Methods for Solutions of Fractional Differential Equations

by

Amna Idrees

Supervised by

Dr. Mujeeb ur Rehman

School of Natural Sciences,

National University of Sciences and Technology, Islamabad

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Examination Committee Members

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2. Name: Dr. Umer Saeed Signature:

3. Name: ____________________ Signature: ____________________

4. Name: Dr. Nasir Rehman Signature:

Supervisor's Name: Dr. Mujeeb ur Rehman Signature:

Head of Department

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Name of Supervisor: Dr. Mujeeb ur Rehman

Date: 12/04/17

Signature (HoD): ____________________

Date: 12-04-17

Signature (Dean/Principal): ____________________

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Dedicated

to

My Beloved Parents

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Abstract

Fractional calculus adds to the beauty of conventional calculus by extending the concept of integer order differentiation and integration to arbitrary order. This thesis directs to existence and uniqueness theory and methods to solve fractional differential equations. This thesis begins with history and introduction to basic concepts related to fractional calculus. We have seen the existence and uniqueness results for initial value problems. Power series method and Laplace transform method for solving fractional differential equations are discussed.

In literature many numerical methods have been established to solve fractional differential equations. In this thesis we have used combination of two quadrature methods to solve fractional differential equations numerically. Particularly we have combined trapezoidal rule and Simpson's rule to solve initial value problems for fractional differential equations. We have transformed the class of fractional differential equations to a system of algebraic equations and then developed operational matrices. Matlab programmes are designed to seek the solution of algebraic system. The proposed method can even work successfully for equations involving weakly singular kernel. The efficiency of this method is revealed by solving a variety of problems. We have compared the results with exact solutions. We have also compared the results for the proposed method with some other numerical methods to check the reliability and efficiency of the method.

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Chapter 1

Introduction

Fractional calculus is a field of mathematics that deals with the derivatives and integrals of non-integer order. It can be marked as old as classical calculus that deals with derivatives and integrals of integer order. Derivatives and integrals of fractional order are considered as non-local operators since both these operators involve integration which is a non-local operator (as it is defined on an interval). This property makes these operators an efficient and powerful tool to define long term memory effects, asymptotic scaling and hereditary properties of various physical phenomena.

Many mathematicians put their interest and efforts in the field of fractional calculus and also developed their own definitions for derivatives and integrals of fractional order. The most famous definitions in the field of fractional calculus are the Riemann-Liouville, Caputo, Weyl and Grunwald-Letnikov definitions. In general, these definitions are not alike except for some special class of functions. The most often used definition of fractional integral and derivative is given by Riemann and Liouville. But the situations in which this definition is not applicable we prefer to use Caputo's approach.

The increasing number of applications of fractional differential equations motivated many researchers and appreciable work has been done to develop efficient methods for exact and numerical solutions of fractional differential equations. Mathematical models are widely used to explain the behavior of physical phenomena. The best way to evaluate these models is by using calculus. These are the analytic methods, because we use analysis to reach the solution. But this tends to work only for simple models. For more complex models, the solution becomes too complicated. Then we turn to numerical methods for solving the problem. Many numerical methods are developed to solve fractional differential equations. These methods include homotype perturbation method [2], Haar Wavelets method [10], Adomian decomposition method [23], extrapolation method [14], predictor-corrector method [16], homotype analysis method [21], fractional linear multi-step method [33], generalized differential transform method [39], variational iteration method [52] and finite difference method [53].

The thesis comprises of five chapters. The first chapter is concerned with history, basic definitions and preliminary concepts related to fractional derivatives and integrals and applications of fractional calculus. Chapter 2 is based on existence and uniqueness results for solution of fractional differential equations. Chapter 3 deals with power series method to solve fractional differential equations. Chapter 4 is about Laplace transform method for fractional differential equations and Chapter 5 is related to solving fractional differential equations using combination of quadrature methods. At the end we have given some further workable contents.

1.1 History

Guillaume de L'Hopital (1661 - 1704) and Gottfried Leibniz (1646 - 1716) are considered as the initiators of fractional calculus that deals with derivatives and integrals of arbitrary order. The key idea behind the development of fractional calculus was the question raised by L'Hopital in 1695. L'Hopital asked Leibniz about the differentiation of order 1/2. Leibniz response was *"an apparent paradox from which one day useful consequences will be drawn."*

Leonhard Euler's (1707 - 1783) also showed his interest in fractional calculus. In 1730 he mentioned that when n is a positive integer and f is a function of x then the ratio $d^n f$ to dx^n can always be achieved. But what will the ratio $d^n f$ to dx^n if n is not an integer. We can see the problem in this case is that when n is a positive integer one can find d^n by continuing the process of differentiation but this is not possible when n is a fraction. Yet one can solve this issue with the help of interpolation.

In 1812 Pierre-Simon Laplace (1749 - 1827) defined a fractional order derivative by using integrals in the definition. Sylvestre Francois Lacroix (1765-1843) was a French mathematician who published a 700 pages text on classical calculus and mentioned fractional derivatives in it by dedicating less than two pages on this issue. Thus the first appearance of fractional derivatives in text was in 1819. It took 279 years after L'Hopital curiosity in fractional calculus to devote a text entirely comprising on fractional calculus.

Leibniz, Euler, Laplace, and Fourier worked on derivatives of arbitrary order, but the first application of fractional operators was given by Niels Henrik Abel (1802 - 1829) in 1823. Abel used the fractional calculus to solve an integral equation which arises in the modeling of the tautochrone problem[1]. In this problem, we get an expression of the form $\int_0^x (x-t)^{-\frac{1}{2}} dt$. The integral under consideration is a particular case of a definite integral that defines fractional integration of order $\frac{1}{2}$, except for a multiplicative constant $\frac{1}{\Gamma(\frac{1}{2})}$.

[1] The problem is connected with the determination of a curve along which a heavy particle, sliding without friction, descends to its lowest position, such that the time of descent is a given function of its initial position.

A French mathematician, Joseph Liouville (1809-1882) was inspired by Abel's solution and gave the first logical definition of a fractional derivative. In 1832 he published three long memoirs and a few more by 1855 that showed his enthusiasm to the theory of fractional calculus. He was also victorious in applying his definitions to problems arising in potential theory.

There was a disagreement from 1835 to 1850 between two definitions of a fractional derivative given by Lacroix and Liouville. George Peacock showed his approval in Lacroix version. While other mathematicians were in favour of Liouville's definition. In 1850 the dispute was resolved by William Center who noticed that the dissimilarity arises between these two definitions from how they define fractional derivative of a constant. If we have a look at the definition given by Lacroix we see that fractional derivative of a constant is not zero. But in accordance with Liouville's formula fractional derivative of a constant is zero since we have $\Gamma(0) = \infty$.

In 1847 Bernhard Riemann (1826-1866) devoted a paper to fractional operations that was published after his death. N. Ya. Sonin in 1869 wrote a paper titled as "On differentiation with arbitrary index" that was comprised of initial work done by Riemann and Liouville in theory of fractional calculus. The starting point of his discussion was Cauchy's formula for repeated integration. A. V. Letnikov wrote four papers on fractional operations. In 1872 he extended the concept of Sonin by writing a paper entitled "An explanation of the fundamental concepts of the theory of differentiation of arbitrary order". Sonin and Letnikov defined a fractional derivative using a closed contour in the definition. Fractional derivatives were used in electromagnetic theory by Oliver Heaviside in 1920 and, later, in 1936 A. Gemant introduced fractional derivatives to problems in elasticity.

Fractional calculus was considered a part of specialized seminars and pacts since the seventies. In June 1974 Bertram Ross assembled the first conference on fractional calculus and its applications after completing his Ph.D. thesis on fractional calculus. In 1974 a chemist Keith B. Oldham and a mathematician Jerome Spanier published the first monograph on this issue. This association between a chemist and a mathematician to use the semi-derivatives and semi-integrals in dealing with problems of mass and heat transfer, surely established the foundations of new era for fractional calculus. In 1987 Stefan G. Samko, Anatoly A. Kilbas and Oleg I. Marichev wrote a book on fractional calculus in Russian. This book was translated to English in 1993 and was titled as "Fractional Integrals and Derivatives: Theory and Applications". In 1991 A. Oustaloup introduced fractional derivatives in frequency response.

Fractional derivatives and integrals were considered as non-local operators till 1996 but in the same year Kolwankar introduced the notion of local fractional derivatives that appears in fractals and were popularized by Mandelbrot in 1970. Mandelbort promoted their significance in modelling of various irregular objects and activities observed in nature. This concept entirely changed the viewpoint of fractional calculus and since then a lot of work has been done to deduce the fractal dimensions in diverse disciplines ranging from biology to astrophysics. A natural continuation of these developments puts our interest to analogous questions which involve fractals, for example, diffusion on fractals, wave equation with fractal boundary conditions etc. As a result, it is required to be able to consolidate fractals in usual calculus or its appropriate generalization. Fractional calculus is considered as a reliable candidate that succeeded in dealing with this deadlock since the theory of ordinary calculus fails to deal with fractals as these are generally non-differentiable.

In 1999 Igor Podlubny showed his great interest in fractional calculus by writing a book entitled "Fractional Differential Equations". This book appeared to be very handy in studying the applications of fractional calculus in automatic control. Fractional calculus has huge applications in linear viscoelasticity as this theory is capable of dealing with hereditary phenomena with long memory. For more detailed knowledge on this topic one can see the books written by R. Hilfer (2000), B. West, M. Bologna and P. Grigolini (2003), A. Kilbas, H. M. Srivastava and J. Trujillo (2006) and the latest one by F. Mainardi (2010).

Fractional calculus was thought to be an abstract discipline of mathematics without applications. But last few decades has changed our opinion after an outburst of research activities on the applications of fractional calculus. It has applications in diverse scientific fields ranging from the physics of diffusion and advection phenomena, to control systems to finance and economics. The increasing number of applications of fractional differential equations motivated many researchers and appreciable work has been done to develop efficient methods for exact and numerical solutions of fractional differential equations. Therefore several methods for the approximate solutions to classical differential equations are extended to solve differential equations of fractional order numerically. This motivates us to use the technique of combining different quadrature rules. We have combined the trapezoidal rule and Simpson $\frac{1}{3}$rd rule as done in [24] and [36]. The reason behind combining these two rules was Simpson rule is only applicable to an even number of intervals and the trapezoidal rule is less accurate. After combining these two rules we can implement it on any number of intervals and also the error estimate can be minimized. Further, we have studied the efficiency of our method by applying the method to different problems that are already done by other researchers.

1.2 Some basic concepts

In this section we discuss some basic definitions and terminologies from analysis that will be helpful in our further work.

Definition 1.2.1. A collection of closed intervals in $\mathbb{R}$ is said to be non overlapping if their interiors are disjoint. The interior of an interval is the largest open interval that is contained in that interval.

Definition 1.2.2. [37] A real-valued function y defined on an interval I is said to be absolutely continuous on I if and only if for every $\epsilon > 0$ there exists $\delta > 0$ such that for any finite collection of non overlapping intervals $\{[a_k, b_k] : k = 1, 2, \cdots, n\}$ contained in I with $\sum_{k=1}^{n}(b_k - a_k) < \delta$, we have $\sum_{k=1}^{n} |y(b_k) - y(a_k)| < \epsilon$.

Example 1.2.3. In this problem we prove $y(x) = \sqrt{x}$ is absolutely continuous on $I = [0, 1]$. For this we take $\{[a_k, b_k] : k = 1, 2, \cdots, n\}$ a collection of non-overlapping closed intervals contained in $[0, 1]$. Also let for $\delta > 0$

$$\sum_{k=1}^{n}(b_k - a_k) < \delta \Rightarrow (b_1 - a_1) + (b_2 - a_2) + \cdots + (b_n - a_n) < \delta. \tag{1.2.1}$$

We know the following relations hold

$$\begin{aligned} &1.\ (\sqrt{u} - \sqrt{v})(\sqrt{u} + \sqrt{v}) = u - v. \\ &2.\ -\sqrt{u} \leq \sqrt{v} \Rightarrow (\sqrt{u} - \sqrt{v}) \leq (\sqrt{u} + \sqrt{v}). \\ &3.\ (\sqrt{v} - \sqrt{u}).^2 = (\sqrt{v} - \sqrt{u})(\sqrt{v} - \sqrt{u}) \leq (\sqrt{v} - \sqrt{u})(\sqrt{v} + \sqrt{u}). \end{aligned} \tag{1.2.2}$$

We use Equation (1.2.2) to prove the absolute continuity of $y(x) = \sqrt{x}$.

$$\begin{aligned} &(\sqrt{b_1} - \sqrt{a_1})^2 + \cdots + (\sqrt{b_n} - \sqrt{a_n})^2 \\ &\leq (\sqrt{b_1} - \sqrt{a_1})(\sqrt{b_1} + \sqrt{a_1}) + \cdots + (\sqrt{b_n} - \sqrt{a_n})(\sqrt{b_n} + \sqrt{a_n}) \\ &= (b_1 - a_1) + (b_2 - a_2) + \cdots + (b_n - a_n). \end{aligned} \tag{1.2.3}$$

Using Equation (1.2.1) in Equation (1.2.3) we get

$$\sum_{k=1}^{n} \left(\sqrt{b_k} - \sqrt{a_k}\right)^2 < \delta.$$

Thus,

$$\sum_{k=1}^{n} |y(b_k) - y(a_k)| < \epsilon \text{ where } \delta = \epsilon.$$

Hence $y(x) = \sqrt{x}$ is absolutely continuous on $[0, 1]$.

Definition 1.2.4. By AC^m or $AC^m[a, b]$ we denote the set of functions with absolutely continuous $(m-1)$ derivatives, i.e, the function y for which there exists (almost everywhere) a function $z \in L_1[a, b]$ such that

$$y^{m-1}(x) = y^{m-1}(a) + \int_a^x z(t)dt.$$

In this case z is the m^{th} derivative of y; and we write $z = y^{(m)}$.

Lemma 1.2.5. *(Grunwald's Lemma) Let $\{y_n\}$ and $\{z_n\}$ be two nonnegative sequences and let a be a nonnegative constant. If*

$$y_n \leq a + \sum_{i=1}^{n} y_i z_i \quad \textit{for} \quad n \geq 0,$$

then

$$y_n \le a \prod_{j=1}^{n}(1+z_j) \le a \exp\left(\sum_{j=1}^{n} z_j\right).$$

Definition 1.2.6. Let function y be defined on a set I. A point $x_0 \in I$ is called fixed point for y if $y(x_0) = x_0$.

Definition 1.2.7. [29] Let $\mathcal{X}$ be a family of functions each of which maps the Banach space (B_1, d_{B_1}) into the Banach space (B_2, d_{B_2}). The family $\mathcal{X}$ is said to be equicontinuous at a point $b_0 \in B_1$ if for each $\epsilon > 0$ there exists a $\delta > 0$ such that for all $y \in \mathcal{X}$ we have

$$d_{B_2}(y(b_0), y(b)) < \epsilon \text{ whenever } d_{B_1}(b_0, b) < \delta.$$

If the above definition is true for all $b_0 \in B_1$, then we say that family $\mathcal{X}$ is equicontinuous on B_1.

Definition 1.2.8. [22] An operator T of Banach space B is said to be compact iff every class of open sets which covers T has a finite subclass which also covers T.

Theorem 1.2.9. *[22](Arzela-Ascoli theorem) Suppose A be a bounded subset of $\mathbb{R}^n$, and let T be a subset of $C(A)$. Then T is relatively compact if and only if it is bounded and equicontinuous.*

Theorem 1.2.10. *(Schauder's fixed point theorem)*
Suppose that S is a nonempty, compact and convex subset of the Banach space B and $J : S \to S$ is a compact operator. Then J has atleast one fixed point in S.

Theorem 1.2.11. *[35](Cauchy's formula for repeated integration) Let y be some continuous function on the interval $[a, b]$. The n^{th} repeated integral of y based at a, is given by single integration*

$$I_a^n y(x) = \int_a^x \int_a^{x_{n-1}} \cdots \int_a^{x_2} \int_a^{x_1} y(x_0) dx_0 dx_1 \cdots dx_{n-1} = \int_a^x \frac{(x-x_0)^{n-1}}{(n-1)!} y(x_0) dx_0. \tag{1.2.4}$$

Proof. A proof is given by induction on order of integration. Consider the case when $n = 1$, we have $(x-t)^{n-1} = (x-t)^0 = 1$. So,

$$I_a^1 y(x) = \int_a^x \frac{(x-x_0)^0}{(0)!} y(x_0) dx_0 = \int_a^x \frac{(x-x_0)^{n-1}}{(n-1)!} y(x_0) dx_0,$$

means Equation (1.2.4) holds for $n = 1$. Now suppose the statement (1.2.4) holds for some arbitrary n and we will prove it for $n + 1$ by changing the order of integration.

$$I_a^{n+1} y(x) = \int_a^x \int_a^{x_n} \cdots \int_a^{x_2} \int_a^{x_1} y(x_0) dx_0 dx_1 \cdots dx_n. \tag{1.2.5}$$

Since the Equation (1.2.4) holds for n, we can write Equation (1.2.5) as

$$I_a^{n+1} y(x) = \frac{1}{(n-1)!} \int_a^x \int_a^{x_n} (x_n - x_0)^{n-1} y(x_0) dx_0 dx_n. \tag{1.2.6}$$

Now we change the order of integration in Equation (1.2.6). The double integral on right side of Equation (1.2.6) is to be first integrated in the $x_0 -$ direction from $x_0 = a$ to $x_0 = x_n$ and then the resulting integral is to be integrated in the $x_n -$ direction from $x_n = a$ to $x_n = x$.

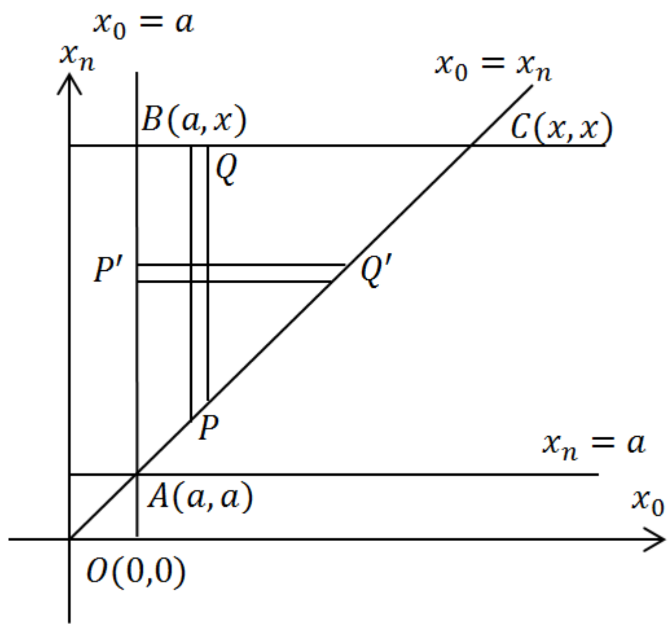


Figure 1.1: Changing order of integration.

The region of integration is the triangular area ABC. In the integral under consideration, the area ABC is divided in strips parallel to $x_0 -$ axis (for example strip $P'Q'$). To reverse the order of integration, we have to first integrate with respect to x_n regarding x_0 as constant and then with respect to x_0. This is done by dividing the above mentioned area ABC in strips parallel to the $x_n -$ axis (for example, PQ). Thus, we note that first we must integrate from $x_n = x_0$ to $x_n = x$ in $x_n -$ direction and afterwards in the $x_0 -$ direction from $x_0 = a$ to $x_0 = x$. Thus, changing the order of integration on right side of Equation (1.2.6) as shown in the Figure (1.1), we obtain

$$I_a^{n+1} y(x) = \frac{1}{(n-1)!} \int_a^x \int_{x_0}^x (x_n - x_0)^{n-1} y(x_0) dx_n dx_0. \tag{1.2.7}$$

Integration of Equation (1.2.7) with respect to x_n yields

$$I_a^{n+1} y(x) = \frac{1}{n!} \int_a^x \{(x - x_0)^n - (x_0 - x_0)^n\} y(x_0) dx_0,$$

$$I_a^{n+1} y(x) = \frac{1}{n!} \int_a^x (x - x_0)^n y(x_0) dx_0.$$

Thus the result follows by induction. □

1.3 The gamma function

The gamma function is defined for all complex numbers except the non-positive integers. The gamma function is defined for complex numbers with positive real part via improper integral

$$\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}dt. \tag{1.3.1}$$

The integral in Equation (1.3.1) is uniformly convergent for $x > 0$. If one integrates by parts the integral

$$\Gamma(x+1) = \int_0^\infty t^x e^{-t}dt,$$

one obtains the functional equation

$$\Gamma(x+1) = x\Gamma(x), \quad x > 0. \tag{1.3.2}$$

From (1.3.2) we get

$$\Gamma(x) = \Gamma(x+1)/x, \quad x > -1, \quad x \neq 0. \tag{1.3.3}$$

The repeated application of (1.3.3) gives us

$$\Gamma(x) = \frac{\Gamma(x+n)}{x(x+1)(x+2)\cdots(x+n-1)}, \quad x \in \mathbb{R}\backslash\{0,-1,-2,-3,\cdots\}. \tag{1.3.4}$$

The Equations (1.3.3) and (1.3.4) are valid even for complex values provided $x \in \mathbb{C}\backslash\{0,-1,-2,\cdots\}$. The gamma function is also an extension of the factorial function with its argument shifted down by 1, to real and complex numbers. That is, if n is a positive integer then $\Gamma(n) = (n-1)!$. The Figure 5.2b shows the plot of gamma and inverse gamma function.

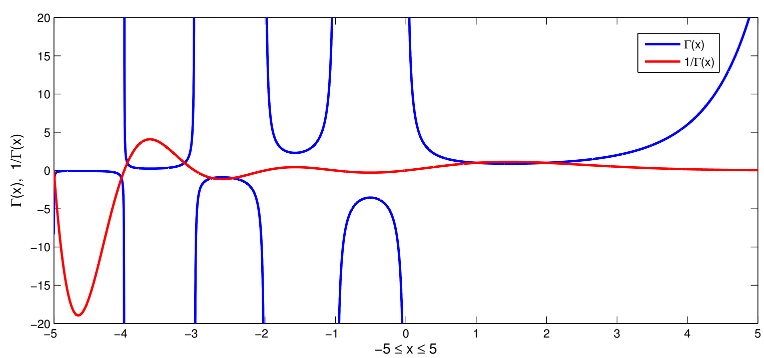


Figure 1.2: Plot of gamma and inverse gamma function.

Definition 1.3.1. [5] The beta function is defined for $Re(x) > 0$ and $Re(y) > 0$ as

$$B(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}dt. \tag{1.3.5}$$

Relation between beta and gamma function: The beta function is related to gamma function as

$$B(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}, \; x, \; y \in \mathbb{C}\backslash\{0,-1,-2,\cdots\}.$$

1.4 Mittag-Leffler function

The Mittag-leffler function is named after the great Swedish mathematician Gosta Magnus Mittag-Leffler (1846 − 1927). This function provides a simple generalization of the exponential function because of the replacement of $k! = \Gamma(k+1)$ by $(\alpha k)! = \Gamma(\alpha k + 1)$ in the denominator of the power terms of the exponential series. Successful applications of the Mittag-Leffler function and its generalizations, has made this better known among scientists. Mittag-Leffler function became very important in the study of fractional differential and integral equations after its involvement in the solution of Abel integral equation of the second kind.

Definition 1.4.1. [19] The one parameter Mittag-Lefler function is defined by the power series

$$E_\alpha(x) = \sum_{k=0}^{\infty} \frac{x^k}{\Gamma(\alpha k + 1)}, \ \ \alpha > 0.$$

The Mittag-Leffler function plays an important role among special functions.

Definition 1.4.2. The two-parametric Mittag-Leffler function is defined as

$$E_{\alpha,\beta}(x) = \sum_{k=0}^{\infty} \frac{x^k}{\Gamma(\alpha k + \beta)}, \quad \alpha > 0, \ \beta \in \mathbb{C}.$$

This is a generalization to one-parameter Mittag-Leffler function since we have $E_{\alpha,1}(x) = E_\alpha(x)$.

Theorem 1.4.3. *[13] Consider the two-parameter Mittag-Leffler function $E_{\alpha,\beta}$ for some $\alpha, \beta > 0$. The power series defining $E_{\alpha,\beta}(x)$ is convergent for all $x \in \mathbb{C}$. In other words, $E_{\alpha,\beta}$ is an entire function.*

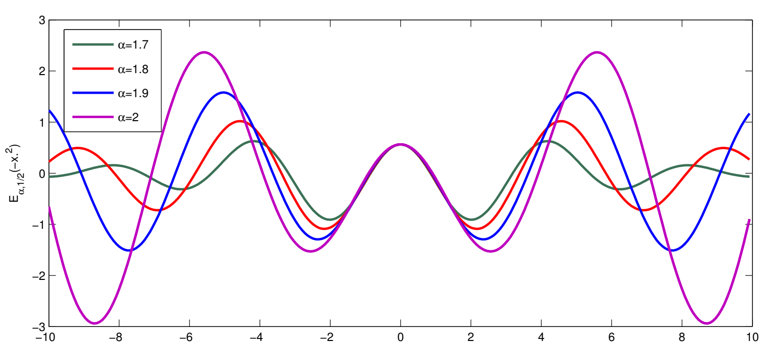


Figure 1.3: The Mittag-Leffler function $E_{\alpha,\frac{1}{2}}(-x^2)$ at different values of α.

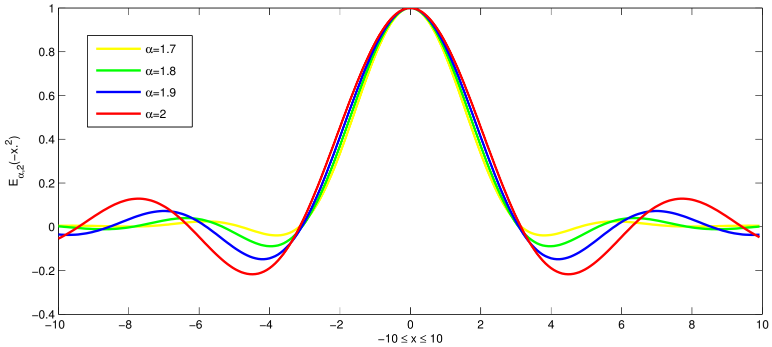


Figure 1.4: The Mittag-Leffler function $E_{\alpha,2}(-x^2)$ at different values of α.

Remark 1.4.4. The continuity of Mittag-Leffler function in $x \geq 0$ imply that if $\mu \geq 0$ then there is a constant C such that

$$E_\alpha(\mu x^\alpha) \leq Ce^{\mu^{\frac{1}{\alpha}} x}, \quad 0 < \alpha < 2.$$

1.5 Fractional derivatives and integrals

In this section, we introduce notations, definitions, and preliminary lemmas concerning fractional calculus theory. Let us start with the development of Riemann-Liouville's formula for integrals with fractional order. We know that

$$\begin{aligned}
I_a^1 y(x) &= \int_a^x y(x_0)dx_0, \\
I_a^2 y(x) &= \int_a^x \int_a^{x_1} y(x_0)dx_0dx_1, \\
I_a^3 y(x) &= \int_a^x \int_a^{x_2} \int_a^{x_1} y(x_0)dx_0dx_1dx_2, \\
&\ \ . \\
&\ \ . \\
&\ \ . \\
I_a^n y(x) &= \int_a^x \int_a^{x_{n-1}} \cdots \int_a^{x_2} \int_a^{x_1} y(x_0)dx_0dx_1 \cdots dx_{n-1}.
\end{aligned}$$

But how can we evaluate $I_a^{\frac{1}{2}} y(x)$ and $I_a^\pi y(x)$? For evaluation of these type of integrals we use Cauchy's formula for repeated integration (1.2.4). If in the Equation (1.2.4), we replace $(n-1)!$ with $\Gamma(n)$ and n with real number α, we get definition for Riemann-Liouville's formula for integrals with fractional order.

Definition 1.5.1. [1] The Riemann-Liouville fractional integral of order $\alpha \in (0, \infty)$ of a function $y \in L_1[a, b]$ is defined by

$$I_a^\alpha y(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} y(t)dt.$$

Theorem 1.5.2. *[13] Let $\alpha,\ \beta \geq 0$ and $y \in L_1[a,b]$. Then,*

$$I_a^{\alpha} I_a^{\beta} y = I_a^{\alpha+\beta} y,$$

holds almost everywhere on $[a,b]$.

Corollary 1.5.3. *[13] Under the assumptions of Theorem 1.5.2 $I_a^{\alpha} I_a^{\beta} y = I_a^{\beta} I_a^{\alpha} y$.*

Lemma 1.5.4. *[13] If $\alpha \geq 0,\ \beta > -1$, then the Riemann-Liouville fractional integral of the function $(x-a)^{\beta}$ is given by*

$$I_a^{\alpha}(x-a)^{\beta} = \frac{\Gamma(\beta+1)}{\Gamma(\beta+\alpha+1)}(x-a)^{\beta+\alpha}. \tag{1.5.1}$$

Proof. The Reimann-Liouville integral of $(x-a)^{\beta}$ is given by

$$I_a^{\alpha}(x-a)^{\beta} = \int_a^x \frac{(x-s)^{\alpha-1}(s-a)^{\beta}}{\Gamma(\alpha)} ds. \tag{1.5.2}$$

Let $u = \frac{s-a}{x-a}$, $u \to 0$ as $s \to a$ and $u \to 1$ as $s \to x$. Also $s = a + (x-a)u$ and $ds = (x-a)du$. Making all these substitutions in (1.5.2) we get

$$I_a^{\alpha}(x-a)^{\beta} = \frac{(x-a)^{\alpha+\beta}}{\Gamma(\alpha)} \int_0^1 (1-u)^{\alpha-1} u^{\beta} du. \tag{1.5.3}$$

Using (1.3.5) in (1.5.3) we get

$$I_a^{\alpha}(x-a)^{\beta} = \frac{\Gamma(\beta+1)}{\Gamma(\alpha+\beta+1)}(x-a)^{\alpha+\beta}. \tag{1.5.4}$$

□

Example 1.5.5. Find $I_1^{\frac{1}{2}}(x-1)^{1/3}$.
For evaluation of above integral we use (1.5.1) and get

$$\begin{aligned} I_1^{\frac{1}{2}}(x-1)^{1/3} &= \frac{\Gamma(\frac{1}{3}+1)}{\Gamma(\frac{1}{2}+\frac{1}{3}+1)}(x-1)^{\frac{1}{2}+\frac{1}{3}} \\ &= \frac{\Gamma(\frac{4}{3})}{\Gamma(\frac{11}{6})}(x-1)^{\frac{5}{6}} \\ &= 0.9464(x-1)^{\frac{5}{6}}. \end{aligned}$$

Now we try to develop formula for Riemann-Liouville fractional derivative. For this we start with fundamental theorem of calculus which states: Let y be continuous on an open interval I containing a, then for every x in the interval,

$$\frac{d}{dx} Y(x) = y(x),$$

where $Y(x) = \int_a^x y(s)ds$. We can write above equation as $y(x) = \frac{d}{dx}\int_a^x y(s)ds$ or $y(x) = DI_a y$, where $D = \frac{d}{dx}$. Repeated application of previous equation gives

$$y = D^n I_a^n y,$$

where $D^n = \frac{d^n}{dx^n},\ n = 0, 1, 2, \cdots$. If we replace n by $m-n$, with $n < m$ and apply D^n on both sides, we will have

$$D^n y = D^n D^{m-n} I_a^{m-n} y \Rightarrow D^n y = D^m I_a^{m-n} y.$$

The above relation is still valid if n is replaced by $\alpha \in \mathbb{R}$, provided $m - \alpha > 0$ and in this way we get formula for Riemann-Liouville fractional derivative of order α.

Definition 1.5.6. [27] The Riemann-Liouville fractional derivative of order $\alpha \in (0, \infty)$ with $n = \lceil \alpha \rceil$, of a function $y \in L_1[a, b]$ is defined by

$$\begin{aligned} {}^{RL}D_a^{\alpha} y(x) &= \frac{d^n}{dx^n} I_a^{n-\alpha} y(x) \\ &= \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dx^n} \int_a^x (x-t)^{n-\alpha-1} y(t) dt, \text{ for almost all } x \in [a, b]. \end{aligned}$$

Theorem 1.5.7. *[13] Let y_1 and y_2 be two functions defined on $[a, b]$ such that ${}^{RL}D_a^{\alpha} y_1$ and ${}^{RL}D_a^{\alpha} y_2$ exist. Moreover, let $c_1,\ c_2 \in \mathbb{R}$. Then, ${}^{RL}D_a^{\alpha}(c_1 y_1 + c_2 y_2)$ exists almost everywhere, and*

$${}^{RL}D_a^{\alpha}(c_1 y_1 + c_2 y_2) = c_1\ {}^{RL}D_a^{\alpha} y_1 + c_2\ {}^{RL}D_a^{\alpha} y_2.$$

Theorem 1.5.8. *[13] If $\alpha \geq 0,\ \beta > -1$, with $n = \lceil \alpha \rceil$ then the Riemann-Liouville fractional derivative of the function $(x-a)^{\beta}$ is given by*

$${}^{RL}D_a^{\alpha}(x-a)^{\beta} = \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)}(x-a)^{\beta-\alpha}. \tag{1.5.5}$$

Example 1.5.9. In this example we develop fractional derivative of $y(x) = c$ where c is constant. We apply Definition 1.5.6 on $y(x) = c$ and we get

$${}^{RL}D_a^{\alpha} y(x) = \frac{d}{dx} I_0^{1-\alpha} c, \quad \alpha > 0. \tag{1.5.6}$$

Applying Definition 1.5.1 on Equation (1.5.6) we get

$${}^{RL}D_a^{\alpha} y(x) = \frac{d}{dx} . \frac{1}{\Gamma(1-\alpha)} \int_0^x (x-t)^{-\alpha} c dt, \quad \alpha > 0. \tag{1.5.7}$$

Making substitution $x - t = u,\ du = -dt$ in Equation (1.5.7) we get

$${}^{RL}D_a^{\alpha} y(x) = \frac{c x^{-\alpha}}{\Gamma(1-\alpha)}, \quad \alpha > 0.$$

The Figure 1.5 shows the fractional derivative of $y(x) = 10$ at $\alpha = 1.1, 2.1, 3.7, 4.3$. From figure it is quite obvious that fractional order derivative of a constant is other than zero.

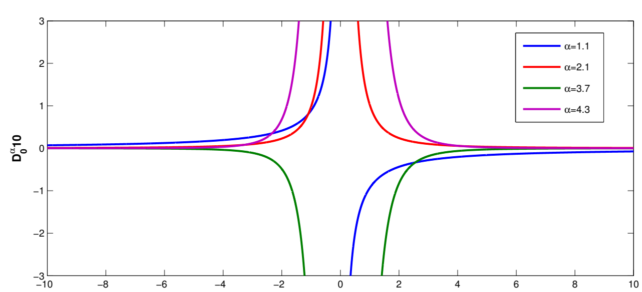


Figure 1.5: Fractional derivative of constant $y(x) = c = 10$ at different values of α.

Example 1.5.10. Now we see fractional order derivative of $y(x) = 1$.

$$
\begin{aligned}
{}^{RL}D_0^{\alpha} 1 &= {}^{RL}D_0^{\alpha} x^0 \\
&= \frac{x^{-\alpha}}{\Gamma(1-\alpha)}.
\end{aligned}
$$

The Figure 1.6 shows the fractional order derivative of $y(x) = 1$ at different values of x. In this example we have taken $x = 1.3, 1.4, 1.6, 1.8$ and $1 \le \alpha \le 4$. We observe that fractional order derivative of 1 at $\alpha = 1, 2, 3, 4$ is zero. A 3D plot of fractional order derivatives and integrals of $y(x) = 1$ is given in Figure 1.7.

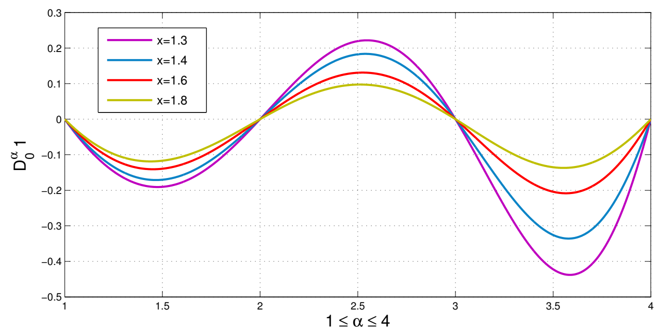


Figure 1.6: Fractional derivative of $y(x) = 1$ at different values of x.

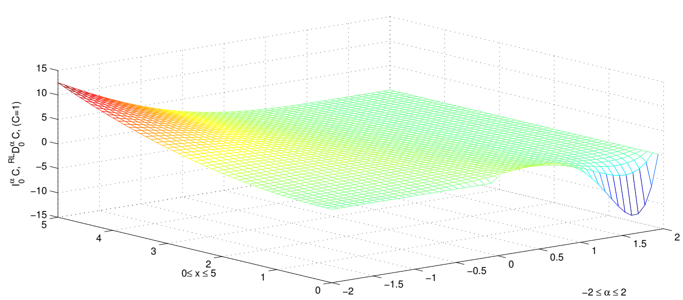


Figure 1.7: 3D representation of fractional derivative and fractional integral of $y(x) = 1$ at different values for $0 \le x \le 5$ and $-2 \le \alpha \le 2$.

Example 1.5.11. In this example we see graphically fractional order derivative and integral of function $y(x) = x$ and $y(x) = x^2$. The Figure 1.8 shows fractional order derivative of $y(x) = x$ with orders $\alpha = 0, 0.5, 1$ and fractional order integral of $y(x) = x$ with orders $\alpha = 0.5, 1$ and x varies between 0 and 3. The Figure 1.9 shows 3D plot of fractional order derivatives and integrals of $y(x) = x^{\beta}$ with $\beta = 1, \; -2 \le \alpha \le 2$ and $0 \le x \le 5$ and Figure 1.10 shows 3D plot of $y(x) = x^2$ with $-2 \le \alpha \le 2$ and $0 \le x \le 5$.

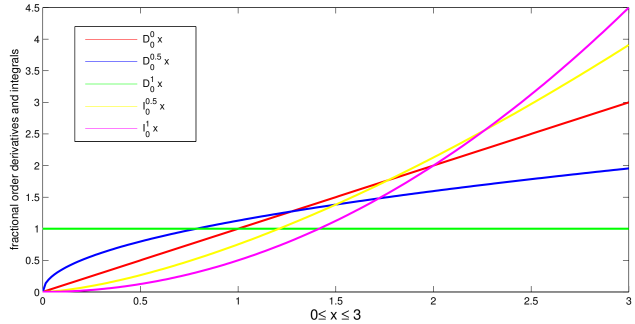


Figure 1.8: Fractional derivative and integral of $y(x) = x$ at different values of α.

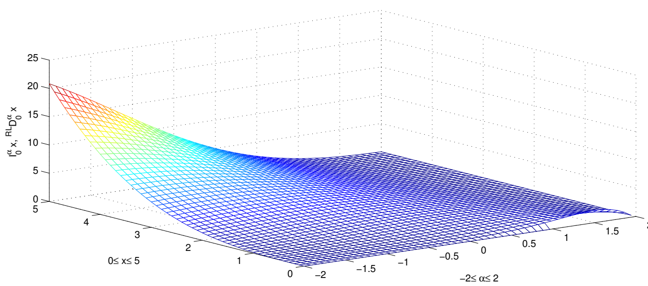


Figure 1.9: Fractional derivative and integral of $y(x) = x$ at different values for $0 \leq x \leq 5$ and $-2 \leq \alpha \leq 2$.

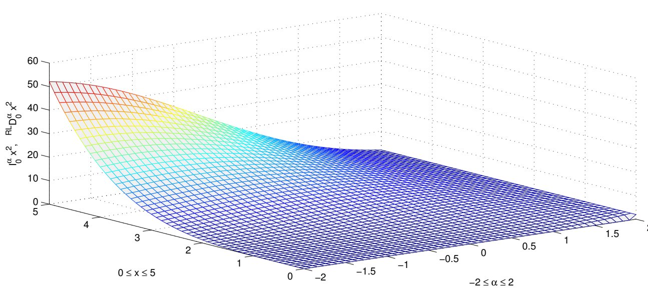


Figure 1.10: Fractional derivative and integral of $y(x) = x^2$ at different values for $0 \leq x \leq 5$ and $-2 \leq \alpha \leq 2$.

Theorem 1.5.12. *[42] Assume that $m-1 < \alpha_1 \leq m,\ n-1 < \alpha_2 \leq n$ and $m, n \in \mathbb{N}$. Both ${}^{RL}D_a^{\alpha_1}$ and ${}^{RL}D_a^{\alpha_2}$ exist. Then,*

$$ {}^{RL}D_a^{\alpha_1}\ {}^{RL}D_a^{\alpha_2} y(x) = {}^{RL}D_a^{\alpha_1+\alpha_2} y(x) - \sum_{j=1}^{n} {}^{RL}D_a^{\alpha_2-j} y(a) \frac{(x-a)^{-\alpha_1-j}}{\Gamma(1-\alpha_1-j)}. \tag{1.5.8} $$

Interchanging α_1 and α_2 also m and n in Equation (1.5.8) we get

$$ {}^{RL}D_a^{\alpha_2}\ {}^{RL}D_a^{\alpha_1} y(x) = {}^{RL}D_a^{\alpha_1+\alpha_2} y(x) - \sum_{j=1}^{m} {}^{RL}D_a^{\alpha_1-j} y(a) \frac{(x-a)^{-\alpha_2-j}}{\Gamma(1-\alpha_2-j)}. \tag{1.5.9} $$

The comparison of Equation (1.5.8) and Equation (1.5.9) shows that in general Riemann-Liouville fractional derivative operators do not commute. These operators can commute only if the sum in the expressions of (1.5.8) and (1.5.9) vanish. In this case we have

$$^{RL}D_a^{\alpha_1}\ ^{RL}D_a^{\alpha_2}y(x) = \ ^{RL}D_a^{\alpha_2}\ ^{RL}D_a^{\alpha_1}y(x) = \ ^{RL}D_a^{\alpha_1+\alpha_2}y(x).$$

Theorem 1.5.13. *[13] Assume that $\alpha_1, \alpha_2 \geq 0$. Moreover let $\phi \in L_1[a,b]$ and $y = I_a^{\alpha_1+\alpha_2}\phi$. Then,*

$$^{RL}D_a^{\alpha_1}\ ^{RL}D_a^{\alpha_2}y = \ ^{RL}D_a^{\alpha_1+\alpha_2}y.$$

Note that in order to apply this identity we do not need to know the function ϕ explicitly. It is sufficient to know that such a function exists.

Example 1.5.14. Let $y(x) = x$ and $\alpha_1 = \alpha_2 = \frac{1}{2}$. To find $^{RL}D_0^{\frac{1}{2}}x$, we use Equation (1.5.5)

$$\begin{aligned} ^{RL}D_0^{\frac{1}{2}}x &= \frac{\Gamma(1+1)}{\Gamma(1-1/2+1)}(x)^{1-1/2} \\ &= \frac{x^{1/2}}{\Gamma(3/2)} \\ &= \frac{2x^{1/2}}{\sqrt{\pi}}. \end{aligned}$$

Now compute $^{RL}D_0^{\frac{1}{2}}(\frac{2x^{\frac{1}{2}}}{\sqrt{\pi}})$ using same procedure and we get $^{RL}D_0^{\frac{1}{2}}(\frac{2x^{\frac{1}{2}}}{\sqrt{\pi}}) = 1$. We know that $^{RL}D_0^{\frac{1}{2}+\frac{1}{2}}x = \ ^{RL}Dx =$ 1. Which means $^{RL}D_0^{\frac{1}{2}}\ ^{RL}D_0^{\frac{1}{2}}x = \ ^{RL}D_0^{\frac{1}{2}+\frac{1}{2}}x = 1$

Example 1.5.15. Now we try to construct counter example for Theorem (1.5.13). For this we take $y(x) = x^{-1/2}$ and $\alpha_1 = \alpha_2 = \frac{1}{2}$. Applying Equation (1.5.5) on $^{RL}D_0^{\frac{1}{2}}x^{-\frac{1}{2}}$ we get $^{RL}D_0^{\frac{1}{2}}x^{-\frac{1}{2}} = 0$ also $^{RL}D_0^{\frac{1}{2}}0 = 0$. We know that $^{RL}D_0^{\frac{1}{2}+\frac{1}{2}}x^{-\frac{1}{2}} = \ ^{RL}Dx^{-\frac{1}{2}} = -\frac{1}{2}x^{-\frac{3}{2}}$.

We observe that Theorem (1.5.13) holds for Example (1.5.14) and does not hold for Example (1.5.15).

Lemma 1.5.16. *If $\alpha, \beta \in \mathbb{R}^+$, $\alpha > \beta$ and $y \in L_1[a,b]$, then $^{RL}D_a^{\beta}I_a^{\alpha}y = I_a^{\alpha-\beta}y$ holds almost everywhere on $[a,b]$.*

Lemma 1.5.17. *[13] Let $\alpha > 0$, then for every $y \in L_1[a,b]$, $^{RL}D_a^{\alpha}I_a^{\alpha}y = y$ almost everywhere.*

Theorem 1.5.18. *[42] Let $y \in AC^m[a,b]$ and $m-1 < \alpha \leq m$. Then, the Riemann-Liouville fractional derivative $^{RL}D_a^{\alpha}$ exists almost everywhere on $[a,b]$. Furthermore, $^{RL}D_a^{\alpha}y \in L_p[a,b]$ for $1 \leq p < \frac{1}{\alpha}$ and*

$$^{RL}D_a^{\alpha}y(x) = \sum_{k=0}^{m-1}\frac{D^k y(a)}{\Gamma(k-\alpha+1)}(x-a)^{k-\alpha} + I_a^{m-\alpha}D^m y(x).$$

1.6 The Caputo fractional derivatives

The definition of fractional derivatives specified by Riemann-Liouville has great significance in expanding theory of fractional derivatives and integrals. But it has many drawbacks when it comes to model real-world phenomena with fractional differential equations. Since applied problems involve initial conditions of type $y(a),\ y'(a)$ etc and Riemann-Liouville definition of fractional derivatives fails to utilize such initial conditions. The best approach to overcome this difficulty is given by Caputo in 1967. In this section we discuss Caputo fractional derivatives and its basic properties.

Definition 1.6.1. [27] The Caputo fractional derivative of order $\alpha \in \mathbb{R}^+$ of a function $y \in AC^m[a,b]$ where $m = \lceil \alpha \rceil$ is defined by

$$^cD_a^\alpha y(x) = I_a^{m-\alpha} D^m y(x).$$

Lemma 1.6.2. *Let $n-1 < \alpha \leq n,\ n \in \mathbb{N},\ \alpha, c_1, c_2 \in \mathbb{C}$, and the functions $y(x)$ and $z(x)$ be such that both $^cD_a^\alpha y(x)$ and $^cD_a^\alpha z(x)$ exist. The Caputo fractional derivative is a linear operator, i. e.,*

$$^cD_a^\alpha (c_1 y(x) + c_2 z(x)) = c_1\ {}^cD_a^\alpha y(x) + c_2\ {}^cD_a^\alpha z(x).$$

Lemma 1.6.3. *Suppose that $n-1 < \alpha \leq n, m,\ n \in \mathbb{N},\ \alpha \in \mathbb{R}$ and the function $y(x)$ is such that $^cD_a^{\alpha_1} y(x)$ exists. Then in general*

$$^cD_a^\alpha D_a^m y(x) = {}^cD_a^{\alpha+m} y(x) \neq D_a^m\ {}^cD_a^\alpha y(x).$$

Theorem 1.6.4. *[13] Assume that $\alpha \geq 0,\ m = \lceil \alpha \rceil$, and $y \in AC^m[a,b]$. Then*

$$I_a^\alpha\ {}^cD_a^\alpha y(x) = y(x) - \sum_{i=0}^{m-1} \frac{D^i y(a)}{i!}(x-a)^i.$$

Corollary 1.6.5. *[13]* ***(Taylor expansion for Caputo derivatives)*** *Under the assumptions of Theorem 1.6.4,*

$$y(x) = I_a^\alpha\ {}^cD_a^\alpha y(x) + \sum_{i=0}^{m-1} \frac{D^i y(a)}{i!}(x-a)^i.$$

The relations shown in Theorem 1.6.4 and Corollary 1.6.5 have major implications when it comes to the solution of differential equations involving the two types of differential operators.

Lemma 1.6.6. *Let $\alpha, \beta \in \mathbb{R}^+, \alpha > \beta$ and y be some continuous function. Then $^cD_a^\beta I_a^\alpha y(x) = I^{\beta-\alpha} y(x)$.*

Lemma 1.6.7. *[13] Let $\alpha, \beta \geq 0$ and $m = \lceil \alpha \rceil$ then Caputo fractional derivative of $y(x) = (x-a)^\beta$ is defined as*

$$^cD_a^\alpha y(x) = \begin{cases} 0, & \beta \in \{0,1,2,...,m-1\}, \\ \frac{\Gamma(\beta+1)}{\Gamma(\beta+1-\alpha)}(x-a)^{\beta-\alpha}, & \beta \in \mathbb{N} \text{ and } \beta \geq m \\ & \text{or } \beta \notin \mathbb{N} \text{ and } \beta > m-1. \end{cases}$$

Theorem 1.6.8. *[13](**Relation between Reimann-Liouville and Caputo's fractional derivative**) Assume $\alpha \geq 0,\ m = \lceil \alpha \rceil$ and for some $y(x)$, both ${}^{RL}D_a^{\alpha} y(x)$ and ${}^{c}D_a^{\alpha} y(x)$ exist. Then*

$$ {}^{c}D_a^{\alpha} y(x) = {}^{RL}D_a^{\alpha} y(x) - \sum_{i=0}^{m-1} \frac{D^i y(a)}{\Gamma(i-\alpha+1)} (x-a)^{i-\alpha}. $$

Comparison between Riemann-liouville and Caputo fractional derivative

The definitions of fractional derivatives specified by Riemann-Liouville and Caputo have played a significant role in expansion of theory of fractional calculus. But every definition have certain limitations and drawbacks. When we compare Riemann-Liouville and Caputo derivatives we see that Riemann-Liouville derivative of a constant is non-zero while Caputo derivative of a constant is zero. Which means Caputo derivative is in line with classical derivative in this sense. Also from the Definition 1.5.6 and Definition 1.6.1 we see that the class of functions for which Riemann-Liouville exists is much larger than the class of functions for which Caputo fractional derivative exists. In problems involving Riemann-Liouville derivative, initial conditions are set by taking fractional derivatives of function at initial point. But when we look at physical models, we need initial conditions of type $y(a),\ y'(a),\ y''(a), \cdots$. Because these initial conditions represent suitable quantities like displacement, velocity, acceleration etc. For this reason Caputo derivative is preferred when dealing with physical models. Now we check the connection of Riemann-Liouville and Caputo derivative with classical derivative. For $\alpha \in (n-1, n),\ n \in \mathbb{Z}^+$, we have

$$ \lim_{\alpha \to (n-1)^+} {}^{RL}D_0^{\alpha} y(x) = y^{(n-1)}(x), $$
$$ \lim_{\alpha \to (n)^+} {}^{RL}D_0^{\alpha} y(x) = y^{(n)}(x). $$

Similarly, for $\alpha \in (n-1, n),\ n \in \mathbb{Z}^+$ we get

$$ \lim_{\alpha \to (n-1)^+} {}^{c}D_0^{\alpha} y(x) = y^{(n-1)}(x) - y^{(n-1)}(0), $$
$$ \lim_{\alpha \to (n)^+} {}^{c}D_0^{\alpha} y(x) = y^{(n)}(x). $$

Remark 1.6.9. Let $y(x)$ be a function for which both Riemann-Liouville and Caputo derivative exist with $n-1 < \alpha < n$ where $n \in \mathbb{N}$. Then in general $D_a^{\alpha} y(x) \neq {}^{c}D_a^{\alpha} y(x)$. But by Theorem 1.6.8 these operators are equal when $\alpha \to n$.

1.7 Applications of fractional calculus

The focus of our study is numerical solution to fractional differential equations. In this section we define what is a fractional differential equation and consider some applications of fractional calculus to determine the handiness of this subject.

Definition 1.7.1. An equation involving fractional order derivatives is called fractional differential equation. For example FDE with constant coefficients and order α is given by

$$\{D^{n\alpha} + a_{n-1}D^{(n-1)\alpha} + ... + a_0 D^{0\alpha}\}y(x) = 0, \quad x \geq 0, \ n \in \mathbb{N}.$$

Abel's Problem:

In 1825 an Italian mathematician Neils Henrik Abel developed an integral equation while modeling of tautochrone problem. We model this problem as done in [46]. For this we consider a smooth curve that is present in a vertical plane. A heavy particle begins its motion from position P as shown in Figure 1.11. We now find the time T of descent to the ending position O, taking in account the role of gravity in it. We choose our coordinates as $x - axis$ vertically upward and $y - axis$ horizontally with O as origin. Also choose coordinates of position P as (x, y), of Q as (ξ, η) and OQ as arc s.

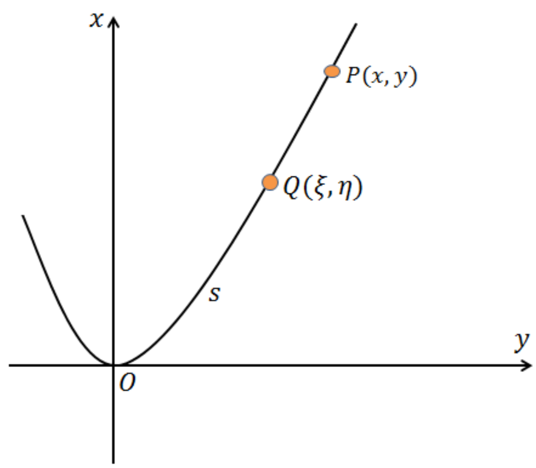


Figure 1.11: Descend of particle.

At any time, the particle under consideration will have the potential energy and kinetic energy at position Q. The sum of these energies is constant. We can write the sum mathematically as

$$K.E. + P.E. = \text{constt.} \tag{1.7.1}$$

Using definition of kinetic energy and potential energy in Equation (1.7.1)

$$\frac{1}{2}mv^2 + mg\xi = \text{constt.} \tag{1.7.2}$$

Rewriting Equation (1.7.2) as

$$\frac{1}{2}v^2 + g\xi = M, \tag{1.7.3}$$

where m denotes the mass of the particle, v is the speed of the particle at Q and it depends on time t, g is the acceleration that is due to action of gravity and ξ is the vertical coordinate of the particle at position Q. Initially, $v(0) = 0$ at P, the vertical coordinate is x, and thus the constant M can be evaluated as $M = gx$. Thus Equation (1.7.3) can be written as

$$\frac{1}{2}v^2 = g(x - \xi). \tag{1.7.4}$$

Hence from Equation (1.7.4) we get

$$v = \pm\sqrt{2g(x - \xi)}. \tag{1.7.5}$$

But $v = \frac{ds}{dt}$ = *speed along the curve* s. Therefore, Equation (1.7.5) takes the form

$$\frac{ds}{dt} = \pm\sqrt{2g(x - \xi)}. \tag{1.7.6}$$

We consider only the negative value of $\frac{ds}{dt}$ and integrate Equation (1.7.6) from P to Q by separating the variables, we get

$$t = -\int_P^Q \frac{ds}{\sqrt{2g(x - \xi)}}. \tag{1.7.7}$$

The total time of descent is given as,

$$T = \int_O^P \frac{ds}{\sqrt{2g(x - \xi)}}. \tag{1.7.8}$$

If the shape of the curve is known, then s can be written in terms of ξ and thus ds can also be written in term of ξ. For this we let $ds = u(\xi)d\xi$, and hence Equation (1.7.8) can be expressed as

$$T = \int_0^x \frac{u(\xi)d\xi}{\sqrt{2g(x - \xi)}}. \tag{1.7.9}$$

From Equation (1.7.9) we observe that time T of descent is a function of x let us call it $y(x)$. The problem in this case is to find the unknown function $u(x)$ from the Euation (1.7.9).

$$y(x) = \int_0^x \frac{u(\xi)d\xi}{\sqrt{2g(x - \xi)}}. \tag{1.7.10}$$

Equation (1.7.10) can be written as

$$y(x) = \int_0^x K(x, \xi)u(\xi)d\xi. \tag{1.7.11}$$

Equation (1.7.11) is a linear integral equation of the first kind for the determination of $u(\xi)$. Here, $K(x, \xi) = \frac{1}{\sqrt{2g(x-\xi)}}$ is the kernel of the integral equation. The Equation (1.7.10) is a specific form of fractional integration of order $\frac{1}{2}$ if we multiply the Equation (1.7.10) with $\frac{1}{\Gamma(\frac{1}{2})}$. Abel's integral equation is the earliest example of fractional integral equation. It is found in many fields of science, such as microscopy, seismology, radio astronomy, electron emission, atomic scattering, radar ranging, plasma diagnostics, X-ray radiography, and optical fiber evaluation.

Fractional oscillator:

Fractional oscillator is a generalization of harmonic oscillator, i.e. we replace ordinary derivatives in equation of harmonic oscillator by fractional order derivatives. The following equation describes fractional oscillator,

$$D^{\alpha}y(x) + \omega^{\alpha-\beta}D^{\beta}y(x) = 0,$$

where $1 < \alpha \leq 2,\ 0 < \beta \leq 1$, and ω is the vibration eigenfrequency.

Bagley-Torvik equation:

Bagley-Torvik equation is a non homogenous multi-term fractional differential equation that arises in the modeling of a rigid plate immersed in a Newtonian fluid. The general form of this equation is given by

$$Ay^{2}(x) + BD^{\frac{3}{2}}y(x) + Cy(x) = h(x),$$

where $A \neq 0$ and $B,\ C \in \mathbb{R}$. We shall discuss Bagley-Torvik equation in detail in Chapter 2 and Chapter 5.

Viscoelasticity:

Viscoelasticity is the property of materials that exhibit both viscous and elastic characteristics when undergoing deformation. Viscous materials, like honey, resist shear flow and strain linearly with time when a stress is applied. Elastic materials strain when stretched and quickly return to their original state once the stress is removed. Viscoclasticity appears to be the field with considerable applications of fractional calculus.

Stress $\sigma(t)$ and strain $\epsilon(t)$ are the two main subjects of viscoelasticity. A number of models vary from each other in the way how we associate stress with strain. We note that for Hooke's law stress is proportional to the zeroth derivative of strain i.e. $\sigma(t) = E\epsilon(t)$ where E is constant and for Newton's law to the first derivative of strain i.e. $\sigma(t) = \eta\frac{d\epsilon(t)}{dt}$. Hence, it is logical to establish a more general model i. e., proportionality of stress to the $\alpha - derivative$ of strain where $\alpha \in (0, 1)$ as done by G. W. Scott Blair

$$\sigma(t) = E_0 D_t^{\alpha}\epsilon(t),$$

where E_0 and α are material-dependent constants. The Hooke's law is a one parameter model and Scott Blair's model is a two parameter model with E and α as the parameters. One can also get a three parameter model known as Voigt model and can be expressed as

$$\sigma(t) = b_0\epsilon(t) + b_1 D^{\alpha}\epsilon(t).$$

Chapter 2

Existence and uniqueness of solutions for fractional differential equations

In this chapter we have seen the existence and uniqueness of solutions of initial value problems for linear and nonlinear fractional differential equations. In the first section operator for fractional differentiation and initial conditions are taken in Caputo sense. The second section deals with the operator for fractional order differentiation in Riemann-Liouville sense and initial conditions are taken as suggested by Caputo. We have seen in particular that the solution depends on the order of the fractional differential equation and also on the initial condition. The existence and uniqueness of the solution to linear and nonlinear fractional differential equations have been thoroughly investigated in [14, 27]. For the latest works on existence and uniqueness of solutions of the initial and boundary value problems for nonlinear fractional differential equation one may see [30–32].

2.1 Existence and uniqueness of solutions for linear fractional differential equations

In this section we investigate the question of existence and uniqueness of solution for the following initial value problem

$$ {}^cD_a^{\alpha_1}y(x) + P\ {}^cD_a^{\alpha_2}y(x) + Qy(x) = h(x), \quad x \in [a,b], \tag{2.1.1}$$

$$ y(a) = y_0, \quad y'(a) = y_0, \tag{2.1.2}$$

where $\alpha_1, \alpha_2 \in \mathbb{R}^+$, $\alpha_1 \geq \alpha_2$, $P, Q \in \mathbb{R}$ and $h(x)$ is known function. It is to be noted that Bagley-Torvik equation and fractional oscillator are special cases of (2.1.1). The questions of existence and uniqueness of the solution to initial value problem (2.1.1) - (2.1.2) have been discussed in [34]. It is to be noted that in [34] the function $h(x)$ in Equation (2.1.1) is assumed to be continuous. We present in [45], the existence result for problem (2.1.1) - (2.1.2) under weaker assumptions.

Integral equations are extensively used to study the qualitative properties of differential equation. At this point we establish an equivalence result between (2.1.1) - (2.1.2) and Volterra integral equation.

Theorem 2.1.1. *Assume $h \in L_1[a,b]$, then a function y is solution of initial value problem* (2.1.1) - (2.1.2) *if and only if y is solution of the integral equation*

$$y(x) + \int_a^x \Upsilon(x,s)y(s)ds = H(x), \tag{2.1.3}$$

where $H(x) := \int_a^x \Upsilon_{\alpha_1}(x,s)h(s)ds + u(x;\alpha_1,\alpha_2)$,
$u(x;\alpha_1,\alpha_2) := \left(1 + \frac{(x-a)^{\alpha_1-\alpha_2}}{\Gamma(\alpha_1-\alpha_2+1)}\right) y_0 + \left(1 + \frac{(x-a)^{\alpha_1-\alpha_2}}{\Gamma(\alpha_1-\alpha_2+2)}\right)(x-a)y_1$,
$\Upsilon_{\alpha_1}(x,s) := \frac{(x-s)^{\alpha_1-1}}{\Gamma(\alpha_1)}$ *and* $\Upsilon(x,s) := P\Upsilon_{\alpha_1-\alpha_2}(x,s) + Q\Upsilon_{\alpha_1}(x,s)$.

Proof. Assume y is solution of (2.1.1), (2.1.2). Applying fractional integral $I_a^{\alpha_1}$ on both sides of Equation (2.1.1) and using Theorem 1.6.4, we get

$$y(x) - y(a) - y'(a)(x-a) + PI_a^{\alpha_1}D_a^{\alpha_2}y(x) + QI_a^{\alpha_1}y(x) = I_a^{\alpha_1}h(x). \tag{2.1.4}$$

Applying Theorem 1.5.2, Equation (2.1.4) reduces to

$$y(x) + PI_a^{\alpha_1-\alpha_2}\left(I_a^{\alpha_2}D_a^{\alpha_2}y(x)\right) + QI_a^{\alpha_1}y(x) = I_a^{\alpha_1}h(x) + y_0 + y_1(x-a). \tag{2.1.5}$$

Using Theorem 1.6.4, we get

$$y(x) + PI_a^{\alpha_1-\alpha_2}\left(y(x) - y_0 - (x-a)y_1\right) + QI_a^{\alpha_1}y(x) = I_a^{\alpha_1}h(x) + y_0 + (x-a)y_1. \tag{2.1.6}$$

Applying Lemma 1.5.4 we get

$$y(x) + PI_a^{\alpha_1-\alpha_2}y(x) + QI_a^{\alpha_1}y(x) = I_a^{\alpha_1}h(x) + u(x;\alpha_1,\alpha_2). \tag{2.1.7}$$

Equation (2.1.7) can be written as

$$y(x) + \int_a^x \Upsilon(x,s)y(s)ds = \int_a^x \Upsilon_{\alpha_1}(x,s)h(s)ds + u(x;\alpha_1,\alpha_2) = H(x).$$

□

Remark 2.1.2. If $h \in L_1[a,b]$ is bounded then the function $H(x)$ is continuous on $[a,b]$.

The function $\Upsilon(x,s)$ is continuous for $x,s \in [a,b]$ provided $\alpha_1 \geq 1$ and weakly singular if $0 < \alpha_1 < 1$. Following theorem guarantees the existence of unique solution for the problem (2.1.1) - (2.1.2).

Theorem 2.1.3. *For $\alpha_1 > 0$ assume $h \in L_1[0,1]$ is bounded. Then there exists a unique continuous solution of the problem* (2.1.1) - (2.1.2).

Proof. Consider the iterations

$$y_n(x) = H(x) - \int_0^x \Upsilon(x,s)u_{n-1}(s)ds, \qquad n = 1, 2, \cdots, \tag{2.1.8}$$

with $y_0(x) = H(x)$. Now

$$\zeta_n(x) = -\int_0^x \Upsilon(x,s)\zeta_{n-1}(s)ds, \qquad n = 1, 2, \cdots, \tag{2.1.9}$$

where

$$\zeta_n(x) := y_n(x) - y_{n-1}(x) \text{ with } \ \psi_0(x) := H(x). \tag{2.1.10}$$

From Equation (2.1.10) we have

$$y_n(x) := \sum_{i=0}^{n} \zeta_i(x).$$

Now, we prove that $\{y_n\}_{n=0}^{\infty}$ converges.

Since H is continuous on $[0, 1]$, so there exists $M > 0$ such that $|H(x)| \leq M$ for all $x \in [0, 1]$.

For $n = 1$, from (2.1.9) we have

$$|\zeta_1(x)| \leq M \int_0^x \Upsilon(x,s)ds = M\frac{(x-0)^{\alpha_1}}{\Gamma(\alpha_1+1)} \leq \frac{M}{\Gamma(\alpha_1+1)}. \tag{2.1.11}$$

By induction, one can easily prove that

$$|\zeta_n(x)| \leq \frac{M}{(\Gamma(\alpha_1+1))^n}. \tag{2.1.12}$$

Note that $\frac{1}{\Gamma(\alpha_1+1)} < 1$. Therefore the sequence of functions $\{y_n(x)\}_{n=0}^{\infty}$ converges uniformly to a continuous limit $y(x)$. That is

$$y(x) = \lim_{n\to\infty} y_n(x) = \sum_{i=0}^{\infty} \zeta_i(x).$$

Now we show that $y(x)$ satisfies the Equation (2.1.1).

Summing equations in (2.1.9)

$$\sum_{i=0}^{\infty} \zeta_{i+1}(x) = -\sum_{i=0}^{\infty} \int_0^x \Upsilon(x,s)\zeta_i(s)ds. \tag{2.1.13}$$

Uniform convergence of $\{\zeta_i\}_{i=0}^{\infty}$ allows us to interchange the sum and integral in (2.1.13). Hence

$$\sum_{i=0}^{\infty} \zeta_i(x) - H(x) = -\int_0^x \Upsilon(x,s) \sum_{i=0}^{\infty} \zeta_i(s)ds,$$

or

$$y(x) = H(x) - \int_0^x \Upsilon(x,s)y(s)ds.$$

Now we prove that $y(x)$ is only solution of (2.1.9). On contrary, assume there exists an other continuous solution $z(x)$ of (2.1.9). For $n = 1$, from Equation (2.1.8) we get

$$|y(x) - z(x)| \leq \int_0^x \Upsilon(x,s)|y(s) - z(s)|ds. \tag{2.1.14}$$

Since $y(x)-z(x)$ is continuous on $[0,1]$. There exists a constant $M>0$ such that $|y(x)-z(x)|\leq M$ for all $x\in[0,1]$. Thus Equation (2.1.14) becomes

$$|y(x)-z(x)|\leq M\int_0^x \Upsilon(x,s)ds=\frac{M}{\Gamma(\alpha_1+1)}. \tag{2.1.15}$$

Repeated application of (2.1.15) gives

$$|y(x)-z(x)|\leq \frac{M}{(\Gamma(\alpha_1+1))^n}. \tag{2.1.16}$$

$$|y(x)-z(x)|\leq \lim_{n\to\infty}\frac{M}{(\Gamma(\alpha_1+1))^n}=0.$$

Hence, $y(x)=z(x)$ for all $x\in[0,1]$. □

2.2 Existence and uniqueness of non-linear fractional differential equations

In this section we check the existence and uniqueness of fractional differential equations by reviewing the work done in [17]. We consider the following initial value problem for fractional differential equation

$$\begin{aligned} &{}^{RL}D_0^{\alpha_1}(y-T_{n-1}[y])(x)=h(x,y(x)),\\ &y^{(i)}(0)=y_0^{(i)},\quad i=0,1,2,\cdots,n-1, \end{aligned} \tag{2.2.1}$$

where $T_{n-1}[y]:=\sum_{k=0}^{n-1}\frac{x^k y^{(k)}(0)}{k!}$ represents the Taylor polynomial of order $(n-1)$ for y which is centered at 0. Now we state the results that will help us in checking the existence and uniqueness of solution for problem (2.2.1).

Lemma 2.2.1. *If the function h in (2.2.1) is continuous then our problem (2.2.1) is equivalent to the nonlinear Volterra integral equation of the second kind:*

$$y(x)=\sum_{i=0}^{n-1}\frac{x^i}{i!}y^{(i)}(0)+\frac{1}{\alpha_1}\int_0^x (x-s)^{\alpha_1-1}h(s,y(s))ds, \tag{2.2.2}$$

with $n-1<\alpha_1\leq n$. In simple words we say that every solution to problem (2.2.2) is also a solution to problem (2.2.1) and vice versa. So we turn our focus to problem (2.2.2).

The kernel in (2.2.2) is weakly singular when $0<\alpha_1<1$ and continuous when $\alpha_1\geq 1$. We consider only the case when $0<\alpha_1<1$ because for $\alpha_1\geq 1$ the proof is similar to classical case. The proof of Lemma 2.2.1 is similar to proof of Theorem 2.1.1.

Theorem 2.2.2. *Let A be a nonempty closed subset of Banach space B, and let $q_m\geq 0$ for every m such that $\sum_{m=0}^{\infty}q_m$ converges. Also, let the mapping $J:A\to A$ satisfy the inequality*

$$\|J^m a-J^m b\|\leq q_m\|a-b\|,$$

for every $m \in \mathbb{N}$ and every $a, b \in A$. Then, J has a uniquely defined fixed point a^*. Furthermore, for any $a_0 \in A$, the sequence $(J^m a_0)_{m=1}^{\infty}$ converges to this fixed point a^*.

Theorem 2.2.3. *(Existence) Assume that $E := [0, \varrho^*] \times [y_0^{(0)} - p, y_0^{(0)} + p]$ with some varrho$^* > 0$ and some $p > 0$, also let the function $h : D \to \mathbb{R}$ be continuous. Further define $\varrho := min\{\varrho^*, (\frac{p\Gamma(\alpha_1+1)}{\|h\|_\infty})^{\frac{1}{\alpha_1}}\}$. Then there exists a function $y : [0, \varrho] \to \mathbb{R}$ that solves the Equation (2.2.1).*

Proof. We will only discuss the case $0 < \alpha_1 < 1$, by Lemma 2.2.1 Equation (2.2.2) reduces to

$$y(x) = y_0^{(0)} + \frac{1}{\Gamma(\alpha_1)} \int_0^x (x - s)^{\alpha_1 - 1} h(s, u(s)) ds. \tag{2.2.3}$$

Now we define the set $A := \{y \in C[0, \varrho] : \|y - y_0^{(0)}\|_\infty \leq p\}$. This is a closed subset of Banach space B of all continuous functions on $[0, \varrho]$ with Chebyshev norm. Sine $y_0^{(0)} \in A$ which means set A is nonempty. Define the operator J on A by

$$(Jy)(x) = y_0^{(0)} + \frac{1}{\Gamma(\alpha_1)} \int_0^x (x - t)^{\alpha_1 - 1} h(s, y(s)) ds. \tag{2.2.4}$$

We see from Equation (2.2.3) and Equation (2.2.4) that $y = Jy$. Now we prove that operator J maps convex and closed set A to itself. For this let $y \in A$ and $x \in [0, \varrho]$, we see

$$\begin{aligned} \left|(Jy)(x) - y_0^{(0)}\right| &= \frac{1}{\Gamma(\alpha_1)} \left| \int_0^x (x - s)^{\alpha_1 - 1} h(s, u(s)) ds \right| \\ &\leq \frac{x^{\alpha_1}}{\alpha_1 \Gamma(\alpha_1)} \|h\|_\infty \\ &\leq \frac{\varrho^{\alpha_1}}{\Gamma(\alpha_1 + 1)} \|h\|_\infty \\ &\leq \frac{p\Gamma(\alpha_1 + 1)\|h\|_\infty}{\|h\|_\infty \Gamma(\alpha_1 + 1)}. \end{aligned}$$

Thus,

$$\left|(Jy)(x) - y_0^{(0)}\right| \leq p. \tag{2.2.5}$$

Equation (2.2.5) implies $Jy \in A$ for $y \in A$. Which means J maps the set A to itself. Now we prove the continuity of operator J. Since a function h is continuous on the compact set E, it is also uniformly continuous on the same set. Thus for an arbitrary $\epsilon > 0$, we can find $\delta > 0$ such that

$$|h(x, y) - h(x, s)| < \frac{\epsilon \Gamma(\alpha_1 + 1)}{\varrho^{\alpha_1}} \text{ whenever } |y - s| < \delta. \tag{2.2.6}$$

Now let $y, \hat{y} \in A$ such that $\|y - \hat{y}\| < \delta$. Then Equation (2.2.6) implies

$$|h(x, y(x)) - h(x, \hat{y}(x))| < \frac{\epsilon \Gamma(\alpha_1 + 1)}{\varrho^{\alpha_1}}, \tag{2.2.7}$$

for all $x \in [0, \varrho]$. Thus,

$$|(Jy)(x) - (J\hat{y})(x)| = \frac{1}{\Gamma(\alpha_1)} \left| \int_0^x (x - t)^{\alpha_1 - 1} \{h(s, y(s)) - h(s, \hat{y}(s))\} ds \right|. \tag{2.2.8}$$

Using Equation (2.2.6) in Equation (2.2.7) we get

$$
\begin{aligned}
|(Jy)(x) - (J\hat{y})(x)| &\leq \frac{\epsilon\Gamma(\alpha_1+1)}{\varrho^{\alpha_1}\Gamma(\alpha_1)}\int_0^x (x-s)^{\alpha_1-1}ds \\
&= \frac{x^{\alpha_1}\epsilon\Gamma(\alpha_1+1)}{\varrho^{\alpha_1}\Gamma(\alpha_1+1)} \\
&\leq \frac{\epsilon\varrho^{\alpha_1}}{\varrho^{\alpha_1}} = \epsilon.
\end{aligned} \tag{2.2.9}
$$

Equation (2.2.9) implies operator J is continuous. Now we show that the set of functions $J(A) := \{Jy : y \in A\}$ is bounded in a point wise sense. For this consider $s \in J(A)$, and for all $x \in [0, \varrho]$ we have

$$
\begin{aligned}
|s(x)| &= |(Jy)(x)| \\
&= \Big| y_0^{(0)} + \frac{1}{\Gamma(\alpha_1)}\int_0^x (x-s)^{\alpha_1-1}h(s,y(s))ds \Big| \\
&\leq |y_0^{(0)}| + \frac{1}{\Gamma(\alpha_1)}\int_0^x (x-s)^{\alpha_1-1}|h(s,y(s))ds| \\
&\leq |y_0^{(0)}| + \frac{\|h\|_\infty}{\Gamma(\alpha_1+1)}x^{\alpha_1} \\
&\leq |y_0^{(0)}| + \frac{\|h\|_\infty}{\Gamma(\alpha_1+1)}\varrho^{\alpha_1} \\
&\leq |y_0^{(0)}| + p.
\end{aligned} \tag{2.2.10}
$$

Equation (2.2.10) shows that $J(A)$ is bounded in a pointwise sense. Now we show that $J(A)$ is equicontinuous. For $0 \leq x_1 \leq x_2 \leq \varrho$, we have

$$
\Big|(Jy)(x_1) - (Jy)(x_2)\Big| = \frac{1}{\alpha_1}\Big|\int_0^{x_1}(x_1-s)^{\alpha_1-1}h(s,y(s))ds - \int_0^{x_2}(x_2-s)^{\alpha_1-1}h(s,y(s))ds\Big|. \tag{2.2.11}
$$

Now in Equation (2.2.11) writing $\int_0^{x_2} = \int_0^{x_1} + \int_{x_1}^{x_2}$, we have

$$
\begin{aligned}
\Big|(Jy)(x_1) - (Jy)(x_2)\Big| &= \frac{1}{\alpha_1}\Big|\int_0^{x_1}\{(x_1-s)^{\alpha_1-1} - (x_2-s)\}h(s,y(s))ds - \int_{x_1}^{x_2}(x_2-s)^{\alpha_1-1}h(s,y(s))ds\Big| \\
&\leq \frac{\|h\|_\infty}{\alpha_1}\Big|\int_0^{x_1}\{(x_1-s)^{\alpha_1-1} - (x_2-s)\}ds - \int_{x_1}^{x_2}(x_2-s)^{\alpha_1-1}ds\Big|.
\end{aligned} \tag{2.2.12}
$$

Integrating Equation (2.2.12) yields

$$
\Big|(Jy)(x_1) - (Jy)(x_2)\Big| \leq \frac{\|h\|_\infty}{\Gamma(\alpha_1+1)}\{2(x_2-x_1)^{\alpha_1} + x_1^{\alpha_1} - x_2^{\alpha_1}\}. \tag{2.2.13}
$$

Since $x_2 \geq x_1$, Equation (2.2.13) can be written as

$$
\Big|(Jy)(x_1) - (Jy)(x_2)\Big| \leq \frac{2\|h\|_\infty}{\Gamma(\alpha_1+1)}(x_2-x_1)^{\alpha_1}. \tag{2.2.14}
$$

Hence,

$$
\Big|(Jy)(x_1) - (Jy)(x_2)\Big| \leq \frac{2\|h\|_\infty}{\Gamma(\alpha_1+1)}(\delta)^{\alpha_1} \quad \text{whenever} \quad |x_2 - x_1| < \delta. \tag{2.2.15}
$$

Equation (2.2.15) shows that the expression on right hand side is independent of y, which means $J(A)$ is equicontinuous. Then the statement of Arzela-Ascoli theorem guarantees that $J(A)$ is relatively compact. Then, by Schauder's fixed point theorem we know that J has a fixed point. By construction of J we know that fixed point of J is solution to our initial value problem which confirms the existence of solution to problem (2.2.1). □

Theorem 2.2.4. *(Uniqueness) Assume that* $E := [0, \varrho^*] \times [y_0^{(0)} - p, y_0^{(0)} + p]$ *with some* $\varrho^* > 0$ *and some* $p > 0$*. Furthermore, let the function* $h : E \to \mathbb{R}$ *be bounded on* E *and fulfill a Lipschitz condition with respect to the second variable; i.e.,*

$$|h(x, y) - h(x, s)| \leq M|y - s|,$$

with some constant $M > 0$ *independent of* x, y *and* s*. Then, denoting* ϱ *as in Theorem 2.2.3, there exists at most one function* $y : [0, \varrho] \to \mathbb{R}$ *solving the initial value problem (2.2.1).*

Proof. We proceed in similar way to Theorem 2.2.3 and define an operator J on $A := \{y \in C[0, \varrho] : \|y - y_0^{(0)}\|_\infty \leq p\}$ in the same way as defined in proof of previous theorem

$$(Jy)(x) = y_0^{(0)} + \frac{1}{\Gamma(\alpha_1)} \int_0^x (x - t)^{\alpha_1 - 1} h(s, y(s))ds. \tag{2.2.16}$$

To prove the uniqueness, we will show that operator J has a unique fixed point. We have seen in Equation (2.2.15) of the previous theorem that Jy is a continuous function for $0 \leq x_1 \leq x_2 \leq \varrho$. Equation (2.2.5) yields that J maps the set A to itself.

Now for every $n \in \mathbb{N}_0$ and every $x \in [0, \varrho]$, we get

$$\|J^n y - J^n \hat{y}\|_{L_\infty[0,x]} = \|J(J^{n-1}y) - J(J^{n-1}\hat{y})\|_{L_\infty[0,x]}. \tag{2.2.17}$$

Using definition of operator J in Equation (2.2.17) we get

$$\|J^n y - J^n \hat{y}\|_{L_\infty[0,x]} = \frac{1}{\Gamma(\alpha_1)} \sup_{0 \leq w \leq x} \left| \int_0^w (w - t)^{\alpha_1 - 1} \{h(s, J^{n-1}y(s)) - h(s, J^{n-1})\hat{y}(s))\} ds \right|. \tag{2.2.18}$$

Using Lipschitz assumption on function h in Equation (2.2.18) we get

$$\begin{aligned} \|J^n y - J^n \hat{y}\|_{L_\infty[0,x]} &\leq \frac{M}{\Gamma(\alpha_1)} \sup_{0 \leq w \leq x} \int_0^w (w - s)^{\alpha_1 - 1} \left| \{J^{n-1}y(s) - J^{n-1}\hat{y}(s))\} \right| ds \\ &\leq \frac{M}{\Gamma(\alpha_1)} \int_0^x (x - s)^{\alpha_1 - 1} \sup_{0 \leq w \leq s} \left| \{J^{n-1}y(w) - J^{n-1}\hat{y}(w))\} \right| ds \\ &= \frac{M}{\Gamma(\alpha_1)\Gamma(\alpha_1(n-1))} \int_0^x (x - s)^{\alpha_1 - 1} \sup_{0 \leq w \leq s} \left| \int_0^w (w - s)^{\alpha_1(n-1) - 1} \{h(s, y(s)) - h(s, \hat{y}(s))\} \right| ds \end{aligned} \tag{2.2.19}$$

Again using Lipschitz assumption on function h we get

$$\begin{aligned}\|J^n y - J^n \hat{y}\|_{L_\infty[0,x]} &\leq \frac{M^n}{\Gamma(\alpha_1)\Gamma(\alpha_1(n-1))} \int_0^x (x-s)^{\alpha_1 - 1} \sup_{0\leq w\leq s} \Big|\{\int_0^w (w-s)^{\alpha_1(n-1)-1}\{|y(s)-\hat{y}(s)|\}\Big| ds \\ &\leq \frac{M^n}{\Gamma(\alpha_1)\Gamma(1+\alpha_1(n-1))} \int_0^x (x-s)^{\alpha_1-1} s^{\alpha_1(n-1)} \sup_{0\leq w\leq s} |y(w)-\hat{y}(w)| ds \\ &\leq \frac{M^n}{\Gamma(\alpha_1)\Gamma(1+\alpha_1(n-1))} \sup_{0\leq w\leq x} |y(w)-\hat{y}(w)| \int_0^x (x-s)^{\alpha_1-1} t^{\alpha_1(n-1)} ds.\end{aligned} \tag{2.2.20}$$

Using definition of beta function in Equation (2.2.20) we get

$$\|J^n y - J^n \hat{y}\|_{L_\infty[0,x]} \leq \frac{M^n}{\Gamma(\alpha_1)\Gamma(1+\alpha_1(n-1))} \|y-\hat{y}\|_{L_\infty[0,x]} \frac{\Gamma(\alpha_1)\Gamma(1+\alpha_1(n-1))}{\Gamma(1+\alpha_1 n)} x^{\alpha_1 n}. \tag{2.2.21}$$

After simplifications Equation (2.2.21) yields

$$\|J^n y - J^n \hat{y}\|_{L_\infty[0,x]} \leq \frac{(M x^{\alpha_1})^n}{\Gamma(1+\alpha_1 n)} \|y-\hat{y}\|_{L_\infty[0,x]}. \tag{2.2.22}$$

Taking Chebyshev norm on our interval $[0,\varrho]$, Equation (2.2.22) yields

$$\|J^n y - J^n \hat{y}\|_{L_\infty[0,x]} \leq \frac{(M \varrho^{\alpha_1})^n}{\Gamma(1+\alpha_1 n)} \|y-\hat{y}\|_{L_\infty[0,x]}. \tag{2.2.23}$$

We have shown that operator J satisfies the conditions of Theorem 2.2.2 with $p_n = \frac{(M\varrho^{\alpha_1})^n}{\Gamma(1+\alpha_1 n)}$. The only thing left here is the convergence of series $\sum_{n=0}^{\infty} p_n$. We see that

$$\sum_{n=0}^{\infty} \frac{(M\varrho^{\alpha_1})^n}{\Gamma(1+\alpha_1 n)} =: E_{\alpha_1}(M\varrho^{\alpha_1}). \tag{2.2.24}$$

Theorem 1.4.4 yields the series in Equation (2.2.24) is convergent. Theorem 2.2.2 yields J has a uniquely fixed point which proves the uniqueness of solution of fractional differential equation. □

We have discussed the case until now in which function h depends on variable x and function $y(x)$. Now we consider the fractional differential equation in which function h depends not only on y but also on fractional derivatives of y. For this we consider the initial value problem for following non-linear fractional differential equation as discussed in [12]

$$\begin{aligned}&{}^cD_a^{\alpha_1} y(x) = h(x, {}^c D_a^{\alpha_2} y(x)), \quad 0 < x \leq 1, \\ &y^{(i)}(0) = \mu_i, \quad i = 0,1,2,...,m-1,\end{aligned} \tag{2.2.25}$$

where $m-1 < \alpha_1 < m,\ n-1 < \alpha_2 < n\ (m,n \in \mathbb{N}, m-1 \geq n)$, ${}^cD_a^{\alpha_1}$, ${}^cD_a^{\alpha_2}$ are the $\alpha_1 st$ and $\alpha_2 nd$ Caputo derivatives of function y and function $h \in C([0,1]\times\mathbb{R})$. Now we state some results for the existence and uniqueness of solution for fractional differential equations from [28].

Lemma 2.2.5. *Let $m, n \in \mathbb{N},\ m-1 < \alpha_1 < m,\ n-1 < \alpha_2 < n,\ n \leq m-1$ and assume that the following two hold*

(a) $h : [0,1] \times \mathbb{R} \to \mathbb{R}$ is continuously differentiable function.

(b) $h(0,0) = 0$ and $h(s,0) \neq 0$ on a compact subinterval of $(0,1]$.

Then $y \in C^m[0,1]$ is a solution of equation (2.2.25) if and only if

$$y(x) = \sum_{i=0}^{n-1} \frac{x^i}{i!}\mu_i + \int_0^x \frac{(x-s)^{n-1}w(s)ds}{(n-1)!},$$

where $w \in C[0,1]$ is a solution of the equation

$$w(x) = \sum_{i=0}^{m-n-1} \frac{x^i}{i!}\mu_{n+i} + \frac{1}{\Gamma(\alpha_1 - n)}\int_0^x (x-t)^{\alpha_1-n-1} h\Big(s, \frac{1}{\Gamma(n-\alpha_2)}\int_0^s (s-t)^{n-\alpha_2-1}w(t)dt\Big)ds.$$

Lemma 2.2.6. *Let $n \in \mathbb{N},\ n-1 < \alpha_2 < \alpha_1 < n$ and assume that (a) and (b) hold. Then, $y \in C^n[0,1]$ is a solution of Equation (2.2.25) if and only if*

$$y(x) = \sum_{i=0}^{n-1} \frac{x^i}{i!}\mu_i + \int_0^x \frac{(x-s)^{\alpha_2-1}w(s)ds}{\Gamma(\alpha_2)}, \quad 0 \leq x \leq 1,$$

where $w \in C[0,1]$ is a solution of the equation

$$w(x) = \frac{1}{\Gamma(\alpha_1-\alpha_2)}\int_0^x (x-s)^{\alpha_1-\alpha_2-1}h(s,w(s))ds, \quad 0 \leq x \leq 1. \tag{2.2.26}$$

Lemma 2.2.7. *[12] If (a), (b) and*

(c) There exists $p > 1,\ 0 < \eta < 1$ and $a(x) \in C([0,1],[0,\infty))$ such that

$$1 - \eta \geq \frac{1}{\Gamma(\alpha_1-\alpha_2)} \sup_{0\leq x\leq 1} \int_0^x (x-s)^{\alpha_1-\alpha_2-1}a(s)ds,$$
$$0 < H := \frac{1}{\Gamma(\alpha_1-\alpha_2)} \sup_{0\leq x\leq 1} \int_0^x (x-s)^{\alpha_1-\alpha_2-1}|h(s,0)|ds < \infty.$$

And for any $y, z \in C([0,\infty))$ with $0 \leq |y(x)|, |z(x)| \leq \frac{H}{\eta}$ for $0 \leq x \leq 1$,

$$|h(x,y(x)) - h(x,z(x))| \leq a(x)|y(x)-z(x)| \quad \text{for} \quad 0 \leq x \leq 1.$$

holds. Then Equation (2.2.26) has a unique solution φ with $\|\varphi\| \leq \frac{pH}{\eta}$.

Lemma 2.2.8. *[12] If (a)-(c) holds and $y(x)$ is a solution of Equation (2.2.26), then $y(x) \equiv \varphi(x)$.*

Theorem 2.2.9. *Let $n-1 < \alpha_2 < \alpha_1 < n$ $(n \in \mathbb{N})$. Assume that (a)-(c) holds then Equation (2.2.25) has a unique solution.*

Proof. Lemmas 2.2.6, 2.2.7 and 2.2.8 can be used to verify the statement of this theorem. □

Theorem 2.2.10. *Let $n-1<\alpha_2<n\leq m-1<\alpha_1<m(n,m\in\mathbb{N})$. Assume that (a), (b) and (d) There exists $p>1,\ 0<\eta<1$ and $a(x)\in C([0,1],[0,\infty))$ such that*

$$1-\eta\geq\frac{1}{\Gamma(\alpha_1-n)\Gamma(n-\alpha_2+1)}\sup_{0\leq x\leq 1}\int_0^x(x-s)^{\alpha_1-n-1}s^{n-\alpha_2}a(s)ds,$$

$$0<H:=\sup_{0\leq x\leq 1}\Big[\Big|\sum_{i=0}^{n-1}\frac{x^i}{i!}\mu_i\Big|+\int_0^x(x-s)^{\alpha_1-n-1}|h(s,0)|ds\Big]<\infty.$$

And for any $y,z\in C([0,\infty))$ with $0\leq|y(x)|,|z(x)|\leq\frac{H}{\eta}$ for $0\leq x\leq 1$,

$$|h(x,y(x))-h(x,z(x))|\leq a(x)|y(x)-z(x)|\quad for\quad 0\leq x\leq 1.$$

hold. Then Equation (2.2.25) has a unique solution.

Chapter 3

Power series method for solving fractional differential equations

Power series method is a powerful tool for solving fractional order differential and integral equations. In this method we look for the solution of fractional order differential or integral equation in the form of power series and coefficients of this series are determined using initial or boundary conditions. For latest works on power series method for fractional differential equation one can see [40] and [48]. In this chapter we have reviewed the work done in [47] and [26]. This chapter deals with solution of fractional differential equations of order α $(0 < \alpha < 1)$ and 2α $(0 < \alpha < 1)$ around α-ordinary and regular α-singular point. We consider the following homogeneous fractional differential equation

$$D_a^{\alpha} y(x) + A(x)y(x) = 0, \tag{3.0.1}$$

where $\alpha \in (0, 1)$ and $A(x)$ is defined on the interval $[a, b]$. D_a^{α} represents both Riemann-Liouville and Caputo fractional derivative.

Definition 3.0.1. [47] Let $\alpha \in (0, 1]$, $y(x)$ is a real function defined on the interval $[a, b]$, and $x_0 \in [a, b]$. Then $y(x)$ is said to be α-analytic at x_0 if there exists an interval $N(x_0)$ such that, for all $x \in N(x_0)$, $y(x)$ can be expressed as a series of natural powers of $(x - x_0)^{\alpha}$. That is, $u(x)$ can be expressed as $\sum_{n=0}^{\infty} c_n (x - x_0)^{n\alpha}$ where $c_n \in \mathbb{R}$ and the series is absolutely convergent for $|x - x_0| < \mu$ with $\mu > 0$.

Definition 3.0.2. [47] A point $x_0 \in [a, b]$ is said to be α-ordinary point of the homogeneous fractional differential Equation (3.0.1) if the functions $A_k(x)$ with $k = 0, 1, \cdots, n - 1$ are α-analytic in x_0. A point $x_0 \in [a, b]$ which is not α-ordinary will be called α-singular.

Definition 3.0.3. [47] Let $x_0 \in [a, b]$ be an α-singular point of the homogeneous fractional differential Equation (3.0.1). The point x_0 is said to be a regular α-singular point of this equation if the functions $(x - x_0)^{(n-k)\alpha} A_k(\alpha)$ are $\alpha - analytic$ in x_0 with $k = 0, 1, \cdots, n - 1$. Otherwise, x_0 is said to be an essential α-singular point.

3.1 Solution around an α-ordinary point

In this section we have evaluated solution of homogeneous fractional differential equation of order α $(0 < \alpha < 1)$ and 2α $(0 < \alpha < 1)$ around a α-ordinary point in the form of power series. The results and method is taken from [26].

3.1.1 Solution around an α-ordinary point of a fractional differential equation of order α

In this section we seek solution of the following fractional differential equation

$$D_a^{\alpha} y(x) + A(x)y(x) = 0, \tag{3.1.1}$$

around an α-ordinary point $x_0 \in [a, b]$ with $\alpha \in (0, 1)$ and $A(x)$ is defined in the interval $[a, b]$. Since x_0 is an α-ordinary point we can represent $A(x)$ in the following form

$$A(x) = \sum A_n (x - x_0)^{n\alpha}. \tag{3.1.2}$$

The series represented in Equation (3.1.2) is convergent for $x \in [x_0, x_0 + \mu]$ with $\mu > 0$.

Theorem 3.1.1. *Let $x_0 \in [a, b]$ be an α-ordinary point of the equation*

$${}^{RL}D_a^{\alpha} y(x) + A(x)y(x) = 0, \tag{3.1.3}$$

with $\alpha \in [0, 1]$ and $q_0 \in \mathbb{R}$. Then there exits a unique function

$$y(x) = (x - x_0)^{\alpha-1} \sum_{n=0}^{\infty} q_n (x - x_0)^{n\alpha}, \tag{3.1.4}$$

which is the solution of the Equation (3.1.3) for $x \in (x_0, x_0 + \mu)$ and which satisfies the initial condition $q_0 = \lim_{x \to x_0} (x - x_0)^{1-\alpha} y(x)$.

Proof. We begin with seeking the solution of the Equation (3.1.3) given in Equation (3.1.4). For this we take fractional order derivative of Equation (3.1.4) and substitute the result in Equation (3.1.3)

$$\sum_{n=1}^{\infty} q_n \frac{\Gamma(n+1)\alpha}{\Gamma(n\alpha)} (x - x_0)^{n\alpha-1} + \sum_{n=0}^{\infty} A(x) q_n (x - x_0)^{(n+1)\alpha-1} = 0. \tag{3.1.5}$$

Replacing n with $n + 1$ in first series of Equation (3.1.5) we get

$$\sum_{n=0}^{\infty} q_{n+1} \frac{\Gamma(n+2)\alpha}{\Gamma(n+1)\alpha)} (x - x_0)^{(n+1)\alpha-1} + \sum_{n=0}^{\infty} A(x) q_n (x - x_0)^{(n+1)\alpha-1} = 0. \tag{3.1.6}$$

Let $\Lambda_n = \frac{\Gamma(n+2)\alpha}{\Gamma(n+1)\alpha}$, and substitute Equation (3.1.2) in Equation (3.1.6) we get

$$\begin{aligned} &\sum_{n=0}^{\infty} q_{n+1} \Gamma_n (x - x_0)^{(n+1)\alpha-1} + \sum_{n=0}^{\infty} A_0 q_n (x - x_0)^{(n+1)\alpha-1} \\ &+ \sum_{n=0}^{\infty} A_1 q_n (x - x_0)^{(n+2)\alpha-1} + \sum_{n=0}^{\infty} A_2 q_n (x - x_0)^{(n+3)\alpha-1} + \cdots = 0. \end{aligned} \tag{3.1.7}$$

Now equate the coefficients of powers of $x - x_0$. When $n = 0$

$$q_1\Lambda_0 + A_0q_0 = 0 \text{ implies } q_1 = -\frac{A_0q_0}{\Lambda_0}.$$

When $n = 1$

$$q_2 = -\frac{A_0q_1 + A_1q_0}{\Lambda_1}.$$

When $n = 2$

$$q_3 = -\frac{A_0q_2 + A_1q_1 + A_2q_0}{\Lambda_2}.$$

Continuing in this manner we get the following recurrence relation

$$q_{n+1} = -\frac{\sum_{k=0}^{n} A_{n-k}q_k}{\Lambda_n}. \tag{3.1.8}$$

Now we show the convergence of the series given in Equation (3.1.4) for $x \in (x_0, x_0 + \mu)$. Since series in Equation (3.1.2) is convergent there exist a constant $M > 0$ such that

$$|A_n(x - x_0)^{n\alpha})| \leq |A_n||(x - x_0)^{n\alpha}| \leq M\mu^{n\alpha}. \tag{3.1.9}$$

Let $r < \mu \Rightarrow \frac{1}{r} > \frac{1}{\mu}$, substitute this in Equation (3.1.9) we get

$$|A_{n-k}| \leq \frac{Mr^{k\alpha}}{r^{n\alpha}}. \tag{3.1.10}$$

Substituting Equation (3.1.10) in Equation (3.1.8) we get

$$|q_{n+1}| \leq \frac{M}{r^{n\alpha}\Lambda_n}\sum_{k=0}^{n} r^{k\alpha}q_k.$$

We define c_n where $n \in \mathbb{N}$ from the above recurrence relation as

$$c_{n+1} = \frac{M}{r^{n\alpha}\Lambda_n}\sum_{k=0}^{n} r^{k\alpha}c_k \quad (n \in \mathbb{N}_0).$$

It is obvious $0 \leq |q_n| \leq |c_n|$ for $n \in \mathbb{N}_0$. The series $\sum_{n=0}^{\infty} c_n(x - x_0)^{n\alpha+1}$ is convergent for $|x - x_0| < r$, because in accordance with the asymptotic representation

$$\frac{\Gamma(z + a)}{\Gamma(z + b)} = z^{a-b}\left\{1 + O(\frac{1}{z})\right\}, \tag{3.1.11}$$

there holds the following estimate

$$\begin{aligned}\lim_{n\to\infty}\left|\frac{c_{n+1}(x - x_0)^{(n+1)\alpha}}{c_n(x - x_0)^{n\alpha}}\right| &= \lim_{n\to\infty}\left|\frac{\frac{M}{(n\alpha)^\alpha r^{n\alpha}}\sum_{k=0}^{n} r^{k\alpha}c_k(x - x_0)^\alpha}{\frac{M}{(n\alpha)^\alpha r^{(n-1)\alpha}}\sum_{k=0}^{n-1} r^{k\alpha}c_k(x - x_0)^\alpha}\right| \\ &= \left|\frac{(x - x_))^\alpha}{r^\alpha}\right|.\end{aligned}$$

Since $|x - x_0| < \mu$ and $r < \mu$, this implies $\frac{|x-x_0|}{r} < 1$

$$\lim_{n\to\infty}\left|\frac{c_{n+1}(x - x_0)^{(n+1)\alpha}}{c_n(x - x_0)^{n\alpha}}\right| = \left|\frac{(x - x_))^\alpha}{r^\alpha}\right| < 1.$$

Which means $|c_n(x - x_0)^{n\alpha+1}| \leq Nr^{n\alpha+1}$ with $N > 0$ and also $|q_n(x - x_0)^{(n+1)\alpha-1}| \leq N_1r^{(n+1)\alpha-1}$ with $N_1 > 0$. So series in Equation (3.1.4) is convergent. □

Theorem 3.1.2. *Let $x_0 \in [a,b]$ be an α-ordinary point of the equation*

$$^cD_a^\alpha y(x) + A(x)y(x) = 0, \tag{3.1.12}$$

with $\alpha \in [0,1]$ and $q_0 \in \mathbb{R}$. Then there exits a unique α-analytic function $y(x)$ in x_0 as a solution to Equation (3.1.12) for $x \in [x_0, x_0+\mu]$ such that the initial condition $y(x_0) = q_0$ is satisfied.

Proof. The proof of this theorem is done on similar pattern of Theorem 3.1.1 by searching the solution in the form $y(x) = \sum_{n=0}^{\infty} q_n(x-x_0)^{n\alpha}$ and using recurrence relation for coefficients q_n as follows

$$q_{n+1} = -\frac{\sum_{k=0}^{n} A_{n-k}q_k}{\Lambda_n},$$

where $\Lambda_n = \frac{\Gamma((n+1)\alpha+1)}{\Gamma(n\alpha+1)}$. □

Example 3.1.3. Consider the following fractional differential equation of order α $(0 < \alpha < 1)$

$$^{RL}D_0^\alpha y(x) + x^\alpha y(x) = 0. \tag{3.1.13}$$

According to Theorem 3.1.1 suppose solution of Equation (3.1.13)

$$y(x) = \sum_{n=0}^{\infty} q_n x^{(n+1)\alpha-1}. \tag{3.1.14}$$

In Equation (3.1.13) $x = 0$ is α-ordinary point and suppose series solution for $A(x) = x^\alpha$ as follows

$$A(x) = x^\alpha = \sum_{n=0}^{\infty} A_n x^{n\alpha}. \tag{3.1.15}$$

Equating coefficients of powers of x^α in Equation (3.1.15) yields

$$A_0 = 0,\ A_1 = 1,\ A_2 = 0,\ A_3 = 0, \cdots .$$

According to Theorem 3.1.1 consider the recurrence relation for coefficients q_n

$$q_{n+1} = -\frac{\sum_{k=0}^{\infty} A_{n-k}q_k}{\Lambda_n}, \tag{3.1.16}$$

where $\Lambda_n = \frac{\Gamma((n+2)\alpha)}{\Gamma((n+1)\alpha)}$.
When $n = 0$ Equation (3.1.16) implies

$$\begin{aligned} q_1 &= -\frac{\sum_{k=0}^{0} a_{0-k}q_k}{\Gamma_0} \\ &= -\frac{\Gamma(\alpha)}{\Gamma(2\alpha)} a_0 q_0 \\ &= 0. \end{aligned}$$

When $n = 1$ Equation (3.1.16) implies

$$q_2 = -\frac{\Gamma(2\alpha)}{\Gamma(3\alpha)} q_0.$$

When $n = 3$ Equation (3.1.16) implies

$$q_3 = 0.$$

When $n = 4$ Equation (3.1.16) implies

$$q_4 = \frac{\Gamma(2\alpha)\Gamma(4\alpha)}{\Gamma(3\alpha)\Gamma(5\alpha)} q_0.$$

Similarly when $n + 1 = 2n$ we get

$$q_{2n} = (-1)^n \prod_{j=1}^{n} \frac{\Gamma(2j\alpha)}{\Gamma((2j+1)\alpha)} q_0. \tag{3.1.17}$$

Substituting Equation (3.1.17) in Equation (3.1.14) results

$$\begin{aligned} y(x) &= \sum_{n=0}^{\infty} q_n x^{(n+1)\alpha-1} \\ &= q_0 x^{\alpha-1} + \sum_{n=1}^{\infty} q_n x^{(n+1)\alpha-1} \\ &= q_0 x^{\alpha-1} + \sum_{n=1}^{\infty} (-1)^n \prod_{j=1}^{n} \frac{\Gamma(2j\alpha)}{\Gamma((2j+1)\alpha)} q_0 x^{(2n+1)\alpha-1} \\ &= q_0 x^{\alpha-1} \left\{ 1 + \sum_{n=1}^{\infty} (-1)^n \prod_{j=1}^{n} \frac{\Gamma(2j\alpha)}{\Gamma((2j+1)\alpha)} x^{2n\alpha} \right\}. \end{aligned}$$

Example 3.1.4. Now we consider the same fractional differential equation of order α $(0 < \alpha < 1)$ with fractional derivative replaced by Caputo fractional derivative

$$^{c}D_0^{\alpha} y(x) + x^{\alpha} y(x) = 0. \tag{3.1.18}$$

$x = 0$ is α-ordinary point of Equation (3.1.18) and we consider the solution of Equation (3.1.18) in accordance with Theorem 3.1.2

$$y(x) = \sum_{n=0}^{\infty} q_n (x - x_0)^{n\alpha}. \tag{3.1.19}$$

The recurrence relation for coefficients q_n is given by

$$q_{n+1} = -\frac{\sum_{k=0}^{n} A_{n-k} q_k}{\Lambda_n}, \tag{3.1.20}$$

where $\Lambda_n = \frac{\Gamma((n+1)\alpha+1)}{\Gamma(n\alpha+1)}$.
When $n = 1$ Equation (3.1.20) yields

$$q_1 = 0.$$

When $n = 2$ Equation (3.1.20) yields

$$q_2 = -\frac{\Gamma(\alpha+1)}{\Gamma(2\alpha+1)} q_0.$$

When $n = 2$ Equation (3.1.20) yields

$$q_3 = 0.$$

When $n = 3$ Equation (3.1.20) yields

$$q_4 = \frac{\Gamma(\alpha+1)\Gamma(3\alpha+1)}{\Gamma(2\alpha+1)\Gamma(4\alpha+1)} q_0.$$

Similarly when $n+1 = 2n$ Equation (3.1.20) yields

$$q_{2n} = (-1)^n \prod_{j=1}^{n} \frac{\Gamma((2j-1)\alpha+1)}{\Gamma(2j\alpha+1)} q_0. \tag{3.1.21}$$

Substituting Equation (3.1.21) in Equation (3.1.19) results

$$y(x) = q_0 \left\{ 1 + \sum_{n=1}^{\infty} (-1)^n \prod_{j=1}^{n} \frac{\Gamma((2j-1)\alpha+1)}{\Gamma(2j\alpha+1)} x^{n\alpha} \right\}.$$

3.1.2 Solution around an α-ordinary point of a fractional differential equation of order 2α

In this section we consider the following fractional order differential equation

$$D_a^{2\alpha} y(x) + A(x) D_a^{\alpha} y(x) + B(x) y(x) = 0, \tag{3.1.22}$$

and find its solution around an α-ordinary point $x_0 \in [a, b]$. Where $A(x)$ and $B(x)$ are defined on an interval [a,b], and $D_a^{2\alpha} y(x)$ and $D_a^{\alpha} y(x)$ represent Riemann-Liouville or Caputo derivatives of order 2α and α respectively of a function $y(x)$. Since x_0 is an α-ordinary point of Equation (3.1.22) we can assume the following series representations of functions $A(x)$ and $B(x)$

$$\begin{aligned} A(x) &= \sum_{n=0}^{\infty} A_n (x-x_0)^{n\alpha}, \qquad \text{for } x \in [x_0, x_0+\mu_1],\ \mu_1 > 0, \\ B(x) &= \sum_{n=0}^{\infty} B_n (x-x_0)^{n\alpha}, \qquad \text{for } x \in [x_0, x_0+\mu_2],\ \mu_2 > 0. \end{aligned} \tag{3.1.23}$$

Theorem 3.1.5. *Let $\alpha \in (0, 1]$, $q_0,\ q_1 \in \mathbb{R}$ and $x_0 \in [a, b]$ be an α-ordinary point of the following equation*

$$^{RL}D_a^{2\alpha} y(x) + A(x)\ ^{RL}D^{\alpha} y(x) + B(x) y(x) = 0. \tag{3.1.24}$$

Then there exists a unique solution of Equation (3.1.24) given by

$$y(x) = (x-x_0)^{\alpha-1} \sum_{n=0}^{\infty} q_n (x-x_0)^{n\alpha},$$

for $x \in (x_0, x_0+\mu)$ with $\mu = min\{\mu_1, \mu_2\}$, which satisfies the following initial conditions

$$\lim_{x \to x_0} [(x-x_0)^{1-\alpha} y(x)] = q_0,$$

and

$$\frac{\Gamma(\alpha)}{\Gamma(2\alpha)} \lim_{x \to x_0} [(x-x_0)^{1-\alpha} D_a^{\alpha} y(x)] = q_1.$$

Proof. We search the solution of the Equation (3.1.24) given by

$$y(x) = \sum_{n=0}^{\infty} q_n (x - x_0)^{(n+1)\alpha - 1}. \tag{3.1.25}$$

Taking fractional derivatives of order α and 2α of Equation (3.1.25) we get

$${}^{RL}D_a^{\alpha} y(x) = \sum_{n=1}^{\infty} q_n \frac{\Gamma((n+1)\alpha)}{\Gamma(n\alpha)} (x - x_0)^{n\alpha - 1}, \tag{3.1.26}$$

and

$${}^{RL}D_a^{2\alpha} y(x) = \sum_{n=2}^{\infty} q_n \frac{\Gamma((n+1)\alpha)}{\Gamma((n-1)\alpha)} (x - x_0)^{(n-1)\alpha - 1}. \tag{3.1.27}$$

After substituting Equation (3.1.26) and Equation (3.1.27) in Equation (3.1.24) we get

$$\begin{aligned} \sum_{n=2}^{\infty} q_n \frac{\Gamma((n+1)\alpha)}{\Gamma((n-1)\alpha)} (x - x_0)^{(n-1)\alpha - 1} + A(x) \sum_{n=1}^{\infty} q_n \frac{\Gamma((n+1)\alpha)}{\Gamma(n\alpha)} (x - x_0)^{n\alpha - 1} \\ + B(x) \sum_{n=0}^{\infty} q_n (x - x_0)^{(n+1)\alpha - 1} = 0. \end{aligned} \tag{3.1.28}$$

Replacing n with $n+2$ in first series and n with $n+1$ in second series of Equation (3.1.28) we get

$$\begin{aligned} \sum_{n=0}^{\infty} q_{n+2} \frac{\Gamma((n+3)\alpha)}{\Gamma((n+1)\alpha)} (x - x_0)^{(n+1)\alpha - 1} + A(x) \sum_{n=0}^{\infty} q_{n+1} \frac{\Gamma((n+2)\alpha)}{\Gamma((n+1)\alpha)} (x - x_0)^{(n+1)\alpha - 1} \\ + B(x) \sum_{n=0}^{\infty} q_n (x - x_0)^{(n+1)\alpha - 1} = 0. \end{aligned} \tag{3.1.29}$$

Let $\Lambda_{n1} = \frac{\Gamma((n+3)\alpha)}{\Gamma((n+1)\alpha)}$ and $\Lambda_{n2} = \frac{\Gamma((n+2)\alpha)}{\Gamma((n+1)\alpha)}$ and substituting these in Equation (3.1.29) we get

$$\begin{aligned} \sum_{n=0}^{\infty} q_{n+2} \Lambda_{n1} (x - x_0)^{(n+1)\alpha - 1} + A(x) \sum_{n=0}^{\infty} q_{n+1} \Lambda_{n2} (x - x_0)^{(n+1)\alpha - 1} \\ + B(x) \sum_{n=0}^{\infty} q_n (x - x_0)^{(n+1)\alpha - 1} = 0. \end{aligned} \tag{3.1.30}$$

Using series for $A(x)$ and $B(x)$ in Equation (3.1.30) we get

$$\begin{aligned} &\sum_{n=0}^{\infty} q_{n+2} \Lambda_{n1} (x - x_0)^{(n+1)\alpha - 1} + A_0 \sum_{n=0}^{\infty} q_{n+1} \Lambda_{n2} (x - x_0)^{(n+1)\alpha - 1} \\ &+ A_1 \sum_{n=0}^{\infty} q_{n+1} \Lambda_{n2} (x - x_0)^{(n+2)\alpha - 1} + A_2 \sum_{n=0}^{\infty} q_{n+1} \Lambda_{n2} (x - x_0)^{(n+3)\alpha - 1} + \cdots \\ &+ B_0 \sum_{n=0}^{\infty} q_n (x - x_0)^{(n+1)\alpha - 1} + B_1 \sum_{n=0}^{\infty} q_n (x - x_0)^{(n+2)\alpha - 1} \\ &+ B_2 \sum_{n=0}^{\infty} q_n (x - x_0)^{(n+3)\alpha - 1} + \cdots = 0. \end{aligned} \tag{3.1.31}$$

Further simplifications transforms Equation (3.1.31) as

$$\begin{aligned}&\sum_{n=0}^{\infty} q_{n+2}\Lambda_{n1}(x-x_0)^{(n+1)\alpha-1} + A_0\sum_{n=0}^{\infty} q_{n+1}\Lambda_{n2}(x-x_0)^{(n+1)\alpha-1}\\ &+ A_1\sum_{n=1}^{\infty} q_n\Lambda_{(n-1)2}(x-x_0)^{(n+1)\alpha-1} + A_2\sum_{n=2}^{\infty} q_{n-1}\Lambda_{(n-2)2}(x-x_0)^{(n+1)\alpha-1} + \cdots\\ &+ B_0\sum_{n=0}^{\infty} q_n(x-x_0)^{(n+1)\alpha-1} + B_1\sum_{n=1}^{\infty} q_{n-1}(x-x_0)^{(n+1)\alpha-1}\\ &+ B_2\sum_{n=2}^{\infty} q_{n-2}(x-x_0)^{(n+1)\alpha-1} + \cdots = 0.\end{aligned} \tag{3.1.32}$$

After equating coefficients of powers of $x - x_0$ in Equation (3.1.32) we get the following recurrence relation with $n \geq 2$

$$q_n = -\frac{1}{\Lambda_{(n-2)1}}\sum_{k=0}^{n-2}\left\{q_{k+1}A_{n-k-2}\Lambda_{k2} + B_{n-k-2}q_k\right\}. \tag{3.1.33}$$

If we replace n by $n + 2$ in Equation (3.1.33) we get

$$q_{n+2} = -\frac{1}{\Lambda_{n1}}\sum_{k=0}^{n}\left\{q_{k+1}A_{n-k}\Lambda_{k2} + B_{n-k}q_k\right\}. \tag{3.1.34}$$

Now we show the convergence of series in Equation (3.1.25) for $x \in (x_0, x_0 + \mu)$ with $\mu = min\{\mu_1, \mu_2\}$ and let $r < \mu \Rightarrow \frac{1}{r} > \frac{1}{\mu}$. Since the series in Equation (3.1.23) are convergent for $x \in [x_0, x_0 + r]$, there exists a constant $M > 0$ such that

$$\begin{aligned}|A_{n-k}| &\leq \frac{Mr^{k\alpha}}{r^{n\alpha}}, \qquad \text{for } n \in \mathbb{N}_0, \quad 0 \leq k \leq n,\\ |B_{n-k}| &\leq \frac{Mr^{k\alpha}}{r^{n\alpha}}, \qquad \text{for } n \in \mathbb{N}_0, \quad 0 \leq k \leq n.\end{aligned} \tag{3.1.35}$$

Using Equation (3.1.35) in Equation (3.1.34) we get

$$|q_{n+2}| \leq \frac{M}{r^{n\alpha}\Lambda_{n1}}\sum_{k=0}^{n}\{q_{k+1}\Lambda_{k2} + q_k\}r^{k\alpha}.$$

Also,

$$|q_{n+2}| \leq \frac{M}{r^{n\alpha}\Lambda_{n1}}\sum_{k=0}^{n}\{q_{k+1}\Lambda_{k2} + q_k\}r^{k\alpha} + M|q_{n+1}|r^{\alpha}.$$

Now let $c_0 = |q_0|$, $c_1 = |q_1|$ and c_k $(k > 1)$ is defined by the following recurrence relation

$$|c_{n+2}| = \frac{M}{r^{n\alpha}\Lambda_{n1}}\sum_{k=0}^{n}\{c_{k+1}\Lambda_{k2} + c_k\}r^{k\alpha} + M|c_{n+1}|r^{\alpha}.$$

The series $\sum_{n=0}^{\infty} c_n(x - x_0)^{(n+1)\alpha-1}$ is convergent for $|x - x_0| < r$, because in accordance with the asymptotic representation (3.1.11) the following estimate holds

$$\lim_{n\to\infty}\left|\frac{c_{n+1}(x-x_0)^{(n+1)\alpha}}{c_n(x-x_0)^{n\alpha}}\right| < 1.$$

Which means $|c_n(x-x_0)^{(n+1)\alpha-1}| \leq Nr^{(n+1)\alpha-1}$ for $N > 0$ and also $|q_n(x-x_0)^{(n+1)\alpha-1}| \leq N_1 r^{(n+1)\alpha-1}$ for $N_1 > 0$. □

Theorem 3.1.6. *Let $\alpha \in (0,1]$, $q_0,\ q_1 \in \mathbb{R}$ and $x_0 \in [a,b]$ be an α-ordinary point of the following equation*

$$^{c}D_a^{2\alpha} y(x) + A(x) D_a^{\alpha} y(x) + B(x) y(x) = 0. \tag{3.1.36}$$

Then there exists a unique solution of Equation (3.1.36) given by

$$y(x) = \sum_{n=0}^{\infty} q_n (x-x_0)^{n\alpha},$$

for $x \in (x_0, x_0+\mu)$ with $\mu = min\{\mu_1, \mu_2\}$. This solution is α-analytic function in x_0 and satisfies the following initial conditions

$$\lim_{x \to x_0} y(x)] = q_0,$$

and

$$\lim_{x \to x_0}\ {}^{c}D_a^{\alpha} y(x)] = q_1.$$

Proof. Proof is done on similar pattern of Theorem 3.1.5 and we get the following recurrence relation

$$q_{n+2} = -\frac{1}{\Lambda_{n1}} \sum_{k=0}^{n} \left\{ \Lambda_{k2} A_{n-k} q_{k+1} + B_{n-k} q_k \right\},$$

where $\Lambda_{n1} = \frac{\Gamma((n+2)\alpha+1)}{\Gamma(n\alpha+1)}$ and $\Lambda_{n2} = \frac{\Gamma((n+1)\alpha+1)}{\Gamma(n\alpha+1)}$. □

Example 3.1.7. Consider the following fractional differential equation of order 2α $(0 < \alpha < 1)$

$$^{RL}D_a^{2\alpha} y(x) + (x-1)^{\alpha} y(x) = 0, \tag{3.1.37}$$

where $x = 1$ is α-ordinary point, $B(x) = (x-1)^{\alpha}$ and $a(x) = 0$. According to Theorem 3.1.5 consider the solution of Equation (3.1.37) as

$$y(x) = \sum_{n=0}^{\infty} q_n (x-1)^{(n+1)\alpha-1}. \tag{3.1.38}$$

Since $A_n\ 's$ are zero, the recurrence relation for coefficients q_n is given by

$$q_{n+2} = -\frac{\sum_{k=0}^{n} B_{n-k} q_k}{\Lambda_{n1}}, \tag{3.1.39}$$

where $\Lambda_{n1} = \frac{\Gamma((n+3)\alpha)}{\Gamma((n+1)\alpha)}$.

When $n = 0$ Equation (3.1.39) yields

$$q_2 = 0.$$

When $n = 1$ Equation (3.1.39) yields

$$q_3 = -\frac{\Gamma(2\alpha)}{\Gamma(4\alpha)} q_0.$$

When $n = 2$ Equation (3.1.39) yields

$$q_4 = -\frac{\Gamma(3\alpha)}{\Gamma(5\alpha)} q_1.$$

When $n = 3$ Equation (3.1.39) yields

$$q_5 = 0.$$

When $n = 4$ Equation (3.1.39) yields

$$q_6 = \frac{\Gamma(2\alpha)\Gamma(5\alpha)}{\Gamma(4\alpha)\Gamma(7\alpha)} q_0.$$

When $n = 5$ Equation (3.1.39) yields

$$q_7 = \frac{\Gamma(3\alpha)\Gamma(6\alpha)}{\Gamma(5\alpha)\Gamma(8\alpha)} q_1.$$

Continuing in this manner we get

$$y(x) = q_0(x-1)^{\alpha-1}\left\{1 + \sum_{k=1}^{\infty}(-1)^k l_k (x-1)^{3k\alpha}\right\} + q_1(x-1)^{\alpha-1}\left\{x^{\alpha} + \sum_{k=1}^{\infty}(-1)^k m_k (x-1)^{(3k+1)\alpha}\right\},$$

where

$$l_k = \prod_{j=1}^{k} \frac{\Gamma((3j-1)\alpha)}{\Gamma((3j+1)\alpha)},$$

and

$$m_k = \prod_{j=1}^{k} \frac{\Gamma(3j\alpha)}{\Gamma((3j+2)\alpha)}.$$

Example 3.1.8. Now consider the following fractional differential equation of order 2α $(0 < \alpha < 1)$

$$^{c}D_0^{2\alpha} y(x) + x^{\alpha}\ {}^{c}D_0^{\alpha} y(x) = 0, \quad (3.1.40)$$

where $x = 0$ is α-ordinary point, $A(x) = x^{\alpha}$ and $B(x) = 0$. According to Theorem 3.1.6 consider the solution of Equation (3.1.40) as

$$y(x) = \sum_{n=0}^{\infty} q_n x^{n\alpha}. \quad (3.1.41)$$

The recurrence relation for coefficients q_n is given by

$$q_{n+2} = -\frac{\sum_{k=0}^{n} \Lambda_{k2} A_{n-k} q_{k+1}}{\Lambda_{n1}}, \quad (3.1.42)$$

where $\Lambda_{n1} = \frac{\Gamma((n+2)\alpha+1)}{\Gamma(n\alpha+1)}$ and $\Lambda_{n2} = \frac{\Gamma((k+1)\alpha+1)}{\Gamma(k\alpha+1)}$.
When $n = 0$ Equation (3.1.42) implies

$$q_2 = 0.$$

When $n = 1$ Equation (3.1.42) implies

$$q_3 = -\frac{(\Gamma(\alpha+1))^2}{\Gamma(3\alpha+1)} q_1.$$

When $n = 2$ Equation (3.1.42) implies

$$q_4 = 0.$$

When $n = 3$ Equation (3.1.42) implies

$$q_5 = \frac{(\Gamma(\alpha+1))^2(\Gamma(3\alpha+1))^2}{\Gamma(2\alpha+1)\Gamma(3\alpha+1)\Gamma(5\alpha+1)} q_1.$$

When $n = 4$ Equation (3.1.42) implies

$$q_6 = 0.$$

When $n = 5$ Equation (3.1.42) implies

$$q_7 = -\frac{(\Gamma(\alpha+1))^2(\Gamma(3\alpha+1))^2(\Gamma(5\alpha+1))^2}{\Gamma(2\alpha+1)\Gamma(3\alpha+1)\Gamma(4\alpha+1)\Gamma(5\alpha+1)\Gamma(7\alpha+1)} q_1.$$

After substituting $q_n's$ in Equation (3.1.41) we get

$$y(x) = q_1 x^\alpha \left\{ 1 - \frac{(\Gamma(\alpha+1))^2}{\Gamma(3\alpha+1)} x^{2\alpha} + \frac{(\Gamma(\alpha+1))^2(\Gamma(3\alpha+1))^2}{\Gamma(2\alpha+1)\Gamma(3\alpha+1)\Gamma(5\alpha+1)} x^{4\alpha} \right.$$
$$\left. - \frac{(\Gamma(\alpha+1))^2(\Gamma(3\alpha+1))^2(\Gamma(5\alpha+1))^2}{\Gamma(2\alpha+1)\Gamma(3\alpha+1)\Gamma(4\alpha+1)\Gamma(5\alpha+1)\Gamma(7\alpha+1)} x^{6\alpha} + - \cdots \right\}.$$

3.2 Solution around an α-singular point

In this section we see how to evaluate solution of homogeneous fractional differential equation of order α $(0 < \alpha < 1)$ and 2α $(0 < \alpha < 1)$ around a regular α-singular point in the form of power series. The results and method is taken from [47].

3.2.1 Solution around an α-singular point to a fractional differential equation of order α

This section is related to fractional power series solution to homogeneous linear fractional differential equation of order α with $0 < \alpha \leq 1$ about a regular α-singular point x_0. For convenience we consider fractional differential equation of the form

$$(x - x_0)^\alpha D_a^\alpha y(x) + A(x) y(x) = 0, \tag{3.2.1}$$

where $A(x)$ is α-analytic function around x_0 since x_0 is regular α-singular point.

Theorem 3.2.1. *Let $x_0 \geq a$ be a regular α-singular point of Equation (3.2.1) of order α, and let $A(x)$ can be written in power series expansion as*

$$A(x) = \sum_{n=0}^{\infty} A_n (x - x_0)^{n\alpha}. \tag{3.2.2}$$

Then there exists the solution

$$y(x;\alpha,s_1) = (x-x_0)^{s_1}\sum_{n=0}^{\infty} q_n (x-x_0)^{n\alpha}, \tag{3.2.3}$$

to the Equation (3.2.1) on a certain interval to the right of x_0. Here q_0 is a non-zero arbitrary constant, $s_1 > -1$ is the unique real solution to the equation

$$\frac{\Gamma(s+1)}{\Gamma(s-\alpha+1)} + A_0 = 0,$$

and the coefficients $q_n (n \geq 1)$ are given by $q_n = -\frac{\Gamma(n\alpha+s-\alpha+1)}{\Gamma(n\alpha+s+1)}\sum_{k=0}^{n-1} q_k A_{n-k}$. Moreover, if the series in Equation (3.2.2) converges for all x in a semi-interval $0 < x - x_0 < R$ $(R > 0)$, then the series solution in Equation (3.2.3) of Equation (3.2.1) is also convergent in the same interval.

Proof. We begin with proving the solution of the Equation (3.2.1) of the form given in Equation (3.2.3). Fractional order derivative of $y(x)$ is given by

$$D_a^{\alpha} y(x) = \sum_{n=0}^{\infty} q_n \frac{\Gamma(n\alpha+s_1+1)}{\Gamma((n-1)\alpha+s_1+1)}(x-x_0)^{(n-1)\alpha+s_1}. \tag{3.2.4}$$

After substituting Equation (3.2.4) in Equation (3.2.1) we get

$$\sum_{n=0}^{\infty} q_n \frac{\Gamma(n\alpha+s_1+1)}{\Gamma((n-1)\alpha+s_1+1)}(x-x_0)^{n\alpha+s_1} + \sum_{n=0}^{\infty} A(x) q_n (x-x_0)^{n\alpha+s_1} = 0. \tag{3.2.5}$$

Let $\Lambda_n = \frac{\Gamma(n\alpha+s_1+1)}{\Gamma((n-1)\alpha+s_1+1)}$, Equation (3.2.5) becomes

$$\sum_{n=0}^{\infty} q_n \Lambda_n (x-x_0)^{n\alpha+s_1} + \sum_{n=0}^{\infty} A(x) q_n (x-x_0)^{n\alpha+s_1} = 0. \tag{3.2.6}$$

After substituting Equation (3.2.2) in Equation (3.2.6) we get

$$\begin{aligned}&\sum_{n=0}^{\infty} q_n \Lambda_n (x-x_0)^{n\alpha+s_1} + \sum_{n=0}^{\infty} A_0 q_n (x-x_0)^{n\alpha+s_1} + \sum_{n=0}^{\infty} A_1 q_n (x-x_0)^{(n+1)\alpha+s_1} \\ &+ \sum_{n=0}^{\infty} A_2 q_n (x-x_0)^{(n+2)\alpha+s_1} + \sum_{n=0}^{\infty} A_3 q_n (x-x_0)^{(n+3)\alpha+s_1} + \cdots = 0.\end{aligned} \tag{3.2.7}$$

Further simplifications reduces the above equation to the following form

$$\begin{aligned}&\sum_{n=0}^{\infty} q_n \Lambda_n (x-x_0)^{n\alpha+s_1} + \sum_{n=0}^{\infty} A_0 q_n (x-x_0)^{n\alpha+s_1} + \sum_{n=1}^{\infty} A_1 q_{n-1} (x-x_0)^{n\alpha+s_1} \\ &+ \sum_{n=2}^{\infty} A_2 q_{n-2} (x-x_0)^{n\alpha+s_1} + \sum_{n=3}^{\infty} A_3 q_{n-3} (x-x_0)^{n\alpha+s_1} + \cdots = 0.\end{aligned} \tag{3.2.8}$$

After equating coefficients of powers of $x - x_0$ we get the following recurrence relation

$$q_n = -\frac{\sum_{k=0}^{n-1} q_k A_{n-k}}{f_0(n\alpha+s_1)},$$

where $f_0(n\alpha+s_1) = \frac{\Gamma(n\alpha+s_1+1)}{\Gamma((n-1)\alpha+s_1+1)} + A_0$. The convergence of the series (3.2.3) is done on similar pattern as done in Theorem 3.1.5 for $0 < |x-x_0| < r < R$. □

Example 3.2.2. Consider the following fractional differential equation of order $\alpha(0 < \alpha < 1)$

$$x^{\alpha} D_0^{\alpha} y(x) - 2y(x) = 0, \tag{3.2.9}$$

where $x = 0$ is α-singular point of Equation (3.2.9). And $A(x) = -2$ which means $A_0 = -2$ and rest of the $A_n's$ are zero for $n \in \mathbb{N}$. According to Theorem 3.2.1 we consider the solution of Equation (3.2.9) as follows

$$y(x; \alpha, s) = \sum_{n=0}^{\infty} q_n x^{n\alpha+s}, \tag{3.2.10}$$

where s is the root of the following indicial equation

$$\frac{\Gamma(s+1)}{\Gamma(s-\alpha+1)} + A_0 = 0.$$

The recurrence relation for coefficients B_n is given by

$$q_n = -\frac{\sum_{k=0}^{n-1} q_k A_{n-k}}{f_0(n\alpha+s)}, \tag{3.2.11}$$

where $f_0(n\alpha + s) = \frac{\Gamma(n\alpha+s+1)}{\Gamma(n\alpha+s-\alpha+1)} + A_0$.
When $n = 1$ Equation (3.2.11) yields

$$q_1 = 0.$$

When $n = 2$ Equation (3.2.11) yields

$$q_2 = 0.$$

When $n = 3$ Equation (3.2.11) yields

$$q_3 = 0.$$

Which means all $q_n\,'s$ are zero except q_0. Equation (3.2.10) takes the form

$$y(x; \alpha, s) = q_0 x^s.$$

Table 3.1 represents different values of s by changing α between 0 and 1.

α	s
0.2	31.5981
0.25	15.6226
0.3	9.7255
0.34	7.3456
0.4	5.3507
0.5	3.7422
0.66	2.6802
0.7	2.5340
0.8	2.2722
0.9	2.1065
0.99	2.0086

Table 3.1: Values of s corresponding to $0 < \alpha < 1$.

3.2.2 Solution around an α-singular point to a fractional differential equation of order 2α

Now consider the following homogeneous fractional differential equation of order 2α with $0 < \alpha \leq 1$

$$(x-x_0)^{2\alpha} D_a^{2\alpha} y(x) + (x-x_0)^{\alpha} A(x) D_a^{\alpha} y(x) + B(x) y(x) = 0, \tag{3.2.12}$$

where x_0 is regular α-singular point of Equation (3.2.12). Then the functions $A(x)$ and $B(x)$ are α-analytic around x_0 and therefore they have the following series expansions

$$A(x) = \sum_{n=0}^{\infty} A_n (x-x_0)^{n\alpha} \quad \text{and} \quad B(x) = \sum_{n=0}^{\infty} B_n (x-x_0)^{n\alpha}, \tag{3.2.13}$$

defined on a semi-interval $0 < x - x_0 < r$ for some $r > 0$ and with $x_0 \geq a$. Our aim is to find a solution to Equation (3.2.12) of the form

$$y(x;\alpha,s) = \sum_{n=0}^{\infty} q_n (x-x_0)^{n\alpha+s}, \tag{3.2.14}$$

with $q_0 \neq 0$ and s being a number to be determined. Differentiating Equation (3.2.14) and substituting the results in Equation (3.2.12) yields

$$\begin{aligned} &\sum_{n=0}^{\infty} q_n \frac{\Gamma(n\alpha+s+1)}{\Gamma((n-2)\alpha+s+1)} (x-x_0)^{n\alpha+s} + A(x) \sum_{n=0}^{\infty} q_n \frac{\Gamma(n\alpha+s+1)}{\Gamma((n-1)\alpha+s+1)} (x-x_0)^{n\alpha+s} \\ &+ B(x) \sum_{n=0}^{\infty} q_n (x-x_0)^{n\alpha+s} = 0. \end{aligned} \tag{3.2.15}$$

Equation (3.2.15) can also be written as

$$\sum_{n=0}^{\infty} q_n \Lambda_{n1}(x-x_0)^{n\alpha+s} + A(x)\sum_{n=0}^{\infty} q_n \Lambda_{n2}(x-x_0)^{n\alpha+s} + B(x)\sum_{n=0}^{\infty} q_n (x-x_0)^{n\alpha+s} = 0. \tag{3.2.16}$$

where $\Lambda_{n1} = \frac{\Gamma(n\alpha+s+1)}{\Gamma((n-2)\alpha+s+1)}$ and $\Lambda_{n2} = \frac{\Gamma(n\alpha+s+1)}{\Gamma((n-1)\alpha+s+1)}$. When $n=1$ Equation (3.2.16) implies

$$q_1\Lambda_{11} + A_0 q_1 \Lambda_{12} + A_1 q_0 \Lambda_{02} + B_0 q_1 + B_1 q_0 = 0. \tag{3.2.17}$$

Equation (3.2.17) ca be rewritten as

$$q_1 f_0(\alpha+s) + (A_1\Lambda_{02} + B_1)q_0 = 0, \tag{3.2.18}$$

where $f_0(n\alpha+s) = \Lambda_{11} + A_0\Lambda_{12} + B_0$. When $n=2$ equation (3.2.16) yields

$$q_2 f_0(2\alpha+s) + q_1(A_1\Lambda_{12} + B_1) + q_0(A_2\Lambda_{02} + B_2) = 0. \tag{3.2.19}$$

Now when $n=n$ then Equation (3.2.16) implies in accordance with Equations (3.2.18) and (3.2.19)

$$q_n = -\frac{\sum_{k=0}^{n-1} q_k f_{n-k}(n\alpha+s)}{f_0(n\alpha+s)}, \tag{3.2.20}$$

where $f_k(s) = A_k \frac{\Gamma(s+1)}{\Gamma(s-\alpha+1)} + B_k$. Since $q_0 \neq 0$ which means $f_0(s) = 0$.
If $f_0(s) = 0$ has two complex roots then they must be conjugates since $\Gamma(\overline{z}) = \overline{\Gamma(z)}$ for all $z \in \mathbb{C}$.
If s_1 and s_2 are two roots of the indicial equation with $s_1 - s_2 \neq n\alpha$ for $n \in \mathbb{N}_0$ then the Equation (3.2.20) yields two solutions by replacing s with s_1 and s_2 to Equation (3.2.12). The proof of the convergence for $0 < x - x_0 < r < R$ of both solutions is done on similar pattern of Theorem 3.1.5.
Now we are left with the problem of finding a solution to Equation (3.2.12) if $s_1 - s_2 = n\alpha$ for $n \in \mathbb{N}_0$. To sort this problem we state the following theorem.

Theorem 3.2.3. *Let $x_0 \geq a$ be a regular $\alpha-$singular point of Equation (3.2.12) and let the series (3.2.13) be convergent on a semi-interval $0 < x - x_0 < R$ with $R > 0$. Let s_1 and s_2 be two real roots of the fractional indicial equation with $s_1,\ s_2 > \alpha - 1$ and $s_1 \geq s_2$. Then, in interval $0 < x - x_0 < R$, Equation (3.2.12) has one solution of the form*

$$y_1(x;\alpha,s_1) = \sum_{n=0}^{\infty} q_n (x-x_0)^{n\alpha+s_1}, \quad q_0(s_1) \neq 0.$$

(a) If $s_1 = s_2$ then the second solution to Equation (3.2.12) has the following form:

$$y_2(x;\alpha,s_1) = y_1(x;\alpha,s_1)\log(x-x_0) + \sum_{n=0}^{\infty} p_n (x-x_0)^{n\alpha+s_1}, \tag{3.2.21}$$

where $p_n = \frac{\partial}{\partial s}(q_n(s))|_{s=s_1}$.
(b) If $s_1 - s_2 = n\alpha$ with $n \in \mathbb{N}$, then a second solution to Equation (3.2.12) is given by

$$y_2(x;\alpha,s_2) = y_1(x;\alpha,s_1).H(x).\log(x-x_0) + \sum_{n=0}^{\infty} p_n(s_2)(x-x_0)^{n\alpha+s_2}, \tag{3.2.22}$$

where $H(x)$ is a function obtained by evaluating the derivative $H(x) = \frac{\partial}{\partial s}\{(s-s_2)y_1(x;\alpha,s)\}|_{s=s_2}$. Moreover, the series (3.2.21) and (3.2.22) are convergent for all x on the semi-interval $0 < x - x_0 < R$.

Example 3.2.4. Consider the following fractional differential equation

$$x^{2\alpha}D_0^{2\alpha}y(x) + x^{\alpha}D_0^{\alpha}y(x) - y(x) = 0. \tag{3.2.23}$$

Here $x = 0$ is α-singular point, $A(x) = 1$ which means $A_0 = 1,\ A_n's = 0$ for $n \in \mathbb{N}$ and $B(x) = -1$ which means $B_0 = -1,\ B_n's = 0$ for $n \in \mathbb{N}$. The general solution of Equation (3.2.23) is given by

$$y(x;\alpha,s) = \sum_{n=0}^{\infty} q_n x^{n\alpha+s}. \tag{3.2.24}$$

s in Equation (3.2.24) is evaluated using the following indicial equation

$$f_0(s) := \frac{\Gamma(s+1)}{\Gamma(s-2\alpha+1)} + \frac{\Gamma(s+1)}{\Gamma(s-\alpha+1)}A_0 + B_0 = 0.$$

The recurrence relation for coefficients q_n is given as

$$q_n = -\frac{\sum_{l=0}^{n-1} q_l f_{n-l}(n\alpha+s)}{f_0(n\alpha+s)}. \tag{3.2.25}$$

When $n = 1$ Equation (3.2.25) yields

$$q_1 = 0.$$

When $n = 2$ Equation (3.2.25) yields

$$q_2 = 0.$$

When $n = 3$ Equation (3.2.25) yields

$$q_3 = 0.$$

Which means all $q_n's$ are zero for $n \in \mathbb{N}$. After substituting $q_n's$ in Equation (3.2.24) we get

$$y(x;\alpha,s) = q_0 x^s.$$

Table 3.2 shows different values of s corresponding to $\alpha \in (0,1)$.

α	s_1	s_2
0.1	-0.6895	-
0.2	-0.3978	-
0.7	0.5828	-
0.8	0.7293	-
0.95	0.9348	-

Table 3.2: Values of s_1 and s_2 corresponding to $0 < \alpha < 1$.

Chapter 4

Laplace transform method for solving fractional differential equations

Laplace transform is an efficient method to solve differential and integral equations. It turns differential and integral equation to algebraic equation that can be solved easily. It transforms a function of real variable to a function of complex variable. Inverse of the Laplace transform exists for a huge class of functions. It transforms a function of complex variable to a function of real variable. In this chapter we will see how Laplace transform method is used to solve fractional integrals and derivatives. Further we will apply it to solve fractional differential equations.

Definition 4.0.1. The Laplace transform of a function $y(x)$, defined for $x \geq 0$, is defined by

$$Y(s) = \mathcal{L}\left\{y(x)\right\} = \int_0^\infty e^{-sx} y(x) dx,$$

where s is real and $\mathcal{L}$ is called the Laplace transform operator. If $Y(s) = \mathcal{L}\left\{y(x)\right\}$ for some function $y(x)$. We define the inverse Laplace transform as

$$\mathcal{L}^{-1}\left\{Y(s)\right\} = y(x).$$

Theorem 4.0.2. *If $y(x)$ is piecewise continuous function on the interval of integration $0 \leq x < A$ for any positive A and $y(x)$ is of exponential order e^{ax} as $x \to \infty$ then the Laplace transform exists and must satisfy $\lim_{s\to\infty} Y(s) = 0$.*

Theorem 4.0.3. *(The Convolution Theorem for Laplace Transform:) Consider two functions $y(x)$ and $z(x)$ that satisfies the conditions which are necessary for the existence of Laplace transform. Then the Laplace transform of the convolution product $(y * z)(x)$ is given by*

$$\mathcal{L}\left\{(y * z)(x)\right\} = \mathcal{L}\left\{\int_0^x y(x-t)z(t)dt\right\} = Y(s)Z(s). \tag{4.0.1}$$

Lemma 4.0.4. *[25] For any $\alpha_1, \alpha_2 > 0$ and $A \in \mathbb{C}^{n\times n}$,*

$$\mathcal{L}\left\{x^{\alpha_2-1}E_{\alpha_1,\alpha_2}(Ax^{\alpha_1})\right\} = \frac{s^{\alpha_1-\alpha_2}}{(s^{\alpha_1} - A)},$$

holds for $Re(s) > \|A\|^{\frac{1}{\alpha_1}}$.

Proof. We know that $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$ with $|x| < 1$. Then for $Re(s) > \|A\|^{\frac{1}{\alpha_1}}$ we have

$$\begin{aligned}\frac{1}{s^{\alpha_1} - A} &= \frac{s^{-\alpha_1}}{1 - \frac{A}{s^{\alpha_1}}} \\ &= s^{-\alpha_1} \sum_{n=0}^{\infty} \left\{ \frac{A}{s^{\alpha_1}} \right\}^n \\ &= \sum_{n=0}^{\infty} A^n s^{-\alpha_1(n+1)}.\end{aligned} \tag{4.0.2}$$

$$\begin{aligned}\mathcal{L}\left\{x^{\alpha_2-1} E_{\alpha_1,\alpha_2}(Ax^{\alpha_1})\right\} &= \mathcal{L}\left\{x^{\alpha_2-1} \sum_{n=0}^{\infty} \frac{(Ax^{\alpha_1})^n}{\Gamma(\alpha_1 n + \alpha_2)}\right\} \\ &= \mathcal{L}\left\{\sum_{n=0}^{\infty} \frac{A^n x^{\alpha_1 n + \alpha_2 - 1}}{\Gamma(\alpha_1 n + \alpha_2)}\right\} \\ &= \sum_{n=0}^{\infty} \frac{A^n \mathcal{L}\left\{x^{\alpha_1 n + \alpha_2 - 1}\right\}}{\Gamma(\alpha_1 n + \alpha_2)} \\ &= \sum_{n=0}^{\infty} \frac{A^n \Gamma(\alpha_1 n + \alpha_2)}{s^{\alpha_1 n + \alpha_2} \Gamma(\alpha_1 n + \alpha_2)} \\ &= \sum_{n=0}^{\infty} A^n s^{-(\alpha_1 n + \alpha_2)} \\ &= s^{\alpha_1 - \alpha_2} \sum_{n=0}^{\infty} A^n s^{-\alpha_1(n+1)}.\end{aligned} \tag{4.0.3}$$

Using Equation (4.0.2) in Equation (4.0.3) we get

$$\mathcal{L}\left\{x^{\alpha_2-1} E_{\alpha_1,\alpha_2}(Ax^{\alpha_1})\right\} = \frac{s^{\alpha_1-\alpha_2}}{(s^{\alpha_1} - A)}.$$

□

4.0.1 Laplace transform of fractional integrals and derivatives

In this section we will discuss Laplace transform of Riemann-Liouville approach and Caputo's approach. First we consider Laplace transform of Riemann-Liouville integral.

$$\mathcal{L}\{I_0^\alpha y(x)\} = \mathcal{L}\left\{\frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} y(t) dt\right\} = \mathcal{L}\left\{\frac{1}{\Gamma(\alpha)} x^{\alpha-1} * y(x)\right\} = \frac{Y(s)}{s^\alpha}.$$

Now we see Laplace transform of Riemann-Liouville derivative which can be developed in the similar pattern.

$$\begin{aligned}\mathcal{L}\left\{{}^{RL}D_0^\alpha y(x)\right\} &= \mathcal{L}\left\{\frac{d^n}{dx^n} I^{n-\alpha} y(x)\right\} \\ &= \mathcal{L}\left\{\frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dx^n} \int_0^x (x-t)^{n-\alpha-1} y(t) dt\right\}.\end{aligned} \tag{4.0.4}$$

Let $F(x) = I_0^{n-\alpha} y(x)$ and applying Laplace transform of derivative on equation (4.0.4) we get

$$\mathcal{L}\left\{ {}^{RL}D_0^{\alpha} y(x) \right\} = s^n \mathcal{L}\left\{ F(x) \right\} - s^{n-1}F(0) - ... - sF^{n-2}(0) - F^{n-1}(0).$$

or

$$\mathcal{L}\left\{ {}^{RL}D_0^{\alpha} y(x) \right\} = s^{\alpha} Y(s) - \sum_{j=1}^{n} s^{n-j} F^{j-1}(0). \tag{4.0.5}$$

Now we construct Laplace transform of Caputo fractional derivative in the following manner:

$$\begin{aligned}
\mathcal{L}\left\{ {}^{c}D_0^{\alpha} y(x) \right\} &= \mathcal{L}\left\{ I_0^{m-\alpha} D^m y(x) \right\} \\
&= \mathcal{L}\left\{ \frac{x^{m-\alpha-1}}{\Gamma(m-\alpha)} \right\} \mathcal{L}\left\{ \frac{d^m}{dt^m} y(t) \right\} \\
&= \frac{1}{s^{m-\alpha}} . \left\{ s^m \mathcal{L}\left\{ y(t) \right\} - s^{m-1} y(0) - ... - s y^{m-2}(0) - y^{m-1}(0) \right\}. \\
&= s^{\alpha} Y(s) - \sum_{j=1}^{m} s^{\alpha-j} y^{j-1}(0).
\end{aligned}$$

Lemma 4.0.5. *[25] Assume $c \geq 0$, $\alpha > 0$ and $a(x)$ is a non-negative function that is locally integrable on $0 \leq x < T$(for some $T \leq \infty$) and also assume $y(x)$ is a non-negative function that is locally integrable on $0 \leq x < T$ with*

$$y(x) \leq a(x) + c \int_0^x (x-t)^{\alpha-1} y(t) dt,$$

on this interval. Then

$$y(x) \leq a(x) + \lambda \int_0^x F'_{\alpha}(\lambda(x-t)) a(t) dt,$$

where $\lambda = (c\Gamma(\alpha))^{\frac{1}{\alpha}}$, $F_{\alpha}(x) = \sum_{k=0}^{\infty} \frac{x^{k\alpha}}{\Gamma(k\alpha+1)}$ and $F'_{\alpha}(x)$ represents derivative of function F with respect to x. Also $F'_{\alpha}(x) \simeq \frac{x^{\alpha-1}}{\Gamma(\alpha)}$ as $x \to 0+$ and $F'_{\alpha}(x) \simeq \frac{e^x}{\alpha}$ as $x \to +\infty$. If $a(x) = a = constant$ then $y(x) \leq aF_{\alpha}(\lambda x)$.

Theorem 4.0.6. *[25] Assume ${}^{c}D_0^{\alpha} y(x) - Ay(x) = h(x)$, with $0 < \alpha < 1, x \geq 0$ and $y(0) = q_1$ has a unique continuous solution $y(x)$, where A is an $n \times n$ constant matrix and $h(x)$ is an $n-$dimensional vector valued continuous function. If $h(x)$ is continuous on $[0, \infty)$ and exponentially bounded, then $y(x)$ and its Caputo derivative ${}^{c}D_a^{\alpha} y(x)$ are both exponentially bounded, thus their Laplace transforms exist.*

Proof. Since $h(x)$ is exponentially bounded then there exist $M,\ m,\ a > 0$ such that $\|h(x)\| \leq me^{ax}$ for all $x \geq M$.
Now consider ${}^{c}D_0^{\alpha} y(x) - Ay(x) = h(x)$ and apply fractional order integral on it

$$I_0^{\alpha}\ {}^{c}D_0^{\alpha} y(x) - I_0^{\alpha} Ay(x) = I_0^{\alpha} h(x), \tag{4.0.6}$$

Equation (4.0.6) can be written as

$$y(x) = q_1 + \int_0^x \frac{(x-t)^{\alpha-1}}{\Gamma(\alpha)} \left\{ Ay(t) + h(t) \right\} dt, \quad 0 \leq x < \infty. \tag{4.0.7}$$

For $x \geq M$ Equation (4.0.7) can be written as

$$y(x) = q_1 + \int_0^M \frac{(x-t)^{\alpha-1}}{\Gamma(\alpha)} \{Ay(t) + h(t)\}\, dt + \int_M^x \frac{(x-t)^{\alpha-1}}{\Gamma(\alpha)} \{Ay(t) + h(t)\}\, dt. \tag{4.0.8}$$

Since the solution $y(x)$ is unique and continuous also A is an $n \times n$ constant matrix and $h(x)$ is bounded. This implies $Ay(x) + h(x)$ is bounded on $[0, M]$. Which means there exists $K > 0$ such that $\|Ay(x) + h(x)\| \leq K$. After applying norm on equation we get

$$\begin{aligned}
\|y(x)\| &= \|q_1\| + \int_0^M \frac{(x-t)^{\alpha-1}}{\Gamma(\alpha)} \|Ay(t) + h(t)\| dt + \int_M^x \frac{(x-t)^{\alpha-1}}{\Gamma(\alpha)} \|Ay(t) + h(t)\| dt \\
&\leq \ \|q_1\| + \frac{K}{\Gamma(\alpha)} \int_0^M (x-t)^{\alpha-1} dt + \int_M^x \frac{(x-t)^{\alpha-1}}{\Gamma(\alpha)} \|A\| \|y(t)\| dt \qquad (4.0.9)\\
&+ \int_M^x \frac{(x-t)^{\alpha-1}}{\Gamma(\alpha)} \|h(t)\| dt.
\end{aligned}$$

Now we multiply Inequality (4.0.9) with e^{-ax}

$$\begin{aligned}
\|y(x)\| e^{-ax} &\leq \|c_1\| e^{-ax} + \frac{Ke^{-ax}}{\Gamma(\alpha)} \int_0^M (x-t)^{\alpha-1} dt + \int_M^x \frac{e^{-ax}(x-t)^{\alpha-1}}{\Gamma(\alpha)} \|A\| \|y(t)\| dt \\
&+ \int_M^x \frac{e^{-ax}(x-t)^{\alpha-1}}{\Gamma(\alpha)} \|h(t)\| dt \\
&\leq \|q_1\| e^{-ax} + \frac{Ke^{-ax}}{\alpha\Gamma(\alpha)} (x^\alpha + (x-M)^\alpha) + \int_M^x \frac{e^{-ax}(x-t)^{\alpha-1}}{\Gamma(\alpha)} \|A\| \|y(t)\| dt \\
&+ \int_M^x \frac{e^{-ax}(x-t)^{\alpha-1}}{\Gamma(\alpha)} \|h(t)\| dt.
\end{aligned}$$

Since $\|h(x)\| \leq me^{ax}$ above Inequality takes the form

$$\begin{aligned}
\|y(x)\| e^{-ax} &\leq \|q_1\| e^{-ax} + \frac{Ke^{-ax}}{\alpha\Gamma(\alpha)} (x^\alpha + (x-M)^\alpha) \\
&+ \int_M^x \frac{e^{-ax}(x-t)^{\alpha-1}}{\Gamma(\alpha)} \|A\| \|y(t)\| dt + \int_M^x \frac{m(x-t)^{\alpha-1}}{\Gamma(\alpha)} e^{a(t-x)} dt. \qquad (4.0.10)
\end{aligned}$$

Since $e^{-ax} \leq e^{-aM}$ and $e^{-ax} \leq e^{-at}$, Inequality (4.0.10) can be written

$$\begin{aligned}
\|y(x)\| e^{-ax} &\leq \|q_1\| e^{-aM} + \frac{Ke^{-aM}}{\alpha\Gamma(\alpha)} (x^\alpha + (x-M)^\alpha) + \int_M^x \frac{e^{-at}(x-t)^{\alpha-1}}{\Gamma(\alpha)} \|A\| \|y(t)\| dt \\
&+ \int_M^x \frac{m(x-t)^{\alpha-1}}{\Gamma(\alpha)} e^{a(t-x)} dt,
\end{aligned}$$

$$\begin{aligned}
\|y(x)\| e^{-ax} &\leq \|c_1\| e^{-aM} + \frac{Ke^{-aM}}{\alpha\Gamma(\alpha)} (x^\alpha + (x-M)^\alpha) \\
&+ \int_0^x \frac{e^{-at}(x-t)^{\alpha-1}}{\Gamma(\alpha)} \|A\| \|y(t)\| dt + \int_0^x \frac{m(x-t)^{\alpha-1}}{\Gamma(\alpha)} e^{a(t-x)} dt. \qquad (4.0.11)
\end{aligned}$$

Since $0 \leq \alpha \leq 1$ and $M \leq x$ which means $x^\alpha - M^\alpha \leq (x-M)^\alpha$ this implies $x^\alpha - (x-M)^\alpha \leq M^\alpha$. Under these conditions Inequality (4.0.11) can be written as

$$\begin{aligned}
\|y(x)\| e^{-ax} &\leq \|q_1\| e^{-aM} + \frac{KM^\alpha e^{-aM}}{\alpha\Gamma(\alpha)} + \int_0^x \frac{e^{-at}(x-t)^{\alpha-1}}{\Gamma(\alpha)} \|A\| \|y(t)\| dt \\
&+ \int_0^x \frac{m(x-t)^{\alpha-1}}{\Gamma(\alpha)} e^{a(t-x)} dt. \qquad (4.0.12)
\end{aligned}$$

Now substitute $x-t=s$ with condition $0 \le x < \infty$ in last integral of Inequality (4.0.12) and we will have

$$\begin{aligned} \|y(x)\|e^{-ax} \le \|q_1\|e^{-aM} + \frac{KM^\alpha e^{-aM}}{\alpha\Gamma(\alpha)} + \int_0^x \frac{e^{-at}(x-t)^{\alpha-1}}{\Gamma(\alpha)}\|A\|\|y(t)\|dt \\ + \int_0^\infty \frac{m(s)^{\alpha-1}}{\Gamma(\alpha)}e^{-as}ds. \end{aligned} \tag{4.0.13}$$

Substitute $s = \frac{u}{a}$ in Inequality (4.0.13)

$$\begin{aligned} \|y(x)\|e^{-ax} \le \|q_1\|e^{-aM} + \frac{KM^\alpha e^{-aM}}{\alpha\Gamma(\alpha)} + \int_0^x \frac{e^{-at}(x-t)^{\alpha-1}}{\Gamma(\alpha)}\|A\|\|y(t)\|dt \\ + \int_0^\infty \frac{mu^{\alpha-1}}{\Gamma(\alpha)a^\alpha}e^{-u}du. \end{aligned} \tag{4.0.14}$$

Using definition of gamma function in Inequality (4.0.14) we get

$$\begin{aligned} \|y(x)\|e^{-ax} \le \|c_1\|e^{-aM} + \frac{KM^\alpha e^{-aM}}{\alpha\Gamma(\alpha)} + \frac{m}{a^\alpha} \\ + \int_0^x \frac{e^{-at}(x-t)^{\alpha-1}}{\Gamma(\alpha)}\|A\|\|y(t)\|dt. \end{aligned} \tag{4.0.15}$$

We denote $a = \|q_1\|e^{-aM} + \frac{KM^\alpha e^{-aM}}{\alpha\Gamma(\alpha)} + \frac{m}{a^\alpha}$, $b = \frac{\|A\|}{\Gamma(\alpha)}$ and $v(x) = \|y(x)\|e^{-ax}$ in Inequality (4.0.15)

$$v(x) \le a + b\int_0^x (x-t)^{\alpha-1}v(t)dt, \quad x \ge M. \tag{4.0.16}$$

Using Lemma 4.0.5 in Inequality (4.0.16) we get

$$v(x) \le aF_\alpha(\lambda x) = a\sum_{k=0}^{\infty}\frac{(b\Gamma(\alpha))^k x^{k\alpha}}{\Gamma(k\alpha+1)}, \quad x \ge M. \tag{4.0.17}$$

Using definition of Mittag-Leffler function in Equation (4.0.17) we get

$$v(x) \le aE_\alpha(b\Gamma(\alpha)x^\alpha), \quad x \ge M. \tag{4.0.18}$$

Using Remark 1.4.4 in Inequality (4.0.18) we get

$$v(x) \le aCe^{(b\Gamma(\alpha))^{\frac{1}{\alpha}}x}, \quad x \ge M. \tag{4.0.19}$$

Since $v(x) = \|y(x)\|e^{-ax}$, Inequality (4.0.19) becomes

$$y(x) \le aCe^{((b\Gamma(\alpha))^{\frac{1}{\alpha}}+a)x}, \quad x \ge M. \tag{4.0.20}$$

Now consider ${}^cD_0^\alpha y(x) = Ay(x) + h(x)$ and after taking norm on it we get

$$\begin{aligned} \|{}^cD_0^\alpha y(x)\| &\le \|A\|\|y(x)\| + \|h(x)\| \\ &\le a\|A\|Ce^{((b\Gamma(\alpha))^{\frac{1}{\alpha}}+a)x} + me^{ax} \\ &\le a\|A\|Ce^{((b\Gamma(\alpha))^{\frac{1}{\alpha}}+a)x} + me^{((b\Gamma(\alpha))^{\frac{1}{\alpha}}+a)x} \\ &\le \{a\|A\|C + m\}\, e^{((b\Gamma(\alpha))^{\frac{1}{\alpha}}+a)x}. \end{aligned} \tag{4.0.21}$$

Thus inequalities (4.0.20) and (4.0.21) show that $u(x)$ and its Caputo derivative are exponentially bounded. □

Example 4.0.7. We consider the following linear differential equation

$$\begin{aligned} D_0^\alpha y(x) + y(x) &= 0, \\ y(0) = 1,\ y'(0) &= 0. \end{aligned} \tag{4.0.22}$$

Applying Laplace transform on Equation (4.0.22) we get

$$\mathcal{L}\{D_0^\alpha y(x)\} + \mathcal{L}\{y(x)\} = 0. \tag{4.0.23}$$

Using Equation (4.0.5) in Equation (4.0.23) we will have

$$s^\alpha Y(s) - s^{\alpha-1} + Y(s) = 0 \text{ implies } Y(s) = \frac{s^{\alpha-1}}{s^\alpha + 1}. \tag{4.0.24}$$

Now consider

$$\begin{aligned} \frac{1}{s^\alpha + 1} &= \frac{s^{-\alpha}}{1 + s^{-\alpha}} \\ &= s^{-\alpha} \sum_{k=0}^{\infty} s^{-\alpha k} \\ &= \sum_{0}^{\infty} s^{-\alpha(k+1)}. \end{aligned} \tag{4.0.25}$$

Using Equation (4.0.25) in Equation (4.0.24) we obtain

$$Y(s) = \sum_{k=0}^{\infty} s^{-\alpha k - 1}. \tag{4.0.26}$$

Applying inverse Laplace transform on Equation (4.0.26) we get

$$y(x) = \sum_{k=0}^{\infty} \frac{x^{\alpha k}}{\Gamma(\alpha k + 1)}.$$

Solutions for different values of α are shown in Figure in 5.5a.

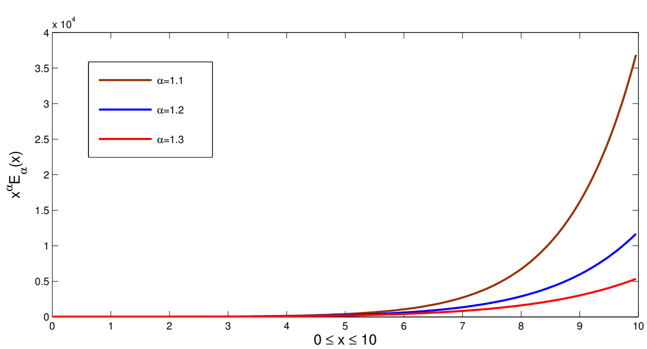


Figure 4.1: Solution of fractional differential equation (4.0.22) at different values of α.

Example 4.0.8. Consider the following fractional order differential equation

$$\begin{aligned} &D_0^2 y(x) + D_0^\alpha y(x) + y(x) = 6x^3 \left\{ \frac{x^{-2}}{\Gamma(2)} - \frac{x^{-\alpha}}{\Gamma(4-\alpha)} \right\}, \\ &0 \le \alpha \le 1,\ y(0) = 0,\ y'(0) = 0. \end{aligned} \tag{4.0.27}$$

Applying Laplace transform on Equation (4.0.27) we get

$$(s^2 + s^\alpha + 1)Y(s) = \frac{6}{s^2} - \frac{6}{s^{4-\alpha}}.$$

Thus,

$$Y(s) = \frac{6}{s^2 + s^\alpha + 1} \left\{ \frac{1}{s^2} - \frac{1}{s^{4-\alpha}} \right\}. \tag{4.0.28}$$

Now consider

$$\begin{aligned} \frac{1}{s^2 + s^\alpha + 1} &= \frac{s^{-\alpha}}{s^{2-\alpha} + 1 + s^{-\alpha}} \\ &= \frac{s^{-\alpha}}{\{s^{2-\alpha+1}\} \left\{1 + \frac{s^{-\alpha}}{s^{2-\alpha}+1}\right\}} \\ &= \frac{s^{-\alpha}}{s^{2-\alpha}+1} \sum_{k=0}^{\infty} \left\{ \frac{s^{-\alpha}}{s^{2-\alpha}+1} \right\}^k \\ &= \sum_{k=0}^{\infty} \frac{s^{-\alpha k - \alpha}}{(s^{2-\alpha}+1)^{k+1}} \\ &= \sum_{k=0}^{\infty} \frac{s^{-2k-2}}{(1 + s^{\alpha-2})^{k+1}} \\ &= \sum_{k=0}^{\infty} s^{-2k-2} \sum_{r=0}^{\infty} (-s^{\alpha-2})^r \binom{k+r}{r}. \end{aligned}$$

Thus

$$\frac{1}{s^2 + s^\alpha + 1} = \sum_{k=0}^{\infty} (-1)^k \sum_{r=0}^{\infty} \binom{k+r}{r} (-1)^r s^{(\alpha-2)r-2k-2}. \tag{4.0.29}$$

Using Equation (4.0.29) in Equation (4.0.28) we obtain,

$$Y(s) = 6 \left\{ \sum_{k=0}^{\infty} (-1)^k \sum_{r=0}^{\infty} \binom{k+r}{r} (-1)^r s^{(\alpha-2)r-2k-4} - \sum_{k=0}^{\infty} (-1)^k \sum_{r=0}^{\infty} \binom{k+r}{r} (-1)^r s^{(\alpha-2)r-2k-6+\alpha} \right\}. \tag{4.0.30}$$

Applying inverse Laplace transform on Equation (4.0.30) we have,

$$\begin{aligned} y(x) =& 6 \sum_{k=0}^{\infty} (-1)^k \sum_{r=0}^{\infty} \binom{k+r}{r} (-1)^r \frac{x^{(2-\alpha)r+2k+3}}{\Gamma((2-\alpha)r+2k+3)} \\ &- 6 \sum_{k=0}^{\infty} (-1)^k \sum_{r=0}^{\infty} \binom{k+r}{r} (-1)^r \frac{x^{(2-\alpha)r+2k-\alpha+5}}{\Gamma((2-\alpha)r+2k+-\alpha+6)}. \end{aligned}$$

Example 4.0.9. Consider the following fractional differential equation

$$\begin{aligned} &\frac{d^2y}{dx^2} + b\ {}^cD^\alpha y(x) + \omega^2 y(x) = h(x), \\ &y(0) = q_0, \qquad y'(0) = q_1. \end{aligned} \tag{4.0.31}$$

We consider three different cases to solve Equation (4.0.31)

Case I: $0 < \alpha < 1$

After applying Laplace transform method on Equation (4.0.31)we get

$$\mathcal{L}\left\{\frac{d^2y}{dx^2}\right\} + b\mathcal{L}\left\{{}^cD_0^\alpha y(x)\right\} + \omega^2\mathcal{L}\left\{y(x)\right\} = \mathcal{L}\left\{h(x)\right\}. \tag{4.0.32}$$

Equation (4.0.32) yields

$$s^2Y(s) - sy(0) - y'(0) + bs^\alpha Y(s) - bs^{\alpha-1}q_0 + \omega^2 Y(s) = H(s). \tag{4.0.33}$$

Equation (4.0.33) can be rewritten as

$$Y(s) = \frac{1}{s^2 + bs^\alpha + \omega^2}\left\{H(s) + (s + bs^{\alpha-1})q_0 + q_1\right\}. \tag{4.0.34}$$

After simplifications Equation (4.0.34) implies

$$Y(s) = \sum_{k=0}^{\infty}(-1)^k\omega^{2k}\sum_{r=0}^{\infty}(-1)^r\binom{k+r}{r}s^{(\alpha-2)r-2(k+1)}\left\{H(s) + (s + bs^{\alpha-1})q_0 + q_1\right\}. \tag{4.0.35}$$

After applying inverse Laplace transform on Equation (4.0.35) we get

$$y(x) = \mathcal{L}^{-1}\left\{\sum_{k=0}^{\infty}(-1)^k\omega^{2k}\sum_{r=0}^{\infty}(-1)^r\binom{k+r}{r}s^{(\alpha-2)r-2(k+1)}\left\{H(s) + (s + bs^{\alpha-1})q_0 + q_1\right\}\right\}. \tag{4.0.36}$$

We can compute Equation (4.0.36) if we know the initial conditions, b, ω and function h. It can be computed directly for the case in which Laplace inverse of function H exits. But there are very few class of functions for which inverse Laplace transform can be computed directly. This is the main disadvantage in applying Laplace transform to the problems. To overcome this situation we look for numerical methods used for inversion of Laplace transform. These methods extend the class of functions that can be solved with Laplace transform method. These methods include Taylor series method for inversion of Laplace transform, Method of Weeks for inversion of Laplace transform etc.

Case II: $1 < \alpha < 2$

After applying Laplace transform on Equation (4.0.31) we get

$$\mathcal{L}\left\{\frac{d^2y}{dx^2}\right\} + b\mathcal{L}\left\{{}^cD_0^\alpha y(x)\right\} + \omega^2\mathcal{L}\left\{y(x)\right\} = \mathcal{L}\left\{h(x)\right\}. \tag{4.0.37}$$

Equation (4.0.37) yields

$$s^2Y(s) - sy(0) - y'(0) + bs^\alpha Y(s) - bs^{\alpha-1}q_0 - s^{\alpha-2}q_1 + \omega^2 Y(s) = H(s). \tag{4.0.38}$$

Equation (4.0.38) can be rewritten as

$$Y(s) = \frac{1}{s^2 + bs^\alpha + \omega^2}\left\{H(s) + (s + bs^{\alpha-1})q_0 + (1 + s^{\alpha-2})q_1\right\}. \tag{4.0.39}$$

Simplifications and inverse Laplace transform converts Equation (4.0.38) to

$$y(x) = \mathcal{L}^{-1}\left\{\sum_{k=0}^{\infty}(-1)^k\omega^{2k}\sum_{r=0}^{\infty}(-1)^r\binom{k+r}{r}s^{(\alpha-2)r-2(k+1)}\left\{H(s) + (s + bs^{\alpha-1})q_0 + (1 + s^{\alpha-2})q_1\right\}\right\}.$$

Case III: $\alpha = 1$

When $\alpha = 1$ Equation (4.0.31) becomes ordinary equation. After applying Laplace transform on it we get

$$\mathcal{L}\left\{\frac{d^2y}{dx^2}\right\} + b\mathcal{L}\left\{D_0^1 y(x)\right\} + \omega^2\mathcal{L}\left\{y(x)\right\} = \mathcal{L}\left\{h(x)\right\}. \tag{4.0.40}$$

Equation (4.0.40) implies

$$s^2Y(s) - sq_0 - q_1 + bsY(s) - bq_0 + \omega^2 Y(s) = H(s). \tag{4.0.41}$$

Equation (4.0.41) can be rewritten as

$$Y(s) = \frac{1}{s^2 + bs + \omega^2}\left\{H(s) + (s + b)q_0 + q_1\right\}. \tag{4.0.42}$$

After applying inverse Laplace transform on Equation (4.0.42) we get

$$y(x) = \mathcal{L}^{-1}\left\{\sum_{k=0}^{\infty}(-1)^k\omega^{2k}\sum_{r=0}^{\infty}(-1)^r\binom{k+r}{r}s^{-r-2(k+1)}\left\{H(s) + (s + b)q_0 + q_1\right\}\right\}.$$

Chapter 5

A quadrature method for numerical solution of fractional differential equations

In this chapter, we have developed a numerical method [45] to solve a class of fractional differential equations. We have combined two quadrature rules by following [4] and [36]. In particular, we have combined trapezoidal rule and Simpson $\frac{1}{3}rd$ rule. After combining these two rules, we have developed the method and we were able to use this method beyond the limitations of these two rules. We have implemented our method on different problems to check its feasibility. Further Matlab code is generated to check it for greater number of intervals. The result is published in [45].
The first section of this chapter deals with quadrature and what is convergence, stability and consistency of a numerical method. Second section deals with development of method, third section is about convergence analysis of presented method and fourth section deals with numerical results. At the end conclusion is given.

5.1 Quadrature

The approximation of integral of a function $y(x)$ on some interval I using some numerical technique is called quadrature method. A quadrature rule is generally defined as

$$\int_a^b y(x)dx = \sum_{k=1}^{m+1} w_k y(x_k) + Error,$$

where w_k represents weights(or coefficients) of a numerical method used and $x_k's$ are nodes at which a function y is evaluated. The interval $I = [a, b]$ can be finite or infinite interval over which integration is done. Whenever we use numerical technique to solve our problem we look for its convergence, stability and consistency.

Definition 5.1.1. A numerical method is said to be convergent if

$$|y - y_m| \to 0 \text{ as } m \to \infty,$$

where y denotes exact solution and y_m denotes numerical solution. In other words we can say a numerical scheme is convergent if the error between exact and numerical solution is approximately zero.

Definition 5.1.2. A numerical scheme is said to be consistent if truncation error vanishes by increasing number of nodes.

Definition 5.1.3. A numerical method is said to be stable if numerical errors does not grow unboundedly.

Theorem 5.1.4. (Convergence theorem) *[20] A quadrature method is said to be convergent if and only if it is consistent and stable.*

5.2 Development of method

In this section we produce the method by combination of quadrature rules. We have used Simpson's $\frac{1}{3}rd$ rule when the number of nodes is odd and when the number of nodes is even we use a combination of trapezoidal rule and Simpson's $\frac{1}{3}rd$rule. We split our interval to deal with the problem of weakly singular kernel as done in [24]. Now consider the following initial value problem for class of fractional differential equations

$$ {}^cD_a^{\alpha_1}y(x) + P\ {}^cD_a^{\alpha_2}y(x) + Qy(x) = h(x), \quad x \in [a,b], \tag{5.2.1}$$

$$ y(a) = y_0, \quad y'(a) = y_0, \quad y^{(k)}(a) = 0, \quad \text{for } \ k = 2,3,\cdots,\lceil\alpha_1\rceil \tag{5.2.2}$$

where $\alpha_1, \alpha_2 \in \mathbb{R}^+$, $\alpha_1 \geq \alpha_2$, $P, Q \in \mathbb{R}$ and $h(x)$ is known functions.

After applying fractional integral $I_a^{\alpha_1}$ on Equation (5.2.1) and using Theorem 1.6.4, we get

$$ y(x) - y(a) - (x-a)y'(a) + PI_a^{\alpha_1}\ {}^cD_a^{\alpha_2}y(x) + QI_a^{\alpha_1}y(x) = I_a^{\alpha_1}h(x). \tag{5.2.3}$$

Using Theorem 1.5.2 in Equation (5.2.3), we get

$$ y(x) + PI_a^{\alpha_1-\alpha_2}\left(I_a^{\alpha_2}D_a^{\alpha_2}y(x)\right) + QI_a^{\alpha_1}y(x) = I_a^{\alpha_1}h(x) + y_0 + (x-a)y_1. \tag{5.2.4}$$

After applying Theorem 1.6.4 on Equation (5.2.4), we get

$$ y(x) + PI_a^{\alpha_1-\alpha_2}\left(y(x) - y_0 - (x-a)y_1\right) + QI_a^{\alpha_1}y(x) = I_a^{\alpha_1}h(x) + y_0 + (x-a)y_1. \tag{5.2.5}$$

Using Lemma 1.5.4, we get

$$ y(x) + PI_a^{\alpha_1-\alpha_2}y(x) + QI_a^{\alpha_1}y(x) = I_a^{\alpha_1}h(x) + u(x;\alpha_1,\alpha_2), \tag{5.2.6}$$

where $y(x;\alpha_1,\alpha_2) := \left(1 + \frac{P(x-a)^{\alpha_1-\alpha_2}}{\Gamma(\alpha_1-\alpha_2+1)}\right)y_0 + \left(1 + \frac{P(x-a)^{\alpha_1-\alpha_2}}{\Gamma(\alpha_1-\alpha_2+2)}\right)(x-a)y_1.$

Equation (5.2.5) can also be written as

$$ y(x) + \int_a^x \Upsilon(x,t)y(s)ds = \int_a^x \Upsilon_{\alpha_1}(x,s)h(s)ds + y(x;\alpha_1,\alpha_2) := H(x), \tag{5.2.7}$$

where $\Upsilon_{\alpha_1}(x,s) := \frac{(x-s)^{\alpha_1-1}}{\Gamma(\alpha_1)}$ and $\Upsilon(x,s) := P\Upsilon_{\alpha_1-\alpha_2}(x,s) + Q\Upsilon_{\alpha}(x,s)$. For approximation of the solution of Equation 5.2.7, we combine trapezoidal and the Simpson's $\frac{1}{3}rd$ rule.

Let $x_k = a + (k-1)h, \quad k = 1, 2, \cdots, m$ where $h = \frac{b-a}{m}, \ m \in \mathbb{N}$. Thus we have $a = x_1 < x_2 < x_3 < \cdots < x_m = b$ a subdivision of $[a,b]$. At $x = a = x_1$ Equation (5.2.7) gives $y_1 = u_1$ and at nodes x_{2i+1}, $i = 1, 2, \cdots, m$ Equation (5.2.7) takes the form

$$y(x_{2i+1}) + \sum_{k=1}^{i} \int_{x_{2k-1}}^{x_{2k+1}} \Upsilon(x_{2i+1}, s) y(s) ds = H(x_{2i+1}; \alpha_1, \alpha_2), \tag{5.2.8}$$

where $H(x_{2i+1}; \alpha_1, \alpha_2) = \int_a^{x_{2i+1}} \Upsilon_{\alpha_1}(x_{2i+1}, s) h(s) ds + u(x_{2i+1}; \alpha_1, \alpha_2)$.
Or $H(x_{2i+1}; \alpha_1, \alpha_2) = \sum_{k=1}^{i} \left\{ \int_{x_{2k-1}}^{x_{2k+1}} \Upsilon_{\alpha_1}(x_{2i+1}, s) h(s) ds + u(x_{2i+1}; \alpha_1, \alpha_2) \right\}$.
After separating $k = i$ from the sum, we get

$$\begin{aligned} \sum_{k=1}^{i} \int_{x_{2k-1}}^{x_{2k+1}} \Upsilon_{\alpha_1}(x_{2i+1}, s) h(s) ds =& \sum_{k=1}^{l-1} \int_{x_{2k-1}}^{x_{2k+1}} \Upsilon_{\alpha_1}(x_{2i+1}, t) h(s) ds \\ &+ \int_{x_{2i-1}}^{x_{2i+1}} \Upsilon_{\alpha_1}(x_{2i+1}, s) h(s) ds. \end{aligned} \tag{5.2.9}$$

Note that at $s = x$, the functions $\Upsilon(x,s)$ is singular when $\alpha_1 - \alpha_2 < 1$ or $\alpha_1 < 1$ and $\Upsilon_{\alpha_1}(x,s)$ is singular for $\alpha_1 < 1$. So we can not directly apply the Simpson's rule to approximate last integrals in (5.2.9). To sort this problem, we add and subtract $h(x_{2i+1})$ in the integrand

$$\begin{aligned} \int_{x_{2i-1}}^{x_{2i+1}} \Upsilon_{\alpha_1}(x_{2i+1}, s) h(s) ds =& h(x_{2i+1}) \int_{x_{2i-1}}^{x_{2i+1}} \Upsilon_{\alpha_1}(x_{2i+1}, s) ds + \int_{x_{2i-1}}^{x_{2i+1}} \Upsilon_{\alpha_1}(x_{2i+1}, t)(h(s) - h(x_{2i+1})) ds \\ =& U_{\alpha_1}(x_{2i-1}, x_{2i+1}) h(x_{2i+1}) + \int_{x_{2i-1}}^{x_{2i+1}} \Upsilon_{\alpha_1}(x_{2i+1}, s)(h(s) - h(x_{2i+1})) ds \\ =& U_{\alpha_1}(x_{2i-1}, x_{2i+1}) h(x_{2i+1}) + \frac{h}{3} \Upsilon_{\alpha_1}(x_{2i+1}, x_{2i-1}) h(x_{2i-1}) \\ &+ \frac{4h}{3} \Upsilon_{\alpha_1}(x_{2i+1}, x_{2i}) h(x_{2i}) - \frac{h}{3} \Upsilon_{\alpha_1}(x_{2i+1}, x_{2i-1}) h(x_{2i+1}) \\ &- \frac{4h}{3} \Upsilon_{\alpha_1}(x_{2i+1}, x_{2i}) h(x_{2i+1}), \end{aligned} \tag{5.2.10}$$

where $U_{\alpha_1}(x_{2i-1}, x_{2i+1}) := \int_{x_{2i-1}}^{x_{2i+1}} \Upsilon_{\alpha_1}(x_{2i+1}, s) ds = \frac{1}{\Gamma(\alpha_1)} \{(x_{2i+1} - x_{2i-1})^{\alpha_1}\}$.
Similarly,

$$\begin{aligned} \int_{x_{2i-1}}^{x_{2i+1}} \Upsilon(x_{2i+1}, s) y(s) ds =& U(x_{2i-1}, x_{2i+1}) y(x_{2i+1}) + \frac{h}{3} \Upsilon(x_{2i+1}, x_{2i-1}) y(x_{2i-1}) \\ &+ \frac{4h}{3} \Upsilon(x_{2i+1}, x_{2i}) y(x_{2i}) - \frac{h}{3} \Upsilon(x_{2i+1}, x_{2i-1}) y(x_{2i+1}) \\ &- \frac{4h}{3} \Upsilon(x_{2i+1}, x_{2i})) y(x_{2i+1}), \end{aligned} \tag{5.2.11}$$

where $U(x_{2i-1}, x_{2i+1}) = PU_{\alpha_1-\alpha_2}(x_{2i-1}, x_{2i+1}) + QU_{\alpha_1}(x_{2i-1}, x_{2i+1})$. Thus Equation (5.2.8) acquires the form

$$\begin{aligned}
y(x_{2k+1}) &+ \frac{h}{3}\sum_{k=1}^{i-1}\Big\{\Upsilon(x_{2i+1}, x_{2k-1})y(x_{2k-1}) + 4\Upsilon(x_{2i+1}, x_{2k})y(x_{2k}) + \Upsilon(x_{2i+1}, x_{2k+1})y(x_{2k+1})\Big\} \\
&+ U(x_{2i-1}, x_{2i+1})y(x_{2i+1}) + \frac{h}{3}\Upsilon(x_{2i+1}, x_{2i-1})y(x_{2i-1}) + \frac{4h}{3}\Upsilon(x_{2i+1}, x_{2i})y(x_{2i}) \\
&- \frac{h}{3}\Upsilon(x_{2i+1}, x_{2i-1})y(x_{2i+1}) - \frac{4h}{3}\Upsilon(x_{2i+1}, x_{2i})y(x_{2i+1}) = H(x_{2i+1}; \alpha_1, \alpha_2).
\end{aligned} \tag{5.2.12}$$

At $x_{2i}, \quad i = 1, 2, \cdots, m$ Equation (5.2.7) takes the form

$$y(x_{2i}) + \sum_{k=1}^{i-1}\int_{x_{2k-1}}^{x_{2k+1}}\Upsilon(x_{2i}, s)y(s)ds + \int_{x_{2i-1}}^{x_{2i}}\Upsilon_{\alpha_1}(x_{2i}, s)y(s)ds = H(x_{2i}; \alpha_1, \alpha_2), \tag{5.2.13}$$

where

$$\begin{aligned}
H(x_{2i}; \alpha_1, \alpha_2) &:= \int_a^{x_{2i}}\Upsilon_{\alpha_1}(x_{2i}, s)h(s)ds \\
&= \int_a^{x_{2i-1}}\Upsilon_{\alpha_1}(x_{2i}, s)h(s)ds + \int_{x_{2i-1}}^{x_{2i}}\Upsilon_{\alpha_1}(x_{2i}, s)h(s)ds + u(x_{2i}; \alpha_1, \alpha_2) \\
&= \sum_{k=1}^{i-1}\int_{x_{2i-1}}^{x_{2i+1}}\Upsilon_{\alpha_1}(x_{2i}, s)h(s)ds + \int_{x_{2i-1}}^{x_{2i}}\Upsilon_{\alpha_1}(x_{2i}, t)f(t)dt + u(x_{2i}; \alpha_1, \alpha_2).
\end{aligned} \tag{5.2.14}$$

Now, here we use a similar argument as in 5.2.11, and apply trapezoidal rule

$$\begin{aligned}
\int_{x_{2i-1}}^{x_{2i}}\Upsilon_{\alpha_1}(x_{2i}, s)h(s)ds &= h(x_{2i})\int_{x_{2i-1}}^{x_{2i}}\Upsilon_{\alpha_1}(x_{2i}, s)ds + \int_{x_{2i-1}}^{x_{2i}}\Upsilon_{\alpha_1}(x_{2i}, s)(h(s) - h(x_{2i}))ds \\
&= \frac{h}{2}\Upsilon_{\alpha_1}(x_{2i}, x_{2i-1})h(x_{2i-1}) + \Big(U_{\alpha}(x_{2i-1}, x_{2i}) - \frac{h}{2}\Upsilon_{\alpha}(x_{2i}, x_{2i-1})\Big)h(x_{2i}).
\end{aligned} \tag{5.2.15}$$

Thus, using (5.2.15) in (5.2.14), we have

$$\begin{aligned}
H(x_{2i}; \alpha_1, \alpha_2) &= \frac{h}{3}\sum_{k=1}^{i-1}\Big\{\Upsilon_{\alpha}(x_{2i}, x_{2i-1})h(x_{2i-1}) + 4\Upsilon_{\alpha_1}(x_{2i}, x_{2k})h(x_{2k}) + \Upsilon_{\alpha_1}(x_{2i}, x_{2k+1})h(x_{2k+1})\Big\} \\
&+ \frac{h}{2}\Upsilon_{\alpha_1}(x_{2i}, x_{2i-1})h(x_{2i-1}) + \Big(U_{\alpha_1}(x_{2i-1}, x_{2i}) - \frac{h}{2}\Upsilon_{\alpha_1}(x_{2i}, x_{2i-1})\Big)h(x_{2i}) + u(x_{2i+1}; \alpha_1, \alpha_2).
\end{aligned} \tag{5.2.16}$$

Therefore, Equation (5.2.13) becomes

$$\begin{aligned}
y(x_{2i}) &+ \frac{h}{3}\sum_{k=1}^{i-1}\Big\{\Upsilon(x_{2i}, x_{2k-1})y(x_{2k-1}) + 4\Upsilon(x_{2i}, x_{2k})y(x_{2k}) + \Upsilon(x_{2i}, x_{2k+1})y(x_{2k+1})\Big\} \\
&+ \frac{h}{2}\Upsilon(x_{2i}, x_{2i-1})y(x_{2i-1}) + \Big(U_{\alpha}(x_{2i-1}, x_{2i}) - \frac{h}{2}\Upsilon(x_{2i}, x_{2i-1})\Big)y(x_{2i}) = H(x_{2i}; \alpha_1, \alpha_2).
\end{aligned} \tag{5.2.17}$$

We have transformed the problem (5.2.1) - (5.2.2) into an algebraic system of equations (5.2.12) and (5.2.17), which can be written as $\mathbf{N}\mathbf{y} = \mathbf{H}$, where, for the matrix $\mathbf{N}$ is given by

$$\mathbf{N} = h \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & \cdots \\ \frac{1}{2}\Upsilon_{21}^{\alpha_1} & \frac{1}{h}\left(U_{12}^{\alpha_1} - 1 - \frac{h}{2}\Upsilon_{21}^{\alpha_1}\right) & 0 & 0 & 0 & \cdots \\ \frac{1}{3}\Upsilon_{31}^{\alpha_1} & \frac{4}{3}\Upsilon_{32}^{\alpha_1} & \frac{1}{h}\left(U_{13}^{\alpha_1} - 1 - \frac{h}{3}(\Upsilon_{31}^{\alpha_1} + 4\Upsilon_{32}^{\alpha_1})\right) & 0 & 0 & \cdots \\ \frac{1}{3}\Upsilon_{41}^{\alpha_1} & \frac{4}{3}\Upsilon_{42}^{\alpha_1} & \frac{5}{6}\Upsilon_{43}^{\alpha_1} & \frac{1}{h}\left(U_{34}^{\alpha_1} - 1 - \frac{h}{2}\Upsilon_{43}^{\alpha_1}\right) & 0 & \cdots \\ \frac{1}{3}\Upsilon_{51}^{\alpha_1} & \frac{4}{3}\Upsilon_{52}^{\alpha_1} & \frac{2}{3}\Upsilon_{53}^{\alpha_1} & \frac{4}{3}\Upsilon_{54}^{\alpha_1} & \frac{1}{h}\left(U_{35}^{\alpha_1} - 1 - \frac{h}{3}(\Upsilon_{53}^{\alpha_1} + 4\Upsilon_{54}^{\alpha_1})\right) & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

where $\Upsilon_{kl}^{\alpha_1} = \Upsilon_{\alpha_1}(x(k), x(l))$ and $U_{kl}^{\alpha_1} = U_{\alpha_1}(x(k), x(l))$. One can compute the numerical results using matlab by following these steps

For $k = 1$ To m

$x[k] = a + (k - 1) * h$

For $k = 1$ To m

$s[l] = a + (l - 1) * h$

$N[1, 1] = 1$

$N[2, 1] = \frac{h}{2} * \Upsilon[2, 1]$

$i = l - \frac{1}{2}(3 - (-1)^l)$

If $(l < 2)$ Then

$N[k, l] = \frac{h}{3} * \Upsilon[k, 1]$

Elseif $(l > k)$ Then

$N[k, l] = 0$

Elseif $(k = l)$ Then

$N[k, l] = U[i, l] - 1 - \frac{2*h}{5-(-1)^l} * (\Upsilon[l, l] + 2 * (1 - (-1)^l) * \Upsilon[l, i + 1])$

Elseif $(l = k1)$ Then

$N[k, l] = \frac{9-(-1)^l}{3(3-(-1)^l)} * h * \Upsilon[k, l]$

Else

$N[k, l] = \frac{h}{3} * (3 + (-1)^l) * \Upsilon[k, l]$

End If

End For k

End For l

5.3 Convergence

In this section we check the convergence of our proposed method. A numerical method is said to be converging if for any specified error tolerance it takes a finite number of steps to arrive at a solution with that error. If this is not the case then a numerical method is divergent.

From equation (5.2.7) we get,

$$y(x) = H(x) - \int_a^x \Upsilon(x, s)y(s)ds. \tag{5.3.1}$$

Above equation can be written as,

$$y_m(x_k) = H(x_k) - \sum_{l=1}^{m} \omega_l \Upsilon(x_k, x_l) y(x_l). \tag{5.3.2}$$

Theorem 5.3.1. *We assume the weight ω_l in* (5.3.2) *satisfy* $|\omega_m| \leq \frac{\Gamma(\alpha_1)}{(b-a)^{\alpha_1 - 1}}$, *then*

$$|y_m(x_k) - y(x_k)| \leq \frac{|\gamma_m|}{1 - |\omega_m|\Upsilon_{km}} \prod_{l=0}^{m-1} \left(1 + \frac{|\omega_l|\Upsilon_{kl}}{1 - |\omega_m|\Upsilon_{km}}\right).$$

where

$$\gamma_m = \sum_{l=1}^{m} \omega_l \Upsilon(x_k, x_l) y(x_l) - \int_a^{x_k} \Upsilon(x_k, t) y(s) ds.$$

Proof. From Equation (5.3.1) and Equation (5.3.2) we get

$$\begin{aligned}
|y_m(x_k) - y(x_k)| &\leq \Big| \sum_{l=1}^{m} \omega_l \Upsilon(x_k, x_l) y_m(x_l) - \int_a^{x_k} \Upsilon(x_k, s) y(s) ds \Big| \\
&= \Big| \sum_{l=1}^{m} \omega_l \Upsilon(x_k, x_l)(y_m(x_l) - y(x_l)) + \sum_{l=1}^{m} \omega_l \Upsilon(x_k, x_l) y(x_l) - \int_a^{x_k} \Upsilon(x_k, s) y(s) ds \Big| \\
&\leq \Big| \sum_{l=1}^{m} \omega_l \Upsilon(x_k, x_l)(y_m(x_l) - y(x_l)) \Big| + |\gamma_m| \\
&\leq |\gamma_m| + \sum_{l=1}^{m-1} |\omega_l| \Upsilon(x_k, x_l) |(y_m(x_l) - y(x_l))| + |\omega_m| \Upsilon(x_k, x_m) |(y_m(x_m) - y(x_m))|,
\end{aligned}$$

Let

$$e_{mk} = |y_m(x_k) - y(x_k)| \quad \text{and} \quad \Upsilon_{kl} = \Upsilon(x_k, x_l).$$

Then

$$e_{mk} \leq |\gamma_m| + \sum_{l=1}^{m} |\omega_l| \Upsilon_{kl} e_{ml} + |\omega_m| \Upsilon_{km} e_{mm}.$$

By Gronwall's lemma

$$e_{mk} \leq \frac{|\gamma_m|}{1 - |\omega_m|\Upsilon_{km}} \prod_{l=0}^{m-1} \left(1 + \frac{|\omega_l|\Upsilon_{kl}}{1 - |\omega_m|\Upsilon_{km}}\right).$$

Thus,

$$\lim_{m \to \infty} e_{mk} = 0.$$

□

5.4 Numerical results

In this section, we apply our method to solve different initial value problems for fractional differential equations. We make comparison of numerical results with exact solutions and results obtained by other methods to check efficiency of the developed method.

Example 5.4.1. Consider the following fractional differential equation

$$^{c}D_0^{\alpha}y(x) + y(x) = x^{\alpha} + \Gamma(1+\alpha),$$

with $y(0) = 0$ if $0 < \alpha \leq 1$ and $y(0) = 0$, $y'(0) = 0$ for $1 < \alpha \leq 2$. The exact solution is $y(x) = x^{\alpha}$. We check the effect of changing fractional order of differentiation on numerical results in this problem. The Table 5.1 shows absolute errors for different step sizes and at different values of α. From readings of Table 5.1 we observe that the method is working efficiently for the given problem. We also see numerical results are less accurate near $\alpha = 0$. The reason behind this is the singularity of Υ and Υ_{α} for $0 < \alpha \leq 1$. Otherwise the results are quite accurate even for small values of m. In Figure 5.1 the numerical results and exact solutions are shown for $m = 50$ and at different values of α.

m	$\alpha = 0.25$	$\alpha = 0.50$	$\alpha = 0.75$	$\alpha = 1$	$\alpha = 1.25$	$\alpha = 1.75$	$\alpha = 2$
10	0.0139	0.0045	9.3711e-004	2.2204e-016	6.9780e-004	0.0013	0.0012
50	0.0071	0.0010	1.1625e-004	4.4409e-016	3.0082e-005	1.8347e-005	1.0300e-005
100	0.0053	5.4287e-004	6.2222e-005	4.4409e-016	9.7015e-006	3.2171e-006	1.2992e-006
150	0.0044	3.6826e-004	4.4260e-005	4.4409e-016	5.2622e-006	1.2144e-006	3.8609e-007
200	0.0039	2.7911e-004	3.5015e-005	8.8818e-016	3.4710e-006	6.2270e-007	1.6312e-007
500	0.0026	1.1452e-004	1.7025e-005	1.1102e-015	9.8930e-007	8.4925e-008	1.0467e-008

Table 5.1: Absolute errors at different values of α.

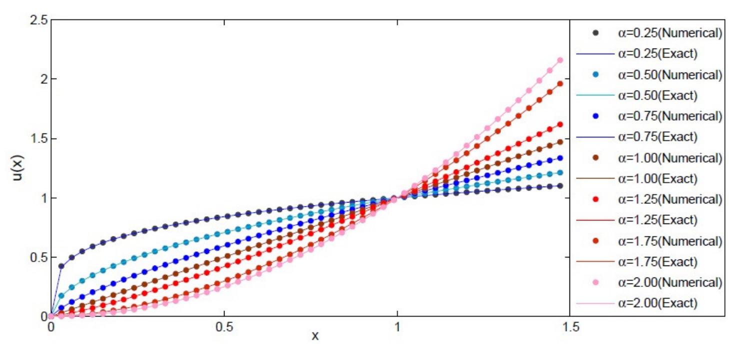


Figure 5.1: Numerical results $m = 50$ at different values of α.

Example 5.4.2. Now consider the following fractional differential equation

$$^{c}D_0^{\alpha}y(x) + x^{\frac{1}{3}}y(x) = x^{\frac{4}{3}} + \frac{3x^{\frac{2}{3}}}{2\Gamma(\frac{2}{3})},$$

with $\alpha = 1/3$, $y(0) = 0$ and exact solution $y(x) = x$. In [10] the problem is solved with the help of Haar wavelet method. We have compared the absolute errors produced by the developed method with absolute errors of Haar wavelet method to check the feasibility of our method. The readings of Table 5.2 show that proposed method converges before Haar wavelet method.

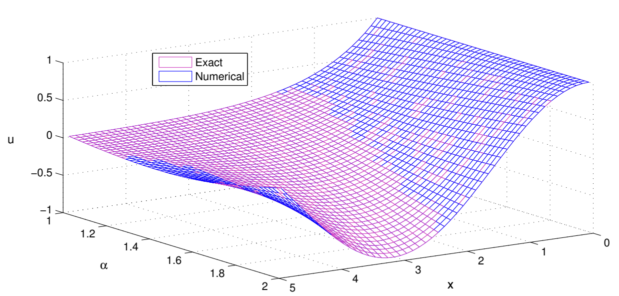
m	Quadrature Method	Haar Wavelets Method in [10]
8	0.0039	0.0095
16	0.0022	0.0051
32	0.0012	0.0027
64	6.2734e-004	0.0014

Table 5.2: Absolute errors at different values of m obtained by presented method and Haar wavelet method [10].

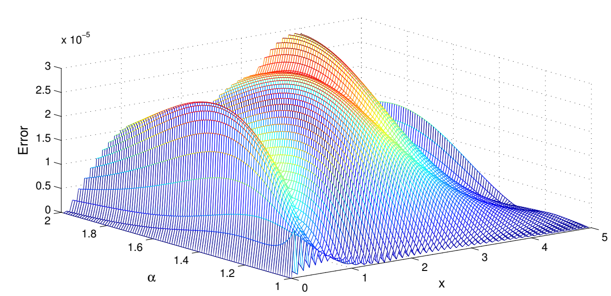
Example 5.4.3. Consider the following initial value problem for fractional differential equation

$$ {}^{c}D_0^{\alpha} y = \lambda y, \qquad \frac{d^i y(x)}{dx^i} = y_0^{(i)}, \tag{5.4.1} $$

with $i = 0, ..., \lceil \alpha \rceil$ and $y_0^{(i)} = 0$ for $i > 0$. The exact solution of Equation (5.4.1) is given by $y(x) = y_0 E_{\alpha,1}(\lambda x^{\alpha})$ where $E_{\alpha,\beta}(z)$ is the Mittag-Leffler function. The Figure 5.2a shows the numerical solutions are quiet similar to exact solutions. In Figure 5.2b maximum absolute error is depicted, where $m = 100$ and $1 \leq \alpha \leq 2$. In Table 5.3 maximum absolute error for $0 \leq x \leq 1$ is shown at different values of m and α. We see from the readings of table that error decreases when m is increased. When we take small values of m error decreases outstandingly but for large m error decreases at very slow rate. In Figure 5.2b, we observe that for $h = 0.05$ the maximum absolute error for $1 \leq \alpha \leq 2$ is bounded by 3×10^{-5}. The level of accuracy for $h = 0.001$ is almost same as in [49].



(a) Comparison of results for $m = 50$ and $1 \leq \alpha \leq 2$.



(b) Absolute error for $m = 100$, $1 \leq \alpha \leq 2$, $\lambda = -1$.

Figure 5.2: Comparison of results and maximum absolute error for $1 \leq \alpha \leq 2$.

m	$\alpha = 0.5$	$\alpha = 1$	$\alpha = 1.5$	$\alpha = 2$	$\alpha = 2.5$	$\alpha = 3$
20	0.0025	9.9123e-006	2.6909e-005	1.4839e-005	3.1779e-006	3.0231e-007
50	9.9938e-004	6.5350e-007	2.3131e-006	9.7401e-007	1.5640e-007	8.4113e-009
100	4.9069e-004	8.2505e-008	1.1330e-006	1.2266e-007	1.6996e-008	5.4001e-010
200	2.4097e-004	1.0365e-008	4.5580e-007	1.5387e-008	2.3249e-009	3.4201e-011
500	1.0579e-004	6.6534e-010	1.2281e-007	9.8678e-010	2.1974e-010	8.8229e-013

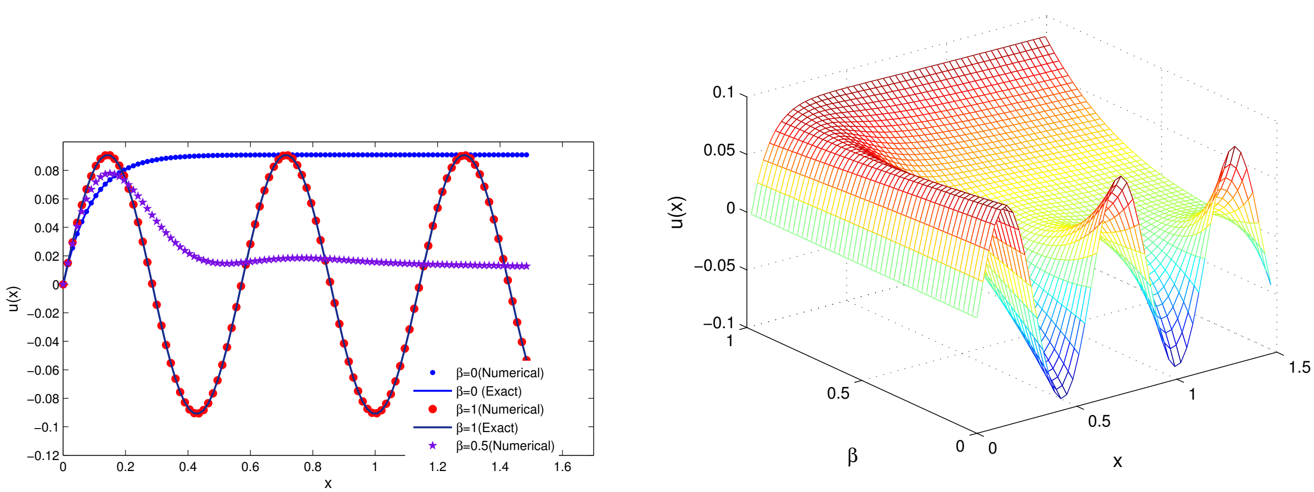
Table 5.3: Maximum absolute errors for $0 \leq x \leq 1$ and at different values of m and α.

Example 5.4.4. We consider the following initial value problem

$$^{c}D_0^{\alpha} y(x) + \omega^{\alpha-\beta}\ ^{c}D_0^{\beta} y(x) = 0, \qquad y(0) = y_0,\ y'(0) = y_1, \tag{5.4.2}$$

where $1 < \alpha \leq 2,\ 0 \leq \beta \leq 1$. When $1 < \alpha \leq 2,\ \beta = 0$ and $y'(0) = 0$ then the Equation (5.4.2) turns into fractional oscillator equation.

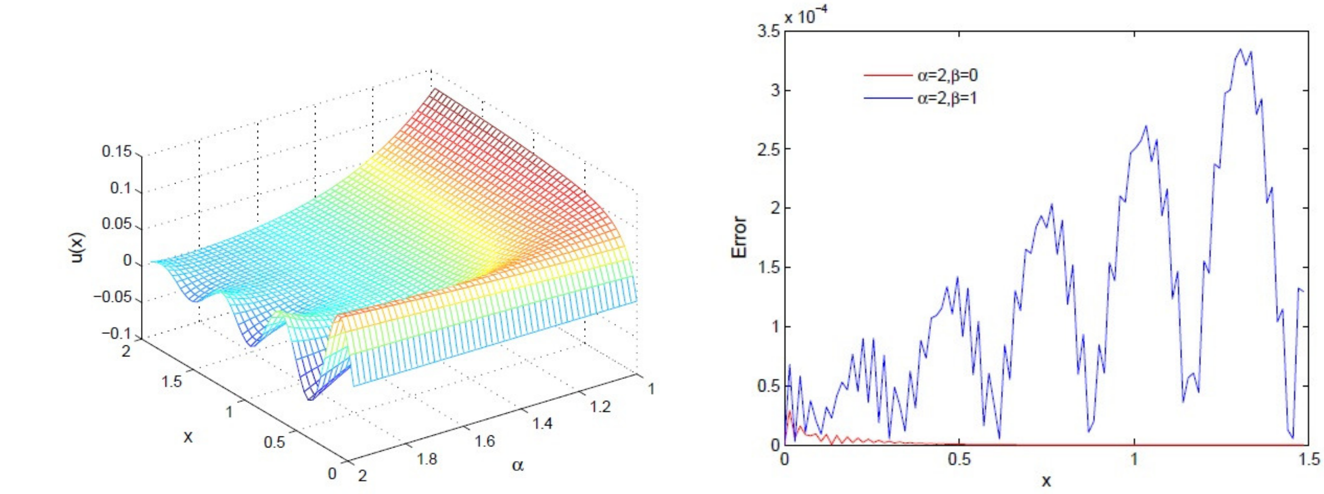
In [3], [8], [18] and [46] fractional oscillator is discussed in detail. Figure 5.3a depicts the numerical and exact solutions at $\alpha = 2$ for different values of β. From Figure 5.3b we see that when β approaches to 0 or 1, the solutions of fractional differential equations are same as solutions of classical differential equation. In Figure 5.4a numerical solutions for $\beta = 0.1$ at different values of α. In Figure 5.4b the maximum absolute error between exact and numerical results is shown for the classical case. When $m = 100$ the upper bound of error is 3.5×10^{-4} on interval $[0, 1.5]$. The error decreases if m is increased.



(a) Comparison of results for $\alpha = 2,\ \beta = 0,\ \beta = 0.5$ and $\beta = 1$.

(b) $3D$ Comparison of results for $\alpha = 2,\ \beta = 0,\ \beta = 0.5$ and $\beta = 1$.

Figure 5.3: Comparison of exact and numerical solution for different values of α and β.



(a) Comparison of results for $m = 100$ at $1 \leq \alpha \leq 2,\ \beta = 0.1$ and $m = 50$.

(b) Maximum absolute error.

Figure 5.4: Comparison of results for different values of α and β and Maximum absolute error.

Example 5.4.5. Now we consider general Bagley-Torvik equation

$$ {}^{c}D_0^2 y(x) + {}^{c}D_0^{\beta} y(x) + y(x) = h(x), \quad y(0) = y_0, \quad y'(0) = y_1. \tag{5.4.3}$$

If we take $\beta = 3/2$ and

$$h(x) = \begin{cases} 8, & 0 \leq x \leq 1, \\ 0, & x > 1, \end{cases}$$

the equation in (5.4.3) turns into Bagley-Torvik equation. Theorem 2.1.3 guarantees the existence of unique solution for initial value problem (5.4.3). Many authors have discussed this equation by different approximate numerical methods. In [9], [15], [38], [44] and [50] analytic and numerical approach for Bagley-Torvik equation is discussed in detail.

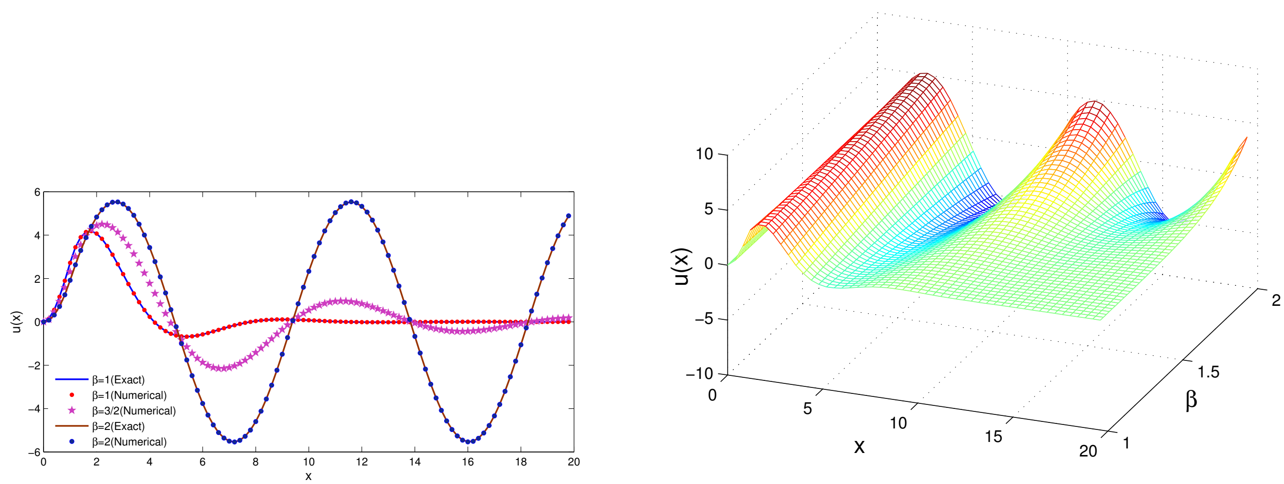
For $\beta = 2,\ y_0 = 0,\ y_1 = 1$, the exact solution of problem (5.4.3) is

$$y(x) = 8 \begin{cases} 1 - \cos\left(\frac{x}{\sqrt{2}}\right), & 0 \leq x \leq 1; \\ \cos\left(\frac{x}{\sqrt{2}}\right) - \cos\left(\frac{1}{\sqrt{2}}\right)\cos\left(\frac{x}{\sqrt{2}}\right) + \sin\left(\frac{1}{\sqrt{2}}\right)\sin\left(\frac{x}{\sqrt{2}}\right), & x > 1. \end{cases}$$

Also for $\beta = 1,\ y_0 = 0,\ y_1 = 1$, the exact solution of the problem (5.4.3) is given by $y(x) = \frac{8}{3}e^{-\frac{x}{2}}q(x)$, where $q(x)$ is given by

$$q(x) = \begin{cases} 3e^{\frac{x}{2}} - 3\cos\left(\frac{\sqrt{3}x}{2}\right) - \sqrt{3}\sin\left(\frac{\sqrt{3}x}{2}\right), & 0 \leq x \leq 1; \\ -3\cos\left(\frac{\sqrt{3}x}{2}\right) + 3\sqrt{e}\cos\left(\frac{\sqrt{3}}{2}\right)\cos\left(\frac{\sqrt{3}x}{2}\right) - \sqrt{3e}\sin\left(\frac{\sqrt{3}}{2}\right)\cos\left(\frac{\sqrt{3}x}{2}\right) & \\ -\sqrt{3}\sin\left(\frac{\sqrt{3}x}{2}\right) + \sqrt{3e}\cos\left(\frac{\sqrt{3}}{2}\right)\sin\left(\frac{\sqrt{3}x}{2}\right) + 3\sqrt{e}\sin\left(\frac{\sqrt{3}}{2}\right)\sin\left(\frac{\sqrt{3}x}{2}\right), & x > 1. \end{cases}$$

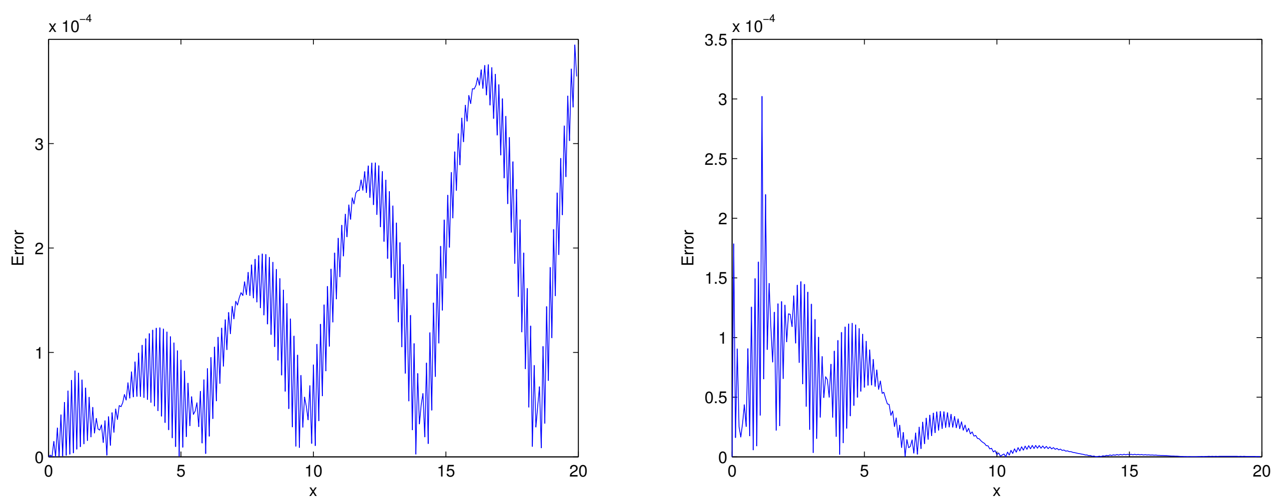
We have compared the exact and numerical solution for $m = 100$ and $\beta = 1, 2$ in Figure 5.5a. The exact solution of the problem (5.4.3) cannot be obtained when β is a fraction. Therefore, for $\beta = 3/2$, we have considered only the numerical solution in Figure 5.5a. From figure we see that the presented method is working effectively. In Figure 5.5b the numerical solutions for $1 \leq \beta \leq 2$ are considered for $x \in [0, 20]$. We see that the proposed method efficiently gives the solutions of the problem (5.4.3) for any $\beta \in (1, 2)$. In Table 5.4 we have compared the results of the presented method, for $\beta = 3/2$ and $m = 200$, with the methods discussed in [6, 38]. The readings of Table 5.4 show that the numerical results and exact solution coincide with each other. The Figure 5.6a and Figure 5.6b show the maximum absolute error for $\beta = 2$ and $\beta = 1$ respectively. The maximum absolute error increases with increase in interval for x and keeping m fixed. The error can be minimized by increasing m.



(a) Numerical solutions for and $\beta = 1, \frac{3}{2}, 2$.

(b) Numerical solutions for $1 \leq \beta \leq 2$.

Figure 5.5: Exact and numerical solutions for $m = 100$,$\alpha = 2$ and $1 \leq \beta \leq 2$.



(a) Error estimate for $m = 200$ and $\alpha = \beta = 2$.

(b) Error estimate for $m = 200$, $\alpha = 2$ and $\beta = 1$.

Figure 5.6: Error estimate for $m = 200$ at different values of α, β.

x	y_{FDM}	y_{ADM}	y_{VIM}	Presented Method	Exact
0.0	0.000000	0.000000	0.000000	0.000000	0.000000
0.1	0.036111	0.036478	0.036478	0.033500	0.033507
0.2	0.139904	0.140640	0.140640	0.125211	0.125221
0.3	0.306402	0.307485	0.307485	0.267598	0.267609
0.4	0.531856	0.533284	0.533284	0.455426	0.455435
0.5	0.812989	0.814757	0.814757	0.684330	0.684335
0.6	1.146733	1.148840	1.148840	0.950396	0.950393
0.7	1.530132	1.532571	1.532571	1.249973	1.249959
0.8	1.960252	1.963033	1.963033	1.579584	1.579557
0.9	2.434223	2.437331	2.437331	1.935875	1.935832
1.0	2.949144	2.952567	2.952567	2.311015	2.315526

Table 5.4: Comparison of present method with FDM, ADM and VIM.

5.5 Conclusion

We have developed an easy and feasible numerical method to deal with a specific class of fractional differential equations. The method works efficiently even when the kernel is singular. The existence and uniqueness results for our problem are discussed in section 1 of chapter 2. We have converted our problem into an algebraic system and developed an algorithm to solve the system. We have checked the capability of our method by applying it on number of problems. We have also checked efficiency of our proposed method by comparing the results produced by our method with the results of different numerical schemes already developed by other authors. The comparison of proposed method with other methods is quite satisfactory for the given problems. We are planning to extend our method in future by using other quadrature rules and get even more accurate results.

Further Workable Content

We have dealt with finding numerical technique by combining quadrature rules. The numerical method is simple to use as it is based on combination of trapezoidal and Simpson's $\frac{1}{3}rd$ rule. These rules are yet easy to apply but both these methods are less accurate then other methods used for numerical integration. For better results one can use methods like Gaussian quadrature, Newton-Cotes method, Gregory's quadrature rule etc. We have not checked this method for nonlinear fractional differential equations and boundary value problems. For these type of problems it need some more calculations. In future we will try to extend this method to such problems.

We are also working on finding numerical technique for fractional differential equations using Chebyshev polynomial. We have developed a method. But yet the method is not applied to any problems for checking it's efficiency. Soon we will get through this problem and come to a nice conclusion.

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