# Behaviour of Timelike Geodesics in the Kerr-Newmann Spacetime

By

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Dedicated to

***My Beloved Parents and Husband***

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# Abstract

In this thesis timelike geodesics in the Kerr-Newmann spacetime are discussed. It is thought that spacetime curvature is responsible for gravitation. Geodesics in General Relativity describe the path of particles under the influence of gravitation. The study of geodesics is of intrinsic significance in the study of the geometry of spacetime.

A. Qadir and A.A. Siddiqui have discussed some aspects of the timelike geodesics in the Reissner-Nordstrom background geometry. In particular, boundary of these geodesics for freely falling observer is obtained at $r = Q^2/2m$, where $Q$ and $m$ are the charge and the mass of the black hole, respectively. Boundaries for the geodesics for observers with positive and negative energies, at infinity, are also given. It is found that boundary moves backward for observer with positive energy and moves forward for observer with negative energy. In this thesis this work is extended for the Kerr-Newmann spacetime geometry. The barrier for the freely falling observers at rest (zero energy at infinity) and with positive and negative energies are obtained. All the results obtained by Qadir and Siddiqui can be recovered from our results by taking the angular momentum, $a$, equal to zero.

# Contents

# Chapter 1

# Some Solutions of the Einstein Field Equations

## 1.1 Introduction

The Einstein theory of General relativity (GR) [1] is the most generally accepted theory of gravitation. According to GR matter-energy alters the geometry of its surroundings and thus the behaviour of nearby objects. GR gives most accurate results for strong gravitational fields and also for weak gravitational fields.

In 1915, Albert Einstein published a set of 10 differential equations known as the Einstein field equations EFEs [2]. Einstein believed that the Universe is static and to balance the gravitational pull he added a cosmological constant to the field equations. These equations relate the curvature of the spacetime with the presence of mass, energy and momentum. The first spherically symmetric static vacuum solution of EFEs was given by Schwarzschild in 1916 [3]. Reissner-Nordstrom (RN) gave [4] another spherically symmetric static solution of EFEs, which represents the geometry around a point massive electric charge, static at origin.

If a pressure gradient force is not sufficiently strong, a body can continue collapsing due to its self-gravity [5]. This phenomenon is known as *gravitational collapse*. Gravitational collapse leads to formation of singularities. A gravitational singularity or spacetime singularity is a location where spacetime curvature becomes infinite. It is considered as missing piece of the spacetime and disrupts the predictability of the spacetime. It is generally assumed that naked singularities must be physically excluded, as they could otherwise introduce unpredictable influences in their future null cones. Considering geodesics for a naked RN singularity, it is found that the singularity is effectively clothed. Regarding electron as a naked singularity, the size of the clothed singularity turns out to be classical electro-magnetic radius of the electron, to an observer initially at rest and falling freely from infinity [6]. In this thesis we extend the study of timelike geodesics in the Kerr-Newmann (KN) geometry, which is a solution of the Einstein-Maxwell equations, describing the spacetime geometry surrounding a charged, rotating mass. The plan of the thesis is as follows:

In the Chapter 1, basic knowledge of differential geometry and derivation of the Einstein field equations are given, then some solutions of Einstein field equations are discussed. Chapter 2 presents review of a paper by Qadir and Siddiqui [6] on some aspects of timelike geodesics in the RN singularity background. This work is extended in Chapter 3 for the KN spacetime and a brief conclusion of the thesis is given.

Differential geometry is a basic tool for deriving EFEs, so some basics of differential geometry are discussed first. Assuming that reader has knowledge of vectors and tensors, a discussion of the metric tensor, Christofell symbols, covariant derivative, the Riemann Curvature tensor and other related tensors are given in this section.

The Minkowski spacetime is a flat four dimensional manifold $(x, y, z, t)$ having three ordinary dimensions of space combined with a dimension of time. Every point on a Minkowski spacetime is called an *event*. An event is represented by a vector. Consider any two events $P$ and $Q$ with coordinates $(t_p, x_p, y_p, z_p)$ and $(t_Q, x_Q, y_Q, z_Q)$ respectively. The distance between these two events is defined as

$$ds^2 = c^2dt^2 - dx^2 - dy^2 - dz^2, \tag{1.1}$$

which is known as the *Minkowski line element*. The interval separating any two events is timelike if $ds^2 > 0$, spacelike if $ds^2 < 0$, and null if $ds^2 = 0$. Eq. (1.1) can be written as

$$ds^2 = g_{ab}dx^a dx^b, \ (a, b = 0, 1, 2, 3) \tag{1.2}$$

where $g_{ab}$ is second rank covariant symmetric tensor called the metric tensor, it maps any two vectors $u$ and $v$ into $\mathbf{R}$ [7] i.e.

$$\mathbf{g}(u, v) = g_{ab}u^a v^b = \underline{u}.\underline{v}.$$

The partial derivatives of a tensor of rank one or higher do not give a tensor. To preserve the invariance of the derivatives we use a complicated rule of differentiation, known as *covariant differentiation*. It is denoted by semicolon. The covariant derivative of a covariant vector $U_a$ is given by

$$U_{a;b} = U_{a,b} - \Gamma^c_{ab}U_c, \tag{1.3}$$

and the covariant derivative of contravariant vector is

$$U^a{}_{;b} = U^a_{,b} + \Gamma^a_{cb}U^c. \tag{1.4}$$

The covariant derivative of a metric tensor is zero.

$$g_{ac;b} = 0, \tag{1.5}$$

For mixed tensor

$$U^a_{b\,;d} = U^a_{b,d} + \Gamma^a_{cd} U^c_b - \Gamma^c_{bd} U^a_c,$$

where $\Gamma^a_{bc}$ is the *Christofell symbol*, and it is defined in terms of the metric tensor and its partial derivatives as

$$\Gamma^a_{cb} = \frac{1}{2} g^{ad}(g_{db,c} + g_{dc,b} - g_{bc,d}), \tag{1.6}$$

and is symmetric in lower indices

$$\Gamma^a_{cb} = \Gamma^a_{bc}. \tag{1.7}$$

The geodesic equations that give shortest path between two events is given by

$$\ddot{x}^a + \Gamma^a_{bc} \dot{x}^b \dot{x}^c = 0. \tag{1.8}$$

The *Riemann Curvature tensor* has great importance in describing the geometry of spacetime, and it also helps to construct other tensors which give a full description of gravitation [3]. In terms of Christofell symbols the Riemann Curvature tensor, $R^a_{bed}$, is given as

$$R^a_{bed} = \Gamma^a_{bd,e} - \Gamma^a_{be,d} + \Gamma^c_{bd}\Gamma^a_{ec} - \Gamma^c_{be}\Gamma^a_{cd}, \tag{1.9}$$

which can also be written as

$$R_{abed} = g_{ac} R^c_{bed}.$$

The curvature tensor satisfies the Bianchi identities, which are

$$R^a_{bed} + R^a_{edb} + R^a_{dbe} = 0, \tag{1.10}$$

and

$$R_{abed;c} + R_{abdc;e} + R_{abce;d} = 0. \tag{1.11}$$

The Ricci tensor, $R_{ab}$, can be obtained by contracting the Curvature tensor as

$$R_{ab} = R^c_{acb} = g^{cd} R_{cadb}. \tag{1.12}$$

The Ricci scalar, $R$, can be obtained by contracting the Ricci tensor as

$$R = R^a_a = g^{ab} R_{ab}. \tag{1.13}$$

### 1.1.1 The Stress-Energy Tensor

The stress-energy tensor is a tensor quantity that describes the flux and density of momentum and energy in spacetime. The stress-energy tensor is the source of the gravitational field in the EFEs, just as mass density is the source of gravitational field in Newtonian physics. The energy-momentum tensor is denoted by $T^{ab}$. It is a tensor of rank two and its components are displayed by square matrix of order 4

$$T^{ab} = \begin{bmatrix} T^{00} & T^{01} & T^{02} & T^{03} \\ T^{10} & T^{11} & T^{12} & T^{13} \\ T^{20} & T^{21} & T^{22} & T^{23} \\ T^{30} & T^{31} & T^{32} & T^{33} \end{bmatrix}, \tag{1.14}$$

where $T^{00}$ is energy density, denoted by $\rho$ i.e

$$T^{00} = \rho.$$

$T^{aa}$ is stress tenor. $T^{0a} = T^{a0}$ represents the momentum density and remaining $T^{ab}$ when $a \neq b$ represents shear stress. The contravariant form, $T_{ab}$, and mixed form, $T^a_b$, of the energy-momentum tensor are

$$T_{ab} = g_{ac} g_{bd} T^{cd}, \tag{1.15}$$

and

$$T^a_b = g_{cb} T^{ac}.$$

In electromagnetic field the stress-energy tensor is given as

$$T^{em}_{ab} = \frac{-1}{\mu_o} \left( -F_{ae} F^e_b + \frac{1}{4} g_{ab} F^{uv} F_{uv} \right), \tag{1.16}$$

where $\mu_o$ is a constant.

### 1.1.2 The Einstein Tensor

The Einstein tensor, $G_{ab}$, is obtained from the second Bianchi identity given by Eq. (1.11). We start from contracting Eq. (1.11) with $g^{ab}$ and using result given in Eq. (1.5) to get

$$R_{bd;e} + R^c_{bde;c} - R_{be;d} = 0. \tag{1.17}$$

Contracting the above Eq. (1.17) with $g^{bd}$, we get

$$(R^{ed} - \frac{1}{2} R g^{ed})_{;d} = 0, \tag{1.18}$$

where the term inside brackets is the Einstein tensor and it tells about the geometric properties of a spacetime. In lower indices the Einstein tensor is given as

$$G_{ab} = R_{ab} - \frac{1}{2} R g_{ab}. \tag{1.19}$$

$G_{ab}$ is symmetric and divergence free, i.e.

$$G_{ab} = G_{ba},$$

and

$$G_{ab;a} = 0. \tag{1.20}$$

## 1.2 Einstein's Theory of General Relativity

GR deals with geometry of gravitational force. Newton's theory and GR give the same answers, but the former tells us to think of gravity as a force, and the latter tells us to think of it as curvature of spacetime. Newton's theory of gravity is an approximation to GR that works only when gravity is relatively weak, but breaks down when gravity is strong [8]. Einstein knew that he had to build a theory that was capable of describing the complicated nature of gravitation, and the universe, utilising his ideas of curved spacetime, but he also had to create this theory with the characteristic that at large distances from mass sources the universe can be explained by Newtonian mechanics. Einstein says, *mass-energy tells spacetime how to curve and curved spacetime tells mass-energy how to move.* In GR we study spacetime in terms of curvature. A number of observational phenomena are successfully explained by GR, such as bending of light, precession of perihelion of Mercury and gravitational red shift [9].

*Precession of Perihelion of Mercury:*

The Mercury's orbit is shifting its position very gradually over time, due to the curvature of space-time around the massive sun. The orientation of Mercury's orbit is found to precess in space over time, This is commonly called the precession of the perihelion, because it causes the position of the perihelion to move. Only part of this can be accounted for by perturbations in Newton's theory. There is an extra 43 seconds of arc per century in this precession that is predicted by GR and observed to occur (a second of arc is 1/3600 of an angular degree). This effect is extremely small, but the measurements are very precise and can detect such small effects very well [10].

*Gravitational redshift:*

GR predicts that light coming from a strong gravitational field should have its wavelength shifted to larger values, detailed observations indicate such a red shift, and that its magnitude is correctly given by Einstein's theory.

*Bending of Light:*

Light bends because of an intrinsic curvature in the spacetime surrounding a massive body [11].

*Gravitational Collapse and Black Hole:*

If a pressure gradient force is not sufficiently strong, a body can continue collapsing due to its self-gravity [5]. This phenomenon is known as *gravitational collapse*. A star is a self-gravitating collection of matter supported by its thermal pressure. Thermal pressure is generated in star by thermonuclear reactions in it. Eventually, these reactions can not continue forever and the pressure reduces to balance gravity and star cools, contracts and the density of star continues to increase, so that the escape velocity exceeds the speed of light. The question arises, what should be the volume of an object so that it becomes dense enough to have escape velocity greater then speed of light, and what would be the geometry of spacetime around such an object?

*Black Hole:*

A black hole is a region of spacetime from which nothing can escape, not even light. Around a black hole there is a mathematically defined surface, that marks the point of no return, and is called an *event horizon* [12] .

*Singularities:*

A gravitational singularity or spacetime singularity is a location where spacetime curvature becomes infinite. They are considered as missing pieces of the spacetime.

They disrupt the predictability of the spacetime, and allow uncontrollable information to enter into it. There are two types of singularities; (i) coordinate singularity and (ii) essential singularity. *Coordinate singularity* is due to bad choice of coordinates and can be removed by using appropriate coordinate transformations, and *Essential singularity* is a physical singularity and can not be removed by coordinates transformation. It is impossible to predict anything about the spacetime at such a singularity.

*Cosmin Censorship Hypothesis:*

In 1969, Penrose gave that all singularities in physically realistic spacetimes are hidden inside event horizon, the horizon is the boundary of the region which is causally connected to a distant observer [13]. Thus it acts like a one way membrane through which energy and information can pass to the interior, but not to the exterior. The singularity has no causal connection to an external observer; he cannot see it.

### 1.2.1 The Einstein Field Equations

In this section we discuss the EFEs and their important solutions. EFEs are a set of 10 partial differential equations. Paths of freely falling bodies in a gravitational field are simply the geodesics of the spacetime [14]. According to Newton's first law of motion *Every object in a state of uniform motion tends to remain in that state of motion unless an external force is applied to it.* Einstein postulated that particle would travel on a geodesic in spacetime. The path's straightness depends on spacetime curvature. In Newtonian viewpoint, gravitational force depends on the presence of matter. In viewpoint of relativity, there is no difference between matter and energy. Thus spacetime curvature must be related to the matter-energy. The symmetric stress energy tensor $T^{ab}$ tells about matter- energy distribution. $T^{ab}$ is divergence free, i.e. we have

$$T^{ab}{}_{;b} = 0. \tag{1.21}$$

As we discussed gravitation is due to curvature in spacetime, and the curvature of spacetime is related to presence of matter-energy. Thus we expect that there must be a relation between the stress energy tensor and the curvature tensor, i.e. we have

$$f^{ab}(R^{e}_{dch}, g_{rp}) = T^{ab}. \tag{1.22}$$

We try to take $f^{ab}$ in the simplest form. We take the Ricci tensor $R_{ab}$ and Ricci scalar $R$ to give

$$R_{ab} - \frac{1}{2}g_{ab}R + \Lambda g_{ab} = \alpha T_{ab}, \tag{1.23}$$

where $\Lambda$ is cosmological constant and $\alpha$ is the Einstein gravitational constant. For $\Lambda = 0$, we get

$$R_{ab} - \frac{1}{2}g_{ab}R = \alpha T_{ab}. \tag{1.24}$$

Equations (1.24) are the EFEs. We take the classical gravitational equations to evaluate the Einstein gravitational constant $\alpha$

$$\nabla^2\varphi = 4\pi G\rho, \tag{1.25}$$

where $\rho$ is matter density and $\varphi$ is the gravitational potential energy. For stress-free material, we write

$$T^{ab} \sim \rho c^2 \dot{x}^a \dot{x}^b. \tag{1.26}$$

Special relativistically in rest frame we have [15]

$$\dot{x}^0 = \frac{dx^0}{dx^0} = 1,$$

and

$$\dot{x}^i = 0, \ (i = 1, 2, 3).$$

Therefore, the geodesic equations

$$\ddot{x}^i + \Gamma^i_{00}\dot{x}^0\dot{x}^0 = 0,$$

give

$$\ddot{x}^i = 1/2 g^{ii} g_{00,i}.$$

Taking the classical limit $g^{ii} \sim -1$, we have

$$\ddot{x}^i = -1/2(\nabla g_{00})c^2. \tag{1.27}$$

Classicaly

$$\ddot{x}^i = -\nabla\varphi. \tag{1.28}$$

Comparing Eqs. (1.27) and (1.28) we get

$$g_{00} = 2\varphi/c^2 + constant. \tag{1.29}$$

As $r \to \infty$, $\varphi \to 0$, therefore, *constant* $\to 1$. Hence,

$$g_{00} = 1 + 2\varphi/c^2. \tag{1.30}$$

The 00-component of the EFEs is

$$R_{00} - \frac{1}{2}Rg_{00} = \alpha T_{00}. \tag{1.31}$$

Using Eq. (1.26) with lower indices in Eq. (1.31) we get

$$R_{00} - \frac{1}{2}Rg_{00} = \alpha\rho c^2 = 2\Gamma^i_{00,i},$$

or

$$R_{00} - \frac{1}{2}Rg_{00} = \nabla^2 g_{00}. \tag{1.32}$$

Using value of $g_{00}$ from Eq. (1.30) in Eq. (1.2.1) we obtain

$$R_{00} - \frac{1}{2}Rg_{00} = 2\nabla^2\varphi/c^2. \tag{1.33}$$

From Eqs. (1.33) and (1.2.1), we get

$$2\nabla^2\varphi/c^2 = \alpha\rho c^2, \tag{1.34}$$

or

$$\alpha = 2\nabla^2\varphi/\rho c^4. \tag{1.35}$$

From Eqs. (1.35) and (1.25), we get

$$\alpha = \frac{8\pi G}{c^4}. \tag{1.36}$$

Putting the value of $\alpha$ in the EFEs (1.24), we get

$$R_{ab} - \frac{1}{2}Rg_{ab} = \frac{8\pi G}{c^4}T_{ab}, \tag{1.37}$$

or

$$g^{ab}R_{ab} - \frac{1}{2}Rg^{ab}g_{ab}R = \frac{8\pi G}{c^4}g^{ab}T_{ab}.$$

Or

$$R - \frac{1}{2}Rg^{ab}g_{ab} = \frac{8\pi G}{c^4}T. \tag{1.38}$$

In 4-dimensions $g^{ab}g_{ab} = 4$, so we have

$$R = -\frac{8\pi G}{c^4}T.$$

Now, putting the value of $R$ in Eq. (1.37), we get

$$R_{ab} = \frac{8\pi G}{c^4}(T_{ab} - \frac{1}{2}g_{ab}T). \tag{1.39}$$

In vacuum trace, $T$, of $T_{ab}$ is zero, therefore, the EFEs in vacuum are

$$R_{ab} = 0. \tag{1.40}$$

### 1.2.2 The Schwarzschild Solution

The Schwarzschild solution is one of the most important and simplest of all exact solutions to EFEs. This is spherically symmetric and time independent solution of Einstein's vacuum equations. the German physicist, Karl Schwarzschild [2] calculated the space around a spherically symmetric body static at the origin. The line element at large distance from origin is required to be Lorentzian. In spherical polar coordinates spherically symmetric static metric can be written as [14]

$$ds^2 = e^{\upsilon(r)}c^2dt^2 - e^{\mu(r)}dr^2 - r^2d\Omega^2, \tag{1.41}$$

where

$$d\Omega^2 = d\theta^2 + r^2\sin^2\theta d\phi^2. \tag{1.42}$$

The metric tensor $g_{ab}$ is

$$g_{ab} = \begin{bmatrix} e^{\upsilon(r)} & 0 & 0 & 0 \\ 0 & -e^{\mu(r)} & 0 & 0 \\ 0 & 0 & -r^2 & 0 \\ 0 & 0 & 0 & -r^2\sin^2\theta \end{bmatrix}, \tag{1.43}$$

and its inverse $g^{ab}$ is

$$g^{ab} = \begin{bmatrix} e^{-\upsilon(r)} & 0 & 0 & 0 \\ 0 & -e^{\mu(r)} & 0 & 0 \\ 0 & 0 & -\frac{1}{r^2} & 0 \\ 0 & 0 & 0 & -\frac{1}{r^2\sin^2\theta} \end{bmatrix}. \tag{1.44}$$

The non zero Christoffel symbols are

$$\Gamma^0_{01} = \frac{\upsilon'(r)}{2},\ \Gamma^2_{33} = -\sin\theta\cos\theta,\ \Gamma^3_{23} = \cot\theta, \tag{1.45}$$

$$\Gamma^2_{12} = \frac{1}{r},\ \Gamma^3_{13} = \frac{1}{r},\ \Gamma^1_{33} = -r\sin^2\theta e^{-\mu(r)}, \tag{1.46}$$

$$\Gamma^1_{11} = \frac{\mu'(r)}{2},\ \Gamma^1_{22} = -re^{-}\mu(r),\ \Gamma^1_{00} = \frac{\upsilon'(r)}{2}e^{\upsilon(r)-\mu(r)}, \tag{1.47}$$

where $'$ denotes derivative with respect to $r$. The surviving EFEs are

$$R_{00} = \upsilon''(r) + \frac{1}{2}\upsilon'(r)(\upsilon'(r) - \mu'(r)) + 2\frac{\upsilon'(r)}{r} = 0, \tag{1.48}$$

$$R_{11} = -\upsilon''(r) - \frac{1}{2}\upsilon'(r)(\upsilon'(r) - \mu'(r)) + 2\frac{\mu'(r)}{r} = 0, \tag{1.49}$$

$$R_{22} = 1 - e^{-\mu(r)} + \frac{1}{2}r(\mu'(r) - \upsilon'(r))e^{-\mu(r)} = 0, \tag{1.50}$$

$$R_{33} = R_{22}\sin^2\theta = 0. \tag{1.51}$$

Adding Eqs. (1.48) and (1.49), we get

$$(\upsilon(r) + \mu(r))' = 0,$$

or

$$\upsilon + \mu = constant. \tag{1.52}$$

If this constant of integration were non-zero we could define a new function $\overline{\upsilon}(r) = \upsilon(r) - constant$ [2]. Therefore, without loss of generality, we can take $constant = 0$, to have

$$\upsilon(r) = -\mu(r). \tag{1.53}$$

Using Eq. (1.53) in Eq. (1.50), we get

$$1 - (e^{-\mu(r)} - r\mu'(r)e^{-\mu(r)}) = 0,$$

or

$$(re^{-\mu(r)})' = 1,$$

or

$$e^{-\mu(r)} = 1 + \frac{\beta}{r}. \tag{1.54}$$

From Eqs. (1.53) and (1.54), we have

$$e^{\upsilon(r)} = (1 + \frac{\beta}{r}). \tag{1.55}$$

Here $\beta$ is constant of integration. Using Eqs. (1.54) and (1.55) in Eq. (1.41), we have

$$ds^2 = c^2\left(1 + \frac{\beta}{r}\right)dt^2 - \frac{dr^2}{1 + \frac{\beta}{r}} - r^2 d\Omega^2. \tag{1.56}$$

Now, to determine the value of $\beta$, we use geodesic equations (1.8), to get

$$\ddot{x}^0 + 2\Gamma^0_{01}\dot{x}^0\dot{x}^1 = 0,$$

$$\ddot{x}^1 + \Gamma^1_{00}(\dot{x}^0)^2 + \Gamma^1_{11}(\dot{x}^1)^2 + \Gamma^1_{22}(\dot{x}^2)^2 + \Gamma^1_{33}(\dot{x}^3)^2 = 0,$$

$$\ddot{x}^2 + 2\Gamma^2_{12}\dot{x}^1\dot{x}^2 + \Gamma^2_{33}(\dot{x}^3)^2 = 0,$$

$$\ddot{x}^3 + 2\Gamma^3_{13}\dot{x}^1\dot{x}^3 + 2\Gamma^3_{23}\dot{x}^2\dot{x}^3 = 0.$$

We can write above system of equations in terms of $t, r, \theta, \phi$ as

$$c\frac{d^2t}{ds^2} + c\upsilon'(r)\frac{dt}{ds}\frac{dr}{ds} = 0. \tag{1.57}$$

$$\frac{d^2r}{ds^2} + \frac{1}{2}c^2\upsilon'(r)e^{\upsilon(r)-\mu(r)}\left(\frac{dt}{ds}\right)^2 + \frac{1}{2}\mu'(r)\left(\frac{dr}{ds}\right)^2 - re^{-\mu(r)}\left(\frac{d\theta}{ds}\right)^2 - r\sin^2\theta e^{-\mu(r)}\left(\frac{d\phi}{ds}\right)^2 = 0, \tag{1.58}$$

$$\frac{d^2\theta}{ds^2} + \frac{2}{r}\frac{dr}{ds}\frac{d\theta}{ds} - \sin\theta\cos\theta\left(\frac{d\phi}{ds}\right)^2 = 0, \tag{1.59}$$

$$\frac{d^2\phi}{ds^2} + \frac{2}{r}\frac{dr}{ds}\frac{d\phi}{ds} + 2\cot\theta\frac{d\theta}{ds}\frac{d\phi}{ds} = 0. \tag{1.60}$$

By choosing $\theta = \pi/2$ and $\dot{\theta} = 0$, above geodesic equations reduce to

$$\frac{d^2t}{ds^2} + \upsilon'(r)\frac{dt}{ds}\frac{dr}{ds} = 0, \tag{1.61}$$

$$\frac{d^2r}{dt^2} + \frac{1}{2}c^2\upsilon'(r)e^{\upsilon(r)-\mu(r)}\left(\frac{dt}{ds}\right)^2 + \frac{1}{2}\mu'(r)\left(\frac{dr}{ds}\right)^2 - re^{-\mu(r)}\left(\frac{d\phi}{ds}\right)^2 = 0, \tag{1.62}$$

$$\frac{d^2\phi}{ds^2} + \frac{2}{r}\frac{dr}{ds}\frac{d\phi}{ds} = 0, \tag{1.63}$$

$$\frac{d\upsilon(r)}{ds} = \frac{d\upsilon(r)}{dr}\frac{dr}{ds} = \upsilon'(r)\frac{dr}{ds}. \tag{1.64}$$

From Eq. (1.61), we get

$$\frac{d}{ds}\left(e^{\upsilon(r)}\frac{dt}{ds}\right) = 0,$$

or

$$e^{\upsilon(r)}\frac{dt}{ds} = \lambda,$$

.

$$\frac{dt}{ds} = \frac{\lambda}{e^{\upsilon(r)}} = \frac{\lambda}{1-\frac{\beta}{r}}, \tag{1.65}$$

where $\lambda$ is constant of integration. When $r \to \infty$, $\frac{dt}{ds} \to \frac{1}{c}$, so we get

$$c\frac{dt}{ds} = e^{-\upsilon(r)}. \tag{1.66}$$

Multiplying Eq. (1.63) with $r^2$, to have

$$\frac{d}{ds}(r^2\frac{d\phi}{ds}) = 0,$$

or

$$r^2\frac{d\phi}{ds} = h_0, \tag{1.67}$$

where $h_0$ is constant of integration. Classically,

$$r^2\frac{d\phi}{ds} = \frac{h}{c}, \tag{1.68}$$

where $c$ is the speed of light and $h$ is Planck's constant. Comparing Eqs. (1.68) and (1.67), we have

$$h_0 = \frac{h}{c}. \tag{1.69}$$

Equation (1.41) can be written as

$$1 = e^{\upsilon(r)}\left(\frac{cdt}{ds}\right)^2 - e^{\mu(r)}\left(\frac{dr}{ds}\right)^2 - r^2\left(\frac{d\phi}{ds}\right)^2. \tag{1.70}$$

Putting values of $c\frac{dt}{ds}$ and $\frac{d\phi}{ds}$ from Eqs. (1.66) and (1.68) in Eq. (1.41), we obtain

$$e^{-\upsilon(r)}\left(\frac{dr}{ds}\right)^2 = e^{\upsilon(r)} - \frac{h^2}{c^2r^2} - 1. \tag{1.71}$$

Using Eqs. (1.71) and (1.66) and $(e^{\upsilon}(r))' = \frac{-\beta}{r^2}$ in Eq. (1.58), we get

$$\frac{dr^2}{ds^2} - \frac{h^2}{c^2r^3} - \frac{\beta}{2r^2} - \frac{3}{2}\frac{\beta h^2}{c^2r^4}. \tag{1.72}$$

From Eq. (1.68), we have

$$r^2d\phi = \frac{h}{c}ds, \tag{1.73}$$

or

$$\frac{d}{ds} = \frac{h}{cr^2}\frac{d}{d\phi}. \tag{1.74}$$

Putting $r = \frac{1}{w}$ in Eq. (1.74), to have

$$\frac{d}{ds} = \frac{hw^2}{c}\frac{d}{d\phi}. \tag{1.75}$$

Using Eq. (1.75) in Eq. (1.72), we get

$$\frac{d^2r}{ds^2} = -\frac{h^2w^2}{c^2}\frac{d^2w}{d\phi^2}. \tag{1.76}$$

From Eqs. (1.72) and (1.77), we have

$$\frac{d^2w}{d\phi^2} + w = -\frac{c^2\beta}{2h} - \frac{3\beta}{2}w^2. \tag{1.77}$$

Classical Newtonian equation for a particle in gravitational field is

$$\frac{d^2w}{d\phi^2} + w = -\frac{Gm}{h^2}. \tag{1.78}$$

For large value of $r$ we can neglect $w^2$, therefore, from Eqs. (1.78) and (1.77) we get

$$\beta = -\frac{2Gm}{c^2}. \tag{1.79}$$

Thus Eq. (1.41) can be written as

$$ds^2 = c^2\left(1 - \frac{2Gm}{c^2r}\right)dt^2 - \frac{dr^2}{\left(1 - \frac{2Gm}{c^2r}\right)} - r^2d\Omega^2. \tag{1.80}$$

This metric (1.80) is known as the *Schwarzschild metric*. In gravitational units ($c = G = 1$) above metric can be written in the simpler form as

$$ds^2 = \left(1 - \frac{2m}{r}\right)dt^2 - \frac{dr^2}{\left(1 - \frac{2m}{r}\right)} - r^2d\Omega^2. \tag{1.81}$$

### 1.2.3 The Reissner-Nordstrom Solution

Writing the most general spherical symmetric static metric given by Eq. (1.41) as

$$ds^2 = A(r)c^2dt^2 - B(r)dr^2 - r^2(d\theta^2 + \sin^2\theta d\phi^2), \qquad (1.82)$$

where $A(r)$ and $B(r)$ are arbitrary functions of $r$. For electromagnetic field, the EFEs are

$$R_{ab} = \alpha T_{ab}{}^{em}, \qquad (1.83)$$

where $T_{ab}{}^{em}$ is electromagnetic stress-energy tensor. Electromagnetic field tensor is taken as

$$F_{ab} = \begin{bmatrix} 0 & E_1(r)/c & 0 & 0 \\ -E_1(r)/c & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \qquad (1.84)$$

and

$$F^{ab} = \begin{bmatrix} 0 & -E_1(r)/A(r)B(r)c & 0 & 0 \\ E_1(r)/A(r)B(r)c & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}. \qquad (1.85)$$

Using Eq. (1.16) we get

$$T_{00}{}^{em} = \frac{-(E_1)^2}{2\mu_o c^2 B(r)}, \qquad (1.86)$$

$$T_{11}{}^{em} = \frac{(E_1)^2}{2\mu_o c^2 A(r)}, \qquad (1.87)$$

$$T_{22}{}^{em} = \frac{-r^2(E_1)^2}{2\mu_o c^2 A(r)B(r)}, \qquad (1.88)$$

$$T_{33}{}^{em} = \frac{-r^2(E_1)^2}{2\mu_o c^2 A(r)B(r)} \sin^2\theta. \tag{1.89}$$

For metric (1.1), using Eqs. (1.86), (1.87) (1.88) and (1.89) in Eq. (1.83) we hve

$$\frac{-A(r)''}{2B(r)} + \frac{A(r)'}{4B(r)}(\frac{A(r)'}{A(r)} + \frac{B(r)'}{B(r)}) - \frac{A(r)'}{rB(r)} = \frac{-\alpha {E_1}^2}{2\mu_o B(r)c^2}, \tag{1.90}$$

$$\frac{A''}{2A} - \frac{A'}{4A}(\frac{A'}{A} + \frac{B'}{B}) - \frac{B'}{rB} = \frac{\alpha {E_1}^2}{2\mu_o A(r)c^2}, \tag{1.91}$$

$$\frac{1}{B(r)} - 1 + \frac{r}{2B(r)}(\frac{A(r)'}{A(r)} - \frac{B(r)'}{B(r)}) - \frac{B(r)'}{rB(r)} = \frac{-\alpha r^2 {E_1}^2}{2\mu_o A(r)B(r)c^2}, \tag{1.92}$$

$$\sin^2\theta\left(\frac{1}{B(r)} - 1 + \frac{r}{2B(r)}(\frac{A(r)'}{A(r)} - \frac{B(r)'}{B(r)}) - \frac{B(r)'}{rB(r)}\right) = \left(\frac{-\alpha r^2 {E_1}^2}{2\mu_o A(r)B(r)c^2}\right). \tag{1.93}$$

Multiplying Eq. (1.90) by $\frac{B(r)}{A(r)}$ and adding in Eq. (1.91), we get

$$A(r)B'(r) + A'(r)B(r) = 0, \tag{1.94}$$

or

$$[A(r)B(r)]' = 0, \tag{1.95}$$

or

$$B(r) = \frac{-c_1}{A(r)}. \tag{1.96}$$

Here $c_1$ is constant of integration. Using Eq. (1.96) in Eq. (1.92), we get

$$A(r) + rA'(r) = c_1 - \frac{\alpha r^2 {E_1}^2}{2\mu_o c^2}, \tag{1.97}$$

where $\alpha = \frac{8\pi G}{c^4}$. Integrating Eq. (1.97) w.r.t $r$, we get

$$A(r) = c_1 + \frac{GQ^2}{4\pi\epsilon_o c^4 r^2} + \frac{c_2}{r}. \tag{1.98}$$

where $c_2$ is constant of integration. To find the arbitrary constants $c_1$ and $c_2$, use the weak field limit

$$g_{00} = 1 - \frac{2GM}{c^2}. \tag{1.99}$$

Comparing Eqs. (1.98) and (1.99), we get $c_1 = 1$ and $c_2 = -\frac{2GM}{c^2}$. Using $c_1$ and $c_2$ in Eq. (1.98), we get

$$A(r) = 1 - \frac{2GM}{c^2 r} + \frac{GQ^2}{4\pi\epsilon_o c^4 r^2}. \tag{1.100}$$

In gravitational units ($c = G = 1$) and taking $\frac{1}{4\pi\epsilon_o} = 1$, $A(r)$ in Eq. (1.100) becomes

$$A(r) = 1 - \frac{2m}{r} + \frac{Q^2}{r^2}, \tag{1.101}$$

and

$$B(r) = (1 - \frac{2m}{r} + \frac{Q^2}{r^2})^{-1}. \tag{1.102}$$

Using Eqs. (1.101) and (1.102) in Eq. (1.82), we get

$$ds^2 = \left(1 - \frac{2m}{r} + \frac{Q^2}{r^2}\right) dt^2 - (1 - \frac{2m}{r} + \frac{Q^2}{r^2})^{-1} dr^2 - r^2(d\theta^2 + \sin^2\theta d\phi^2). \tag{1.103}$$

The above metric (1.103) is known as the Riessner-Nordstrom solution of the EFEs. It describes the spherically symmetric static geometry due to a charged massive point. The event horizons for this metric are located where $g_{11} \to \infty$, and are at

$$r_\pm = m + \sqrt{m^2 - Q^2}. \tag{1.104}$$

$r_\pm$ are coordinate singularities. $r_-$ defines the inner horizon of the black hole and $r_+$ defines the outer horizon of the black hole. At $r = 0$ there is an essential singularity.

*Naked RN Solution:*

If $Q > m$ then there is no real root of Eq. (1.104), so there is no coordinate singularity, and in this case solution is known as naked RN Solution.

*Extreme RN Solution:*

When $Q = m$ then we have repeated roots $r_{\pm} = m$, and solution is called extreme RN Solution.

*Usual RN Solution:*

If $m > Q$ then there are two distinct real roots, and solution is called usual RN Solution.

### 1.2.4 The Kerr Solution

Solution of the EFEs due to a point mass, $m$, and angular momentum per unit mass, $a$, called the Kerr solution, is given in Boyer-Lindquist coordinates as

$$ds^2 = c^2\left(1 - \frac{2mr}{\rho^2}\right)dt^2 + \frac{4macr}{\rho^2}\sin^2\theta dt d\phi - \frac{\rho^2}{\Delta}dr^2 - \rho^2 d\theta^2 \tag{1.105}$$

$$- \left(r^2 + a^2 + \frac{2ma^2 r}{\rho^2}\sin^2\theta\right)\sin^2\theta d\phi^2, \tag{1.106}$$

where

$$\rho^2 = r^2 + a^2\cos^2\theta,$$

and

$$\Delta = r^2 - 2mr + a^2.$$

Horizons exist when $g_{11} \to \infty$, and are at

$$r_{\pm} = m \pm \sqrt{m^2 - a^2}. \tag{1.107}$$

For $m > a$, there are two horizons. This case is called regular Kerr black hole. When $m < a$, there is no horizon and we have naked singularity. For $m = a$ one horizon exists and it is called extreme Kerr Solution.

# Chapter 2

# Behaviour of Timelike Geodesics in the Reissner-Nordstrom Spacetime

A *geodesic* is a generalization of a straight line to curved spaces. A *geodesic* in Euclidean space is a straight line. The *geodesic* is the straightest available path between two points in the spacetime [2]. The term *geodesic* comes from geodesy, the science of measuring the size and shape of Earth, in the original sense, a *geodesic* was the shortest path between two points on the Earth's surface, namely, a segment of a great circle. The term has been generalized to include measurements in much more general spaces. Geodesics in $GR$ describe the path of particles under the influence of gravitational force. The shortest path between two points in a curved space can be found mathematically by equation of the length of a curve, and then minimizing this length by the *calculus of variations.* Geodesics are commonly studied in metric geometry. In particular, the path followed by, an orbiting satellite, or orbits of planets are all *geodesics* in curved spacetime.

To derive *geodesic* equations, let us begin by considering a geodesic as a curve $x^a(u)$. The equations satisfied by $x^a(u)$ are determined by the requirement that, along the curve

$$\frac{dt}{du} = f(u)t, \tag{2.1}$$

The components $t^a = \frac{dx^a}{du}$ of the tangent vector in the coordinate basis must satisfy

$$\frac{Dt^a}{Du} = \frac{dt^a}{du} + \Gamma^a_{bc} t^b \frac{dx^c}{du} = f(u)t^a. \tag{2.2}$$

Also Equations satisfied by geodesics are

$$\frac{d^2x^a}{du^2} + \Gamma^a_{bc}\frac{dx^b}{du}\frac{dx^c}{du} = f(u)\frac{dx^a}{du}. \tag{2.3}$$

If the curve is parameterized in such a way that $f(u)$ vanishes, then $u$ is a privileged parameter called an affine parameter [16]. From Eq. (2.1) we see that this corresponds to a parametrization in which the tangent vector $t^a$ remains same at all points along the curve, so we have

$$\frac{dt}{du} = 0 \rightarrow \frac{Dt^a}{Du} = 0. \tag{2.4}$$

Therefore, from Eq. (2.2), an affinely parameterized geodesic satisfies the equation as

$$\frac{d^2x^a}{du^2} + \Gamma^a_{bc}\frac{dx^b}{du}\frac{dx^c}{du} = 0. \tag{2.5}$$

If we change the parameter from an affine parameter $u$ to some other parameter $v$ then the functions $x^a(v)$ describing curve in terms of $v$ will differ from the original functions $x^a(u)$. If, for some arbitrary new parameter $v$, we rewrite Eq.(2.5) in terms of derivatives with respect to $v$ then the geodesic equation becomes

$$\frac{d^2x^a}{dv^2} + \Gamma^a_{bc}\frac{dx^b}{dv}\frac{dx^c}{dv} = \left(\frac{d^2u/dv^2}{du/dv}\right)\frac{dx^a}{dv}. \tag{2.6}$$

It is clear from above Eq. (2.6) that if $u$ is an affine parameter then so too is any linearly related parameter $v = pu + q$, where $p$ and $q$ are constants.

### 2.0.5 Geodesics Equations through Lagrangian

In classical mechanics, a system can be described in terms of generalised coordinates $x^a$ that are functions of time $t$. A line element in these coordinates is defined as given in Eq. (1.2). The Lagrangian for the system can be formed from the potential and the kinetic energies as

$$L = K - V = \frac{1}{2} g_{ab}\dot{x}^a\dot{x}^b - V(x). \tag{2.7}$$

By demanding the action

$$S = \int_{t_i}^{t_j} L dt,$$

to be stationary for small variations in the functions $x^a(t)$, the equations of motion for this system are then found as the Euler-Lagrange equations [16]

$$\frac{d}{ds}\left(\frac{\partial L}{\partial \dot{x}^a}\right) - \frac{\partial L}{\partial x^a} = 0. \tag{2.8}$$

Ignoring term $\frac{1}{2}$ in Eq. (2.7) and substituting in Eq. (2.8), we get

$$\ddot{x^a} + \Gamma^a_{bc}\dot{x}^b\dot{x}^c = 0. \tag{2.9}$$

Thus the Euler-Lagrange equations provide a useful and easy way of generating the geodesic equations. We note that, in finding solutions of the geodesic equations, it is often helpful to find the first integral of the equations. For null geodesics the first integral is simply

$$g_{ab}\dot{x}^a\dot{x}^b = 0,$$

and for time-like geodesics, we have

$$|g_{ab}\dot{x}^a\dot{x}^b| = 1.$$

If $g_{ab}$ do not depend on some particular coordinate $x^c$ then

$$\frac{d}{ds}\left(\frac{\partial L}{\partial \dot{x}^c}\right) = 0,$$

or

$$\frac{\partial L}{\partial \dot{x}^c} = constant,$$

or

$$g_{cb}\dot{x}^b = constant,$$

where $\dot{x}^b = t^b$ and $t^b$ is tangent vector, and we have

$$t_c = constant.$$

If the metric coefficients $g_{ab}$ do not depend on the coordinate $x^c$ then the covariant component $t_c$ of the tangent vector is a conserved quantity along an affinely parameterised geodesic.

### 2.0.6 Stationary property of non-null geodesics

Let us consider non-null geodesics as curves of extremal length between two points $P$ and $Q$ in the manifold. Consider a curve $x^a(u)$, where $u$ is some general parameter. The length along the curve is

$$L = \int_P^Q ds, \tag{2.10}$$

or

$$L = \int_P^Q |g_{ab}\dot{x}^a\dot{x}^b|^{\frac{1}{2}} du,$$

where the dot represents derivative with respect to $u$. Now consider the variation in path as $x^a(u) \to x^a(u) + \delta x^a(u)$. The requirement for $x^a(u)$ to be a geodesic is that $\delta L = 0$ with respect to the variation in the path. This is a calculus of variations problem, in which the integrand is $F = \dot{s} = |g_{ab}\dot{x}^a\dot{x}^b|^{\frac{1}{2}}$. Substituting $F$ in Euler-Lagrange equation (2.8) we get

$$\frac{d}{du}\left(\frac{\partial}{\partial \dot{x}^e}|g_{ab}\dot{x}^a\dot{x}^b|^{\frac{1}{2}}\right) - \frac{\partial}{\partial x^e}|g_{ab}\dot{x}^a\dot{x}^b|^{\frac{1}{2}} = 0,$$

or

$$\frac{d}{du}\left(\frac{1}{\dot{s}}g_{ae}\dot{x}^a\right) - \frac{1}{2\dot{s}}(\partial_e g_{ab}\dot{x}^a\dot{x}^b) = 0. \tag{2.11}$$

Using $\dot{g}_{ae} = (\partial_b g_{ae})\dot{x}^b$ in Eq. (2.11), we get

$$g_{ae}\ddot{x}^a + (\partial_b g_{ae})\dot{x}^b\dot{x}^a - \frac{1}{2}(\partial_e g_{ab}\dot{x}^a\dot{x}^b) = \frac{\ddot{s}}{\dot{s}}g_{ae}\dot{x}^a. \tag{2.12}$$

Interchanging $a$ and $c$ we can write $(\partial_b g_{ae})\dot{x}^a\dot{x}^b = \frac{1}{2}(\partial_b g_{ae} + \partial_a g_{be})\dot{x}^a\dot{x}^b$ and substituting in Eq. (2.12), we get

$$\frac{1}{2}(\partial_b g_{ae} + \partial_a g_{be})\dot{x}^a\dot{x}^b + g_{ae}\ddot{x}^a - \frac{1}{2}(\partial_e g_{ab}\dot{x}^a\dot{x}^b) = \frac{\ddot{s}}{\dot{s}}\dot{x}^a g_{ae}. \tag{2.13}$$

Now, multiplying Eq. (2.13) by $g^{de}$, we have

$$\ddot{x}^d + \frac{1}{2}g^{de}(\partial_b g_{ae} + \partial_a g_{be} - \partial_c g_{ab})\dot{x}^b\dot{x}^c = \frac{\ddot{s}}{\dot{s}}\dot{x}^d. \tag{2.14}$$

We can see that $\frac{1}{2}g^{de}(\partial_b g_{ae} + \partial_a g_{be} - \partial_c g_{ab}) = \Gamma^d_{bc}$, so we can write above equation as

$$\ddot{x}^d + \Gamma^d_{bc}\dot{x}^a\dot{x}^b = \frac{\ddot{s}}{\dot{s}}\dot{x}^d, \tag{2.15}$$

and relabeling gives

$$\ddot{x}^a + \Gamma^a_{bc}\dot{x}^b\dot{x}^c = \frac{\ddot{s}}{\dot{s}}\dot{x}^a. \tag{2.16}$$

Comparing Eq. (2.16) with Eq. (2.6), we can see that the two are equivalent. For a non null geodesic, an affine parameter $u$ is related to the distance $s$ measured along the curve by $u = as + b$, where $a$ and $b$ are constants and $a$ is non zero.

## 2.1 Behaviour of Timelike Geodesics in the Reisnerr-Nordstrom Metric

In this subsection we review the work of Qadir and Siddiqui [6] on timelike geodesics in the RN black hole spacetime. Consider Lagrangian for the metric (1.103)

$$L = g_{ab}\dot{x}^a\dot{x}^b,$$

where dot represents derivative with respect to $s$. Using Eq. (1.103), we have

$$L = e^{\nu(r)}\dot{t}^2 - e^{-\nu(r)}\dot{r}^2 - r^2\dot{\theta}^2 - r^2\sin^2\theta\dot{\phi}^2. \tag{2.17}$$

Geodesic equations for $t$, $r$, $\theta$ and $\phi$ are

$$\frac{d}{ds}(2e^{\nu(r)}\dot{t}) = 0, \tag{2.18}$$

or

$$e^{\nu(r)}\dot{t} = K, \tag{2.19}$$

$$\ddot{r}e^{-\nu(r)} + v'\dot{t}^2 - e^{-2\nu}v'\dot{r}^2 - r(\dot{\theta}^2 + \sin^2\theta\dot{\phi}^2) = 0, \tag{2.20}$$

$$r^2\ddot{\theta} + 2r\dot{r}\dot{\theta} - r^2\sin\theta\cos\theta\dot{\phi}^2 = 0, \tag{2.21}$$

and

$$\frac{d}{ds}(r^2 \sin^2\theta\dot{\phi}) = 0,$$

or

$$r^2 \sin^2\theta\dot{\phi} = J. \tag{2.22}$$

Here $K$ and $J$ are constants of integration. $K$ and $J$ are related to energy and angular momentum respectively. From Eq. (2.19), we have

$$\dot{t} = Ke^{-\nu(r)}. \tag{2.23}$$

Because of the spherical symmetry of the $RN$ metric, take $\theta = \phi = \frac{\pi}{2}$. And dividing both sides of Eq. (1.103) by $ds^2$, we have

$$1 = e^{\nu(r)}\dot{t}^2 - e^{-\nu(r)}\dot{r}^2. \tag{2.24}$$

Using Eq. (2.23) in Eq. (2.24), to get

$$1 = K^2 e^{-\nu(r)} - e^{-\nu(r)}\dot{r}^2,$$

or

$$\dot{r} = \pm\sqrt{K^2 - e^{\nu(r)}}. \tag{2.25}$$

Using Eq. (2.23) and Eq. (2.25), we get

$$\frac{dr}{dt} = \pm\frac{\sqrt{K^2 - e^{\nu(r)}}}{Ke^{-\nu(r)}}. \tag{2.26}$$

It is clear from above equation that the geodesics will be defined for

$$K^2 \geq e^{\nu}(r).$$

Using re-scaled radial parameter

$$r^* = \int e^{-\nu(r)} dr,$$

the differential equation for the geodesics given by Eq. (2.26) can be written as

$$\frac{dr*}{dt} = K\sqrt{K^2 - e^{\nu(r)}}. \tag{2.27}$$

For RN metric $e^{\nu(r)} = 1 - \frac{2m}{r} + \frac{Q^2}{r^2}$. And considering following three cases

(i) $K = 1$

(ii) $K > 1$

(iii) $K < 1$

**Case 1:** Taking $K = 1$, for an observer falling freely from infinity initially at rest. In this case there is a boundary beyond which geodesics are not defined. This boundary is obtained from Eq. (2.27) at $r = r_b$ as

$$r_b = \frac{Q^2}{2m}. \tag{2.28}$$

These geodesics are shown in Figure 2.1. **Case 2:** For $K > 1$, taking $K^2 = 1 + \epsilon$, geodesics correspond to an observer having positive energy at infinity. In this case geodesics have barrier at

$$r_c = \frac{-m \pm \sqrt{m^2 + \epsilon Q^2}}{\epsilon}. \tag{2.29}$$

Notice that $r_c \approx r_b - \epsilon Q^4/8m^3$. This shows that for an observer with positive energy at infinity size of the barrier is smaller. Behaviour of these geodesics is given in Figure 2.2.

**Case 3:** For $K < 1$, taking $K^2 = 1 - \epsilon$, correspond to an observer having negative energy at infinity. In this case there are two barriers

$$r_\pm = \frac{-m \pm \sqrt{m^2 + \epsilon Q^2}}{\epsilon}. \tag{2.30}$$

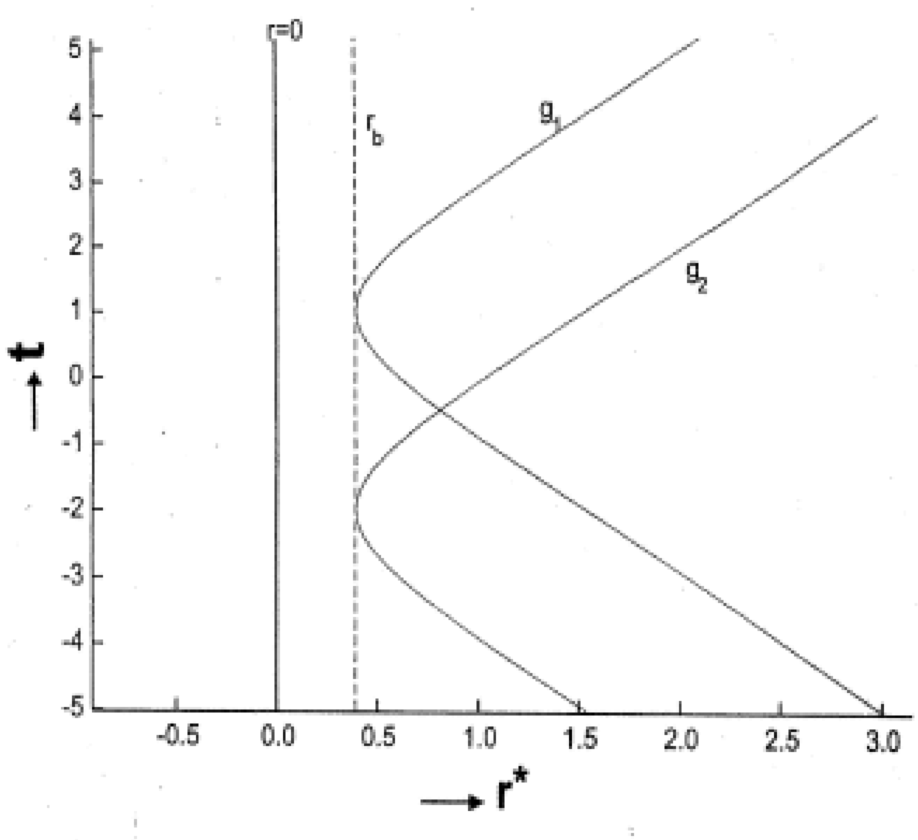


Figure 2.1: Two geodesics, $g_1$ and $g_2$, in $(t, r^*)$ coordinates, for $K = 1$. Both have the same behaviour, coming in, hitting $r = r_b$ and going back. Also note that they are parallel. The only difference between them is the time at which they hit the classical electromagnetic radius, $t = 1$, and $t = 2$. ($Q = 2m$ is taken)

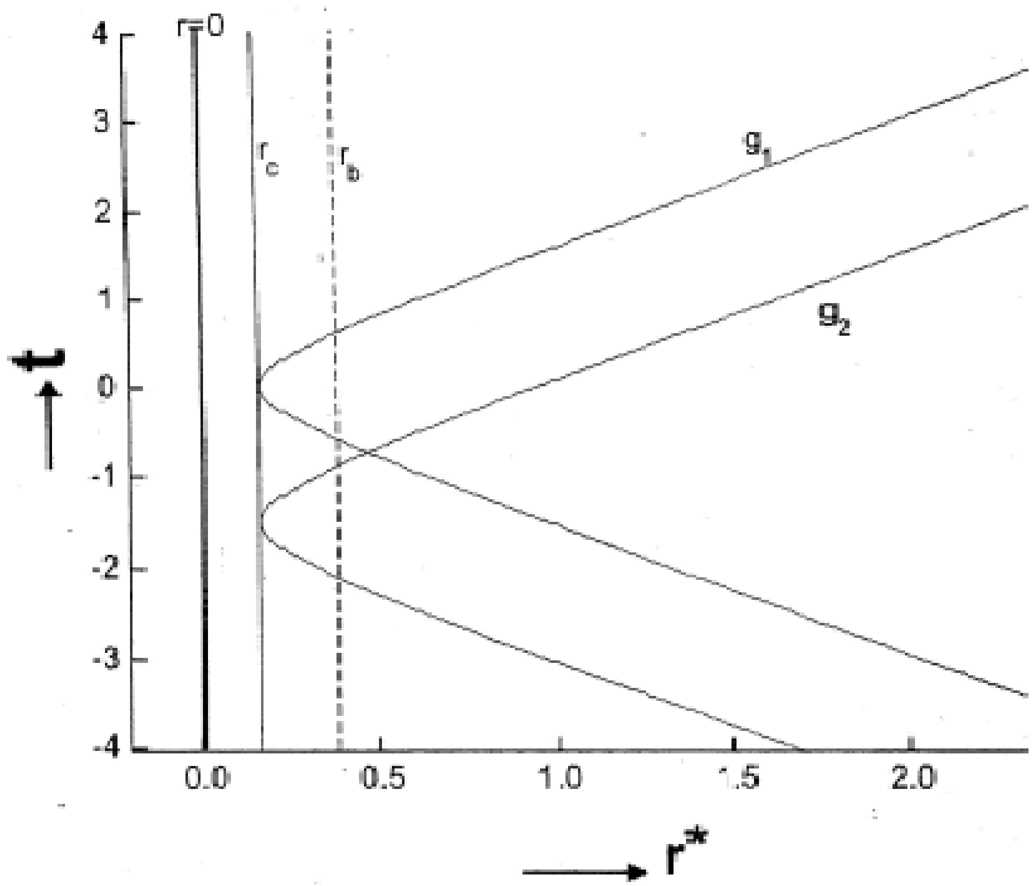


Figure 2.2: Two geodesics in the $(t, r^*)$ coordinates for $K^2 = 1.5$, touching the new barrier, $r_c$, at $t = 0$ for $(g_1)$ and $t = 1.5$ for $(g_2)$. The geodesics are again parallel. Notice that the barrier moves inwards to $r = rc$ compared with $r = r_b$.

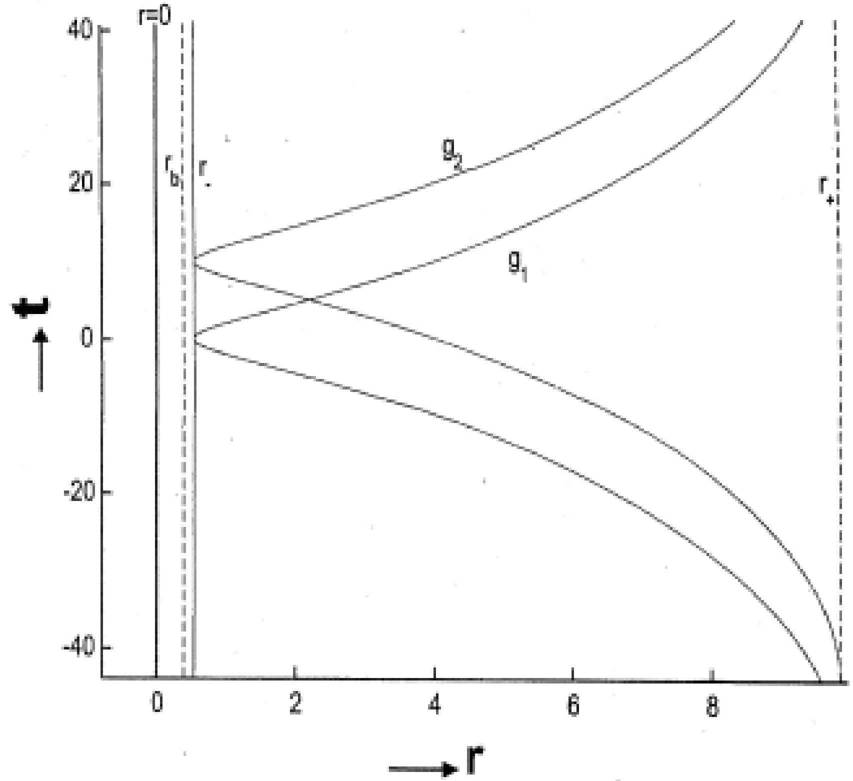


Figure 2.3: Two geodesics in the $(t, r^*)$ coordinates for $K^2 = 0.9$. Now the geodesics start at $r = r_+$ in the infinite past and go in to $r = r$ before going back out to $r = r_+$. We have taken one geodesic, $g_1$, touching the inner boundary at $t = 0$, and the other, $g_2$, at $t = 10$. Notice that the inner boundary lies outside the classical electromagnetic radius.

Here the inner boundary moves forward from $r_b$ to $r_- \approx r_b + \frac{\epsilon Q^4}{8m^3}$. The outer barrier indicates that the observer with negative energy at infinity actually do not reach to infinity. These geodesics are displayed in Figure 2.3.

### 2.1.1 Electron as a Naked Singularity

In gravitational units ($c = G = 1$), $m = 6.8 \times 10^{59}$cm and $Q = 1.4 \times 10^{34}$cm for the electron. As $Q > m$ for electron, it is considered as a naked $RN$ singularity. Of course, the electron should be treated as a Kerr-Newmann (KN) singularity and not a RN singularity. It is seen that $r_b$ is the classical electromagnetic radius, $1.4 \times 10^{13}$cm. This is the radius of the electron that would appear to an observer with zero energy at infinity. However an observer with positive energy would see that $r_b$ shrunk arbitrarily to $r_c$. In the high energy limit, the size decreases to $\frac{Q}{\sqrt{\epsilon}}$.

# Chapter 3

# TimeLike Geodesics in the Kerr-Newmann Spacetime

In Chapter 2, timelike geodesics for the RN black hole have been discussed. It is seen that there is a barrier beyond which these geodesics are not defined. Therefore, there is no timelike path available beyond the barrier. Treating electron as a source of naked RN geometry, it is found that the size of the boundary of timelike geodesics turns out to be the classical electromagnetic radius of the electron [6]. However, it is felt that spin angular momentum of the electron was not considered in this analysis. In order to incorporate the angular momentum also one should consider KN geometry. Here, in this Chapter we calculate the timelike geodesics for the KN black hole and obtain the boundaries for these geodesics.

## 3.1 The Kerr-Newmann Solution

The KN solution, representing the geometry of a charged rotating object, is an exact solution of the Einstein-Maxwell equations. Its line element in Boyer-Lindquist coordinates is written as [17]

$$ds^2 = \frac{\rho^2}{\Delta}dr^2 + \rho^2 d\theta^2 + \frac{\sin^2\theta}{\rho^2}[adt - (r^2 + a^2)d\phi]^2 - \frac{\Delta}{\rho^2}(dt - a\sin^2\theta d\phi)^2, \qquad (3.1)$$

where

$$\rho^2 = r^2 + a^2\cos^2\theta,\ \Delta = r^2 - 2mr + a^2 + Q^2. \qquad (3.2)$$

Here $m$, $Q$ and $a$ are mass, charge and angular momentum per unit mass respectively. For this metric we have two event horizons at

$$r_\pm = m \pm \sqrt{m^2 - Q^2 - a^2}.$$

If the condition $m^2 < a^2 + Q^2$ is satisfied, then no horizons are present and the KN spacetime represents the exterior field of a naked singularity.

## 3.2 Timelike Geodesics in the Kerr-Newmann Spacetime

Lagrangian for the KN metric is

$$L = \frac{\rho^2}{\Delta}\dot{r}^2 + \rho^2\dot{\theta}^2 + \frac{\sin^2\theta}{\rho^2}[a\dot{t} - (r^2 + a^2)\dot{\phi}]^2 - \frac{\Delta}{\rho^2}(\dot{t} - a\sin^2\theta\dot{\phi})^2. \qquad (3.3)$$

As the above Lagrangian (3.3) is independent of $t$, so the Euler Lagrange Eq. (2.8) for $t$ reduces to

$$\frac{d}{ds}\frac{\partial L}{\partial \dot{t}} = 0, \qquad (3.4)$$

or

$$\frac{\partial L}{\partial \dot{t}} = K, \qquad (3.5)$$

where $K$ is a constant of integration. This constant is conserved and corresponds to the energy of the observer at infinity. Putting the value of $L$ (with $\theta = \frac{\pi}{2}$) in Eq. (3.5), we have

$$\frac{1}{r^2}[a^2\dot{t} - a(r^2+a^2)\dot{\phi} - \Delta\dot{t} + a\Delta\dot{\phi}] = K. \tag{3.6}$$

The Euler-Lagrange Eq. (2.8) for $\phi$, gives

$$\frac{d}{ds}\frac{\partial L}{\partial\dot{\phi}} - \frac{\partial L}{\partial\phi} = 0.$$

As $L$ is also independent of $\phi$, so second term is zero and we get

$$\frac{\partial L}{\partial\dot{\phi}} = J,$$

where $J$ is a constant of integration and it is related to angular momentum. Again using $L$ from Eq. (2.8) with $\theta = \frac{\pi}{2}$, we get

$$\frac{1}{r^2}[(r^2+a^2)^2\dot{\phi} - a(r^2+a^2)\dot{t} - \Delta a^2\dot{\phi} + a\Delta\dot{t}] = J. \tag{3.7}$$

Multiplying Eq. (3.6) by $a$ and adding in Eq. (3.7), we get

$$(r^2+a^2)\dot{\phi} - a\dot{t} = aK + J, \tag{3.8}$$

or

$$\dot{\phi} = \frac{aK + J + a\dot{t}}{(r^2+a^2)}. \tag{3.9}$$

Using Eqs. (3.9) and (3.6) and simplifying, we get

$$\dot{t} = \frac{K[r^2(r^2+a^2) + a^2(2mr - Q^2)] - aJ(2mr - Q^2)}{r^2\Delta}. \tag{3.10}$$

Using Eq. (3.10) in Eq. (3.9), we get

$$\dot{\phi} = \frac{aK(2mr - Q^2) + J(r^2 - 2mr + Q^2)}{r^2(r^2 - 2mr + a^2 + Q^2)}. \tag{3.11}$$

For timelike geodesics taking $L = 1$ in Eq. (3.3) and using Eqs. (3.11) and (3.10), we get

$$\dot{r} = \pm\frac{\sqrt{r^2[2mr - Q^2 - J^2 - (1-K^2)(r^2+a^2)] + (J-aK)^2(2mr-Q^2)}}{r^2}. \quad (3.12)$$

From Eqs. (3.12) and (3.10), we finally obtain the required equation for the timelike geodesics as

$$\frac{dr}{dt} = \pm\Delta\frac{\sqrt{(J-aK)^2(2mr-Q^2) + r^2[-J^2 + 2mr - Q^2 - (1-K^2)(r^2+a^2)]}}{K[r^2(r^2+a^2) + a^2(2mr-Q^2)] - aJ(2mr-Q^2)}. \quad (3.13)$$

The geodesic given by the above Eq. (3.13) are defined if the term inside the square root is non-negative, i.e

$$(J-aK)^2(2mr-Q^2) + r^2[2mr - Q^2 - J^2 - (1-K^2)(r^2+a^2)] \geqslant 0. \quad (3.14)$$

Taking $J = a$ in Eq. (3.14), we have

$$a^2(1-K)^2(2r-Q^2) + r^2[-a^2 + 2r - Q^2 - (1-K^2)(r^2+a^2)] \geqslant 0. \quad (3.15)$$

Now, we have three cases $K = 1$, $K > 1$ and $K < 1$.

**Case 1:** For an observer falling freely from infinity initially at rest, taking $K = 1$ in Eq. (3.15), we get

$$r^2(2r - a^2 - Q^2) \geqslant 0.$$

We get barrier at $r = r_b$ as

$$r_b = \frac{a^2+Q^2}{2}. \quad (3.16)$$

Notice that for $a = 0$, this barrier is same as that for the RN case given by Eq. (2.28).

**Case 2:** For $K > 1$, taking $K^2 = 1 + \epsilon$ in Eq. (3.15), we obtain

$$a^2(1-\sqrt{1+\epsilon})^2(2r-Q^2) + r^2(2r - a^2 - Q^2 + \epsilon(r^2+a^2)) \geqslant 0. \quad (3.17)$$

$K > 1$ corresponds to an observer with positive velocity at infinity. Note that if we take $a = 0$ in Eq. (3.17), then we get same result as for RN in Eq. (2.29).

**Case 3:** $K < 1$, corresponds to an observer with negative energy at infinity. we take $K^2 = 1 - \epsilon$ in Eq. (3.15), to have

$$a^2(1 - \sqrt{1+\epsilon})^2(2r - Q^2) + r^2(2r - a^2 - Q^2 - \epsilon(r^2 + a^2)) \geqslant 0. \tag{3.18}$$

Taking $a = 0$ in Eq. (3.18), we get same results as obtained in Eq. (2.30).

## 3.3 Conclusion

The study of black hole spacetime has always been interesting, due to their special geometries. The study of such geometries helps to understand the dynamics of these objects. In this thesis, timelike geodesics in the Kerr-Newmann black hole geometry have been studied.

In chapter one, some basic concepts like, the metric tensor, Christofell symbols, covariant derivative, the Riemann Curvature tensor and other related tensors have been given. In chapter two, some basic concepts and equations related to the timelike geodesics and review of the work by A. Qadir and A.A Siddiqui on timelike geodesics in the RN geometry are given. In their paper, they have shown that timelike geodesics do not go up to the essential singularity. They find that, there is a barrier at

$$r_b = \frac{Q^2}{2m},$$

for freely falling observer initially at rest, at infinity. For an observer with positive energy, at infinity, the geodesics will not turn back at $r_b$, and have barrier at

$$r_c \approx r_b - \epsilon Q^4/8m^3,$$

geodesics for an observer with negative energy, at infinity, have two barriers, $r_-$ and $r_+$.

$$r_- \approx r_b + \frac{\epsilon Q^4}{8m^3}, \tag{3.19}$$

$$r_+ \approx \frac{2m}{\epsilon}. \tag{3.20}$$

In this thesis we have extended their work for the KN spacetime. We have obtained the barrier for an observer, freely falling at rest, with positive and negative energies, at infinity, respectively, at

$$r = \frac{Q^2 + a^2}{2m}, \tag{3.21}$$

$$2r^3 - r^2(a^2 + Q^2 + \epsilon^2 + 2\epsilon) - a^2(\epsilon^2 + 2\epsilon + Q^2\epsilon^2) + 2r\epsilon^2 = 0, \tag{3.22}$$

and

$$2r^3 - r^2(a^2 + Q^2 - 2\epsilon + \epsilon^2) + 2ra^2(\epsilon)^2 + a^2(2\epsilon - \epsilon^2) - \frac{Q^2a^2}{(\epsilon)^2} = 0. \tag{3.23}$$

Notice that the barrier for positive energy at infinity moves closer to the essential singularity, whereas for negative energy at infinity it moves away from the essential singularity and is outside the barrier for the freely falling observer.

**Effect of Rotation:**

The effect of rotation is same as effect of charge in case of RN metric. Greater values of $a$ and $Q$ both move barrier away from the essential singularity. Also notice that all the results for the RN metric can be obtained as a special case of the KN metric by taking $a = 0$.

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