Fluid Flow and Heat Transfer

In a thin liquid film

by

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Dedicated

to

My parents and beloved wife

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Abstract

This dissertation is a mathematical analysis of fluid flow and heat transfer to a laminar thin liquid film of a viscoelastic fluid over a horizontal stretching sheet. An appropriate similarity transformation has been used to investigate the flow of a thin liquid film and subsequent heat transfer from the stretching sheet. The similarity transformation enables one to reduce the unsteady Prandtl's boundary layer equations to a system of non–linear ordinary differential equations. The resulting non–linear differential equations are solved via the homotopy decomposition method. The results obtained have a higher degree of accuracy as compared with the RK–4 method with shooting technique used by different authors. Boundary layer thickness is explored for some typical values of the unsteadiness parameter and magnetic parameter. Some general results of the present analysis show the effects of the Prandtl number, Eckert number, unsteadiness and magnetic parameters on the flow and the heat transfer parameters. Film thickness is decreased by increasing values of the magnetic parameter and unsteadiness parameter and vice versa. Both the parameters decrease the temperature profile and increase the shear stress by increasing their values. High values of the Prandtl number decrease the temperature profile. The effect of increasing values of the Eckert number is to increase the temperature distribution in flow region.

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S.No.	Symbol	Purpose of use	Unit	Dimension
1	x	horizontal coordinate	m	$[L]$
2	y	vertical coordinate	m	$[L]$
3	t	time	s	$[T]$
4	$U(x,t)$	sheet velocity	m/s	$[L\,T^{-1}]$
5	b	stretching rate	1/s	$[T^{-1}]$
6	α	stretching constant	1/s	$[T^{-1}]$
7	$T_s(x,t)$	temperature of sheet	Kelvin degree	$[K]$
8	T_0	temperature of the slit	Kelvin degree	$[K]$
9	T_{ref}	constant reference temperature	Kelvin degree	$[K]$
10	ν	kinematic viscosity	m^2/s	$[L^2\,T^{-1}]$
11	u	horizontal velocity component	m /s	$[L\,T^{-1}]$
12	v	vertical velocity component	m /s	$[L\,T^{-1}]$
13	ρ	mass density	kg/m^3	$[ML^{-3}]$
14	$1/\rho$	volume density	m^3/kg	$[L^3\,M^{-1}]$
15	C_p	specific heat	$m^2/(s^2.K)$	$[J\,kg^{-1}\,K^{-1}]$
16	h	film thickness	m	$[L]$
17	$\psi(x,y,t)$	stream function	m^2/s	$[L^2\,T^{-1}]$
18	μ	dynamic viscosity	kg/(m.s)	$[M/L.T]$
19	B_0	strength of applied magnetic field	A/m	$[A/L]$

Table 2: Nomenclature of non dimensional quantities

S.No.	Symbol	Purpose of use	Unit	Dimension
20	σ	electrical conductivity	$s^3.A^2/(m^3.kg)$	$[T^3.A^2/(L^3.kg]$
21	η	similarity variable	-	$\eta = \sqrt{\frac{b}{\nu(1-\alpha t)}}y$
22	$f(\eta)$	similarity function	-	$f(\eta) = \frac{\psi(x,y,t)}{\sqrt{\frac{\nu b}{(1-\alpha t)}}x}$
23	$\theta(\eta)$	temperature distribution function	-	$\theta(\eta) = \frac{T_0 - T(x,y,t)}{T_{ref}(\frac{bx^2}{2\nu})(1-\alpha t)^{\frac{-3}{2}}}$
24	β	dimensionless film thickness	-	$\beta = \sqrt{\frac{b}{\nu(1-\alpha t)}}h$
25	S	unsteadiness parameter	-	$\frac{\alpha}{b}$
26	P_r	Prandtl number	-	$P_r = \frac{\nu\rho C_p}{k}$
27	E_c	Eckert number	-	$E_c = \frac{U^2}{C_p(T_s - T_0)}$
28	M_n	magnetic parameter	-	$M_n = \frac{\sigma B_0^2}{\rho b}$
29	C_f	skin friction	-	$C_f \equiv -2R_e^{-\frac{1}{2}}f''(0)$
30	Nu_x	Local Nusselt number	-	$\frac{1}{2}(1-\alpha t)^{-\frac{1}{2}}\theta'(0)R_e^{\frac{3}{2}}$
31	R_e	Reynolds number	-	$R_e = \frac{Ux}{\nu}$
32	$T(x,y,t)$	temperature of fluid particle	-	$T_0 - \frac{T_{ref}(\frac{bx^2}{2\nu})\theta(\eta)}{(1-\alpha t)^{\frac{3}{2}}}$
33	$\frac{\partial}{\partial x}$	local change	-	-
34	$\frac{\partial}{\partial t}$	Local change	-	-
35	$\frac{D}{Dt}$	Particle derivative or Total derivative	-	-

Chapter 1

Introduction

1.1 Motivation

Thin liquid films have many applications in different branches of science and technology. They have attracted the attention of number of researchers. The knowledge of fluid flow and heat transfer within a thin liquid film is crucial to understand the coating process, design of various heat exchangers, chemical processing equipments, food stuff processing, wire and fiber coating and cooling of plastic sheets. The prime aim in almost every extrusion application is to maintain the surface quality of the extrudate. All the coating processes demand a smooth glossy surface to meet the requirements for best appearance and optimum service properties such as low friction, transparency and strength.

The problem of extrusion of thin surface layers needs special attention to gain some knowledge for cooling the coating product efficiently. Heat transfer may alter the results appreciably due to viscous dissipation if the fluid is very viscous in the extrusion of plastic. In most problems of polymer extrusion, the flow is induced by the stretching motion of the elastic sheet. For example, in a melt spinning process the extrudate from the die is generally drawn and simultaneously stretched into a filament or sheet. That sheet is solidified with coolant liquid or by direct contact of water by gradual cooling. The quality of the final product greatly depends on the rate of cooling and the rate of stretching. The choice of an appropriate cooling liquid has a vital role as it has a direct impact on the rate of cooling. Care must be taken to exercise optimum stretching rate. Sudden stretching may spoil the properties desired for the final outcome. Some important liquids like synthetic oils, dilute polymeric solutions such as 5.4 % of polyisobutelene in cetane can be used as effective coolant liquids [1]. The flow and heat transfer characteristics of a thin liquid film over a stretching sheet considerably affect the quality of the final product in such extrusion processes. So the analysis and fundamental understanding of the momentum and thermal transports for such processes are very important.

1.2 Literature survey

B.C. Sakiadis [2] was the pioneer researcher who work on various aspects of the stretching problem involving Newtonian and non–Newtonian fluids. Crane [3] was the first among others who consider the steady two dimensional flow of a Newtonian fluid driven by a stretching elastic flat sheet which moves in its own plane with a velocity varying linearly with a distance from a fixed point. Prandtl's boundary layer theory proved to be of great use in Newtonian fluids as Navier-Stokes equations can be converted into simplified boundary layer equations that are easier to handle. Many authors extended Crane's work to explore various aspects of flow and heat transfer occurring in an infinite domain of the fluid surrounding the stretching sheet [4–17]. Sarpakaya [18] was the pioneer researcher to study the magnetohydrodynamic (MHD) flow of a non-Newtonian fluid.

Wang [19] was first who considered the hydrodynamics of a flow in a thin liquid film driven by an unsteady stretching surface. Wang himself [20, 21] reduced the unsteady Navier–Stokes equation to a non–linear ordinary differential equation with the help of a similarity transformation and solved the same using a kind of multiple shooting method (see [22]). Lio [23] has used homotopy analysis method to re-investigate the thin film flow over a stretching sheet. Of late the works of Wang [24] on finite fluid domain are extended by several authors [25–30] for fluid of both Newtonian and non–Newtonian kinds using various velocity and thermal boundary conditions. There are extensive works in literature concerning the production of thin fluid film either on vertical wall achieved through the action of gravity or that over a rotating disk achieved through the action of centrifugal forces. Sparrow and Gregg [31] considered the problem of laminar film condensation on a vertical plate. They were pioneer to solve this problem. The solution given by them is based on the boundary layer theory and similarity transformation. Dandput *et al.* [32] have investigated the liquid film over an unsteady stretching sheet. Hayyat *et al.* [33] have studied the flow of second grade fluid film over an unsteady stretching sheet. Chin [34, 35] has discussed heat transfer in a power law fluid film over an unsteady stretching sheet and the effects of viscous dissipation on heat transfer in a non-Newtonian liquid film over an unsteady stretching sheet. Abel and Tawade [36] analyzed the heat transfer in a liquid film over an unsteady stretching surface with viscous dissipation in presence of external magnetic field. Mamaloukas and Abel [37] discussed the effect of the Prandtl number and magnetic parameter on viscous flow parameters. They have not discussed the effect of the Eckert number for the different values of unsteadiness parameter and magnetic parameter.

In this dissertation we have solved the problem of fluid flow and heat transfer in a liquid film over an unsteady stretching sheet by the Decomposition method connected with homotopy. The important observation in this study is that the sheet temperature is reduced for increasing values of the Prandtl number and the Eckert number. Our results have a higher degree of accuracy than the results of Wang [24], Anderson [26], Subhas [36] and Mamaloukas [37]. The effects of the Eckert number and Parandtl number on the flow and heat transfer have been discussed for different values of the unsteadiness parameter and magnetic parameter.

1.3 Contribution of this dissertation

The main contribution of this dissertation is to solve a system of boundary value problems consisting of a third order ordinary differential equation for flow problem and a second order differential equation for heat transfer via the homotopy decomposition method. The effects of the four parameters, namely unsteadiness parameter, magnetic parameter, the Prandtl number and the Eckert number on the flow and heat transfer have been analyzed.

1.4 Plan of the dissertation

In chapter 2, the preliminaries along with examples have been given which are the pillars of the later work. The terminology which is used in chapter 2, will work throughout this dissertation. This chapter demands the basic knowledge of description of flow, general theory of stress and strain, conservation of mass, conservation of momentum, constitutive equations, Navier–Stokes equations, boundary layer theory and thermal boundary layer equations. In chapter 3.1, we have described a problem of magnetohydrodynamic (MHD) flow and heat transfer in a liquid film of viscoelastic incompressible fluid over a horizontal stretching sheet and have analyzed the problem with four parameters. The effects of the unsteadiness parameter, magnetic parameter, Prandtl number and Eckert number on the flow and heat transfer have been discussed.

Chapter 2

Preliminaries

Fluid motion has been analyzed by the assumption that fluid under consideration forms a physical continuum. A physical continuum is a medium filled with a continuous matter such that every part of the medium, however small, is itself a continuum and entirely filled with matter. While considering the motion of fluids, it is helpful to keep an infinitesimal volume of fluid as a geometrical point in a mathematical continuum of numbers. The laws of fluid motion can be described by the Euclidean space. Euclidean space is sufficient to describe the laws of fluid motion because this space is a curvature free space in which set of rectangular cartesian coordinates can always be introduced on a global scale. One can introduce any other system of coordinates in this space without altering the nature of space itself. In motion of fluid, the speed encounter is much smaller than the speed of light so that the relativistic effects are negligibly small. To describe any kind of motion, a reference coordinate system is needed. Since fluid motion does not require any relativistic considerations, one can take time as an absolute quantity common to any frame of reference at rest [38]. In this dissertation we have discussed Navier–Stokes equation for motion of fluid. Boundary layer theory have been used to derive boundary layer equations with the help of order of magnitude approach. Model analysis and the Reynold's number have been explained for the requirement of the problem. Basics of heat transfer, energy conservation, thermal boundary layer equations for heat transfer have also been discussed.

The following part of this chapter deals with some basic definitions and notions about the fluid flow and heat transfer. All the definitions and notions have been taken from the Literature given in references [38–44].

2.1 Basic definitions and notions

Configuration: The set of coordinates that describes the position of all the particles of a substance is known as a configuration of that substance.

Deformation: If the substance undergoes some change from any initial configuration then the change in the substance from an initial configuration to the current configuration is defined as deformation. The deformation can take place because of external forces, body forces and temperature variations.

Flow: When a force is exerted on a substance, its configuration changes and thus it undergoes deformation. If the deformation takes place continuously then this phenomenon is called a flow.

Fluid: In every day life, three states of matter are recognized: solid, liquid and gas. Liquids and gases have flow property while solids do not have that characteristic. On the basis of flow property, liquids and gases are known as fluids. Fluids flow under the action of some internal forces and flow continuously as long as these forces are in action. These forces are known as shearing forces. Fluids flow under its own weight and take the shape of any body with which it comes into contact. Hence fluid is defined as "a substance which deforms continuously or we say it is flowing when some shearing forces are applied, no matter how these forces are" [38].

Fluid mechanics: It is a branch of engineering science which deals with the behavior of fluids under the conditions of rest and motion. One can study the fluid behavior with and without influence of the forces. If the forces are absent the study is characterized as fluid kinematics and otherwise fluid dynamics. Furthermore, the effect of forces could be discussed in both the ways whether fluid is at rest or in motion. However, in this dissertation we are dealing with the fluid in motion.

Magnetohydrodynamics (MHD): The subject which deals with the mutual interaction of fluid flow and magnetic field is called magnetohydrodynamics [38].

Continuum assumption: It is well known that matter is made up of molecules or atoms which are always in the state of random motion. In fluid dynamics the study of individual molecule is neither necessary nor appropriate from the point of view of use of mathematical methods. Hence one can consider the macroscopic (bulk) behavior of fluids by assuming that fluids are continuously distributed over a given space [38]. This assumption is known as continuum assumption. This continuum concept of matter allows one to subdivide a fluid into fluid elements indefinitely.

Fluid element or fluid parcel or material element: It is a very small amount of fluid. Fluid element is defined as the fluid contained within the infinitesimal volume [38].

Thin liquid film: A microscopically thin layer of material that is deposited onto a metal, ceramic, semiconductor or plastic base. Typically less than one micron thick, thin films can be conductive or dielectric (non-conductive) and are used in myriad applications.

For example: The top metallic layer on a chip and the coating on magnetic disks are thin films. Thin films of photovoltaic material using silicon, cadmium telluride and other elements are used to make solar panels and solar roof shingles.

Since Kinematics is the branch of science that deals with the motion apart from the consideration of mass and forces so we describe the kinematics of a fluid.

2.1.1 Kinematics of fluid

Lagrangian and Euler method of description: In Lagrangian method of description of fluid motion, a particular fluid particle is identified and changes in velocity, acceleration etc are studied as that fluid move onwards. On the other hand, in the Eulerian method of description, the individual fluid particle is not identified. Instead, a point in a fluid is chosen and changes in velocity, acceleration etc are studied as the fluid passes through the chosen fixed point. We have adopted here the Lagrangian method of describing fluid motion. We take a fluid particle and discuss its velocity and acceleration.

Velocity of fluid particle: Let us consider the fluid particle be at a point $P(x, y, z)$ at any time t and vector $\mathbf{r}$ denotes its position. This particle is displaced at another point $Q(x+\delta x, y+\delta y, z+\delta z)$ at time $t+\delta t$. Then the movement of particle in the time interval δt is described by change of displacement PQ and denoted by $\delta\mathbf{r}$. The rate of change of displacement is called velocity and one can denote this velocity of fluid particle by $\mathbf{q}$. Mathematically we write

$$\mathbf{q} = \lim_{\delta t\to 0}\frac{\delta\mathbf{r}}{\delta t} = \frac{d\mathbf{r}}{dt}.$$

Clearly $\mathbf{q}$ is a function of position vector $\mathbf{r}$ and time t. Hence $\mathbf{q}$ can be expressed as

$$\mathbf{q} = \frac{dx}{dt}\mathbf{i} + \frac{dy}{dt}\mathbf{j} + \frac{dz}{dt}\mathbf{k} = u\mathbf{i} + v\mathbf{j} + w\mathbf{k},$$

which implies that

$$u = \frac{dx}{dt},\ v = \frac{dy}{dt},\ w = \frac{dz}{dt},$$

where u, v, w are called components of velocity of fluid particle along the coordinate axes.

Material, local and convective derivatives: Suppose $P(x, y, z)$ is any point within the fluid and the velocity components of the fluid element are the function of position and time t, that is,

$$u = f(x, y, z, t), \quad v = g(x, y, z, t), \quad w = h(x, y, z, t).$$

In a short interval of time δt, let particle which is at P moves to the point $Q(x, y, z, t)$ by covering a distance: $u\delta t$ in the x-direction, $v\delta t$ in the y-direction and $w\delta t$ in the z-direction. Then coordinates of Q are $(x+u\delta t, y+v\delta t, z+w\delta t)$. If $u+\delta u$ be the x-component , $v+\delta v$ be the y-component, $w+\delta w$ be the z-component of velocity at Q. Then,

$$\begin{aligned} u + \delta u &= f(x + u\delta t, y + v\delta t, z + w\delta t, t + \delta t),\\ v + \delta v &= g(x + u\delta t, y + v\delta t, z + w\delta t, t + \delta t),\\ w + \delta w &= h(x + u\delta t, y + v\delta t, z + w\delta t, t + \delta t).\end{aligned} \tag{2.1.1}$$

Applying Taylor's series to first component of equation (2.1.1) up to first order partial derivative

$$\begin{aligned} u+\delta u =& f(x,y,z,t)+\delta t\frac{\partial f(x,y,z,t)}{\partial t}+u\delta t\frac{\partial f(x,y,z,t)}{\partial x}+v\delta t\frac{\partial f(x,y,z,t)}{\partial y} \\ &+w\delta t\frac{\partial f(x,y,z,t)}{\partial z}+\cdots, \\ u+\delta u =& u+(\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z})\delta t+O(\delta t)^2. \end{aligned}$$

Let a_x, a_y and a_z be the components of the acceleration of the element of fluid at P. Then we have

$$a_x=\lim_{\delta t\to 0}\frac{\delta u}{\delta t}=(\frac{\partial}{\partial t}+u\frac{\partial}{\partial x}+v\frac{\partial}{\partial y}+w\frac{\partial}{\partial z})u=\frac{Du}{Dt},$$

where $\frac{D}{Dt}=(u\frac{\partial}{\partial x}+v\frac{\partial}{\partial y}+w\frac{\partial}{\partial z}+\frac{\partial}{\partial t})$, which is known as Particle derivative, Material derivative or Substantial derivative. Similarly, we have:

$$a_y=(\frac{\partial}{\partial t}+u\frac{\partial}{\partial x}+v\frac{\partial}{\partial y}+w\frac{\partial}{\partial z})v=\frac{Dv}{Dt},\quad a_z=(\frac{\partial}{\partial t}+u\frac{\partial}{\partial x}+v\frac{\partial}{\partial y}+w\frac{\partial}{\partial z})w=\frac{Dw}{Dt}.$$

One can represent this derivative in vector notation as

$$\frac{Du}{Dt}=\frac{\partial u}{\partial t}+(\mathbf{q}\cdot\nabla)u, \tag{2.1.2}$$

where $\mathbf{q}=u\mathbf{i}+v\mathbf{j}+w\mathbf{k}$, $\nabla=\frac{\partial}{\partial x}\mathbf{i}+\frac{\partial}{\partial y}\mathbf{j}+\frac{\partial}{\partial z}\mathbf{k}$. The term $\frac{\partial}{\partial t}$ is called local derivative and it is associated with time variation at a fixed position, the term $(\mathbf{q}\cdot\nabla)$ is called convective derivative and it is associated with the change of a physical quantity (fluid particle) due to motion. Similarly, we can write for the components v, w

$$\frac{Dv}{Dt}=\frac{\partial v}{\partial t}+(\mathbf{q}\cdot\nabla)v \text{ and } \frac{Dw}{Dt}=\frac{\partial w}{\partial t}+(\mathbf{q}\cdot\nabla)w. \tag{2.1.3}$$

Combining equations (2.1.2) and (2.1.3) with vector notation

$$\frac{D\mathbf{q}}{Dt}=\frac{\partial\mathbf{q}}{\partial t}+(\mathbf{q}\cdot\nabla)\mathbf{q}.$$

Here $\frac{D\mathbf{q}}{Dt}$ shows the acceleration of fluid particle of a fixed identity which is rate of change of velocity.

2.1.2 Types of force

Force: It is an influence that causes a change of direction, change of speed or change of shape of an object. In fluid mechanics we come across different forces. In this subsection we briefly explain them one by one.

Surface force: It is a type of force which acts on the inner and outer surface of the fluid element and which is proportional to the surface area of the element on which it acts. Surface force arises due to action of surrounding fluid (through direct contact) on the element under consideration. Thus it is a boundary or surface action. This force is expressed as "force per unit surface area of the chosen element". For example, stress force on fluid.

Stress force on fluid: A fluid has the property that it is deformable, then the stress is a measure of internal forces acting on a fluid. In the quantitative terms, it is the amount of average force per unit area of a control volume within the fluid. A fluid is continuum by continuum assumption. So the stress forces are distributed continuously in all possible directions. The intensity of stress is expressed in unit of force divided by unit of area. In a three dimensional space the stress force has nine components and is represented by a tensor of rank 2 which is denoted by τ_{ij}, where $i, j = x, y, z$.

Shear stress: A shear stress is defined as a stress which is applied on parallel or tangential direction to a surface of a material (fluid). Usually shear stresses are denoted by $\tau_{xy}, \tau_{xz}, \tau_{yx}, \tau_{yz}, \tau_{zx}, \tau_{zy}$.

Normal stress: The stress applied in normal direction to the surface of a material (fluid) is known as normal stress. Usually normal stresses are denoted by $\tau_{xx}, \tau_{yy}, \tau_{zz}$.

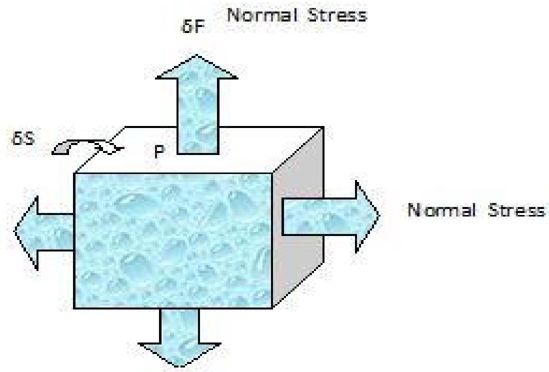


Figure 2.1: Stress force on surface of fluid body.

Strain: Whenever a stress is applied to a body (fluid) it deforms. This deformation is called strain. The tensor that measures this deformation is called strain tensor. In solids deformation takes place in one step, however, for fluids the deformation is continuous and we deal with rate of strain.

Body force: It is a type of force which is proportional to mass (or possibly the volume) of the fluid on which it acts. The body force is distributed throughout the volume of the body and this force is usually expressed as "force per unit mass of the element".

For example: gravity and inertia forces. Body forces may arise from other physical reasons such as electric and magnetic force.

Inertial force: This force arises due to the acceleration of the body itself and therefore not require any physical interaction. It is also called fictitious or pseudo or d'Alembert force. Fictitious forces are always proportional to mass and according to Newton second law

$$\mathbf{F} = m\mathbf{a}, \tag{2.1.4}$$

where $\mathbf{a}$ is acceleration.

2.1.3 Some basic properties of fluids

Mass density: For any material, mass density is defined as the quantity of matter (mass of the fluid) in a unit volume at a given temperature and pressure [40]. It is denoted by ρ. Here the concept of continuum will be useful and not the properties of individual molecule.

Consider a point P in the fluid having coordinates (x, y, z) in an Euclidean space. Take a small volume δv about the point P. Denote the mass of this small volume element by δm. Let x denote the linear dimension of the volume element which is large compared with the mean distance between molecules. The mass density ρ at a point P is the limiting value as the unit volume δv tends to x^3, that is

$$\rho = \lim_{\delta v \to x^3} \frac{\delta m}{\delta v}, \tag{2.1.5}$$

its unit in system international (SI) is kg/m^3. The fluid having less density will float over the fluid having more density provided we have two different fluids having different densities and mixing does not occur. Mass density varies with variation in temperature and pressure. However, for solids and liquids this variation is negligible and we take it as constant.

Pressure: When fluid is contained in a vessel, it exerts a force at each point of the inner side of the vessel. This normal force per unit area acting on a real or imaginary surface in the fluid is defined as pressure, that is,

$$\text{Pressure} = \frac{\text{force exerted}}{\text{area of boundary}}.$$

If the force F exerted on each unit area S of the boundary is the same, the pressure is said to be uniform. It is denoted by p. Symbolically we can write $p = \frac{F}{S}$. In most cases of fluid problems, it is observed that the pressure changes from point to point. Mathematically, consider the element of force δF normal to a small area δS surrounding the point under consideration: Mean pressure $= \frac{\delta F}{\delta S}$. Take limit $\delta S \to 0$ but δS remains large enough to preserve the concept of the fluid as continuum. Pressure at a point $P(x, y, z)$ is

$$p = \lim_{\delta S \to 0} \frac{\delta F}{\delta S} = \frac{dF}{dS}, \tag{2.1.6}$$

its unit in system international (SI) is N/m^2.

Incompressibility: The changes in pressure also occurs in every liquid problem. These changes are sufficiently large. They are not so large to cause appreciable changes in density. So these changes are usually ignored and liquids are treated as incompressible. A fluid is said to be incompressible if it requires large change in pressure to produce some appreciable change in density [40]. We have used this concept in remark given in section 2.2.

Viscosity: One can observe that flow of water and air is much easier than syrup and heavy oils. This demonstrate the existence of a property in the fluid, which controls its rate of flow. This property of fluids is said to be viscosity or internal resistance (friction). Newton says that viscosity is due to molecular diffusion between layers in the fluid. When a molecule leaves one layer, it transfers its momentum to the adjoining layer. That transfer creates an acceleration and that acceleration creates shear forces which are related to the viscosity. Hence viscosity is defined as “it is a representative of internal fluid friction occurs due to the motion of inter connected layers of fluid and thus causing a resistance to the fluid flow” [40].

There are two types of viscosity.

(a) **Dynamic viscosity:** In a particular context of fluid dynamics, dynamic viscosity is the ratio of shearing stress to the velocity gradient in a fluid. This definition comes from Newton’s law of viscosity which states that the resulting shear stress is directly proportional to the deformation rate. The constant of proportionality in this case is known as dynamic viscosity or absolute viscosity or coefficient of viscosity. It is denoted by symbol μ and defined as

$$\mu = \frac{\tau_{xy}}{\frac{du}{dy}}, \tag{2.1.7}$$

where τ is a shear stress and $\frac{du}{dy}$ is a velocity grad.

(b) **Kinematic viscosity:** In a continuum description, we are interested in diffusion of momentum which is characterized by the ratio of dynamic viscosity μ to the density ρ of the fluid. It is denoted by ν and expressed as

$$\nu = \frac{\mu}{\rho}. \tag{2.1.8}$$

2.1.4 Types of fluids

Viscous fluid or real fluid: When normal as well as shear forces (stresses) exist in a fluid, then the fluid is said to be viscous fluid. A viscous fluid has non-zero coefficient of viscosity $\mu \neq 0$. For example: syrup and heavy oil are treated as viscous fluids.

Viscoelastic fluids: When the applied stress is released, some fluids after deformation partially return to their original shape, such fluids are known as viscoelastic fluids.
For example: paint, crude oil (engine oils), asphalt, cosmetics, biological fluids (blood, protein solutions), toothpaste, grease, foodstuffs (ketckup, dough, salad dressing, egg white), plastics (polymer melts, rubbers and polymeric liquids) are viscoelastic fluids.

In solids, shear stress is a function of strain but in fluids, shear stress is a function of rate of strain. Depending on this relationship, fluids can be characterized for modeling as:

(a) **Newtonian fluid:** A Newtonian fluid is a fluid in which viscosity remains constant for all shear rates when the constant conditions of temperature and pressure are maintained. For such fluids we say that applied shear stress is directly and linearly proportional to rate of deformation. Mathematically,

$$\tau = \mu \frac{du}{dy}.$$

This law is known as Newton's law of viscosity. For example: water, air, ethnol, benzene etc are Newtonian fluids.

(b) **Non–Newtonian fluid:** A non-Newtonian fluid is a fluid whose flow properties are not described by a single value of viscosity. In such cases viscosity is not constant but depends upon the shear stress. If the viscosity decreases with an increase in applied shear stress we call the fluid as shear thinning and if the viscosity increases the fluid is termed as shear thickening. For non-Newtonian fluids we say that applied shear stress is not proportional to rate of strain but its higher powers and its derivatives. In mathematical terms

$$\tau = k(\frac{du}{dy})^n,$$

where k is consistency index and n is flow behavior index. Above equation can also be written as

$$\tau = k(\frac{du}{dy})^{n-1}\frac{du}{dy} = \eta\frac{du}{dy},$$

where η is apparent viscosity. This relation is also known as constitutive relation. Due to complexity of fluids that exists in nature, a single constitutive relation is not possible that describes all the features of non–Newtonian fluids. For example: Tooth paste, shampoo, gel, greases, lubricating oils, paints, blood, molten polymers and polymer solutions are non-Newtonian fluids.

Viscoelastic materials behave in a manner similar to Newtonian fluids under time-invariant conditions.

2.1.5 Types of flow

Laminar flow: If each fluid particle traces out a definite curve and curves traced out by any two different fluid particles do not overlap (intersect) each other then this type of flow is said to be laminar flow.

Unsteady flow: A flow in which properties and condition ϕ (say) associated with motion of fluid depend on time so that flow pattern varies with time is said to be unsteady flow. Symbolically,

$$\frac{\partial \phi}{\partial t} \neq 0. \tag{2.1.9}$$

ϕ may be velocity, density, pressure, temperature etc.

Incompressible flow: A flow in which volume density of the flowing fluid does not change with respect to time and space coordinates during the flow is said to be incompressible flow. All common liquids are generally incompressible and all gases are compressible. Mathematically, if $\mathbf{q}$ is the velocity vector of fluid particle then

$$\nabla.\mathbf{q} = 0. \tag{2.1.10}$$

2.2 Law of conservation of mass (equation of continuity)

The equation of continuity is the mathematical form of the law of conservation of mass. The law states that "fluid mass can neither be created nor destroyed". Thus in continuous motion, equation of continuity expresses the fact " the rate at which the mass enters into the system is equal to the rate at which mass leaves the system". The continuity equation expresses the fact that the flow of fluid is continuous; it has no break in it. Equation of continuity (law of conservation of mass) in mathematical form is written as

$$\frac{D\rho}{Dt} + \rho\nabla\cdot\mathbf{q} = 0. \tag{2.2.1}$$

In cartesian coordinate system, this equation holds at all points of fluids.

Remark

- If the fluid is incompressible then ρ is constant, that is $\frac{D\rho}{Dt} = 0$. Then equation (2.2.1) in cartesian coordinate system implies $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$.
- If there is motion in two dimensional xy-plane (say), then equation of continuity in two dimensional flow is

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0. \tag{2.2.2}$$

2.3 Pattern of flow

Streamlines provide a fundamental and valuable tool for visualizing two dimensional (2D) or three dimensional (3D) flow fields. The concept of streamlines is useful because it enables the fluid flow in patterns of stream lines.

Streamlines: A stream line is a curve drawn in a fluid so that its tangent at each point is in the direction of motion (i.e fluid velocity) at that point. Since the tangent is taken as straight line, consider a point $P(x, y, z)$ on a straight line. Let $\mathbf{r}$ be the position vector of a point P. Let $\mathbf{q} = u\mathbf{i} + v\mathbf{j} + w\mathbf{k}$ be the fluid velocity at P. In case of stream line, tangent $\mathbf{dr} = dx\mathbf{i} + dy\mathbf{j} + dz\mathbf{k}$ is parallel to the fluid velocity $\mathbf{q}$. Then

$$\mathbf{q} \times \mathbf{dr} = 0 \text{ implies that } \frac{dx}{u} = \frac{dy}{v} = \frac{dz}{w}.$$

In 2-dimensional flow

$$\frac{dx}{u(x,y)} = \frac{dy}{v(x,y)}. \tag{2.3.1}$$

The equation (2.3.1) is a double infinite set of solutions that constitutes stream lines. Stream function ψ is a scalar field whose relationship to velocity $\mathbf{q}$ is carefully selected to automatically satisfy continuity.

Stream function: A stream function is an integral solution of the streamline equation.

$$\frac{dx}{dt} = u(x,y), \quad \frac{dy}{dt} = v(x,y).$$

By eliminating the time variable we obtain:

$$\frac{dx}{u(x,y)} = \frac{dy}{v(x,y)}. \tag{2.3.2}$$

The streamlines $v(x,y)dx - u(x,y)dy = 0$ are described by a differential of a function i.e $d\psi = 0$, whose integral gives a stream function $\psi(x,y) = c$, where c is a constant of integration. The two characteristic properties of the stream function are: (i) the value of ψ is constant on each streamline. (ii) the mass flow between two streamlines is $\psi_2 - \psi_1$, where ψ_i is the value of ψ at the i- th streamline.

Use of stream function to satisfy mass–conservation equation: In case of flow of an incompressible fluid, mass–conservation equation (2.2.2) reduces to the statement that a vector divergence is zero, the divergence being of $\mathbf{q}$ or $\rho\mathbf{q}$ respectively [39]. If we impose the further restriction that the flow field is two dimensional, this vector divergence is actually the sum of only two partial derivatives, and the mass conservation equation can then be regarded as defining a scalar function $\psi(x,y)$ from which the components of $\mathbf{q}$ are obtained by differentiation. The procedure is described here for the case of an incompressible fluid.

Consider two dimensional flow of a fluid. Let u, v be the components of velocity. Then from equation (2.3.1), we have:

$$v\,dx - u\,dy = 0. \tag{2.3.3}$$

The equation of continuity (2.2.2) gives $-\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$, which suggests the introduction of a function $\psi(x,y)$ called a stream function such that the differential equation (2.3.3) must be exact differential $d\psi$ (say). We can write $d\psi = v\,dx - u\,dy = 0$, also $d\psi$ can be written as

$$d\psi = \frac{\partial \psi}{\partial x}dx + \frac{\partial \psi}{\partial y}dy = 0. \tag{2.3.4}$$

Comparing the equations (2.3.3) and (2.3.4), we obtain:

$$v = \frac{\partial \psi}{\partial x}, \quad u = -\frac{\partial \psi}{\partial y},$$

which satisfy the equation of continuity (2.2.2).

2.4 Navier–Stokes equation of motion

The Navier–Stokes equation describes the motion of a fluid. Equation of motion for viscous incompressible flow with constant viscosity can be written from the literature given in references [39].

$$\overbrace{\rho\Big(\underbrace{\frac{\partial \mathbf{q}}{\partial t}}_{\substack{\text{unsteady}\\\text{acceleration}}} + \underbrace{(\mathbf{q}\cdot\nabla)\mathbf{q}}_{\substack{\text{convective}\\\text{acceleration}}}\Big)}^{\text{Inertia(per volume)}} = \overbrace{\underbrace{-\nabla p}_{\substack{\text{pressure}\\\text{gradient}}} + \underbrace{\mu\nabla^2\mathbf{q}}_{\text{viscosity}}}^{\text{Divergence of stress}} + \underbrace{\rho\mathbf{B},}_{\text{body force}} \tag{2.4.1}$$

which may be written as

$$(\frac{\partial \mathbf{q}}{\partial t} + (\mathbf{q}\cdot\nabla)\mathbf{q}) = -\frac{1}{\rho}\nabla p + \nu\nabla^2\mathbf{q} + \mathbf{B},$$

where $\nu = \frac{\mu}{\rho}$ is kinematic viscosity. In cartesian coordinate system, we obtain:

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}) + B_x,$$
$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial y} + \nu(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2}) + B_y,$$
$$\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial z} + \nu(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2}) + B_z.$$

For 2-dimensional plane flow, the equations

$$\begin{aligned}
&\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}) + B_x,\\
&\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial y} + \nu(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}) + B_y,\\
&0 = B_z.
\end{aligned} \tag{2.4.2}$$

are known as Navier Stoke's equation of motion for two dimensional flow of viscous incompressible fluid.

2.5 Fluid model and dynamical similarity

In order to construct the model which should have all the characteristics of the actual object (prototype) and should give the required information about the prototype, the following similarity must be ensured between the model and the prototype:

Reynold's law of dynamical similarity: Dynamical similarity is the similarity of forces. The flow in the model and its prototype are dynamical similar if at all the corresponding points, identical type of forces are parallel and bear the same ratio. For example, if F_{vm}, F_{im}, F_{gm} and F_{vp}, F_{ip}, F_{gp} denote the viscous, inertia, gravity forces at a point in the model and its prototype. Then, force ratio must be

$$\frac{F_{vm}}{F_{vp}} = \frac{F_{im}}{F_{ip}} = \frac{F_{gm}}{F_{gp}} = F_r.$$

Reynold's number: Reynold's number denoted by R_e ensures dynamic similarity at corresponding points near the boundaries where viscous effects are more important. We know that the inertia force (product of mass and acceleration) always exist in all flow problems. Besides the inertia force, there always exist some additional forces (viscous force, gravity force, pressure force, elastic force and so on) which are responsible for fluid motion. The Reynold's number is defined as the ratio of inertia force to the viscous force.

$$R_e = \frac{\text{inertia force}}{\text{viscous force}} = \frac{\text{mass} \cdot \text{acceleration}}{\text{stress} \cdot \text{cross sectional area}}$$
$$= \frac{(\text{ cross sectional area } \cdot \text{ linear dimension } \cdot \text{ density }) \cdot \text{ velocity/time}}{\mu(\frac{du}{dy}) \text{ cross sectional area}}$$

$$R_e = \frac{\text{linear dimension } \cdot \text{ density } \cdot \text{ velocity}}{\mu(\frac{du}{dy}) \cdot \text{ time}} = \frac{\text{linear dimension } \cdot \rho \cdot \text{ velocity}}{\mu(\frac{U}{L})(\text{ linear dimension/velocity })}$$
$$= \frac{\rho \cdot (\text{ velocity })^2}{\mu(\frac{U}{L})} = \frac{UL}{\frac{\mu}{\rho}} = \frac{UL}{\nu}.$$

2.6 Prandtl boundary layer theory

The boundary layer is a thin layer in which the effect of viscosity is important however high the Reynold's number may be [39]. As the Reynold's number of a flow increases, the effect of convection at any point becomes more important. In cases of some flow systems involving a rigid boundary, the region in which viscosity has any effect on the flow shrinks to a thin layer at the boundary as viscosity $\nu \to 0$. For example: flow due to an oscillating plane wall, flow near a stagnation point at a plane wall, convergent flow in a channel, the MHD flow over a stretching sheet etc, needs the concept of boundary layer. Now we explain these approximations.

For convenience, consider a laminar two dimensional flow of a fluid. The fluid is of low viscosity (large Reynold's number) over a fixed semi infinite sheet. Unlike an ideal fluid, it does not slide over the sheet but "sticks" to it. Assume that the sheet is at rest. The fluid in contact with it will also be at rest. When the fluid is flowing on the surface of sheet,the effect of surface motion is maximum at the adjacent fluid layer. The resistance or viscosity is maximum near the surface. As one moves outwards (away from the sheet) along the normal, the effect of viscosity reduces, that is, the velocity of the fluid will gradually increase and will attain a full stream velocity U. After a thin region above the sheet this effect is practically negligible (the transition from zero velocity at the sheet to the full magnitude U takes place within the thin layer of the fluid in contact with the sheet). That thin layer adjacent to the surface is known as a boundary layer. There is no definite line between potential flow region (when friction is negligible) and the boundary layer. Therefore in practice, we define the boundary layer as that region where the fluid velocity is parallel to the surface and less than 99 % of the free stream velocity which is described by potential flow theory. The distance from the surface to the fluid layer which attains a free stream velocity is termed as boundary layer thickness. It is denoted by δ.

The thickness of the boundary layer(δ) grows along a surface (over which fluid is flowing) from the leading edge. The shape of the velocity profile and rate of increase of the boundary layer thickness depend on the pressure gradient $\frac{\partial p}{\partial x}$. Thus, if the pressure increases in the direction of flow, the boundary layer thickness increases rapidly. When the adverse pressure gradient is large, separation will occur followed by a region of reversed flow. Separation point S is defined as the point at which

$$(\frac{\partial u}{\partial y})_{y=0} = 0$$

where u is the x component of velocity. Due to the reversal of flow, there is considerable thicknening of the boundary layer, and associated with it, there is a flow of boundary layer fluid into the outside region. The exact location of the *point of separation* can be determined only with the help of integration of the boundary layer equations.
The method of dividing the fluid in two regions was first proposed by Prandtl in 1905. He suggested that the entire field of flow can be divided (for the sake of mathematical analysis) into the following two regions:

(i) A very thin layer (boundary layer) in the vicinity of sheet in which the velocity gradient normal to wall (i.e. $\frac{\partial u}{\partial y}$) is very large. Accordingly, the viscous stress $\mu\frac{\partial u}{\partial y}$ becomes important even when μ is small. Thus the viscous and inertial forces are of the same order within the boundary layer.

(ii) In the remaining region (i.e. outside the boundary layer) $\frac{\partial u}{\partial y}$ is very small and so the viscous forces may be ignored completely. Outside the boundary layer, the flow can be regarded non-viscous and hence the theory of non-viscous fluids offers a very good approximation there.

The following conditions are usually imposed on the distribution of velocity in the boundary layer:

- the no slip condition
- that no mass shall flow through the wall
- that the velocity at the outer edge of the boundary layer shall approach that predicted by an appropriate non-viscous theory.

Across the boundary layer the flow velocity changes from value zero at the boundary to some finite value characteristic of an inviscid fluid, and derivatives w.r.t. y of any flow quantity are in general much larger than those with respect to x. Thus at points within the boundary layer we may use the approximations [39]

$$\mid \frac{\partial u}{\partial x} \mid \leqslant \mid \frac{\partial u}{\partial y} \mid, \quad \mid \frac{\partial^2 u}{\partial x^2} \mid \leqslant \mid \frac{\partial^2 u}{\partial y^2} \mid .$$

Importance of boundary layer theory: A thin boundary layer plays a vital role in fluid dynamics. It has become a powerful method of analyzing the complex behavior of real fluids. The concept of boundary layer can be utilized to simplify the Navier-Stokes equations to such an extent that it becomes possible to tackle many practical problems of great importance. The drag on ships and missiles, the efficiency of compressors and turbines in jet engines, the effectiveness of air intakes for ram-and turbojets and so on depend on the concept of boundary layer and its effects on the main flow. The boundary layer theory can predict flow separation. It can explain the existence of a wake. The pressure distribution produces a net force in the direction in which stream flows.

2.6.1 Prandtl boundary layer equations for 2-dimensional flow

Order of magnitude approach:

To apply the order of magnitude approach, consider the flow over a semi-infinite sheet. We take rectangular cartesian coordinates with x measured in the sheet in the direction of the laminar two dimensional incompressible flow, and y measured normal to the sheet. Let viscosity of the fluid and thickness δ of boundary layer be small. Let U be the velocity in the main stream just outside the boundary layer. We now determine the order of magnitude of each term of the equations (2.4.2) and (2.2.2) to enable us to drop small terms and thus to arrive at the simplified boundary layer equations. One can designate the order of any quantity (q, say) by $O(q)$.

Let $O(u) = 1$ and $O(\frac{\partial u}{\partial x}) = 1$ within the boundary layer. Then $\frac{\partial u}{\partial y}$ is large as $\frac{\partial u}{\partial y} = \frac{1}{\delta}$, u decreasing from a finite value U at the outer boundary of the layer to zero at the flat sheet. Again $O(\frac{\partial^2 u}{\partial y^2}) = \frac{1}{\delta^2}$ in the boundary layer. Further $O(\frac{\partial u}{\partial t}) = O(\frac{\partial^2 u}{\partial x^2}) = 1$. The equation of continuity after labeling the order of magnitude shows that $O(\frac{\partial v}{\partial y}) = 1$.

$$\begin{array}{c} \dfrac{\partial u}{\partial x} + \dfrac{\partial v}{\partial y} = 0, \\ 1 \qquad 1 \end{array}$$

Since $v = 0$ when $y = 0$, $O(v) = \delta$. Again $O(\frac{\partial v}{\partial t}) = O(\frac{\partial v}{\partial x}) = O(\frac{\partial^2 v}{\partial x^2}) = \delta$ whereas $O(\frac{\partial^2 v}{\partial y^2}) = \frac{\delta}{\delta^2} = \frac{1}{\delta}$. All these values have been inserted in equations of momentum in absence of body force. The terms of body forces can not be dropped as they are not small terms. The momentum equations after labeling the order of magnitude implies

$$\begin{array}{l} \dfrac{\partial u}{\partial t} + u\dfrac{\partial u}{\partial x} + v\dfrac{\partial u}{\partial y} = -\dfrac{1}{\rho}\dfrac{\partial p}{\partial x} + \nu(\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2}), \\ (\ 1 \qquad 1\ 1 \qquad \delta\ \dfrac{1}{\delta}) \qquad\qquad\qquad (1 \qquad \dfrac{1}{\delta^2}) \end{array} \tag{2.6.1}$$

$$\begin{aligned} &\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial y} + \nu(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}). \\ &(\ \delta \quad 1\ \delta \quad \delta\ 1) \qquad\qquad (\delta \quad \frac{1}{\delta}) \end{aligned} \tag{2.6.2}$$

Since the viscous force is taken as of the same order of magnitude as the inertia forces within the boundary layer, the equation (2.6.1) implies that we must have $O(\frac{\nu}{\delta^2}) = 1$ so that $O(\delta) = \sqrt{\nu} = \sqrt{\frac{\mu}{\rho}}$ showing that smaller the viscosity of the fluid, the thinner the boundary layer.

By neglecting the terms of the order δ and smaller from mass and momentum equation, we obtain:

$$\begin{aligned} &\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\frac{\partial^2 u}{\partial y^2}, \\ &0 = -\frac{\partial p}{\partial y}, \end{aligned} \tag{2.6.3}$$

with equation of continuity

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0.$$

The equation (2.6.3) shows that the pressure distribution is a function of x only, that is, for a given x, pressure p is constant throughout the boundary layer. If there is no pressure gradient then $\nabla p = 0$.

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \nu\frac{\partial^2 u}{\partial y^2}, \qquad \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0. \tag{2.6.4}$$

For the flow outside the boundary layer

$$\frac{\partial U}{\partial t} + U\frac{\partial U}{\partial x} = -\frac{1}{\rho}\frac{dp}{dx}. \tag{2.6.5}$$

If the body force is present then equation (2.6.4) takes the form

$$\begin{aligned} &\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \nu\frac{\partial^2 u}{\partial y^2} + B_x, \\ &\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0. \end{aligned} \tag{2.6.6}$$

The above equations (2.6.6) are known as Prandtl's boundary layer equations.

2.7 Basics of heat transfer

In simple terms, the discipline of heat transfer is concerned with only two things [42]:

(i) **Temperature:** It is a number that is related to average kinetic energy of the molecules of a substance. If temperature is measured in kelvin degrees then this number is proportional to the average kinetic energy of the molecules.

(ii) **Heat:** In physics, heat is energy which is spontaneously flowing from an object with a high temperature to an object with a lower temperature. It is a measurement of the total energy in a substance. That total energy is made up of the kinetic energy and potential energy of the molecules of a fluid.

A clear structure: When heat comes into a fluid, energy comes into the fluid which can be used to increase the kinetic energy of the molecules, would cause an increase in temperature, or that heat could be used to increase the potential energy of the molecules causing a change in state that is not accompanied by an increase in temperature.

Similar deal is with polymeric solutions. On a microscopic scale, thermal energy is related to the kinetic energy of the molecules. The greater a material temperature, the greater the thermal agitation of its constituent molecules (manifested both in linear motion and vibrational modes). It is natural for regions containing greater molecular kinetic energy to pass this energy to regions with less kinetic energy. Several material properties serve to modulate the heat transferred between two regions at differing temperature. For example thermal conductivities, specific heat, material densities, fluid velocities, surface emissivity and more. Taken together, these properties serve to make the solution of many heat transfer problems.

Heat transfer: It is a mechanism of transfer of thermal energy between two places having different temperatures. It occurs until the system reaches to thermal equilibrium. Heat transfer can take place through three basic modes, conduction, convection and radiation.

Conduction: In conduction process the thermal energy is transferred in two ways: molecular interaction and by free electron. The equation which describes heat transfer in this mechanism is known as Fourier's law and mathematically

$$\frac{\mathbf{q}}{A} = -k\nabla T,$$

where $\mathbf{q}$ is the heat flux vector, A is the area normal to the direction of heat flow, ∇T is the temperature gradient, k is the thermal conductivity and negative sign indicates that heat flow is in the direction of a negative gradient.

Convection: The transfer of heat due to mixing of moments of different parts of the fluid caused by density differences. In case of laminar flow all the energy transfer is by molecular means. The rate equation for convective heat transfer was first given by Newton in 1701, that is

$$\frac{\mathbf{q}}{A} = h\triangle T,$$

in which $\triangle T$ is the temperature difference and h is the convective heat transfer coefficient. Convective heat transfer is of two types.

Forced convection: In this type the fluid is made to flow past a surface by an external agent. This classification describes those convective situation in which fluid circulation is produced due to some surface force or external force.

Natural or free convection: In free convection, the fluid motion result from the density difference which are due to temperature difference.

Mixed convection: A convection in which both natural and forced convection occur at a time.

2.7.1 Conservation of energy (energy equation)

We consider conservation of energy on the basis of first law of thermodynamics which states that " the total energy added to the system (both by heat and by work done on the fluid) increase the internal energy per unit mass of fluid". Let Q be the heat added per unit mass of fluid through conduction and E be the internal energy per unit mass of fluid. Then the rate of work done W by the normal and shearing stresses on a unit volume of the fluid is

$$W = -\, p\nabla \cdot \mathbf{q} + \Phi, \tag{2.7.1}$$

where the dissipation function Φ [39] is given by

$$\begin{aligned}\Phi =&\mu\Big[2\{(\frac{\partial u}{\partial x})^2 + (\frac{\partial v}{\partial y})^2 + (\frac{\partial w}{\partial z})^2\} - \frac{2}{3}(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z})^2 \\ &+ (\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y})^2 + (\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y})^2 + (\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z})^2)\Big].\end{aligned} \tag{2.7.2}$$

Then the first law of thermodynamics (in terms of variation of energy) [40] may be written as

$$\rho\frac{dQ}{dt} + W = \rho\frac{DE}{Dt}. \tag{2.7.3}$$

Using equation (2.7.1) in equation (2.7.3), we obtain:

$$\rho\frac{dQ}{dt} - p\nabla \cdot \mathbf{q} + \Phi = \rho\frac{DE}{Dt} \quad \text{which implies} \quad \rho\frac{dQ}{dt} + \Phi = \rho\frac{DE}{Dt} + p\nabla \cdot \mathbf{q}.$$

Also from equation of continuity (2.2.1)

$$\frac{D\rho}{Dt} + \rho\nabla \cdot \mathbf{q} = 0 \quad \text{which implies} \quad p\nabla \cdot \mathbf{q} = -\frac{p}{\rho}\frac{D\rho}{Dt}.$$

Now borrow a result

$$\frac{D}{Dt}(\frac{p}{\rho}) = \frac{1}{\rho}\frac{Dp}{Dt} - (\frac{p}{\rho^2})\frac{D\rho}{Dt} \quad \text{which implies} \quad p\nabla \cdot \mathbf{q} = \rho\frac{D}{Dt}(\frac{p}{\rho}) - \frac{Dp}{Dt}.$$

Using above equations, first law of thermodynamics gives

$$\rho\frac{dQ}{dt} + \Phi = \rho\frac{D}{Dt}(E + \frac{p}{\rho}) - \frac{Dp}{Dt}.$$

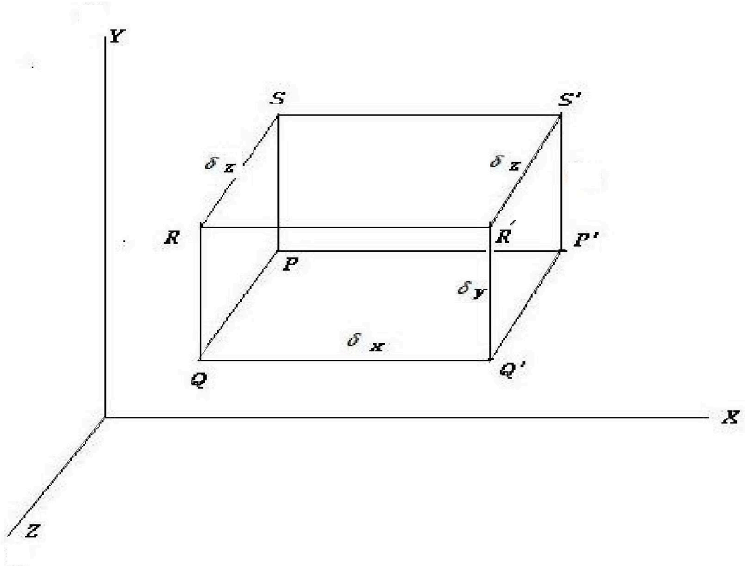
Put $(E + \frac{p}{\rho}) = h$ (enthalpy)

$$\rho\frac{dQ}{dt} + \Phi = \rho\frac{Dh}{Dt} - \frac{Dp}{Dt}. \tag{2.7.4}$$

Evaluation of heat Q: Let f be the rate of flow of heat across a given surface and $\frac{\partial T}{\partial n}$ be the temperature gradient along the surface. According to the Fourier heat conduction law " the heat flux f crossing an area (i.e quantity of heat per unit time) is proportional to the temperature gradient along the surface" [1]

$$f = -k\frac{\partial T}{\partial n},$$

where k is the thermal conductivity of the fluid and the negative sign signifies that the direction of the flux is opposite to that of temperature gradient. Let there be a fluid



particle at $P(x, y, z)$. T and ρ be the temperature and density of the fluid at $P(x, y, z)$ respectively. Construct a small parallelepiped with edges of length $\delta x, \delta y, \delta z$, parallel to their respective coordinate axes, have P at one of the angular points. Then the heat flow through the face $PQRS$ per unit time is

$$f(x, y, z) = -k\frac{\partial T}{\partial x}\delta y\delta z.$$

The heat flow through the opposite face $\acute{P}\acute{Q}\acute{R}\acute{S}$ per unit time can be obtained by Taylor's theorem

$$f(x + \delta x, y, z) = f(x, y, z) + \delta x\frac{\partial}{\partial x}f(x, y, z) + \cdots.$$

Hence, the net gain in energy per unit time within the fluid element in the x-direction (due to flow through faces) $PQRS$ and $\acute{P}\acute{Q}\acute{R}\acute{S}$ is

$$\begin{aligned}
\text{Net gain in energy in } x\text{- direction} &= f(x, y, z) - [f(x, y, z) + \delta x\frac{\partial}{\partial x}f(x, y, z) + \cdots] \\
&= -\delta x\frac{\partial}{\partial x}f(x, y, z) \quad \text{to the first order of approximation} \\
&= -\delta x\frac{\partial}{\partial x}(-k\frac{\partial T}{\partial x}\delta y\delta z) \\
&= \delta x\delta y\delta z\frac{\partial}{\partial x}(k\frac{\partial T}{\partial x}).
\end{aligned}$$

Similarly,

$$\text{Net gain in energy in } y\text{- direction} = \delta x \delta y \delta z \frac{\partial}{\partial y}(k\frac{\partial T}{\partial y}),$$
$$\text{Net gain in energy in } z\text{- direction} = \delta x \delta y \delta z \frac{\partial}{\partial z}(k\frac{\partial T}{\partial z}).$$

Hence the total quantity of heat introduced in the fluid element during time δt is

$$\text{Net gain in energy} = \delta t \delta x \delta y \delta z [\frac{\partial}{\partial x}(k\frac{\partial T}{\partial x}) + \frac{\partial}{\partial y}(k\frac{\partial T}{\partial y}) + \frac{\partial}{\partial z}(k\frac{\partial T}{\partial z})].$$

Hence the rate of heat added by conduction per unit volume is given by

$$\rho\frac{dQ}{dt} = [\frac{\partial}{\partial x}(k\frac{\partial T}{\partial x}) + \frac{\partial}{\partial y}(k\frac{\partial T}{\partial y}) + \frac{\partial}{\partial z}(k\frac{\partial T}{\partial z})],$$
$$\text{implies that } \rho\frac{dQ}{dt} = \nabla(k\nabla T). \tag{2.7.5}$$

Using above equation (2.7.5) in (2.7.4) with the assumption that there is no direct heating from chemical reaction and radiation heating, the required energy equation (2.7.4) becomes

$$\nabla(k\nabla T) + \Phi = \rho\frac{Dh}{Dt} - \frac{Dp}{Dt}.$$

In cartesian coordinates the energy equation for viscous compressible fluid reduces to the form

$$[\frac{\partial}{\partial x}(k\frac{\partial T}{\partial x}) + \frac{\partial}{\partial y}(k\frac{\partial T}{\partial y}) + \frac{\partial}{\partial z}(k\frac{\partial T}{\partial z})] + \Phi = \rho\frac{Dh}{Dt} - \frac{Dp}{Dt}.$$

If $h = C_pT$ and C_p is specific heat at constant pressure.

$$[\frac{\partial}{\partial x}(k\frac{\partial T}{\partial x}) + \frac{\partial}{\partial y}(k\frac{\partial T}{\partial y}) + \frac{\partial}{\partial z}(k\frac{\partial T}{\partial z})] + \Phi = \rho\frac{D}{Dt}(C_pT) - \frac{Dp}{Dt}. \tag{2.7.6}$$

Case of viscous incompressible fluids: When the fluid is taken incompressible viscous fluid, $k =$ constant and $\mu =$ constant. Further more, the equation of continuity for such a fluid is given by

$$\nabla\cdot\mathbf{q} = 0 = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}.$$

The dissipation function Φ for the present problem is given by [39]

$$\Phi = \mu\Big[2\{(\frac{\partial u}{\partial x})^2 + (\frac{\partial v}{\partial y})^2 + (\frac{\partial w}{\partial z})^2\} + (\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y})^2 + (\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y})^2 + (\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z})^2)\Big].$$

If C_p be the specific heat at constant volume, then $C_p = C_v$ for an incompressible fluid. With the above mentioned discussion, the energy equation assumes the form

$$k(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2}) + \Phi = \rho C_p\frac{DT}{Dt} - \frac{Dp}{Dt}.$$

If pressure is kept constant that is $\frac{Dp}{Dt} = 0$, then

$\rho C_p\frac{DT}{Dt} = k(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2}) + \Phi,$ from definition of material derivative we obtain

$$\rho C_p(\frac{\partial T}{\partial t} + u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} + w\frac{\partial T}{\partial z}) = k(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2}) + \Phi,$$

where $\Phi = 2\mu\Big[(\frac{\partial u}{\partial x})^2 + (\frac{\partial v}{\partial y})^2\Big] + \mu\{(\frac{\partial v}{\partial x})^2 + (\frac{\partial u}{\partial y})^2 + 2\frac{\partial v}{\partial x}\frac{\partial u}{\partial y}\}.$

In two dimensional plane xy-plane (say)

$$\rho C_p(\frac{\partial T}{\partial t}+u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y})=k(\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2})+\Phi,$$
$$\text{where } \Phi=\mu\Big[2\{(\frac{\partial u}{\partial x})^2+(\frac{\partial v}{\partial y})^2\}+(\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y})^2\Big]. \tag{2.7.7}$$

2.7.2 Thermal boundary layer theory

When a fluid past a heated or cooled body, the heat is transferred by conduction, convection, and radiation. Heat transfer by radiation is negligible unless the temperature is very high. Accordingly, we shall confine our present discussion to heat transfer by conduction and convection only. The conductivity of ordinary fluids is small. For such fluids the heat transport due to conduction is comparable to that due to convection only across a thin layer near the surface of the body. It follows that the temperature field which spreads from the body extends only over a narrow zone in the immediate vicinity of its surface, where as the fluid at a larger distance from the surface is not materially effected by the heated body. This thin layer (narrow region) near the surface of the body is called *thermal boundary layer*. There are two types of thermal boundary layer problems, namely

Forced convection: When fluid is forced past a body and change in temperature is not too large. This is called forced convection. In this convection heat transfer coefficient is independent of change in temperature. It is the flow in which the velocity arises from the variable density (i.e due to force of buoyancy) is negligible in comparison with the velocity of the main or forced flow.

Free or natural convection: When fluid buoys up from a hot body or down from a cold one, the heat transfer coefficient varies as some weak powers of change in temperature. This is free or natural convection. The free convection is the flow in which the motion is essentially caused by the effect of gravity on the heated fluid of variable density.

2.7.3 Thermal boundary layer equation in 2-dim flow

Order of magnitude approach:

To apply the order of magnitude approach, consider the flow over a semi-infinite sheet. We derive the thermal boundary layer equation. We take rectangular cartesian coordinates with x measured in the sheet in the direction of the laminar two dimensional incompressible flow, and y measured normal to the sheet. Let u, v be the velocity components in x- and y-directions respectively. Let viscosity of the fluid be small and δ be the small thickness of velocity boundary layer. Let δ_t be the small thermal boundary layer. Let U be the velocity in the main stream just outside the boundary layer. Then the equation of energy for unsteady flow of viscous incompressible fluid in two dimension is given as: we shall follow a method similar to that used in velocity boundary layer equation.

We now determine the order of magnitude of each term in momentum equation to enable us to drop small terms and thus to arrive at the simplified thermal boundary layer equations. We shall designate the order of any quantity (q, say) by $O(q)$.

Let δ and δ_t be the velocity boundary layer thickness and thermal boundary layer thickness respectively. Let $O(T) = 1$, $O(x) = 1$, $O(u) = 1$, $O(y) = \delta_t$. As we have proved in section 2.6.1, $O(v) = \delta$, $O(\nu) = \delta^2$. The orders of magnitudes are shown in equations (2.7.8) and (2.7.9) under the individual terms.

$$\begin{aligned} &\rho C_p(\frac{\partial T}{\partial t} + u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y}) = k(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2}) + \Phi, \\ &1 \quad (1 \quad\quad 1\ 1 \quad\quad \delta\ \frac{1}{\delta t}) \quad\quad \delta_t^2(1 \quad\quad\quad \frac{1}{\delta_t^2}) \end{aligned} \tag{2.7.8}$$

where the dissipation function Φ is given by

$$\begin{aligned} \Phi =& 2\mu\Big[(\frac{\partial u}{\partial x})^2 + (\frac{\partial v}{\partial y})^2\Big] + \mu\{(\frac{\partial v}{\partial x})^2 + (\frac{\partial u}{\partial y})^2 + 2\frac{\partial v}{\partial x}\frac{\partial u}{\partial y}\}. \\ & \delta_t^2[1 \quad\quad \delta^2/\delta_t^2] \quad\quad \delta^2\{\delta^2 \quad\quad\quad \frac{1}{\delta_t^2} \quad\quad\quad \delta\frac{1}{\delta_t}\} \end{aligned} \tag{2.7.9}$$

From equation(2.7.8) it follows that the term $\frac{\partial^2 T}{\partial x^2}$ may be neglected in comparison with $\frac{\partial^2 T}{\partial y^2}$. The conduction terms become of same order of magnitude as the convection term, only, if $O(k) = \delta_t^2$ and δ and δ_t are of the same order of magnitude. $O(\frac{\frac{k}{\rho C_p}}{\nu}) = \frac{\delta_t^2}{\delta^2}$, that is, $O(\frac{1}{\sqrt{P_r}}) = \frac{\delta_t}{\delta}$ as $O(\rho C_p) = 1$, where P_r= Prandtl number = $(\mu C_p/k)$ and $\nu = \mu/\rho$. For liquids, $O(\delta_t) < O(\delta)$. Suppose that $O(\delta_t)$ and $O(\delta)$ have the same order of magnitude. By neglecting the terms of the order δ and smaller, we observe the equations (2.7.8), (2.7.9) and note that only term which retained in Φ is $\mu(\frac{\partial u}{\partial y})^2$. Hence the equation of the boundary layer for an incompressible fluid with constant properties in two dimensional unsteady flow with equation of continuity is

$$\begin{aligned} &\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \\ &\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \nu\frac{\partial^2 u}{\partial y^2} + B_x, \\ &\frac{\partial T}{\partial t} + u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} = \frac{k}{\rho C_p}(\frac{\partial^2 T}{\partial y^2}) + \frac{\mu}{\rho C_p}(\frac{\partial u}{\partial y})^2. \end{aligned} \tag{2.7.10}$$

Above equations (2.7.10) are the basic equations which govern the velocity and temperature distribution in a boundary layer past a solid sheet, in forced convection. If the solid sheet is replaced by unmixed fluid then the equations (2.7.10) also govern the mass, momentum and heat transfer in a boundary layer past a unmixed fluid sheet, in forced convection.

Chapter 3

Fluid flow and heat transfer in a liquid film over a stretching sheet

3.1 Description of the physical problem

Thin liquid films have many applications in different branches of sciences and technology. Knowledge of thin liquid film has main role to understand the coating process, design of various heat exchangers, chemical processing, wire and fibre coating, food stuff processing, extrusion of plastic sheets and transpiration cooling. Best appearance with a smooth glossy surface and attractive service properties such as low friction, transparency and strength is the market demand in all coating process. The knowledge for cooling the coating product efficiently solve the problem of extrusion of thin surface layer. The rate of heat transfer and cooling procedures improve the quality of the final product. If the fluid is very viscous in the extrusion of plastic sheet, heat transfer may alter the results due to viscous dissipation [36].

In most problems of polymer extrusion, the flow of fluid is induced by stretching of elastic sheet. For example, in a melting process, the extrudate from the die is generally drawn and simultaneously stretched into a filament or sheet. That sheet is solidified by gradual cooling with coolant liquid. The rate of cooling and rate of stretching usually affects the quality of the final product. The choice of an appropriate cooling liquid is very important as it has a direct impact on the rate of cooling. Synthetic oil, dilute polymeric solutions such as 5.4 % of isobutelene in cetane are some important cooling liquids [1]. Sudden stretching may spoil the properties desired for the final outcome, so the optimum stretching rate needs a great care.

As the flow and heat transfer characteristics directly affects the quality of the final product, so the analysis and fundamental understanding of the momentum and thermal transports for such extrusion processes are very important.

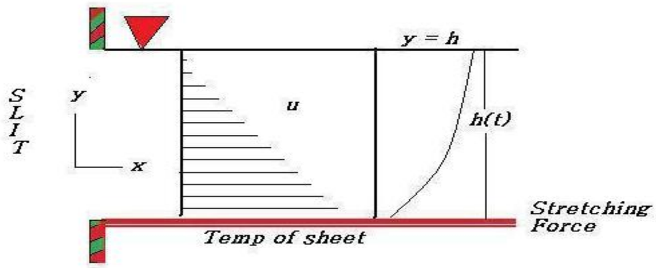


Figure 3.1: Fluid flow over a horizontal plastic sheet.

3.1.1 Mathematical formulation

In a problem of polymer extrusion, consider the flow of a thin viscoelastic liquid film over an elastic sheet which is stretching horizontally. The sheet issues from a narrow slit. At the slit we fix origin of cartesian coordinate system. Naturally the sheet is like a plane. The x- axis is chosen in the direction of motion of sheet and the y-axis is taken normal to the sheet. The stretching velocity is proportional to x. If b is the initial stretching rate with dimension per time, then velocity of stretching sheet is assumed to be of the form $U(x,t) = b\,x$ at time $t = 0$. The stretching rate is controlled by a factor $(1 - \alpha\,t)$ to ensure the quality of the final product, where α is a positive constant $0 \leq \alpha < 1$ with dimension per time. Then the velocity of stretching sheet takes the form [19, 27]

$$U(x,t) = b\,x\,(1 - \alpha\,t)^{-1},\; t < \alpha^{-1}. \tag{3.1.1}$$

In the context of polymer extrusion, the material properties and the elasticity of the extruded sheet may vary with the pulling force. If the sheet is stretched by the action of magnetic force along the x-axis. This force is the body force in the form of applied transverse magnetic field which is assumed to be of variable kind. Let B_0 be the strength of magnetic field. The magnetic force denoted by $B(x,t)$ may be taken in the special form to stretch the elastic sheet with rate $(1 - \alpha t)^{-\frac{1}{2}}$

$$B(x,t) = (1 - \alpha t)^{-\frac{1}{2}} B_0. \tag{3.1.2}$$

Now we dissolve a high-molecular-weight polymer (viscoelastic) into a simple Newtonian fluid and consider a thin liquid film of uniform thickness $h(t)$, that is, independent of position, lying on the horizontal stretching sheet. The fluid motion within the film is initially caused by stretching of sheet and then by viscous shearing arising from the stretching of the elastic sheet. The flow field is exposed on the influence of an external transverse magnetic field of strength B as defined in equation (3.1.2). For the viscous fluid of kinematic viscosity ν, we recognize the local Reynolds number R_e based on the surface velocity $U(x,t)$ as

$$R_e = U\,x\,\nu^{-1} = bx^2(\nu(1 - \alpha t))^{-1}. \tag{3.1.3}$$

Let T_0 be temperature at the slit and temperature of surrounding T_{ref} can be taken either as constant reference temperature with $0 \leq T_{ref} < T_0$. The surface temperature T_s of the stretching sheet varies with the distance x from the slit and time t and is defined in particular form as

$$\begin{aligned} T_s &= T_0 - \frac{R_e T_{ref}}{2(1-\alpha t)^{\frac{1}{2}}} \\ &= T_0 - \frac{b\,x^2}{2\,\nu}(1-\alpha t)^{-\frac{3}{2}} T_{ref}. \end{aligned} \tag{3.1.4}$$

The equation(3.1.4) for the temperature $T_s(x,t)$ of the sheet represents a situation in which sheet temperature decreases from T_0 at the slit in proportion to x^2 and amount of temperature reduction along the sheet increases with time. Assume that the liquid we are using is non-volatile. So we neglect the effect of latent heat due to evaporation. Further the buoyancy is neglected due to relatively thin liquid film.

Assumptions:

The pressure in the surrounding gas phase is assumed to be uniform and the gravity force gives rise to a hydrostatic pressure variation in the liquid film. Further it is assumed that the induced magnetic field is negligibly small.

In order to justify the boundary layer approximation, the length scale in the primary flow direction must be significantly larger than the length scale in the cross stream direction. Choose the representative measure of film thickness to be $(\frac{b}{\nu})^{\frac{1}{2}}$ so that the scale ratio is large enough, that is, $\frac{x}{(\frac{b}{\nu})^{\frac{1}{2}}} \gg 1$. This choice of length scale enables us to employ the boundary layer approximations.

3.1.2 Governing equations

The temperature and velocity fields within the thin liquid film are governed by two dimensional boundary layer equations for mass, momentum and thermal energy [36], [See also section (2.7.3)]

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \tag{3.1.5}$$

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \nu\frac{\partial^2 u}{\partial y^2} - \frac{\sigma B^2}{\rho}u, \tag{3.1.6}$$

$$\frac{\partial T}{\partial t} + u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} = \frac{k}{\rho C_p}\frac{\partial^2 T}{\partial y^2} + \frac{\mu}{\rho C_p}(\frac{\partial u}{\partial y})^2. \tag{3.1.7}$$

where $\frac{\sigma B^2}{\rho}$ is the magnetic parameter, σ is the electric conductivity, B is a variable magnetic field along the y-axis, $\frac{k}{\rho C_p}$ is thermal diffusivity and C_p is the specific heat, u and v are the horizontal and vertical components of velocity of thin liquid film, T is the temperature of thin liquid film.

3.1.3 Boundary conditions

The statement of the problem may be completed by giving the boundary conditions as follows:

$$\begin{aligned} &\text{At } y=0,\ u=U,\ v=0, T=T_s, \\ &At\, y=h,\ \frac{\partial u}{\partial y}=0,\ \frac{\partial T}{\partial y}=0. \end{aligned} \tag{3.1.8}$$

Initially, the horizontal component of velocity of thin liquid film is same as the velocity of horizontal stretching sheet (solidified plastic sheet) and there is no vertical component of velocity of thin liquid film. Also temperature of the thin liquid film is same as the temperature of elastic sheet. The sheer stress ($\tau = \mu\frac{\partial u}{\partial y}$) and heat flux ($q = -k\frac{\partial T}{\partial y}$) vanish at the free surface.

The thin liquid film is assumed to have smooth planner surface at the adiabatic free surface (at film thickness y = h) so as to avoid the complications due to surface waves. Also it is assumed that the upper boundary surface of thin liquid film is thermally insulated.

3.2 Non-dimensionalization

Introducing the dimensionless quantities η, $f(\eta)$ and $\theta(\eta)$ as follows:

$$\begin{aligned} &\eta = \Big(\frac{b}{\nu(1-\alpha t)}\Big)^{\frac{1}{2}} y, \\ &f(\eta) = \psi(x,y,t)\, x^{-1}\Big(\frac{\nu b}{1-\alpha t}\Big)^{-\frac{1}{2}}, \\ &\theta(\eta) = \frac{T_0 - T(x,y,t)}{T_{ref}\Big(\frac{bx^2}{2\nu(1-\alpha t)^{-\frac{3}{2}}}\Big)}. \end{aligned} \tag{3.2.1}$$

The fluid tangential velocity and normal velocity can be expressed in terms of stream function as follows

$$u = \frac{\partial \psi}{\partial y} = \frac{b}{1-\alpha t} x f'(\eta), \quad v = -\frac{\partial \psi}{\partial x} = -(\frac{\nu b}{1-\alpha t})^{\frac{1}{2}} f(\eta). \tag{3.2.2}$$

Taking first order partial derivative of u with respect to x, of v with respect to y

$$\frac{\partial u}{\partial x} = \frac{b}{1-\alpha t} f'(\eta), \quad \frac{\partial v}{\partial y} = -\frac{b}{1-\alpha t} f'(\eta), \tag{3.2.3}$$

which implies that $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$, that is, the equation of continuity is satisfied.

3.2.1 Fluid flow problem (velocity boundary layer problem)

To convert the momentum equation (3.1.6) into ordinary differential equation, we proceed as

$$\begin{aligned}
\frac{\partial u}{\partial t} &= \frac{b\alpha}{(1-\alpha t)^2}xf'(\eta) + \frac{b}{(1-\alpha t)}xf''(\eta)\frac{\partial \eta}{\partial t} \\
&= \frac{b\alpha}{(1-\alpha t)^2}xf'(\eta) + \frac{b}{(1-\alpha t)}xf''(\eta)\frac{\partial}{\partial t}\sqrt{\frac{b}{\nu(1-\alpha t)}}y \\
&= \frac{b\alpha}{(1-\alpha t)^2}xf'(\eta) + \frac{b}{(1-\alpha t)}xf''(\eta)\sqrt{\frac{b}{\nu}}\frac{\alpha}{2}(1-\alpha t)^{\frac{-3}{2}}y \qquad (3.2.4)\\
&= \frac{b\alpha}{(1-\alpha t)^2}xf'(\eta) + b\sqrt{\frac{b}{\nu}}\frac{\alpha}{2}(1-\alpha t)^{\frac{-5}{2}}xyf''(\eta) \\
&= b\alpha(1-\alpha t)^{-2}xf'(\eta) + \sqrt{\frac{b}{\nu}}\frac{b\alpha}{2}(1-\alpha t)^{\frac{-5}{2}}xyf''(\eta).
\end{aligned}$$

Taking partial derivative of u with respect to y

$$\begin{aligned}
\frac{\partial u}{\partial y} &= \frac{\partial}{\partial y}[\frac{b}{(1-\alpha t)}xf'(\eta)] = \frac{b}{(1-\alpha t)}xf''(\eta)\frac{\partial \eta}{\partial y} \\
&= \frac{b}{(1-\alpha t)}xf''(\eta)\frac{\partial}{\partial y}\sqrt{\frac{b}{\nu(1-\alpha t)}}y = \frac{b}{(1-\alpha t)}xf''(\eta)\sqrt{\frac{b}{\nu(1-\alpha t)}}y \qquad (3.2.5)\\
&= \frac{b}{(1-\alpha t)}xf''(\eta)\frac{1}{\sqrt{\nu}}\sqrt{\frac{b}{(1-\alpha t)}}y = \frac{1}{\sqrt{\nu}}(\frac{b}{1-\alpha t})^{\frac{3}{2}}xf''(\eta).
\end{aligned}$$

Taking second partial derivative of u with respect to y, we get:

$$\begin{aligned}
\frac{\partial^2 u}{\partial y^2} &= \frac{\partial}{\partial y}[\frac{1}{\sqrt{\nu}}(\frac{b}{1-\alpha t})^{\frac{3}{2}}xf''(\eta)] = \frac{1}{\sqrt{\nu}}(\frac{b}{1-\alpha t})^{\frac{3}{2}}xf'''(\eta)\frac{\partial}{\partial y}\eta \\
&= \frac{1}{\sqrt{\nu}}(\frac{b}{1-\alpha t})^{\frac{3}{2}}xf'''(\eta)\frac{\partial}{\partial y}\sqrt{\frac{b}{\nu(1-\alpha t)}}.y = \frac{1}{\sqrt{\nu}}(\frac{b}{1-\alpha t})^{\frac{3}{2}}xf'''(\eta)\sqrt{\frac{b}{\nu(1-\alpha t)}} \\
&= \frac{1}{\nu}(\frac{b}{1-\alpha t})^2xf'''(\eta) = \frac{1}{\nu}b^2(1-\alpha t)^{-2}xf'''(\eta).
\end{aligned} \qquad (3.2.6)$$

Using equations (3.2.2 - 3.2.6) in equation of momentum (3.1.6), we obtain:

$$\begin{aligned}
b\alpha(1-\alpha t)^{-2}xf'(\eta) + b\sqrt{\frac{b}{\nu}}\frac{\alpha}{2}(1-\alpha t)^{\frac{-5}{2}}yxf''(\eta) + xf'(\eta)(\frac{b}{(1-\alpha t)})^2f'(\eta) \\
-f(\eta)(\frac{b}{1-\alpha t})^2xf''(\eta) = b^2(1-\alpha t)^{-2}xf'''(\eta) - (\frac{\sigma B^2}{\rho})\frac{b}{(1-\alpha t)}xf'(\eta).
\end{aligned}$$

Dividing by $b^2(1-\alpha t)^{-2}$, above equation takes the form

$$\frac{\alpha}{b}f'(\eta) + \frac{\alpha}{2b}\sqrt{\frac{b}{\nu(1-\alpha t)}}yf''(\eta) + f'(\eta)f'(\eta) - f(\eta)f''(\eta) = f'''(\eta) - (\frac{\sigma B^2}{\rho b})\frac{1}{(1-\alpha t)^{-1}}f'(\eta)$$

using equation (3.1.2)

$$\frac{\alpha}{b}f'(\eta)+\frac{\alpha}{2b}\eta f''(\eta)+f'(\eta)f'(\eta)-f(\eta)f''(\eta)=f'''(\eta)-(\frac{\sigma B_0^2}{\rho b})f'(\eta). \qquad (3.2.7)$$

The quantity $\frac{\alpha}{b}$ is known as unsteadiness parameter which is denoted by S. The parameter $S\equiv\frac{\alpha}{b}$ is the dimensionless measure of the unsteadiness.

The quantity $\frac{\sigma B_0^2}{\rho b}$ is known as magnetic parameter which is denoted by M_n. The magnetic parameter $M_n\equiv\frac{\sigma B_0^2}{\rho b}$ reflects electrically conducting fluid with magnetic field or the Hartman number.

Substitute $S\equiv\frac{\alpha}{b}$ and $M_n\equiv\frac{\sigma B_0^2}{\rho b}$ in equation (3.2.7), we obtain:

$$S[f'(\eta)+\frac{\eta}{2}f''(\eta)]+f'^2(\eta)-f(\eta)f''(\eta)=f'''(\eta)-M_nf'(\eta).$$

One can reorganize this equation as the flow model in the form

$$[\frac{d^3}{d\eta^3}-(M_n+S)\frac{d}{d\eta}]f(\eta)=[\{\frac{S\eta}{2}-f(\eta)\}\frac{d^2}{d\eta^2}+(\frac{d}{d\eta})^2]f(\eta) \text{ or}$$

$$Lf(\eta)=Nf(\eta). \qquad (3.2.8)$$

Here $L=[\frac{d^3}{d\eta^3}-(M_n+S)\frac{d}{d\eta}]$ is a linear differential operator and $N=[\{\frac{S\eta}{2}-f(\eta)\}\frac{d^2}{d\eta^2}+(\frac{d}{d\eta})^2]$ is a non-linear.

3.2.2 Heat flow problem (thermal boundary layer problem)

Differentiate the equation of temperature (3.2.1) with respect to t, x and y, we obtain:

$$\frac{\partial T}{\partial t}=-T_{ref}(\frac{b^2x^2}{2\nu})(1-\alpha t)^{\frac{-5}{2}}\frac{S}{2}[3\theta(\eta)+\eta\theta'(\eta)],\ \frac{\partial T}{\partial x}=-T_{ref}(\frac{2bx}{2\nu})(1-\alpha t)^{\frac{-3}{2}}\theta(\eta). \qquad (3.2.9)$$

$$\begin{aligned}\frac{\partial T}{\partial y}&=\frac{\partial}{\partial y}[(T_0-T_{ref}(\frac{bx^2}{2\nu})(1-\alpha t)^{\frac{-3}{2}}\theta(\eta))]=-T_{ref}(\frac{bx^2}{2\nu})(1-\alpha t)^{\frac{-3}{2}}\theta'(\eta)\frac{\partial\eta}{\partial y}\\&=-T_{ref}(\frac{bx^2}{2\nu})(1-\alpha t)^{\frac{-3}{2}}\theta'(\eta)\sqrt{\frac{b}{\nu(1-\alpha t)}}=-T_{ref}(\frac{bx^2}{2\nu})\sqrt{\frac{b}{\nu}}(1-\alpha t)^{-2}\theta'(\eta),\end{aligned} \qquad (3.2.10)$$

$$\begin{aligned}\frac{\partial^2T}{\partial y^2}&=-T_{ref}(\frac{bx^2}{2\nu})\sqrt{\frac{b}{\nu}}(1-\alpha t)^{-2}\theta''(\eta)\frac{\partial\eta}{\partial y}=-T_{ref}(\frac{bx^2}{2\nu})\sqrt{\frac{b}{\nu}}(1-\alpha t)^{-2}\theta''(\eta)\frac{\partial}{\partial y}\sqrt{\frac{b}{\nu(1-\alpha t)}}y\\&=-T_{ref}(\frac{bx^2}{2\nu})(\sqrt{\frac{b}{\nu}})(1-\alpha t)^{-2}\theta''(\eta)\sqrt{\frac{b}{\nu(1-\alpha t)}}=-T_{ref}(\frac{bx^2}{2\nu})(\frac{b}{\nu})(1-\alpha t)^{\frac{-5}{2}}\theta''(\eta).\end{aligned} \qquad (3.2.11)$$

Taking square of both sides of equation (3.2.5), we get:

$$(\frac{\partial u}{\partial y})^2=\frac{1}{\nu}(\frac{b}{1-\alpha t})^3x^2f''^2(\eta). \qquad (3.2.12)$$

Substituting these equations (3.2.9 - 3.2.11, 3.2.12) in the heat equation (3.1.7), we obtain:

$$[\frac{S}{2}(3\theta(\eta)+\eta\theta'(\eta))+2\theta(\eta)f'(\eta)-\theta'(\eta)f(\eta)]$$

$$= \frac{k}{\nu\rho C_p}\theta''(\eta))+\frac{\mu}{\rho C_p}2b\frac{(1-\alpha t)^{-3}(\frac{b^2x^2}{2\nu})f''^2(\eta))}{-T_{ref}(\frac{b^2x^2}{2\nu})(1-\alpha t)^{\frac{-5}{2}}}$$

$$= \frac{k}{\nu\rho C_p}\theta''(\eta))-\frac{\mu}{\rho C_p}2b\frac{(1-\alpha t)^{\frac{-1}{2}}}{-T_{ref}}f''^2(\eta)$$

$$= \frac{k}{\nu\rho C_p}\theta''(\eta))+\frac{\mu}{\rho C_p}2b\frac{(1-\alpha t)^{\frac{-1}{2}}}{T_{ref}}f''^2(\eta).$$

Using the equation (3.1.4) for reference temperature $T_{ref} = \frac{T_s-T_0}{(\frac{bx^2}{2\nu})(1-\alpha t)^{\frac{-3}{2}}}$, leads to

$$[\frac{S}{2}(3\theta(\eta)+\eta\theta'(\eta))+2\theta(\eta)f'(\eta)-\theta'(\eta)f(\eta)] = \frac{k}{\nu\rho C_p}\theta''(\eta))+\frac{\mu}{\rho C_p}2b(\frac{bx^2}{2\nu})$$

$$(1-\alpha t)^{\frac{-3}{2}}\frac{(1-\alpha t)^{\frac{-1}{2}}}{(T_s-T_0)}f''^2(\eta)$$

$$= \frac{k}{\nu\rho C_p}\theta''(\eta))+\frac{\mu}{\rho C_p}b(\frac{bx^2}{\nu})\frac{(1-\alpha t)^{-2}}{(T_s-T_0)}f''^2(\eta)$$

$$= \frac{k}{\nu\rho C_p}\theta''(\eta))+\frac{\mu}{\nu\rho C_p}b^2x^2\frac{(1-\alpha t)^{-2}}{(T_s-T_0)}f''^2(\eta)$$

$$= \frac{k}{\nu\rho C_p}\theta''(\eta))+\frac{\mu}{\nu\rho C_p}\frac{U^2}{(T_s-T_0)}f''^2(\eta)$$

$$= \frac{k}{\nu\rho C_p}\theta''(\eta))+\frac{\mu k C_p}{\nu\rho C_p}\frac{U^2}{kC_p(T_s-T_0)}f''^2(\eta).$$

Substitute $\mu = \nu\rho$ known as dynamic viscosity in the above equation, where $\nu = \frac{\mu}{\rho}$ is kinematic viscosity, we have:

$$[\frac{S}{2}(3\theta(\eta)+\eta\theta'(\eta))+2\theta(\eta)f'(\eta)-\theta'(\eta)f(\eta)] = \frac{k}{\nu\rho C_p}\theta''(\eta))+\frac{\nu\rho k C_p}{\nu\rho C_p}\frac{U^2}{kC_p(T_s-T_0)}f''^2(\eta)$$

$$= \frac{k}{\nu\rho C_p}\theta''(\eta))+\frac{k}{\nu\rho C_p}\frac{\nu\rho C_p}{k}\frac{U^2f''^2(\eta)}{C_p(T_s-T_0)}.$$

Multiplying both sides by $\frac{\nu\rho C_p}{k}$, we get:

$$\frac{\nu\rho C_p}{k}[\frac{S}{2}(3\theta(\eta)+\eta\theta'(\eta))+2\theta(\eta)f'(\eta)-\theta'(\eta)f(\eta)] = \theta''(\eta))+\frac{\nu\rho C_p}{k}\frac{U^2}{C_p(T_s-T_0)}f''^2(\eta).$$

The quantity $\frac{\nu\rho C_p}{k}$ is known as dimensionless Prandtl number and denoted by P_r which is ratio of momentum diffusivity to thermal diffusivity. The Prandtl number is actually the ratio of viscous force to the thermal force. It throws light on the relative importance of viscous dissipation to the thermal dissipation. Substitute $P_r = \frac{\nu\rho C_p}{k}$ in above equation we have

$$P_r[\frac{S}{2}(3\theta(\eta)+\eta\theta'(\eta))+2\theta(\eta)f'(\eta)-\theta'(\eta)f(\eta)] = \theta''(\eta))+P_r\frac{U^2}{C_p(T_s-T_0)}f''^2(\eta). \tag{3.2.13}$$

The quantity $\frac{U^2}{C_p(T_s-T_0)}$ is known as Eckert number and denoted by E_c which is the ratio of kinetic energy to the enthalpy. Substitute $E_c = \frac{U^2}{C_p(T_s-T_0)}$ in equation (3.2.13), we obtain:

$$P_r[\frac{S}{2}(3\theta(\eta)+\eta\theta'(\eta))+2\theta(\eta)f'(\eta)-\theta'(\eta)f(\eta)] = \theta''(\eta)+P_rE_cf''(\eta). \tag{3.2.14}$$

This equation is linear in θ and can be expressed as

$$\theta''(\eta) = \Big[\frac{3P_rS}{2}+2P_rf'(\eta)\Big]\theta(\eta)+\Big[\frac{P_rS\eta}{2}-P_rf(\eta)\Big]\theta'(\eta)-P_rE_cf''(\eta) \quad \text{or}$$

$$\theta''(\eta) = a(\eta)\theta(\eta)+b(\eta)\theta'(\eta)+c(\eta), \tag{3.2.15}$$

where $a(\eta) = \frac{3P_rS}{2}+2P_rf'(\eta),\ b(\eta) = \frac{P_rS\eta}{2}-P_rf(\eta),\ c(\eta) = -P_rE_cf''(\eta).$

The equation (3.2.15) is linear second order differential equation with variable coefficients containing unknown function $f(\eta)$ and its derivatives. The associated boundary conditions are given below.

3.2.3 Non-dimensionalization of film thickness:

Let β denotes the dimensionless film thickness which is defined by the value of similarity variable η at the free surface $y = h$.

$$\text{At } y = h \qquad \eta = \Big(\frac{b}{\nu(1-\alpha t)}\Big)^{\frac{1}{2}}y \text{ implies that } \beta = \Big(\frac{b}{\nu(1-\alpha t)}\Big)^{\frac{1}{2}}h(t). \tag{3.2.16}$$

Physically, $h(t)$ is the original film thickness. Yet β is an unknown constant which should be determined by the integral part of the boundary value problem. The rate at which film thickness varies can be obtained by differentiating the original film thickness $h(t)$ in the above equation (3.2.16) with respect to t in the form

$$\frac{d}{dt}h(t) = -\frac{\alpha\beta}{2}\sqrt{\frac{\nu}{b(1-\alpha t)}}. \tag{3.2.17}$$

Free surface condition: The kinematic constraint at $y = h(t)$ given by equation $v = \frac{dh}{dt}$ transforms into the free surface condition which is

$$v = \frac{d}{dt}h(t) = -\frac{\alpha\beta}{2}\sqrt{\frac{\nu}{b(1-\alpha t)}}. \tag{3.2.18}$$

Note that $h(t)$ decreases monotonically when time increases and β is constant depending only upon S and M_n. Then β can be explored by the formula $\beta = \Big(\frac{\alpha}{\nu S(1-\alpha t)}\Big)^{\frac{1}{2}}h(t)$ and $\beta = \Big(\frac{\alpha B^2}{\nu\rho M_n(1-\alpha t)}\Big)^{\frac{1}{2}}h(t)$. The relation between S and M_n can be derived by solving these above two expressions. These scenarios have been discussed in detail in the references [11, 26, 27].

3.2.4 Non-dimensionalization of boundary conditions

Initial and boundary conditions for fluid flow problem:

(i) From equation (3.1.8) we note, at $y = 0, u = U$ and from equation (3.2.1) $\eta = 0$, and using equation (3.1.1), (3.2.5) we note:
$(\frac{bx}{1-\alpha t})f'(\eta) = \frac{b}{(1-\alpha t)}x$, for $(1-\alpha t) > 0$ and for all $x, b > 0$ the last equation reduces to

$$f'(0) = 1. \tag{3.2.19}$$

(ii) From equation (3.1.8) we note: at $y = 0, v = 0$, then from equation(3.2.1) $y = 0$ implies that $\eta = 0$, using equation (3.2.5) we note: $\sqrt{\frac{\nu b}{1-\alpha t}}f(0) = 0$, for $\nu > 0, \quad b > 0, \quad (1-\alpha t) > 0$, which becomes

$$f(0) = 0. \tag{3.2.20}$$

(iii) From equation (3.1.8) we note: at the free boundary $y = h, \frac{\partial u}{\partial y} = 0$. Then from equation (3.2.1) $y = h$ implies that $\eta = \sqrt{\frac{b}{\nu(1-\alpha t)}}h$, we denote this value of η by β. Now $\frac{\partial u}{\partial y} = 0$ implies that $\frac{1}{\sqrt{\nu}}(\frac{b}{1-\alpha t})^{\frac{3}{2}}xf''(\beta) = 0, \quad \nu > 0, \quad b > 0, \quad (1-\alpha t) > 0$. We can write

$$f''(\beta) = 0. \tag{3.2.21}$$

Initial and boundary conditions for heat transfer problem:

(i) From equation (3.1.8) we note: at $y = 0, \quad T = T_s$. Using equation (3.2.1) $y = 0$ implies $\eta = 0$, from equation (3.1.4) & (3.2.1), we have:
$T_0 - T_{ref}\frac{bx^2}{2\nu(1-\alpha t)^{\frac{3}{2}}} = T_0 - T_{ref}(\frac{bx^2}{2\nu})(1-\alpha t)^{\frac{-3}{2}}\theta(0)$ which implies

$$\text{At } \eta = 0, \ \ \theta(0) = 1. \tag{3.2.22}$$

(ii) From equation (3.1.8) we note: at the free boundary $y = h, \ \ \frac{\partial T}{\partial y} = 0$.
Using equation (3.2.1) $y = h$ implies that $\eta = \beta$ and from equation (3.2.1) we write:
$\frac{\partial T}{\partial y} = -T_{ref}(\frac{bx^2}{2\nu})\sqrt{\frac{b}{\nu}}(1-\alpha t)^{-2}\theta'(\beta) = 0, \quad$ where $\nu > 0, \ \ b > 0, \ \ 1-\alpha t > 0$, $x > 0, \ \ T_{ref} > 0$ which implies

$$\text{At } \eta = \beta, \ \ \theta'(\beta) = 0. \tag{3.2.23}$$

The momentum boundary layer problem defined by equation (3.2.8) with boundary conditions (3.2.19),(3.2.20),(3.2.21) is decoupled from the thermal boundary layer problem, while the temperature field $\theta(\eta)$ is on the other hand coupled to the velocity field.

Model:

To solve equation (3.2.8) with boundary conditions (3.2.19),(3.2.20),(3.2.21) and the equation (3.2.15) using boundary conditions (3.2.22) and (3.2.18).

$$Lf(\eta) = Nf(\eta), \quad \text{where } L = [\frac{d^3}{d\eta^3} - (M_n + S)\frac{d}{d\eta}], \;\; N = [\{\frac{S\eta}{2} - f(\eta)\}\frac{d^2}{d\eta^2} + (\frac{d}{d\eta})^2],$$
$$f'(0) = 1, \;\; f(0) = 0, \;\; \theta(0) = 1, \tag{3.2.24}$$

$$\theta''(\eta) = a(\eta)\theta(\eta) + b(\eta)\theta'(\eta) + c(\eta),$$
$$\text{where } a(\eta) = \frac{3P_rS}{2} + 2P_rf'(\eta), \;\; b(\eta) = \frac{P_rS\eta}{2} - P_rf(\eta), \;\; c(\eta) = -P_rE_cf''(\eta), \tag{3.2.25}$$
$$f''(\beta) = 0, \;\; \theta'(\beta) = 0.$$

The equation (3.2.24) and (3.2.25) is the model of our problem to solve.

3.3 Solution of the problem

3.3.1 Method of solution

Now we have to find the solution of boundary value problem (3.2.24) and (3.2.25) with boundary conditions. Different authors used different numerical and analytical methods to solve this type of problem. For example:

- **Finite difference method:**
  Finite difference method works reasonably well for linear boundary value problems. Difference methods can be adapted for non–linear problems but they require guessing at a tentative solution and then improving this by an iterative process. In addition to complexity of the programming required, there is no guarantee of convergence of iterations.

- **Shooting technique with RK- method:**
  Shooting method applies equally well to linear and non–linear problems. Again, there is no guarantee of convergence, but this method is easy to apply, and when it does converge, it is usually more efficient than other other iterative methods. Shooting technique with RK-method have a drawback of lengthy calculation and guessing for $f''(0)$ and $\theta'(0)$ in our problem. Generally, the numerical methods such as RungKutta method are based on discretization techniques, and they only permit us to calculate the approximate solutions for some values of time and space variables, which causes us to overlook some important phenomena such as chaos and bifurcation, in addition to the intensive computer time required to solve the problem. The above drawbacks of linearization and numerical methods arise the need to search for an alternative

techniques to solve the nonlinear differential equations, namely, the analytic solution methods, such as the perturbation method and the Adomian decomposition method.

- **Adomian decomposition method:**
  The Adomian decomposition method is quantitative rather than qualitative, analytic, requiring neither linearization nor perturbation and continuous with no resort to discretization. It consists of splitting the given equation into linear and nonlinear parts, inverting the highest-order derivative operator contained in the linear operator on both sides, identifying the initial and/or boundary conditions and the terms involving the independent variables alone as initial approximation, decomposing the unknown function into a series whose components are to be determined, decomposing the nonlinear function in terms of special polynomials called Adomians polynomials, and finding the successive terms of the series solution by recurrent relation using Adomian polynomials.

Procedure of Adomian's decomposition method: Adomian decomposition method (ADM) depends on decomposing the nonlinear differential equation

$$\digamma(\eta, f(\eta)) = 0$$

into the two components

$$(L + N)f(\eta) = 0,$$

where where L and N are the linear and the non–linear parts of $\digamma$ respectively. The operator L is assumed to be an invertible operator. Solving for $Lf(\eta)$ leads to

$$Lf(\eta) = -Nf(\eta) \tag{3.3.1}$$

where Applying the inverse operator L^{-1} on both sides of Eq. (3.3.1) yields

$$f(\eta) + \phi(\eta) = -L^{-1}Nf(\eta) \tag{3.3.2}$$

where $\phi(\eta)$ is the constant of integration satisfies the condition $L\phi = 0$
Now assuming that the solution $f(\eta)$ can be represented as infinite series of the form

$$f(\eta) = \sum_{n=0}^{\infty} f_n(\eta) \tag{3.3.3}$$

Furthermore, suppose that the nonlinear term $Nf(\eta)$ can be written as infinite series in terms of the Adomian polynomials A_n of the form

$$Nf(\eta) = \sum_{n=0}^{\infty} A_n(\eta) \tag{3.3.4}$$

where the Adomian polynomials A_n of $Nf(\eta)$ are evaluated using the formula

$$A_n = \frac{1}{n!}[\frac{d^n}{d\lambda^n} N \sum_{n=0}^{\infty} f_n(\eta)]_{\lambda=0}.$$

Then substituting Eqs. (3.3.3) and (3.3.4) in Eq. (3.3.2) gives

$$\sum_{n=0}^{\infty} f_n(\eta) = \phi(\eta) - L^{-1} \sum_{n=0}^{\infty} A_n(\eta) \tag{3.3.5}$$

Then equating the terms in the linear system of Eq. (3.3.5) gives the recurrent relation

$$f_0(\eta) = \phi(\eta) f_{n+1} = -L^{-1} A_n(\eta), \quad n \geq 07$$

However, in practice all the terms of series (3.3.5) cannot be determined, and the solution is approximated by the truncated series $\sum_{n=0}^{N} f_n(\eta)$. This method has been proven to be very efficient in solving various types of nonlinear boundary and initial value problems. However, ADM does not converge in general, in particular, when the method is applied to linear operator equations.

Homotopy: The homotopy is a continuous mapping from one deformation to other such that topological properties are preserved.

So we have adopted the homotopy decomposition method.

3.3.2 Homotopy decomposition method

To explain the basic idea, consider a general non–linear differential equation

$$\digamma u = f, \tag{3.3.6}$$

where $\digamma$ represents a general non–linear differential operator involving both linear and non–linear parts and f is continuous function. Identify linear L and non–linear N parts of $\digamma$. The linear part is decomposed as $L + R$, where L is invertible linear part and R is remainder of the linear operator. Thus equation (3.3.6) can be written as

$$[(L + R) + N]u = f, \quad \text{or} \quad Lu = f + \mathbf{N}u, \tag{3.3.7}$$

where $\mathbf{N}u = -(R + N)u$ represents non–linear terms. Consider a parametric family of non–linear differential equation by introducing homotopy parameter λ

$$Lu = f + \lambda \mathbf{N}u, \quad \lambda \epsilon [0, 1]. \tag{3.3.8}$$

For $\lambda = 0$ we have:

$$Lu = f,$$

a simple linear problem which can be easily solved. Applying L^{-1} if it exist, it follows

$$L^{-1} Lu = L^{-1}(f).$$

If L is of order n, then L^{-1} is the n-fold integral. Thus $L^{-1}Lu = u + \phi$ where ϕ is the term emerging from the integration and one gets

$$u + \phi = L^{-1}(f) \quad \text{or} \quad u = L^{-1}(f) - \phi.$$

Identifying u as the zeroth component of the solution i.e. $u_0 = L^{-1}f - \phi$.
For $\lambda = 1$ we have the original problem (3.3.7). Hence equation (3.3.8) is a homotopy between

$$Lu = f \text{ and } Lu = f + \mathbf{N}u.$$

For $\lambda \neq 0$ we have:

$$Lu = f + \lambda \mathbf{N}u \quad \text{implies that} \quad u = L^{-1}(f) - \phi + \lambda L^{-1}(\mathbf{N}u). \tag{3.3.9}$$

Assume the solution $u(\lambda, \eta)$ of the problem (3.3.9) can be expressed as an infinite series of the type

$$u(\lambda, \eta) = \Sigma \lambda^n u_n, \qquad n = 0, 1, 2, 3, \cdots, \tag{3.3.10}$$

and assume the non–linear part $\mathbf{N}u$ can be expressed as an infinite series of polynomials of the type

$$\mathbf{N}u(\lambda, \eta) = \Sigma \lambda^n A_n, \qquad n = 0, 1, 2, 3, \cdots, \tag{3.3.11}$$

where A_n are called Adomian's polynomials, which will be found later. Substituting equation (3.3.10) and (3.3.11) in (3.3.9), it follows that

$$\Sigma \lambda^n u_n = L^{-1}f - \phi - \lambda L^{-1} \Sigma \lambda^n A_n = L^{-1}f - \phi + L^{-1} \Sigma \lambda^{n+1} A_n.$$

Identifying the zeroth component u_0 as $L^{-1}f - \phi$ for $n = 0$ and $\lambda = 1$, the remaining components u_n, $n = 1, 2, 3, \cdots$ can be determined with comparing coefficients of powers of λ by the following relations

$$u_0 = L^{-1}f - \phi, \qquad u_1 = L^{-1}\mathbf{N}u_0 = L^{-1}A_0,$$
$$u_2 = L^{-1}\mathbf{N}u_1 = L^{-1}A_1, \quad u_3 = L^{-1}\mathbf{N}u_2 = L^{-1}A_2,$$
$$\cdots\cdots\cdots \qquad \cdots\cdots\cdots$$
$$u_n = L^{-1}\mathbf{N}u_{n-1} = L^{-1}A_{n-1}, \qquad u_{n+1} = L^{-1}\mathbf{N}u_n = L^{-1}A_n.$$

The polynomials $A_0, A_1, A_2, A_3, ..., A_{n-1}, A_n$ can be computed as follows :
Choose a parameter λ and set $u(\lambda, \eta) = \Sigma \lambda^n u_n, \qquad n = 0, 1, 2, 3, \cdots$
Then

$$\mathbf{N}(u(\lambda, \eta)) = \Sigma \lambda^n A_n, \qquad n = 0, 1, 2, 3, \cdots$$
$$\mathbf{N}(u(\lambda, \eta)) = A_0 + \lambda A_1 + \lambda^2 A_2 + \lambda^3 A_3 + \cdots + \lambda^{n-1} A_{n-1} + \lambda^n A_n,$$
$$\frac{d}{d\lambda}\mathbf{N}(u(\lambda, \eta)) = A_1 + 2\lambda A_2 + 3\lambda^2 A_3 + \cdots + (n-1)\lambda^{n-2} A_{n-1} + n\lambda^{n-1} A_n,$$
$$\frac{d^2}{d\lambda^2}\mathbf{N}(u(\lambda, \eta)) = 2!A_2 + 3!\lambda A_3 + \cdots + (n-1)(n-2)\lambda^{n-3} A_{n-1} + n(n-1)\lambda^{n-2} A_n,$$
$$\frac{d^3}{d\lambda^3}\mathbf{N}(u(\lambda, \eta)) = 3!A_3 + \cdots + (n-1)(n-2)(n-3)\lambda^{n-4} A_{n-1} + n(n-1)(n-2)\lambda^{n-3} A_n,$$

....

....

....

$$\frac{d^n}{d\lambda^n}\mathbf{N}(u(\lambda,\eta)) = n(n-1)(n-2)(n-3)\cdots(n-n+2)(n-n+1)\lambda^{n-n}A_n,$$
$$\frac{d^n}{d\lambda^n}\mathbf{N}(u(\lambda,\eta)) = n!A_n,$$
$$A_n = \frac{1}{n!}[\frac{d^n}{d\lambda^n}\mathbf{N}(u(\lambda,\eta))]_{\lambda=0}.$$

Define M-term approximant to the solution $u(\lambda,\eta)$ by

$$\phi_M[u] = \sum_{n=0}^{M} u_n$$

with

$$u = \lim_{M\to\infty}\phi_M[u],$$

where

$$Lu_0 = 0 \text{ implies } u_0, \qquad Lu_1 = A_0 \text{ implies } u_1,$$
$$Lu_2 = A_1 \text{ implies } u_2, \qquad Lu_3 = A_2 \text{ implies } u_3,$$
$$Lu_4 = A_3 \text{ implies } u_4, \qquad Lu_5 = A_4 \text{ implies } u_5,$$
$$Lu_6 = A_5 \text{ implies } u_6, \qquad Lu_7 = A_6 \text{ implies that } u_7,$$

$$u = u_0 + u_1 + u_2 + u_3 + u_4 + u_5 + u_6 + u_7 + u_8, \text{ up to 8th order}$$

is the solution of non–linear ordinary differential equation.

Convergence of the Decomposition scheme was established by many authors by using fixed point theorems [44].

3.3.3 Analytical solution of the flow problem

Fluid flow equation (3.2.8) is a third order ordinary non–linear differential equation. So we need at least three boundary conditions (3.2.19 – 3.2.21) to solve (3.2.8). The Decomposition method connected with homotopy is being adopted to solve the fluid flow equation (3.2.8). Keeping in view the fluid flow equation (3.2.8) put

$$\mathbf{N}f(\eta) = g(\eta,f,f',f'') = [\frac{S\eta}{2} - f(\eta)]f''(\eta) + f'^2(\eta), \quad \text{we obtain:}$$

$$Lf(\eta) = g(\eta,f,f',f'').$$

Choose a parameter λ such that

$$Lf(\lambda,\eta) = \lambda g(\eta,f,f',f''). \tag{3.3.12}$$

Case I : For $\lambda = 0$, the equation (3.3.12) becomes $Lf(\eta) = 0$, which implies

$$f'''(\eta) - (S + M_n)f'(\eta) = 0.$$

Put $a = (S + M_n)$ implies that $f'''(\eta) - af'(\eta) = 0$ with boundary conditions given by equation (3.2.19 – 3.2.20) is a simple linear problem and homogenous part of our problem which can be easily solved.

Homogeneous solution or complementary solution:

$$f(\eta) = \frac{e^{-\sqrt{a}\eta}(1 + e^{\sqrt{a}\eta})(e^{2\sqrt{a}\beta} + e^{\sqrt{a}\eta})}{\sqrt{a}(1 + e^{2\sqrt{a}\beta})}.$$

Here we assume that

$$f_0(\eta) = f(\eta) = \frac{e^{-\sqrt{a}\eta}(1 + e^{\sqrt{a}\eta})(e^{2\sqrt{a}\beta} + e^{\sqrt{a}\eta})}{\sqrt{a}(1 + e^{2\sqrt{a}\beta})}.$$

Case II: For $\lambda = 1$, we obtain original problem of MHD flow from equation (3.2.8),

$$f'''(\eta) - af'(\eta) = g(\eta, f, f', f'') = \mathbf{N}f(\eta) = [\frac{S\eta}{2} - f(\eta)]f''(\eta) + f'^2.$$

Non–homogeneous solution: We assume here the Decomposition series solution in terms of solution components as

$$f(\eta) = \Sigma\lambda^i f_i(\eta) \qquad \text{while} \qquad g(\eta, f, f', f'') = \Sigma\lambda^i A_i, \;\; i = 0, 1, 2, 3, \cdots. \tag{3.3.13}$$

The equation (3.3.12) becomes

$$\Sigma\lambda^i f_i'''(\eta) - a\Sigma\lambda^i f_i'(\eta) = \lambda\Sigma\lambda^i A_i \quad \text{which implies} \quad \lambda^i[\Sigma f_i'''(\eta) - a\Sigma f_i'(\eta)] = \lambda^{i+1}\Sigma A_i,$$

with boundary conditions $\Sigma\lambda^i f_i(0) = 0$, $\Sigma\lambda^i f_i'(0) = 1$, $\Sigma\lambda^i f_i''(\beta) = 0$ for $i = 0, 1, 2, 3, \cdots$.
On comparing coefficient of λ, we get:

$$\begin{array}{lll}
f_0'''(\eta) - af_0'(\eta) = 0, \;\; \text{for} & \lambda = 0, & f_0(0) = 0, f_0'(0) = 1, f_\beta''(0) = 0, \\
f_1'''(\eta) - af_1'(\eta) = A_0 & \lambda = 1, & f_1(0) = 0, f_1'(0) = 0, f_1''(\beta) = 0, \\
f_2'''(\eta) - af_2'(\eta) = A_1 & , & f_2(0) = 0, f_2'(0) = 0, f_2''(\beta) = 0, \\
f_3'''(\eta) - af_3'(\eta) = A_2 & , & f_3(0) = 0, f_3'(0) = 0, f_3''(\beta) = 0, \\
\cdots & , & \cdots \\
f_7'''(\eta) - af_7'(\eta) = A_6 & , & f_7(0) = 0, f_7'(0) = 0, f_7''(\beta) = 0. \\
\cdots & , & \cdots.
\end{array}$$

Now we need to seek components of non–linear operator $A_i's$. For this, we have:

$$g(\eta, f, f', f'') = A_0 + \lambda A_1 + \lambda^2 A_2 + \lambda^3 A_3 + \cdots + \lambda^{n-1} A_{n-1} + \cdots. \tag{3.3.14}$$

One can find these components with the help of Taylor's series of $g(\eta, f, f', f'')$ at the initial point

$$g(\eta, f, f', f'') = [\frac{S\eta}{2} - f(\eta)]f''(\eta) + f'^2.$$

Taking higher partial derivatives of $g(\eta, f, f', f'')$ with respect to f, f', f'', we obtain:

$$\begin{aligned}
&g_f(\eta, f, f', f'') = -f''(\eta), \quad g_{f'}(\eta, f, f', f'') = 2f'(\eta), \quad g_{f''}(\eta, f, f', f'') = [\frac{S\eta}{2} - f(\eta)],\\
&g_{ff}(\eta, f, f', f'') = 0, \quad g_{ff'}(\eta, f, f', f'') = 0, \quad g_{ff''}(\eta, f, f', f'') = -1,\\
&g_{f'f}(\eta, f, f', f'') = 0, \quad g_{f'f'}(\eta, f, f', f'') = 2, \quad g_{f'f''}(\eta, f, f', f'') = 0,\\
&g_{f''f}(\eta, f, f', f'') = -1, \quad g_{f''f'}(\eta, f, f', f'') = 0, \quad g_{f''f''}(\eta, f, f', f'') = 0.
\end{aligned}$$

Computing all these partial derivatives at the initial point (η, f_0, f'_0, f''_0) or "0"

$$\begin{aligned}
&g(0) = [\frac{S\eta}{2} - f_0(\eta)]f''_0(\eta) + f'^2_0, \; g_f(0) = -f''_0(\eta), \; g_{f'}(0) = 2f'_0\eta), \; g_{f''}(0) = [\frac{S\eta}{2} - f_0(\eta)],\\
&g_{ff}(0) = 0, \quad g_{ff'}(0) = 0, \quad g_{ff''}(0) = -1, \quad g_{f'f}(0) = 0, \quad g_{f'f'}(0) = 2,\\
&g_{f'f''}(0) = 0, \quad g_{f''f}(0) = -1, \quad g_{f''f'}(0) = 0, \quad g_{f''f''}(0) = 0.
\end{aligned}$$

Applying Taylor's series, we have:

$$\begin{aligned}
g(\eta, f, f', f'') &= g(0) + (f - f_0)g_f(0) + (f' - f'_0)g_{f'}(0) + (f'' - f''_0)g_{f''}(0)\\
&+ \frac{1}{2!}[(f - f_0)(f'' - f''_0)2g_{f''f}(0) + (f' - f'_0)^2 g_{f'f'}(0)],\\
&= -f''_0 f_0 + f'^2_0 + \frac{S\eta}{2}f''_0 + (f - f_0)(-f''_0) + (f' - f'_0)(2f'_0)\\
&+ (f'' - f''_0)(\frac{S\eta}{2} - f_0) + \frac{1}{2!}[(f - f_0)(f'' - f''_0)2(-1) + (f' - f'_0)^2(2)].
\end{aligned}$$

From equation (3.3.13) $\quad f(\eta) = \Sigma\lambda^i f_i(\eta), \; i = 0, 1, 2, 3, \cdots$.

$$\begin{aligned}
g(\eta, f, f', f'') &= -f''_0 f_0 + f'^2_0 + \frac{S\eta}{2}f''_0 + (\Sigma\lambda^i f_i - f_0)(-f''_0) + (\Sigma\lambda^i f'_i - f'_0)(2f'_0)\\
&+ (\Sigma\lambda^i f''_i - f''_0)(\frac{S\eta}{2} - f_0) + \frac{1}{2!}[(\Sigma\lambda^i f_i - f_0)(\Sigma\lambda^i f''_i - f''_0)2(-1)\\
&+ (\Sigma\lambda^i f'_i - f'_0)^2(2)],\\
&= -f''_0 f_0 + f'^2_0 + \frac{S\eta}{2}f''_0 + (\Sigma\lambda^i f_i)(-f''_0) + (\Sigma\lambda^i f'_i)(2f'_0)\\
&+ (\Sigma\lambda^i f''_i)(\frac{S\eta}{2} - f_0) + \frac{1}{2!}[(\Sigma\lambda^i f_i)(\Sigma\lambda^i f''_i)2(-1) + (\Sigma\lambda^i f'_i)^2(2)].
\end{aligned}$$

$$\begin{aligned}g(\eta, f, f', f'') &= -f_0''f_0 + f_0'^2 + \frac{S\eta}{2}f_0'' + \lambda[-f_0''f_1 + 2f_0'f_1' + (\frac{S\eta}{2} - f_0)f_1''] \\
&+ \lambda^2[-f_0''f_2 + 2f_0'f_2' + (\frac{S\eta}{2} - f_0)f_2'' - f_1''f_1 + (f_1')^2] \\
&+ \lambda^3[-f_0''f_3 + 2f_0'f_3' + (\frac{S\eta}{2} - f_0)f_3'' - (f_1''f_2 + f_2''f_1) + 2f_1'f_2'] \\
&+ \lambda^4[-f_0''f_4 + 2f_0'f_4' + (\frac{S\eta}{2} - f_0)f_4'' - (f_1''f_3 + f_2''f_2 + f_3''f_1) + (2f_1'f_3' + (f_2')^2] \\
&+ \lambda^5[-f_0''f_5 + 2f_0'f_5' + (\frac{S\eta}{2} - f_0)f_5'' - (f_1''f_4 + f_2''f_3 + f_3''f_2 + f_4''f_1) \\
&+ (f_5'f_1 + f_4'f_2 + f_3'f_3 + f_2'f_4 + f_1'f_5] \\
&+ \lambda^6[-f_0''f_6 + 2f_0'f_6' + (\frac{S\eta}{2} - f_0)f_6'' - (f_1''f_5 + f_2''f_4 + f_3''f_4 + f_2''f_4 + f_1''f_5) \\
&+ (2f_5'f_1 + 2f_4'f_2 + (f_3')^2)].\end{aligned}$$

Comparing the coefficient of λ with following equation

$$g(\eta, f, f', f'') = A_0 + \lambda A_1 + \lambda^2 A_2 + \lambda^3 A_3 + \cdots + \lambda^{n-1}A_{n-1} + \cdots.$$

we get

$$\begin{aligned}&A_0 = -f_0''f_0 + f_0'^2 + \frac{S\eta}{2}f_0'', \quad A_1 = -f_0''f_1 + 2f_0'f_1' + (\frac{S\eta}{2} - f_0)f_1'', \\
&A_2 = -f_0''f_2 + 2f_0'f_2' + (\frac{S\eta}{2} - f_0)f_2'' - f_1''f_1 + (f_1')^2, \\
&A_3 = -f_0''f_3 + 2f_0'f_3' + (\frac{S\eta}{2} - f_0)f_3'' - (f_1''f_2 + f_2''f_1) + 2f_1'f_2', \\
&\cdots \\
&A_6 = -f_0''f_6 + 2f_0'f_6' + (\frac{S\eta}{2} - f_0)f_6'' - (f_1''f_5 + f_2''f_4 + f_3''f_4 + f_2''f_4 + f_1''f_5) \\
&\quad + (2f_5'f_1 + 2f_4'f_2 + (f_3')^2).\end{aligned}$$

Then

$$\begin{aligned}&f_0'''(\eta) - af_0'(\eta) = 0, \\
&f_1'''(\eta) - af_1'(\eta) = -f_0''f_0 + f_0'^2 + \frac{S\eta}{2}f_0'', \\
&f_2'''(\eta) - af_2'(\eta) = -f_0''f_1 + 2f_0'f_1' + (\frac{S\eta}{2} - f_0)f_1'', \\
&f_3'''(\eta) - af_3'(\eta) = -f_0''f_2 + 2f_0'f_2' + (\frac{S\eta}{2} - f_0)f_2'' - f_1''f_1 + (f_1')^2, \\
&f_4'''(\eta) - af_4'(\eta) = -f_0''f_3 + 2f_0'f_3' + (\frac{S\eta}{2} - f_0)f_3'' - (f_1''f_2 + f_2''f_1) + 2f_1'f_2', \\
&f_5'''(\eta) - af_5'(\eta) = -f_0''f_4 + 2f_0'f_4' + (\frac{S\eta}{2} - f_0)f_4'' - (f_1''f_3 + f_2''f_2 + f_3''f_1) + (2f_1'f_3' + (f_2')^2, \\
&f_6'''(\eta) - af_6'(\eta) = -f_0''f_5 + 2f_0'f_5' + (\frac{S\eta}{2} - f_0)f_5'' - (f_1''f_4 + f_2''f_3 + f_3''f_2 + f_4''f_1) \\
&\qquad + (f_5'f_1 + f_4'f_2 + f_3'f_3 + f_2'f_4 + f_1'f_5), \\
&f_7'''(\eta) - af_7'(\eta) = -f_0''f_6 + 2f_0'f_6' + (\frac{S\eta}{2} - f_0)f_6'' - (f_1''f_5 + f_2''f_4 + f_3''f_4 + f_2''f_4 + f_1''f_5) \\
&\qquad + (2f_5'f_1 + 2f_4'f_2 + (f_3')^2).\end{aligned}$$

We obtain the dimensionless function $f(\eta)$ to find stream function $\psi(x, y, t)$

$$f(\eta) = f_0 + f_1 + f_3 + f_4 + f_5 + f_6 + f_7. \tag{3.3.15}$$

The equation (3.3.15) will be used to solve the coupling equation of the model (3.2.24).

3.3.4 Solution of the heat flow problem

The dimensionless heat flow problem given in the model(3.2.24):

$$\begin{aligned} &\theta''(\eta) = a(\eta)\theta(\eta) + b(\eta)\theta'(\eta) + c(\eta), \\ &\text{where } a(\eta) = \frac{3P_r S}{2} + 2P_r f'(\eta),\ b(\eta) = \frac{P_r S\eta}{2} - P_r f(\eta),\ c(\eta) = -P_r E_c f''(\eta), \\ &\theta(0) = 1,\ \theta'(\beta) = 0, \end{aligned} \tag{3.3.16}$$

is a coupling equation containing $f(\eta)$, $f'(\eta)$, $f''(\eta)$. It is a second linear order differential equation with variable coefficients. We can easily solve it with the help of Mathematica. The code of Mathematica is given in the Appendix A. The tables obtained have been explained in detail in the Appendix.

3.4 Results and discussion

3.4.1 Parametric effect of M_n and S on the film thickness

We can adjust the dimensionless film thickness β by variation of the unsteadiness parameter S and magnetic parameter M_n. The numerical results are obtained for small values of S and M_n. Some typical values of β are shown in the table(A.4) given in the Appendix A. Figure 3.2 shows the asymptotic graph of β verses S and Figure 3.3 shows the decreasing behavior of the dimensionless film thickness as $S \geq 0.8$. From these two figures we note that β decreases strictly for $S \geq 0.8$ and increases rapidly as $S < 0.8$. As $S \to 0$ the solution approaches to analytical solution obtained by Crane [3] with infinitely thick layer of fluid ($\beta \to \infty$). The other limiting solution corresponding to $S \to \infty$ represents a thin liquid film of infinitesimal thickness ($\beta \to 0$).

The effect of magnetic parameter on the film thickness is clear in Figure 3.4 which shows that increasing values of M_n decrease the film thickness. The variation of film thickness β with respect to the magnetic parameter M_n is projected in Figure 3.5 for two different values of $S = 0.8$ and $S = 1.2$. The combined effect of S and M_n is shown in Figure 3.5. It is evident from the graph in Figure 3.5 that both parameters decrease the film thickness monotonically by increasing their positive values. It is clear from this plot that film thickness decreases as increasing values of unsteadiness and magnetic parameter. The particular value of M_n, say $M_n \geq 10$ changes the fluid layer into thin film.

Local skin friction coefficient or drag coefficient: The local skin friction coefficient, which is practically important to estimate the heat losses, is denoted by C_f and is given by

$$C_f \equiv \frac{\text{shearing stress}}{\text{kinetic energy}} = \frac{\tau_0}{\frac{1}{2}\rho U^2} = \frac{-2\mu(\frac{\partial u}{\partial y})_{y=0}}{\rho U^2} = -2R_e^{-\frac{1}{2}} f''(0). \tag{3.4.1}$$

It is also known as drag coefficient per unit area.

Nusselt number: The local Nusselt number is used to calculate the heat transfer between the surface (sheet) and the fluid, which is conventionally expressed in dimensionless form as

$$Nu_x \equiv -\frac{x}{T_{ref}}(\frac{\partial T}{\partial y})_{y=0} = \frac{1}{2}(1-\alpha t)^{-\frac{1}{2}}\theta'(0)R_e^{\frac{3}{2}}. \tag{3.4.2}$$

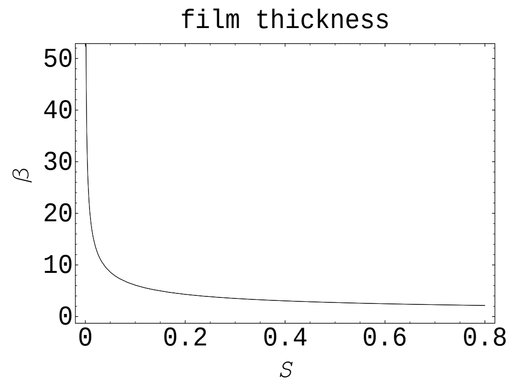


Figure 3.2: Variation of film thickness β with variation of $S = 0 - 0.8$.

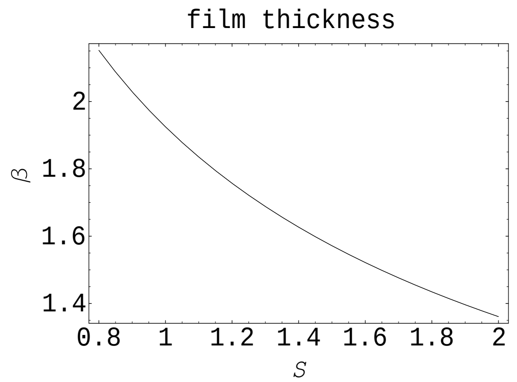


Figure 3.3: Variation of film thickness β with variation of $S = 0.8 - 2.0$.

3.4.2 Effect of M_n on fluid flow and heat transfer

The variation of free surface velocity $f'(\beta)$ with respect to magnetic parameter M_n for two different values of $S = 0.8, 1.26$, is shown in Figure 3.6. Diagram shows that the free surface velocity initially increases by increasing unsteadiness parameter S but behaves almost as

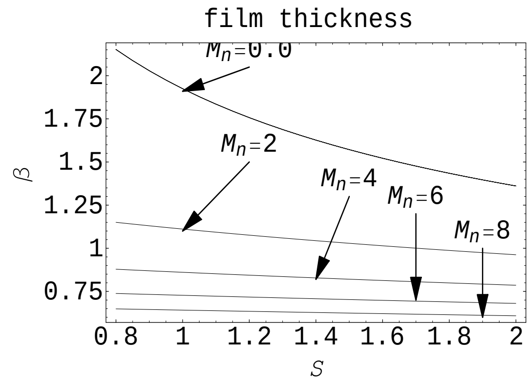


Figure 3.4: Variation of β with variation of $S = 0.0 - 2.0$ for $M_n = 0.0, 2, 4, 6, 8$.

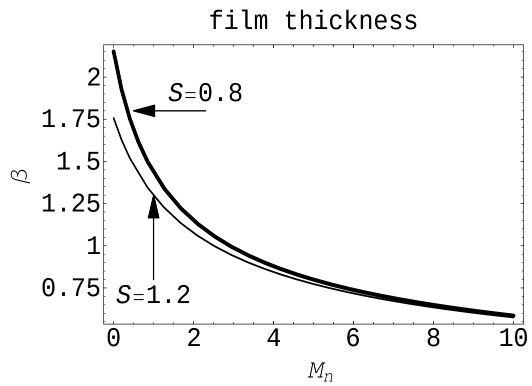


Figure 3.5: Variation of film thickness β with variation of M_n

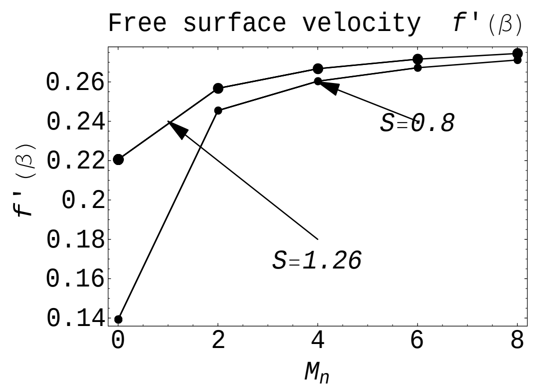


Figure 3.6: Variation of surface velocity $f'(\beta)$ verses M_n for $P_r = 10$, $E_c = 0.8$

a constant function of M_n. The effect of M_n on the wall shear stress parameter $-f''(0)$ is illustrated in Figure 3.7. Clearly increasing values of M_n result in increasing the wall shear stress. Also wall shear stress decreases by decreasing the unsteadiness parameter. Figure 3.8 demonstrates the effect of M_n on the free surface temperature $\theta(\beta)$. This figure has two graphs, one for $S = 0.8$ containing dots of small thickness and other for $S = 1.2$ containing thicker dots. From this plot it is evident that the free surface temperature

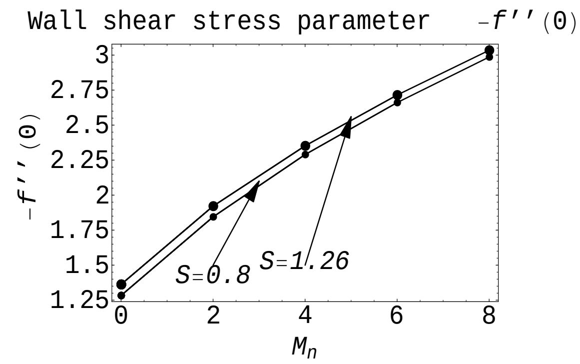


Figure 3.7: Variation of wall shear stress $-f''(0)$ verses M_n for $P_r = 10$, $E_c = 0.8$

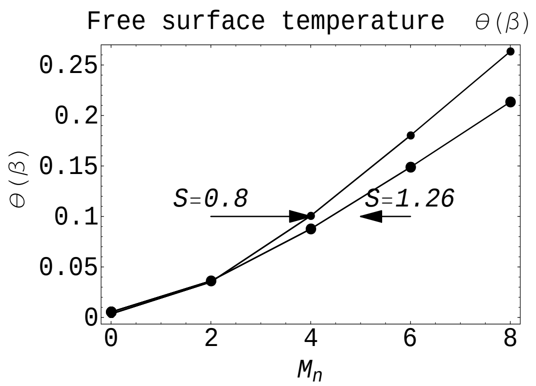


Figure 3.8: Variation of free surface temperature $\theta(\beta)$ verses M_n for $P_r = 10$, $E_c = 0.8$

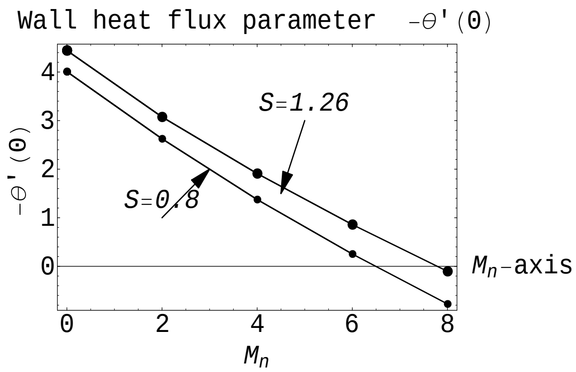


Figure 3.9: Variation of wall dimensionless heat flux $-\theta'(0)$ verses M_n for $P_r = 10$, $E_c = 0.8$

increases monotonically with M_n but temperature decreases with increase of unsteadiness parameter S. Figure 3.9 highlights the effect of M_n on the dimensionless wall heat flux $-\theta'(0)$. It is found from this plot that the dimensionless heat flux $-\theta'(0)$ decreases with the increasing value of M_n and S.

The effect of M_n on the axial velocity is depicted in Figure 3.10 and 3.11 for two different values of S. Mathematically we can analyze that velocity profile of thin liquid

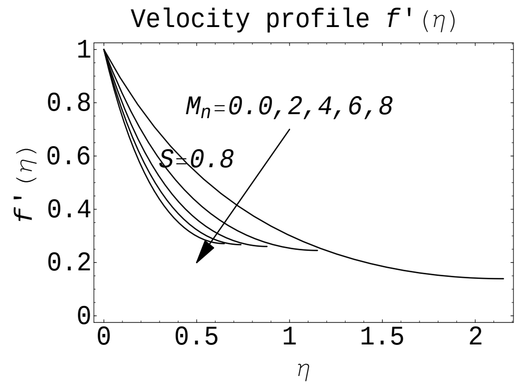


Figure 3.10: Variation of velocity profile $f'(\eta)$ verses η for $P_r = 10$, $E_c = 0.8$, $S = 0.8$

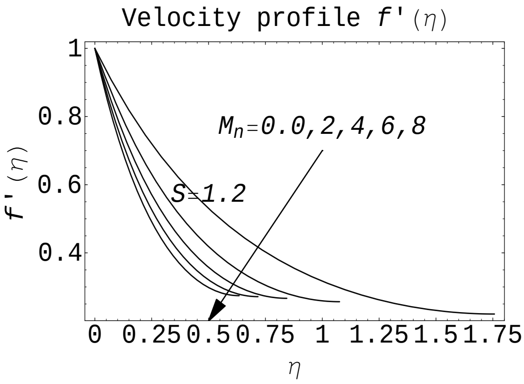


Figure 3.11: Variation of velocity profile $f'(\eta)$ verses η for $P_r = 10$, $E_c = 0.8$, $S = 1.2$

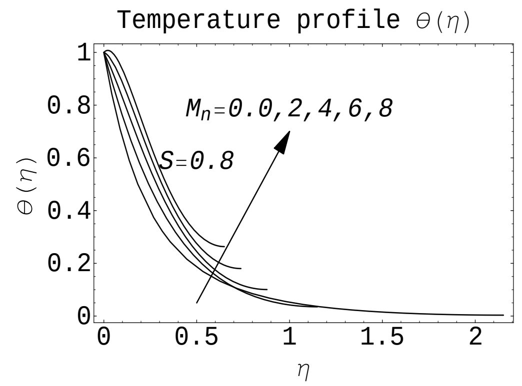


Figure 3.12: Variation of temperature profile $\theta(\eta)$ verses η for $P_r = 10$, $E_c = 0.8$, $S = 0.8$

film is decreasing with increasing values of dimensionless variable. The rate of decrease of velocity becomes zero after a particular value of magnetic and unsteadiness parameter. The same behavior is with temperature profile. From these plots it is clear that the increasing values of M_n decreases the axial velocity. This is due to fact that applied transverse magnetic field produces a drag in the form of Lorentz force thereby decreasing the magnitude of velocity. The drop in horizontal velocity as a consequence of increase

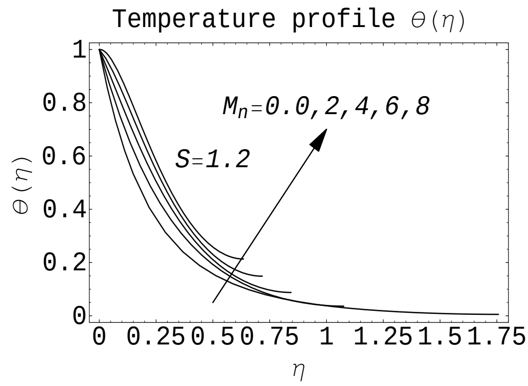


Figure 3.13: Variation of temperature profile $\theta(\eta)$ verses η for $P_r = 10$, $E_c = 0.8$, $S = 1.2$

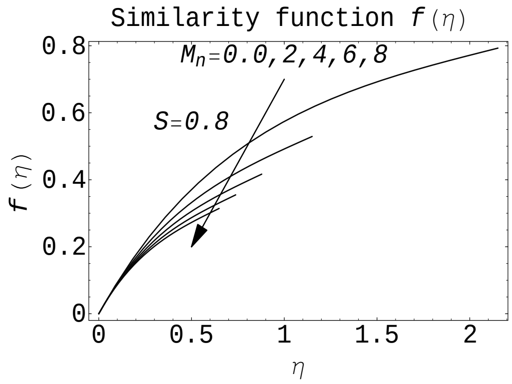


Figure 3.14: Variation of similarity function $f(\eta)$ verses η for $P_r = 10$, $E_c = 0.8$, $S = 0.8$

in the strength of magnetic field can be observed for both the values of of $S = 0.8$ and $S = 1.2$. Figure 3.12 and Figure 3.13 depict the effect of M_n on temperature profiles for two different values of S. The results show that the thermal boundary layer thickness increases with the increasing values of M_n. The increasing frictional drag due to Lorentz force is responsible for increasing the thermal boundary layer thickness. Figure 3.14 shows the graph of similarity function $f(\eta)$ is increasing with η which reveals the fact that concentration of the fluid (quantity of fluid) is increasing on the sheet.

3.4.3 Effect of P_r on the fluid flow and heat transfer

The effects of the Prandtl number P_r and the magnetic parameter M_n on the surface temperature $\theta(\beta)$ are respectively illustrated in the next figures. We have observed from previous graphs and tables in Appendix A that, increasing values of the magnetic parameter M_n cause the surface temperature to blow up monotonically. The opposite effect is exhibited in case of the Prandtl number P_r, that is, increasing values of P_r decrease the surface temperature as shown in Figure 3.15. For the Parandtl number of order unity and below the surface temperature $\theta(\beta)$ attains a finite value below "1" and the temperature gradients extend all the way to the free surface. In the limiting case $P_r \to 0$, however, the dimensionless surface temperature tends to unity that is, the temperature T of liquid becomes uniform in the vertical direction and equals temperature T_s of sheet. This is consistent with the trivial solution $\theta(\beta) = 1$ obtained from the thermal energy equation (3.2.15) when $P_r = 0$. At sufficiently high Prandtl number, that is, low thermal diffusivity, the surface temperature remained practically equal to zero.

Figures (3.16, 3.17) demonstrate the effect of the Prandtl number P_r on temperature profiles $\theta(\eta)$ for two different values of the unsteadiness parameter S. These plots reveals the fact that for a particular value of P_r the temperature increases monotonically from the free surface temperature T_s to wall temperature T_0 as observed by Anderson [27]. The thermal boundary layer thickness decreases drastically for high values of P_r that is, low thermal diffusivity. Figures 3.18 reveals that the magnetic parameter and unsteadiness parameter increases the wall dimensionless heat flux for all Parandtl numbers in absence of the Eckert number. Figures 3.19 shows that the wall shear stress becomes constant for the particular value of the Parandtl number $P_r \geq 10$.

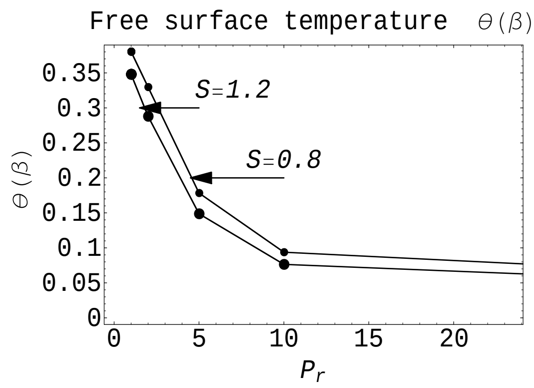


Figure 3.15: Variation of free surface temperature $\theta(\beta)$ verses P_r for $M_n = 2-10$, $E_c = 0.0$

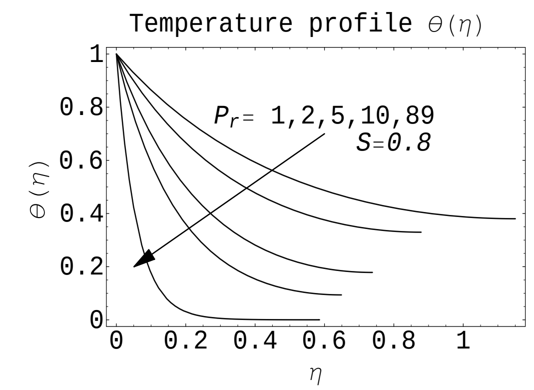


Figure 3.16: Variation of temperature profile $\theta(\eta)$ verses η for $M_n = 2 - 10$, $E_c = 0.0$, $S = 0.8$

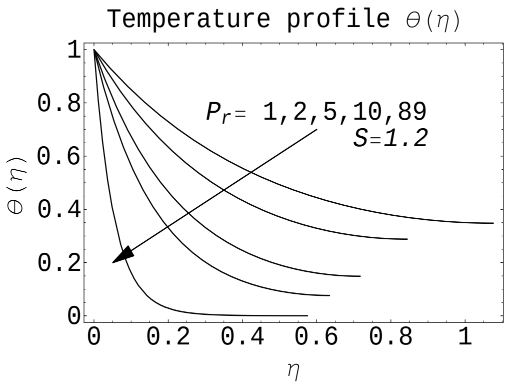


Figure 3.17: Variation of $\theta(\eta)$ verses η for $M_n = 2 - 10$, $E_c = 0.0$, $S = 1.2$

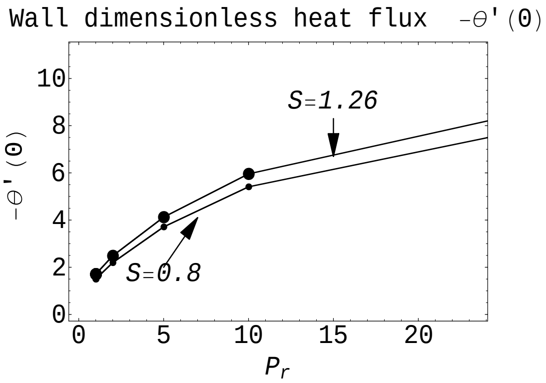


Figure 3.18: Variation of $-\theta'(0)$ verses P_r for $M_n = 2 - 10$, $E_c = 0.0$

3.4.4 Effect of E_c on the fluid flow and heat transfer

The Eckert number characterize the viscous dissipation. When the fluid is being heated ($E_c > 0$), the dimensionless temperature will increase but decrease when the fluid is being cooled ($E_c < 0$). The dimensionless fluid temperature decreases with η monotonically for a positive E_c; while for a negative E_c, θ initially decreases rapidly with η, attains a minimum value and then increases more gradually to its free surface value $\theta(\beta)$. ($E_c = 0$) shows the case of no viscous dissipation.

Figure 3.20 shows the temperature distribution verses η for $S = 0.8$ while Figure 3.21 for $S = 1.2$. By analyzing the graphs individually it reveals that the effect of increasing values of E_c is to increase the temperature distribution in flow region. These graphs also demonstrates the variation of temperature profiles for various values of E_c. It is observed from these graphs that the effect of viscous dissipation is to amplify the temperature. This is due to fact that heat energy is stored in the liquid due to frictional heating. The effect of $E_c > 0$ on the free surface temperature $\theta(\beta)$ has been shown in Figure 3.22. Also the decreasing behavior of wall dimensionless heat flux $-\theta'(0)$ may be noted in Figure 3.23. At particular values of S and M_n Figure 3.24 reveals that the dimensionless free surface velocity $f'(\beta)$ increases for increasing values of Eckert E_c. Figure 3.25 looks the behavior of shear stress, that is, the shear stress increases in the thermal boundary layer whose distance from the sheet is less. As the liquid layer move away from the sheet, the shear stress rate becomes zero.

By increasing unsteadiness parameter similarity function decreases and flow velocity decreases with negligible shear stress (Table A.3). The temperature of the boundary layer increases by increasing of magnetic parameter at a certain level of unsteadiness parameter(Table A.3). Keeping constant magnetic parameter and increasing unsteadiness parameter, temperature of boundary layer initially increases and then cools down.

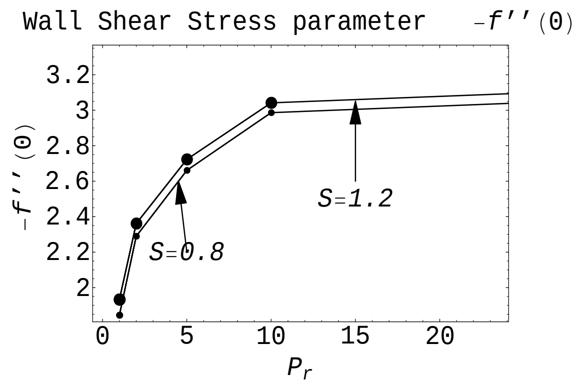


Figure 3.19: Variation of $-f''(0)$ verses P_r for $M_n = 2 - 10$, $E_c = 0.0$

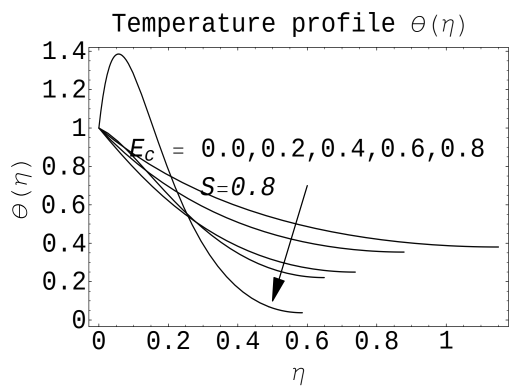


Figure 3.20: Variation of $\theta(\eta)$ verses η for $M_n = 0.0 - 8.0$, $P_r = 1, 2, 5, 10, 89$, $S = 0.8$

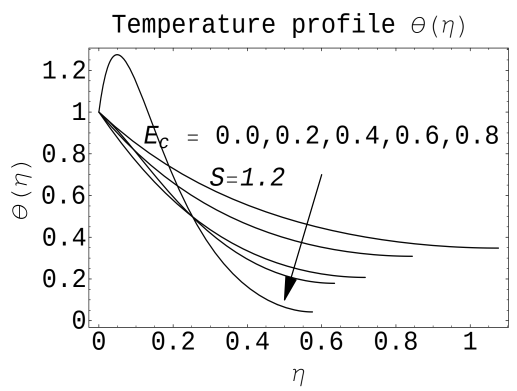


Figure 3.21: Variation of $\theta(\eta)$ verses η for $M_n = 0.0 - 8.0$, $P_r = 1, 2, 5, 10, 89$, $S = 1.2$

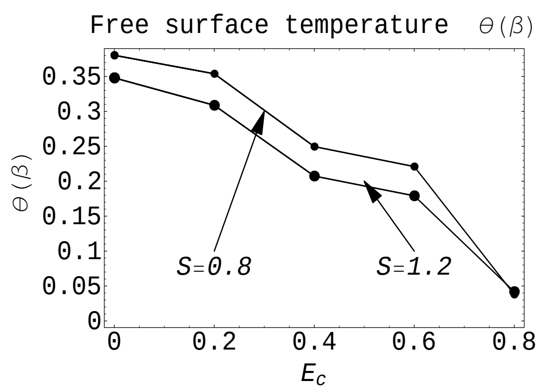


Figure 3.22: Variation of $\theta(\beta)$ verses E_c for $M_n = 0.0 - 8.0$, $P_r = 1, 2, 5, 10, 89$

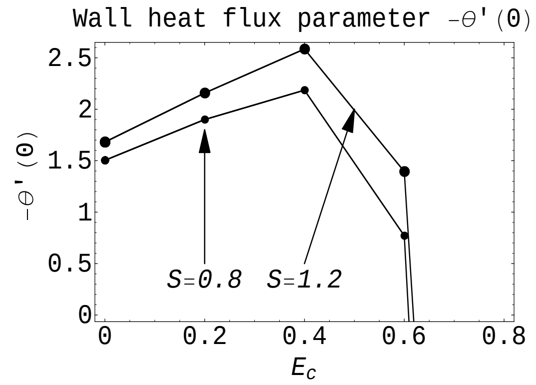


Figure 3.23: Variation of $\theta'(0)$ verses E_c for $M_n = 0.0 - 8.0$, $P_r = 1, 2, 5, 10, 89$

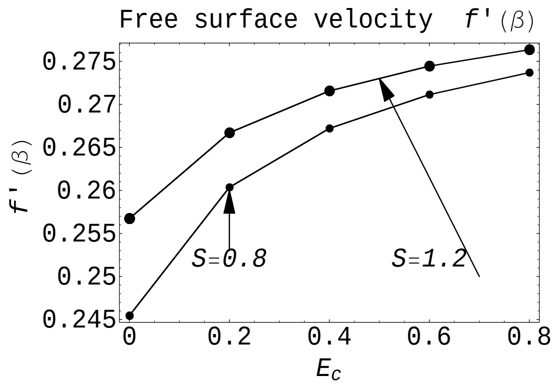


Figure 3.24: Variation of $f'(\beta)$ verses E_c for $M_n = 0.0 - 8.0$ and $P_r = 1, 2, 5, 10, 89$

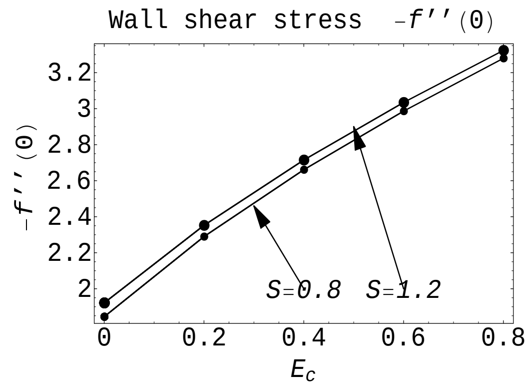


Figure 3.25: Variation of $f''(\beta)$ verses E_c for $M_n = 0.0 - 8.0$ and $P_r = 1, 2, 5, 10, 89$

Chapter 4

Conclusion

This dissertation aims to present the analysis of a class of non–linear problems of the mass, momentum and heat transfer of unsteady flow of thin liquid film over a horizontal elastic stretching sheet. We use the homotopy decomposition method to solve the system. With the help of this method, One can control four parameters (the unsteady parameter, magnetic parameter, Parandtl number and Eckert number) to obtain best quality of product. The analytic and purely numerical solutions agree very well. Based on the case investigated, the magnetic parameter M_n has a large effect on the velocity as compared to the temperature. It is hoped that this dissertation is helpful to understand the flow and heat transfer mechanism of a liquid film and would find applications in different technologies and manufacturing industries, such as polymer extrusion. The effect of the Eckert number in presence of heat source/sink on temperature profile will be dealt in future work.

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Appendix A

A.1 Mathematica Code

Taking different values of $i = 1, 2, 3, ...$; with different parameters, we will check the effect of parameters on the behavior of fluid flow and heat transfer in a liquid film over a stretching sheet.

S=0.3 i; $h = 0.01$; $\sigma = 0.05$; $\rho = 0.1$; $B = 0.84528$; $M_n = 2.0$; $a = M_n + S$; $Pr = 3$; $E_c = 0.1$; $\beta = 2.0$;

$de = NDSolve[\{u'''[\eta] - au'[\eta]] == 0, u[0] == 0, u'[0] == 1, u''[\beta] == 0\}, u[\eta], \{\eta, 0, \beta\}];$

$f_0[\eta\_] := Evaluate[u[\eta]]/.First[de]]$

$A_0[\eta\_] = -f_0''[\eta]f_0[\eta] + f_0'[\eta]f_0'[\eta] + \frac{\eta S}{2}f_0''[\eta];$

$de1 = NDSolve[\{u'''[\eta] - au'[\eta] == A_0[\eta], u[0] == 0, u'[0] == 0, u''[\beta] == 0\}, u[\eta], \{\eta, 0, \beta\}];$

$f_1[\eta\_] := Evaluate[u[\eta]/.First[de1]]$

$A_1[\eta\_] = -f_0''[\eta]f_1[\eta] + 2f_0'[\eta]f_1'[\eta] + (\frac{\eta S}{2} - f0[\eta])f_1''[\eta]$

$de2 = NDSolve[\{u'''[\eta] - au'[\eta] == A_1[\eta], u[0] == 0, u'[0] == 0, u''[\beta] == 0\}, u[\eta], \{\eta, 0, \beta\}];$

$f_2[\eta\_] := Evaluate[u[\eta]/.First[de2]];$

$A_2[\eta\_] = -f_0''[\eta]f_2[\eta] + 2f_0'[\eta]f_2'[\eta] + (\frac{\eta S}{2} - f_0[\eta])f_2''[\eta] - f_1[\eta]f_1''[\eta] + (f_1'[\eta])^2;$

$de4 = NDSolve[\{u'''[\eta] - au'[\eta] == A_2[\eta], u[0] == 0, u'[0] == 0, u''[\beta] == 0\}, u[\eta], \{\eta, 0, \beta\}];$

$f_3[\eta\_] := Evaluate[u[\eta]/.First[de4]]$

$$A_3[\eta\_] = -f_0''[\eta]f_3[\eta] + 2f_0'[\eta]f_3'[\eta] + (\frac{\eta S}{2} - f_0[\eta])f_3''[\eta] - f_1[\eta]f_2''[\eta] - f_2[\eta]f_1''[\eta] + f_1'[\eta]f_2'[\eta] + f_1'[\eta]f_2'[\eta]$$

$de5 = NDSolve[\{u'''[\eta] - au'[\eta] == A_3[\eta], u[0] == 0, u'[0] == 0, u''[\beta]] == 0\}, u[\eta], \{\eta, 0, \beta\}];$

$f_4[\eta\_] := Evaluate[u[\eta]/.First[de5]];$

$A_4[\eta\_] = -f_0''[\eta]f_4[\eta] + 2f_0'[\eta]f_4'[\eta] + (\frac{\eta S}{2} - f_0[\eta])f_4''[\eta] - f_1[\eta]f_3''[\eta] - f_3[\eta]f_1''[\eta]$
$\quad - f_2[\eta]f_2''[\eta] + 2f_1'[\eta]f_3'[\eta] + f_2'[\eta]^2;$

$de6 = NDSolve[\{u'''[\eta] - au'[\eta] == A_4[\eta], u[0] == 0, u'[0] == 0, u''[\beta]] == 0\}, u[\eta], \{\eta, 0, \beta\}];$

$f_5[\eta\_] := Evaluate[u[\eta]/.First[de6]]$

$g[\eta\_] = f_0[\eta] + f_1[\eta];$

$h[\eta\_] = f_0[\eta] + f_1[\eta] + f_2[\eta];$

$k[\eta\_] = f_0[\eta] + f_1[\eta] + f_2[\eta] + f_3[\eta];$

$l[\eta\_] = f_0[\eta] + f_1[\eta] + f_2[\eta] + f_3[\eta] + f_4[\eta];$

$m[\eta\_] = f_0[\eta] + f_1[\eta] + f_2[\eta] + f_3[\eta] + f_4[\eta] + f_5[\eta];$

$m[i] = m[\eta\_];, f[i] = m'[\eta\_];,$

$a[\eta\_] = \frac{3P_rS\eta}{2} + 2P_rm'[\eta],\ b[\eta\_] = \frac{P_rS\eta}{2} - P_rm[\eta],\ c[\eta\_] = -P_rE_c(m''[\eta])^2;$

$de8 = NDSolve[\{u''[\eta] == a[\eta]u[\eta] + b[\eta]u'[\eta] + c[\eta], u[0] == 1, u'[\beta] == 0\}, u[\eta], \{\eta, 0, \beta\}];$

$\theta][\eta\_] := Evaluate[u[\eta]]/.First[de8]]$

$q[i] = \theta[\eta]; step[i];$

Print[" i","","$m[i]$","","$m'[i]$ "," "," $m''[i]$ " "," $\theta[i]$ "," "," $\theta'[i]$ "]
Table[$i, m[i], m'[i], m''[i]$, $\theta[i]$, $\theta'[i]$, i, 0, β, 0.2] // TableForm

A.2 Tables

The following tables have been calculated with the help of Mathematica using computer Code given in Appendix. The table A.1 & A.2 give the comparison of present results with that of Wang [24] and M. S. Abel [36]. With out any doubt, from these tables we can claim that our results are in well agreement with that of Wang [24] and M. S. Abel [36] under some limiting cases. The values of $f(\beta)$, $f'(\beta)$, $f''(\beta)$, at the free surface, obtained through the actual computation, are tabulated in Table A.3. The values tabulated in this table are very important as they serve the purpose of validating the momentum equation 3.1.6 in dimensionless form, at the free surface $\eta = \beta$. The table A.4 is given for information about film thickness for different values of unsteadiness and magnetic parameters.

Table A.1: Comparison of values of skin friction coefficient $f''(0)$ with $M_n = 0.0$

	Wang [24]			M. S. Abel [36]		Present Results	
S	β	$f''(0)$	$f''(0)/\beta$	β	$f''(0)$	β	$f''(0)$
0.4	5.122490	-6.699120	-1.307785	4.981455	-1.134098	4.981455	-2.03163
0.6	3.131250	-3.742330	-1.195155	3.131710	-1.195128	3.131710	-1.32446
0.8	2.151990	-2.680940	-1.245795	2.151990	-1.245805	2.151990	-1.28315
1.0	1.543620	-1.972380	-1.277762	1.543617	-1.277769	1.543617	-1.29167
1.2	1.127780	-1.442631	-1.279177	1.127780	-1.279171	1.127780	-1.28397
1.4	0.821032	-1.012784	-1.233549	0.821033	-1.233545	0.821033	-1.23467
1.6	0.576173	-0.642397	-1.114937	0.576176	-1.114941	0.576176	-1.11507
1.8	0.356389	-0.309137	-0.867414	0.356390	-0.867414	0.356389	-0.867417

Comparison of values of skin friction coefficient $f''(0)$ with $M_n = 0.0$

Table A.2: Comparison of values of surface temperature $\theta(\beta)$ and temperature gradient $\theta'(0)$

	Wang [24]			M. S. Abel [36]		Present Results	
P_r	$\theta(\beta)$	$-\theta'(0)$	$-\theta'(0)/\beta$	$\theta(\beta)$	$-\theta'(0)$	$\theta(\beta)$	$-\theta'(0)$
$S = 0.8$	$\beta = 2.151990$						
0.01	0.960480	0.090474	0.042042	0.960438	0.042120	0.962218	0.407631
1.0	0.097884	3.595790	1.670913	0.097825	1.671919	0.105976	0.166003
2.0	0.024941	5.244150	2.436884	0.024869	2.443914	0.028200	2.434040
3.0	0.008785	6.514440	3.027170	0.008324	3.034915	0.009828	3.026840
$S = 1.2$	$\beta = 1.127780$						
0.01	0.982331	0.037734	0.033458	0.982312	0.033515	0.982366	0.033410
1.0	0.286717	1.999590	1.773032	0.286634	1.773772	0.28729	1.772040
2.0	0.128124	2.975450	2.638324	0.128174	2.638431	0.128523	2.637770
3.0	0.067658	3.698830	3.279744	0.067737	3.280329	0.067920	3.281030

Comparison of values of $\theta(\beta)$ and $\theta'(0)$ with $M_n = 0.0$

Table A.3: Values of $f(\beta)$, $f'(\beta)$, $f''(\beta)$, $\theta(\beta)$, $\theta'(\beta)$ for different values of unsteadiness S and magnetic parameter M_n with increasing values of the Prandtl and Eckert number

S	M_n	β	P_r	E_c	$f(\beta)$	$f'(\beta)$	$f''(\beta)$	$\theta(\beta)$	$\theta'(\beta)$
0.84	5	0.816158	1.0	0.1	2.103810	1.26047	0.000000	0.440266	0.000000
0.84	10	0.599054	2.0	0.2	1.634830	1.42183	0.000000	0.866285	0.000000
0.84	15	0.495568	3.0	0.3	1.382060	1.48664	0.000000	1.579600	0.000000
0.84	20	0.432048	4.0	0.4	1.218740	1.52156	0.000000	2.561430	0.000000
0.84	25	0.388002	5.0	0.5	1.102230	1.54338	0.000000	3.807520	0.000000
0.84	30	0.355159	6.0	0.6	1.013770	1.55831	0.000000	5.316630	0.000000
0.84	35	0.329455	7.0	0.7	0.943654	1.56917	0.000000	7.088180	0.000000
0.84	40	0.308630	8.0	0.8	0.686318	1.57742	0.000000	9.121890	0.000000
0.84	45	0.291312	9.0	0.9	0.838301	1.58390	0.000000	11.41760	0.000000
S	M_n	β	P_r	E_c	$f(\beta)$	$f'(\beta)$	$f''(\beta)$	$\theta(\beta)$	$\theta'(\beta)$
0.21	10	0.617259	1.0	0.1	1.66773	1.39082	0.000000	0.792689	0.000000
0.42	10	0.611008	2.0	0.2	1.65659	1.40153	0.000000	0.938928	0.000000
0.63	10	0.604942	3.0	0.3	1.64562	1.41186	0.000000	1.030600	0.000000
0.84	10	0.599054	4.0	0.4	1.63483	1.42183	0.000000	1.038890	0.000000
1.05	10	0.593334	5.0	0.5	1.62422	1.43145	0.000000	0.999654	0.000000
1.26	10	0.587775	6.0	0.6	1.61377	1.44075	0.000000	0.942386	0.000000
1.47	10	0.582370	7.0	0.7	1.60349	1.44973	0.000000	0.883304	0.000000
1.68	10	0.577111	8.0	0.8	1.59338	1.45842	0.000000	0.829751	0.000000
1.89	10	0.571992	9.0	0.9	1.58344	1.46683	0.000000	0.784323	0.000000

Table A.4: Values of dimensionless film thickness β for different values of unsteadiness parameter S and magnetic parameter M_n

$Part$	$-I$		$Part$	$-II$		$Part$	$-III$	
S	M_n	β	S	M_n	β	S	M_n	β
0.0	0.0	∞	0.8	0.0	2.15199	1.2	0.0	1.757090
0.001	0.0	60.8675	0.8	1.0	1.43466	1.2	1.0	1.297700
0.01	0.0	19.2480	0.8	2.0	1.15029	1.2	2.0	1.075990
0.1	0.0	6.08675	0.8	3.0	0.987401	1.2	3.0	0.939205
0.2	0.0	4.30398	0.8	4.0	0.878546	1.2	4.0	0.844080
0.4	0.0	3.04338	0.8	5.0	0.799229	1.2	5.0	0.773017
0.6	0.0	2.48491	0.8	6.0	0.738126	1.2	6.0	0.717330
0.8	0.0	2.15199	0.8	7.0	0.689188	1.2	7.0	0.672169
1.0	0.0	1.92480	0.8	8.0	0.648849	1.2	8.0	0.634587
1.2	0.0	1.75709	0.8	9.0	0.614854	1.2	9.0	0.602678
10.0	0.0	0.608675	0.8	10.0	0.585697	1.2	10.0	0.575143
100.0	0.0	0.192480	0.8	11.0	0.560330	1.2	11.0	0.551068

Appendix B

Thermal conductivity: k is called thermal conductivity of fluid.
Momentum diffusivity: μ/ρ is known as momentum diffusivity.
kinematic viscosity: The contribution to acceleration of the material element due to viscous stresses arising from a given rate of strain is evidently determined by the ratio μ/ρ and not by the viscosity μ alone. The momentum diffusivity μ/ρ is given the special name kinematic viscosity and is denoted by ν [39].
Prandtl number:

$$P_r \equiv \frac{\mu C_p}{k} = \frac{\frac{\mu}{\rho}}{\frac{k}{\rho C_p}} = \frac{\text{momentum diffusivity}}{\text{energy diffusivity}} = \frac{\nu}{\alpha}$$

$$P_r = \frac{\mu C_p}{k} = \frac{\text{Molecular diffusivity of momentum}}{\text{Molecular diffusivity of heat}}$$