Abelian Sandpile group of Square cycle graph C_n^2

by

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Abstract

Sandpile group of graph G is a finite abelian group it is closely related with the laplacian matrix. In this thesis, we have discussed abelian sandpile model. The structure of square cycle graph C_n^2 is determined and also it is shown that the Smith normal form of sandpile group is always the direct sum of two or three cyclic groups.

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To My Parents

Preface

The Abelian sandpile model could be described informally as a cellular automaton on a graph as follows: The cells of such automata are the vertices of a rooted graph and each cell contains a certain number of grains of sand. The transitions of the automaton follow the toppling rule, which applies to any cell except the root: a cell v_i containing at least as many grains as its degree d_i transfers a grain of sand through each edge to each of its neighbors v_j . After a toppling of the vertex v_i, the number of grains in this cell decreases by its degree, while the number of those neighbors increases by the number of edges that are adjacent. The root r does not topple and could be considered as collecting all the grains leaving the system. If the graph is connected it is easy to see that from any initial configuration the system reaches a stable configuration in which the number of grains in each cell is less than its degree. Dhar shows also that some configurations, the so-called recurrent configurations, play an important role and possess some interesting mathematical properties: they form a finite abelian group (called the sandpile group) whose order is equal to the number of spanning trees of the graph.

The thesis has four chapters. Chapter one covers the introduction and basic about groups and finitely generated abelian groups and also discusses some results related to these.

Chapter two is on Graph theory. Here we define the concepts related to the Graph theory and also provide different representation of Graph.

Chapter three is on the literature review in which we define the concept related to Abelian Sandpile Model. Here we also discuss The Smith normal form of a matrix.

Chapter four gives the review of [14] in which the structure of square cycle graph C_n^2 is determined. The aim is to compute the structure of sandpile group by determine its smith normal form . In first section we define square cycle graph. In second section using toppling rule we find the system of relation for the generators of $S(C_n^2)$ and also it is shown that there are at most three generators for $S(C_n^2)$. In Next section relation matrix between these generators is calculated we also prove proposition and then some well-known identities of Fibonacci number are listed in remark which we will use later. Next there is a theorem in which we prove using row and column operation that the relation matrix between the generators of abelain sandpile group on C_n^2 is equivalent to matrix A_n. In last section we prove some lemma which will help us in finding the coefficient of Smith normal form of C_n^2 and then using these result we compute the smith normal of relation matrix and prove that $S(C_n^2) \cong \mathbb{Z}_{(n,F_n)} \oplus \mathbb{Z}_(F_n) \oplus \mathbb{Z}_{\frac{nF_n}{(n,F_n)}}$.

Contents

Chapter 1

Group Theory

1.1 Introduction

In this chapter, basic concepts of group theory are given. Especially abelian group and finitely genertared abelian group. These definition and results will be used in the results of next chapters.

Definition 1.1.1. A set A under a binary operation * is called a group if it satisfies following axioms:

- * is associative, that is

$$(a * b) * c = a * (b * c),\ for\ all\ a,\ b,\ c \in A.$$

- There exists $e \in A$ such that for all $a \in A$

$$e * a = a * e = a,$$

  and e is called an identity element in A.

- For each $a \in A$ there exists $a^{'} \in A$ such that

$$a * a^{'} = a^{'} * a = e,$$

  $a^{'}$ is called an inverse of a.

A group G under the binary relation $*$ is denoted by $(G, *)$. The number of elements in a group is called order of group if the number of element in G is finite then group is called finite otherwise infinite.
A group G is called abelian if G is commutative under binary operation * that is

$$\text{for all}\ \ a,\ b \in G, \quad a * b = b * a.$$

Example 1.1.1. $(\mathbb{R},+)$, $(\mathbb{Q},+)$, $(\mathbb{Z},+)$, $(\mathbb{C},+)$ *are groups under the ordinary addition of numbers.*

Example 1.1.2. *The set of all* $m\times n$ *matrices with real entries denoted by* $M_{m\times n}(\mathbb{R})$ *is a group under the matrix addition.*

Example 1.1.3. *The group of integer modulo* n *denoted by* $\mathbb{Z}_n = \{0,1,2\cdots,n-1\}$ *under the operation of addition modulo* n *is an abelian group.*

Definition 1.1.2. Let G be a group, a non empty subset K of G is called a subgroup of G, if K itself is a group under the same binary operation of G.

Proposition 1.1.1. *[12]. A non empty subset* K *of a group* G *is a subgroup iff*

$$\text{for all } a,b \in K, \quad ab^{-1}\in K.$$

Example 1.1.4. *Let* n *be an integer then,* $n\mathbb{Z} = \{nr : r\in\mathbb{Z}\}$ *is a subgroup of* $(\mathbb{Z},+)$.

$$(n\mathbb{Z},+) < (\mathbb{Z},+) < (\mathbb{Q},+) < (\mathbb{R},+) < (\mathbb{C},+)$$

Definition 1.1.3. A group G, is cyclic if it can be generated by a single element say $a\in G$ that is,

$$G = \langle a \rangle = \{a^n | n\in\mathbb{Z}\}.$$

All subgroups of a cyclic group are also cyclic. Cyclic groups are abelian group.

Example 1.1.5. $\mathbb{Z}$ *is cyclic because* $\mathbb{Z} = \langle 1 \rangle$, $\mathbb{Z}_n$ *is also cyclic as* $\mathbb{Z}_n = \langle 1 \rangle$.

Corollary 1.1.1. *[12]. All groups of prime order are cyclic.*

Theorem 1.1.1. *[12]. The order of cyclic group is equal to the order of its generator.*

Definition 1.1.4. Let G be a group and H be a subgroup of G for an element $a\in G$, we define a subset of G

$$aH = \{ah : h\in H\},$$

this subset aH of G is called left coset of G, and the subset

$$Ha = \{ha : h\in H\},$$

is called right coset of G.

Remark 1.1.1. *[12]. Let* G *be a group and* H *be a subgroup of* G*, then*

- *If $a \in H$ then $aH = H = Ha$.*
- *Usually $aH \neq Ha$ but when G is an abelian group the left and right coset of a subgroup by a same element are equal.*
- *A subgroup H is always a left and right coset of itself $eH = H = He$.*

Lemma 1.1.1. *[12]. Let G be a group for $g \in G$, $gH = H$ iff $g \in H$.*

Theorem 1.1.2. *[12]. Consider G be a group and H be any subgroup then the set of all left or right cosets of H in G defines a partition of G.*

Definition 1.1.5. Let H be a subgroup of G. The index of H in G is the number of distinct left or right cosets of H in G. This number, which is positive and sometime may be infinite is denoted by $[G : H]$.

Example 1.1.6. *The index of subgroup $(n\mathbb{Z}, +)$ in $(\mathbb{Z}, +)$ is n.*

Example 1.1.7. *The index of $(\mathbb{Z}, +)$ in $(\mathbb{R}, +)$ is infinite.*

Theorem 1.1.3. *[12] Let H be a subgroup of a group G, then $o(G) = o(H)[G : H]$. In particular, if G is finite then $o(H)$ divides $o(G)$ and*

$$[G : H] = \frac{o(G)}{o(H)}.$$

Corollary 1.1.2. *[12]. If G is finite group and $g \in G$, then the order of g divides the order of G.*

Definition 1.1.6. Let G be a group and H be a subgroup of G if $Ha = aH$ for all $a \in G$ then H is called normal subgroup G. Symbolically,

$$H \trianglelefteq G.$$

G and $\{e\}$ are improper normal subgroup of G.

Remark 1.1.2. *[12]. If G is an abelian group then its any subgroup H of G will be normal as,*

$$\text{for all } g \in G, \quad gH = Hg.$$

Definition 1.1.7. Let G be a group the center of G is the set

$$Z(G) = \{x \in G : gx = xg, \quad \text{for all } g \in G\}.$$

It is an abelian subgroup of G and clearly $Z(G)$ is a normal subgroup of G.

Proposition 1.1.2. *[12]. Let G be a group and H be a subgroup of index 2 in G then H must be normal in G.*

Definition 1.1.8. Let G be a group and fix $g \in G$ the element gxg^{-1} is called conjugate of x by g and the set $gHg^{-1} = \{gHg^{-1} : h \in H\}$ is the conjugate of H by g, where H is the subgroup of G.

Theorem 1.1.4. *[12] The subgroup H of G is normal in G if and only if*

$$g^{-1}Hg = H, \quad \textit{for all } g \in G.$$

Definition 1.1.9. Let H be a normal subgroup of G. The set $G/H = \{aH \mid a \in G\}$ the set of left cosets becomes a gruop under well defined coset multiplication,

$$(aH)(bH) = (ab)H.$$

In this case G/H is called factor or quotient group of G by H.

The order of above factor or quotient group of G by H is $o(G/H) = [G : H]$.

Example 1.1.8. *The group $(\mathbb{Q}, +)$ has normal subgroup $(\mathbb{Z}, +)$ then $(\mathbb{Q}/\mathbb{Z}, +)$ is a factor group under the operation*

$$(a + \mathbb{Z}) + (b + \mathbb{Z}) = (a + b) + \mathbb{Z}, \quad \textit{for all } a, b \in \mathbb{Q}.$$

Remark 1.1.3. *[12]. If G is an abelian group and G/H be a factor group of G by H, then G/H is also abelian.*

Theorem 1.1.5. *[12] Every quotient group of a cyclic group is cyclic.*

1.2 Group Isomorphism

In this section we define the idea of two groups having same structure by concept of isomorphism. Isomorphism of two groups means that elements of one group can be renammed as the element of other group.

Definition 1.2.1. The homomophrism between groups $(G, *)$ and $(H, \bullet)$ is a mapping or a function $\phi : G \longrightarrow H$ such that

$$\phi(a * b) = \phi(a) \bullet \phi(b), \quad \text{for all} \quad a, b \in G,$$

if this map is one to one then it is called a monomorphism. If this map is onto then it is called an epimorphsim and if this map is one to one and also onto then ϕ is called an isomorphism, then we say G is isomorpic to H and we write

$$(G, *) \cong (H, \bullet).$$

Isomorphism is an equivalence relation. If G, H, K are groups then

- $G \cong G$.
- If $G \cong H$ then $H \cong G$.
- If $G \cong H$ and $H \cong K$ then $G \cong K$.

Theorem 1.2.1. *[12]. Consider G be cyclic group if*

- *If $o(G) = n$, then $G \cong \mathbb{Z}_n$.*
- *If $o(G) = \infty$, then $G \cong \mathbb{Z}$.*

So, the complete list of cyclic groups is $\{0\}, \mathbb{Z}_2, \mathbb{Z}_3, \mathbb{Z}_4, \ldots, \mathbb{Z}$ in this list no any two cyclic groups are isomorphic because these all are of different order.

Example 1.2.1. *The group of real number under addition is isomorpic to the group of positive real number under multiplication under the isomorphisim $f(x) = e^x$.*

Definition 1.2.2. Let ϕ: $G \to H$ be a homomorphism. If e_H is an identity element in H then kernel is a subset of G and it contains all the elements of G which are mapped to e_H,

$$ker\ \phi = \{x \in G : \phi(x) = e_H\},$$

kernel is the normal subgroup of G.

Theorem 1.2.2. *[12]. **(The first isomorphism theorem)** Let $f : G \to G'$ be a group homoorphism and let $K = ker f$, then $G/K \cong f(G)$.*

Theorem 1.2.3. *[12]. **(The second isomorphism theorem)***
Let G be a group, let A and B be subgroups of G and assume $A \leqslant N_G(B)$ then AB is a subgroup of G and $B \trianglelefteq AB$, $(A \cap B) \trianglelefteq A$, $AB/B \cong \frac{A}{A \bigcap B}$

Theorem 1.2.4. *[12]. **(The third isomorphism)***
Let G be a group and let H and K be normal subgroups of G with $H \leqslant K$ then $K/H \triangleright G/H$ and

$$G/H/K/H \cong G/K.$$

1.3 Internal direct product

Definition 1.3.1. Consider H and K are two subgroup of a group G by HK we mean the set $\{hk : h \in H, k \in K\}$ and it is not necessary that HK is also a subgroup of G.

Example 1.3.1. *Let* $G = S_3$ *,*$H =< (1,2) >$*,* $K =< (1,3) >$*.Then* $HK = \{(1),(1,2),(1,3),(1,3,2)\}$ *which is not subgroup of* G*.*

Proposition 1.3.1. *[12]. Consider* H, K *are two subgroups of a group* G *and assume that* H, K *are normal in* G *then* HK *is a subgroup of* G*.*

Definition 1.3.2. Let G be a group and N, N' be subgroups of G then G is termed the internal direct product of N, N' if the following conditions hold,

- N, N' are normal in G
- The intersection of N and N' is trivial that is $N \cap N' = \{e\}$
- The product of N, N' is equal to G, that is $G = NN'$.

Theorem 1.3.1. *[12]. Consider* G *be a group and* $H_i \trianglelefteq G$ *for* $i \in \{1, \ldots, n\}$ *then* G *is the internal direct product of* $H_1, H_2, \ldots, H_i$ *if*

- $G = H_1 H_2 H_3 \cdots H_n$
- $(H_1 H_2 \cdots H_i) \cap H_{i+1} = \{e\}$.

1.4 External direct product of groups

Definition 1.4.1. External direct product of the groups $G_1,\ G_2,\ ,\ \ldots,\ G_n$ is the set of all n-tuples

$$\{(g_1,\ g_2,\ \ldots, g_n)\ \mid\ g_i\ \in G_i\}\,.$$

External direct product of $G_1,\ G_2,\ \ldots,\ G_n$ is denoted by $G_1 \oplus G_2 \oplus \cdots \oplus G_n$. It is also a group under the comopnentwise operation.

If $G_1, G_2, \ldots, G_n$ are finite and $o(G_1) = m_1,\ o(G_2) = m_2$, $o(G_n) = m_n$ then $o(G_1 \times G_2 \ldots \times G_n) = m_1 m_2 \ldots m_n$. Clearly $o(G_1 \times G_2 \ldots \times G_n)$ is finite.

Example 1.4.1. $\mathbb{Z}_2 = \{0,1\}$ *is an abelian group.* $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ *is also an abelian group of order* 4*. Its four elements are* $(0,0),(0,1),(1,0),(1,1)$*.*

Proposition 1.4.1. *[12]. Let G be a group and suppose G is the internal direct product of $H_1, H_2, \ldots, H_n$. Let T be the external direct product of $H_1, H_2, \ldots, H_n$ then G and T are isomorphic.*

Proposition 1.4.2. *[12]. Suppose $G_1, G_2, \ldots G_k$ are groups and $G_1 \oplus G_2 \oplus \ldots \oplus, G_k$ be the direct sum of these groups, then for $(g_1, g_2, \ldots, g_k) \in G_1 \oplus G_2 \oplus \cdots \oplus G_k$*

$$o(g_1, g_2, \ldots, g_k) = lcm\left\{o(g_1), o(g_2), \ldots, o(g_k)\right\}, \textit{where } o(g_i) < \infty \textit{ for all } i.$$

Lemma 1.4.1. *[12]. Let G and H be groups of finite order such that $G \oplus H$ is cyclic then G and H are cyclic and $o(G)$ and $o(H)$ are relatively prime.*

Definition 1.4.2. Let M be any subset of a group G.

$$< M >= \bigcap_{M \subseteq H} H, \qquad \text{where } H \text{ is subgroup of } G.$$

Then $< M >$ is called a subgroup of G generated by M.

Definition 1.4.3. Consider M be any subset of G then we define a set $\overline{M}$ by

$$\overline{M} = \left\{m_1^{\delta} m_2^{\delta} \cdots m_n^{\delta} \quad | \; n \in \mathbb{Z}, \;\; n \geq 0, \;\; m_i \in M \quad \delta_i = \pm 1, \;\; \text{for each } \; i\right\}$$

$\overline{M}$ is the set of all finite products of elements of M and inverse of elements of M. If $M = \phi$ then $\overline{M} = 1$

Proposition 1.4.3. *[12]. Let M be a non empty subset of a group G, and let*

$$\overline{M} = \left\{m_1^{\delta} m_2^{\delta} \cdots m_n^{\delta} \quad | \; n \in \mathbb{Z}, \quad n \geq 0, \quad m_i \in M \quad \delta_i = \pm 1 \;\; \textit{for each } \; i\right\} \textit{ That is,}$$
$$\overline{M} = \left\{m_1^{\delta} m_2^{\delta} \cdots m_n^{\delta} \quad | \; n \in \mathbb{Z}, \quad n \geq 0 \quad \textit{and } \; \delta_i \in \mathbb{Z}\right\}$$

and $< M >$ is the subgroup of G generated by M then $< M >= \overline{M}$.
If M is finite and $< M >= G$, then we say that G is finitely generated.

1.5 Finitely generated abelian group

An abelian group G is finitely generated if for all $x \in G$ there are elements $x_1, \ldots, x_n$ belongs to G such that x can be written as a linear combination of these x_i $i \in \{1, \ldots, n\}$

$$x = a_1 x_1 + a_2 x_2 + \ldots + a_n x_n, \qquad a_i \in \mathbb{Z}$$

and we say $\{x_1, x_2, \ldots, x_n\}$ is a generating set of G or $x_1, \ldots, x_n$ generate G.

Remark 1.5.1. *[12]. Let G be a group*

- *If G is finite then it is clearly finitely generated.*
- *If G is cyclic group then it is finitely generated. As cyclic groups are generated by single element. So $\mathbb{Z}$ and $\mathbb{Z}_n$ are finitely generated.*
- *Finitely generated groups are always countable.*

Theorem 1.5.1. *[12]. (Fundamental theorem of finitely generated abelian group) Let G be a finitely generated abelian group then it is isomorphic to the one of the form*

$$G \cong \mathbb{Z}^r \oplus \mathbb{Z}_{n_1} \oplus \mathbb{Z}_{n_2} \oplus \cdots \oplus \mathbb{Z}_{n_s}, \tag{1.1}$$

for some integers $r, n_1, n_2 \cdots, n_s$ satisfy the given conditions.

- *$r \geq 0$ and $\quad n_j \geq 2$ for all j.*
- *$n_{i+1} | n_i$ for $\quad 1 \leq i \leq s-1$.*
- *Moreover the representation of G in (1.1) is unique.*

Definition 1.5.1. To descirbe G in form of Theorem 1.5.1 is known as invariant factor decomposition of G, where integer r is called free rank or betti number of G and integers $n_1, \ldots, n_s$ are called invariant factors of G.

Remark 1.5.2. *The free rank is zero if and only if the finitely generated abelian group is finite.*
If G is finite then product of invariant factors is equal to the order of finitely generated abelian group.

Example 1.5.1. *Consider a group G to be an abelian group of order 180 then its all non isomorphic groups are $\mathbb{Z}_{180}$, $\mathbb{Z}_{90} \oplus \mathbb{Z}_2$, $\mathbb{Z}_{60} \oplus \mathbb{Z}_3$, $\mathbb{Z}_{30} \oplus \mathbb{Z}_6$.*

Proposition 1.5.1. *[12]. If $GCD(m, n) = 1$ then $\mathbb{Z}_m \oplus \mathbb{Z}_n \cong \mathbb{Z}_{mn}$*

Theorem 1.5.2. *[12]. (Primary decomposition theorem)*
Let G be a finitely generated abelian and of order $n > 1$, factorize n into the distinct prime powers and this factorization must be unique, $n = p_1^{\alpha_1} \cdots p_k^{\alpha_k}$, then

$$G \cong A_1 \oplus A_2 \oplus \cdots \oplus A_k, \quad where \quad |A_i| = P_i^{\alpha_i}. \tag{1.2}$$

Moreover, for each $A \in \{A_1, \cdots, A_k\}$, with $|A| = P^{\alpha}$,

$$A \cong \mathbb{Z}_{p^{\beta_1}} \oplus \cdots \oplus \mathbb{Z}_{p^{\beta_t}} \quad for \quad some \quad partition \quad \beta_1 \geq \cdots \geq \beta_t \geq 1 \quad of \ \alpha.$$

Finally these decompositions are unique.

Definition 1.5.2. In theorem 1.5.2 the integer P^{β_j} are called elementary divisor of G and to describe G in 1.2 is known as elementary divisor decomposition of G.

Example 1.5.2. $\mathbb{Z}_2 \oplus \mathbb{Z}_3 \cong \mathbb{Z}_6$ *as* $GCD(2,3) = 1$

1.6 Generators and relations

In abstract algebra, there is small part of theory of abelian groups but an important part and has important features. One of the most important features is underlying patterns. One way to define an abelian group is to construct a group table and through this we can find the sum of two elements of group without having to be bothered what element actullay are. It is cleared that it is not possible to construct the table in case of infinite group. One way to define abelian groups absractly the another way is to use of generators and relations.
Consider $x_1, x_2 \ldots, x_n$ are n symbols, and representing nothing in particular. The expression given below known as formal linear combination of x_i's ,

$$\alpha_1 x_1 + \alpha_2 x_2 + \ldots + \alpha_n x_n, \text{ where the coefficients } \alpha_i \in \mathbb{Z}.$$

If two formal linear combinations of these x_i are given say $\alpha_1 x_1 + \alpha_2 x_2 + \ldots + \alpha_n x_n$ and $\beta_1 x_1 + \beta_2 x_2 + \ldots + \beta_n x_n$ then we can add these two by common way that is

$$(\alpha_1 x_n + \ldots + \alpha_n x_n) + (\beta_1 x_1 \ldots + \beta_n x_n) = (\alpha_1 + \beta_1)x_1 + \ldots + (\alpha_n + \beta_n)x_n.$$

So by adding these two formal linear combinations of x_i we get another formal linear combination of x_i. So set of all formal linear combinations of x_i is closed under addition. As the coefficients of these formal linear combinations belong to integers so, associative law holds. $0x_1 + 0x_2 + \ldots + 0x_n$ is identity formal linear combination. For $\alpha_1 x_1 + \alpha_2 x_2 + \ldots + \alpha_n x_n$ the inverse formal linear combinations is $(-\alpha_1)x_1 + (-\alpha_2)x_2 + \ldots + (-\alpha_n)x_n$. So set of all formal linear combination of $x_1, x_2 \ldots, x_n$ is a group and is denoted by $< x_1, x_2, \ldots, x_n >$ and as the coefficient α_i's belongs to integers so this group is also an abelian group.

Example 1.6.1. $< x_1, x_2, x_3 >= \{mx_1 + nx_2 + qx_3 : \quad m, n, q \in \mathbb{Z}\}$ *is a group. It is clear that* $< x_1, x_2, x_3 > \cong \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}$ *under the isomorphism* $\phi(mx_1 + ny_1 + qz_1) = (m, n, q)$.
Similarly if we have more gernators then these are isomorphic to more copies of $\mathbb{Z}$.

Consider a finite set of these formal linear combinations and put all these linear combinations equal to zero

$$\begin{aligned} \alpha_{11}x_1 + \ldots + \alpha_{1n}x_n &= 0 \\ \alpha_{21}x_1 + \ldots + \alpha_{2n}x_n &= 0 \\ \ldots\ldots\ldots\ldots\ldots\ldots & \\ \ldots\ldots\ldots\ldots\ldots\ldots & \\ \ldots\ldots\ldots\ldots\ldots\ldots & \\ \alpha_{m1}x_1 + \ldots + \alpha_{mn}x_n &= 0 \end{aligned}$$

from these relations we have to identify how many formal linear equations in group to one another.

Example 1.6.2. *Consider two generators, one is* $3x + 5y = 0$ *then second is* $7x + 10y = 0$*, now add these two relation* $7x + 10y + (3x + 5y) = 10x + 15y$ *but it is not necessary* $10x + 15y = 0$ *that* $2x + 3y = 0$*, because division is not possible as this work is on integers while addition and subtraction is possible.*
From $3x + 5y = 0$*, we can write* $3x = -5y$*. So in all formal linear combinations we can replace* $3x$ *by the multiple of* y*, so abelian can be split into disjoint subsets that are:*

$$\{ay \mid a \in \mathbb{Z}\}, \{x + ay \mid a \in \mathbb{Z}\}, \{2x + ay \mid a \in \mathbb{Z}\}.$$

These subsets of abelian group are disjoint.

So if $R_1 = R_2 = \cdots = R_n = 0$ are given relations so the abelian group generated by $x_1, x_2, \cdots, x_n$ subject to these relations denoted by

$$< x_1, x_2, \ldots, x_n \mid R_1, R_2, \ldots, R_n >$$

Example 1.6.3. *Consider* $< x \mid 6x = 0 >$*. A linear combination of one generators is same as multiple of this generator that is why the element of this group must be of form* nx *where* $n \in \mathbb{Z}$ *and distinct elements are* $0,\ x,\ 2x,\ 3x,\ 4x,\ 5x$ *as* $6x = 0$ *and no other element of this group for example if we have* $7x$ *then* $7x = x+6x = x+0 = x$ *so* $7x \equiv x$ *so this group is same as the group of integer mod 6 so,* $< x \mid 6x = 0 >\cong \mathbb{Z}_6$*.*

Example 1.6.4. $< x, y \mid 2x = 0\ , 5y = 0 > \ \cong \mathbb{Z}_2 \oplus \mathbb{Z}_5$*.*
$< x, y \mid 2x = 0\ , 5y = 0 >= \{0, y, 2y, 3y, 4y, x + y, x + 2y, x + 3y, x + 4y\}$*. So* $< x, y \mid 2x = 0\ , 5y = 0 >\cong \mathbb{Z}_2 \oplus \mathbb{Z}_5$ *under the map* ϕ *which is defined as* $\phi(mx + ny) = (m, n)$*.*

1.6.1 Relation Matrices.

If one or two generators with one or two relations are given then it is easy to find its isomorphic group which is also abelian but in case when we have given m relations in n variable it is difficult. In this case we just manipulate their coefficients, as a row of a matrix rather than manipulate the relation. So if we have m relations with n variables we associate a matrix $m \times n$ whose rows are coefficients of successive relation and this matrix is known as relation matrix.
Consider A is $m \times n$ integers matrix, $[A] = [\alpha_{ij}]$ then $[A]$ denotes the abelian group,

$$< x_1, \ldots, x_n \mid \alpha_{11}x_1 + \ldots + \alpha_{1n}x_n = 0, \ldots, \alpha_{m1}x_1 + \ldots + \alpha_{mn}x_n = 0 > . \tag{1.3}$$

If we change the name of variables in above abelian group for example

$$< y_1, \ldots, y_n \mid \alpha_{11}y_1 + \ldots + \alpha_{1n}y_n = 0, \ldots, \alpha_{m1}y_1 + \ldots + \alpha_{mn}y_n = 0 > . \tag{1.4}$$

So (1.3) and (1.4) are isomorphic.

Example 1.6.5. *The set of relations:*

$$4x + 3y + 2z = 0, \quad 2x + 9z = 0,$$

can be written as, $\begin{pmatrix} 4 & 3 & 2 \\ 2 & 0 & 9 \end{pmatrix}$.

Example 1.6.6. *Let* $< x, y \mid 2x + 4y = 0, 8x + 3y = 0 > \; = \begin{bmatrix} 2 & 4 \\ 8 & 3 \end{bmatrix}$.
If we change C_1 *by* C_2 *then* $\begin{bmatrix} 2 & 4 \\ 8 & 3 \end{bmatrix} \cong \begin{bmatrix} 4 & 2 \\ 3 & 8 \end{bmatrix}$ *by changing two columns mean changing two variables, in one group* $2x + 4y = 0$ *and in second group* $4x + 2y = 0$ *and this give a group that is isomorphic to the original one. So*

$$\begin{bmatrix} 2 & 4 \\ 8 & 3 \end{bmatrix} =< x, y \mid 2x + 4y = 0, 8x + 3y = 0 >\cong \begin{bmatrix} 4 & 2 \\ 3 & 8 \end{bmatrix} =< x, y \mid 4x + 2y = 0, 3x + 8y = 0 > .$$

under the map $\phi(ax + by) = bx + ay$.

By above example it is clear that by changing a row or column in relation matrix we get the group that is isomorphic to the original one.

Similarly in relation matrix, by adding or subtracting the integer multiple of one row (column) to another row (column) we get the group that is isomorphic to the original group.

Example 1.6.7. *Let* $\begin{bmatrix} 3 & 7 \\ 2 & 4 \end{bmatrix}$, *if we add the 10 times the second column to the first then*

$$\begin{bmatrix} 3 & 7 \\ 2 & 4 \end{bmatrix} \cong \begin{bmatrix} 73 & 7 \\ 42 & 4 \end{bmatrix}$$

Let us see how, $\begin{bmatrix} 3 & 7 \\ 2 & 4 \end{bmatrix} =< x, y \mid 3x + 7y = 0, 2x + 4y = 0 >$ *now, let* $z = y - 10x$ *or* $y = z + 10x$ *put* $y = z + 10x$ *in formal linear combination of* x *and* y

$$3x + 7y = 3x + 7(z + 10x) = 73x + 7z = 0$$
$$2x + 4y = 2x + 4(z + 10x) = 42x + 4z = 0$$

$\begin{bmatrix} 3 & 7 \\ 2 & 4 \end{bmatrix} \cong< x, y \mid 3x + 7y = 0, 2x + 4y = 0 >\cong< x, z \mid 73x + 7z = 0, 42x + 4z = 0 >\cong< x, y \mid 73x + 7y = 0, 42x + 4y = 0 >\cong \begin{bmatrix} 73 & 7 \\ 42 & 4 \end{bmatrix}$.

So abelian groups of $\begin{bmatrix} 3 & 7 \\ 2 & 4 \end{bmatrix}$ *and* $\begin{bmatrix} 73 & 7 \\ 42 & 4 \end{bmatrix}$ *are identical.*

So in relation matrix we can interchange a row or column also can multiply a row(column) by -1 and can add or subtract the integer multiple of row(column). So elementary row operations are

Elementary integers row(column) operation.

- $R_i \leftrightarrow R_j$ (Interchange row i with j).
- $R_i \to -R_i$ (Change the sign of row i).
- $R_i \pm kR_j$ (Add or subtract the k times row j from row i).

Theorem 1.6.1. *Let a matrix* P *which is obtained by applying the row and column operations on matrix* Q *then* $[P] \cong [Q]$.

So we can associate an abeilan group, which is given in terms of finitely many generators and relations, by a matrix, by simplifying the matrix using above elementary rows and column operation.

Example 1.6.8. *Let us simplify* $\begin{bmatrix} 3 & 7 \\ 2 & 4 \end{bmatrix}$

$$\begin{aligned}\begin{bmatrix}3 & 7\\ 2 & 4\end{bmatrix} &\cong \begin{bmatrix}1 & 3\\ 2 & 4\end{bmatrix} \quad C_1 \leftrightarrow C_2\\ &\cong \begin{bmatrix}1 & 1\\ 2 & 0\end{bmatrix} \quad C_2 - 2C_1\\ &\cong \begin{bmatrix}1 & 0\\ 0 & 2\end{bmatrix} \quad C_1 \leftrightarrow C_2\\ &\cong [2] \cong \mathbb{Z}_2\end{aligned}$$

In Theorem 1.2.1 complete list of cyclic groups are given so if relation matrix R is diagonal then the abelian group $[R]$ is the direct sum of cyclic groups.

Example 1.6.9. $\begin{bmatrix}2 & 0 & 0\\ 0 & 3 & 0\\ 0 & 0 & 5\end{bmatrix} \cong \mathbb{Z}_2 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_5$

If one of the diagonal entry is 1 it represent a variable that is equated to zero. So it can be ignored by removing the row and column of that entry.

Example 1.6.10. $\begin{bmatrix}2 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 5\end{bmatrix} \cong \mathbb{Z}_2 \oplus \mathbb{Z}_5$ *we ignore* R_2 *and* C_2 *because diagonal entry* 1 *lies in it.*

If the rows are more than columns in a diagonal matrix then one or more columns must be zero at the right end. Each one represents a variables that enter into no relation and contibute a summand of $\mathbb{Z}$ to the direct sum decomposition.

Example 1.6.11. $\begin{bmatrix}2 & 0 & 0 & 0 & 0\\ 0 & 3 & 0 & 0 & 0\\ 0 & 0 & 5 & 0 & 0\end{bmatrix} \cong \begin{bmatrix}2 & 0 & 0\\ 0 & 3 & 0\\ 0 & 0 & 5\end{bmatrix} \oplus \mathbb{Z} \oplus \mathbb{Z} \cong \mathbb{Z}_2 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_5 \oplus \mathbb{Z} \oplus \mathbb{Z}$

In case when the matrix has more columns then rows then by using elementary row (column) operations we can get one column $\begin{pmatrix}m\\ 0\\ \cdots\\ 0\end{pmatrix}$ thus it will give $\mathbb{Z}_m$ if $m > 0$ nothing if $m = 1$ and $\mathbb{Z}$ if $m = 0$ for example $[4] \cong \mathbb{Z}_4, [1] \cong [0], [0] \cong \mathbb{Z}$

Example 1.6.12. $< \alpha, \beta, \gamma \mid 4\alpha - \beta + 5\gamma = 0, 14\alpha + 7\beta + 7\gamma = 0 >$ *the relation matrix is* $\begin{bmatrix}4 & -1 & 5\\ 14 & 7 & 7\end{bmatrix}$.

$$
\begin{aligned}
\begin{bmatrix} 4 & -1 & 5 \\ 14 & 7 & 7 \end{bmatrix} &\cong \begin{bmatrix} -1 & 4 & 5 \\ 7 & 14 & 7 \end{bmatrix} C_1 \leftrightarrow C_2 \\
&\cong \begin{bmatrix} 1 & 4 & 5 \\ -7 & 14 & 7 \end{bmatrix} - C_1 \\
&\cong \begin{bmatrix} 1 & 4 & 5 \\ 0 & 42 & 42 \end{bmatrix} 7R_1 + R_2 \\
&\cong \begin{bmatrix} 1 & 0 & 0 \\ 0 & 42 & 42 \end{bmatrix} C_2 - 4C_1 \ \textit{and}\ C_3 - 5C_1 \\
&\cong \begin{bmatrix} 1 & -0 & 0 \\ 0 & 42 & 42 \end{bmatrix} C_3 - C_2 \\
&\cong \mathbb{Z}_{42} \oplus \mathbb{Z}.
\end{aligned}
$$

Example 1.6.13.

$$
\begin{aligned}
\textit{Let} \quad \begin{bmatrix} 2 & 2 & 2 \\ 0 & 8 & 0 \\ 0 & 0 & 8 \\ 8 & 0 & 0 \end{bmatrix} &\cong \begin{bmatrix} 2 & 2 & 2 \\ 0 & 8 & 0 \\ 0 & 0 & 8 \\ 0 & -8 & -8 \end{bmatrix} \\
&\cong \mathbb{Z}_2 \oplus \begin{bmatrix} 8 & 0 \\ 0 & 8 \\ -8 & -8 \end{bmatrix} \\
&\cong \mathbb{Z}_2 \oplus \begin{bmatrix} 8 & 0 \\ 0 & 8 \\ 0 & -8 \end{bmatrix} \\
&\cong \mathbb{Z}_2 \oplus \mathbb{Z}_8 \oplus \begin{bmatrix} 8 \\ -8 \end{bmatrix} \\
&\cong \mathbb{Z}_2 \oplus \mathbb{Z}_8 \oplus \begin{bmatrix} 8 \\ 0 \end{bmatrix} \\
&\cong \mathbb{Z}_2 \oplus \mathbb{Z}_8 \oplus \mathbb{Z}_8.
\end{aligned}
$$

Chapter 2

Basic concept and Definition of Graph Theory

2.1 Introduction.

Graph theory is a branch of mathematics and it is started by Euler as early as 1736 when he proposed problem to travel the seven bridges of the city Konigsberg in one round and every bridge is traveled only once. He modeled this problem in graph and found out the solution. Graph theory has huge applications in engineering and science especially in chemical engineering, mechanical engineering, architecture, operational research, technology, combinatorics, and computer science. Therefore many books has been published on graph theory such as Bondy and Murty, D.B West. In this chapter basic definition and concept of graph theory are given. This chapter gives a detailed overview of different types of graphs, different representation of graph and results which we will use in our last chapter.

2.2 What is graph?

A graph G is a structure of two sets V and E and denoted by $G = (V, E)$, where V is called vertex set and elements of V are called vertices (nodes, point) of graph and E which is the adjenceny relation between vertices is called edge set and elements of E are called edges or lines. We can represent the graph with different ways. One usual way is by drawing a picture of graph using different points and lines. We locate a dot for each vertex and if there is a link or an edge between two vertices, then join these two vertices by a line (line may be straight or curved). The graph can be drawn in different ways all just matter where two vertices from an edge or link and where does not. The order of a graph is the number of vertices of a graph and it is denoted by $|V|$ and number of edges of graph is the size of graph and denoted by

$|E|$. If the graph G has order 0 or 1 then G is called trivial graph.
Adjacency and Incidence. Two vertices v_1 and v_2 are called adjacent if there is an edge e between these two vertices. Two edges $e_1 \neq e_2$ are called incident if these two edges have a common end vertex v.
Degree of a vertex of graph. The degree of a vertex v of a graph G is the number of edges incident with v which is denoted by $d(v)$. The minimum degree vertex of a graph G is denoted by $\delta(G)$ and maximum degree vertex of a graph G is denoted by $\Delta(G)$. A vertex is said to be an isolated vertex if the degree of that vertex is 0. If the degree of vertex is 1 then vertex is called leaf. vertex is called even(odd) if the degree of that vertex is even(odd).

Lemma 2.2.1. *[4] The graph $G = (V, E)$ where $V = \{v_1, v_2, \ldots, v_n\}$ and $E = \{e_1, e_2, \ldots, e_m\}$ satisfies $\sum_{i=1}^{n} d(v_i) = 2m$. Or in other words, the sum of degree of vertices of a graph is twice the number of its edges.*
This lemma is known as hand shaking lemma.

Corollary 2.2.1. *[4] Every graph has an even number of vertices of odd degree.*

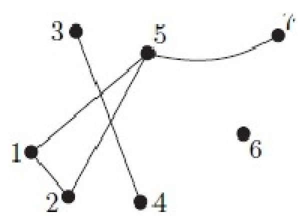

Figure 2.1: Graph G

Example 2.2.1. *In figure* 2.1 *which is given above the graph G is given with vertices* $1, 2, \cdots, 7$ *and edge set* $\{(1,2), (1,5), (2,5), (3,4), (5,7)\}$.

- *The order of this graph G is* 7.
- *The size of this graph G is* 5.
- *The vertices* $1, 2, 5$ *and* $5, 7$ *and* $3, 4$ *are adjacent.*
- *The edges* $(1,2)$, $(2,5)$ *and* $(1,5)$ *are incident.*
- *The degree of vertices,* $d(1) = 2, d(2) = 2, d(3) = 1, d(4) = 1, d(5) = 3, d(6) = 0, d(7) = 1$ *while* $\Delta(G) = 3$ *and* $\delta(G) = 0$.
- *. The vertex* 6 *is an isolated vertex.*

Two or more than two edges of a graph G are called **multiple edges** if these are connected by a same pair of vertices and an edge joining a vertex to itself is called

loop and a graph G which has no multiple edge or loop is called **simple graph.** In figure 2.2 graph with loop is given while in figure 2.3 simple graph is given.

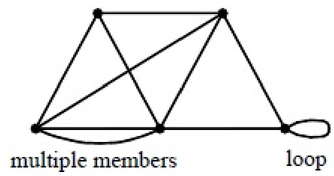

Figure 2.2: Graph G with loop and multiple edges(member).

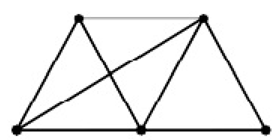

Figure 2.3: Simple graph.

Subgraphs. A graph $G' = (V', E')$ is said to be a subgraph of a graph $G = (V, E)$, if $V \subseteq V'$ and $E \subseteq E'$ and edges of G' have sames endpoints as that in G, and we write $G' \subseteq G$.

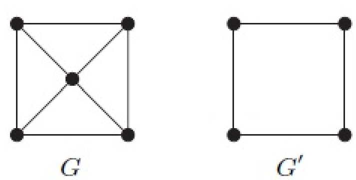

Figure 2.4: Graph G with subgraph G'

Isomorphic graphs. Two graphs $G = (V, E)$ and $G^{'} = (V^{'}, E^{'})$, where $|V| = |V'|$ and $|E| = |E'|$ are isomrphic if there exists one to one correspondence between their vertices set but adjacency must be preserved and we write $G \cong G'$.

Walk, Trail, path. A non empty graph $G = (V, E)$ of the form $V = \{v_0, v_1, \cdots, v_k\}$ $E = \{v_0v_1, \cdots, \cdots, v_{k-1}, v_k\}$ is called a **walk** of length k where number of edges in

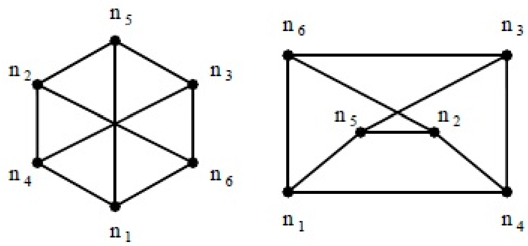

Figure 2.5: Two isomorphic graph.

walk is called length of walk. If $v_0 = v_k$ then walk is closed. A walk in graph G is called **trail** if no edge is repeated more than once. A walk in graph G is called **path** in which no vertex is repeated and hence no edge. A path of length n is denoted by P_n. A trail in graph G is called **circuit** if trail is closed. A trial must be of length three or more. A closed trail that repeat no vertex except first and last is called **cycle**. A cycle of n length is denoted by C_n if length of edges are odd then cycle is odd if length of edges is even then cycle is even.

2.3 Matrix representation of graph.

In this section, we introduce the matrix representation of graph, that is, adjacency matrix. Suppose G be a graph with n vertices $v_1, \ldots, v_n$. The adjacency matrix $A(G)$ of a graph G is a $n \times n$ square matrix in which the entry $a_{ij} = 1$, if vertex v_i is adjacent to v_j and 0 otherwise. $A(G)$ is a square matrix and the sum of rows of $A(G)$ is equal to the degree vertex of graph G. The adjacency matrix for figure 2.1 is given below.

$$A(G) = \begin{bmatrix} 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & \end{bmatrix}.$$

Another way to represent a graph by matrix is incident matrix. Suppose G be a graph with n vertices $v_1, \ldots, v_n$ and m edges $e_1, \ldots, e_m$ then incident matrix is $m \times n$ matrix in which ij(ith row and jth column) is equal to 1 if vertex v_1 is incident to edge e_j and zero otherwise.

Degree matrix of graph. Suppose G be a graph with n vertices $v_1, \ldots, v_n$ then degree matrix $D(G)$ is $n \times n$ diagonal matrix in with d_{ij} is equal to degree of v_i if $i = j$ and zero otherwise. This matrix contains the information of each vertex of graph. The degree matrix for $D(G)$ for figure 2.1 is given below.

$$D(G) = \begin{bmatrix} 2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

Laplacian matrix. Laplacian matrix is defined as $L(G) = D(G) - A(G)$. It is used to find the number of spanning trees of graph. We will define spanning tree of graph in next section.

2.4 Types of graph

There are many types of graphs depending on the number of edges and vertices of graph, inter connectivity and overall structure of graph. We will discuss only few important types of graphs.

Null graph. A graph which has no edge is called null graph.

Trivial graph. A graph which has only one vertex and hence no edge is called trivial graph.

Connected graph. A graph G is called connected if for each pair $v_1, v_2 \in V(G)$ there exist some v_1v_2 path or in simple word there must be atleast one edge for every vertex in graph.

Disconnected graph. If there is no path for some pair $v_1, v_2 \in V(G)$ then G is called disconnected.

Regular graph. In a graph if degree of each vertex is same then graph is called regular graph. If in a graph degree of each vertex is k then graph is called $k-$regualr graph.

Complete graph. A simple graph in which each pair of vertices of graph is connected by a unique edge is called complete graph. A complete graph which has n vertices is denoted by K_n.

Cyclic graph. A simple graph that contains atleast one cycle is called cyclic graph. In cyclic graph number vertices of graph is equal to number of edges and degree of each vertex is exactly 2. A cyclic graph with n vertices is denoted by C_n.

Acyclic graph. A simple graph that contains no cycle is called acyclic.

Bipartite graph. A simple graph $G = (V, E)$ in which vertex set $V(G)$ can be partitioned into two set $V_1(G), V_2(G)$ such that every edge E connect a vertex $V_1(G)$ to one in $V_2(G)$.

Theorem 2.4.1. *[4] A graph G is called bipartite if and only if it contains no odd cycle.*

Complete bipartite Graph. A bipartite graph is called complete bipartite if every vertex of first partition (vertex set) is connected by every vertex of second partition(vertex set). If number of vertices in first set in m and number of vertices in second partitioning set is n then complete bipartite set is denoted by $K_{m,n}$.
Star graph. Star is a special form of complete bipartite graph. It is of form $K_{1,n-1}$ with n vertices i.e in one first partition set there is a single vertex and remaining all vertices are in second partitioning set.
Tree. An acyclic graph which is also connected is called tree. Tree is minimal connected that is if we remove a single edge graph will become disconnected. A tree of order n has exactly $n-1$ edges.
Spanning tree of a graph. Spanning tree of a graph is its subgraph that contains all vertices, and is a tree. A graph may have many spanning trees. The number of spanning trees of a graph G is denoted by $T(G)$. We can find number of spanning trees of a graph using different methods. Number of spanning trees of a tree itself is 1. Spanning tree of cycli graph with n vertices is n. There is also Kirchhoff's theorem. We take the Laplacian matrix of the graph then delete an arbitrary row and its corresponding column, and then find the determinant of the matrix. The value of the determinant will be the number of spanning trees for the graph.
Examples.

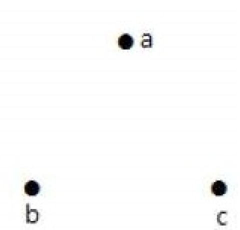

Figure 2.6: A graph with three vertices and no edge hence null graph.

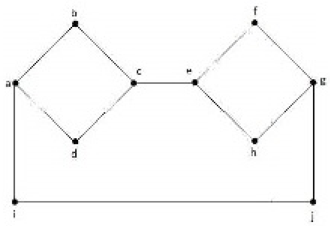

Figure 2.7: A connected graph.

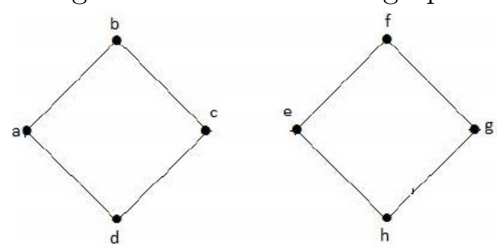

Figure 2.8: A disconnected graph.

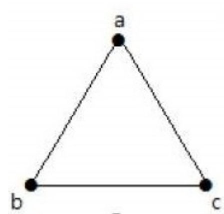

Figure 2.9: Each vertex is of degree 2 hence a regular graph

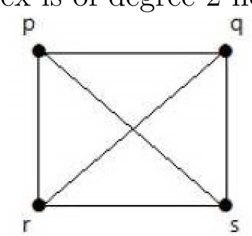

Figure 2.10: A complete graph K_4.

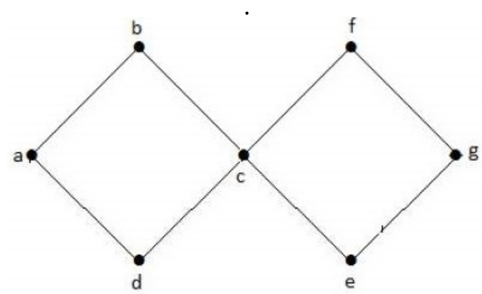

Figure 2.11: A Graph with two cycle *abcda* and cfgec hence it is a cyclic graph.

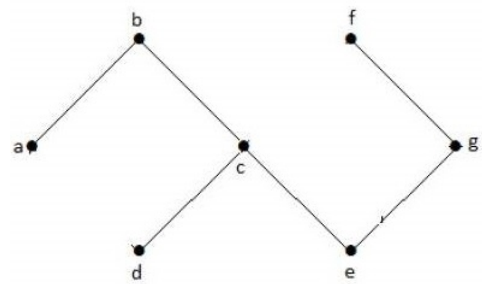

Figure 2.12: A graph with no cycle hence it is acyclic.

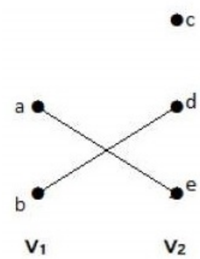

Figure 2.13: A bipartite graph with vertex set V_1 and V_2.

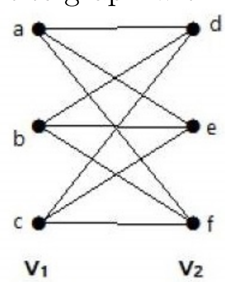

Figure 2.14: Each vertex of set V_1 is connected by an edge wueh vertex set V_2 hence complete bipartite graph.

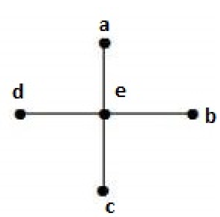

Figure 2.15: All vertices a, b, c, d
are connected by single vertex e hence it is star $K_{1,4}$.

Chapter 3

Abelian Sandpile Model

3.1 Introduction

In 1987 Per Bak, Chao Tang introduced a model known as "Self-Organized Criticality" using the examples of sandpile in [1, 11] then Deepak Dhar generalized the above model and introduced Abelian Sandpile Model. In 2010 Lionel Levine, James propp considered finite connected graph G for which they associate an abelian group $S(G)$.
The Abelian Sandpile Model is a diffusion process on a finite directed multi-graph that has been used to show self organized criticality. The model was defined on square grids with cells that randomly, with breaks in between, received sand grains. These had a maximum capacity of three sand grains; once this capacity was exceeded, the sand would topple into adjacent cells or fall off the edge of the grid. These square grids are a type of graph, with the cells as the vertices and edges connecting adjacent cells and allowing sand grains to pass from one cell to a neighboring cell.

Definition 3.1.1. Let $G = (V \cup \{s\}, E)$ be a graph consists of finitely many vertices and finitely many edges. These edges might be multiple, there may be two or more edges between two vertices. We also consider a special vertex s which is acting as a sink s.
An abelian sandplie Γ is a sequence of indistinguishable particle or chips on the vertices of graph. More precisely, it is an assignment of non negative integer to the vertices of graph, where non negative integer indicates how many particles or chips are at each vertex or it is specified by a map $\Gamma : V \to \{0, 1, 2 \cdots\}$. The sequence of above non negative integers is called configuration.

Definition 3.1.2. A sandpile Γ is stable at $v \in V$, if $\Gamma(v) < deg_G(v)$ and Γ is called a stable sandpile if it is stable for all $v \in V$. A sandpile Γ is called unstable if for some $v \in V$, $\Gamma(v) \geq deg_G(v)$.

3.1.1 Toppling

When a vertex of a graph is unstable or when a vertex accumulates as many chips as its degree then the vertex is allowed to topple which means that it fires, sending one chip along each vertex adjacent to it. When a vertex v topples , the chips are re-distributed are follows:

$$\begin{aligned} &\Gamma(v) \to \Gamma(v) - deg_G(v) \\ &\Gamma(w) \to \Gamma(w) + a_{vw}, \qquad w \in V, \quad w \neq v, \end{aligned}$$

where a_{vw} is the number of edges between v and w. Toppling a vertex may in turn cause other vertices unstable, then these vertices are also toppled simultaneously. This process repeats until all the vertices becomes stable. Consider $\eta_j \in S_G$ which is row vector it can be written as

$$\eta_j \to \eta_j - \triangle_{ij}, \text{ where } j \in V \text{ and } \triangle_{ij} \text{ is the toppling matrix.}$$

The toppling matrix is the graph laplacian:

$$(\triangle_G)_{ij} = \begin{cases} deg_{(G)}i - a_{ii}, & if \;\; i = j \in V \\ -a_{ij}, & if \;\; i \neq j \;\; i \,, j \in V \\ 0 & otherwise. \end{cases}$$

Toppling rule can be be defined by the mapping $T_z : \mathbb{N} \to \mathbb{N}$

$$T_z(\eta_j) = \begin{cases} \eta_j - \triangle_{ij}, & if \;\; \eta_j \geq deg_G i; \\ \eta_j = \eta_j, & \quad otherwise. \end{cases}$$

Remark 3.1.1. *[24].*

The toppling rule commutes on the unstable configurations that is for all $y, \; z \in V$ *and* η *such that* $\eta(y) \geq \triangle_{yy}$ *and* $\eta(z) \geq \triangle_{zz}$*, we have*

$$T_y \circ T_z(\eta) = T_z \circ T_x(\eta)$$

After a finitely many toppling the configuration reached to a stable configuration. The result of final configuration does not depends on the the order in which sequence of toppling is chosen.

Definition 3.1.3. A sink s is a special vertex that never topples. We require that from every vertex it is possible to travel via the edges get to the sink. It is one of the vertex that collect the particle falling off to the system. The presence of sink assures that the process of toppling terminate after stability of graph.

Definition 3.1.4. Recurrent configuration are configuration if there is a sequence of toppling that leads to the same configuration.

Definition 3.1.5. . The set of all stable recurrent configuration forms the sandpile group with given binary operation,

$$C_1 \oplus C_2 = \overline{C_1 + C_2}.$$

Here $C_1 + C_2$ is pointwise addition of chip assignments to vertices and $\overline{C}$ denotes the unique critical configuration reachable from C.

3.2 Sandpile group

Now in this section Sandpile group of graph G is defined. Sandpile group can be defined as a quotient group of $\mathbb{Z}^{n-1}$ by a subgroup generated by the elements that are express the toplling rule.

Definition 3.2.1. Let $G = (V \cup \{s\}, E)$ be a finite connected multigraph where $|V| = n$ and s is sink that doesnot topple. Sandpile group of graph G is closely connected with the laplacian matrix $L(G)$ as follow: Thinking of $L(G)$ as an linear map $\mathbb{Z}^n \to \mathbb{Z}^n$, its cokernel has the form [14].

$$\text{coker} L(G) = \mathbb{Z}^n / L(G)\mathbb{Z}^n \cong \mathbb{Z} \oplus S(G)$$

Let G be a graph with n vertices with sink or rooted vertex x_s then $S(G)$ on graph G is a quotient group of $\mathbb{Z}^{n-1}$ by the subgroup spanned by $n-1$ element that are $\triangle_1, \triangle_2, \ldots, \triangle_{s-1}, \triangle_{s+1}, \ldots, \triangle_n$, where $\triangle_j$ expressing the toppling rule that is

$$\triangle_j = d_j x_j - \sum_{v_k \text{ is adjacent to } v_i} a_{jk} x_k,$$

where $x_k = (0, \ldots, 0, 1, 0, \ldots, 0) \in \mathbb{Z}^n$ whose unique non zero entry 1 is in the position k, and the element x_s, so,

$$S(G) \cong \mathbb{Z}^n / \text{span}(\triangle_1, \triangle_2, \ldots, \triangle_{s-1}, x_s, \triangle_{s+1}, \ldots, \triangle_n) \text{ for more detail see [7].}$$

Theorem 3.2.1. *[1] The order of sandplie group on graph G is equal to number of spanning tree of graph.*

Theorem 3.2.2. ***Kirchhoffs Matrix Tree Theorem***

For any connected graph G, the number of spanning trees is equal to $det(\widehat{L(G)})$*, where* $\widehat{L(G)}$ *is obtained from* $L(G)$ *by deleting the row and column corresponding to any chosen vertex.*

3.3 Smith Normal Form of a Matrix.

3.3.1 Introduction

In 1861, H.J.Smith introduced the Smith normal form of a matrix. Its plays an important role in the study of algebraic group theory, matrix equivalence, homology group theory. Due to its wide application many researcher worked on the problem of smith normal form.

Definition 3.3.1. For every integer matrix A, there exist unimodular matrices P and Q and also a diagonal matrix S where diagonal entries $(s_{11}, s_{22}, \ldots, s_{nn})$ of S are non-negative, for $i = 1, \cdots, n$ such that $A = PSQ$, $s_{ii}|s_{ii+1}$. S is called smith normal form of A, s_i are called the invariant factors of matrix A, s_{ii} are called the invariant factors of matrix A.

In above definition, the smith normal form S of a matrix A is unique while the choices of P and $Q$are not unique.

Remark 3.3.1. *i)If A is a given $n \times n$ integer matrix with invariant factors $s_1, s_2, \cdots, s_n$ then*

$$det(A) = \pm \prod_{0 \leq i \leq n} s_{ii} \quad and \quad s_{ii} = 0, \; if \;\; i \; > rank(A)$$

ii) For each i, product $s_{11}, \;\; s_{22}, \;\; \cdots, s_{ii}$ is the greatest common divisor of all $i \times i$ minor determinant of A [14].

Smith normal form of a matrix can be computed via a series of elementary column and row operations instead of abstract definition of smith normal form of a matrix. The elementary column and row operations consisting of
1. Adding an integer multiple of one row (column) to another.
2. Multiplying a row (columns) by -1.
3. Swapping two rows (columns).

Example 3.3.1. *Consider a 2×2 matrix:*

$$\begin{aligned} M &= \begin{bmatrix} 1 & 5 \\ 1 & 3 \end{bmatrix} \\ &\sim \begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix} \quad C_2 - 5C_1 \\ &\sim \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \quad R_2 - R_1 \end{aligned}$$

$1, 2$ *are the invariant factors of M. det(M)*$= 1(3) - 5(1) = 2 = 1 \times 2$.

Chapter 4

Sandpile group of square cycle graph

4.1 Introduction

There are infinite numbers of graphs whose sandpile group have been completely determined. In [5, 6, 7, 8] sandpile group sturcture of different graph have been completely determined. In this chapter the structure of sandpile group of square cycle graph C_n^2 is computed by determining its smith normal form. We also show that sandpile group of square cycle graph C_n^2 is direct sum of two or three cyclic groups. Number of spanning tree of C_n^2 is $\tau(C_n^2) = nF_n^2$ which is equal to order of sandpile group of C_n^2. Hence the order of sandpile group of C_n^2 is equal to nF_n^2. First we define square cycle graph.

Definition 4.1.1. Square cycle, C_n^2 is a graph that has n vertices. Square cycle graph C_n^2 is a 4-regular graph and each vertex i is adjacent to the vertices $i \pm 1$, $i \pm 2$ (mod n). See fig. 4.1 for some examples of C_n^2

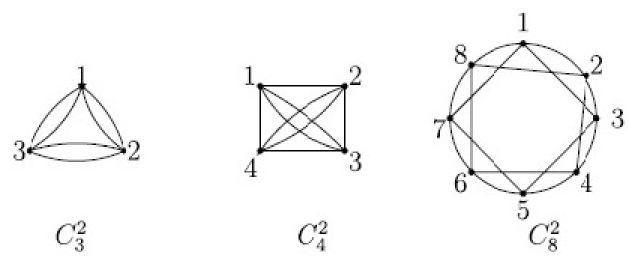

Figure 4.1: Some examples of the graphs C_n^2.

4.2 System of relation for the generators of $S({C_n}^2)$

In this section, we will first show that there are at most three generator for Sandpile group $S({C_n}^2)$ of square cycle graph and then we reduce these generator to the relation matrix and then we will give some properties of the sequences concerning the entries of relation matrix.
Now we first start work on the system of relation. It is assumed that $n \geq 5$. we have chosen vertex 4 as the root, such that $x_4 = 0$. Applying the toppling rule (that is , if $v_i \neq v_r$ is a vertex degree d_i, a generator of this group is.

$$\Delta_i = d_i x_i - \sum_{v_j \ is \ adjacent \ v_i} a_{ij} x_j,$$

where a_{ij} is the number of edges between vertices v_i and v_j, and $x_i = (0, \cdots, 1, 0, \cdots, 0) \in \mathbb{Z}_n$. By applying this on each vertex except root vertex we get,

$$
\begin{aligned}
&for\ i = 1,\ \Delta_1 = 4x_1 - x_2 - x_3 - x_{n-1} - x_n, && (4.1)\\
&for\ \ i = 2,\ \ \Delta_2 = 4x_2 - x_1 - x_3 - x_4 - x_n, && (4.2)\\
&for\ \ i = 3,\ \ \Delta_3 = 4x_3 - x_2 - x_1 - x_4 - x_5, && (4.3)\\
&\qquad\qquad \vdots\\
&for\ i = n-1,\ \Delta_{n-1} = 4x_{n-1} - x_1 - x_{n-2} - x_{n-3} - x_n, && (4.4)\\
&for\ i = n,\ \Delta_n = 4x_n - x_1 - x_2 - x_{n-1} - x_{n-2}. && (4.5)
\end{aligned}
$$

Since vertex 4 has been chosen as the rooted vertex (Sink), therefore $x_4 = 0$. If $\overline{x}_i$ is the image of x_i in $\mathbb{Z}^n / \triangle \mathbb{Z}^n$. The following system of relation is obtained

$$
\begin{cases}
\overline{x}_1 = 4\overline{x}_{n-1} - \overline{x}_{n-2} - \overline{x}_{n-3} - \overline{x}_n, & \text{by (4.4)}\\
\overline{x}_2 = 4\overline{x}_n - \overline{x}_1 - \overline{x}_{n-2} - \overline{x}_{n-1}, & \text{by (4.5)}\\
\overline{x}_3 = 4\overline{x}_1 - \overline{x}_2 - \overline{x}_{n-1} - \overline{x}_n, & \text{by (4.1)}\\
\overline{x}_i = 4\overline{x}_{i-2} - \overline{x}_{i-3} - \overline{x}_{i-4} - \overline{x}_{i-1}. & \text{for each } 5 \leq i \leq n
\end{cases}
$$

From this system it is cleared that there are at most three generators for $S(C_n^2)$. Indeed each $\overline{x}_1, \overline{x}_2, \overline{x}_3$. For each $5 \leq i \leq n$, a_i, b_i, c_i are defined such that,

$$\overline{x}_i = (-1)^i (a_i \overline{x}_1 + b_i \overline{x}_2 + c_i \overline{x}_3).$$

Proposition 4.2.1. *[14]. If the sequence $(c_i)_{i\geq 5}$ is extended by $c_o = 1$, $c_1 = c_2 = c_4 = 0$, $c_3 = 1$, then the following equalities hold:*

$$c_i = 4c_{i-2} + c_{i-1} + c_{i-3} - c_{i-4}, \tag{4.6}$$

$$a_i = \frac{c_i - c_{i-1} - c_{i-2} + (-1)^i}{2}, \qquad b_i = c_{i-2}. \tag{4.7}$$

Proof. From system of generators of $S(C_n^2)$ relations relations are obtained for $a_i,\ b_i,\ c_i$ with $5 \leq i \leq n$ that are,

$$a_i = 4a_{i-2} - a_{i-1} - a_{1-3} - a_{i-4}.$$
$$b_i = 4b_{i-2} - b_{i-1} - b_{i-3} - b_{i-4}.$$
$$c_i = 4c_{i-2} + c_{i-1} + c_{i-3} - c_{i-4}.$$

from these recurrence relation and initial values of c_i, $a_i's$ and $b_i's$ can be expressed in terms of the $c_i's$. □

4.2.1 Relation matrix between the the generators $\overline{x}_1, \overline{x}_2, \overline{x}_3$

System of equation after applying the toppling rule are:

$$x_1 = 4\overline{x}_{n-1} - \overline{x}_{n-2} - \overline{x}_{n-3} - \overline{x}_n \tag{4.8}$$
$$\overline{x}_2 = 4\overline{x}_n - \overline{x}_1 - \overline{x}_{n-2} - \overline{x}_{n-1}, \tag{4.9}$$
$$\overline{x}_3 = 4\overline{x}_1 - \overline{x}_2 - \overline{x}_{n-1} - \overline{x}_n, \tag{4.10}$$
$$\overline{x}_i = 4\overline{x}_{i-2} - \overline{x}_{i-3} - x_{i-4} - \overline{x}_{i-1}. \tag{4.11}$$
$$\tag{4.12}$$

From this system it can been seen each $\overline{x}_i$ can be expressed in terms of $\overline{x}_1,\ \overline{x}_2,\ \overline{x}_3$ such that

$$\overline{x}_i = (-1)^i(a_i\overline{x}_1 + b_i\overline{x}_2 + c_i\overline{x}_3), \quad for \quad 5 \leq i \leq n. \tag{4.13}$$

Using (4.13),

$$\overline{x}_{n-1} = (-1)^{n-1}(a_{n-1}\overline{x}_1 + b_{n-1}\overline{x}_2 + c_{n-1}\overline{x}_3),$$
$$\overline{x}_{n-2} = (-1)^{n-2}(a_{n-2}\overline{x}_1 + b_{n-2}\overline{x}_2 + c_{n-2}\overline{x}_3),$$
$$\overline{x}_{n-3} = (-1)^{n-3}(a_{n-3}\overline{x}_1 + b_{n-3}\overline{x}_2 + c_{n-3}\overline{x}_3).$$

Put these value in (4.8)

$$\begin{aligned}\overline{x}_1 =& 4\left[(-1)^n - 1(a_{n-1}\overline{x}_1 + b_{n-1}\overline{x}_2 + c_{n-1}\overline{x}_3\right] - \left[(-1)^n - 2(a_{n-2}\overline{x}_1 + b_{n-2}\overline{x}_2 + c_{n-2}\right] \\ & - \left[(-1)^n - 3(a_{n-3}\overline{x}_1 + b_{n-3}\overline{x}_2 + c_{n-3}\overline{x}_3\right] - \left[(-1)^n(a_n\overline{x}_1 + b_n\overline{x}_2 + c_n\overline{x}_3\right] \\ \\ =& \left[(-1)^{n-1}4a_{n-1} - (-1)^{n-2}a_{n-2} - (-1)^{n-3}a_{n-3} - (-1)^n a_n\right]\overline{x}_1 + \\ & \left[(-1)^{n-1}4b_{n-1} - (-1)^{n-2}b_{n-2} - (-1)^{n-3}b_{n-3} - (-1)^n b_n\right]\overline{x}_2 + \\ & \left[(-1)^{n-1}4c_{n-1} + (-1)^{n-2}c_{n-2} + (-1)^{n-3}c_{n-3} + (-1)^n c_n\right]\overline{x}_3.\end{aligned}$$

By putting values of $a,\ b,\ c$ we get,

$$\begin{aligned}\overline{x}_1 &= (-1)^{n+1}a_{n+1}\overline{x}_1 + (-1)^{n+1}b_{n+1}\overline{x}_2 - (-1)^{n+1}c_{n+1}\overline{x}_3 \\ \overline{x}_1 &= (-1)^{n+1}(a_{n+1}\overline{x}_1 + b_{n+1}\overline{x}_2 - c_{n+1}\overline{x}_3).\end{aligned} \tag{4.14}$$

Now for $\overline{x}_2$ we put the values of $x_n,\ \overline{x}_{n-1},\ \overline{x}_{n-2},\ \overline{x}_1$ in (4.9)

$$\begin{aligned}\overline{x}_2 =& 4\left[(-1)^n(a_n\overline{x}_1 + b_n\overline{x}_2 + c_n\overline{x}_3)\right] - \left[(-1)^{n-1}(a_{n-1}\overline{x}_1 + b_{n-1}\overline{x}_2 + c_{n-1}\overline{x}_3)\right] \\ & - \left[(-1)^n - 2(a_{n-2}\overline{x}_1 + b_{n-2}\overline{x}_2 + c_{n-2}\overline{x}_3)\right] + \left[(-1)^{n+1}(a_{n+1}\overline{x}_1 + b_{n+1}\overline{x}_2 + c_{n+1}\overline{x}_3)\right] \\ \\ =& \left[4(-1)^n a_n - (-1)^{n-1}a_{n-1} - (-1)^{n-2}a_{n-2} - (-1)^{n+1}a_{n+1}\right]\overline{x}_1 + \\ & \left[4(-1)^n b_n - (-1)^{n-1}b_{n-1} - (-1)^{n-2}b_{n-2} - (-1)^{n+1}b_{n+1}\right]\overline{x}_2 + \\ & \left[-4(-1)^n c_n + (-1)^{n-1}c_{n-1} + (-1)^{n-2}c_{n-2} + (-1)^{n+1}c_{n+1}\right]\overline{x}_3\ .\end{aligned}$$

By putting values of $a,\ b,\ c$ we get,

$$\begin{aligned}\overline{x}_2 &= (-1)^{n+2}a_{n+2}\overline{x}_1 + (-1)^{n+2}b_{n+2}\overline{x}_2 - (-1)^{n+2}c_{n+2}\overline{x}_3 \\ \overline{x}_2 &= (-1)^{n+2}(a_{n+2}\overline{x}_1 + b_{n+2}\overline{x}_2 - c_{n+2}\overline{x}_3).\end{aligned} \tag{4.15}$$

Now for $\overline{x}_3$ we put the values of $\overline{x}_1,\ ,\overline{x}_{n-1},\ \overline{x}_n,\ ,\overline{x}_1$ in (4.10)

$$\begin{aligned}\overline{x}_3 =& \left[(-1)^{n+1}(a_{n+1}\overline{x}_1 + b_{n+1}\overline{x}_2 - c_{n+1}\overline{x}_3)\right] - \left[(-1)^{n-1}(a_{n-1}\overline{x}_1 + b_{n-1}\overline{x}_2 - c_{n-1}\overline{x}_3)\right] \\ & - \left[(-1)^n(a_n\overline{x}_1 + b_n\overline{x}_2 - c_n\overline{x}_3)\right] - \left[(-1)^{n+2}(a_{n+2}\overline{x}_1 + b_{n+2}\overline{x}_2 - c_{n+2}\overline{x}_3)\right] \\ \\ =& \left[4(-1)^{n+1}a_{n+1} - (-1)^{n-1}a_{n-1} - (-1)^n a_n - (-1)^{n+2}a_{n+2}\right]\overline{x}_1 + \\ & \left[4(-1)^{n+1}b_{n+1} - (-1)^{n-1}b_{n-1} - (-1)^n b_n - (-1)^{n+2}b_{n+2}\right]\overline{x}_2 + \\ & \left[-4(-1)^{n+1}c_{n+1} + (-1)^{n-1}c_{n-1} + (-1)^n c_n + (-1)^{n+2}c_{n+2}\right]\overline{x}_3.\end{aligned}$$

By putting values of $a,\ b,\ c$ we get,

$$\overline{x}_3 = (-1)^{n+3}a_{n+3}\overline{x}_1 + (-1)^{n+3}b_{n+3}\overline{x}_2 - (-1)^{n+3}c_{n+3}\overline{x}_3$$
$$\overline{x}_3 = (-1)^{n+3}(a_{n+3}\overline{x}_1 + b_{n+3}\overline{x}_2 - c_{n+3}\overline{x}_3). \tag{4.16}$$

Using (4.14)

$\overline{x}_1 = (-1)^{n+1}(a_{n+1}\overline{x}_1 + b_{n+1}\overline{x}_2 - c_{n+1}\overline{x}_3)$
$\overline{x}_1(-1)^{-n-1} = (a_{n+1}\overline{x}_1 + b_{n+1}\overline{x}_2 - c_{n+1}\overline{x}_3)$
$a_{n+1}\overline{x}_1 - \overline{x}_1(-1)^{-n-1} + b_{n+1}\overline{x}_2 - c_{n+1}\overline{x}_3) = 0$
$(a_{n+1} - (-1)^n)\overline{x}_1 + b_{n+1} + c_{n+1} = 0$
Similarly, Using (4.15) and (4.16) we get,
$a_{n+2}\overline{x}_1 + (-b_{n+2} + (-1)^n)\overline{x}_2 - c_{n+2} = 0$
$a_{n+3}\overline{x}_1 + b_{n+3}\overline{x}_2 + (-c_{n+3} + (-1)^n)\overline{x}_3 = 0.$
Thus, the relation matrix between the generators $\overline{x}_1,\ \overline{x}_2,\ \overline{x}_3$ is

$$B_n = \begin{pmatrix} a_{n+1} + (-1)^n & b_{n+1} & -c_{n+1} \\ -a_{n+2} & -b_{n+2} + (-1)^n & c_{n+2} \\ a_{n+3} & b_{n+3} & -c_{n+3} + (-1)^n \end{pmatrix}.$$

Proposition 4.2.2. *[14]. For $n \geq 0$, the following equations hold:*

$$c_{n+1} + c_{n+2} = F_n^2, \quad c_{n+2} - c_n = F_n^2 - F_{n-1}^2, \quad a_{n+1} + a_{n+2} = F_{n-1}Fn - 2. \tag{4.17}$$

Proof. From eq. ,we get

$$c_{n+1} = 4c_{n-1} + c_n + c_{n-2} - c_{n-3}.$$
$$c_{n+2} = 4c_n + c_{n+1} + c_{n-1} - c_{n-2}.$$

Adding c_{n+1}, c_{n+2} we get,

$$c_{n+1} + c_{n+2} = 2(c_n + c_{n+1}) + 2(c_{n-1+c_n}) - c_{n-2} + c_{n-1} \ \text{ For } n \ \geq \ 3. \tag{4.18}$$

with initial condition $c_1 + c_2 = 0,\ c_2 + c_3 = 1, c_3 + c_4 = 1$, writing explicit form of the sequence in (4.18), The characteristic equation will be,

$$r^3 - 2r^2 - 2r + 1 = 0, \quad \text{by solving we get,}$$
$$r_1 = \frac{3+\sqrt{5}}{2}, \quad r_2 = \frac{3-\sqrt{5}}{2}, \quad r_3 = -1.$$

Using these values of r_1, r_2, r_3 the sequence in (4.18) becomes,

$$c_{n+1} + c_{n+2} = \alpha_1 \left(\frac{3+\sqrt{5}}{2} \right)^n + \alpha_2 \left(\frac{3-\sqrt{5}}{2} \right)^n + \alpha_3(-1)^n.$$

Using initial values of (4.18), we get the value of $\alpha_1 = \frac{1}{5}$, $\alpha_2 = \frac{1}{5}$, $\alpha_3 = \frac{-2}{5}$ by putting these value we get,

$$c_{n+1} + c_{n+2} = \frac{\left[\left(\frac{3+\sqrt{5}}{2} \right)^n + \left(\frac{3-\sqrt{5}}{2} \right)^n - 2(-1)^n \right]}{5} = F_n^2.$$

Where $F_n = \frac{1}{\sqrt{5}} \left[\left(\frac{1+\sqrt{5}}{2} \right)^n - \left(\frac{1-\sqrt{5}}{2} \right)^n \right]$ is the nth Fibonacci number. Hence,

$$c_{n+1} - c_n = (c_{n+2} + c_{n+1} - c_{n+1} + c_n) = F_n^2 - F_{n-1}.$$

As, $a_n = \frac{c_n - c_{n-1} - c_{n-2} + (1)^n}{2}$ then $a_{n+1} = \frac{c_{n+1} - c_n - c_{n-1} + (-1)^{n+1}}{2}$,

$a_{n+2} = \frac{c_{n+2} - c_{n+1} - c_n + (-1)^{n+2}}{2}$.

$$\begin{aligned} \text{So, } a_{n+1} + a_{n+2} &= \frac{c_{n+2} + c_{n+1} - c_n}{2} \\ &= \frac{F_n^2 - F_{n-1}^2 - F_{n-2}^2}{2} \\ &= F_{n-2}F_{n-1}. \end{aligned}$$

□

Some well-known identities of Fibonacci numbers are listed in next remark which will be used later.

Remark 4.2.1. *[17]. For all n,*

$$F_{2n} = F_n(F_{n+1} + F_{n-1}), \tag{4.19}$$
$$F_{n+1}^2 - F_n^2 = F_{n+2}F_{n-1}, \tag{4.20}$$
$$F_{n+1}F_{n-1} = F_n^2 + (-1), \tag{4.21}$$
$$F_nF_{n-3} = F_{n-1}F_{n-2} + (-1)^n, \tag{4.22}$$
$$F_1^2 + F_3^2 + \cdots + F_{2m+1}^2 = \frac{2(F_{4m} + 2m)}{5}, \tag{4.23}$$
$$F_1^2 + F_2^2 + \cdots + F_n^2 = F_{n+1}F_{n+2}, \tag{4.24}$$

$$F_{2n+2}-1=\begin{cases} F_nF_{n+1}+F_nF_{n+3}, & n \textit{ is even},\\ F_{n+1}F_{n+2}+F_{n1}F_{n+2}, & n \textit{ is odd}, \end{cases}$$

$$F_{2n-2}+1=\begin{cases} F_nF_{n-1}+F_nF_{n-3}, & n \textit{ is even},\\ F_{n+1}F_{n-2}+F_{n1}F_{n-2}, & n \textit{ is odd}. \end{cases}$$

Lemma 4.2.1. *[14]. For $m \geq 0$, then*

$$c_{2m+2}=F_{2m}F_{2m+1}-\frac{2(F_{4m}+2m)}{5}, \tag{4.25}$$

$$c_{2m+1}=\frac{2(F_{4m}+2m)}{5}-F_{2m-1}F_{2m}, \tag{4.26}$$

$$a_{2m+2}-1=F_{2m-2}F_{2m+1}-\frac{F_{4m+2m}}{5}, \tag{4.27}$$

$$a_{2m+1}+1=\frac{F_{4m+2m}}{5}+F_{2m-3}F_{2m}. \tag{4.28}$$

Proof. The first identity will be proven using (4.17)

$$\begin{aligned} c_{2m+2} &= F_{2m}^2-c_{2m+1}\\ &= F_{2m}^2-(F_{2m-1}^2-c_{2m})\\ &= \cdots\\ &= (F_2^2+F_4^2+\cdots+F_{2m-2}^2+F_{2m}^2)-(F_1^2+F_3^2+\cdots+F_{2m-1}^2)\\ &= (F_1^2+F_2^2+\cdots+F_{2m-1}^2+F_{2m}^2)-2(F_1^2+F_3^2+\cdots+F_{2m-1}^2)\\ &= F_{2m}F_{2m+1}-\frac{2(F_{4m}+2m)}{5}. \qquad \text{by (4.23), (4.24)} \end{aligned}$$

The second identity will be proven using (4.17)

$$\begin{aligned} c_{2m+1} &= F_{2m-1}^2-c_{2m}\\ &= F_{2m-1}^2-(F_{2m-2}-c_{2m-1})\\ &= \cdots\\ &= (F_1^2+F_3^2+\cdots+F_{2m-1}^2)-(F_2^2+F_4^2+\cdots+F_{2m-2}^2+F_{2m}^2)\\ &= 2(F_1^2+F_3^2+\cdots+F_{2m-1}^2)-(F_1^2+F_2^2+\cdots+F_{2m-1}^2+F_{2m}^2)\\ &= \frac{2(F_{4m}+2m)}{5}-F_{2m}F_{2m-1}. \qquad \text{by (4.23), (4.24)} \end{aligned}$$

The third identity,

$$
\begin{aligned}
a_{2m+2} &= \frac{c_{2m+2} - c_{2m+1} - c_{2m}}{2} \\
a_{2m+2} - 1 &= \frac{c_{2m+2} - c_{2m+1} - c_{2m}}{2} - 1 \\
&= \frac{c_{2m+2} - F_{2m-1}^2}{2} - 1 \qquad \text{by}(4.17) \\
&= \frac{F_{2m}F_{2m+1} - \frac{2(F_{4m}+2)}{5} - F_{2m-1}^2}{2} - 1 \qquad \text{by}(4.25) \\
&= \frac{5F_{2m}F_{2m+1} - 2F_{4m} + 4m - 5F_{2m-1}^2 - 5}{10} \\
&= \frac{F_{2m}F_{2m+1} - F_{2m-3}F_{2m+1}}{2} - \frac{F_{4m} + 2m}{5} \\
&= F_{2m-2}F_{2m+1} - \frac{F_{4m} + 2m}{5}.
\end{aligned}
$$

The fourth identity,

$$
\begin{aligned}
a_{2m+1} &= \frac{c_{2m+1} - c_{2m} - c_{2m-1} - 1}{2} \\
a_{2m+1} + 1 &= \frac{c_{2m+1} - c_{2m} - c_{2m-1} - 1}{2} + 1 \\
&= \frac{c_{2m+1} - F_{2m-2}^2 + 1}{2} \\
&= \frac{c_{2m+1} - F_{2m-2}F_{2m-4}}{2} \\
&= \frac{F_{4m} + 2m}{5} - \frac{F_{2m}F_{2m-1} - F_{2m-4}F_{2m}}{2} \\
&= \frac{F_{4m} + 2m}{5} - \frac{F_{2m}(F_{2m-1} + F_{2m-4})}{2} \\
&= \frac{F_{4m+2m}}{5} + F_{2m}F_{2m-3}.
\end{aligned}
$$

□

Theorem 4.2.1. *[14]. For $n \geq 5$, the relation matrix between the relation matrix $\overline{x}_1$, $\overline{x}_2$, $\overline{x}_3$ of the abelian sandile group on C_n^2 is equivalent to*

$$
A_n = \begin{pmatrix} F_n & 0 & 0 \\ 0 & F_n & 0 \\ 0 & \frac{F_{4m}+2m}{5} & n \end{pmatrix},
$$

where $n = 2m$ is even or $n = 2m+1$ is odd. In other words, for $n \geq 5$, one has

$$
S(C_n^2) \cong \mathbb{Z}^3 / A_n \mathbb{Z}^3.
$$

Proof. In order to prove the theorem, it suffices to prove the relation matrix B_n is unimodular equivalent to A_n.
The odd case: $n = 2m+1$, $b_{n+1} = b_{2m+2} = c_{2m}$ and Proposition and Remark.

$$B_n = \begin{pmatrix} a_{2m+2}-1 & c_{2m} & -c_{2m+2} \\ -a_{2m+3} & -c_{2m+1}-1 & c_{2m+3} \\ a_{2m+4} & c_{2m+2} & -c_{2m+4}-1 \end{pmatrix}$$
$$\sim \begin{pmatrix} a_{2m+2}-1 & c_{2m} & -c_{2m+2} \\ -(a_{2m+2}+a_{2m+3}+1 & -(c_{2m}+c_{2m+1})-1 & c_{2m+2}+c_{2m+3} \\ a_{2m+3}+a_{2m+4} & c_{2m+1}+c_{2m+2}+1 & -(c_{2m+3}+c_{2m+4}-1) \end{pmatrix} R_1-R_2,\ R_3-R_2$$

$$\sim \begin{pmatrix} a_{2m+2}-1 & -(c_{2m+2}-c_{2m}) & -c_{2m+2} \\ -(a_{2m+2}+a_{2m+3}+1) & -(c_{2m}+c_{2m+1}-1+(c_{2m+2}+c_{2m+3})) & c_{2m+2}+c_{2m+3} \\ a_{2m+3}+a_{2m+4} & (c_{2m+1}+c_{2m+2})-(c_{2m+3}+c_{2m+4}) & -(c_{2m+3}+c_{2m+4})-1 \end{pmatrix}, \tag{4.29}$$

using identities from Remark 4.2.1 we calculate the values of entries β_{ij} of above matrix (4.29).

$\beta_{12} = -(c_{2m+2} - c_{2m})$
$= -(F_{2m}^2 - F_{2m-1}^2)$ by using (4.17)
$= -F_{2m-2}F_{2m+1}$,
$\beta_{21} = -(a_{2m+2} + a_{2m+3}) + 1$
$= -F_{2m}F_{2m-1} + 1$, by using (4.17)

$\beta_{22} = -(c_{2m} + c_{2m+1}) - 1 + (c_{2m+2} + c_{2m+3})$
As, $c_{2m} + c_{2m+1} = F_{2m-1}^2$ and $c_{2m+2} + c_{2m+3} = F_{2m+1}^2$ by using (4.17)
$= F_{2m+1}^2 - F_{2m-1}^2 - 1$,
$\beta_{31} = a_{2m+3} + a_{2m+4}$
As, $a_{n+1} + a_{n+2} = F_{n-1}F_{n-2}$ by (4.17) then
$= F_{2m}F_{2m+1}$,
$\beta_{32} = (c_{2m+1} + c_{2m+2}) - (c_{2m+3} + c_{2m+4})$
As, $c_{2m+1} + c_{2m+2} = F_{2m}^2$ and $c_{2m+3} + c_{2m+4} = F_{2m+2}^2$ by using (4.17)
$= F_{2m}^2 - F_{2m+2}^2$,
$\beta_{33} = -(c_{2m+3} + c_{2m+4}) - 1$
As, $c_{n+1} + c_{n+2} = F_n^2$
$=-F_{2m+2}^2 - 1$.
By substituting the values of β_{ij} in matrix (4.29), the matrix will be equal to,

$$\begin{pmatrix} a_{2m+2}-1 & -F_{2m-2}F_{2m+1} & -c_{2m+2} \\ -F_{2m-1}F_{2m}+1 & F_{2m+1}^2-F_{2m-1}^2-1 & F_{2m+1}^2 \\ F_{2m}F_{2m+1} & F_{2m}^2-F_{2m+2}^2 & -F_{2m+2}^2-1 \end{pmatrix}$$

$$\sim \begin{pmatrix} F_{2m}F_{2m+1} & F_{2m}^2-F_{2m+2}^2 & -F_{2m+2}^2-1 \\ -F_{2m-1}F_{2m}+1 & F_{2m+1}^2-F_{2m-1}^2-1 & F_{2m+1}^2 \\ a_{2m+2}-1 & -F_{2m-2}F_{2m+1} & -c_{2m+2} \end{pmatrix} R_1 \leftrightarrow R_3$$

$$\sim \begin{pmatrix} -F_{2m+2}^2-1 & F_{2m}^2-F_{2m+2}^2 & F_{2m}F_{2m+1} \\ F_{2m+1}^2 & F_{2m+1}^2-F_{2m-1}^2-1 & -F_{2m-1}F_{2m}+1 \\ -c_{2m+2} & -F_{2m-2}F_{2m+1} & a_{2m+2}-1 \end{pmatrix} C_1 \leftrightarrow C_3$$

$$\sim \begin{pmatrix} F_{2m+1}^2 & F_{2m+1}^2-F_{2m-1}^2-1 & -F_{2m-1}F_{2m}+1 \\ -F_{2m+2}^2-1 & F_{2m}^2-F_{2m+2}^2 & F_{2m}F_{2m+1} \\ -c_{2m+2} & -F_{2m-2}F_{2m+1} & a_{2m+2}-1 \end{pmatrix} R_1 \leftrightarrow R_2$$

$$\sim \begin{pmatrix} F_{2m+1}^2 & F_{2m+1}^2-F_{2m-1}^2-1 & -F_{2m-1}F_{2m}+1 \\ F_{2m+2}^2+1 & F_{2m+2}^2-F_{2m}^2 & -F_{2m}F_{2m+1} \\ -c_{2m+2} & -F_{2m-2}F_{2m+1} & a_{2m+2}-1 \end{pmatrix} (-1)R_2. \tag{4.30}$$

Now again we calculate the values of entries β_{ij} of above matrix (4.30).

$$\begin{aligned}
\beta_{12} =& F_{2m+1}^2 - F_{2m-1}^2 - 1 \\
=& F_{2m+1}^2 - (F_{2m-1}^2 + 1) \\
=& F_{2m+1}^2 - (F_{2m-1}^2 - F_{2m-2}^2 + F_{2m-2}^2 + 1) \\
=& F_{2m+1}^2 - (F_{2m}F_{2m-3} + F_{2m-1}F_{2m-3}) \\
=& F_{2m+1}^2 - F_{2m-3}(F_{2m} + F_{2m-1}) \quad \text{Using (4.20), (4.21)} \\
=& F_{2m+1}^2 - F_{2m-3}F_{2m+1} \\
=& F_{2m+1}(F_{2m+1} + F_{2m-3}) \\
=& F_{2m+1}(F_{2m} + F_{2m-1} - F_{2m-3}) \\
=& F_{2m+1}(F_{2m} + F_{2m-2} + F_{2m-3} - F_{2m-3}) \\
=& F_{2m+1}(F_{2m} + F_{2m-2}),
\end{aligned}$$

$$\begin{aligned}
\beta_{13} =& - F_{2m-1}F_{2m} + 1 \\
=& - (F_{2m-1}F_{2m} + (-1)^{2m+1}) \\
=& - F_{2m-2}F_{2m+1}, \quad \text{Using (4.22)}
\end{aligned}$$

$$
\begin{aligned}
\beta_{21} =& F_{2m+2}^2 + 1 \\
=& F_{2m+2}^2 + (-1)^{2m+2} \\
=& F_{2m+1}F_{2m+3}, \quad \text{Using (4.21)}
\end{aligned}
$$

$$
\begin{aligned}
\beta_{22} =& F_{2m+2}^2 - F_{2m}^2 \\
=& F_{2m+2}^2 - 1 - F_{2m}^2 + 1 \\
=& F_{2m+2}^2 + 1 - (F_{2m}^2 + 1) \\
=& F_{2m+3}F_{2m+1} - F_{2m+1}F_{2m-1} \quad \text{Using (4.21)} \\
=& F_{2m+1}(F_{2m+3} - F_{2m-1}) \\
=& F_{2m+1}(F_{2m+2} + F_{2m+1} - F_{2m-1}) \\
=& F_{2m+1}(F_{2m+2} + F_{2m} + F_{2m-1} - F_{2m-1}) \\
=& F_{2m+1}(F_{2m+2} + F_{2m}).
\end{aligned}
$$

By substituting the values in above matrix (4.30) will be equal to

$$
\sim \begin{pmatrix} F_{2m+1}^2 & F_{2m+1}(F_{2m} + F_{2m-2}) & -F_{2m+1}F_{2m-2} \\ F_{2m+1}F_{2m+3} & F_{2m+1}(F_2m + F_{2m+2}) & -F_{2m}F_{2m+1} \\ -c_{2m+2} & -F_{2m-2}F_{2m+1} & a_{2m+2} - 1 \end{pmatrix}
$$

$$
\sim \begin{pmatrix} F_{2m+1}^2 & F_{2m+1}^2 + F_{2m+1}(F_{2m} + F_{2m-2}) & -F_{2m+1}F_{2m-2} \\ F_{2m+1}F_{2m+3} & F_{2m+1}F_{2m+3} + F_{2m+1}(F_{2m} + F_{2m+2}) & -F_{2m}F_{2m+1} \\ -c_{2m+2} & -c_{2m+2} - F_{2m-2}F_{2m+1} & a_{2m+2} - 1 \end{pmatrix} C_1 + C_2.
$$

As, the sequence F_n of Fibonacci numbers is defined by the recurrence relation $F_n = F_{n-1} + F_{n-2},$ then
$F_{2m+1} = F_{2m} + F_{2m-1} \; and \; F_{2m} = F_{2m} - F_{2m-1}$
$F_{2m-1} = F_{2m} - F_{2m-1}.$

Put these values in $F_{2m+1}^2 + F_{2m+1}(F_{2m} + F_{2m-2})$
$= F_{2m+1}(F_{2m+1} + F_{2m} + F_{2m-2})$
$= F_{2m+1}(F_{2m} + F_{2m-1} + F_{2m} + F_{2m} - F_{2m-1})$
$= 3F_{2m}F_{2m+1}.$
Now $F_{2m+1}F_{2m+3} + F_{2m+1}(F_{2m} + F_{2m+2})$
$= F_{2m+1}(F_{2m+3} + (F_{2m} + F_{2m+2})$
$= F_{2m+1}(F_{2m+1} + F_{2m+2} + F_{2m} + F_{2m+1} + F_{2m})$
$= F_{2m+1}(2F_{2m+1} + 2F_{2m} + F_{2m+1} + F_{2m})$ By putting the value of F_{2m+2}
$= F_{2m+1}(3F_{2m} + F_{2m+1})$
$= 3F_{2m+1}(F_{2m} + F_{2m+1}$
$= 3F_{2m+1}F_{2m+2}.$

By substituting value of $F_{2m+1}^2 + F_{2m+1}(F_{2m} + F_{2m-2})$ and $F_{2m+1}F_{2m+3} + F_{2m+1}(F_2 m + F_{2m+2})$ above matrix becomes,

$$\begin{pmatrix} F_{2m+1}^2 & 3F_{2m+1}F_{2m} & -F_{2m+1}F_{2m-2} \\ F_{2m+1}F_{2m+3} & 3F_{2m+1}F_{2m+2} & -F_{2m}F_{2m+1} \\ -c_{2m+2} & -c_{2m+2} - F_{2m-2}F_{2m+1} & a_{2m+2} - 1 \end{pmatrix}$$

$$= \begin{pmatrix} F_{2m+1} & F_{2m} & 0 \\ F_{2m+3} & F_{2m+2} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} F_{2m+1} & 0 & F_{2m+1} \\ 0 & 3F_{2m+1} & -2F_{2m+1} \\ -c_{2m+2} & -c_{2m+2} - F_{2m-2}F_{2m+1} & a_{2m+2} - 1 \end{pmatrix}.$$

Since

$$\begin{aligned} det \begin{pmatrix} F_{2m+1} & F_{2m} & 0 \\ F_{2m+3} & F_{2m+2} & 0 \\ 0 & 0 & 1 \end{pmatrix} &= det \begin{pmatrix} F_{2m+1} & F_{2m} \\ F_{2m+2} & F_{2m+1} \end{pmatrix} \\ &= F_{2m+1}^2 - F_{2m}F_{2m+2} \\ &= F_{2m}F_{2m+2} - (-1)^{2m+1} - F_{2m}F_{2m+2} \\ &= -(-1) = 1, \end{aligned}$$

$$B_n \sim \begin{pmatrix} F_{2m+1} & 0 & F_{2m+1} \\ 0 & 3F_{2m+1} & -2F_{2m+1} \\ -c_{2m+2} & -c_{2m+2} - F_{2m-2}F_{2m+1} & a_{2m+2} - 1 \end{pmatrix}$$

$$\sim \begin{pmatrix} F_{2m+1} & 0 & 0 \\ 0 & 3F_{2m+1} & -2F_{2m+1} \\ -c_{2m+2} & -c_{2m+2} - F_{2m-2}F_{2m+1} & a_{2m+2} + c_{2m+2} - 1 \end{pmatrix} C_3 - C_1$$

$$\sim \begin{pmatrix} F_{2m+1} & 0 & 0 \\ 0 & F_{2m+1} & -2F_{2m+1} \\ -c_{2m+2} & -F_{2m-2}F_{2m+1} + a_{2m+2} & a_{2m+2} + c_{2m+2} - 1 \end{pmatrix} C_2 + C_3$$

$$\sim \begin{pmatrix} F_{2m+1} & 0 & 0 \\ 0 & F_{2m+1} & 0 \\ -c_{2m+2} & y & x \end{pmatrix} 2C_2 + C_3,$$

where

$$\begin{aligned} x &= a_{2m+2} + c_{2m+2} - 1 + 2(-F_{2m-2}F_{2m+1} + a_{2m+2} - 1) \\ &= 3(a_{2m+2} - 1) + c_{2m+2} - 2F_{2m-2}F_{2m+1} \\ &= 3(F_{2m-2}F_{2m+1} - \tfrac{F_{4m}+2m}{5}) + (F_{2m}F_{2m+1} - \tfrac{2(F_{4m}+2m)}{5} - 2F_{2m-2}F_{2m+1}) \\ &= 3F_{2m-2}F_{2m+1} - \tfrac{3(F_{4m}+2m)}{5} + F_{2m}F_{2m+1} - \tfrac{2(F_{4m}+2m)}{5} - 2F_{2m-2}F_{2m+1}) \\ &= F_{2m}F_{2m+1} + F_{2m-2}F_{2m+1} - F_{4m} - 2m \\ &= F_{2m}F_{2m+1} + F_{2m}F_{2m-1} - 1 - F_{4m} - 2m \\ &= F_{2m}(F_{2m+1} + F_{2m-1}) - F_{4m} - 2m - 1 \\ &= -(2m+1), \end{aligned}$$

And $\quad y = -F_{2m-2}F_{2m+1} + a_{2m+2} + 1$

$= -F_{2m-2}F_{2m+1} + (F_{2m-2}F_{2m+1} - \frac{F_{4m+2m}}{5}),$

$= -\frac{F_{4m+2m}}{5}.$

Hence

$$B_n \sim \begin{pmatrix} F_{2m+1} & 0 & 0 \\ 0 & F_{2m+1} & 0 \\ -c_{2m+2} & -\frac{F_{4m+2m}}{5} & -(2m+1) \end{pmatrix}$$

$$\sim \begin{pmatrix} F_{2m+1} & 0 & 0 \\ 0 & F_{2m+1} & 0 \\ \frac{2(F_{4m+2m})}{5} & -\frac{F_{4m+2m}}{5} & -(2m+1) \end{pmatrix}$$

$$\sim \begin{pmatrix} F_{2m+1} & 0 & 0 \\ 0 & F_{2m+1} & 0 \\ \frac{-2(F_{4m+2m})}{5} & \frac{F_{4m+2m}}{5} & 2m+1 \end{pmatrix} (-1)R_3$$

$$\sim \begin{pmatrix} F_{2m+1} & 0 & 0 \\ 0 & F_{2m+1} & 0 \\ 0 & \frac{F_{4m+2m}}{5} & 2m+1 \end{pmatrix} 2C_1 + 2C_2.$$

The even case: $n = 2m$, by $b_{n+1} = b_{2m+2} = c_{2m}$ and Proposition and Remark 4.2.1,

$$B_n = \begin{pmatrix} a_{2m+1}+1 & c_{2m-1} & -c_{2m+1} \\ -a_{2m+2} & -c_{2m}+1 & c_{2m+2} \\ a_{2m+3} & c_{2m+1} & -c_{2m+3}+1 \end{pmatrix}$$

$$\sim \begin{pmatrix} a_{2m+1}+1 & c_{2m-1} & -c_{2m+1} \\ -(a_{2m+1}+a_{2m+2})-1 & -(c_{2m-1}+c_{2m})+1 & c_{2m+1}+c_{2m+2} \\ a_{2m+2}+a_{2m+3} & c_{2m}+c_{2m+1} & -(c_{2m+2}+c_{2m+3})+1 \end{pmatrix} R_1 - R_2,\ R_3 - R_2$$

$$\sim \begin{pmatrix} a_{2m+1}+1 & -(c_{2m+1}-c_{2m-1}) & -c_{2m+1} \\ -(a_{2m+1}+a_{2m+2})-1 & -(c_{2m-1}+c_{2m})+1+c_{2m+1}+c_{2m+2} & c_{2m+1}+c_{2m+2} \\ a_{2m+2}+a_{2m+3} & c_{2m}+c_{2m+1}-(c_{2m+2}+c_{2m+3})+1 & -(c_{2m+2}+c_{2m+3})+1 \end{pmatrix} C_2 + C_3.$$

Using identities from Remark 4.2.1 we calculate the values of entries α_{ij} of above matrix.

$\alpha_{12} = -(c_{2m+1} - c_{2m-1})$
$\quad = -(F^2_{2m-1} - F^2_{2m-2})$ by using (4.17)
$\quad = -F_{2m-3}F_{2m},$
$\alpha_{21} = -(a_{2m+1} + a_{2m+2}) - 1$
$\quad = -F_{2m-1}F_{2m-2} - 1,$ by using (4.17)

$\alpha_{22} = -(c_{2m-1} + c_{2m}) + 1 + (c_{2m+1} + c_{2m+2})$
As, $c_{2m-1} + c_{2m} = F^2_{2m-2}$ and $c_{2m+1} + c_{2m+2} = F^2_{2m}$ by using (4.17)
$\quad = F^2_{2m} - F^2_{2m-2} + 1,$
$\alpha_{31} = a_{2m+2} + a_{2m+3}$
As, $a_{n+1} + a_{n+2} = F_{n-1}F_{n-2}$ by (4.17) then
$\quad = F_{2m-1}F_{2m}$

$\alpha_{32} = (c_{2m} + c_{2m+1}) - (c_{2m+2} + c_{2m+3})$
As, $c_{2m} + c_{2m+1} = F^2_{2m-1}$ and $c_{2m+2} + c_{2m+3} = F^2_{2m+1}$ by using (4.17)
$= F^2_{2m-1} - F^2_{2m+1}$,
$\beta_{33} = -(c_{2m+2} + c_{2m+3}) + 1$
As, $c_{n+1} + c_{n+2} = F^2_n$
$=-F^2_{2m+1} + 1$.
By substituting the values of α_{ij} in matrix (**??**), the matrix will be equal to,

$$\begin{pmatrix} a_{2m+1}+1 & -F_{2m-3}F_{2m} & -c_{2m+1} \\ -F_{2m-2}F_{2m-1}-1 & F^2_{2m}-F^2_{2m-2}+1 & F^2_{2m} \\ F_{2m-1}F_{2m} & F^2_{2m-1}-F^2_{2m+1} & -F^2_{2m+1}+1 \end{pmatrix}$$

$$\sim \begin{pmatrix} F_{2m-1}F_{2m} & F^2_{2m-1}-F^2_{2m+1} & -F^2_{2m+1}+1 \\ -F_{2m-2}F_{2m-1}-1 & F^2_{2m}-F^2_{2m-2}+1 & F^2_{2m} \\ a_{2m+1}+1 & -F_{2m-3}F_{2m} & -c_{2m+1} \end{pmatrix} R_1 \leftrightarrow R_3$$

$$\sim \begin{pmatrix} -F^2_{2m+1}+1 & F^2_{2m-1}-F^2_{2m+1} & F_{2m-1}F_{2m} \\ F^2_{2m} & F^2_{2m}-F^2_{2m-2}+1 & -F_{2m-2}F_{2m-1}-1 \\ -c_{2m+1} & -F_{2m-3}F_{2m} & a_{2m+1}+1 \end{pmatrix} C_1 \leftrightarrow C_3$$

$$\sim \begin{pmatrix} F^2_{2m} & F^2_{2m}-F^2_{2m-2}+1 & -F_{2m-2}F_{2m-1}-1 \\ -F^2_{2m+1}+1 & F^2_{2m-1}-F^2_{2m+1} & F_{2m-1}F_{2m} \\ -c_{2m+1} & -F_{2m-3}F_{2m} & a_{2m+1}+1 \end{pmatrix} R_1 \leftrightarrow R_2$$

$$\sim \begin{pmatrix} F^2_{2m} & F^2_{2m}-F^2_{2m-2}+1 & -F_{2m-2}F_{2m-1}-1 \\ F^2_{2m+1}-1 & F^2_{2m+1}-F^2_{2m-1} & -F_{2m-1}F_{2m} \\ -c_{2m+1} & -F_{2m-3}F_{2m} & a_{2m+1}+1 \end{pmatrix} (-1)R_2.$$

Now again we calculate the values of entries α_{ij} of above matrix .

$\alpha_{12} =F^2_{2m} - F^2_{2m-2} + 1$
$=F^2_{2m} - (F^2_{2m-2} - 1)$
$=F^2_{2m} - (F^2_{2m-2} - F^2_{2m-3} + F^2_{2m-3} - 1)$
$=F^2_{2m} - (F_{2m-1}F_{2m-4} + F_{2m-2}F_{2m-4})$
$=F^2_{2m} - F_{2m-4}(F_{2m-1} + F_{2m-2})$ Using (4.20), (4.21)

$$\begin{aligned}
&= F_{2m}^2 - F_{2m-4}F_{2m} \\
&= F_{2m}(F_{2m} - F_{2m-4}) \\
&= F_{2m}(F_{2m-1} + F_{2m-2} - F_{2m-4}) \\
&= F_{2m}(F_{2m-1} + F_{2m-3} + F_{2m-4} - F_{2m-4}) \\
&= F_{2m}(F_{2m-1} + F_{2m-3})\alpha_{13} \\
&= - F_{2m-2}F_{2m-1} - 1 \\
&= - (F_{2m-2}F_{2m-1} + (-1)^{2m}) \\
&= - F_{2m-3}F_{2m}, \quad \text{Using (4.22)}
\end{aligned}$$

$$\begin{aligned}
\alpha_{21} &= F_{2m+1}^2 - 1 \\
&= F_{2m+1}^2 + (-1)^{2m+1} \\
&= F_{2m}F_{2m+2}, \quad \text{Using (4.21)}
\end{aligned}$$

$$\begin{aligned}
\alpha_{22} &= F_{2m+1}^2 - F_{2m-1}^2 \\
&= F_{2m+1}^2 - 1 - F_{2m-1}^2 + 1 \\
&= F_{2m+1}^2 - 1 - (F_{2m-1}^2 - 1) \\
&= F_{2m+2}F_{2m} - F_{2m-2}F_{2m} \quad \text{Using (4.21)} \\
&= F_{2m}(F_{2m+2} - F_{2m-2}) \\
&= F_{2m}(F_{2m+1} + F_{2m} - F_{2m-2}) \\
&= F_{2m}(F_{2m+1} + F_{2m-1} + F_{2m-2} - F_{2m-2}) \\
&= F_{2m}(F_{2m+1} + F_{2m-1}).
\end{aligned}$$

By substituting the values the above matrix will be equal to

$$\sim \begin{pmatrix} F_{2m}^2 & F_{2m}(F_{2m-1} + F_{2m-3}) & -F_{2m}F_{2m-3} \\ F_{2m}F_{2m+2} & F_{2m}(F_{2m-1} + F_{2m+1}) & -F_{2m-1}F_{2m} \\ -c_{2m+1} & -F_{2m-3}F_{2m} & a_{2m+1} + 1 \end{pmatrix}$$

$$\sim \begin{pmatrix} F_{2m}^2 & F_{2m}^2 + F_{2m}(F_{2m-1} + F_{2m-3}) & -F_{2m}F_{2m-3} \\ F_{2m}F_{2m+2} & F_{2m}F_{2m+2} + F_{2m}(F_{2m-1} + F_{2m+1}) & -F_{2m-1}F_{2m} \\ -c_{2m+1} & -c_{2m+1} - F_{2m-3}F_{2m} & a_{2m+1} + 1 \end{pmatrix} C_1 + C_2.$$

As, the sequence F_n of Fibonacci numbers is defined by the recurrence relation $F_n = F_{n-1} + F_{n-2}$, using this recurrence relation,

$$\begin{aligned}
& F_{2m}^2 + F_{2m}(F_{2m-1} + F_{2m-3}) \\
&= F_{2m}(F_{2m} + F_{2m-1} - F_{2m-3}) \\
&= F_{2m}(F_{2m-1} + F_{2m-2} + F_{2m-1} + F_{2m-1} - F_{2m-2}) \\
&= 3F_{2m-1}F_{2m}.
\end{aligned}$$

$$\begin{aligned}
\text{Now}\quad & F_{2m}F_{2m+2} + F_{2m}(F_{2m-1} + F_{2m+1}) \\
=& F_{2m}(F_{2m+2} + F_{2m-1} + F_{2m+1}) \\
=& F_{2m}(F_{2m} + F_{2m+1} + F_{2m-1} + F_{2m} + F_{2m-1}) \\
=& F_{2m}(2F_{2m} + 2F_{2m-1} + F_{2m+1}) \\
=& F_{2m}(2F_{2m} + 2F_{2m-1} + F_{2m} + F_{2m-1}) \\
=& 3F_{2m}(F_{2m} + F_{2m-1}) \\
=& 3F_{2m}F_{2m+1}.
\end{aligned}$$

By putting the value of $F_{2m}^2 + F_{2m}(F_{2m-1}$ and $F_{2m}F_{2m+2} + F_{2m}(F_{2m-1} + F_{2m+1}$ the above matrix becomes,

$$\begin{pmatrix} F_{2m}^2 & 3F_{2m-1}F_{2m} & -F_{2m}F_{2m-3} \\ F_{2m}F_{2m+2} & 3F_{2m}F_{2m+1} & -F_{2m-1}F_{2m} \\ -c_{2m+1} & -c_{2m+1} - F_{2m-3}F_{2m} & a_{2m+1} + 1 \end{pmatrix}$$

$$= \begin{pmatrix} F_{2m} & F_{2m-1} & 0 \\ F_{2m+2} & F_{2m+1} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} F_{2m} & 0 & F_{2m} \\ 0 & 3F_{2m} & -2F_{2m} \\ -c_{2m+1} & -c_{2m+1} - F_{2m-3}F_{2m} & a_{2m+1} + 1 \end{pmatrix}.$$

Since

$$\begin{aligned}
det \begin{pmatrix} F_{2m} & F_{2m-1} & 0 \\ F_{2m+2} & F_{2m+1} & 0 \\ 0 & 0 & 1 \end{pmatrix} =& det \begin{pmatrix} F_{2m} & F_{2m-1} \\ F_{2m+1} & F_{2m} \end{pmatrix} \\
=& F_{2m}^2 - F_{2m-1}F_{2m+1} \\
=& F_{2m-1}F_{2m+1} - (-1)^{2m} - F_{2m-1}F_{2m+1} \\
=& -1,
\end{aligned}$$

$$\begin{aligned}
B_n \sim & \begin{pmatrix} F_{2m} & 0 & F_{2m} \\ 0 & 3F_{2m} & -2F_{2m} \\ -c_{2m+1} & -c_{2m+1} - F_{2m-3}F_{2m} & a_{2m+1} + 1 \end{pmatrix} \\
\sim & \begin{pmatrix} F_{2m} & 0 & 0 \\ 0 & 3F_{2m} & -2F_{2m} \\ -c_{2m+1} & -c_{2m+1} - F_{2m-3}F_{2m} & a_{2m+1} + c_{2m+2} + 1 \end{pmatrix} C_3 - C_1 \\
\sim & \begin{pmatrix} F_{2m} & 0 & 0 \\ 0 & F_{2m} & -2F_{2m} \\ -c_{2m+1} & -F_{2m-3}F_{2m} + a_{2m+1} + 1 & a_{2m+1} + c_{2m+1} + 1 \end{pmatrix} C_2 + C_3
\end{aligned}$$

$$\sim \begin{pmatrix} F_{2m} & 0 & 0 \\ 0 & F_{2m} & 0 \\ -c_{2m+1} & y & x \end{pmatrix} C_3 + 2C_2,$$

where

$$x = a_{2m+1} + c_{2m+1} + 1 + 2(-F_{2m-3}F_{2m} + a_{2m+1} + 1)$$

$$= 3(a_{2m+1} + 1) + c_{2m+1} - 2F_{2m-3}F_{2m}$$

$$= 3(\tfrac{F_{4m}+2m}{5} + F_{2m-3}F_{2m}) + \tfrac{2(F_{4m}+2m)}{5} - F_{2m}F_{2m-1} - 2F_{2m-3}F_{2m}$$

$$= F_{4m} + 2m - F_{2m-3}F_{2m} - F_{2m-1}F_{2m}$$

$$= F_{4m} + 2m - F_{2m}(F_{2m-1} + F_{2m-3})$$

$$= 2m,$$

and $y = -F_{2m-3}F_{2m+1} + a_{2m+1}$

$$= -F_{2m-3}F_{2m} + \tfrac{F_{4m+2m}}{5} + F_{2m-3} + F_{2m}$$

$$= \tfrac{F_{4m+2m}}{5}.$$

Hence

$$B_n \sim \begin{pmatrix} F_{2m} & 0 & 0 \\ 0 & F_{2m} & 0 \\ -c_{2m+1} & \frac{F_{4m+2m}}{5} & 2m \end{pmatrix}$$

$$\sim \begin{pmatrix} F_{2m} & 0 & 0 \\ 0 & F_{2m} & 0 \\ \frac{2(F_{4m+2m})}{5} & \frac{F_{4m+2m}}{5} & 2m \end{pmatrix}$$

$$\sim \begin{pmatrix} F_{2m} & 0 & 0 \\ 0 & F_{2m} & 0 \\ 0 & \frac{F_{4m+2m}}{5} & 2m \end{pmatrix} C_1 - 2C_2.$$

□

Corollary 4.2.1. *[16]. The number of spanning trees in C_n^2 is*

$$\tau(C_n^2) = nF_n^2.$$

4.3 Analysis of the coefficient of the Smith normal form of $S(C_n^2)$

In this section structure of sandpile group $S(C_n^2)$ will be determined. It is known that the sequence $\{F_n \mod 5^e\}_{n \geqslant 0}$ is periodic and the least period of this sequence is $4 \cdot 5^e$ for each $e \geqslant 1$ [18]. Hence

$$F_{x+4y5^e} \equiv F_x mod 5^e.$$

As $F_{-n} = (1)^{n+1} F_n$. The following identity can be found in [13] , which can be proven by induction.

$$F_{kn+r} = \sum_{h=0}^{k} \binom{k}{h} F_n^h F_{n-1}^{k-h} F_{r+h}. \tag{4.31}$$

for $k \geqslant 0$ and every integer r.
From the above identity and the facts $F_1 = F_1 = F_2 = 1, F_0 = 0, F_3 = 2, F_4 = 3, F_5 = 5$, one has

$$F_{5n} = F_n \left(5F_{n-1}^4 + 10F_n F_{n-1}^3 + 20F_n^2 F_{n-1}^2 + 15F_n^3 F_{n-1} + 5F_n^4\right), \tag{4.32}$$
$$F_{5n-1} = F_{n-1}^5 + 10F_n^2 F_{n-1}^3 + 15F_n^3 F_{n-1} + 10F_n^4 F_{n-1} + 3F_n^5, \tag{4.33}$$
$$F_{5n+1} = F_{n-1}^5 + 5F_n F_{n-1}^4 + 20F_n^2 F_{n-1}^3 + 30F_n^3 F_{n-1} + 25F_n^4 F_{n-1} + 8F_n^5. \tag{4.34}$$

Now identities of Fibonacci numbers modulo 5^{e+1} will be shown, which will be used in the proof of the main result of this section.

Lemma 4.3.1. *[14]. Let $e \geqslant 0$. Then*

$$F_{4 \cdot 5^e} \equiv 3 \cdot 5^e \mod 5^{e+1},$$
$$F_{4 \cdot 5^{e-1}} \equiv 5_{e+1} \mod 5^{e+1},$$
$$F_{4 \cdot 5^{e+1}} \equiv 4 \cdot 5^{e+1} \mod 5^{e+1}.$$

Proof. This lemma can be proven by induction on e. If $e = 0$ the lemma is clearly true, and when $e = 1$, $F_{20} = 6765 \equiv 15 \mod 5^2$, $F_{19} = 4181 \equiv 6 \mod 52$ and $F_{21} = 10,946 \equiv 21 \mod 52$.
Suppose the lemma is true for some $e \geqslant 1$. Since $2e + 1 \geqslant e + 2$ for $e \geqslant 1$ and by

virtue of the induction assumption, the following relations can be obtained:

$$\begin{aligned} 5F_{4\cdot 5^e} &= 5(3\cdot 5^e + a5^{e+1}) \qquad \text{for some } a \in \mathbb{Z} \\ &= 5^{2e+1}(3+5a)^2 \\ &\equiv 0 \mod 5^{e+2}, \\ 5F_{4\cdot 5^e}F^4_{4\cdot 5^2-1} &= 5(3\cdot 5^e + a5^{e+1})(5^e+1+b5^{e+1})^4 \qquad \text{for some } a,\ b \in \mathbb{Z} \\ &\equiv 5^{e+1}(3+5a) \mod 5^{e+2} \\ &\equiv 3\cdot 5^{e+1} \mod 5^{e+2}, \\ F_{4\cdot 5^{e1}} &= (5_e + 1 + b5^{e+1})^5 \\ &\equiv 5^{e+1} + 1 \mod 5^{e+2}. \end{aligned}$$

Thus, by Eqs. (4.33) to (4.34) and the induction assumption, the following relations can be obtained:

$$\begin{aligned} F_{4\cdot 5^{e+1}-1} &= F_{5\cdot 4\cdot 5^e - 1} \equiv F^5_{4\cdot 5^e-1} \mod 5^{e+2} \\ &\equiv 5^{e+1} + 1 \mod 5^{e+2}, \\ F_{4\cdot 5^{e+1}} &= F_{5\cdot 4\cdot 5^e} \equiv 5F_{4.5^e}F^4_{4.5^e-1} \\ &\equiv 3\cdot 5^{e+1} \mod 5^{e+2}, \\ F_{4\cdot 5^{e+1}+1} &= F_{5\cdot 4\cdot 5^e+1} \\ &\equiv F^5_{c\cdot 5^e-1}5 + F_{4\cdot 5^e}F^4_{4\cdot 5^e-1} \mod 5^{e+2} \\ &\equiv 4\cdot 5^{e+1} + 1 \mod 5^{e+2}. \end{aligned}$$

This completes the proof. □

Lemma 4.3.2. *[14] Let $e \geqslant 0$. Then*

$$\begin{aligned} &F_{2\cdot 5^e-2} \equiv 4\cdot 5^e + 1 \mod 5^{e+1}, \\ &F_{6\cdot 5^e-2} \equiv 2\cdot 5^e + 1 \mod 5^{e+1}, \\ &F_{14\cdot 5^e-2} \equiv 3\cdot 5^e + 1 \mod 5^{e+1}, \\ &F_{18\cdot 5^e-2} \equiv 5^e + 1 \mod 5^{e+1}. \end{aligned}$$

If $5^e|m$ then $5^e|F_{4m}$ and thus $5^e|F_{4m} + 2m$. The next two lemmas are more.

Lemma 4.3.3. *[14]. Let $5^e|m$. Then $5^e|\frac{F_{4m+2m}}{5}$.*

Proof. If $5^{e+1}|m$ then $5^{e+1}|F_{2m} + 2m$ and the result follows. Let $5^e|m$ but $5^{e+1} \nmid m$. Then $m = 5^{e+1}y + x5^e$, for some $y, x \in \mathbb{Z}$ and $1 \leqslant x \leqslant 4$. Hence $2m = 2x5^e \mod 5^{e+1}$ and by Lemma 6, then

$$\begin{aligned} F_{4m} &= F_{4x5^e+4y5^{e+1}} \\ &\equiv F_{4x5^e} \mod 5^{e+1} \end{aligned}$$

$$\equiv \begin{cases} 3\cdot 5^e \mod 5^{e+1}, & x=1 \\ 5^e \mod 5^{e+1}, & x=2 \\ 4\cdot 5^e \mod 5^{e+1}, & x=3 \\ 2\cdot 5^e \mod 5^{e+1}, & x=4. \end{cases}$$

Hence $5^e | \frac{F_{4m}+2m}{5}$. □

Lemma 4.3.4. *[14]. Let $5^e | 2m+1$. Then $5^e | \frac{F_{4m}+2m}{5}$.*

Proof.

$$\frac{F_{4m}+2m}{5} = \frac{F_{4m}1+2m+1}{5} = \frac{F_{2m+1}(F_{2m}+F_{2m2})+2m+1}{5}.$$

As in the proof of the above lemma, let $5^e | 2m+1$ but $5^{e+1} \nmid 2m+1$. Thus $2m+1 = 5^{e+1}y + x5^e$, $y, x \in \mathbb{Z}$ and $1 \leqslant x \leqslant 4$, and just one of x and y is odd. Then $4m = 2y5^{e+1} + 2x5^e - 2$, $2m = x5^e - 1 \mod 5^{e+1}$. If $y = 2a$ and x is odd, then

$$\begin{aligned} F_{4m} &= F_{2x5^e+4a5^{e+1}-2} \\ &\equiv F_{2x5^e-2} \mod 5^{e+1} \end{aligned}$$

$$\equiv \begin{cases} 4\cdot 5^e+1 \mod 5^{e+1}, & x=1 \\ 2\cdot 5^e+1 \mod 5^{e+1}, & x=3. \end{cases}$$

If $y = 2a+1$ and x is even, then

$$\begin{aligned} F_{4m} &= F_{(2x+10)5^e+4a5^{e+1}-2} \\ &\equiv F_{(2x+10)5^e-2} \mod 5^{e+1} \end{aligned}$$

$$\equiv \begin{cases} 3\cdot 5^e+1 \mod 5^{e+1}, & x=2 \\ 5^e+1 \mod 5^{e+1}, & x=4. \end{cases}$$

Hence $5^e | \frac{F_{4m}+2m}{5}$. □

Theorem 4.3.1. *Let $n \geqslant 3$. Then*

$$S(C_n^2) \cong \mathbb{Z}_{(n,F_n)} \oplus \mathbb{Z}_{F_n} \oplus \mathbb{Z}_{\frac{nF_n}{(n,F_n)}}.$$

Proof. For $n = 3$ or 4, the result follows from that $S(C_3^2) \cong \mathbb{Z}_2 \oplus \mathbb{Z}_6$ and $S(C_4^2) \cong \mathbb{Z}_3 \oplus \mathbb{Z}_{12}$. For $n \geqslant 5$, it suffices to prove the Smith normal form $S = \mathrm{diag}(S_{11}, S_{22}, S_{33})$ of the matrix A_n in Theorem (4.2.1) is the diagonal matrix:

$$\begin{pmatrix} (n,F_n) & 0 & 0 \\ 0 & F_n & 0 \\ 0 & 0 & \frac{nF_n}{(n,F_n)} \end{pmatrix}.$$

For finding the coefficient of smith normal form of matrix A_n we use remark (3.3.1)
Computation of S_{11}.
S_{11}= Greatest common divisor of all 1×1 determinants.
For $n = 2m$ or $n = 2m + 1$ so $S_{11} = (n,\ F_n,\ \frac{F_{4m}+2m}{5})$,
Computation of S_{22}.
S_{22}=Greatest common divisor of all 2×2 determinants.
For $n = 2m$ or $n = 2m+1$ so $S_{22} = (nFn,\ F_n^2,\ F_n\frac{F_{4m}+2m}{5}) = F_n(n,\ F_n,\ \frac{F_{4m}+2m}{5})$ $= S_{11}F_n$, $S_{22} = F_n$,
Computation of S_{33}.
$S_{11}S_{22}S_{33}$= determinant of A_n.
For $n = 2m$ or $n = 2m + 1$ so $S_{11}S_{22}S_{33} = nF_n^2$. $S_{33} = \frac{nF_n}{(n,\ F_n)}$.

In order to prove theorem it is sufficient to prove that $S_{11} = (n,\ F_n,\ \frac{F_{4m+2m}}{5}) = (n,\ F_n)$.

The even case: $n = 2m$. For each common factor d of $2m$ and F_{2m}, if $(d,\ 5) = 1$ then d is also a factor of $\frac{F_{2m}(F_{2m-1}+F_{2m+1}+2m)}{5} = \frac{F_{4m}+2m}{5}$; if $d = 5^e$ for some $e \geq 1$, then d is also a factor of $\frac{F_{4m}+2m}{5}$ by lemma 4.3. Hence, $(2m,\ F_{2m},\ \frac{F_{4m}+2m}{5}) = (2m,\ F_{2m})$.

The odd case: $n = 2m + 1$. As $\frac{F_{4m}+2m}{5} = \frac{F_{2m+1}(F_{2m}+F_{2m-2}+2m+1)}{5}$, for each common factor d of $2m + 1$ and F_{2m+1}, if $(d, 5) = 1$ then d is also a factor of $\frac{F_{4m}+2m}{5}$; if $d = 5^e$ for some $e \geq 1$, then d is also a factor of $\frac{F_{4m}+2m}{5}$ by lemma 4.3. Hence, $(2m + 1,\ F_{2m+1},\ \frac{F_{4m}+2m}{5}) = (2m + 1,\ F_{2m+1})$.

□

Hence, the sandpile group of square cycle graph $S(C_n^2)$ is,

$$S(C_n^2) \cong \mathbb{Z}_{(n,F_n)} \oplus \mathbb{Z}_{F_n} \oplus \mathbb{Z}_{\frac{nF_n}{(n,F_n)}}.$$

4.4 Concluding remarks

The main objective of the work was to study the structure of abelian sandpile group of square cycle graph C_n^2 and also to give the explicit Smith normal form of this group. Pointing out that that sandpile group is never cyclic but is always the direct sum of two or three cycles. As by-product this implies that cyclic group $\mathbb{Z}_{F_n}$ is always a summand of sandpile group of square cycle graph C_n^2.

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