Numerical Solution of Burger's Equation With Random Initial Conditions Using Wiener Chaos Expansion and Lax-Wendroff Method

By

SADIA SAEED

Centre for Advanced Mathematics and Physics (CAMP)

National University of Sciences and Technology (NUST)

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Prof. MUNEER AHMAD RASHID

Centre for Advanced Mathematics and Physics (CAMP)

National University of Sciences and Technology (NUST)

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Dedicated

To My Mother.

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List of Figures

Contents

0.1 Abstract

In this dissertation, we study a numerical method that is based on Wiener Chaos expansion (WCE) and the Lax-Wendroff method for solving Burger's equation. Modified form of Lax-Wendroff scheme is obtained when Lax-Wendroff scheme is expanded using Wiener Chaos expansion, this introduces an infinite system of deterministic equations with respect to non random Fourier-Hermite coefficients. One of the important property of the modified form of Lax-Wendroff scheme is that all the statistical moments of the solution can be computed using simple formulas obtained by this scheme.

The stability of Wiener Chaos expansion approach to computing statistical moments have been numerically tested as well.

0.2 Introduction

This dissertation is mainly concerned with the solution of a Burger's differential equation using non-deterministic initial conditions whereas for deterministic continuous initial conditions, the solution is unique which is also well-behaved. Non-deterministic initial conditions gives fluctuation and discontinuity in the results. In this dissertation, behavior in the first two moments is examined using Burger's differential equation as a model.

Burger's differential equation was initially used to study turbulence which exhibits chaotic behavior in the fluid.

To study the random behavior we used the modified form of Lax-Wendroff numerical scheme which obtained from Wiener chaos expansion (WCE). In this dissertation, WCE is expanded with respect to the Gaussian random variable. $v(x, t, \xi)$ is the WCE of a solution to the Burger's equation and assumed as

$$v(x, t, \xi) = \sum_{n=0}^{\infty} \frac{1}{\sqrt{n!}} v^n(x, t)\xi_n(\xi). \tag{1}$$

Where $\xi_n(x)$ is the nth-order Wick polynomial [5]. As Wick polynomials forms complete orthogonal basis in the Wiener space and the series in the right-hand side of Eq. (1) converges in $L^2(\Re, \xi)$, where the Gaussian measure is defined as

$$\mu(dx) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx[5]. \tag{2}$$

Like every Fourier expansion, WCE also separates variables. More precisely WCE separates non-random variables (x, t) from the random variables ξ. Then non-random Fourier-Hermite coefficients satisfy an infinite system of deterministic equations with nonlinearity like to the one in Burger's equation. This system for the Fourier-Hermite coefficients is commonly indicated as propagator because it controls the propagation of randomness by the deterministic dynamics of the equation. An important property of the propagator is that all statistical moments of the solution to the Burger's equation can be calculated by simple formulae that include only the solution of the propagator [5]. Moreover since the propagator is deterministic, so it only needs to be solved once.

Chapter 1 consists of the theoretical background of not only the Burger's equation but also the properties of Hermite polynomials. They are used to expand the random conditions as a series of these polynomials.

Chapter 2 is devoted to the development of Lax-Wendroff scheme for the numerical solution of the problem. Also stability of Lax-Wendroff is discussed in this chapter. This is the most useful part of the dissertation. This scheme is then used to solve the differential equation in the presence of random discontinuous initial conditions, these content is in chapter 3. Conclusion of scheme is given in chapter 4.

Chapter 1

Theoretical Background

This chapter deals with some basic definitions, equations and theorems which are used in the remaining chapters.

1.1 Burger's Equation

Burger's equation is a fundamental partial differential equation in fluid mechanics. Burger's equation was initially proposed as a turbulence model until further studies, including the work of Hopf and Cole, showed that this equation did not have the adequate properties [1].

For a given velocity v and viscosity coefficient ν with x and t as spatial and temporal variables respectively, the general form of *Viscous* Burger's equation in one dimension is [1]

$$\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial x} = \nu\frac{\partial^2 v}{\partial x^2}. \tag{1.1}$$

Nonlinear term in Eq. (1.1) still represents a great challenge for analytical as well as for numerical solving. By putting $\nu = 0$ in Eq. (1.1) gives *Inviscous* Burger's equation. Conservation form is defined as

$$\frac{\partial v}{\partial t} + \frac{\partial f}{\partial x} = \nu\frac{\partial^2 v}{\partial x^2}, \tag{1.2}$$

where $f = v^2/2$ [1].
The second and third term in Eq. (1.1) is called convective and diffusive/viscous respectively.

Burger's equation is parabolic when viscous term is included. In the absence of viscous term from the Burger's equation, the remaining equation becomes hyperbolic whose solution is known to show shock formulation. Also, by dropping the viscous term from Burger's equation, the nonlinearity allows discontinuous solutions.

1.2 Hermite Polynomials

Probabilistic Hermite polynomials are orthogonal polynomials over the interval $(-\infty, \infty)$ with respect to the weight function $e^{-x^2/2}$ defined as [2]

$$H_n(x) = (-1)^n e^{x^2/2} \frac{d^n}{dx^n} e^{-x^2/2}. \tag{1.3}$$

1.3 Properties of Hermite Polynomials

Following are the few properties of Hermite polynomials

1.3.1 Generating Function

Generating function of Hermite polynomials is defined as [2]

$$\psi(x, z) = e^{2zx - z^2} = \sum_{n=0}^{\infty} \frac{H_n(x)}{n!} z^n. \tag{1.4}$$

1.3.2 Even/Odd Function

The Hermite polynomials satisfy the property

$$H_n(-x) = (-1)^n H_n(x). \tag{1.5}$$

In Eq. (1.5), the Hermite polynomial $H_n(x)$ is an even function when n is even and is odd when n is odd.

1.3.3 Relations Satisfied by Hermite Polynomials

$$H_n'(x) = nH_{n-1}(x) = xH_n(x) - H_{n+1}(x), \quad n = 1, 2, 3, \ldots \tag{1.6}$$

$$H_n''(x) - xH_n'(x) + nH_n(x) = 0; \quad n = 0, 1, 2, \ldots \tag{1.7}$$

Proof. By expanding $\psi(x,z)$ as a Taylor series in z gives

$$\psi(x,z) = \sum_{n=0}^{\infty} \frac{\partial^n \psi(x,z)}{\partial z^n}|_{z=0} \frac{z^n}{n!}. \tag{1.8}$$

On the other hand Eq. (1.4) can be written as

$$\psi(x,z) = e^{-z^2/2+xz+x^2/2-x^2/2}, \tag{1.9}$$

$$= e^{x^2/2} e^{\frac{-1}{2}(z-x)^2}. \tag{1.10}$$

Substituting Eq. (1.10) in Eq. (1.8) gives

$$\psi(x,z) = e^{x^2/2} \sum_{n=0}^{\infty} \frac{\partial^n e^{\frac{-1}{2}(z-x)^2}}{\partial z^n}|_{z=0} \frac{z^n}{n!}. \tag{1.11}$$

But

$$\frac{\partial \psi}{\partial z}(z-x) = -\frac{\partial \psi}{\partial x}(z-x). \tag{1.12}$$

Then Eq. (1.11) becomes

$$\psi(x,z) = e^{x^2/2} \sum_{n=0}^{\infty} (-1)^n \frac{\partial^n}{\partial x^n} e^{\frac{-1}{2}(z-x)^2}|_{z=0} \frac{z^n}{n!}, \tag{1.13}$$

or

$$\psi(x,z) = \sum_{n=0}^{\infty} H_n(x) \frac{z^n}{n!}. \tag{1.14}$$

Hence, the coefficients of Taylor series of $\psi(x,z)$ are exactly the Hermite polynomials. By comparing Eq. (1.8) and Eq. (1.14), we have

$$H_n(x) = \frac{\partial^n \psi(x,z)}{\partial z^n}|_{z=0}. \tag{1.15}$$

To prove the relation given in Eq. (1.10) differentiating Eq. (1.3) w.r.t $'x'$

$$H_n'(x) = (-1)^n x e^{x^2/2} \frac{d^n}{dx^n} e^{-x^2/2} + (-1)^n e^{x^2/2} \frac{d^{n+1}}{dx^{n+1}} e^{-x^2/2}, \tag{1.16}$$

$$= xH_n(x) - (-1)^{n+1} e^{x^2/2} \frac{d^{n+1}}{dx^{n+1}} e^{-x^2/2}. \tag{1.17}$$

Again using Eq. (1.3) in Eq. (1.17) yields

$$H_n'(x) = xH_n(x) - H_{n+1}(x). \tag{1.18}$$

By differentiating Eq. (1.14) with respect to $''x''$ gives

$$\frac{\partial}{\partial x}\psi(x,z) = \frac{\partial}{\partial x}\left[\sum_{n=0}^{\infty}((-1)^n e^{x^2/2}\frac{d^n}{dx^n}e^{-x^2/2})\frac{z^n}{n!}\right], \tag{1.19}$$

$$= \sum_{n=0}^{\infty}(-1)^n \frac{z^n}{n!}\left[xe^{x^2/2}\frac{d^n}{dx^n}e^{-x^2/2} - (-1)e^{x^2/2}\frac{d^{n+1}}{dx^{n+1}}e^{-x^2/2}\right], \tag{1.20}$$

$$= \sum_{n=0}^{\infty}\frac{z^n}{n!}\left[x(-1)^n e^{x^2/2}\frac{d^n}{dx^n}e^{-x^2/2} - (-1)^{n+1}e^{x^2/2}\frac{d^{n+1}}{dx^{n+1}}e^{-x^2/2}\right]. \tag{1.21}$$

Using Eq. (1.3) in Eq. (1.21), we get

$$\frac{\partial}{\partial x}\psi(x,z) = \sum_{n=0}^{\infty}\frac{z^n}{n!}\Big[xH_n(x) - H_{n+1}(x)\Big]. \tag{1.22}$$

Differentiating Eq. (1.4) w.r.t $''x''$ gives

$$\frac{\partial}{\partial x}\psi(x,z) = ze^{-z^2/2+xz}, \tag{1.23}$$

$$\frac{\partial}{\partial x}\psi(x,z) = z\psi(x,z). \tag{1.24}$$

Comparing Eq. (1.22) and Eq. (1.24), we find

$$z\psi(x,z) = \sum_{n=0}^{\infty}\frac{z^n}{n!}\Big[xH_n(x) - H_{n+1}(x)\Big]. \tag{1.25}$$

Substituting Eq. (1.11) and Eq. (1.14) in Eq. (1.25)

$$\sum_{n=0}^{\infty} H_n(x)\frac{z^{n+1}}{n!} = \sum_{n=0}^{\infty} \frac{z^n}{n!}H_n'(x). \tag{1.26}$$

By shifting the summation index in Eq. (1.26), we obtain

$$\sum_{n=1}^{\infty} nH_{n-1}(x)\frac{z^n}{n!} = \sum_{n=1}^{\infty} \frac{z^n}{n!}H_n'(x). \tag{1.27}$$

On comparing the coefficients of $z^n/n!$ in Eq. (1.27) gives

$$H_n'(x) = nH_{n-1}(x), \qquad n = 1, 2, \ldots \tag{1.28}$$

From Eq. (1.18) and Eq. (1.28), we arrive at [3]

$$H_n'(x) = xH_n(x) - H_{n+1}(x) = nH_{n-1}(x). \quad n = 1, 2, 3, \ldots \tag{1.29}$$

Hermite polynomials $\{H_n(x)\}$, satisfies a second order recursion relation or a first order difference differential equation.

Differentiating Eq. (1.18), we have

$$H_{n+1}'(x) - H_n(x) - xH_n'(x) + H_n''(x) = 0. \tag{1.30}$$

Using Eq. (1.28) in Eq. (1.30) gives

$$(n+1)H_n(x) - H_n(x) - xH_n'(x) + H_n''(x) = 0, \tag{1.31}$$

or

$$H_n''(x) - xH_n'(x) + nH_n(x) = 0, \quad n = 0, 1, 2, \ldots \tag{1.32}$$

This is a second order linear differential equation satisfied by the Hermite polynomials $\{H_n(x)\}$, called Hermite's differential equation.

1.3.4 Orthogonality

The family of Hermite polynomial's $\{H_n(x)\}$ for $n = 0, 1, 2, \ldots$ is orthogonal with respect to the weight function $e^{-x^2/2}$ [4]

$$\int_{-\infty}^{\infty} H_m(x) H_n(x) e^{\frac{-x^2}{2}} dx = \begin{cases} 0 & \text{if } n \neq m, \\ n!\sqrt{(2\pi)} & \text{if } n = m. \end{cases} \tag{1.33}$$

Proof. Let u_n be a particular solution defined as

$$u_n = H_n(x) e^{\frac{-x^2}{4}} \tag{1.34}$$

be satisfying the second order linear differential equation

$$u_n'' + (n + \frac{1}{2} - \frac{x^2}{4}) u_n = 0. \tag{1.35}$$

Let

$$u_m = H_m(x) e^{\frac{-x^2}{4}} \tag{1.36}$$

be a particular solution satisfying the second order linear differential equation

$$u_m'' + (m + \frac{1}{2} - \frac{x^2}{4}) u_m = 0. \tag{1.37}$$

Multiplying Eq. (1.35) by u_m and Eq. (1.37) by u_n gives

$$u_n'' u_m + n u_n u_m + \frac{1}{2} u_n u_m - \frac{x^2}{4} u_n u_m = 0, \tag{1.38}$$

$$u_m'' u_n + m u_n u_m + \frac{1}{2} u_n u_m - \frac{x^2}{4} u_n u_m = 0. \tag{1.39}$$

Subtracting Eq. (1.39) from Eq. (1.38), we get

$$u_n'' u_m - u_m'' u_n + (n - m) u_n u_m = 0, \tag{1.40}$$

or

$$\frac{d}{dx}(u_n' u_m - u_m' u_n) + (n - m) u_n u_m = 0. \tag{1.41}$$

Integrating Eq. (1.41) over $(-\infty, \infty)$, we get

$$(n-m)\int_{-\infty}^{\infty} u_n u_m dx = 0. \tag{1.42}$$

Since u_n, u_m, $\to 0$ as x $\to \pm\infty$. So by substituting the values of u_n, u_m from Eq. (1.34) and Eq. (1.36) respectively in Eq. (1.42) gives

$$(n-m)\int_{-\infty}^{\infty} H_m(x)H_n(x)e^{-\frac{x^2}{2}} dx = 0. \tag{1.43}$$

Thus if $n \neq m$ then

$$\int_{-\infty}^{\infty} H_m(x)H_n(x)e^{-\frac{x^2}{2}} dx = 0. \tag{1.44}$$

If $n = m$ then using Eq. (1.28) in Eq. (1.30) gives

$$H_{n+1}(x) - xH_n(x) + nH_{n-1}(x) = 0. \quad n = 1, 2, \ldots \tag{1.45}$$

Replacing n by $(n-1)$ in Eq. (1.45), we have

$$H_n(x) - xH_{n-1}(x) + (n-1)H_{n-2}(x) = 0, \qquad n = 2, 3, \ldots \tag{1.46}$$

Multiplying Eq. (1.45) by $H_{n-1}(x)$ and Eq. (1.46) by $H_n(x)$ respectively, we obtain

$$H_{n+1}(x)H_{n-1}(x) - xH_n(x)H_{n-1}(x) + nH_{n-1}^2(x) = 0, \qquad n = 1, 2, \ldots \tag{1.47}$$

$$H_n^2(x) - xH_{n-1}(x)H_n(x) + (n-1)H_{n-2}(x)H_n(x) = 0. \qquad n = 2, 3, \ldots \tag{1.48}$$

Subtracting Eq. (1.47) from Eq. (1.48), we get

$$H_n^2(x) + (n-1)H_{n-2}(x)H_n(x) - nH_{n-1}^2(x) - H_{n+1}(x)H_{n-1}(x) = 0. \qquad n = 2, 3, \ldots \tag{1.49}$$

Multiplying Eq. (1.49) by $e^{-x^2/2}$ and integrating over $(-\infty, \infty)$

$$\int_{-\infty}^{\infty} H_n^2(x)e^{\frac{-x^2}{2}} dx + (n-1)\int_{-\infty}^{\infty} H_{n-2}(x)H_n(x)e^{\frac{-x^2}{2}} dx - n\int_{-\infty}^{\infty} H_{n-1}^2(x)e^{\frac{-x^2}{2}} dx - \int_{-\infty}^{\infty} H_{n+1}(x)H_{n-1}(x)e^{\frac{-x^2}{2}} dx = 0, \quad n = 2, 3, \ldots \tag{1.50}$$

Using Eq. (1.44) in Eq. (1.50), we have

$$\int_{-\infty}^{\infty} H_n^2(x)e^{\frac{-x^2}{2}}\,dx = n\int_{-\infty}^{\infty} H_{n-1}^2(x)e^{\frac{-x^2}{2}}\,dx, \qquad n = 2, 3, \ldots \tag{1.51}$$

$$= n(n-1)\int_{-\infty}^{\infty} H_{n-2}^2(x)e^{\frac{-x^2}{2}}\,dx, \tag{1.52}$$

$$= n(n-1)(n-2)\int_{-\infty}^{\infty} H_{n-3}^2(x)e^{\frac{-x^2}{2}}\,dx \tag{1.53}$$

and so on.
In general

$$\int_{-\infty}^{\infty} H_n^2(x)e^{\frac{-x^2}{2}}\,dx = n!\int_{-\infty}^{\infty} H_0^2(x)e^{\frac{-x^2}{2}}\,dx, \qquad n = 0, 1, 2, \ldots \tag{1.54}$$

$$= n!\int_{-\infty}^{\infty} e^{-x^2/2}dx, \tag{1.55}$$

$$= n!\sqrt{2\pi}. \qquad n = 2, 3, \ldots \tag{1.56}$$

Direct calculation shows that this result is also valid for $n = 0$ and 1.
Hence if $n = m$ then

$$\int_{-\infty}^{\infty} H_n^2(x)e^{\frac{-x^2}{2}}\,dx = n!\sqrt{2\pi}, \tag{1.57}$$

Eq. (1.44) and Eq. (1.57) have the required results. The inner product of two real valued function $F(x)$ and $G(x)$ is defined as [5]

$$< F, G >_w = \int_{-\infty}^{\infty} F(x)G(x)w(x)dx, \tag{1.58}$$

where the weight function $w(x)$ is the Gaussian probability density function

$$w(x) = \frac{1}{\sqrt{2\pi}}e^{\frac{-x^2}{2}}. \tag{1.59}$$

Let

$$\|G(x)\|_w = \sqrt{< G, G >_w} \tag{1.60}$$

be the induced norm from the inner product [5] . The Gram Schmidt process with respect to this norm applied to the sequences of functions defined by $1, x, x^2, \ldots$ gives the orthonormal sequences ξ_n called the Wick polynomial of order n defined as

$$\xi_n(x) = \frac{1}{\sqrt{n!}} H_n(x) \tag{1.61}$$

with Hermite polynomials defined as in Eq. (1.3) [5].

Theorem 1.3.1. *Wick polynomial $\{\xi_n\}$ defined by Eq. (1.61) is a complete set of orthonormal basis in $L^2(-\infty, \infty)$ with respect to the weight function $w(x)$ [5]:*
where

$$w(x) = \frac{1}{\sqrt{2\pi}} e^{\frac{-x^2}{2}}. \tag{1.62}$$

Theorem 1.3.2. *Let $f(x)$ be defined for all finite values of x and integrable in any finite interval;*
Let

$$F(x) = F(0) + \int_0^x f(t)dt. \qquad -\infty < x < \infty. \tag{1.63}$$

If

$$F(x) = O(e^{kx^2/2}), \qquad 0 < k < 1, \quad and \quad |x| \longrightarrow \infty \tag{1.64}$$

and

$$\| - xF(x) + f(x)\|_w^2 \tag{1.65}$$

is convergent, then

$$\sum_{n=0}^{N} < F, \xi_n >_w \xi_n(x) w(x) \tag{1.66}$$

converges uniformly to the product of $F(x)$ and $\omega(x)$ in $(-\infty, \infty)$ [5].

Lemma 1. *Since the set of Hermite polynomials of order n ($\{H_n\}$) is a basis, thus the product of Hermite polynomials of different order $H_\alpha(x)$ and $H_\beta(x)$ can be represented in terms of H_n, where α and β are any nonnegative integers, i.e.,*

$$H_\alpha(x) H_\beta(x) = \sum_{\gamma=0}^{\alpha\Lambda\beta} \gamma! \binom{\alpha}{\gamma} \binom{\beta}{\gamma} H_{\alpha+\beta-2\gamma}(x), \tag{1.67}$$

where $\alpha\Lambda\beta = min(\alpha, \beta)$.

Proof. $\psi(x, z)$ is the generating function of $H_n(x)$ defined in Eq. (1.4) can be written as

$$\psi(x, z)\psi(x, w) = \sum_{\alpha=0}^{\infty} \sum_{\beta=0}^{\infty} \left(\frac{H_\alpha(x)}{\alpha!} z^\alpha \right) \left(\frac{H_\beta(x)}{\beta!} w^\beta \right), \tag{1.68}$$

or

$$\psi(x,z)\psi(x,w) = \sum_{\alpha=0}^{\infty}\sum_{\beta=0}^{\infty}\frac{H_\alpha(x)}{\alpha!}\frac{H_\beta(x)}{\beta!}z^\alpha w^\beta, \tag{1.69}$$

$$\psi(x,z)\psi(x,w) = e^{(-\frac{z^2}{2}+xz)}e^{(-\frac{w^2}{2}+xw)}, \tag{1.70}$$

$$\psi(x,z)\psi(x,w) = e^{(\frac{x^2}{2})}e^{(zw)}e^{-\frac{1}{2}(z+w-x)^2}. \tag{1.71}$$

or

$$\psi(x,z)\psi(x,w) = \sum_{p=0}^{\infty}\frac{(zw)^p}{p!}e^{x^2/2}e^{-(z+w-x)^2/2}. \tag{1.72}$$

Using Eq. (1.4) in Eq. (1.72), we obtain

$$\psi(x,z)\psi(x,w) = \sum_{p=0}^{\infty}\frac{(zw)^p}{p!}\sum_{k=0}^{\infty}\frac{H_k(x)}{k!}(z+w)^k. \tag{1.73}$$

For some values z and w and a nonnegative number k, definition of binomial theorem is

$$(z+w)^k = \sum_{0\le m\le k}\binom{k}{m}z^m w^{k-m}. \tag{1.74}$$

Using Eq. (1.74) in Eq. (1.73), we have

$$\psi(x,z)\psi(x,w) = \sum_{p=0}^{\infty}\frac{(zw)^p}{p!}\sum_{k=0}^{\infty}\frac{H_k(x)}{k!}\sum_{0\le m\le k}\binom{k}{m}z^m w^{k-m}, \tag{1.75}$$

$$= \sum_{p=0}^{\infty}\sum_{k=0}^{\infty}\frac{H_k(x)}{p!}\sum_{0\le m\le k}\frac{1}{k!}\binom{k}{m}z^{(m+p)}w^{(k-m+p)}. \tag{1.76}$$

Let $k = m+v$, then $m \le k$ is equivalent to $v \ge 0$.
Thus we can rewrite the above equation as

$$\psi(x,z)\psi(x,w) = \sum_{p=0}^{\infty}\sum_{k=0}^{\infty}\frac{H_k(x)}{p!}\sum_{v=0}^{\infty}\frac{1}{k!}\binom{k}{m}z^{(m+p)}w^{(k-m+p)}, \tag{1.77}$$

$$= \sum_{p=0}^{\infty}\sum_{m=0}^{\infty}\sum_{v=0}^{\infty}\frac{H_{m+v}(x)}{p!m!v!}z^{m+p}w^{v+p}. \tag{1.78}$$

Let $\alpha = m+p$ and $\beta = v+p$ satisfying $m = \alpha - p \ge 0$ and $v = \beta - p \ge 0$ or $p \le \alpha \wedge \beta$, Then Eq. (1.78) becomes

$$\psi(x,z)\psi(x,w)=\sum_{\alpha=0}^{\infty}\sum_{\beta=0}^{\infty}\left(\sum_{m+p=\alpha}\sum_{v+p=\beta}\frac{H_{m+v}(x)}{p!m!v!}\right)z^{\alpha}w^{\beta}, \tag{1.79}$$

$$\psi(x,z)\psi(x,w)=\sum_{\alpha=0}^{\infty}\sum_{\beta=0}^{\infty}\sum_{p\leq\alpha\wedge\beta}\frac{H_{\alpha+\beta-2p}(x)}{p!(\alpha-p)!(\beta-p)!}z^{\alpha}w^{\beta}. \tag{1.80}$$

Comparing Eq. (1.69) and Eq. (1.80)

$$H_{\alpha}(x)H_{\beta}(x)=\sum_{p\leq\alpha\wedge\beta}\frac{\alpha!\beta!}{p!(\alpha-p)!(\beta-p)!}H_{\alpha+\beta-2p}(x). \tag{1.81}$$

Taking $p=\gamma$

$$H_{\alpha}(x)H_{\beta}(x)=\sum_{\gamma\leq\alpha\wedge\beta}\frac{\alpha!}{\gamma!(\alpha-\gamma)!}\ \frac{\beta!\gamma!}{\gamma!(\beta-\gamma)!}H_{\alpha+\beta-2p}(x), \tag{1.82}$$

or

$$H_{\alpha}(x)H_{\beta}(x)=\sum_{\gamma\leq\alpha\wedge\beta}\binom{\alpha}{\gamma}\binom{\beta}{\gamma}\gamma!H_{\alpha+\beta-2\gamma}(x). \tag{1.83}$$

□

Lemma 2.

$$\left(\sum_{\alpha=0}^{\infty}\frac{u_{\alpha}}{\alpha!}H_{\alpha}(x)\right)\left(\sum_{\beta=0}^{\infty}\frac{v_{\beta}}{\beta!}H_{\beta}(x)\right)=\sum_{n=0}^{\infty}\frac{1}{n!}\sum_{m=0}^{\infty}\frac{1}{m!}\sum_{k=0}^{n}\binom{n}{k}u_{m+k}v_{m+n-k}H_{n}(x). \tag{1.84}$$

Proof. For constants u_{α} and v_{β} $(\alpha=\beta=0,1,2,\ldots)$, Lemma 1 can be written as

$$\left(\sum_{\alpha=0}^{\infty}\frac{u_{\alpha}}{\alpha!}H_{\alpha}(x)\right)\left(\sum_{\beta=0}^{\infty}\frac{v_{\beta}}{\beta!}H_{\beta}(x)\right)=\sum_{\alpha=0}^{\infty}\frac{u_{\alpha}}{\alpha!}\sum_{\beta=0}^{\infty}\frac{v_{\beta}}{\beta!}\sum_{\gamma=0}^{\alpha\wedge\beta}\binom{\alpha}{\gamma}\binom{\beta}{\gamma}\gamma!H_{\alpha+\beta-2\gamma}(x). \tag{1.85}$$

Letting $(\alpha, \beta, \gamma) = (m+k, m+n-k, m)$ in Eq. (1.85) gives

$$\left(\sum_{\alpha=0}^{\infty} \frac{u_\alpha}{\alpha!} H_\alpha(x)\right)\left(\sum_{\beta=0}^{\infty} \frac{v_\beta}{\beta!} H_\beta(x)\right) \tag{1.86}$$

$$= \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\sum_{k=0}^{n} \frac{u_{m+k}}{(m+k)!}\frac{v_{m+n-k}}{(m+n-k)!} m! \binom{m+k}{m}\binom{m+n-k}{m} H_n(x), \tag{1.87}$$

$$= \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\sum_{k=0}^{n} \frac{u_{m+k} v_{m+n-k}}{k! m! (n-k)!} H_n(x), \tag{1.88}$$

$$= \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{m=0}^{\infty} \frac{1}{m!} \sum_{k=0}^{n} \binom{n}{k} u_{m+k} v_{m+n-k} H_n(x), \tag{1.89}$$

which is the required result [3]. $\square$

Chapter 2

Derivation of Numerical Algorithm

2.1 Lax-Wendroff scheme (LW) for Deterministic Conservation Law

We first derive Lax-Wendroff scheme (LW) for deterministic conservation law. The conservative form of Burger's equation in one dimension can be expressed as

$$\frac{\partial v}{\partial t} + \frac{\partial}{\partial x} f(v) = 0, \tag{2.1}$$

as $f(v) = v^2/2$.
Differentiating Eq. (2.1) w.r.t $''t''$ gives

$$\frac{\partial}{\partial t}\left(\frac{\partial v}{\partial t}\right) = \frac{\partial}{\partial t}\left(-\frac{\partial f}{\partial x}\right), \tag{2.2}$$

$$= \frac{\partial}{\partial x}\left(-\frac{\partial f}{\partial v}\frac{\partial v}{\partial t}\right). \tag{2.3}$$

Using Eq. (2.1) in Eq. (2.3), we get

$$\frac{\partial^2 v}{\partial t^2} = \frac{\partial}{\partial x}\left(A\frac{\partial f}{\partial x}\right), \tag{2.4}$$

where

$$A = \frac{\partial f}{\partial v}. \tag{2.5}$$

Consider Taylor series expansion in time variable for $v(x, t + \Delta t)$ is defined by

$$v(x, t + \Delta t) = v(x, t) + \frac{\partial v}{\partial t}\Delta t + \frac{\partial^2 v}{\partial t^2}\frac{(\Delta t)^2}{2!} + O(\Delta t)^3. \tag{2.6}$$

Replacing $\partial v/\partial t$ and $\partial^2 v/\partial t^2$ in Eq. (2.6) by Eq. (2.1) and Eq. (2.4), we have

$$v(x, t + \Delta t) = v(x, t) - \frac{\partial f}{\partial x}\Delta t + \frac{\partial}{\partial x}\left(A\frac{\partial f}{\partial x}\right)\frac{(\Delta t)^2}{2!} + O(\Delta t)^3, \tag{2.7}$$

$$v(x, t + \Delta t) - v(x, t) = -\frac{\partial f}{\partial x}\Delta t + \frac{\partial}{\partial x}\left(A\frac{\partial f}{\partial x}\right)\frac{(\Delta t)^2}{2!} + O(\Delta t)^3. \tag{2.8}$$

Eq. (2.8) can be rewritten as

$$\frac{v(x, t + \Delta t) - v(x, t)}{\Delta t} = -\frac{\partial f}{\partial x} + \frac{\partial}{\partial x}(A\frac{\partial f}{\partial x})\frac{(\Delta t)}{2!} + O(\Delta t)^2. \tag{2.9}$$

By using central difference formula for spatial variable x in Eq. (2.9) gives

$$\left(\frac{\partial f}{\partial x}\right)_{j,i} = \frac{f_{j+1,i} - f_{j-1,i}}{2\Delta x}, \tag{2.10}$$

$$\left(\frac{\partial}{\partial x}\left(A\frac{\partial f}{\partial x}\right)\right)_{j,i} = \frac{\left(A\frac{\partial f}{\partial x}\right)_{j+1/2,i} - \left(A\frac{\partial f}{\partial x}\right)_{j-1/2,i}}{\Delta x}. \tag{2.11}$$

Using forward and backward difference formula's for spatial variable x in Eq. (2.11), we have

$$\left(\frac{\partial}{\partial x}\left(A\frac{\partial f}{\partial x}\right)\right)_{j,i} = \frac{A_{j+1/2,i}\left(\frac{f_{j+1,i}-f_{j,i}}{\Delta x}\right) - A_{j-1/2,i}\left(\frac{f_{j,i}-f_{j-1,i}}{\Delta x}\right)}{\Delta x}. \tag{2.12}$$

By averaging the function A defined in Eq. (2.5)

$$A_{j+1/2,i} = \frac{A_{j+1} + A_j}{2}, \tag{2.13}$$

$$A_{j-1/2,i} = \frac{A_j + A_{j-1}}{2}. \tag{2.14}$$

Thus Eq. (2.12) becomes

$$\frac{\partial}{\partial x}\left(A\frac{\partial f}{\partial x}\right)=\frac{\frac{1}{2\Delta x}\left(\left(A_{j+1,i}+A_{j,i}\right)\left(f_{j+1,i}-f_{j,i}\right)-\left(A_{j-1,i}+A_{j,i}\right)\left(f_{j,i}-f_{j-1,i}\right)\right)}{\Delta x}. \quad (2.15)$$

Putting Eq. (2.10) and Eq. (2.15) in Eq. (2.9) and after simplification, we get

$$v_{j,i+1}=v_{j,i}-\frac{1}{2}\left(\frac{\Delta t}{\Delta x}\right)\left(f_{j+1,i}-f_{j-1,i}\right)+\frac{1}{4}\left(\frac{\Delta t}{\Delta x}\right)^2\left[\left(A_{j+1,i}+A_{j,i}\right)\left(f_{j+1,i}-f_{j,i}\right)-\left(A_{j-1,i}+A_{j,i}\right)\left(f_{j,i}-f_{j-1,i}\right)\right], \quad (2.16)$$

where $v_{j,i}=v(x_j,t_i)$.
Using

$$f_{j+1,i}-f_{j,i}=\Delta_{+x}F_{j,i}, \quad (2.17)$$

$$f_{j,i}-f_{j-1,i}=\Delta_{-x}F_{j,i}, \quad (2.18)$$

$$\frac{A_{j+1,i}+A_{j,i}}{2}=A_{j+1/2,i}, \quad (2.19)$$

$$\frac{A_{j-1,i}+A_{j,i}}{2}=A_{j-1/2,i}, \quad (2.20)$$

then Eq. (2.16) becomes [6]

$$v_{j,i+1}=v_{j,i}-\frac{1}{2}\left(\frac{\Delta t}{\Delta x}\right)\left(\Delta_{+x}F_{j,i}+\Delta_{-x}F_{j,i}\right)+\frac{1}{4}\left(\frac{\Delta t}{\Delta x}\right)^2\left[A_{j+1/2,i}\Delta_{+x}F_{j,i}-A_{j-1/2,i}\Delta_{-x}F_{j,i}\right]. \quad (2.21)$$

2.2 Modified Form of the Lax-Wendroff Scheme Using Wiener Chaos Expansion

Now we derive modified form of the *Lax-Wendroff* scheme using *Wiener Chaos Expansion*. Let us consider Eq. (2.1) for $v(x,t,\xi)$ with random initial condition $v(x,0,\xi)=g(x,\xi)$. $g(x,\xi)$ is a function of x and ξ. ξ is standard Gaussian random variable i.e., it has zero mean and unit variance [5].

The Wiener Chaos expansion is the Fourier expansion with respect to the randomness [7]. In this dissertation, WCE represents the solution $v=v(x,t,\xi)$ of Eq. (2.1) as a series with respect to the Wick polynomials $\xi_n(\xi)$

$$v(x,t,\xi) = \sum_{n=0}^{\infty} \frac{1}{\sqrt{n!}} v^n(x,t)\xi_n(\xi), \tag{2.22}$$

where

$$v^n(x,t) = \sqrt{n!} < v(x,t,\xi), \xi_n(\xi) >_w . \tag{2.23}$$

Therefore, v^n is called *Fourier-Hermite Coefficients* of order n of the function v. For notational simplicity we write $H_n(\xi) = H_n$ and $\xi_n(\xi) = \xi_n$.
Eq. (2.1) can be written as

$$v_t + vv_x = 0. \tag{2.24}$$

By using the series Eq. (2.22) for vv_x of Eq. (2.1), we get

$$vv_x = \left(\sum_{\alpha=0}^{\infty} \frac{1}{\sqrt{\alpha!}} v^{\alpha}(x,t)\xi_{\alpha} \right) \left(\sum_{\beta=0}^{\infty} \frac{1}{\sqrt{\beta!}} v_x^{\beta}(x,t)H_{\beta} \right). \tag{2.25}$$

Using Eq. (1.61) in Eq. (2.25) gives

$$vv_x = \left(\sum_{\alpha=0}^{\infty} \frac{1}{\alpha!} v^{\alpha}(x,t)H_{\alpha} \right) \left(\sum_{\beta=0}^{\infty} \frac{1}{\beta!} v_x^{\beta}(x,t)H_{\beta} \right). \tag{2.26}$$

By using Lemma 2 on the right side of Eq. (2.26)

$$vv_x = \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{m=0}^{\infty} \frac{1}{m!} \sum_{k=0}^{n} \binom{n}{k} v^{m+k} v_x^{m+n-k} H_n. \tag{2.27}$$

Again using Eq. (1.61) gives

$$vv_x = \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{m=0}^{\infty} \frac{1}{m!} \sum_{k=0}^{n} \binom{n}{k} v^{m+k} v_x^{m+n-k} \sqrt{n!}\xi_n. \tag{2.28}$$

After putting Eq. (2.28) in Eq. (2.1) becomes

$$\frac{\partial v}{\partial t} = -\sum_{n=0}^{\infty} \frac{1}{n!} \sum_{m=0}^{\infty} \frac{1}{m!} \sum_{k=0}^{n} \binom{n}{k} v^{m+k} v_x^{m+n-k} \sqrt{n!}\xi_n. \tag{2.29}$$

Using Eq. (2.22) in Eq. (2.29) gives

$$\frac{\partial}{\partial t}\sum_{n=0}^{\infty}\frac{1}{\sqrt{n!}}v^n(x,t)\xi_n = -\sum_{n=0}^{\infty}\frac{1}{n!}\sum_{m=0}^{\infty}\frac{1}{m!}\sum_{k=0}^{n}\binom{n}{k}v^{m+k}v_x^{m+n-k}\sqrt{n!}\xi_n, \tag{2.30}$$

or

$$\frac{\partial}{\partial t}v^n(x,t) = -\sum_{m=0}^{\infty}\frac{1}{m!}\sum_{k=0}^{n}\binom{n}{k}v^{m+k}(x,t)v_x^{m+n-k}(x,t). \tag{2.31}$$

Substituting $t=0$ in Eq. (2.23) and using $v(x,0,\xi)=g(x,\xi)$ gives

$$v^n(x,0) = \sqrt{n!} < v(x,0,\xi), \xi_n(\xi) >_w, \tag{2.32}$$

$$= \sqrt{n!} < g(x,\xi), \xi_n(\xi) >_w, \tag{2.33}$$

$$= E\Big[g(x,\xi), \xi_n(\xi)\Big]. \tag{2.34}$$

Now by using orthogonality of Theorem 1.3.1 i.e., for every $n<\infty$, the Fourier-Hermite coefficient v^n is a solution of following system

$$\frac{\partial}{\partial t}v^n(x,t) = -\sum_{m=0}^{\infty}\frac{1}{m!}\sum_{k=0}^{n}\binom{n}{k}v^{m+k}(x,t)\frac{\partial}{\partial x}v^{m+n-k}(x,t) \tag{2.35}$$

with

$$v^n(x,0) = E\Big[g(x,\xi), \xi_n(\xi)\Big], \tag{2.36}$$

where $t\in(0,T]$ and $x\in\Re$ [5]. Also E is an expectation operator defined as

$$E\Big[f(x)\Big] = \int_{-\infty}^{\infty} f(x)w(x)dx. \tag{2.37}$$

The system of Eq. (2.35) and Eq. (2.36) is known as *Propagator system* because they governs the propagation of randomness. It separates non-random variable (x,t) from random variable ξ. Since this propagator system is a deterministic, so it need to be solved only once.

When $v_{j,i}$ is expanded using a series in Eq. (2.21), the propagator system for each Hermite Fourier coefficient $v^n(j,i)=v^n(x_j,t_i)$ can be obtained by a following modified version of the

Lax-Wendroff scheme

$$v^n_{j,i+1} = v^n_{j,i} - \frac{1}{4}\left(\frac{\Delta t}{\Delta x}\right)\Big(w^n_{j+1,i} + w^n_{j-1,i}\Big)$$
$$+ \frac{1}{8}\left(\frac{\Delta t}{\Delta x}\right)^2 \sum_{\alpha=0}^{\infty}\Bigg[\Big(z^{n,\alpha}_{j,i} + z^{n,\alpha}_{j+1,i}\Big)\Big(w^{\alpha}_{j+1,i} - w^{\alpha}_{j,i}\Big) - \Big(z^{n,\alpha}_{j,i} + z^{n,\alpha}_{j-1,i}\Big)\Big(w^{\alpha}_{j,i} - w^{\alpha}_{j-1,i}\Big)\Bigg], \quad (2.38)$$

where

$$w^{\alpha}_{j,i} = \sum_{m=0}^{\infty}\frac{1}{m!}\left(2\sum_{0\le k<\alpha/2}\binom{\alpha}{k}v^{m+k}_{j,i}v^{m+\alpha-k}_{j,i} + \chi_{\big(\alpha/2=[\alpha/2]\big)}\binom{\alpha}{\alpha/2}\left(v^{m+\alpha/2}_{j,i}\right)^2\right) \quad (2.39)$$

and

$$z^{n,\alpha}_{j,i} = \sum_{q=max(0,n-\alpha)}^{n}\frac{1}{(\alpha-n+q)!}\binom{n}{q}v^{\alpha-n+2q}_{j,i}, \quad (2.40)$$

where $\chi_{(a=[b])}$ is one, when a and b in χ is one otherwise equal to zero. Also $[x]$ is the smallest integer which is not smaller than x.

Eq. (2.38), Eq. (2.39) and Eq. (2.40) can be obtained from Eq. (2.21) by substituting $\Delta_{+x}F_{j,i}$, $\Delta_{-x}F_{j,i}$, $A_{j+1/2,i}$, $A_{j-1/2,i}$ in it which can be derived as follows:
Using $F_{j,i} = f(v_{j,i})$ and $f(v_{j,i}) = v^2_{j,i}/2$ in Eq. (2.17) gives

$$\Delta_{+x}F_{j,i} = f(v_{j+1,i}) - f(v_{j,i}), \quad (2.41)$$
$$= \frac{1}{2}\Big(v_{j+1,i}v_{j+1,i} - v_{j,i}v_{j,i}\Big). \quad (2.42)$$

Using Eq. (2.28) in Eq. (2.42) gives

$$\Delta_{+x}F_{j,i} = \frac{1}{2}\sum_{n=0}^{\infty}\frac{1}{\sqrt{n!}}\sum_{m=0}^{\infty}\frac{1}{m!}\sum_{k=0}^{n}\binom{n}{k}\xi_n\Big[v^{m+k}_{j+1,i}v^{m+n-k}_{j+1,i} - v^{m+k}_{j,i}v^{m+n-k}_{j,i}\Big], \quad (2.43)$$
$$= \frac{1}{2}\sum_{n=0}^{\infty}\frac{1}{\sqrt{n!}}\sum_{m=0}^{\infty}\frac{1}{m!}\sum_{k=0}^{n}\binom{n}{k}\psi^{m,n,k}_{j+1/2,i}\xi_n, \quad (2.44)$$

where

$$\psi^{m,n,k}_{j+1/2,i} = v^{m+k}_{j+1,i}v^{m+n-k}_{j+1,i} - v^{m+k}_{j,i}v^{m+n-k}_{j,i}. \quad (2.45)$$

Eq. (2.18) gives

$$\Delta_{-x}F_{j,i} = f(v_{j,i}) - f(v_{j-1,i}), \tag{2.46}$$

$$= \frac{1}{2}\Big(v_{j,i}v_{j,i} - v_{j-1,i}v_{j-1,i}\Big). \tag{2.47}$$

Using Eq. (2.28) in Eq. (2.47)

$$\Delta_{-x}F_{j,i} = \frac{1}{2}\sum_{n=0}^{\infty}\frac{1}{\sqrt{n!}}\sum_{m=0}^{\infty}\frac{1}{m!}\sum_{k=0}^{n}\binom{n}{k}\xi_n\left[v_{j,i}^{m+k}v_{j,i}^{m+n-k} - v_{j-1,i}^{m+k}v_{j-1,i}^{m+n-k}\right], \tag{2.48}$$

$$= \frac{1}{2}\sum_{n=0}^{\infty}\frac{1}{\sqrt{n!}}\sum_{m=0}^{\infty}\frac{1}{m!}\sum_{k=0}^{n}\binom{n}{k}\psi_{j-1/2,i}^{m,n,k}\xi_n, \tag{2.49}$$

where

$$\psi_{j-1/2,i}^{m,n,k} = v_{j,i}^{m+k}v_{j,i}^{m+n-k} - v_{j-1,i}^{m+k}v_{j-1,i}^{m+n-k}. \tag{2.50}$$

Also

$$\Delta_{+x}F_{j,i} + \Delta_{-x}F_{j,i} = F_{j+1,i} - F_{j-1,i}, \tag{2.51}$$

$$= f(v_{j+1,i}) - f(v_{j-1,i}), \tag{2.52}$$

$$= \frac{1}{2}\Big(v_{j+1,i}v_{j+1,i} - v_{j-1,i}v_{j-1,i}\Big). \tag{2.53}$$

Using Eq. (2.28) in Eq. (2.53) gives

$$\Delta_{+x}F_{j,i} + \Delta_{-x}F_{j,i} = \frac{1}{2}\sum_{n=0}^{\infty}\frac{1}{\sqrt{n!}}\sum_{m=0}^{\infty}\frac{1}{m!}\sum_{k=0}^{n}\binom{n}{k}\xi_n\left[v_{j+1,i}^{m+k}v_{j+1,i}^{m+n-k} - v_{j-1,i}^{m+k}v_{j-1,i}^{m+n-k}\right], \tag{2.54}$$

$$= \frac{1}{2}\sum_{n=0}^{\infty}\frac{1}{\sqrt{n!}}\sum_{m=0}^{\infty}\frac{1}{m!}\sum_{k=0}^{n}\binom{n}{k}\phi_{j,i}^{m,n,k}\xi_n, \tag{2.55}$$

where

$$\phi_{j,i}^{m,n,k} = v_{j+1,i}^{m+k}v_{j+1,i}^{m+n-k} - v_{j-1,i}^{m+k}v_{j-1,i}^{m+n-k}. \tag{2.56}$$

Replacing $A_{j,i}$ from Eq. (2.5) in Eq. (2.19) gives

$$A_{j+1/2,i} = \frac{1}{2}\Big[\frac{\partial}{\partial v}f(v_{j,i}) + \frac{\partial}{\partial v}f(v_{j+1,i})\Big], \tag{2.57}$$

$$A_{j+1/2,i} = \frac{1}{2}\Big(v_{j,i} + v_{j+1,i}\Big). \tag{2.58}$$

Substituting Eq. (2.22) in Eq. (2.58)

$$A_{j+1/2,i} = \frac{1}{2}\sum_{\alpha=0}^{\infty}\frac{1}{\sqrt{\alpha!}}\phi^{\alpha}_{j+1/2,i}\xi_{\alpha}, \tag{2.59}$$

where

$$\phi^{\alpha}_{j+1/2,i} = v^{\alpha}_{j,i} + v^{\alpha}_{j+1,i}. \tag{2.60}$$

Again replacing $A_{j,i}$ from Eq. (2.5) in Eq. (2.20) gives

$$A_{j-1/2,i} = \frac{1}{2}\left[\frac{\partial}{\partial v}f(v_{j,i}) + \frac{\partial}{\partial v}f(v_{j-1,i})\right], \tag{2.61}$$

$$= \frac{1}{2}\Big(v_{j,i} + v_{j-1,i}\Big). \tag{2.62}$$

By putting Eq. (2.22) in Eq. (2.62), we obtain

$$A_{j-1/2,i} = \frac{1}{2}\sum_{\alpha=0}^{\infty}\frac{1}{\sqrt{\alpha!}}\phi^{\alpha}_{j-1/2,i}\xi_{\alpha}, \tag{2.63}$$

where

$$\phi^{\alpha}_{j-1/2,i} = v^{\alpha}_{j,i} + v^{\alpha}_{j-1,i}. \tag{2.64}$$

Taking product of Eq. (2.44) and Eq. (2.59), we get

$$A_{j+1/2,i}\Delta_{+x}F_{j,i} = \frac{1}{4}\sum_{\alpha=0}^{\infty}\frac{1}{\alpha!}\phi^{\alpha}_{j+1/2,i}\sum_{\beta=0}^{\infty}\frac{1}{\beta}\sum_{m=0}^{\infty}\frac{1}{m!}\sum_{k=0}^{\beta}\binom{\beta}{k}\psi^{m,\beta,k}_{j+1/2,i}\Big(\sqrt{\alpha!}\xi_{\alpha}\Big)\Big(\sqrt{\beta!}\xi_{\beta}\Big). \tag{2.65}$$

Using Eq. (1.61) in Eq. (2.65) yields

$$A_{j+1/2,i}\Delta_{+x}F_{j,i} = \frac{1}{4}\sum_{\alpha=0}^{\infty}\frac{1}{\alpha!}\phi^{\alpha}_{j+1/2,i}\sum_{\beta=0}^{\infty}\frac{1}{\beta}\sum_{m=0}^{\infty}\frac{1}{m!}\sum_{k=0}^{\beta}\binom{\beta}{k}\psi^{m,\beta,k}_{j+1/2,i}\Big(H_{\alpha}(x)H_{\beta}(x)\Big). \quad (2.66)$$

By using Lemma 1 in Eq. (2.65), we have

$$A_{j+1/2,i}\Delta_{+x}F_{j,i} = \frac{1}{4}\sum_{\alpha=0}^{\infty}\frac{1}{\alpha!}\phi^{\alpha}_{j+1/2,i}\sum_{\beta=0}^{\infty}\frac{1}{\beta!}\sum_{m=0}^{\infty}\frac{1}{m!}\sum_{k=0}^{\beta}\binom{\beta}{k}\psi^{m,\beta,k}_{j+1/2,i}\sum_{\gamma=0}^{\alpha\Lambda\beta}\gamma!\binom{\alpha}{\gamma}\binom{\beta}{\gamma}H_{\alpha+\beta-2\gamma}(x). \quad (2.67)$$

Taking $\alpha = p+q$, $\beta = p+n-q$ and $\gamma = p$ in Eq. (2.67) gives

$$A_{j+1/2,i}\Delta_{+x}F_{j,i} = \frac{1}{4}\sum_{n=0}^{\infty}\sum_{p=0}^{\infty}\sum_{q=0}^{n}\frac{1}{(p+q)!}\phi^{p+q}_{j+1/2,i}\frac{1}{(p+n-q)!}\sum_{m=0}^{\infty}\frac{1}{m!}$$
$$\sum_{k=0}^{p+n-q}\binom{p+n-q}{k}\psi^{m,p+n-q,k}_{j+1/2,i}p!\binom{p+q}{p}\binom{p+n-q}{p}H_n(x), \quad (2.68)$$

or

$$A_{j+1/2,i}\Delta_{+x}F_{j,i} = \frac{1}{4}\sum_{n=0}^{\infty}\frac{1}{\sqrt{n!}}\sum_{p=0}^{\infty}\frac{1}{p!}\sum_{q=0}^{n}\binom{n}{q}\phi^{p+q}_{j+1/2,i}\sum_{m=0}^{\infty}\frac{1}{m!}\sum_{k=0}^{p+n-q}\binom{p+n-q}{k}\psi^{m,p+n-q,k}_{j+1/2,i}\xi_n. \quad (2.69)$$

Similarly, taking product of Eq. (2.49) and Eq. (2.63) gives

$$A_{j-1/2,i}\Delta_{-x}F_{j,i} = \frac{1}{4}\sum_{n=0}^{\infty}\frac{1}{\sqrt{n!}}\sum_{p=0}^{\infty}\frac{1}{p!}\sum_{q=0}^{n}\binom{n}{q}\phi^{p+q}_{j-1/2,i}\sum_{m=0}^{\infty}\frac{1}{m!}\sum_{k=0}^{p+n-q}\binom{p+n-q}{k}\psi^{m,p+n-q,k}_{j-1/2,i}\xi_n. \quad (2.70)$$

Now substituting Eq. (2.22),(2.55),(2.69) and (2.70) in Eq. (2.21)

$$\sum_{n=0}^{\infty}\frac{1}{\sqrt{n!}}v_{j,i+1}^{n}\xi_n=\sum_{n=0}^{\infty}\frac{1}{\sqrt{n!}}v_{j,i}^{n}\xi_n-\frac{1}{2}\left(\frac{\Delta t}{\Delta x}\right)\left[\frac{1}{2}\sum_{n=0}^{\infty}\frac{1}{\sqrt{n!}}\sum_{m=0}^{\infty}\frac{1}{m!}\sum_{k=0}^{n}\binom{n}{k}\phi_{j,i}^{m,n,k}\xi_n\right]$$

$$+\frac{1}{2}\left(\frac{\Delta t}{\Delta x}\right)^2\left[\frac{1}{4}\sum_{n=0}^{\infty}\frac{1}{\sqrt{n!}}\sum_{p=0}^{\infty}\frac{1}{p!}\sum_{q=0}^{n}\binom{n}{q}\phi_{j+1/2,i}^{p+q}\sum_{m=0}^{\infty}\frac{1}{m!}\sum_{k=0}^{p+n-q}\binom{p+n-q}{k}\psi_{j+1/2,i}^{m,p+n-q,k}\xi_n\right.$$

$$\left.-\frac{1}{4}\sum_{n=0}^{\infty}\frac{1}{\sqrt{n!}}\sum_{p=0}^{\infty}\frac{1}{p!}\sum_{q=0}^{n}\binom{n}{q}\phi_{j-1/2,i}^{p+q}\sum_{m=0}^{\infty}\frac{1}{m!}\sum_{k=0}^{p+n-q}\binom{p+n-q}{k}\psi_{j-1/2,i}^{m,p+n-q,k}\xi_n\right]. \quad (2.71)$$

or

$$\sum_{n=0}^{\infty}\frac{1}{\sqrt{n!}}v_{j,i+1}^{n}\xi_n=\sum_{n=0}^{\infty}\frac{1}{\sqrt{n!}}\xi_n\left[v_{j,i}^{n}-\frac{1}{4}\left(\frac{\Delta t}{\Delta x}\right)\sum_{m=0}^{\infty}\frac{1}{m!}\sum_{k=0}^{n}\binom{n}{k}\phi_{j,i}^{m,n,k}\right.$$

$$+\frac{1}{8}\left(\frac{\Delta t}{\Delta x}\right)^2\sum_{p=0}^{\infty}\frac{1}{p!}\sum_{q=0}^{n}\binom{n}{q}\phi_{j+1/2,i}^{p+q}\sum_{m=0}^{\infty}\frac{1}{m!}\sum_{k=0}^{p+n-q}\binom{p+n-q}{k}\psi_{j+1/2,i}^{m,p+n-q,k}$$

$$\left.-\frac{1}{8}\left(\frac{\Delta t}{\Delta x}\right)^2\sum_{p=0}^{\infty}\frac{1}{p!}\sum_{q=0}^{n}\binom{n}{q}\phi_{j-1/2,i}^{p+q}\sum_{m=0}^{\infty}\frac{1}{m!}\sum_{k=0}^{p+n-q}\binom{p+n-q}{k}\psi_{j-1/2,i}^{m,p+n-q,k}\right]. \quad (2.72)$$

Comparing the coefficient of ξ on both sides of Eq. (2.72), we obtain

$$v_{j,i+1}^{n} = v_{j,i}^{n} - \frac{\Delta t}{4\Delta x}\sum_{m=0}^{\infty}\frac{1}{m!}\sum_{k=0}^{n}\binom{n}{k}\phi_{j,i}^{m,n,k}$$

$$+\frac{1}{8}\left(\frac{\Delta t}{\Delta x}\right)^{2}\sum_{p=0}^{\infty}\frac{1}{p!}\sum_{q=0}^{n}\binom{n}{q}\phi_{j+1/2,i}^{p+q}\sum_{m=0}^{\infty}\frac{1}{m!}\sum_{k=0}^{p+n-q}\binom{p+n-q}{k}\psi_{j+1/2,i}^{m,p+n-q,k}$$

$$-\frac{1}{8}\left(\frac{\Delta t}{\Delta x}\right)^{2}\sum_{p=0}^{\infty}\frac{1}{p!}\sum_{q=0}^{n}\binom{n}{q}\phi_{j-1/2,i}^{p+q}\sum_{m=0}^{\infty}\frac{1}{m!}\sum_{k=0}^{p+n-q}\binom{p+n-q}{k}\psi_{j-1/2,i}^{m,p+n-q,k}. \quad (2.73)$$

As Eq. (2.73) contains nested loops, so it is practically not possible to compute $v_{j,i}^{n}$ directly from Eq. (2.73), we need to simplify, the procedure is as follows:

Eq. (2.73) can be written as

$$v_{j,i+1}^{n} = v_{j,i}^{n} - \frac{\Delta t}{4\Delta x}E_1 + \frac{1}{8}\left(\frac{\Delta t}{\Delta x}\right)^{2}E_2 - \frac{1}{8}\left(\frac{\Delta t}{\Delta x}\right)^{2}E_3, \quad (2.74)$$

where

$$E_1 = \sum_{m=0}^{\infty}\frac{1}{m!}\sum_{k=0}^{n}\binom{n}{k}\phi_{j,i}^{m,n,k} \quad (2.75)$$

$$E_2 = \sum_{p=0}^{\infty}\frac{1}{p!}\sum_{q=0}^{n}\binom{n}{q}\phi_{j+1/2,i}^{p+q}\sum_{m=0}^{\infty}\frac{1}{m!}\sum_{k=0}^{p+n-q}\binom{p+n-q}{k}\psi_{j+1/2,i}^{m,p+n-q,k}, \quad (2.76)$$

$$E_3 = \sum_{p=0}^{\infty}\frac{1}{p!}\sum_{q=0}^{n}\binom{n}{q}\phi_{j-1/2,i}^{p+q}\sum_{m=0}^{\infty}\frac{1}{m!}\sum_{k=0}^{p+n-q}\binom{p+n-q}{k}\psi_{j-1/2,i}^{m,p+n-q,k}. \quad (2.77)$$

Using Eq. (2.56) in Eq. (2.75)

$$E_1 = \sum_{m=0}^{\infty}\frac{1}{m!}\sum_{k=0}^{n}\binom{n}{k}\left[v_{j+1,i}^{m+k}v_{j+1,i}^{m+n-k} - v_{j-1,i}^{m+k}v_{j-1,i}^{m+n-k}\right], \quad (2.78)$$

$$E_1 = \omega_{j+1,i} - \omega_{j-1,i}, \quad (2.79)$$

where

$$\omega_{j+1,i} = \sum_{m=0}^{\infty} \frac{1}{m!} \sum_{k=0}^{n} \binom{n}{k} v_{j+1,i}^{m+k} v_{j+1,i}^{m+n-k} \tag{2.80}$$

and

$$\omega_{j-1,i} = \sum_{m=0}^{\infty} \frac{1}{m!} \sum_{k=0}^{n} \binom{n}{k} v_{j-1,i}^{m+k} v_{j-1,i}^{m+n-k}. \tag{2.81}$$

Let $\alpha = p + n - q$, then Eq. (2.76) becomes

$$E_2 = \sum_{\alpha=0}^{\infty} \left(\sum_{q=max(0,n-\alpha)}^{n} \frac{1}{(\alpha - n + q)!} \binom{n}{q} \phi_{j+1/2,i}^{\alpha+2q-n} \right) \left(\sum_{m=0}^{\infty} \frac{1}{m!} \sum_{k=0}^{\alpha} \binom{\alpha}{k} \psi_{j+1/2,i}^{m,\alpha,k} \right). \tag{2.82}$$

Now in the above equation, we have summation over α, in which first term is independent of m and k and the second term is independent of q. This saves computations by removing unnecessary nested loops. In addition Eq. (2.82) can also be simplified as

$$E_2 = \sum_{\alpha=0}^{\infty} (G_1 G_2), \tag{2.83}$$

where

$$G_1 = \sum_{q=max(0,n-\alpha)}^{n} \frac{1}{(\alpha - n + q)!} \binom{n}{q} \phi_{j+1/2,i}^{\alpha+2q-n} \tag{2.84}$$

and

$$G_2 = \sum_{m=0}^{\infty} \frac{1}{m!} \sum_{k=0}^{\alpha} \binom{\alpha}{k} \psi_{j+1/2,i}^{m,\alpha,k}. \tag{2.85}$$

Now using Eq. (2.60) in Eq. (2.84)

$$G_1 = \sum_{q=max(0,n-\alpha)}^{n} \frac{1}{(\alpha - n + q)!} \binom{n}{q} \left[v_{j,i}^{\alpha-n+2q} + v_{j+1,i}^{\alpha-n+2q} \right], \tag{2.86}$$

$$= \sum_{q=max(0,n-\alpha)}^{n} \frac{1}{(\alpha - n + q)!} \binom{n}{q} v_{j,i}^{\alpha-n+2q} + \sum_{q=max(0,n-\alpha)}^{n} \frac{1}{(\alpha - n + q)!} \binom{n}{q} v_{j+1,i}^{\alpha-n+2q}. \tag{2.87}$$

If in Eq. (2.87) we say

$$z_{j,i}^{n,\alpha} = \sum_{q=max(0,n-\alpha)}^{n} \frac{1}{(\alpha-n+q)!} \binom{n}{q} v_{j,i}^{\alpha-n+2q} \tag{2.88}$$

and

$$z_{j+1,i}^{n,\alpha} = \sum_{q=max(0,n-\alpha)}^{n} \frac{1}{(\alpha-n+q)!} \binom{n}{q} v_{j+1,i}^{\alpha-n+2q}. \tag{2.89}$$

Then, Eq. (2.76) becomes

$$G_1 = z_{j,i}^{n,\alpha} + z_{j+1,i}^{n,\alpha}. \tag{2.90}$$

Now substituting Eq. (2.45) in Eq. (2.85)

$$\begin{aligned} G_2 = \sum_{m=0}^{\infty} \frac{1}{m!} \Bigg(2 \sum_{0\le k<\alpha/2} \binom{\alpha}{k} \left[v_{j+1,i}^{m+k} v_{j+1,i}^{m+\alpha-k} - v_{j,i}^{m+k} v_{j,i}^{m+\alpha-k} \right] \Bigg) \\ + \chi_{\left(\frac{\alpha}{2}=[\frac{\alpha}{2}]\right)} \binom{\alpha}{\alpha/2} \left[v_{j+1,i}^{m+\alpha/2} v_{j+1,i}^{m+\alpha-\alpha/2} - v_{j,i}^{m+\alpha/2} v_{j,i}^{m+\alpha-\alpha/2} \right], \end{aligned} \tag{2.91}$$

or

$$\begin{aligned} G_2 = \sum_{m=0}^{\infty} \frac{1}{m!} \Bigg(2 \sum_{0\le k<\alpha/2} \binom{\alpha}{k} v_{j+1,i}^{m+k} v_{j+1,i}^{m+\alpha-k} - 2 \sum_{0\le k<\alpha/2} \binom{\alpha}{k} v_{j,i}^{m+k} v_{j,i}^{m+\alpha-k} \Bigg) \\ + \chi_{\left(\frac{\alpha}{2}=[\frac{\alpha}{2}]\right)} \binom{\alpha}{\alpha/2} \left(v_{j+1,i}^{m+\alpha/2} \right)^2 - \chi_{\left(\frac{\alpha}{2}=[\frac{\alpha}{2}]\right)} \binom{\alpha}{\alpha/2} \left(v_{j,i}^{m+\alpha/2} \right)^2, \end{aligned} \tag{2.92}$$

or

$$G_2 = \sum_{m=0}^{\infty} \frac{1}{m!}\Big(2 \sum_{0\le k<\alpha/2} \binom{\alpha}{k} v_{j+1/2,i}^{m+k} v_{j+1/2,i}^{m+\alpha-k} + \chi_{\left(\frac{\alpha}{2}=[\frac{\alpha}{2}]\right)} \binom{\alpha}{\alpha/2} \left(v_{j+1,i}^{m+\alpha/2}\right)^2\Big) - \sum_{m=0}^{\infty} \frac{1}{m!}\Big(2 \sum_{0\le k<\alpha/2} \binom{\alpha}{k} v_{j,i}^{m+k} v_{j,i}^{m+\alpha-k} + \chi_{\left(\frac{\alpha}{2}=[\frac{\alpha}{2}]\right)} \binom{\alpha}{\alpha/2} \left(v_{j,i}^{m+\alpha/2}\right)^2\Big). \tag{2.93}$$

If in Eq. (2.93) we say

$$\omega_{j+1,i}^{\alpha} = \sum_{m=0}^{\infty} \frac{1}{m!} \left(2 \sum_{0\le k<\alpha/2} \binom{\alpha}{k} v_{j+1/2,i}^{m+k} v_{j+1/2,i}^{m+\alpha-k} + \chi_{\left(\frac{\alpha}{2}=[\frac{\alpha}{2}]\right)} \binom{\alpha}{\alpha/2} \left(v_{j+1,i}^{m+\alpha/2}\right)^2\right) \tag{2.94}$$

$$\omega_{j,i}^{\alpha} = \sum_{m=0}^{\infty} \frac{1}{m!} \left(2 \sum_{0\le k<\alpha/2} \binom{\alpha}{k} v_{j,i}^{m+k} v_{j,i}^{m+\alpha-k} + \chi_{\left(\frac{\alpha}{2}=[\frac{\alpha}{2}]\right)} \binom{\alpha}{\alpha/2} \left(v_{j,i}^{m+\alpha/2}\right)^2\right). \tag{2.95}$$

Then Eq. (2.93) becomes

$$G_2 = \omega_{j+1,i}^{\alpha} - \omega_{j,i}^{\alpha}. \tag{2.96}$$

Substituting Eq. (2.90) and Eq. (2.96) in Eq. (2.83), we get

$$E_2 = \sum_{\alpha=0}^{\infty} \left[(z_{j,i}^{n,\alpha} + z_{j+1,i}^{n,\alpha})(\omega_{j+1,i}^{\alpha} - \omega_{j,i}^{\alpha}) \right]. \tag{2.97}$$

Similarly Eq. (2.77) becomes

$$E_3 = \sum_{\alpha=0}^{\infty} \left[(z_{j,i}^{n,\alpha} + z_{j-1,i}^{n,\alpha})(\omega_{j,i}^{\alpha} - \omega_{j-1,i}^{\alpha}) \right]. \tag{2.98}$$

Substituting Eq. (2.79), (2.97) and (2.98) in Eq. (2.74), we get

$$v_{j,i+1}^{n} = v_{j,i}^{n} - \frac{1}{4}\left(\frac{\Delta t}{\Delta x}\right)\left(\omega_{j+1,i} - \omega_{j-1,i}\right) + \frac{1}{8}\left(\frac{\Delta t}{\Delta x}\right)^{2} \sum_{\alpha=0}^{\infty} \left[(z_{j,i}^{n,\alpha} + z_{j+1,i}^{n,\alpha})(\omega_{j+1,i}^{\alpha} - \omega_{j,i}^{\alpha})\right] - \frac{1}{8}\left(\frac{\Delta t}{\Delta x}\right)^{2} \sum_{\alpha=0}^{\infty} \left[(z_{j,i}^{n,\alpha} + z_{j-1,i}^{n,\alpha})(\omega_{j,i}^{\alpha} - \omega_{j-1,i}^{\alpha})\right], \tag{2.99}$$

or

$$v_{j,i+1}^{n} = v_{j,i}^{n} - \frac{1}{4}\left(\frac{\Delta t}{\Delta x}\right)\left(\omega_{j+1,i} - \omega_{j-1,i}\right) + \frac{1}{8}\left(\frac{\Delta t}{\Delta x}\right)^{2} \sum_{\alpha=0}^{\infty} \left[(z_{j,i}^{n,\alpha} + z_{j+1,i}^{n,\alpha})(\omega_{j+1,i}^{\alpha} - \omega_{j,i}^{\alpha}) - (z_{j,i}^{n,\alpha} + z_{j-1,i}^{n,\alpha})(\omega_{j,i}^{\alpha} - \omega_{j-1,i}^{\alpha})\right]. \tag{2.100}$$

Thus we complete the derivation of Eq. (2.38), Eq. (2.39) and Eq. (2.40).

Having the propagator system, (Eq. (2.35) and Eq. (2.36)) we can compute the statistical moments as follows:
Let $n = 0$ in Eq. (1.61)

$$\xi_0 = H_0(x). \tag{2.101}$$

Taking expectation on both sides of Eq. (2.101) and using the definition of expectation from Eq. (2.37) on right hand side of Eq. (2.101) gives

$$E\Big[\xi_0\Big] = \int_{-\infty}^{+\infty} H_0(x) w(x) dx. \tag{2.102}$$

By using the definition of weight function from Eq. (1.59) and $H_0(x) = 1$, we have

$$E\Big[\xi_0\Big] = \int_{-\infty}^{+\infty} H_0(x) \frac{1}{2\sqrt{\pi}} e^{-x^2/2} dx. \tag{2.103}$$

After simplifying Eq. (2.103), we get

$$E\Big[\xi_0\Big] = 1. \tag{2.104}$$

Now we examine the cases when $n \geq 0$. Putting $n = 1$ and $H_1(x) = x$ in (2.102), it becomes

$$E\Big[\xi_1\Big] = \int_{-\infty}^{+\infty} x \frac{1}{2\sqrt{\pi}} e^{-x^2/2} dx. \tag{2.105}$$

After simplify Eq. (2.105), we get

$$E\Big[\xi_1\Big] = 0. \tag{2.106}$$

Let $n = 2$ and $H_1(x) = x^2 - 1$ in Eq. (2.102) becomes

$$E\Big[\xi_2\Big] = \int_{-\infty}^{+\infty} (x^2 - 1) \frac{1}{2\sqrt{\pi}} e^{x^2/2} dx. \tag{2.107}$$

After simplify Eq. (2.107), it gives

$$E\Big[\xi_2\Big] = 0. \tag{2.108}$$

In General

$$E\Big[\xi_n\Big] = \begin{cases} 1 & ifn = 0, \\ 0 & ifn \neq 0. \end{cases} \tag{2.109}$$

Thus the *First Moment* can be obtained by putting $n = 0$ and taking expectation on both sides of Eq. (2.22) and then using (2.109) we can say that

$$E\Big[v(x, t, \xi)\Big] = v^0(x, t), \tag{2.110}$$

where expectation and limit can be interchanged because of uniform convergence in Theorem 1.61. From Theorem 1.3.1 and Parseval's identity, *Second Moment* can be computed as follows: Squaring Eq. (2.22)

$$v^2(x, t, \xi) = \left[\sum_{n=0}^{\infty} \frac{1}{\sqrt{n!}} v^n(x, t) \xi_n\right] \left[\sum_{m=0}^{\infty} \frac{1}{\sqrt{m!}} v^m(x, t) \xi_m\right], \tag{2.111}$$

or

$$v^2(x, t, \xi) = \sum_{n=0}^{\infty} \frac{1}{\sqrt{n!}} v^n(x, t) \sum_{m=0}^{\infty} \frac{1}{\sqrt{m!}} v^m(x, t) \Big[\xi_n \xi_m\Big]. \tag{2.112}$$

Taking expectation on both sides of Eq. (2.112)

$$E\Big(v^2(x,t,\xi)\Big) = \sum_{n=0}^{\infty} \frac{1}{\sqrt{n!}} v^n(x,t) \sum_{m=0}^{\infty} \frac{1}{\sqrt{m!}} v^m(x,t) E\Big[\xi_n \xi_m\Big]. \tag{2.113}$$

Using Eq. (1.61) to find $E\Big[\xi_n\xi_m\Big]$ of Eq. (2.113) gives

$$E\Big[\xi_n\xi_m\Big] = E\left[\frac{1}{\sqrt{n!}}H_n(x)\frac{1}{\sqrt{m!}}H_m(x)\right], \tag{2.114}$$

$$= \frac{1}{\sqrt{n!}}\frac{1}{\sqrt{m!}}E\Big[H_n(x)H_m(x)\Big]. \tag{2.115}$$

But as $E\Big[H_n(x)H_m(x)\Big] = \sqrt{n!}\delta_{n,m}$ [3], Eq. (2.113) becomes

$$E\Big(v^2(x,t,\xi)\Big) = \sum_{n=0}^{\infty} \frac{1}{n!}|v^n(x,t)|^2. \tag{2.116}$$

2.3 Stability of Modified Lax-Wendroff Scheme

Any numerical scheme is said to be stable if disturbance does not grows as the calculations proceed.

Let us assume that there is no explicit dependance of f on x except through v in Eq. (2.1) describe as

$$\frac{\partial v}{\partial t} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial x}, \tag{2.117}$$

where

$$A = \frac{\partial f}{\partial v} = v \tag{2.118}$$

is constant Jacobian which depends on v itself. As in Eq. (2.1), $\partial f/\partial x = vv_x$ which on using Eq. (2.118) becomes

$$\frac{\partial f}{\partial x} = Av_x. \tag{2.119}$$

Using Eq. (2.119) in Eq. (2.1) gives

$$v_t = Av_x, \tag{2.120}$$

$$v_{tt} = A^2 \frac{\partial^2 v}{\partial x^2}. \tag{2.121}$$

Consider Taylor series expansion in terms of indices as

$$v_{j,i+1} = v_{j,i} + \frac{\partial v}{\partial t}\Delta t + \frac{\partial^2 v}{\partial t^2}\frac{(\Delta t)^2}{2!} + O(\Delta t)^3. \tag{2.122}$$

Substituting Eq. (2.120) and Eq. (2.121) in Eq. (2.122), we obtain

$$v_{j,i+1} = v_{j,i} - A\Delta t\frac{\partial v}{\partial x} + \frac{(\Delta t)^2}{2!}A^2\frac{\partial^2 v}{\partial x^2}. \tag{2.123}$$

Using central difference formula of second order for spatial derivatives in Eq. (2.120) [6]

$$v_{j,i+1} = v_{j,i} - A\Delta t\left[\frac{v_{j+1,i} - v_{j-1,i}}{2\Delta x}\right] + \frac{1}{2}(\Delta t)^2 A^2\left[\frac{v_{j+1,i} - 2v_{j,i} + v_{j-1,i}}{(\Delta x)^2}\right], \tag{2.124}$$

or

$$v_{j,i+1} = v_{j,i} - \frac{1}{2}\left(\frac{A\Delta t}{\Delta x}\right)\left[v_{j+1,i} - v_{j-1,i}\right] + \frac{1}{2}\left(\frac{A\Delta t}{\Delta x}\right)^2\left[v_{j+1,i} - 2v_{j,i} + v_{j-1,i}\right]. \tag{2.125}$$

The stability of modified Lax-Wendroff in Eq. (2.100) can be obtained by discussing the stability of scheme when Jacobian of Eq. (2.1) can assumed to be a constant [5]. Each Fourier-Hermite coefficient v^n, then satisfies deterministic Lax-Wendroff in Eq. (2.21). Thus Eq. (2.125) can be written as [5]

$$v^n_{j,i+1} = v^n_{j,i} - \frac{1}{2}\left(\frac{A\Delta t}{\Delta x}\right)\left[v^n_{j+1,i} - v^n_{j-1,i}\right] + \frac{1}{2}\left(\frac{A\Delta t}{\Delta x}\right)^2\left[v^n_{j+1,i} - 2v^n_{j,i} + v^n_{j-1,i}\right]. \tag{2.126}$$

To check stability criteria of Eq. (2.126), we have to check the stability of Eq. (2.125) by using Von-Neumann stability analysis, which is used to check out the stability requirements of any finite difference equation [6].

Let us assume that the solution of Eq. (2.1) at time step i and space step $x = x_j$ is of complex exponential type defined as $v(j,i) = e^{\iota k x_j}$, where $\iota = \sqrt{-1}$, k is real number. Then let

$$v_{j,i} = e^{\iota k x_j}, \tag{2.127}$$

also

$$v_{j+1,i} = e^{\iota k x_{(j+1)}}, \tag{2.128}$$

$$v_{j-1,i} = e^{\iota k x_{(j-1)}}. \tag{2.129}$$

Substituting Eq. (2.127), (2.128) and (2.129) in Eq. (2.125), we get

$$V_{j,i+1} = e^{\iota k x_j} - \frac{1}{2}\left(\frac{A\Delta t}{\Delta x}\right)\left[e^{\iota k x_{(j+1)}} - e^{\iota k x_{(j-1)}}\right] + \frac{1}{2}\left(\frac{A\Delta t}{\Delta x}\right)^2\left[e^{\iota k x_{(j+1)}} - 2e^{\iota k x_j} + e^{\iota k x_{(j-1)}}\right]. \tag{2.130}$$

As $x_{(j+1)} = x_j + \Delta x$ and $x_{(j-1)} = x_j - \Delta x$ then Eq. (2.130) becomes

$$V_{j,i+1} = e^{\iota k x_j} - \frac{1}{2}\left(\frac{A\Delta t}{\Delta x}\right)\left[e^{\iota k(x_j+\Delta x)} - e^{\iota k(x_j-\Delta x)}\right] + \frac{1}{2}\left(\frac{A\Delta t}{\Delta x}\right)^2\left[e^{\iota k(x_j+\Delta x)} - 2e^{\iota k x_j} + e^{\iota k(x_j-\Delta x)}\right]. \tag{2.131}$$

or

$$V_{j,i+1} = e^{\iota k x_j}\left[1 - \frac{A\Delta t}{2\Delta x}\left[e^{\iota k\Delta x} - e^{-\iota k\Delta x}\right] + \frac{1}{2}\left(\frac{A\Delta t}{\Delta x}\right)^2\left[e^{\iota k\Delta x} - 2 + e^{-\iota k\Delta x}\right]\right]. \tag{2.132}$$

Let $\theta = k\Delta x$ then

$$V_{j,i+1} = e^{\iota k x_j}\left[1 - \left(\frac{A\Delta t}{\Delta x}\right)\iota\left[\frac{e^{\iota\theta} - e^{-\iota\theta}}{2\iota}\right] + \left(\frac{A\Delta t}{\Delta x}\right)^2\left[\frac{e^{\iota\theta} + e^{-\iota\theta}}{2} - 1\right]\right]. \tag{2.133}$$

Defining trigonometric identities as

$$cos\theta = \frac{e^{\iota\theta} + e^{-\iota\theta}}{2} \tag{2.134}$$

and

$$sin\theta = \frac{e^{\iota\theta} - e^{-\iota\theta}}{2\iota}. \tag{2.135}$$

Substituting Eq. (2.134) and Eq. (2.135) in Eq. (2.133) gives

$$V_{j,i+1} = e^{\iota k x_j}\left[1 - \left(\frac{A\Delta t}{\Delta x}\right)\iota sin\theta + \left(\frac{A\Delta t}{\Delta x}\right)^2\left(cos\theta - 1\right)\right], \tag{2.136}$$

or

$$V_{j,i+1} = \lambda e^{\iota k x_j}, \tag{2.137}$$

where λ is the amplification factor defined as

$$\lambda = 1 - \Big(\frac{A\Delta t}{\Delta x}\Big)\iota sin\theta + \Big(\frac{A\Delta t}{\Delta x}\Big)^2\Big(cos\theta - 1\Big), \tag{2.138}$$

As λ has real and imaginary parts, so for stable solution, the absolute values of λ should be bounded for all values of θ. Mathematically

$$\mid \lambda \mid^2 = \lambda\bar{\lambda} \leq 1. \tag{2.139}$$

Putting the value of Eq. (2.138) in Eq. (2.139) gives

$$\mid \lambda \mid^2 = \Bigg(\Bigg(1 + \Big(\frac{A\Delta t}{\Delta x}\Big)^2\Big(cos\theta - 1\Big)\Bigg) - \iota\Big(\frac{A\Delta t}{\Delta x}sin\theta\Big)\Bigg)\Bigg(\Bigg(1 + \Big(\frac{A\Delta t}{\Delta x}\Big)^2\Big(cos\theta - 1\Big)\Bigg) + \iota\Big(\frac{A\Delta t}{\Delta x}sin\theta\Big)\Bigg), \tag{2.140}$$

$$= \Bigg(1 + \Big(\frac{A\Delta t}{\Delta x}\Big)^2(cos\theta - 1)\Bigg)^2 + \Bigg(\Big(\frac{A\Delta t}{\Delta x}sin\theta\Big)\Bigg)^2. \tag{2.141}$$

Let

$$c = \frac{A\Delta t}{\Delta x}. \tag{2.142}$$

Then Eq. (2.141) becomes

$$\mid \lambda \mid^2 = \Big(1 + c^2\Big(cos\theta - 1\Big)\Big)^2 + c^2 sin^2\theta, \tag{2.143}$$

or

$$\mid \lambda \mid^2 = \Big(1 - 2c^2 sin^2\frac{\theta}{2}\Big)^2 + 2c^2 sin^2\frac{\theta}{2}cos^2\frac{\theta}{2}. \tag{2.144}$$

After simplification, we get

$$\mid \lambda \mid^2 = 1 - 4c^2\Big(1 - c^2\Big)sin^4\frac{\theta}{2}. \tag{2.145}$$

If

$$g(\theta) = 4c^2\Big(1 - c^2\Big)sin^4\frac{\theta}{2}, \tag{2.146}$$

then Eq. (2.145) becomes

$$\mid \lambda \mid^2 = 1 - g(\theta). \tag{2.147}$$

For $\lambda \leq 1$, we must have $0 \leq g(\theta) \leq 2$, in Eq. (2.147). This gives us stability condition i.e.,

$$| c | \leq 1, \tag{2.148}$$

where

$$c = \frac{\Delta t}{\Delta x} max[v], \tag{2.149}$$

is the Courant number or CFL (Courant, Friedrichs, Lewy) condition, which is necessary (not sufficient) condition for the stability of explicit hyperbolic equation. $Max[v]$ is the maximum eigen value of jacobian matrix.

2.4 Significance of Courant Number

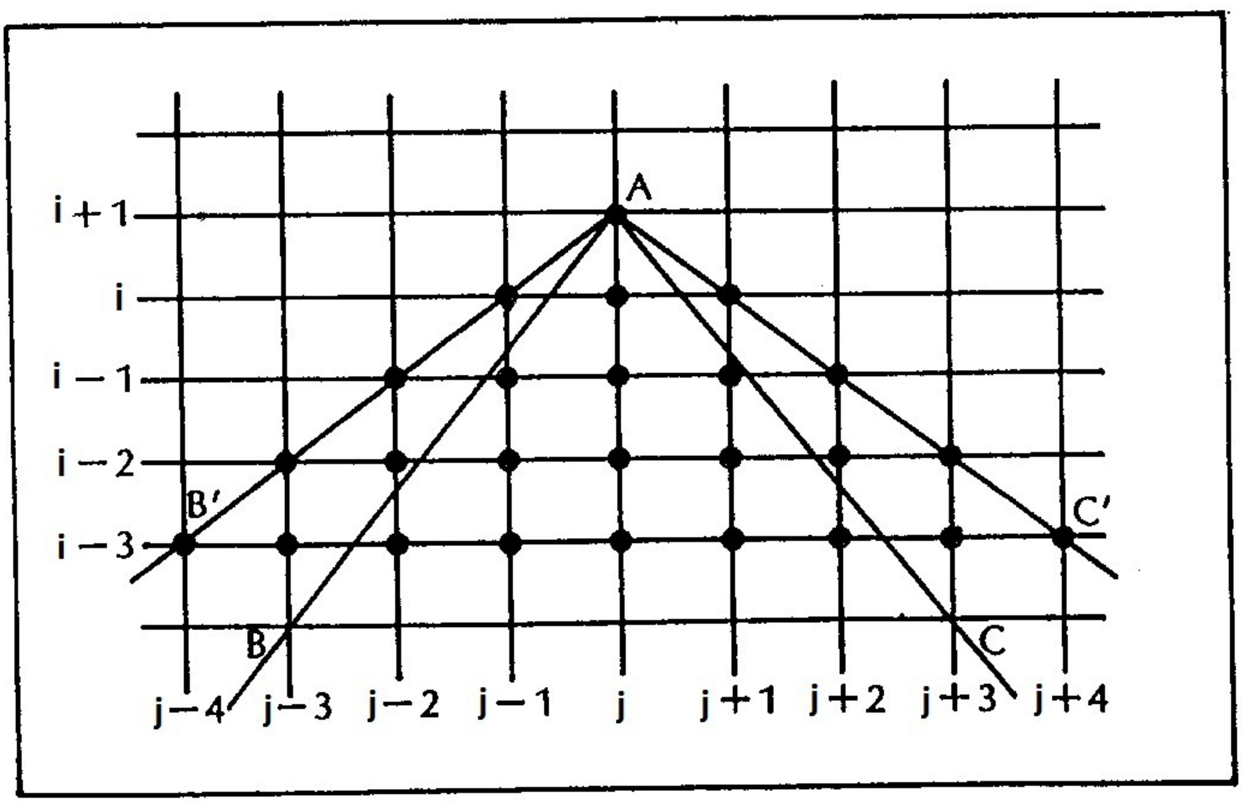

Figure 2.1: *Analytical and computational characteristic lines.*

In Fig. 2.1 lines AB and AC are characteristic of Eq. (2.1). Point A is influenced by the data of the region BAC which is called domain of dependance of point A [6]. Now consider Eq. (2.126) which is finite difference representation of Eq. (2.1). Eq. (2.126) shows that a newly calculated value at $(j, i+1)$ depends up on the values $j-1$, j, $j+1$ at time level $i+1$ and it proceeds at time level $i+2$ in a similar manner.

Eq. (2.149) indicates that for a stable solution, the domain of dependance in the finite difference scheme shown by $B'AC'$ in Fig. 2.1 must include the domain of dependance of partial differential equation shown by BAC.

Result given in Fig. 2.1 is considered to be inaccurate because it includes some unnecessary information propagating from region $B'AB$ and $C'AC$. If $c = 1$ then the numerical solution is considered to be stable because in that case both analytical and computational domain are identical. On the other hand, if the numerical domain does not contain the analytical domain then some necessary information for computation is deleted which results as an unstable solution [6].

Chapter 3

Numerical Results

Before analyzing the stability of Lax-Wendroff with Wiener Chaos expansion, we have to change the initial and boundary conditions for v^n (Fourier-Hermite coefficient) as follows:

By using initial condition $v(x, 0, \xi) = g(x, \xi)$, in Eq. (2.22) gives

$$g(x, 0, \xi) = \sum_{n=0}^{\infty} \frac{1}{\sqrt{n!}} g^n(x, 0)\xi_n, \tag{3.1}$$

where

$$g^n = \sqrt{n!} E\Big[g\xi_n\Big]. \tag{3.2}$$

Thus $g^n(x, 0)$ becomes the initial condition of $v^n(x, t)$.

Similarly, a left boundary condition $v(x_0, t, \xi) = h(t, \xi)$, can be written as

$$h(t, \xi) = \sum_{n=0}^{\infty} \frac{1}{\sqrt{n!}} h^n \xi_n, \tag{3.3}$$

where

$$h^n = \sqrt{n!} E\Big[h\xi_n\Big]. \tag{3.4}$$

Thus $h^n(t)$ is the left boundary condition for$v^n(x, t)$ at $x = x_0$.

In ref. [8], Leveque remarks that for general systems of equations with arbitrary initial data, no numerical method has been proved to be stable or convergent in general. The stability of Lax-Wendroff with Wiener Chaos expansion in Eq. (2.38) is obtained by analyzing the stability of the scheme when Jacobian of Eq. (2.1) is assumed to be constant. Now let us consider a nonlinear inviscid Burger's equation in term of randomness as

$$v_t(x,t,\xi) + \frac{\partial}{\partial x} f(v(x,t,\xi)) = 0, \tag{3.5}$$

$$v_t(x,t,\xi) + v(x,t,\xi)v_x(x,t,\xi) = 0 \tag{3.6}$$

with the boundary condition as $v(-1,t,\xi) = 1$ and *random discontinuous initial condition* as

$$v(x,0,\xi) = \begin{cases} 1 & if\ x < 0, \\ sin(x\xi) & if\ x > 0. \end{cases} \tag{3.7}$$

Where $x \in [-1,1]$ and ξ is normally distributed with zero mean and unit variance. The initial condition can be obtained from Eq. (3.1) as follows:
For $x < 0$ and $v(x,0,\xi) = 1$, then Eq. (2.23) becomes

$$v^n = \sqrt{n!}E\Big[\xi_n\Big]. \tag{3.8}$$

Using Eq. (2.109) in Eq. (3.8) we have for $x < 0$

$$v^n(x,0) = \begin{cases} 1 & if\ n = 0, \\ 0 & if\ n \neq 0. \end{cases} \tag{3.9}$$

Now for $x > 0$, Eq. (2.23) becomes

$$v^n = E\Big[sin(x\xi)\xi_n\Big], \tag{3.10}$$

and by definition of expectation, Eq. (3.10) becomes

$$E\Big[sin(x\xi)\xi_n\Big] = \int_{-\infty}^{\infty} sin(x\xi)\xi_n w(\xi)d\xi \tag{3.11}$$

using Eq. (1.61) and Eq. (1.62) in Eq. (3.11) gives

$$E\Big[sin(x\xi)\xi_n\Big] = \int_{-\infty}^{\infty} sin(x\xi)\frac{1}{\sqrt{n!}}\Big((-1)^n e^{x^2/2}\frac{d^n}{d\xi^n}e^{-\xi^2/2}\Big)\frac{1}{\sqrt{2\pi}}e^{\xi^2/2}d\xi, \tag{3.12}$$

$$= \frac{(-1)^n}{\sqrt{n!}\sqrt{2\pi}}\int_{-\infty}^{\infty} sin(x\xi)\frac{d^n}{d\xi^n}e^{-\xi^2/2}d\xi. \tag{3.13}$$

Integrating by parts, we get

$$E\Big[sin(x\xi)\xi_n\Big] = \frac{(-1)^{n+1}x}{\sqrt{n!}\sqrt{2\pi}}\int_{-\infty}^{\infty} cos(x\xi)\frac{d^{n-1}}{d\xi^{n-1}}e^{-\xi^2/2}d\xi. \tag{3.14}$$

Again integrating by parts

$$E\Big[sin(x\xi)\xi_n\Big] = \frac{(-1)^{n+1}x^2}{\sqrt{n!}\sqrt{2\pi}}\int_{-\infty}^{\infty} sin(x\xi)\frac{d^{n-2}}{d\xi^{n-2}}e^{-\xi^2/2}d\xi. \tag{3.15}$$

Now for $n = 2m+1$, $sin(x\xi)$ has $(-1)^{2m+1}$ signs.
Also

$$\frac{d^{2m+1}}{d\xi^{2m+1}}sin(x\xi) = (-1)^m x^{(2m+1)}cos(x\xi). \qquad m = 0, 1, 2, \ldots \tag{3.16}$$

Thus (3.13) becomes

$$E\Big[sin(x\xi)\xi_n\Big] = \frac{(-1)^m x^{2m+1}}{\sqrt{(2m+1)!}\sqrt{2\pi}}\int_{-\infty}^{\infty} cos(x\xi)e^{-\xi^2/2}d\xi. \qquad m = 0, 1, 2, \ldots \tag{3.17}$$

Considering

$$\int_{-\infty}^{\infty} cos(x\xi)e^{-\xi^2/2}d\xi = Re\int_{-\infty}^{\infty} e^{\iota x\xi}e^{-\xi^2/2}d\xi, \tag{3.18}$$

$$= Re\int_{-\infty}^{\infty} e^{\iota x\xi-\frac{\xi^2}{2}}d\xi, \tag{3.19}$$

$$= e^{-x^2/2}Re\int_{-\infty}^{\infty} e^{\frac{-1}{2}(\xi-\iota x)^2}d\xi. \tag{3.20}$$

Then using the result from [9] in Eq. (3.20), we obtain

$$\int_{-\infty}^{\infty} cos(x\xi)e^{-\xi^2/2}d\xi = e^{-x^2/2}\sqrt{2\pi}. \tag{3.21}$$

Using Eq. (3.21) in Eq. (3.17) gives

$$E\Big[sin(x\xi)\xi_n\Big] = \frac{(-1)^m x^{2m+1}}{\sqrt{(2m+1)!}} e^{-x^2/2}. \tag{3.22}$$

Then substituting Eq. (3.22) in Eq. (3.10) gives

$$v^n(x,0) = \frac{(-1)^m x^{2m+1}}{\sqrt{(2m+1)!}} e^{-x^2/2}, \qquad if\ n = 2m+1. \tag{3.23}$$

If $n = 2m$, then Eq. (3.10) becomes

$$v^n(x,0) = 0. \tag{3.24}$$

Thus from Eq. (3.23) and Eq. (3.24), we get for $x > 0$

$$v^n(x,0) = \begin{cases} \frac{(-1)^m x^{2m+1}}{\sqrt{(2m+1)!}} e^{-x^2/2} & if\ n = 2m+1, \\ 0 & if\ n = 2m. \end{cases} \tag{3.25}$$

For left boundary condition at $x_0 = -1$, we have

$$v(-1,t) = 1. \tag{3.26}$$

Also

$$h(t,\xi) = 1. \tag{3.27}$$

Using Eq. (3.27)

$$h^n = \sqrt{n!}E\Big[\xi_n\Big]. \tag{3.28}$$

Using Eq. (2.109) in Eq. (3.4) gives

$$h^n(t,\xi) = \begin{cases} 0 & if\ n \neq 0, \\ \\ 1 & if\ n = 0. \end{cases} \tag{3.29}$$

Thus $h^n(t,\xi)$ is the left boundary condition for $v^n(x,t)$ at $x = -1$.

For the right boundary condition at $x = 1$, we can use the following possibilities [10]

$$v_{j,i+1} = v_{j-1,i+1}, \tag{3.30}$$

$$v_{j,i+1} = 2v_{j-1,i+1} - v_{j-2,i+1}, \tag{3.31}$$

$$v_{j,i+1} = v_{j-1,i}, \tag{3.32}$$

$$v_{j,i+1} = 2v_{j-1,i+1} - v_{j-2,i-1}. \tag{3.33}$$

Eq. (3.30) and Eq. (3.31) are simple extrapolations and Eq. (3.32) and Eq. (3.33) are called quasi-characteristics extrapolations. Any error at boundary will effect the entire solution. The use of wrong boundary conditions will cause oscillations in the solution. These oscillations are usually away from the boundaries.
The only way to pinpoint the problem is to change the boundary conditions and then observe the solution. Also the incorrect boundary conditions will effect the stability of the whole scheme. Here in this scheme we use boundary conditions Eq. (3.31) on right side, at point 1 of given interval.

Using *MATLAB* code, Fig. 3.1 shows *First moment* when $n = 0$ of the solution for Eq. (3.6) computed by Lax-Wendroff with WCE when Courant number is 0.6 or 1. These solutions obtained are at $t = 1.80$ sec.

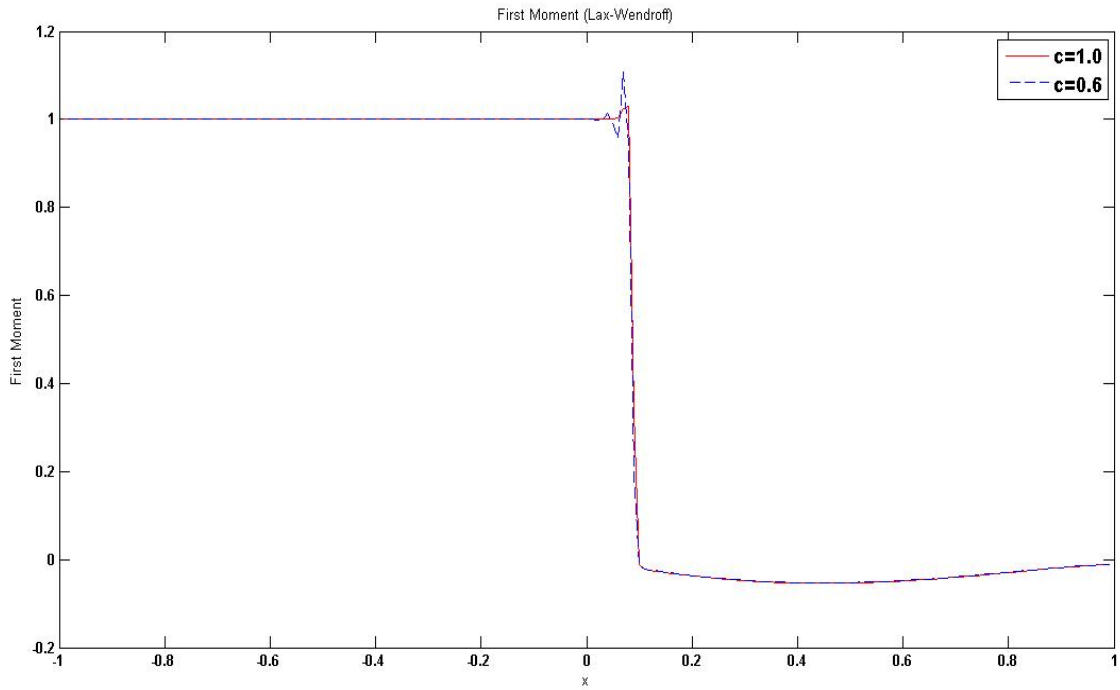

Figure 3.1: *First Moment from LW with WCE when* $c = 0.6$ *or 1.*

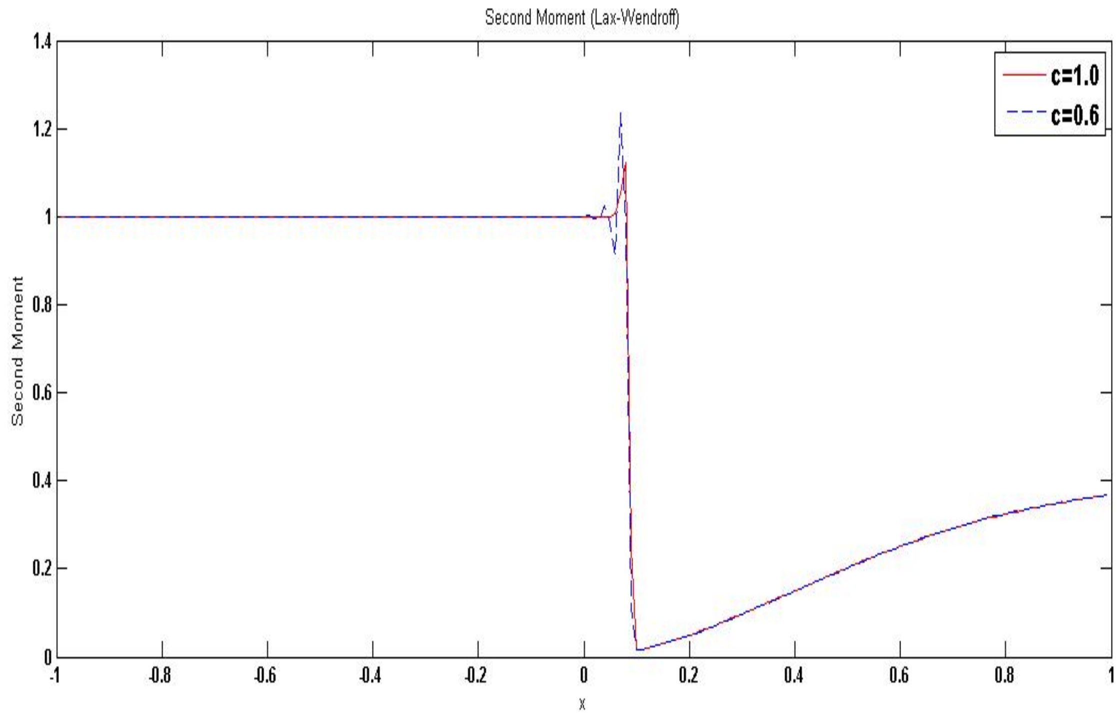

Figure 3.2: *Second Moment from LW with WCE when* $c = 0.6$ *or 1.*

Fig. 3.2 shows *second moment* for $n \geq 0$ and $t = 1.80$ sec for Eq. (3.6).

As we see that both solutions are stable but the one obtained with Courant number 0.6 is much less accurate and shows fluctuations near discontinuities. Also Lax-Wendroff is dissipative (dissipation is damping out high frequency waves which makes the solution too oscillatory) because if $c < 1$, then there is rapid exponential decay of oscillations. So, the best solution is obtained when Courant number is 1.

The solution for step sizes $\Delta x = 0.01$ and $\Delta t = 0.01$ with various $t's$ are shown in Fig. 3.3 and Fig. 3.4 for *first* and *second moments* respectively.

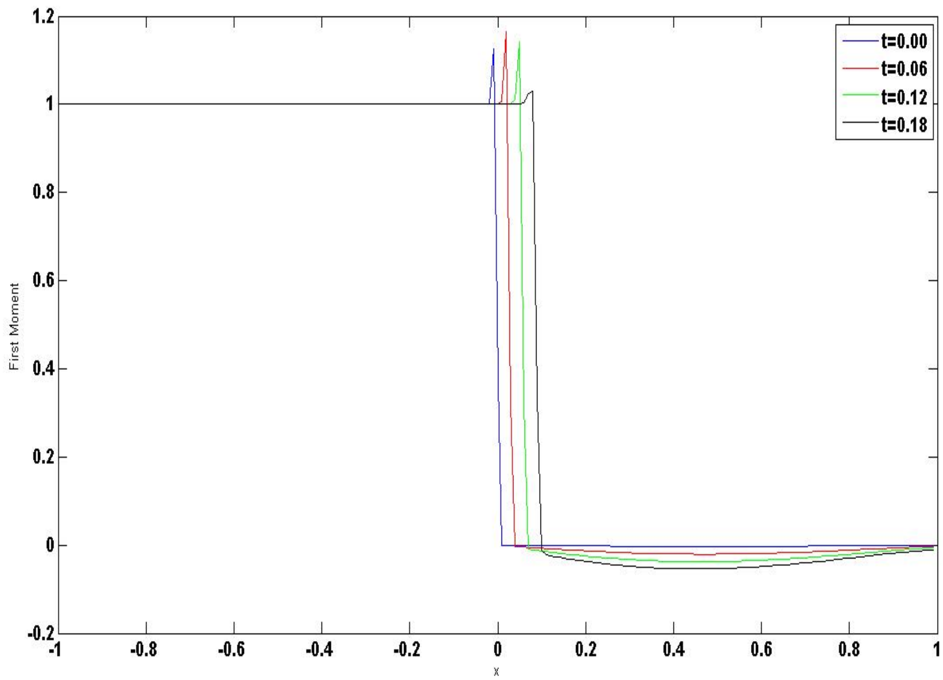

Figure 3.3: *First Moment from LW with WCE with* $\Delta x = 0.01$ *and* $\Delta t = 0.01$.

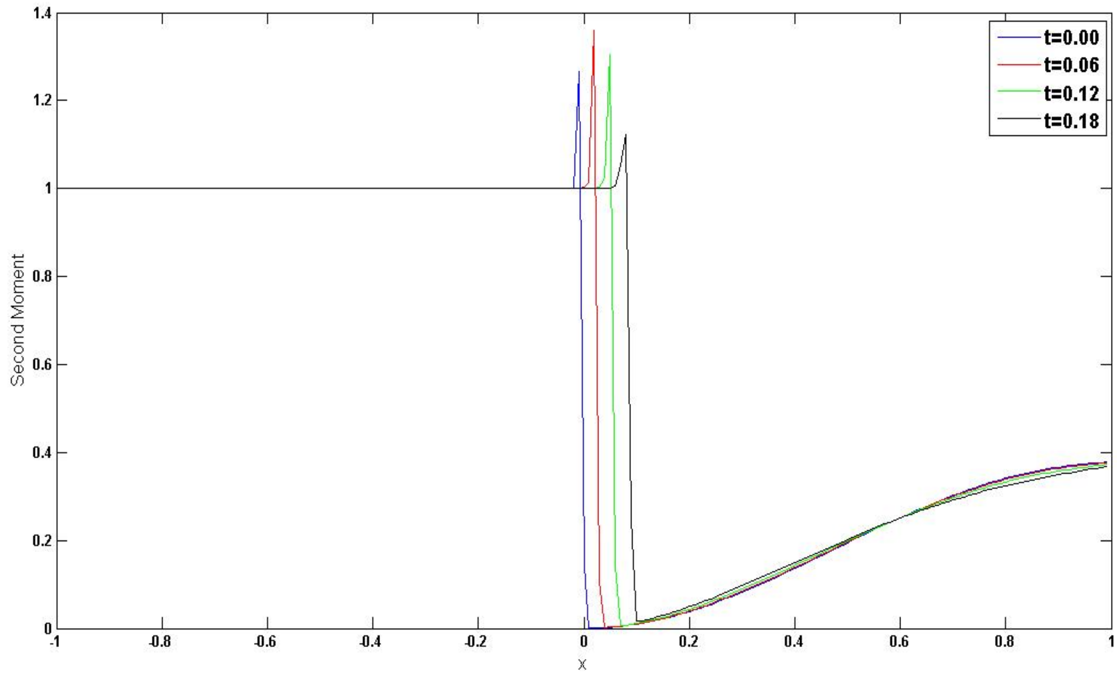

Figure 3.4: *Second Moment from LW with WCE with* $\Delta x = 0.01$ *and* $\Delta t = 0.01$.

Clearly these solutions show the propagation in horizontal direction as the time passes.

Let us consider Eq. (3.6) with the boundary condition at $x_0 = -1$ is $v(-1, t, \xi) = 1$ and *random continuous initial condition* at $t = 0$

$$v(x, 0, \xi) = \begin{cases} 1 & if\ x \leq 0, \\ cos(x\xi) & if\ x > 0, \end{cases} \tag{3.34}$$

where $x \in [-1, 1]$ and ξ is normally distributed with zero mean and unit variance. We solve Eq. (3.6) to find out *First moment* and *Second moment*.

The initial condition for $x \leq 0$ remains same as in Eq. (3.9). Now for $x > 0$, Eq. (2.23) becomes

$$v^n = E\Big[cos(x\xi)\xi_n\Big], \tag{3.35}$$

and by using the definition of expectation from Eq. (2.37) in Eq. (3.35) gives

$$E\Big[cos(x\xi)\xi_n\Big] = \int_{-\infty}^{\infty} cos(x\xi)\xi_n w(\xi)d\xi. \tag{3.36}$$

Using Eq. (1.59) and Eq. (3.36) gives

$$E\Big[cos(x\xi)\xi_n\Big] = \int_{-\infty}^{\infty} cos(x\xi)\frac{1}{\sqrt{n!}}\Big((-1)^n e^{x^2/2}\frac{d^n}{d\xi^n}e^{-\xi^2/2}\Big)\frac{1}{\sqrt{2\pi}}e^{\xi^2/2}d\xi, \tag{3.37}$$

$$= \frac{(-1)^n}{\sqrt{n!}\sqrt{2\pi}}\int_{-\infty}^{\infty} cos(x\xi)\frac{d^n}{d\xi^n}e^{-\xi^2/2}d\xi. \tag{3.38}$$

Integrating by parts, we get

$$E\Big[cos(x\xi)\xi_n\Big] = \frac{(-1)^{n+1}x}{\sqrt{n!}\sqrt{2\pi}}\int_{-\infty}^{\infty} sin(x\xi)\frac{d^{n-1}}{d\xi^{n-1}}e^{-\xi^2/2}d\xi. \tag{3.39}$$

Again integrating by parts

$$E\Big[cos(x\xi)\xi_n\Big] = \frac{(-1)^{n+1}x^2}{\sqrt{n!}\sqrt{2\pi}}\int_{-\infty}^{\infty} cos(x\xi)\frac{d^{n-2}}{d\xi^{n-2}}e^{-\xi^2/2}d\xi. \tag{3.40}$$

For $n = 2m$, $cos(x\xi)$ has $(-1)^{2m}$ signs.
Also

$$\frac{d^{2m}}{d\xi^{2m}}cos(x\xi) = (-1)^m x^{2m} sin(x\xi). \qquad m = 0, 1, 2, \ldots \tag{3.41}$$

Thus Eq. (3.38) becomes

$$E\Big[cos(x\xi)\xi_n\Big] = \frac{(-1)^m x^{2m}}{\sqrt{(2m)!}\sqrt{2\pi}}\int_{-\infty}^{\infty} sin(x\xi)e^{-\xi^2/2}d\xi. \qquad m = 0, 1, 2, \ldots \tag{3.42}$$

Considering

$$\int_{-\infty}^{\infty} sin(x\xi)e^{-\xi^2/2}d\xi = Img\int_{-\infty}^{\infty} e^{\iota x\xi}e^{-\xi^2/2}d\xi, \tag{3.43}$$

$$= Img\int_{-\infty}^{\infty} e^{\iota x\xi-\frac{\xi^2}{2}}d\xi, \tag{3.44}$$

$$= e^{-x^2/2}Img\int_{-\infty}^{\infty} e^{\frac{-1}{2}(\xi-\iota x)^2}d\xi. \tag{3.45}$$

Using the result from [9] in Eq. (3.45), we have

$$\int_{-\infty}^{\infty} sin(x\xi)e^{-\xi^2/2}d\xi = e^{-x^2/2}\sqrt{2\pi}. \tag{3.46}$$

Now using Eq. (3.46) in Eq. (3.42) becomes

$$E\Big[cos(x\xi)\xi_n\Big] = \frac{(-1)^m x^{2m}}{\sqrt{(2m)!}}e^{-x^2/2}. \tag{3.47}$$

Putting Eq. (3.47) in Eq. (3.35) gives

$$v^n(x,0) = \frac{(-1)^m x^{2m}}{\sqrt{(2m)!}}e^{-x^2/2}, \qquad if\ n = 2m. \tag{3.48}$$

If $n = 2m + 1$, Eq. (3.35) becomes

$$v^n(x,0) = 0. \tag{3.49}$$

Thus from Eq. (3.48) and Eq. (3.49), we get for x >0,

$$v^n(x,0) = \begin{cases} \frac{(-1)^m x^{2m}}{\sqrt{(2m)!}}e^{-x^2/2} & if\ n = 2m, \\ 0 & if\ n = 2m. \end{cases} \tag{3.50}$$

Fig. 3.5 and Fig. 3.6 shows the graphs of *First moment* and *Second moment* respectively when $t = 1.8$ sec.

Both graph does not show any dispersion or dissipation.

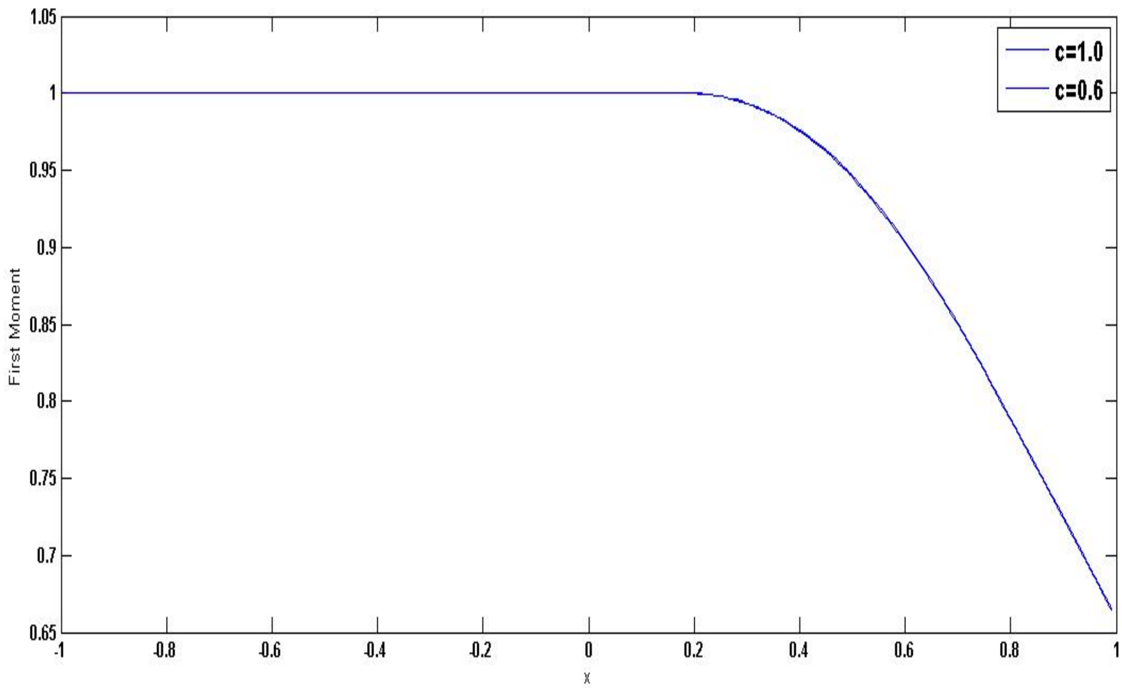

Figure 3.5: *First Moment from LW with WCE with* $c = 0.6$ *or 1.*

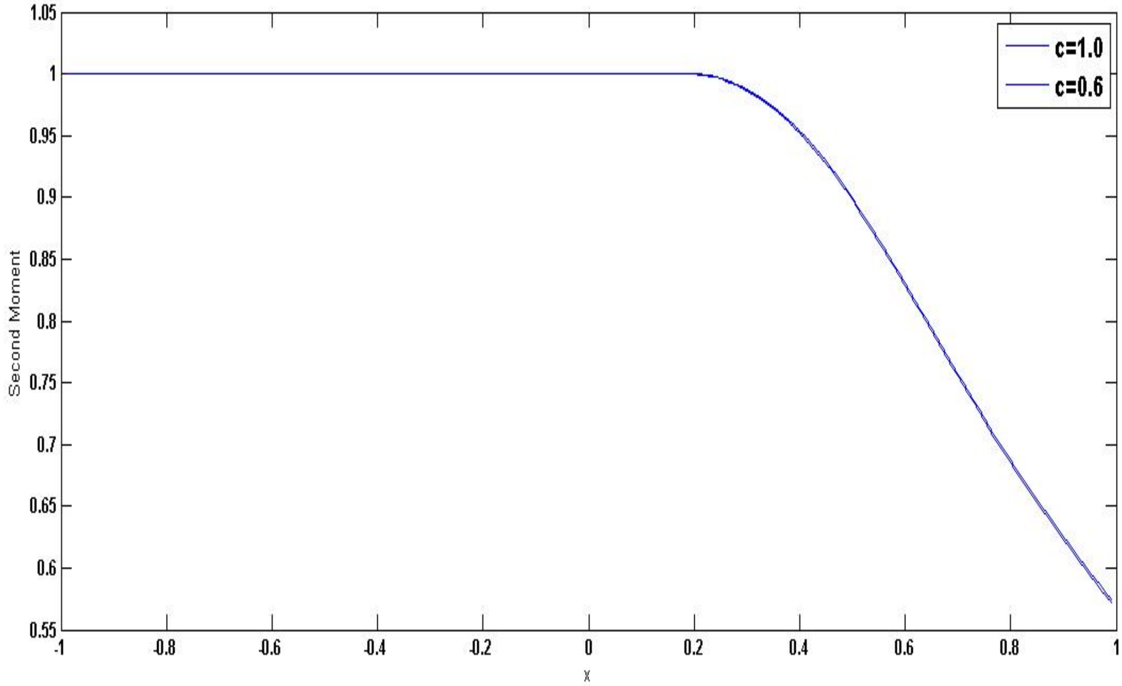

Figure 3.6: *Second Moment from LW with WCE with* $\Delta x = 0.01$ *and* $\Delta t = 0.01$.

Chapter 4

Concluding Remarks

First we derived modified Lax-Wendroff scheme for nonlinear Burger's equation with random initial conditions using Wiener Chaos expansion. Then we obtained first and second statistical moments for the problem

We see that when the initial conditions are discontinuous, the modified Lax-Wendroff scheme will exhibit oscillatory behavior. In order to implement the method with a non-oscillatory numerical scheme, predictor-corrector methods such as MacCormack method or Total Variation Diminishing (TVD) approach can be applied.

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