satisfied by different structures that crystals posses, which belong to tetragonal, hexagonal, cubic and orthorhombic crystal systems. The equation of velocity and mechanical displacements are derived.

3.1 Velocity equation for Rayleigh wave

The components of stress tensor T_{ij} of elastic solids are defined in terms of components of particle displacement u_i and stiffness tensor components C_{ijkl} given in Eq. (2.7.5). The elastic stiffness components transform from C^R_{ijkl} (where R represents reference configuration) to C_{ijkl} which are obtained by using transformation from XYZ to $x_1x_2x_3$ frame. An elastic surface wave traveling in semi-infinite medium unbounded in directions x_1 and x_3, satisfies the equation of motion that is Eq. (2.7.6). On the free surface $x_2 = 0$, the mechanical boundary conditions are

$$T_{i2} = C_{i2kl}\frac{\partial u_l}{\partial x_k} = 0, \quad \text{for } i = \{1, 2\}. \tag{3.1.1}$$

A component of particle displacement u_i is expressed as

$$u_i = {}^0d_i \exp^{(-iqkx_2)} \exp^{i(\omega t - kx_1)} \quad \text{with} \quad \mathrm{Im}[q] < 0. \tag{3.1.2}$$

The above equation defines a wave that propagates with phase velocity $V = \omega/k$ along direction x_1 and where k is wave number. Its amplitude decreases when move deep down to the surface. Substituting Eq. (3.1.2) into equation of motion Eq. (2.7.6) gives the following Christoffel equation.

$$(\Gamma_{il} - \xi\delta_{il})\,{}^0d_l = 0, \quad \text{with} \quad \xi = \rho V^2, \tag{3.1.3}$$

By varying $i, l = \{1, 2, 3\}$ in Eq. (3.1.3), we get a matrix of Christoffel symbol components and general formula for finding out each component is

$$\Gamma_{il} = C_{i11l} + (C_{i12l} + C_{i21l})\,q + C_{i22l}q^2 \tag{3.1.4}$$

The system of equation will split into two parts if Γ_{13} and Γ_{23} vanish, resulting the Rayleigh wave to polarize into x_1x_2-plane and a shear horizontal wave that is polarized along x_3. This is the situation when the elastic constants with a single index of that number are zero which can be shown by solving in detail.

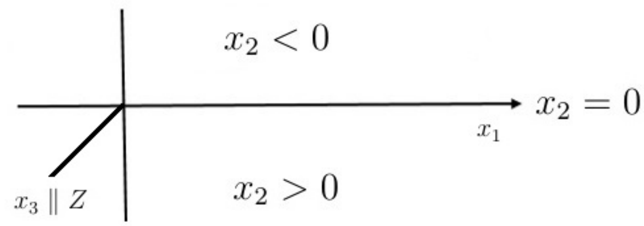

Figure 3.1: Half space with $x_2 > 0$

For example if x_3 is parallel to the axis Z as shown in Figure 3.1, the 6 constants of elasticity having a single index of 3 will vanish to zero.

$$\begin{aligned} &C_{1123}\,(C_{14}) = 0, C_{2223}\,(C_{24}) = 0, C_{1113}\,(C_{15}) = 0, \\ &C_{2213}\,(C_{25}) = 0, C_{1223}\,(C_{64}) = 0, C_{1213}\,(C_{65}) = 0. \end{aligned} \tag{3.1.5}$$

Particularly, this case is when x_3 is parallel to direct or inverse dyad axis of symmetry. In this case, Eq. (3.1.1) at $x_2 = 0$, gives

$$C_{i211}\frac{\partial u_1}{\partial x_1} + C_{i212}\left(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}\right) + C_{i222}\frac{\partial u_2}{\partial x_2} = 0, \quad \text{for } x_2 = 0 \tag{3.1.6}$$

It is identically satisfied when $i = 3$. The other two equations that is for $i = 1, 2$ can also be simplified if we assume

$$C_{1211}\,(C_{61}) = 0 \quad \text{and} \quad C_{1222}(C_{62}) = 0. \tag{3.1.7}$$

Particularly, this case occurs when x_1 or x_2 are parallel to inverse or the direct symmetry axis. If we suppose that the condition(3.1.6) and (3.1.7) are satisfied, then we attain the characteristic equation of system Eq. (3.1.3). For finding the non trivial solution of Eq. (3.1.3), $\det[\Gamma_{il} - \xi\delta_{il}] = 0$. Substituting the values of Γ_{il} from Eq. (3.1.4) in Eq. (3.1.3).

$$(C_{11} - \xi)(C_{66} - \xi) + C_{22}(C_{11} - \xi)q^2 + C_{66}(C_{66} - \xi)q^2 + C_{66}C_{22}q^4 = 0. \tag{3.1.8}$$

Rearranging Eq. (3.1.8), we have

$$C_{22}C_{66}q^4 + \left[C_{22}(C_{11}-\xi) + C_{66}(C_{66}-\xi) - (C_{12}+C_{66})^2\right] q^2$$
$$+(C_{11}-\xi)(C_{66}-\xi) = 0. \tag{3.1.9}$$

As this equation is bi-quadratic in q, we get the roots of this quadratic equation in q^2. We further discuss the cases of roots being real and imaginary. If roots of Eq. (3.1.9) turn out to be real, they should be negative such that imaginary part of two solutions q_r is negative where r=1, 2. We may choose $q_r = -i\chi_r$ where $\chi_r > 0$.
If roots are complex, then are conjugate of each other, so the solutions are $q_2 = -q_1^*$ where q_2 is the solution of one root and q_1^* is one solution of other root. Product of q_1q_2 is negative and real in above cases and

$$P = q_1^2q_2^2 = \frac{(C_{11}-\xi)(C_{66}-\xi)}{C_{22}C_{66}}, \tag{3.1.10}$$

is positive, and ξ should not be between C_{66} and C_{11}:

$$q_1q_2 = -\left(\frac{(C_{11}-\xi)(C_{66}-\xi)}{C_{22}C_{66}}\right)^{\frac{1}{2}}. \tag{3.1.11}$$

with $\xi \notin [C_{66}, C_{11}]$. The component ${}^0d_l^r$ of eigenvector relating to q_r are shown and given in Eq. (3.1.3) ${}^0d_1^r = 1$, ${}^0d_2^r = p_r$, ${}^0d_3^r = 0$ with

$$p_r = -\frac{C_{11}-\xi+C_{66}q_r^2}{(C_{12}+C_{66})q_r}, \quad r = 1, 2. \tag{3.1.12}$$

Let the two components of displacement u_1 and u_2 are assumed as

$$u_1 = Au_1^r \exp^{iqkx_2} \exp^{ik(\omega t - x_1)}, \tag{3.1.13}$$
$$u_2 = Bu_2^r \exp^{iqkx_2} \exp^{ik(\omega t - x_1)}. \tag{3.1.14}$$

where ${}^0d_1^r = 1$ and ${}^0d_2^r = p_r$. Using Eq. (3.1.13) and Eq. (3.1.14) in following equations given as

$$\mu\left(\frac{\partial^2 u_1}{\partial x_1^2} + \frac{\partial^2 u_1}{\partial x_2^2}\right) + (\lambda+\mu)\frac{\partial}{\partial x_1}\left(\frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2}\right) = \rho\frac{\partial^2 u_1}{\partial t^2}, \tag{3.1.15}$$
$$\mu\left(\frac{\partial^2 u_2}{\partial x_1^2} + \frac{\partial^2 u_2}{\partial x_2^2}\right) + (\lambda+\mu)\frac{\partial}{\partial x_2}\left(\frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2}\right) = \rho\frac{\partial^2 u_2}{\partial t^2}. \tag{3.1.16}$$

we get

$$A\left[c_L^2 - c_T^2 q^2 - v^2\right] + qBp\left[c_L^2 - c_T^2\right] = 0. \tag{3.1.17}$$

$$Aq(c_L^2 - c_T^2) - Bp[c_L^2 q^2 - c_T^2 + v^2] = 0. \tag{3.1.18}$$

where c_L is longitudinal wave velocity, c_T denotes transverse wave velocity and v is the velocity of plane waves. From Eq. (3.1.17) and (3.1.18), for non trivia solution of the system, we require

$$\begin{vmatrix} (c_L^2 - c_T^2 q^2 - v^2) & pq(c_L^2 - c_T^2) \\ (c_L^2 - c_T^2)q & -p[c_L^2 q^2 + c_T^2 - v^2] \end{vmatrix} = 0, \tag{3.1.19}$$

which simplifies to

$$q^4 + q^2(\frac{v^2}{c_L^2} + \frac{v^2}{c_T^2} - 2) + (1 - \frac{v^2}{c_L^2})(1 - \frac{v^2}{c_T^2}) = 0. \tag{3.1.20}$$

In bi-quadratic equation Eq. (3.1.20), we can evaluate product and sum of roots as

$$\text{Product of roots } = (1 - \frac{v^2}{c_T^2})(1 - \frac{v^2}{c_L^2}). \tag{3.1.21}$$

$$\text{Sum of roots } = 2 - \frac{v^2}{c_L^2} - \frac{v^2}{c_T^2}. \tag{3.1.22}$$

Say

$$q_1^2 = 1 - \frac{v^2}{c_L^2}, \;\; q_2^2 = 1 - \frac{v^2}{c_T^2}, \tag{3.1.23}$$

which implies

$$q_1 = \sqrt{1 - \frac{v^2}{c_L^2}}, \quad q_2 = \sqrt{1 - \frac{v^2}{c_T^2}}. \tag{3.1.24}$$

The linear combination of two displacements is here in the form of general solution that were propagating at same velocity $V = \omega/k$:

$$u_1 = \left[A_1 e^{-iq_1 kx_2} + A_2 e^{-iq_2 kx_2}\right] e^{i(\omega t - kx_1)}, \tag{3.1.25}$$

$$u_2 = \left[A_1 p_1 e^{-iq_1 kx_2} + A_2 p_2 e^{-iq_2 kx_2}\right] e^{i(\omega t - kx_1)}, \tag{3.1.26}$$

where $A_1,\ A_2$ are weighing factors. The boundary condition (3.1.6) at $x_2 = 0$ for $i = 1$ is

$$C_{1211}\frac{\partial u_1}{\partial x_1} + C_{1212}\left(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}\right) + C_{1222}\frac{\partial u_2}{\partial x_2} = 0, \tag{3.1.27}$$

From Eq. (3.1.7), we set C_{16} and C_{26} equal to zero. so we get,

$$C_{66}\left(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}\right) = 0. \tag{3.1.28}$$

As $C_{66} \neq 0$ so we take

$$\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1} = 0. \tag{3.1.29}$$

Differentiating Eqs. (3.1.25) and (3.1.26) with respect to x_2 and x_1 respectively and substitute them in Eq. (3.1.29), gives

$$(q_1 + p_1)A_1 + (q_2 + p_2)A_2 = 0. \tag{3.1.30}$$

Putting $i = 2$ in boundary condition (3.1.6), we have

$$C_{21}\frac{\partial u_1}{\partial x_1} + C_{22}\frac{\partial u_2}{\partial x_2} = 0 \tag{3.1.31}$$

Using Eq. (3.1.25) and (3.1.26) in Eq. (3.1.31), we get

$$(C_{21} + p_1 q_1 C_{22})A_1 + (C_{21} + p_2 q_2 C_{22})A_2 = 0. \tag{3.1.32}$$

Taking determinant of Eq. (3.1.30) and (3.1.32) equal to zero for a non trivial solution, we get

$$(p_1 - p_2)(C_{12} - C_{22}q_1 q_2) + (q_1 - q_2)(C_{12} - C_{22}p_1 p_2) = 0. \tag{3.1.33}$$

Substituting the values of p_1 and p_2 from Eq. (3.1.12) to deduce the expression of $p_1 - p_2$ from Eq. (3.1.33), which is

$$p_1 - p_2 = \frac{(q_1 - q_2)[C_{11} - \xi + C_{66}q_1 q_2]}{(C_{12} + C_{66})q_1 q_2}. \tag{3.1.34}$$

Now using the expression of $p_1 - p_2$ given in Eq. (3.1.34) in Eq. (3.1.33) yields

$$\begin{aligned}&(C_{11} - \xi - C_{66}q_1q_2)(C_{12} - C_{22}q_1q_2) + \\ &(C_{12} + C_{66})(C_{12}q_1q_2 - C_{22}p_1p_2q_1q_2) = 0.\end{aligned} \tag{3.1.35}$$

Expressing the factor $C_{22}p_1p_2q_1q_2$ of Eq. (3.1.35) as a function of sum which is

$$S = q_1^2 + q_2^2 = \frac{[C_{11}^2 + 2C_{12}C_{66} + C_{66}\xi - C_{22}(C_{11} - \xi)]}{C_{22}C_{66}}, \tag{3.1.36}$$

and the product

$$\begin{aligned}C_{22}p_1p_2q_1q_2 &= C_{22}\frac{(C_{11} - \xi + C_{66}q_1^2)(C_{11} - \xi + C_{66}q_2^2)}{(C_{12} + C_{66})^2 q_1q_2} \times q_1q_2, \\ &= C_{11}\frac{(C_{12} + C_{66})^2 - \xi(C_{12} + C_{66})^2}{(C_{12} + C_{66})^2}, \\ &= C_{11} - \xi.\end{aligned} \tag{3.1.37}$$

As shown in Eq. (3.1.11) that the product is negative so Eq. (3.1.35) will reduce as follows

$$C_{66}q_1^2q_2^2 - \left(C_{11} - \xi - \frac{C_{12}^2}{C_{22}}\right) q_1q_2 + \frac{C_{66}}{C_{22}} = 0, \tag{3.1.38}$$

Putting the value of $q_1^2q_2^2$ from Eq. (3.1.10) in Eq. (3.1.38), we get

$$q_1q_2 = -\frac{\xi(\xi - C_{11})}{C_{22}(c - \xi)}. \tag{3.1.39}$$

Using Eq. (3.1.37) for finding the value of p_1p_2, we have

$$p_1p_2 = 1 - \frac{c}{\xi}. \tag{3.1.40}$$

where

$$c \equiv C_{11} - \frac{C_{12}^2}{C_{22}} > \xi. \tag{3.1.41}$$

Equating both expressions of q_1q_2 from Eq. (3.1.39) and Eq. (3.1.11), we get

$$-\sqrt{\frac{(C_{11} - \xi)(C_{66} - \xi)}{C_{22}C_{66}}} = -\frac{\xi(C_{11} - \xi)}{C_{22}(c - \xi)}, \tag{3.1.42}$$

Squaring Eq. (3.1.42) and after simplification, we define the expression as a function of ξ that is $f(\xi)$ as

$$f(\xi) \equiv \xi - \left(\frac{C_{22}}{C_{66}} \frac{C_{66} - \xi}{C_{11} - \xi} \right)^{\frac{1}{2}} (c - \xi) = 0. \tag{3.1.43}$$

It is noted that when we set $\xi = 0$, $f(\xi)$ is negative since $C_{11}C_{22}>C_{12}^2$. When $\xi = \xi_m \equiv \min[C_{66}, c]$, $f(\xi) = \xi_m$ is positive. There will always be a root ξ_R such that

$$0<\xi_R = \rho V_R^2<\xi_m = \min[C_{66}, c]. \tag{3.1.44}$$

where V_R is the velocity of Rayleigh wave and this is only solution when ξ is greater than C_{66} and C_{11}, $f(\xi)$ is always positive. The conditions shown in Eq. (3.1.44) for ξ_R are satisfied, the velocity equation of Rayleigh wave can be written as

$$2C_{22}C_{66}\xi(C_{11} - \xi) = (C_{66} - \xi)[C_{22}(C_{11} - \xi) - C_{12}^2]^2. \tag{3.1.45}$$

For cubic material, when we set $C_{11} = C_{22}$, it include two cases which are satisfied by [7]. It is worthy to look for different configurations.

3.2 Configurations and compositions satisfying required conditions

Inclination and disposition of direction of propagation x_1 and of normal x_3 to sagittal plane regarding crystallographic axes X,Y,Z for which the preceding analysis is valid which is now to be decided and concluded. These configurations must satisfy the conditions (3.1.5) and (3.1.7) that are imposed on elastic constants. The crystals that belong to triclinic system do not satisfy these configurations because of the availability of three Eulerian angles only that are not enough. The solid must possess at least monoclinic symmetry, so that if one of the axis x_j is parallel to inverse or direct binary axis of symmetry, for example if it is Z axis, six elastic constants with odd number of indices j will be identically zero whatever the disposition of other two axis is. We examine the cases one by one where we start from x_3 then $x_1$or (x_2) is parallel to Z.

3.2.1 The x_3 axis is parallel to crystallographic Z-axis

As Z is the binary (direct or inverse) axis of symmetry that can be seen in Figure 3.2, the equations in (3.1.5) are satisfied whatever angle ϕ is taken. Here ϕ is describing the direction of propagation of wave x_1 in XY-plane.

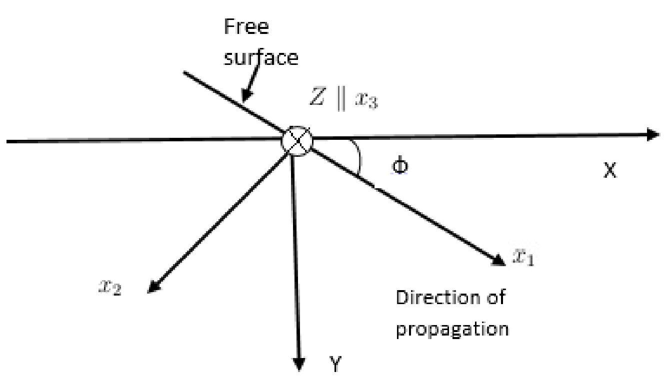

Figure 3.2: Disposition of propagation direction $x_1$and of normal x_3 to the sagittal plane with respect to crystallographic axes X, Y, Z.

The matrix of rotation for Figure 3.2 when x_3 is parallel to Z is denoted as

$$\alpha = \begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{3.2.1}$$

Now the six elastic constants with single index of 3 will be zero and the other two elastic constants in condition (3.1.5) that are C_{16} and C_{26} should be zero that are further shown in detail.

$$C_{16} = \alpha_1^p \alpha_1^q \alpha_1^r \alpha_2^s C_{pqrs}^R = 0, \tag{3.2.2}$$

$$C_{26} = \alpha_2^p \alpha_2^q \alpha_1^r \alpha_2^s C_{pqrs}^R = 0. \tag{3.2.3}$$

As we know that α_1^3 *and* α_2^3 are zero and the constants C_{pqrs}^R without index 3 will appear in expansion. Expanding p, q, r, s from 1 to 3 for Eq. (3.2.2) and Eq. (3.2.3), we get following expressions.

$$(-C_{11}^R \cos^2\phi + C_{22}^R \sin^2\phi)\frac{\sin 2\phi}{2} + (C_{66}^R + \frac{C_{12}^R}{2})\frac{\sin 4\phi}{2} + (C_{16}^R \cos^2\phi + C_{26}^R \sin^2\phi)$$
$$\cos 2\phi - \frac{C_{16}^R - C_{26}^R}{2} \sin 2\phi = 0, \tag{3.2.4}$$

$$(-C_{11}^R \sin^2\phi + C_{22}^R \cos^2\phi)\frac{\sin 2\phi}{2} - (C_{66}^R + \frac{C_{12}^R}{2})\frac{\sin 4\phi}{2} + (C_{16}^R \sin^2\phi + C_{26}^R \cos^2\phi)$$
$$\cos 2\phi - \frac{C_{16}^R - C_{26}^R}{2} \sin^2 2\phi = 0. \tag{3.2.5}$$

The difference and the sum of these two equations shows that we can simultaneously satisfy them only in following cases:

(i) $C_{16}^R = C_{26}^R = 0$, then the angle ϕ is 0 or $\frac{\pi}{2}$.

(ii) $C_{22}^R = C_{11}^R$ and $C_{26}^R = -C_{16}^R$, then the equations Eq. (3.2.4) and Eq. (3.2.5) reduce to single equation. Subtracting Eq. (3.2.4) and (3.2.5) , we get

$$\left(-C_{11}^R \sin^2\phi + C_{22}^R \cos^2\phi\right)\frac{\sin 2\phi}{2} - \left(C_{66}^R + \frac{C_{12}^R}{2}\right)\frac{\sin 4\phi}{2} + (C_{16}^R \sin^2\phi +$$
$$C_{26}^R \cos^2\phi)\cos 2\phi - \frac{C_{16}^R - C_{26}^R}{2} \sin^2 2\phi = 0, \tag{3.2.6}$$

Simplifying Eq. (3.2.6), we get

$$\gamma \sin 4\phi + 2\cos^2 4\phi = 0, \tag{3.2.7}$$

where

$$\gamma = C_{66}^R - \frac{(C_{11}^R - C_{12}^R)}{2}. \tag{3.2.8}$$

The case (i) relates to crystals that belong to orthorhombic system which consists of three perpendicular binary axis. The direction of wave x_1 is either parallel to X or Y. The general stiffness tensor matrix of orthorhombic system is shown in Eq. (2.6.6).

The case (ii) relates to the crystals that belongs to tetragonal, hexagonal and cubic system. A single matrix is built by taking into account the symmetries of all three crystal systems. Final Matrix of above 3 is

$$C_{\alpha\beta}^R = \begin{bmatrix} C_{11}^R & C_{12}^R & C_{13}^R & 0 & 0 & C_{16}^R \\ C_{12}^R & C_{11}^R & C_{13}^R & 0 & 0 & -C_{16}^R \\ C_{13}^R & C_{13}^R & C_{33}^R & 0 & 0 & 0 \\ 0 & 0 & 0 & C_{44}^R & 0 & 0 \\ 0 & 0 & 0 & 0 & C_{44}^R & 0 \\ C_{16}^R & -C_{16}^R & 0 & 0 & 0 & C_{66}^R \end{bmatrix}. \tag{3.2.9}$$

3.2.2 The x_1 or x_2 axis is parallel to crystallographic Z- axis

When x_j for (j= 1 or 2) is either direct or inverse symmetry axis, the constants possessing odd number of indices for j are zero. Denoting the third index with k which is different from 3 and j that is $k = 2$ if $j = 1$ and $k = 1$ if $j = 2$, the result applies to C_{jjjk} and C_{jkkk}, that is to the constants without any index 3 and to the constants having single index of 3 in condition (3.1.5) that are

$$C_{jkk3}, \quad C_{kkj3} \text{ and } C_{jjj3} = 0. \tag{3.2.10}$$

Thus the condition reduces to cancellation of two constants with index 3 and with two indices of j i.e. C_{jjk3} and C_{jkj3} and the constant without any single index of j which is C_{kkk3}. Figure 3.3 and 3.4 shows the x_1 and x_2 axis when parallel to Z axis.

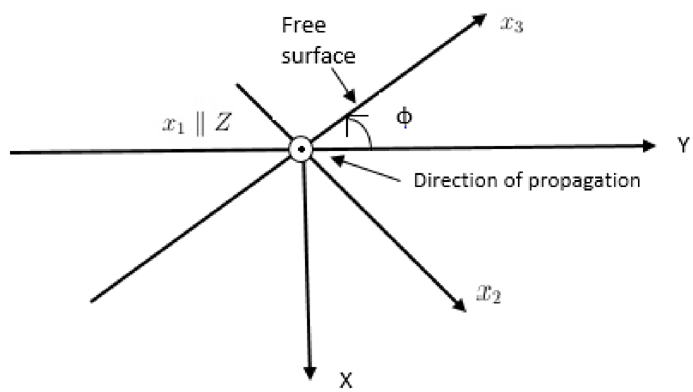

Figure 3.3: Disposition of propagation direction $x_1$and of normal x_3 to the sagittal plane with respect to crystallographic axes X, Y, Z.

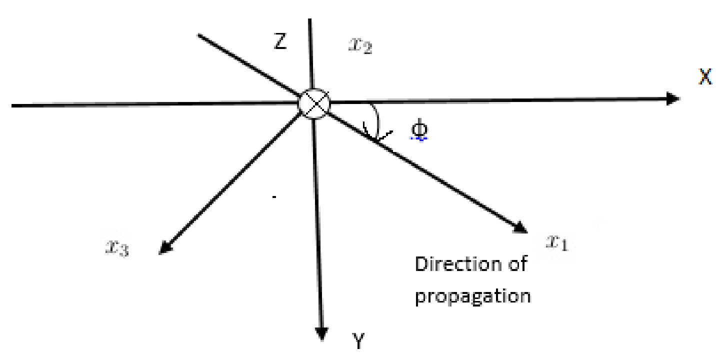

Figure 3.4: [2] Disposition of propagation direction $x_1$and of normal x_3 to the sagittal plane with respect to crystallographic axes X, Y, Z.

Expressing a matrix β of rotation ϕ around $x_j \parallel Z$.

$$\beta = \begin{bmatrix} \beta_j^1 = 0 & \beta_j^2 = 0 & \beta_j^3 = (-1)^{j+1} \\ \beta_k^1 = \cos\phi & \beta_k^2 = \sin\phi & \beta_k^3 = 0 \\ \beta_3^1 = -\sin\phi & \beta_3^2 = 0 & \beta_3^3 = 0 \end{bmatrix}. \tag{3.2.11}$$

$$C_{jjk3} = \beta_k^1\beta_3^1 C_{31}^R + \beta_k^2\beta_3^3 C_{32}^R + (\beta_k^1\beta_3^2 + \beta_k^2\beta_3^1)C_{36}^R,$$

which gives

$$\left[(C_{32}^R - C_{31}^R)/2\right]\sin 2\phi + C_{36}^R\cos 2\phi = 0, \tag{3.2.12}$$

$$C_{jkj3} = \beta_k^1\beta_3^1 C_{55}^R + \beta_k^2\beta_3^2 C_{44}^R + (\beta_k^1\beta_3^2 + \beta_k^2\beta_3^1)C_{54}^R,$$

which gives

$$\left[\frac{C_{44}^R - C_{55}^R}{2}\right]\sin 2\phi + C_{54}^R\cos 2\phi = 0. \tag{3.2.13}$$

Conditions (3.2.12) and (3.2.13) are compatible in the following two cases only

(i) $C_{36}^R = C_{54}^R = 0$, the angle is then 0 or $\pi/2$. It is case of crystals from orthorhombic system.

(ii) $C_{32}^R = C_{31}^R = 0$ and $C_{55}^R = C_{44}^R = 0$. This is case of crystals belonging to tetragonal, hexagonal and cubic systems, then C_{36}^R and C_{54}^R are also zero so that equalities (3.2.12) and (3.2.13) are met whatever ϕ is. As $\beta_k^p = \alpha_1^p$ and $\beta_3^s = \alpha_2^s$ whatever p and s, the last condition

$$C_{kkk3} = \beta_k^p\beta_k^q\beta_k^r\beta_3^s C_{pqrs}^R = 0. \tag{3.2.14}$$

This condition is identical to Eq. (3.2.2) which is expanded and shown in Eq. (3.2.4). Thus, for the configuration (3.3) and (3.4), there is a single equation for ϕ that is Eq. (3.2.4). The crystals that belong to orthorhombic system that is $C_{26}^R = C_{16}^R = 0$, the only two possible angles are $\phi = 0$ or $\pi/2$. The solutions for the tetragonal crystals system evaluated by Eq.(3.2.7)

$$\phi_o = -\frac{1}{4}\arctan\left(\frac{2C_{16}^R}{\gamma}\right) \tag{3.2.15}$$

and (3.2.16)

$$\phi_1 = \phi_o + \frac{\pi}{4}.$$

For crystals of hexagonal system that is $C_{16}^R = 0$ and $C_{16}^R = \frac{(C_{11}^R - C_{12}^R)}{2}$ so the Eq. (3.2.7) is verified whatever the ϕ is.

Chapter 4

Explicit secular equation of Rayleigh waves in monoclinic elastic crystals

In this chapter, the work is on monoclinic elastic crystals. The stiffness matrix for monoclinic is already defined in Chapter 2. The notations for axis that are going to be used are x_1, x_2 and x_3 and the plane of material symmetry is supposed to be at $x_3 = 0$. The system of equations in terms of stress α_{ij}, strain S_{ij} and elastic stiffness components C_{ij} are given

$$
\begin{aligned}
T_{11} &= C_{11}S_{11} + C_{12}S_{22} + C_{13}S_{33} + 2C_{16}S_{12},\\
T_{22} &= C_{12}S_{11} + C_{22}S_{22} + C_{23}S_{33} + 2C_{26}S_{12},\\
T_{33} &= C_{13}S_{11} + C_{22}S_{22} + C_{36}S_{33} + 2C_{36}S_{12},\\
T_{23} &= 2C_{44}S_{23} + 2C_{45}S_{31},\\
T_{31} &= 2C_{45}S_{23} + 2C_{55}S_{31},\\
T_{12} &= C_{16}S_{11} + C_{26}S_{22} + C_{36}S_{33} + 2C_{66}S_{12},
\end{aligned}
\tag{4.0.1}
$$

Here, S_{ij}'s are the strain components which are defined in Eq. (2.3.2). The equation of motion is given by Eq. (2.7.4) with the body forces B_i which are absent here.

4.1 Surface waves

In this section, we consider the surface wave propagation in a monoclinic elastic half space. The amplitude of displacement that is associated is supposed to vary sinusoidally with time "t" in propagation direction x_1, but variation in x_2 is not defined explicitly. Thus we write the equation of displacement components as

$$u_j(x_1, x_2, x_3, t) = U_j(x_2)e^{ik(x_1 - vt)}, \qquad j = \{1, 2, 3\}. \tag{4.1.1}$$

Here U's are the amplitudes that depend on x_2 and v is the speed and k is wave number. By examining these waves, it is seen that the constant phase planes are perpendicular to x_1 -axis. The planes having constant amplitude are perpendicular to x_2 - axis. The relation between stress and strain in Eq. (4.0.1) is reduced by using Eq. (4.1.1) as

$$\begin{aligned}
t_{11} &= C_{12}U_2' + iC_{11}U_1 + C_{16}(U_1' + iU_2),\\
t_{22} &= C_{22}U_2' + iC_{12}U_1 + C_{26}(U_1' + iU_2),\\
t_{33} &= C_{23}U_2' + iC_{13}U_1 + C_{36}(U_1' + iU_2),\\
t_{32} &= C_{44}U_3' + iC_{45}U_3, \quad t_{13} = C_{45}U_3' + iC_{55}U_3,\\
t_{12} &= C_{26}U_2' + iC_{16}U_1 + C_{66}(U_1' + iU_2),
\end{aligned} \tag{4.1.2}$$

where t_{ij}'s are defined as

$$T_{ij}(x_1, x_2, x_3, t) = kt_{ij}(x_2)e^{ik(x_1 - vt)}, \quad i, j = \{1, 2, 3\}. \tag{4.1.3}$$

and prime is the differentiation of displacement components with kx_2. At $x_2 = 0$, we consider a free surface that is free of tractions and also displacement vanishes as

x_2 moves to infinity. The boundary conditions of problem are modeled as

$$t_{i2}(0) = 0, \quad U_i(\infty) = 0, \qquad i = \{1, 2, 3\}. \tag{4.1.4}$$

The equation of motion (2.7.6) reduced in terms of U_i and t_{ij} as below

$$it_{11} + t'_{12} = -\rho v^2 U_1, \tag{4.1.5}$$

$$t'_{22} + it_{12} = -\rho v^2, \tag{4.1.6}$$

$$t'_{32} + it_{13} = -\rho v^2 U_3. \tag{4.1.7}$$

A sextic formalism can be developed at this moment for displacement components U_1, U_2, U_3 and traction components t_{12}, t_{22}, t_{32}. However, it turns out that one of these traction components is identically zero, as is now proved.

4.2 Plane stress

It is very well known from [10] and [11] that for the two dimensional deformation of monoclinic crystal with the axis of symmetry at $x_3 = 0$, u_1 and u_2 are decoupled from displacement component u_3. Putting $u_3 = 0$, the stress strain relation in Eq. (4.1.2) follows that $t_{13} = t_{32} = 0$. The alternate way to prove this result is also given below.

The two differential equations of first order for U_3 and t_{32} are obtained by using Eq. $(4.1.2)_4$, $(4.1.2)_5$ and (4.1.7)

$$t_{32} = iC_{45}U_3 + C_{44}U'_3, \qquad t'_{32} = (C_{55} - \rho v^2)U_3 - iC_{45}U'_3. \tag{4.2.1}$$

The above equations (4.2.1) may be inverted to give U_3 and U'_3 as

$$(C_{44}C_{55} - C_{45}^2 - C_{44}\rho v^2)U'_3 = (C_{55} - \rho v^2)t_{32} - iC_{45}t'_{32}. \tag{4.2.2}$$

$$(C_{44}C_{55} - C_{45}^2 - C_{44}\rho v^2)U_3 = iC_{45}t_{32} + C_{44}t'_{32}. \tag{4.2.3}$$

Differentiating Eq. (4.2.3) with respect to kx_2 and then subtracting the new expression from Eq. (4.2.2) gives a second order differential equation in t_{32} that is

$$C_{44}t''_{32} + 2iC_{45}t'_{32} - (C_{55} - \rho v^2)t_{32} = 0. \qquad (4.2.4)$$

Equation $(4.1.2)_5$ and the boundary conditions (4.1.4) imply that the stress tensor component t_{32} satisfies the conditions, that are, $t_{32}(0) = t_{32}(\infty) = 0$. Equation (4.2.4) is the boundary value problem with these conditions and the solution of this boundary value problem is trivial. Consequently

$$t_{32}(x_2) = 0, \qquad \text{for all } x_2. \qquad (4.2.5)$$

And now it is clear that in monoclinic crystals, as far as the surface wave propagation is concerned with plane of symmetry at $x_3 = 0$, the generalized plain strain transforms to plane stress.

4.3 Equations of motion

Equations of motion will be derived here in terms of U_1, U_2, t_1 and t_2. First, equations of motion are written as a system of four differential equations of first order and then these equations are written as a system of two second order differential equations for the tractions.
Using the stress-strain relation (4.1.2) and equations of motion (4.1.5)-(4.1.7) for displacement and traction components defined as

$$t_1 = t_{12}, \qquad t_2 = t_{22}, \qquad (4.3.1)$$

The system is as follows

$$\begin{bmatrix} \mathbf{u}' \\ \mathbf{t}' \end{bmatrix} = \begin{bmatrix} iN_1 & N_2 \\ -(N_3 + X1) & iN_1^T \end{bmatrix} \begin{bmatrix} \mathbf{u} \\ \mathbf{t} \end{bmatrix}, \qquad (4.3.2)$$

where, in this system $\mathbf{u} = [U_1, U_2]^T$, $\mathbf{t} = [t_1, t_2]^T$and $X = \rho v^2$. Here, N_1, N_2 and N_3 are 2×2 matrices. These matrices are the submatrices of elasticity fundamental matrix N which in introduced by Ingebrigtsen and Tonning [12]. These $2{\times}2$ matrices are represented as

$$-N_1 = \begin{bmatrix} r_6 & 1 \\ r_2 & 0 \end{bmatrix}, \quad N_2 = \begin{bmatrix} s_{22} & -s_{26} \\ -s_{26} & s_{66} \end{bmatrix} = N_2^T,$$

$$-N_3 = \begin{bmatrix} \eta & 0 \\ 0 & 0 \end{bmatrix} = -N_3^T, \quad N_3 + X1 = \begin{bmatrix} -\eta + X & 0 \\ 0 & X \end{bmatrix}, \tag{4.3.3}$$

The quantities $r_2,\ r_6,\ s_{22},\ s_{66},\ s_{26}$ and η are expressed in the from of elastic stiffness constants as

$$\Delta = \begin{vmatrix} C_{22} & C_{26} \\ C_{26} & C_{66} \end{vmatrix} = C_{22}C_{66} - C_{26}^2,$$

$$r_2 = \frac{1}{\Delta}(C_{12}C_{66} - C_{16}C_{26}), \quad r_6 = \frac{1}{\Delta}(C_{22}C_{16} - C_{12}C_{26}),$$

$$s_{ij} = \frac{1}{\Delta}C_{ij} \quad (i, j = 2, 6), \tag{4.3.4}$$

$$\eta = \frac{1}{\Delta}\begin{vmatrix} C_{11} & C_{12} & C_{16} \\ C_{12} & C_{22} & C_{26} \\ C_{16} & C_{26} & C_{66} \end{vmatrix},$$

$$= C_{11} - \frac{C_{66}C_{12}^2 + C_{22}C_{16}^2 - 2C_{12}C_{16}C_{26}}{C_{22}C_{66} - C_{26}^2}.$$

Throughout this problem, it is supposed that matrix $(N_3 + X1)$ is not singular, which depicts that surface wave travels at the speed different from that is given by $\rho v^2 = \eta$. If this is so, the matrix becomes singular, so for this reason, it is assumed to be non singular. From this supposition, the vector line of the system Eq. (4.3.2) that is second in number yields an expression that is

$$\mathbf{u} = i(N_3 + X1)^{-1}N_1^T\mathbf{t} - (N_3 + X1)^{-1}\mathbf{t}'. \tag{4.3.5}$$

Now, differentiating the first vector line of Eq. (4.3.2), we get

$$\mathbf{u}'' = iN_1\mathbf{u}' + N_2\mathbf{t}', \tag{4.3.6}$$

Putting the value of u' and t' from Eq. (4.3.2) in Eq. (4.3.6), we get

$$\mathbf{u}'' = [-N_1N_1 - N_2(N_3 + X1)]\mathbf{u} + i[N_1N_2 + N_1^TN_2]\mathbf{t}. \tag{4.3.7}$$

Now taking the second derivative of Eq. $(4.3.2)_2$ gives

$$\mathbf{t}'' = -i[N_1(N_3 + X1) + N_1^T(N_3 + X1)]\mathbf{u} + [-(N_3 + X1)N_2 - N_1^TN_1^T]\mathbf{t}. \tag{4.3.8}$$

The two equations (4.3.7) and (4.3.8) can be written in matrix form as

$$\begin{bmatrix}\mathbf{u}'' \\ \mathbf{t}''\end{bmatrix} = \begin{bmatrix}-N_1N_1 - N_2(N_3 + X1) & i[N_1N_2 + N_1^TN_2] \\ -i[N_1(N_3 + X1) + N_1^T(N_3 + X1)] & -(N_3 + X1)N_2 - N_1^TN_1^T\end{bmatrix}\begin{bmatrix}\mathbf{u} \\ \mathbf{t}\end{bmatrix}. \tag{4.3.9}$$

Using Eq. (4.3.5) in the second vector line of Eq. (4.3.9) yields an expression

$$\mathbf{t}'' = iN_1\mathbf{t}' + N_1N_1^T\mathbf{t} + iN_1^T\mathbf{t}' - N_2(N_3 + X1)\mathbf{t}, \tag{4.3.10}$$

Diving the whole expression with $-(N_3 + X1)$, we get

$$\begin{aligned}&-(N_3 + X1)^{-1}\mathbf{t}'' = i[-N_1(N_3 + X1)^{-1} - N_1^T(N_3 + X1)^{-1}]\mathbf{t}' + \\ &[N_2 - N_1(N_3 + X1)^{-1}N_1^T]\mathbf{t},\end{aligned} \tag{4.3.11}$$

Naming the coefficients of $\mathbf{t}''$, $\mathbf{t}'$, $\mathbf{t}$ as α, β and γ respectively which are 2×2 symmetric matrices, given by

$$\begin{aligned}\boldsymbol{\alpha} &= -(N_3 + X1)^{-1}, \\ \boldsymbol{\beta} &= -N_1(N_3 + X1)^{-1} - (N_3 + X1)^{-1}N_1^T, \\ \boldsymbol{\gamma} &= N_2 - N_1(N_3 + X1)^{-1}N_1^T.\end{aligned} \tag{4.3.12}$$

Equation (4.3.11) in terms of $\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}$ is written as

$$\alpha_{ik}t_k'' - i\beta_{ik}t_k' - \gamma_{ik}t_k = 0. \tag{4.3.13}$$

Showing the values of $\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}$ components, we Equations (4.3.3) and (refp316) which gives

$$\boldsymbol{\alpha} = \begin{bmatrix} \frac{1}{\eta - X} & 0 \\ 0 & \frac{-1}{X} \end{bmatrix}, \quad \boldsymbol{\beta} = \begin{bmatrix} \frac{-2r_6}{\eta - X} & \frac{1}{X} - \frac{r_2}{\eta - X} \\ \frac{1}{X} - \frac{r_2}{\eta - X} & 0 \end{bmatrix},$$
$$\boldsymbol{\gamma} = \begin{bmatrix} s_{22} + \frac{\gamma_6^2}{\eta - X} - \frac{1}{X} & \frac{r_2 r_6}{\eta - X} - s_{26} \\ \frac{r_2 r_6}{\eta - X} - s_{26} & s_{66} + \frac{r_2^2}{\eta - X} \end{bmatrix}. \tag{4.3.14}$$

It is favorable to work with differential equation for traction components rather than displacement components due to the simple and easily calculated boundary conditions from Eqs. (4.1.2), (4.1.4) and (4.3.1) as

$$t_i(0) = t_i(\infty) = 0, \qquad \{i = 1, 2.\} \tag{4.3.15}$$

4.4 Secular equation for Rayleigh wave in monoclinic elastic half space

For finding the secular equation in monoclinic elastic half space, we use method of first integral [5] on Eq. (4.3.13). Applying the definition of inner product using Eq. (2.7.7). Multiplying Eq. (4.3.13) by it_j and applying the definition of inner product yields the form

$$\alpha_{ik}(t_k'', it_j) - i\beta_{ik}(t_k', it_j) - \gamma_{ik}(t_k, it_j) = 0. \tag{4.4.1}$$

From Eq. (2.7.7), it can be shown that inner product satisfies

$$(u, cw) = \bar{c}(u, w), \tag{4.4.2}$$

where c is the constant which comes out of integral with a conjugate as $\bar{c}$. Using Eq. (4.4.2) in Eq. (4.4.1), we get

$$\alpha_{ik}(t_k'', it_j) - i(\bar{i})\beta_{ik}(t_k', t_j) - \gamma_{ik}(t_k, it_j) = 0. \tag{4.4.3}$$

Applying the definition of inner product suggested by [5]

$$\alpha_{ik}\left[\int_0^\infty(-it_k''\overline{t_j}+t_j i\overline{t_k''})dx_2\right]+\beta_{ik}\left[\int_0^\infty-(t_k'\overline{t_j}+t_j\ \overline{t_k'})dx_2\right]$$
$$+\ \gamma_{ik}\left[\int_0^\infty(+t_k i\overline{t_j}-it_j\ \overline{t_k})dx_2\right]=0. \tag{4.4.4}$$

It gives

$$\alpha_{ik}D_{kj}+\beta_{ik}E_{kj}+\gamma_{ik}F_{kj}=0, \tag{4.4.5}$$

where the 2×2 matrices $\mathbf{D}$, $\mathbf{E}$, $\mathbf{F}$. and the components D_{kj}, E_{kj} and F_{kj} are defined as

$$D_{kj}=(it_k'',t_j),\qquad E_{kj}=(t_k',t_j),\qquad F_{kj}=(t_k,it_j). \tag{4.4.6}$$

It is convenient to check the property of antisymmetry of matrix $\mathbf{D}$ when we write $D_{kj}+D_{jk}$ with integration by parts.

$$D_{kj}+D_{jk}=(it_k'',t_j)+(it_j'',t_k), \tag{4.4.7}$$

Using the definition of inner product and by putting the values of boundary conditions from Eq. (4.3.15) yields

$$D_{kj}+D_{jk}=-i\int_0^\infty\overline{t_j'}t_k'dx_2+i\int_0^\infty t_j'\overline{t_k'}dx_2-i\int_0^\infty\overline{t_k'}t_j'dx_2$$
$$+i\int_0^\infty t_k'\overline{t_j'}dx_2, \tag{4.4.8}$$
$$D_{kj}+D_{jk}=0. \tag{4.4.9}$$

From Eq. (4.4.8), it can be seen that the terms cancel out and the property of antisymmetry is satisfied. Similarly

$$E_{kj}+E_{jk}=0, \tag{4.4.10}$$

whic shows that $\mathbf{E}$ is also antisymmetric.

Likewise to check the antisymmetric property for the components of $\mathbf{F}$, we again apply the property of inner product on its components, and get

$$F_{kj}+F_{jk}=(t_k,it_j)+(t_j,it_k)=0, \tag{4.4.11}$$

So, now $\mathbf{D}, \mathbf{E}, \mathbf{F}$ can be written in form of

$$\mathbf{D} = \begin{bmatrix} 0 & D \\ -D & 0 \end{bmatrix}, \quad \mathbf{E} = \begin{bmatrix} 0 & E \\ -E & 0 \end{bmatrix}, \quad \mathbf{F} = \begin{bmatrix} 0 & F \\ -F & 0 \end{bmatrix}. \tag{4.4.12}$$

Now, Eq. (4.4.5) gives the system that consists of three linearly independent equations having three unknowns that are D, E, F which re yet to be determined. The system is as follows

$$\begin{aligned} \alpha_{11}D + \beta_{11}E + \gamma_{11}F = 0, \\ \alpha_{12}D + \beta_{12}E + \gamma_{12}F = 0, \\ \alpha_{22}D + \beta_{22}E + \gamma_{22}F = 0. \end{aligned} \tag{4.4.13}$$

The system is homogeneous and linear algebraic that gives nontrivial solutions for D, E, F only in the case when determinant is zero which happens when $\alpha_{12} = \beta_{22} = 0$,

$$\alpha_{11}(\beta_{12}\gamma_{22} - \gamma_{12}\beta_{22}) - \beta_{11}(\alpha_{12}\gamma_{22} - \gamma_{12}\alpha_{22}) + \gamma_{11}(\alpha_{12}\beta_{22} - \alpha_{22}\beta_{12}) = 0, \tag{4.4.14}$$

Simplifying Eq. (4.4.14), we get

$$\beta_{12}(\alpha_{11}\gamma_{22} - \alpha_{22}\gamma_{11}) = -\alpha_{22}\beta_{11}\gamma_{12}. \tag{4.4.15}$$

Equivalently, if the expressions written in Eq.(4.3.14) are used in Eq. (4.4.15) and further multiply it by $X^3(\eta - X)^3$, the expression for secular equation is calculated

$$\begin{aligned} &[\eta - (1 + r_2)X]\,(\eta - X)\left[(\eta - X)(s_{22}X - 1) + r_6^2 X\right] \\ &+X^2\left[(\eta - X)s_{66} + r_2^2\right] = 2r_6X^2(\eta - X)\left[(\eta - X)s_{26} - r_2r_6\right]. \end{aligned} \tag{4.4.16}$$

This is the secular equation that is derived explicitly as quartic in $X = \rho v^2$ having coefficients expressed in terms of elastic stiffness constants from Eq. (4.3.4).

Particularly, considering orthorhombic case for which we take $C_{16} = C_{26} = C_{45} = 0$.

The coefficients in Eq. (4.3.4) reduce to

$$r_6 = 0, \quad r_2 = \frac{C_{12}}{C_{22}}, \quad s_{22} = \frac{1}{C_{66}},$$
$$s_{26} = 0, \quad s_{66} = \frac{1}{C_{22}}, \quad \eta = C_{11} - \frac{C_{12}^2}{C_{22}}. \tag{4.4.17}$$

The right hand-side of Eq. (4.4.16) becomes zero while the left side is transformed and the secular equation for orthorhombic is as follows

$$[\eta - (1 + r_2)X] \left\{(\eta - X)^2(s_{22}X - 1) + X^2 \left[(\eta - X)s_{66} + r_2^2\right]\right\} = 0. \tag{4.4.18}$$

In Eq. (4.4.18), the nullity of first factor corresponds to $\beta_{12} = 0$ that is

$$\eta - (1 + r_2)X = 0, \tag{4.4.19}$$

which implies

$$\beta_{12} = \frac{1}{X} - \frac{r_2}{\eta - x} = 0, \tag{4.4.20}$$

As now we are dealing with orthorhombic case so some of the entities in Eq. (4.3.4) reduce to zero. These are $\alpha_{12} = \gamma_{12} = \beta_{11} = \beta_{22} = 0$ and then equations of motion (4.3.13) decouple into

$$\alpha_{11}t_1'' + \gamma_{11}t_1 = 0, \quad \alpha_{22}t_2'' + \gamma_{22}t_2 = 0. \tag{4.4.21}$$

and its solutions satisfying the boundary conditions (4.3.15) are the trivial ones. The nullity of later factor in Eq. (4.4.18) gives us the secular equation of orthorhombic crystals that is as

$$\left(C_{11} - \frac{C_{12}^2}{C_{22}} - \rho v^2\right)^2 \left(\frac{\rho v^2}{C_{66}} - 1\right) + (\rho v^2)^2 \left[\left(C_{11} - \frac{C_{12}^2}{C_{22}} - \rho v^2\right)\left(\frac{1}{C_{22}}\right) + \frac{C_{12}^2}{C_{22}^2}\right] = 0,$$

Dividing Eq. (4.4.22) by C_{66}^2 on both sides, we get

$$\left(\frac{C_{22}}{C_{11}}\right)\left(1 - \frac{\rho v^2}{C_{66}}\right)\left(\frac{C_{11}C_{22} - C_{12}^2}{C_{22}C_{66}} - \frac{\rho v^2}{C_{66}}\right)^2 - \left(\frac{\rho v^2}{C_{66}}\right)^2\left(1 - \frac{\rho v^2}{C_{11}}\right) = 0. \tag{4.4.22}$$

Eq. (4.4.22) corresponds to a well-studied secular equation for the surface waves in orthorhombic crystals. Concrete examples are given in Table (4.1). Eq. (4.4.16) has either 2 or 4 positive real roots, out of which there is only one that corresponds to subsonic wave, while other roots are compared with homogeneous body wave that is propagating in x_1 direction of material axis. For this body wave, functions $U_i(x_2)$, $t_i(x_2)$, $(i = 1, 2)$, are constant, and it implies that determinant of 4×4 matrix in Eq. (4.3.2) is zero, which is the condition from which the body wave speed can be found. For instance, η is of order 3×10^7 for tin fluoride, the secular equation (4.4.16) has roots 1339, 2350, 2513, and 3403. The slowest body wave in x_1 direction travels at 1504 ms^{-1}; hence a subsonic surface wave travels in tin fluoride at 1339 ms^{-1}.

Table 4.1: *Values of relevant density (kgm^{-3}), elastic stiffnesses (GPa) and surface wave speed (ms^{-1}) for 12 monoclinic crystals.*

Material	Density	C_{11}	C_{22}	C_{12}	C_{16}	C_{26}	C_{66}	v
aegirite-augite	216	156	66	19	25	46.5	3420	3382
augite	218	182	72	25	20	51.1	3320	3615
diallage	211	154	37	12	15	62.2	3300	4000
diopside	238	204	88	-34	-19	58.8	3310	3799
diphenyl	14.6	5.95	2.88	2.02	0.40	2.26	1114	1276
epidote	202	212	45	-14.3	0	43.2	3400	3409
gypsum	50.2	94.5	28.2	-7.5	-11.0	32.4	2310	3011
hornblende	192	116	61	10	4	31.8	3120	3049
microcline	122	66	26	-13	-3	23.8	2561	2816
oligoclase	124	81	54	-7	16	27.4	2638	2413
tartaric acid	46.5	93	36.7	-0.4	-12.0	8.20	1760	1756
tin fluoride	33.6	47.9	5.3	6.5	-5.1	12.9	4875	1339

Velocities for 12 monoclinic material are re-calculated through MATHEMATICA using equation (4.4.16).

4.5 Mathematica script to calculate Rayleigh wave speed in monoclinic materials

From Table 4.1. the values of elastic stiffness constants and ρ are taken and recalculated in detail using *Mathematica* for **tartaric acid**. As discussed in section **4.4**, we get 4 positive roots and here the smallest root is considered as a Rayleigh velocity. The four positive roots for tartaric acid are 1755, 3657, 4282 and 4282. A subsonic surface wave in tartaric acid travels at 1755m/s. On next page, a Mathematica file 4.5 can be seen where first of all, the coefficients and expressions $r_2, r_6, s_{22}, s_{26}, s_{66}, \eta$ and $\chi = \rho v^2$ in Eq. (4.3.4) are calculated and then put in Eq. (4.4.16) to find the possible roots.

```
In[31]:= c[11] = 46.5 * 10^9
         c[22] = 93 * 10^9
         c[12] = 36.7 * 10^9
         c[16] = -0.4 * 10^9
         c[26] = -12.0 * 10^9
         c[66] = 8.20 * 10^9
         p = 1760
         g = c[22] c[66] - (c[26])^2
Out[31]= 4.65 × 10^10
Out[32]= 93 000 000 000
Out[33]= 3.67 × 10^10
Out[34]= -4. × 10^8
Out[35]= -1.2 × 10^10
Out[36]= 8.2 × 10^9
Out[37]= 1760
Out[38]= 6.186 × 10^20

In[39]:= r[6] = (c[22] c[16] - c[12] c[26]) / g
Out[39]= 0.651794

In[40]:= r[2] = (c[12] c[66] - c[16] c[26]) / g
Out[40]= 0.478726

In[43]:= s[26] = c[26] / g
Out[43]= -1.93986 × 10^-11

In[44]:= s[22] = c[22] / g
Out[44]= 1.50339 × 10^-10

In[45]:= s[66] = c[66] / g
Out[45]= 1.32557 × 10^-11

In[46]:= n = (c[11] c[22] c[66] - c[11] (c[26])^2 -
            c[66] (c[12])^2 - c[22] (c[16])^2 + 2 c[12] c[16] c[26]) / g
Out[46]= 2.91915 × 10^10

In[47]:= Roots[(n - (1 + r[2]) p v^2) ((n - p v^2) ((n - p v^2) (s[22] p v^2 - 1) + (r[6])^2 p v^2) +
              (p v^2)^2 ((n - p v^2) s[66] + (r[2])^2)) ==
           2 r[6] (p v^2)^2 (n - p v^2) ((n - p v^2) s[26] - r[2] r[6]), v]
Out[47]= v == 1755.77 || v == -1755.77 || v == 3657.42 || v == -3657.42 || v == 4282.75 - 694.332 i ||
           v == -4282.75 + 694.332 i || v == 4282.75 + 694.332 i || v == -4282.75 - 694.332 i
```

Chapter 5

Surface waves in orthotropic materials

In this chapter, the focus and aim is to derive the secular equation for surface wave propagating in orthotropic materials by the method of first integral. The motivation is to find the velocity equation primarily using the method suggested in [5] having traction components as variable.

5.1 Basic equations for orthotropic material

Some basic equations are recalled here. In Chaper 2, the stiffness matrix for orthotropic material is given in Eq. (2.6.6). The system of equations consisting of stress tensor T_{ij}, strain tensors S_{ij} and also the additional term of (isotropic) pressure p are as follows

$$\begin{aligned}
T_{11} &= -p + C_{11}S_{11} + C_{12}S_{22} + C_{13}S_{33}, \\
T_{22} &= -p + C_{12}S_{11} + C_{22}S_{22} + C_{23}S_{33}, \\
T_{33} &= -p + C_{13}S_{11} + C_{23}S_{22} + C_{33}S_{33}, \\
T_{32} &= 2C_{44}S_{32}, \quad T_{13} = 2C_{55}S_{31}, \quad T_{12} = 2C_{66}S_{12},
\end{aligned} \tag{5.1.1}$$

where C_{ijkl} are the elastic components. The strain components S_{ij} are expressed in terms of displacement components defined in Eq. (2.3.2). The incompressibility constraints are expressed as

$$u_{1,1} + u_{2,2} + u_{3,3} = 0. \tag{5.1.2}$$

Th author also noted that for plane strain, the strain energy function density is positive definite when the following inequalities are satisfied.

$$C_{66} \geq 0, \quad C_{11} + C_{22} - 2C_{12} \geq 0. \tag{5.1.3}$$

For a semi-infinite body that is made up of incompressible orthotropic material, we establish equations of motion. The modeling of surface wave follows that of [5]. Surface waves propagating with speed v, wave number k, and the pressure p and displacement components are of the form

$$[u_j(x_1, x_2, x_3), p(x_1, x_2, x_3)] = [U_j(x_2), kP(x_2)] \exp^{ik(x_1 - vt)}, \qquad j = \{1, 2, 3\}. \tag{5.1.4}$$

for here $U's$ and P are the unknown functions of x_2 alone. The waves posses the constant phase planes that are orthogonal to x_1- axis and constant amplitude planes that are orthogonal to x_2-axis. We use Eq. (4.1.3) and Eq. (5.1.4) in Eq. (5.1.1) to reduce it into the following system of equations.

$$\begin{aligned} t_{11} &= -P + iC_{11}U_1 + C_{12}U_2', \\ t_{22} &= -P + iC_{12}U_1 + C_{22}U_2', \\ t_{33} &= -P + iC_{13}U_1 + C_{23}U_2', \\ t_{32} &= C_{44}U_3', \quad t_{13} = C_{55}U_3, \quad t_{12} = C_{66}\left(U_1' + iU_2\right), \end{aligned} \tag{5.1.5}$$

where t_{ij}'s are stress tensors. Surface $x_2 = 0$ is supposed to be free of tractions and also the pressure and mechanical displacement components vanish as x_2 goes to infinity. This condition leads to the following boundary conditions.

$$t_i = t_{i2}(0) = 0, \qquad U_i(\infty) = 0 \quad (i = 1, 2, 3), \qquad P(\infty) = 0. \tag{5.1.6}$$

Incompressibility constraints in Eq. (5.1.2) and equations of motion (2.7.4) in terms of $t's$ and $U's$ is expressed as

$$it_{11} + t'_{12} = -\rho v^2 U_1, \quad it_{12} + t'_{22} = -\rho v^2 U_2, \tag{5.1.7}$$
$$it_{13} + t'_{32} = -\rho v^2 U_3, \quad iU_1 + U'_2 = 0. \tag{5.1.8}$$

A classical approach can be used by substituting in last equations that is, the expressions that are obtained earlier in Eq. (5.1.5) for stress tensor components, leads to a system of four differential equations of second order for U_1, U_2, U_3, P, which are to be determined. Here, instead, Stroh formalism [10] is used for deriving the six first order differential equations by assuming the traction on surface $x_2 = 0$. Further, the notation for traction components is used that was suggested for earlier problem which is mentioned in Eq. (4.3.1). Using Eqs. (5.1.5) - (5.1.8), the system is

$$\begin{aligned} &U'_1 = -iU_2 + (\frac{1}{C_{66}})t_1, \quad -iU_1 = U'_2, \quad U'_3 = (\frac{1}{C_{44}})t_3, \\ &t'_1 = (C_{11} + C_{22} - 2C_{12} - \rho v^2)U_1 - it_2, \\ &t'_2 = -\rho v^2 U_2 - it_1, \quad t'_3 = (C_{55} - \rho v^2)U_3. \end{aligned} \tag{5.1.9}$$

Now Eqs. $(5.1.9)_{4-6}$ are differentiated with respect to kx_2 which give

$$\begin{aligned} &t''_1 = -i(C_{11} + C_{22} - 2C_{12} - \rho v^2)U'_1 - it'_2, \\ &t''_2 = -\rho v^2 U'_2 - it'_1, \\ &t''_3 = (C_{55} - \rho v^2)U'_3. \end{aligned} \tag{5.1.10}$$

Substitute U'_1, U'_2 and U'_3 from Eq. $(5.1.9)_{1-3}$ to convert Eq. (5.1.10) in terms of t_j's and its derivatives, we have

$$(\rho v^2)t''_1 - i(C_{11} + C_{22} - 2C_{12} - 2\rho v^2)t'_2 + (C_{11} + C_{22} - 2C_{12} - \rho v^2)\left(1 - \frac{\rho v^2}{C_{66}}\right)t_1 = 0, \tag{5.1.11}$$
$$(C_{11} + C_{22} - 2C_{12} - \rho v^2)t''_2 + i(C_{11} + C_{22} - 2C_{12} - 2\rho v^2)t'_1 + \rho v^2 t_2 = 0, \tag{5.1.12}$$
$$C_{44}t''_3 - (C_{55} - \rho v^2)t_3 = 0, \tag{5.1.13}$$

subject to the boundary conditions

$$t_j(0) = t_j(\infty) = 0, \qquad j = \{1, 2, 3\}. \tag{5.1.14}$$

It can be seen that Eq. (5.1.13) is already decoupled so it can easily be solved. Using the boundary condition (5.1.14) for $j = 3$ with Eq. (5.1.13), it is concluded that

$$t_3(x_2) = 0, \quad \text{for all } x_2. \tag{5.1.15}$$

Now we are left with two second order coupled differential equations. To solve these equations we use method of first integral as suggested by [5].

5.2 Secular equation for Rayleigh wave in orthotropic incompressible material

Mozhaev [5] used the method of first integral on a system of two second order differential equations on two non zero components of mechanical displacement but here, the method is used non zero traction components t_1, t_2. The coefficients of Eqs. (5.1.11, 5.1.12) are reduced to

$$\begin{aligned} &\xi t_1'' - i(\delta - 2\xi)t_2' + (\delta - \xi)(1 - \xi)t_1 = 0, \\ &(\delta - \xi)t_2'' + i(\delta - 2\xi)t_1' + \xi t_2 = 0, \end{aligned} \tag{5.2.1}$$

where δ and ξ are defined as

$$\delta = \frac{(C_{11} + C_{22} - 2C_{12})}{C_{66}} \ , \ \xi = \frac{(\rho v^2)}{C_{66}}. \tag{5.2.2}$$

Now multiply Eq. $(5.2.1)_1$ by t_1' and Eq. $(5.2.1)_2$ by t_2' and then integrate from $x_2 = 0$ to $x_2 = \infty$, we get

$$\begin{aligned} &\xi \int_0^\infty t_1' t_1'' dx_2 - i(\delta - 2\xi) \int_0^\infty t_1' t_2' dx_2 + (\delta - \xi)(1 - \xi) \int_0^\infty t_1' t_1 dx_2 = 0, \\ &(\delta - \xi) \int_0^\infty t_2' t_2'' dx_2 + i(\delta - 2\xi) \int_0^\infty t_2' t_1' + \xi \int_0^\infty t_2' t_2 dx_2 = 0, \end{aligned} \tag{5.2.3}$$

Applying the boundary conditions while solving yields

$$\xi[t_1'(0)]^2 + 2i(\delta - 2\xi)\int t_1't_2' = 0,$$
$$(\delta - \xi)[t_2'(0)]^2 - 2i(\delta - 2\xi)\int t_1't_2' = 0. \quad (5.2.4)$$

Adding the above equations $(5.2.4)_1$ and $(5.2.4)_2$ gives

$$\xi t_1'(0)^2 + (\delta - \xi)t_2'(0)^2 = 0. \quad (5.2.5)$$

Similarly, multiply Eq. $(5.2.1)_1$ by $\xi t_1' + i(\delta - 2\xi)t_2$ and Eq. $(5.2.1)_2$ by $(\delta - \xi)t_2' - i(\delta - 2\xi)t_1$ and then integrate from $x_2 = 0$ to $x_2 = \infty$, we get

$$\xi^2[t_1'(0)]^2 + 2i(\delta - 2\xi)(\delta - \xi)(1 - \xi)\int t_1t_2 = 0,$$
$$(\delta - \xi)^2[t_2'(0)]^2 - 2i(\delta - 2\xi)\xi\int t_1t_2 = 0. \quad (5.2.6)$$

so that

$$\xi^3[t_1'(0)]^2 + (\delta - \xi)^3(1 - \xi)[t_2'(0)]^2 = 0. \quad (5.2.7)$$

Now Eq. (5.2.5) and Eq. (5.2.7) provide homogeneous system for unknowns $t_1'(0)^2$ and $t_2'(0)^2$ and for non trivial solution, its determinant must be zero

$$\xi(\delta - \xi)[(\delta - \xi)^2(1 - \xi) - \xi^2] = 0. \quad (5.2.8)$$

The factor $\xi(\delta - \xi)$ is equal to zero and gives the trivial answer so the secular equation from the second factor is evaluated which is

$$(\delta - \xi)^2(1 - \xi) = \xi^2. \quad (5.2.9)$$

Substituting back the values of ξ and δ, the secular equation becomes

$$(C_{11} + C_{22} - 2C_{12} - \rho v^2)^2(C_{66} - \rho v^2) = C_{66}(\rho v^2)^2. \quad (5.2.10)$$

Equation (5.2.10) represents the explicit derivation of a secular equation for the subsonic surface wave that is propagating in semi-infinite body which is composed of orthotropic incompressible elastic material. It is significant to mention that this result can also be used for many other types of anisotropy: Royer and Dieulesaint [2] has indeed showed that results drawn from orthotropic case can be applied for 16 different configurations of cubic, tetragonal and hexagonal anisotropy.

In order to justify existence of real wave speed, the above equation (5.2.10) is written as

$$f(\xi) = 0, \quad \text{where} \quad f(\xi) = \xi^2 - (\delta - \xi)^2(1 - \xi). \tag{5.2.11}$$

For the traveling subsonic waves, the above secular equation (5.2.10) is subject to

$$0 \leq \xi \leq 1. \tag{5.2.12}$$

Between this range, it is convenient to show that f is function of ξ and it is increasing monotonically as it can be seen

$$f(0) = -\delta^2, \quad f(1) = 1. \tag{5.2.13}$$

For the purpose of consistency, the major result that is drawn in the problem is linked with previous studies. Like, if we draw the attention to isotropic limits, when $C_{11} = C_{22} = \lambda + 2\mu$, $C_{12} = \lambda$, $C_{66} = \mu$, where μ and λ are Lame constants. By using these values in Eq. (5.2.9), the secular equation can be written as

$$(4 - \xi)^2(1 - \xi) = \xi^2, \quad \text{or} \quad \xi^3 - 8\xi^2 + 24\xi - 16 = 0, \tag{5.2.14}$$

which is well-known secular equation that was derived in [1].

Chapter 6

Conclusion

The analysis is done on propagation of Rayleigh wave in anisotropic material which is based on fact that equation of propagation and boundary conditions are simplified under the condition that some of the elastic constants vanish. Rayleigh wave displacement has only two components left. The equation for wave velocity is also established. The configuration of crystals fulfilling the conditions of simplification have been sought for the number of crystals that belongs to orthorhombic, tetragonal, hexagonal, and cubic systems. In [3], the motion of surface wave in monoclinic material with symmetry at $x_3 = 0$ shows very close similarity to plane stress and plain strain motion. We attain a system of two differential equations for traction components from the equation of motion. Once we apply the method of first integral, we get a system of three equations that are linearly independent having three unknowns. If the variables in equations are displacement components then the system that we get is of 18 equations and 18 unknowns but instead of this we used the traction components as variable which helped us reduce the number of equations to 9 having 9 unknowns. However, Eq. (4.4.13) are not linearly independent. It is stressed again that this method can not be called a general method to calculate the secular equation. For triclinic or monoclinic materials, the method of first integrals can not be used in case of three dimensional displacement. It is only applicable when the wave is propagating in x_1 direction with the plane of symmetry at $x_3 = 0$ of monoclinic crystals whereas in [4], due to the orthotropic properties and symmetries, the secular equation is calculated directly through the method of first integral. This method proved to be a powerful method by [5], by using constraint of incompressibility into account.

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