Sandpile Group of Cartesian Product of Some Graphs

by

Shaista Khan

A thesis submitted in partial fulfillment of the requirements

for the degree of Master of Science in Mathematics

Supervised by

Dr.Muhammad Ishaq

School of Natural Sciences

National University of Sciences and Technology

Islamabad, Pakistan

FORM TH-4

National University of Sciences & Technology

MS THESIS WORK

We hereby recommend that the dissertation prepared under our supervision by: MS. SHAISTA KHAN, Regn No. 00000117036 Titled: Sandpile Group of Cartesian Product of Some Graphs be accepted in partial fulfillment of the requirements for the award of **MS** degree.

Examination Committee Members

1. Name: DR. RASHID FAROOQ Signature: ____________

2. Name: DR. MUJEEB-UR-REHMAN Signature: ____________

External Examiner: DR. WAQAS MAHMOOD Signature: ____________

Supervisor's Name DR. MUHAMMAD ISHAQ Signature: ____________

Head of Department

28/11/18
Date

COUNTERSINGED

Date: 28/11/18

Dean/Principal

THESIS ACCEPTANCE CERTIFICATE

Certified that final copy of MS thesis written by **Ms. Shaista Khan** (Registration No. **00000117036**), of **School of Natural Sciences** has been vetted by undersigned, found complete in all respects as per NUST statutes/regulations, is free of plagiarism, errors, and mistakes and is accepted as partial fulfillment for award of MS/M.Phil degree. It is further certified that necessary amendments as pointed out by GEC members and external examiner of the scholar have also been incorporated in the said thesis.

Signature: ______________________

Name of Supervisor: Dr. Muhammad Ishaq

Date: 28 /11 / 2018

Signature (HoD): ______________________

Date: 28/11/18

Signature (Dean/Principal): ______________________

Date: 28/11/18

Acknowledgment

In the name of Allah, the most benevolent, the most merciful, I am thankful to Allah for his graciousness for making me able to accomplish this thesis.

Seeking higher education has not only been my dream but my parent's earnest desire and the conclusion of this thesis harbingers the translation of that dream into reality. I extend heartfelt gratitude to my parents for their prayers and support.

My elder sister and brother in law have been my inspiration and motivation in the pursuit of higher education. My husband Arif's constant encouragement is also a great asset for me.

I would be failing in my duty if I don't mention my supervisor Dr.Muhammad Ishaq with whose guidance, support and mentorship, I was able to complete thesis on this apparently difficult subject. Besides my supervisor, my sincere thanks to my other thesis committee, Dr.Rashid Farooq and Dr.Mujeeb Ur Rehman.

I would like to express gratitude to my friends for helping me out whenever their assistance was required.

The end of this thesis harbingers the completion of my MS Mathematics degree, which is a big mile stone in my life and it gives me the courage to dream about further education in this field. May Allah be my protector and guide, Ameen.

Abstract

This dissertation focuses on the objective of sandpile group of cartesian product of some graphs. We aim to explore the algebraic structure, combinatorical structure and algebraic geometry of abelian sandpile models.

To determine the structure of sandpile group, one can relate an abelian group known as sandpile group to every connected graph. The structure of sandpile group associated to any graph can be expressed as the direct sum of four of five cyclic groups. It is concluded that order of sandpile group associated to any graph is the number of spanning trees of the graph.

Contents

Chapter 1

Group Theory

This chapter covers some fundamental definitions of abstract algebra. The aim of this chapter is to recall the basic concepts that are essential to know in order to solve complex problems in the further chapter coming ahead. For futher studies [2,5,6,12] can be pursued.

1.0.1 Groups

Binary operation

For a non-empty set A, a mapping from $A \times A$ into A is called a binary operation in A.

Group

Let A be a non-empty set. A group is a pair $(A, \star)$ that comprises a non-empty set A accompanied by a binary operation $\star$ that satisfies the following axioms:

- **Associativity**: $(A, \star)$ is associative if for all ϑ_1, ϑ_2, $\vartheta_3 \in A$ we have

$$(\vartheta_1 \star \vartheta_2) \star \vartheta_3 = \vartheta_1 \star (\vartheta_2 \star \vartheta_3).$$

- **Identity Element**: There exists an element $\widetilde{e}$ (neutral element) in A such that for every $\vartheta \in A$ we have

$$\vartheta \star \widetilde{e} = \widetilde{e} \star \vartheta = \vartheta.$$

  $\widetilde{e}$ is known as the identity element of A.

- **Inverse Element**: If for every $\vartheta \in A$ there exists ϑ' such that

$$\vartheta' \star \vartheta = \vartheta \star \vartheta' = \widetilde{e}.$$

  Both ϑ and ϑ' are inverses of each other. ϑ' is usually denoted by ϑ^{-1}. All the three axioms ensure the existence of a group.

Some Important Results

- There exists only one identity element in a group.
- There exists a unique inverse for every element in a group..
- For all $\vartheta \in A$, $(\vartheta^{-1})^{-1} = \vartheta$.
- For all ϑ_1, $\vartheta_2 \in A$, $(\vartheta_1 \star \vartheta_2)^{-1} = \vartheta_2^{-1} \star \vartheta_1^{-1}$.
- Concellation laws hold in groups, that is for all ϑ, b, $c \in A$ we have $\vartheta_1 \star \vartheta_2 = \vartheta_3 \star \vartheta_2 \implies \vartheta_1 = \vartheta_3$ and $\vartheta_2 \star \vartheta_1 = \vartheta_2 \star \vartheta_3 \implies \vartheta_1 = \vartheta_3$

Abelian group

A group $(A, \star)$ that satisfies the commutative property is abelian. For all ϑ_1, $\vartheta_2 \in A$,

$$\vartheta_1 \star \vartheta_2 = \vartheta_2 \star \vartheta_1.$$

Order of element

Let $(A, \star)$ be a group, for $\vartheta \in A$, the least positive integer n such that $\underbrace{\vartheta \star \vartheta \star \cdots \star \vartheta}_{n\text{-times}} = \widetilde{e}$ (the identity element) is the order of an element ϑ in a group. It is denoted by $\mid \vartheta \mid$. If such specific n for ϑ cannot be found then $\mid \vartheta \mid$ is infinite.

Order of group

The cardinality (the number of elements) in a group $(A, \star)$ denoted as $\mid A \mid$ is the order of a group. The trivial group $A = \{\widetilde{e}\}$ is a group with $\mid A \mid = 1$.

Subgroup

Let A be a group under binary operation $\star$. A non-empty subset I is the subgroup of group A if it itself satisfies the axioms of a group under the same binary operation $\star$ as defined in A. $\{\widetilde{e}\}$ and A are obviously subgroups of A. The remaining existing subgroups are the proper subgroups of A.

Theorem 1.0.1. Let $(A, \star)$ be a group and $I \subseteq A$. Then I is a subgroup of A if and only if for every i, $j \in I$ we have $i \star j^{-1} \in I$.

Product of subgroups

Let $(A, \star)$ be a group and I and J be any subgroups of A. We define the product IJ of A as follow,

$$I \star J = \{i \star j\} \mid \check{i} \in I, j \in J\}.$$

The product IJ is not necessarily a subgroup of A.

Theorem 1.0.2. The product of subgroups I and J is a subgroup if and only if $IJ = JI$.

Lagrange Theorem

Let $(I, \star)$ be a subgroup of finite group $(A, \star)$. Then for a subgroup I of finite group A, $\mid I \mid$ divides $\mid A \mid$.

Corollary 1.0.1. Let $(A, \star)$ be a finite group. Then for all $\vartheta \in A$, $\mid a \mid$ divides $\mid A \mid$.

Homomorphism

Let $(A, \star)$ and $(A', \circ)$ be two groups. A homomorphism from the group $(A, \star)$ to $(A', \circ)$ is a mapping $\alpha : A \to A'$; satisfying

$$\alpha(\vartheta \star \vartheta') = \alpha(\vartheta) \circ \alpha(\vartheta').$$

Isomorphism

A bijective (one-one and onto) homomorphism is termed as isomorphism denoted by $(A, \star) \cong A', \circ)$ or $A \cong A'$; where $(A, \star)$ and $(A', \circ)$ are two groups.

Cyclic Group

A group $(A, \star)$ turns out to be a cyclic group if a single element in A is capable of generating the entire group. If there exists $\vartheta \in A$ such that $A = \{\vartheta^n : n \in \mathbb{Z}\}$. In this case we write $A =< \vartheta >$. A cyclic group with order n is denoted by C_n. Hence we have

$$C_n =< \vartheta >= \{\widetilde{e}, \vartheta, \vartheta \star \vartheta, \cdots, \underbrace{\vartheta \star \vartheta \star \vartheta \star \cdots \star \vartheta}_{(n-1)\text{-times}}\}.$$

with $\underbrace{\vartheta \star \vartheta \cdots \star \vartheta}_{n\text{-times}} = \widetilde{e}$. Cyclic groups are always abelian. A cyclic group of finite order n is found out to be isomorphic to $\mathbb{Z}_n$ (where $\mathbb{Z}_n$ additive group of integers modulo n).

Remark 1.0.1. If $(A, \star)$ and $(A', \circ)$ are cyclic groups of same order then they are isomorphic.

Remark 1.0.2. Cyclic groups have cyclic subgroups.

Coset

Let $(A, \star)$ has subgroup $(I, \star)$. For subgroup I of group A and for $\vartheta \in A$, we have;

$$I \star \vartheta = \{i \star \vartheta; i \in I\}.$$
$$\vartheta \star I = \{\vartheta \star i; i \in I\}.$$

$I \star \vartheta$ is called the right coset and $\vartheta \star I$ is called the left coset of I in A.

Theorem 1.0.3. For a subgroup $(I, \star)$ of a group $(A, \star)$, the left respectively the right cosets partition A.

Index

Consider the subgroup $(I, \star)$ of a group $(A, \star)$. Then the number of left respectively the right cosets of I formed in A is called the index, denoted by $[A{:}I]$.

Lemma 1.0.1.1. The order of the two left respectively the right cosets of subgroup I in the group A is the same.

Normal Subgroup

Let $(A, \star)$ be a group. A subgroup N of a group A is a normal subgroup (also known as invariant subgroup) if for all $\vartheta \in A$ we have,

$$\vartheta \star N \star \vartheta^{-1} = N.$$

The normal subgroup N of A is denoted by $N \trianglelefteq A$.

Proposition 1.0.1. A necessary and sufficient condition for a subgroup $(N, \star)$ of group $(A, \star)$ to be normal is that the left coset also appears to be the right coset of N in A, that is for all $\vartheta \in A$,

$$N \star \vartheta = \vartheta \star N.$$

Remark 1.0.3. Every subgroup pertaining to index 2 is normal.

Remark 1.0.4. If $(A, \star)$ is an abelian group, then its every subgroup is normal. Since $(N, \star)$ is a subgroup of $(A, \star)$, then for $\vartheta \in A$ and $n \in N$

$$\vartheta \star n \star \vartheta^{-1} = \vartheta \star \vartheta^{-1} \star n = n \in N.$$

But the converse may not be true.

Proposition 1.0.2. The product of normal subgroups is normal in a group.

Kernel

The kernel of homomorphism $\alpha : A \to A'$ is

$$ker(\alpha) = \{\vartheta \in A : \alpha(\vartheta) = \widetilde{e}\}.$$

The kernel of homomorphisms is a normal subgroup of a group.

Quotient Group

Let $(N, \star)$ be any normal subgroup of a group $(A, \star)$, then the quotient group A/N (A over N) is actually the set of distinct cosets (left or right) of normal subgroup N in the group A.

$$A/N = \{\vartheta \star N : \vartheta \in A\}.$$

Example 1.0.1. $\mathbb{Z}/n\mathbb{Z}$ is an example of quotient group. For $n \in \mathbb{Z}$, let $(\mathbb{Z}, +)$ be a group and $n\mathbb{Z}$ be its subgroup. Since $(\mathbb{Z}, +)$ is abelian so its every subgroup is normal. Thus $n\mathbb{Z}$ is normal. For all $(z_1 + n\mathbb{Z})$, $(z_2 + n\mathbb{Z}) \in \mathbb{Z}/n\mathbb{Z}$ we have

$$(z_1 + n\mathbb{Z}) \star (z_2 + n\mathbb{Z}) = (z_1 + z_2) + n\mathbb{Z}.$$

So $\mathbb{Z}/n\mathbb{Z} \cong \mathbb{Z}_n$.

Proposition 1.0.3.

- An infinite cyclic group is isomorphic to $\mathbb{Z}$.
- A finite cyclic group of order n is isomorphic to $\mathbb{Z}/n\mathbb{Z}$.

Normalizer of a subgroup

Let $(A, \star)$ be a group and I be its any subgroup such that $I \neq \emptyset$. The normalizer of I in A is defined as

$$N_A(I) = \{\vartheta \in A \mid \vartheta \star I \star \vartheta^{-1} = I\}.$$

1.0.2 Theorems of Isomorphisms

First theorem of Isomorphism

Theorem 1.0.4. Let $\alpha : A \to A'$ be a group homomorphism and K be the kernel of α. Then

$$A/K \cong \alpha(A).$$

Second Theorem of Isomorphism

Theorem 1.0.5. Consider $(A, \star)$ be a group. I and J are subgroups of A and assume $I \leq N_A(J)$. Then IJ is a subgroup of A, $J \trianglelefteq IJ$ and $I \cap J \trianglelefteq I$. Then

$$I/I \cap J \cong I \star J/J.$$

Third Theorem of Isomorphism

Theorem 1.0.6. Consider $(A, \star)$ be a group and let I and J are normal subgroups of A with $I \leq J$. Then $(J/I) \trianglelefteq (A/I)$ and $(A/I)/(J/I) \cong A/J$.

1.0.3 Direct Products

Let $(I, \star_1)$ and $(J, \star_2)$ be two groups, then $(A, \star)$ is a group formed by the Cartesian product of I and J with binary operation defined component wise.

$$A = I \times J = \{(i, j) | i \in I, j \in J\},$$

For $(i_1, j_1), (i_2, j_2) \in I \times J$ with binary operation $\star_1$ and $\star_2$ then the direct product is given as

$$(i_1, j_1) \star (i_2, j_2) = (i_1 \star_1 i_2, j_1 \star_2 j_2) \in I \times J.$$

$A = I \times J$ is a group since it satisfies all the axioms of a group.
Direct product is the collection of smaller groups, which results in the formation of larger groups. If I_1, I_2, I_3, $\cdots$, I_k are the groups under binary operation $\star_1$, $\star_2$, $\cdots$, $\star_k$ respectively then the direct product $I_1 \times I_2 \times I_3 \times \cdots \times I_k$ of the groups I_i's is a group endowed with the binary operations $\star$ can be given as

$$(i_1, i_2, \cdots, i_k) \star (j_1, j_2, \cdots, j_k) = (i_1 \star_1 j_1, i_2 \star_2 j_2, \cdots, i_k \star_k j_k).$$

Example: Consider $I = \mathbb{R}$ and $J = \mathbb{R}$ then the direct product $I \times J = \mathbb{R} \times \mathbb{R}$ under the operation

$$(i_1, i_2) + (j_1, j_2) = (i_1 + j_1, i_2 + j_2).$$

The direct product is $\mathbb{R}^2$, which is also abelian.

Proposition 1.0.4. For the groups I_1, I_2, I_3, $\cdots$, I_k their direct product appears to be a group whose order is equal to $\mid I_1 \mid . \mid I_2 \mid . \mid I_3 \mid \cdots \mid I_k \mid$. If any I_i is infinite so is the direct product group.

1.0.4 Finitely Generated Abelian Groups

Finitely Generated Abelian Groups

An abelian group A is finitely generated if there exists a finite subset $\mathcal{B}$ of A such that $\mathcal{B}$ generates the entire group A. Mathematically $A =< \mathcal{B} >$.

Free Abelian Groups

The direct product of n copies of $\mathbb{Z}$ given as $\mathbb{Z}^n \cong \mathbb{Z} \oplus \mathbb{Z} \oplus \cdots \oplus \mathbb{Z}$ is called the free abelian group of rank n where $n \in \mathbb{Z}$ with $n \geq 0$ and $\mathbb{Z}^0$=1.

1.0.5 Fundamental Theorem of Finitely Generated Abelian Groups

Theorem 1.0.7. If A is a finitely generated abelian group, then the decomposition is as follow

$$A = \mathbb{Z}^n \oplus \mathbb{Z}_{t_1} \oplus \mathbb{Z}_{t_2} \oplus \cdots \oplus \mathbb{Z}_{t_r},$$

for some integer n, $t_1 \geq \cdots \geq t_i \geq 2$ such that $n \geq 0$ and $t_{i+1} \mid t_i$. Moreover this expression is unique.

Free rank or Betti number

The integer n in the decomposition of finitely generated abelian groups is called the free rank or Betti number.

Invariant factors

The integers t_1, t_2, $\cdots$, t_i are called the invariant factors of A, where as the description is termed as invariant factor decomposition.

Remark 1.0.5. [5] Any two finitely generated abelian groups are isomorphic if and only if they have same rank and same list of invariant factores.

Remark 1.0.6. [5] A finitely generated group is a finite group if and only if the free rank is zero.

Remark 1.0.7. [5] The order of finite abelian group is just the product of its invarient factors.

Corollary 1.0.2. Every subgroup of finitely generated abelian group is also finitely generated.

Remark 1.0.8. By calculating all finite sequence of integers t_1, t_2, $\cdots$, t_r all finite abelian groups of a given order t are obtained where

1. $t_j \geq 2$,
2. $t_{i+1} \mid t_i$,
3. $t_1 \cdot t_2 \cdots t_r = t$.
4. Every prime divisor of t must divide t_1.

Corollary 1.0.3. If A is an abelian group with order n equal to the product of distinct primes then A is isomorphic to $\mathbb{Z}_n$ the cyclic group of order n.

Theorem 1.0.8. Let A be a finite abelian group with order $t = p_1^{r_1} \cdot p_2^{r_2} \cdots p_k^{r_k}$, where all p_i's are primes and all r_i's are natural numbers then

$$A \cong \mathbb{Z}^n \oplus \mathbb{Z}_{{p_1}^{r_1}} \oplus \mathbb{Z}_{{p_2}^{r_2}} \oplus \cdots \oplus \mathbb{Z}_{{p_k}^{r_k}}.$$

Elementary divisors

The integers ${p_j}^{r_j}$ in the above theorem are called elementary divisors.

Lemma 1.0.5.1. $\mathbb{Z}_{mn} \cong \mathbb{Z}_m \oplus \mathbb{Z}_n$ iff $gcd(m, n) = 1$.

Chapter 2

Graph Theory and Combinatorics

This chapter deals with graph theory and combinatorics that are relevant to the research being carried out. Basic definitions that comprises graph, different kinds of graph in graph theory and recurrence relation and different kinds of recurrence relations in combinatorics are taken into account. Detail of the subject in hand is available in [6, 8, 9].

2.1 Graph Theory

2.1.1 Graphs

Graph

A graph G is basically collection of two sets, the vertex set $V(G)$ and the edge set $E(G)$, where

- The members of vertex set $V(G)$ are the vertices or nodes of graph G.
- The edge set $E(G)$ formed by pair of vertices defines an assocaition between two vertices.

The graph G is symbolically represented as $G(V, E)$. The most common representation of a graph is through diagram where the edges act as line segments joining the vertices represented by points. If a and b are any two vertices of graph G and e is an edge, then e joins a and b.

Order and size of the graph

The number of vertices is the order in a graph, whereas the size of the graph is related to the number of its edges.

Degree of vertex

In a graph G, $d(v)$ is the degree of vertex v which is the number of edges of graph G incident to v.

Proposition 2.1.1. (Degree-sum formula) In a graph G $\sum_{k=1}^{n} d(v_i) = 2e$ that is the summation of degrees of all the vertices of graph G count twice the number of all the edges of graph G.

Basic Terminologies: We have the following terminologies:

- The edge (a, b) has two end vertices a and b.
- A loop is an edge (a, a).
- A multigraph has parallel edges.
- Parallel edges are those edges with same end vertices.
- A tirvial graph has one vertex.
- In a null graph, the graph has no verterx.
- A graph with no edges is empty.

- Adjacent edges share a common end vertex.
- A loop counts twice.
- Two vertices say a and b are adjacent if there lies an edge between them. The pair (a, b) represents an edge.
- For any pair of vertices, $(a, b) = (b, a)$
- A vertex with degree 1 is known as pendant vertex.
- $d(a)$ represents the degree of a vertex a which is basically the number of edges with a as an end point. A loop is always counted twice and parallel edges once.
- A vertex with degree is 0 is an isolated vertex.

Simple Graph

A loop-less graph which has no parallel edges is a simple graph.

Connected and Disconnected graphs

A graph is connected if its every vertex is reachable by any other vertex through an edge, otherwise disconnected.

Proposition 2.1.2. Every connected graph G with n vertices has atleast $n - 1$ edges.

Corollary 2.1.1. Every graph has an even number of vertices of odd degree.

Subgraph

The graph H is a subgraph of a graph G if the vertex set and edge set of H are the subsets of vertex set and edge set of G. One can obtain a subgraph of G by deleting its vertices and edges.

- Every graph is the subgraph of itself.
- A single vertex can be the subgraph of any graph.

Path

A path is a simple graph in which no vertex appears more than once or there is no repetition in vertices.

Walk

A walk starts with vertex consisting of finite alternating sequence of vertices and edges.

Cyclic and acyclic graph

An undirected simple graph that contains a cycle is a cyclic graph. It is a closed path that cannot be defined for a graph with vertices less than 3, where each vertex must have degree atleast 2. A cycle graph is basically set of consecutive vertices, having the same first and last vertex. A graph with no cycle is acyclic graph.

Lemma 2.1.1.1. A graph contains a cycle if each of its vertex has degree atleast 2.

Regular graph

If each vertex in a simple graph has same degree then it is a regular graph. Regular graph is $k - regular$ graph if the degree of each of its vertex is k.

Complete graph

A complete graph denoted by K_n is a simple graph with n vertices where each vertex is connected to every other vertex by an edge.

Bipartite graph

A simple graph is said to be bipartite graph if its vertices can be partitioned into two vertex sets say V_1 and V_2 such that every edge of the graph connects a vertex in V_1 to one vertex in V_2 and there is no edge between vertices of same set.

Complete bipartite graph

A bipartite graph, denoted by $K_{m,n}$ if there is an edge from every m vertices of V_1 to every n vertices of V_2.

2.1.2 Trees

Tree

A connected and acyclic graph is known as tree.

Forest

A graph with no cycle is a forest. A forest is a disjoint collection of trees. Singleton graph, empty graph and trees are all example of forest.
Since acyclic graph is forest thus its each component is considered as a tree and any tree is considered to be a connected forest.

Theorem 2.1.1. A graph is a tree if and only if a single path exists between every two vertices.

Remark 2.1.1. Path is a special case of tree.

Theorem 2.1.2. If T is a tree with n vertices, then $E(T) = n - 1$.

Theorem 2.1.3. A connected graph is a tree if it has n vertices and $n - 1$ edges.

Conclusion 2.1.1. From different definitions we concluded that a graph on n vertices is a tree if

- It is acyclic and connected.
- It is acyclic and has $n-1$ edges.
- It is connected and has $n-1$ edges
- There is exactly one path connecting each pair of vertices.

Theorem 2.1.4. In any non-trivial tree there exists at least two leaves (end vertices).

Remark 2.1.2.
- Only one cycle is formed if an edge is added to an acyclic graph.
- Every tree is bipartite.

Spanning tree

In a connected graph G, a subtree T' is a spanning tree of G if

- T' itself is a tree.
- T' is a subgraph of G (it contains all vertices of G).

Theorem 2.1.5. A connected graph G with m edges and n vertices has atleast one spanning tree.

Remark 2.1.3. The subgraph T' of graph G with m edges and n vertices is a spanning tree if it follows the following axioms:

- T' has same vertices as G that is n.
- T' must be connected and acyclic.
- T' has edges $m = n - 1$.

Theorem 2.1.6. T' is a spanning tree of G if and only if G is connected.

Remark 2.1.4. If a graph is connected and acyclic, then it is itself a spanning tree.

Direct product

Tensor product, also called direct product of any two simple graphs G and G' denoted by $G \times G'$ is the graph such that

- Vertex set is the cartesian product $(g, g') \in V(G) \times V(G')$.
- Any pair of vertices say $(g_1, {g'}_1).(g_2, {g'}_2) \in V(G) \times V(G')$ are adjacent if $g_1 g_2 \in E(G)$ and ${g'}_1 {g'}_2 \in E(G')$.

Remark 2.1.5. Direct product is commutative as well as associative.

2.2 Combinatorics

2.2.1 Recurrence Relation ($\hat{R}.\hat{R}$)

For a sequence a_n the $\hat{R}.\hat{R}$ is an equation expressed using earlier terms of the sequence together with certain initial condition.
a_0, a_1, $\cdots$, a_n are the terms that define a $\hat{R}.\hat{R}$ for a_n. It means we can find each subsequent term if previous term in a sequence along with some initial conditions are known. If the earlier terms a_n, a_{n+1}, $\cdots$, a_{n+k} of $\hat{R}.\hat{R}$ of order k are linear then it is termed as linear $\hat{R}.\hat{R}$ otherwise non-linear. A linear $\hat{R}.\hat{R}$ of order k is a sequence satisfying

$$a(n) = A_1 a^{n-1} + A_2 a^{n-2} + \cdots + A_k a^{n-k}.$$

Solution of $\hat{R}.\hat{R}$

A function fully satisfying the $\hat{R}.\hat{R}$ is said to be the solution of $\hat{R}.\hat{R}$.

Order of $\hat{R}.\hat{R}$

The order of $\hat{R}.\hat{R}$ is the difference of its higher and lower superscripts of the members in the equation.

Remark 2.2.1. The required number of initial conditions depends on the order of $\hat{R}.\hat{R}$.

Examples:

- $a^n = a^{n+1} + 2$ is a $\hat{R}.\hat{R}$ of order 1.
- $2(a^n)^2 + (a^{n+1})^5 = 2$ is a $\hat{R}.\hat{R}$ of order 1.
- $a^n = 2a^{n+4} + a^{n+1}$ is a $\hat{R}.\hat{R}$ of order 4.
- $(a^{n+2} + a^n)^{1/2} = 1$ is a $\hat{R}.\hat{R}$ of order 2.

Homogeneous recurrence relation $\hat{H}.\hat{R}.\hat{R}$

A linear $\hat{R}.\hat{R}$ of order k of the form

$$A_0x^n + A_1x^{n+1} + \cdots + A_kx^{n+k} = 0,$$

is called a $\hat{H}.\hat{L}.\hat{R}.\hat{R}$ where A_0, A_1, $\cdots$, A_k are constant.

Non-homogeneous recurrence relation $\hat{N}.\hat{H}.\hat{R}.\hat{R}$

A linear $\hat{R}.\hat{R}$ of order k of the form

$$A_0x^n + A_1x^{n+1} + \cdots + A_kx^{n+k} = A(n),$$

is called a $\hat{N}.\hat{H}.\hat{L}.\hat{R}.\hat{R}$ where $A(n) \neq 0$ and A_0, A_1, $\cdots$, A_k serve as constants.

2.2.2 Homogeneous Linear Recurrence Relation $\hat{H}.\hat{L}.\hat{R}.\hat{R}$

Solution to a $\hat{H}.\hat{L}.\hat{R}.\hat{R}$

Consider a $\hat{H}.\hat{L}.\hat{R}.\hat{R}$

$$x_n = a_1x_{n-1} + a_2x_{n-2},$$

where a_1 and a_2 can be any real numbers. The characteristic equation of the $\hat{H}.\hat{L}.\hat{R}.\hat{R}$ is

$$\gamma^2 = a_1\gamma + a_2,$$
$$\gamma^2 - a_1\gamma - a_2 = 0.$$

On the basis of characteristic equation the following cases may occur:

- **Case 1** : From the characteristic equation $\gamma^2 - a_1\gamma - a_2 = 0$, if $(\gamma - \gamma_1)(\gamma - \gamma_2) = 0$ appears to be the factors and $\gamma_1 \neq \gamma_2$ two distinct roots then solution of $\hat{H}.\hat{L}.\hat{R}.\hat{R}$ according to the distinct roots is

$$x_n = c_1\gamma_1^n + c_2\gamma_2^n.$$

- **Case 2** : From the characteristic equation $\gamma^2 - a_1\gamma - a_2 = 0$ if $(\gamma - \gamma_1)^2 = 0$ appears to be the factors and γ_1 a single real root, then the solution of $\hat{H}.\hat{L}.\hat{R}.\hat{R}$ according to the single real root is

$$x_n = c_1\gamma_1^n + c_2 n\gamma_2^n.$$

- **Case 3** : From the characteristic equation $\gamma^2 - a_1\gamma - a_2 = 0$ if $(\gamma - \gamma_1)(\gamma - \gamma_2) = 0$ appears to be the factors, and $\gamma_1 = a + \iota b(\theta)$ and $\gamma_2 = a - \iota b(-\theta)$ are two distinct complex roots then solution of $\hat{H}.\hat{L}.\hat{R}.\hat{R}$ according to the distinct complex roots is

$$x_n = a^n(c_1 cos(n\theta) + c_2 sin(n\theta)).$$

Example 1. Find the solution of the $\hat{R}.\hat{R}$ $a^k - 4a^{k-2} = 0$ with initial conditions $a^0 = 1$ and $a^1 = 1$.

Solution: The above $\hat{R}.\hat{R}$ has characteristic equation,

$$r^2 - 4 = 0.$$

By factoring the characteristic equation, we get two distinct roots $r_1 = 2$ and $r_2 = -2$. Thus general solution of the sequence is,

$$a^k = c_1(2)^k + c_2(-2)^k.$$

Using the given initial values of a^k to find c_1 and c_2 we get,

$$c_1 + c_2 = 1,$$
$$2c_1 - 2c_2 = 1.$$

Upon solving we get $c_1 = \frac{3}{4}$ and $c_2 = \frac{1}{4}$. Substituting values of c_1 and c_2 in general solution resulting formula is,

$$a^k = \frac{3}{4}(2)^k + \frac{1}{4}(-2)^\cdot$$

Example 2: Solve the $\hat{R}.\hat{R}$ $a^k = 4a^{k-1} - 4a^{k-2}$, for all integers $k \geq 2$, with initial conditions $a^0 = 1$, $a^1 = 3$.

Solution: Constructing characteristic equation for the $\hat{R}.\hat{R}$ $a^k = 4a^{k-1} - 4a^{k-2}$, which is,

$$r^2 - 4r + 4 = 0.$$

So its only root is 2 with multiplicity 2. It follows that $\hat{R}.\hat{R}$ has a general solution,

$$a^k = c_1(2)^k + c_2k(2)^k.$$

Finding values arbitrary constants c_1 and c_2. Using initial conditions, we get $c_1 = 1$ and $c_2 = \frac{1}{2}$. Hence, solution to the $\hat{R}.\hat{R}$ based on initial values appears to be,

$$a^k = (2)^k + \frac{1}{2}k(2)^k.$$

Example 3: Consider a $\hat{R}.\hat{R}$ $x^k = 2x^{k-1} - 2x^{k-2}$ with initial conditions, $x^0 = 1$ and $x^1 = 1$. Solve it using characteristic equation.

Solution: The characteristic polynomial $r^2 - 2r + 2 = 0$ cannot be factored. We get eigen values $1 + \iota$ and $1 - \iota$. For complex roots the general solution of recurrence is

$$x^k = c_1(1)^k cos(k) + c_2(1)^k sin(-k).$$

We need to determine the arbitrary constants c_1 and c_2 from initial conditions. Thus using initial conditions,

$$x^0 = 1 = c_1 cos(0) + c_2 sin(0),$$
$$x^1 = 1 = c_1 cos(1) + c_2 sin(1).$$

So we get $c_1 = 1$ and $c_2 = 1$. Thus solution the of $\hat{R}.\hat{R}$ is,

$$x^k = cos(k) - sin(k)$$

2.2.3 Non-Homogeneous Linear Recurrence Relation $\hat{N}.\hat{H}.\hat{L}.\hat{R}.\hat{R}$

Solution to a $\hat{N}.\hat{H}.\hat{L}.\hat{R}.\hat{R}$

Consider a $\hat{N}.\hat{H}.\hat{L}.\hat{R}.\hat{R}$,

$$x^n = a_1 x^{n-1} + a_2 x^{n-2} + \cdots + a_k x^{n-k} + F(n),$$

where $F(n)$ is the $\hat{N}.\hat{H}$ part.

The general solution of $\hat{N}.\hat{H}.\hat{L}.\hat{R}.\hat{R}$ is the sum of general solution of the homogeneous $\hat{H}$ part $a_h^{(n)}$ and particular solution of the non-homogeneous $\hat{N}.\hat{H}$ part $a_p^{(n)}$. So general solution for $\hat{N}.\hat{H}$ problem would be $a^n = a_h^{(n)} + a_p^{(n)}$. The initial conditions are later on considered in order to find the arbitrary constants. The general solution of $\hat{H}$ problem is in a manner as discussed earlier; however, for particular solution of the $\hat{N}.\hat{H}$ problem we consider the following cases.

If the $\hat{N}.\hat{H}$ part $F(n)$ is of the form

$$F(n) = (c_r n^r + c_{r-1} n^{r-1} + \cdots + c_1 n + c_0) \cdot t^n,$$

where $c_0, \cdots, c_r$ and m are real numbers.

- **Case 1**: The particular solution for the $\hat{N}.\hat{H}$ problem takes the form

$$(\beta_r n^r + \beta_{r-1} n^{r-1} + \cdots + \beta_1 n + \beta_0) \cdot t^n.$$

  If t does not appear to be the characteristic root.

- **Case 2**: The particular solution for the $\hat{N}.\hat{H}$ problem takes the form

$$(\beta_r n^r + \beta_{r-1} n^{r-1} + \cdots + \beta_1 n + \beta_0) \cdot t^n \cdot n^m.$$

  If t appears to be the characteristic root with multiplicity m.

Example: Solve the $\hat{N}.\hat{H}.\hat{R}.\hat{R}$ $a^k = 2a^{k-1} + 3^k$ with initial condition $a^1 = 5$.

Solution: Consider the $\hat{H}.\hat{R}.\hat{R}$ problem $a^k = 2a^{k-1}$. Characteristic equation for the $\hat{R}.\hat{R}$ is $r - 2 = 0$ and the characteristic root is 2. Hence, general solution is $a_{(h)}^k = c_1 2^k$. Now proceeding towards $\hat{N}.\hat{H}$ part which is $F(k) = 3^k$. Thus particular solution appears to be $a_{(p)}^k = c.3^k$, which implies $c.3^k = 2c.3^{k-1} + 3^k$. So $c = 3$ which follows $a_k^{(p)} = 3.3^k = 3^{k+1}$.

The general solution is $\hat{H}$ solution sums the particular solution.

$$a^k = c_1 2^k + 3^{k+1}.$$

Using initial condition $a^1 = 5$,

$$5 = c_1 2^1 + 3^{1+1},$$

$$5 = 2c_1 + 9,$$

$$c_1 = -2.$$

The required solution of the $\hat{N}$. $\hat{H}$. $\hat{L}.\hat{R}$.

Chapter 3

Abelian Sandpile Model

This chapter envisages background of abelian sandpile apart from the basic concepts and definition of sandpile group. Detail of the subject in question can be search in [1, 3, 4, 10, 13].

3.1 Abelian Sandpile Model

3.1.1 Background

The Abelian Sandpile Model abbreviated as ASM was classically proposed by Bak, Tang and Wiesenfled [10], using theory displaying self-organized criticality (SOC). The concept of SOC has been invoked to describe abelian sandpile model that is certainly the most simplest theoretical model of SOC. Later on, it was mathematically described by Dhar [3]. By linking an abelian group $S(G)$ to a finite connected graph Lionel Levine and James Propp concluded that abelian group is an isomorphism invariant of the finite connected graph, some sort of combinatorial properties can also be known about the graph. The abelian group features a finite abelian group generated by the operators corresponding to particle added up at several sites. Chip firing game in computer science and dollar-game is identical to ASM.

3.1.2 Abelian Sandpile

Abelian sandpile consists of a sandpile graph that is a finite connected multigraph (or a finite directed multigraph) with vertices known as ordinary vertices such that each vertex is assigned non-negative integer value and a distinguishable single vertex termed as sink vertex accessible from all other ordinary vertices. The integer-value assigned to each vertex is the degree of each vertex, which is actually the number of particles (or sand grains or chips) present at each vertex and each vertex is a site accommodating the particles. We say two vertices are adjacent if there is atleast one edge that connects both of them.

All non-sink vertices are allowed to have required quantity of particles. The sink collects the particles arriving at it and are lost. The track of ignored particles falling off the sink is not kept.

3.1.3 Configuration

A configuration is assigning an integer value to each ordinary vertex, which is in actual the number of particles being placed at the non sink vertices.

Stable Configuration

In a stable configuration each ordinary vertex holds a required amount of particles and does not exceeds the limit.

Unstable Configuration

If any of non-sink vertex exceeds the required number of particles then it is characterized to be unstable configuration. It suggests that any of the vertex that holds an integer value exceeds the required limit. In order to make the unstable configuration, a stable configuration the unstable vertices need to topple.

3.1.4 Toppling

In order to make unstable vertices stable toppling occurs. Toppling is the basic rule to stabilized an unstable configuration. In toppling unstable vertices lose some particles, which either fall off the system or are either added to the neighboring sites. The unstable vertex loses one particle each to every vertex adjacent to it. An unstable vertex can be created due to toppling. If the previous toppling results in the unstability of another site then toppling process continues until the time all the unstable vertices become stable. Toppling rule is applied to all vertices except single out sink vertex.

The toppling process does not depend on the sequence it is carried out since the final configuration is the same, irrespective of the sequence carried out in order to stabilize an unstable configuration. That is why the model is termed as abelian sandpile model. Sending particles to neighbouring vertices may reduce the quantity of particles, since there is an edge between every ordinary vertex and a sink vertex. The toppling process stops after certain finite toppling. The choice of which vertex to topple first does not effect the process of toppling. For further specification consider a map for sandpile:

$$\xi : V \to \{0, 1, 2, \cdots\}.$$

Let v be any vertex, ξ can only be stable at v if $\xi(v)$ <deg(v) otherwise ξ is unstable and v will topple by sending one particle to each incident vertex. On toppling vertex v, the particles will redistributed as follows:

$$\xi(v) \to \xi(v) - deg(v)$$
$$\xi(u) \to \xi(u) + \alpha_{uv} \text{ ,where } u \in V \text{ and } u \neq v$$

where α_{vu} is the number of edges between v and u [1].

Toppling matrix

When a vertex v is topple and ξ to be a row vector, one can write

$$\xi = \xi - \Delta_{vu},$$

where

$$\Delta_{vu} = \begin{cases} deg(v), & \text{if } v = u \in V; \\ -\alpha_{vu}, & \text{if } v \neq u,\ v,\ u \in V; \\ 0, & \text{otherwise.} \end{cases}$$

Δ is the Reduced Laplacian, since row and column of sink vertex is eliminated. Δ (Laplacian) can be given as,

$$L = D - A.$$

where D is the degree matrix and A is the adjacency matrix of vertices, which is zero in case of no edges between vertices.

3.1.5 Avalanche

The addition of particles to the pile that leads to a collection of rapid topplings that occur in order to stabilize the unstable system is called the avalanche. Different factors can be taken into an account to measure the strength of an avalanche. These factors involve

- time of an avalanche
- area cover by the avalnche
- the total count of topplings

- The count of different size toppled

3.1.6 The Sandpile Group

One can relate an abelian group known as sandpile group to every finite connected graph. A commutative monoid is formed as a result of set of stable states (configuration) under operation of an ordinary addition followed by stabilization. Let this operation be denoted by $\oplus$ and h_i, h_j be any two configuration such that

$$h_i \oplus h_j = \xi(h_i + h_j).$$

($\xi(h)$ is unique stabilization for any configuration h). This commutative monoid is sandpile monoid. For any two configuration say a and b: if $a \oplus b = c$ (also a configuration) then it is called recurrent Configuration. A configuration that belongs to a cycle is said to be recurrent. Thus, one can say recurrent configuration is the one in which a stable configuration is accesssible from every configuration otherwise transient. According to W.Chen T.Schedler [13] and Dhar [4] defined these recurrent configuration to be stable configuration forming an abelian group under $\oplus$ called the sandpile group $S(G)$.

Fixing order for all ordinary vertices say $n =\mid V \mid -1$ (sink vertex is eliminated) we take into account subgroup of $\mathbb{Z}^n$. The elements of $\mathbb{Z}^n$ may be considered as configuration in which number of grains of sand (particles) in each cell (site) may be positive or negative. The subgroup is generated by n elements expressing the toppling rule [11]. The quotient of $\mathbb{Z}^n$ by this subgroup is abelian group, which is the sandpile group. The abstract group structure for this sandpile group $S(G)$ can be given as

$$S(G) \approx \mathbb{Z}^n / \Delta \mathbb{Z}^n$$

where Δ is the Reduced Laplacian whose det gives order of sandpile group; which, according to *Matrix-tree theorem*, is the number of spanning trees of the graph.

Chapter 4

On the sandpile group of the graph $K_3 \times C_n$

4.1 On the Sandpile Group of the Graph $K_3 \times C_n$

The abelian group model was introduced by Bak et al. in [10], based on the concept of self-organized criticality. A finite connected multi-graph $G = (V, E)$ on n vertices is considered and by determining its Laplacian matrix $L = D - A$ where D is the degree matrix and A is the adjacency matrix, one can define a finite abelian group known as sandpile group.

Let v_r be a vertex (call root) of the graph G with n vertices, then the sandpile group $S(G)$ on the graph G with n vertices is the quotient of $\mathbb{Z}^n$ by the subgroup spanned by $n-1$ elements expressing the toppling rule (that is, if $v_i \neq v_r$ is a vertex of degree $\mathfrak{d}_i$, a generator of this subgroup is

$$\Delta_i = d_i x_i - \sum_{v_i \text{ is adjacent } v_i} a_{ij} x_j,$$

where a_{ij} is the number of edges between vertices v_i and v_j, and $x_i = (0, \cdots, 0, 1, 0, \cdots, 0) \in$

$\mathbb{Z}^n$, whose unique non-zero is in the position i and the element x_r, that is, $S(G) \cong \mathbb{Z}^n/span(\Delta_1, \cdots, \Delta_{r-1}, x_r, \Delta_{r+1}, \cdots, \Delta_n)$ [14].

This paper focuses on the objective of determining the structure of the sandpile group on the graph of $C_3 \times C_n$, the cartesian product of C_3 and C_n. The main tool is the computation of Smith normal (SNF) form of an integer matrix say A, which is the unique diagonal matrix $\mathcal{S}(A) = diag(\mathcal{S}_{11}, \mathcal{S}_{22}, \cdots, \mathcal{S}_{nn})$ whose entries are nonnegative and $\mathcal{S}_{ii}$ divides $\mathcal{S}_{i+1,i+1}$, and for each i, the product $\mathcal{S}_{11}\mathcal{S}_{22}, \cdots, \mathcal{S}_{ii}$ is the greatest common divisor (GCD) of all $i \times i$ minor determinants of A [15].

Here sandpile group on the graph $K_3 \times C_n$ is determined which is the direct product of four or five cyclic groups. Same method can be applied to figure out sandpile groups of $K_4 \times C_n$ and $K_5 \times C_n$.

4.1.1 The Relations Matrix for Generators of $K_3 \times C_n$

In order to obtain a relation matrix for $K_3 \times C_n$, consider vertex set of K_3 that is $V_k = \{0, 1, 2\}$ and C_n which is $V_c = \{0, 1, 2, \cdots, n-1\}$. Then the vertex set of $K_3 \times C_n$ is the Cartesian product $V_k \times V_c$, where $V_k \times V_c = \{(k, c) \mid k = \{0, 1, 2\}, c = \{0, 1, 2, .., n-1\}\}$. The pair (k, c) can be adjacent only to $(k, (c+1) \mod n)$, $(k, (c-1) \mod n)$, $((k+1) \mod 3, c)$ and $((k-1) \mod 3, c)$. Obtaining system of equations by applying toppling rule

$$\Delta_i = d_i x_i - \sum_{0 \le j \le n-1} e_{ij} x_j$$

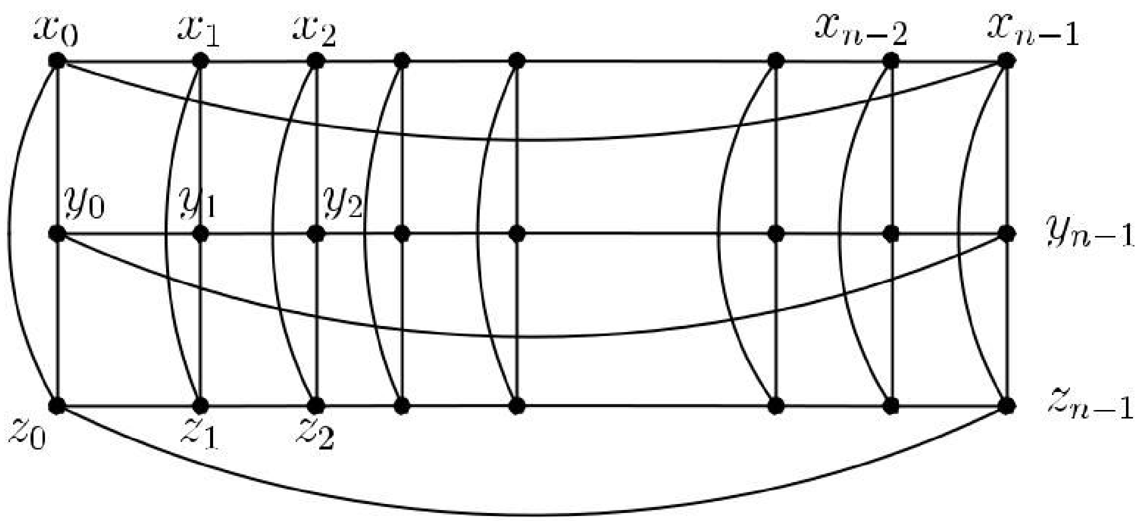

Figure 4.1: The graph of $K_3 \times C_n$.

$$\Delta x_0 = 4x_0 - x_1 - x_{n-1} - y_0 - z_0,$$

$$\Delta y_0 = 4y_0 - x_0 - y_1 - y_{n-1} - z_0,$$

$$\Delta z_0 = 4z_0 - x_0 - y_0 - z_{n-1} - z_1,$$

$$\Delta x_1 = 4x_1 - x_0 - x_2 - y_1 - z_1,$$

$$\Delta y_1 = 4y_1 - x_1 - y_0 - y_2 - z_1,$$

$$\Delta z_1 = 4z_1 - x_1 - y_1 - z_0 - z_2.$$

$$\vdots$$

$$\Delta x_{n-2} = 4x_{n-2} - x_{n-1} - x_{n-3} - y_{n-2} - z_{n-2},$$

$$\Delta y_{n-2} = 4y_{n-2} - y_{n-1} - y_{n-3} - x_{n-2} - z_{n-3},$$

$$\Delta z_{n-2} = 4z_{n-2} - z_{n-1} - z_{n-3} - x_{n-2} - y_{n-2},$$

$$\Delta x_{n-1} = 4x_{n-1} - x_0 - x_{n-2} - y_{n-1} - z_{n-1},$$

$$\Delta y_{n-1} = 4y_{n-1} - y_0 - y_{n-2} - x_{n-1} - z_{n-1},$$

$$\Delta z_{n-1} = 4z_{n-1} - z_0 - z_{n-2} - x_{n-1} - y_{n-1}.$$

By equating all the equations, we get

$$4x_0 - x_1 - x_{n-1} - y_0 - z_0 = 0,$$

$$4y_0 - x_0 - y_1 - z_0 - yn - 1 = 0,$$

$$4z_0 - x_0 - y_0 - z_1 - z_{n-1} = 0,$$

$$4x_1 - x_0 - x_2 - y_1 - z_1 = 0,$$

$$4y_1 - x_1 - y_0 - y_2 - z_1 = 0,$$

$$4z_1 - x_1 - y_1 - z_0 - z_2 = 0.$$

$$\vdots$$

$$4x_{n-2} - x_{n-1} - x_{n-3} - y_{n-2} - z_{n-2=0},$$

$$4y_{n-2} - y_{n-1} - y_{n-3} - x_{n-2} - z_{n-3=0},$$

$$4z_{n-2} - z_{n-1} - z_{n-3} - x_{n-2} - y_{n-2} = 0,$$

$$4x_{n-1} - x_0 - x_{n-2} - y_{n-1} - z_{n-1} = 0,$$

$$4y_{n-1} - y_0 - y_{n-2} - x_{n-1} - z_{n-1} = 0,$$

$$4z_{n-1} - x_0 - z_{n-2} - x_{n-1} - y_{n-1} = 0.$$

let $x_r = (0, r)$, $y_r = (1, r)$ and $z_r = (2, r)$, $r = 0, 1, 2, \cdots, n-1$ and the image of x_r, y_r and z_r in the cokernal $\mathbb{Z}^{3n}/ImL(K_3 \times C_n)$ be x_r, y_r and z_r, respectively. We have:

$$\bar{x}_r = 4\bar{x}_{r-1} - \bar{x}_{r-2} - \bar{y}_{r-1} - \bar{z}_{r-1},$$

$$\bar{y}_r = 4\bar{y}_{r-1} - \bar{y}_{r-2} - \bar{z}_{r-1} - \bar{x}_{r-1},$$

$$\bar{z}_r = 4\bar{z}_{r-1} - \bar{z}_{r-2} - \bar{x}_{r-1} - \bar{y}_{r-1}.$$

For $r \geq 2$, we get a system of at most six generators for the cokernel $\mathbb{Z}^{3n}/ImL(K_3 \times C_n)$. All x_r, y_r and z_r can be expressed in terms of x_0, y_0, z_0, x_1, y_1 and z_1.

$$\bar{x}_0 = 4\bar{x}_{n-1} - \bar{x}_{n-2} - \bar{y}_{n-1} - \bar{z}_{n-1},$$

$$\bar{y}_0 = 4\bar{y}_{n-1} - \bar{y}_{n-2} - \bar{z}_{n-1} - \bar{x}_{n-1},$$

$$\bar{z}_0 = 4\bar{z}_{n-1} - \bar{z}_{n-2} - \bar{x}_{n-1} - \bar{y}_{n-1},$$

$$\bar{x}_1 = 4\bar{x}_0 - \bar{x}_{n-1} - \bar{y}_0 - \bar{z}_0,$$

$$\bar{y}_1 = 4\bar{y}_0 - \bar{y}_{n-1} - \bar{z}_0 - \bar{x}_0,$$

$$\bar{z}_1 = 4\bar{z}_0 - \bar{z}_{n-1} - \bar{x}_0 - \bar{y}_0.$$

For $2 \leq r \leq n-1$, $\mathfrak{a}_r, \mathfrak{b}_r, \mathfrak{c}_r, \mathfrak{d}_r, \mathfrak{e}_r$ and $\mathfrak{f}_r$ are defined as

$$\bar{x}_r = \mathfrak{a}_r\bar{x}_1 - \mathfrak{b}_r\bar{x}_0 - \mathfrak{c}_r\bar{y}_1 + \mathfrak{d}_r\bar{y}_0 - \mathfrak{e}_r\bar{z}_1 + \mathfrak{f}_r\bar{z}_0, \tag{4.1}$$

$$\bar{y}_r = \mathfrak{a}_r\bar{y}_1 - \mathfrak{b}_r\bar{y}_0 - \mathfrak{c}_r\bar{z}_1 + \mathfrak{d}_r\bar{z}_0 - \mathfrak{e}_r\bar{x}_1, +\mathfrak{f}_r\bar{x}_0, \tag{4.2}$$

$$\bar{z}_r = \mathfrak{a}_r\bar{z}_1 - \mathfrak{b}_r\bar{z}_0 - \mathfrak{c}_r\bar{x}_1 + \mathfrak{d}_r\bar{x}_0 - \mathfrak{e}_r\bar{y}_1 + \mathfrak{f}_r\bar{y}_0. \tag{4.3}$$

Equating equations

$$\bar{x}_r = 4\bar{x}_{r-1} - \bar{x}_{r-2} - \bar{y}_{r-1} - \bar{z}_{r-1} \tag{4.4}$$

and

$$\bar{x}_r = 4\bar{x}_{r-1} - \bar{x}_{r-2} - \bar{y}_{r-1} - \bar{z}_{r-1} \tag{4.5}$$

we get

$$4\bar{x}_{r-1} - \bar{x}_{r-2} - \bar{y}_{r-1} - \bar{z}_{r-1} = \mathfrak{a}_r\bar{x}_1 - \mathfrak{b}_r\bar{x}_0 - \mathfrak{c}_r\bar{y}_1 + \mathfrak{d}_r\bar{y}_0 - \mathfrak{e}_r\bar{z}_1 + \mathfrak{f}_r\bar{z}_1.$$

putting values of $\bar{x}_{r-1}, \bar{x}_{r-2}, \bar{y}_{r-1}$ and $\bar{z}_{r-1}$ by using equation 4.1, 4.2 and 4.3,

$$4(\mathfrak{a}_{r-1}\bar{x}_1-\mathfrak{b}_{r-1}\bar{x}_0-\mathfrak{c}_{r-1}\bar{y}_1+\mathfrak{d}_{r-1}\bar{y}_0-\mathfrak{e}_{r-1}\bar{z}_1+\mathfrak{f}_{r-1}\bar{z}_0)-(\mathfrak{a}_{r-2}\bar{x}_1-\mathfrak{b}_{r-2}\bar{x}_0-\mathfrak{c}_{r-2}\bar{y}_1+\mathfrak{d}_{r-2}\bar{y}_0-\mathfrak{e}_{r-2}\bar{z}_1+\mathfrak{f}_{r-2}\bar{z}_0)$$
$$-(\mathfrak{a}_{r-1}\bar{y}_1-\mathfrak{b}_{r-1}\bar{y}_0-\mathfrak{c}_{r-1}\bar{z}_1+\mathfrak{d}_{r-1}\bar{z}_0-\mathfrak{e}_{r-1}\bar{x}_1+\mathfrak{f}_{r-1}\bar{x}_0)-(\mathfrak{a}_{r-1}\bar{z}_1-\mathfrak{b}_{r-1}\bar{z}_0-\mathfrak{c}_{r-1}\bar{x}_1+\mathfrak{d}_{r-1}\bar{x}_0-\mathfrak{e}_{r-1}\bar{y}_1+\mathfrak{f}_{r-1}\bar{y}_0)$$
$$= \mathfrak{a}_r\bar{x}_1 - \mathfrak{b}_r\bar{x}_0 - \mathfrak{c}_r\bar{y}_1 + \mathfrak{d}_r\bar{y}_0 - \mathfrak{e}_r\bar{z}_1 + \mathfrak{f}_r\bar{z}_r$$

Comparing the coefficients we get

$$\begin{aligned}
\mathfrak{a}_r &= 4\mathfrak{a}_{r-1} - \mathfrak{a}_{r-2} + \mathfrak{e}_{r-1} + \mathfrak{c}_{r-1},\\
\mathfrak{b}_r &= 4\mathfrak{b}_{r-1} - \mathfrak{b}_{r-2} + \mathfrak{f}_{r-1} + \mathfrak{d}_{r-1},\\
\mathfrak{c}_r &= 4\mathfrak{c}_{r-1} - \mathfrak{c}_{r-2} + \mathfrak{a}_{r-1} - \mathfrak{e}_{r-1},\\
\mathfrak{d}_r &= 4\mathfrak{d}_{r-1} - \mathfrak{d}_{r-2} + \mathfrak{b}_{r-1} - \mathfrak{f}_{r-1},\\
\mathfrak{e}_r &= 4\mathfrak{e}_{r-1} - \mathfrak{e}_{r-2} - \mathfrak{c}_{r-1} + \mathfrak{a}_{r-1},\\
\mathfrak{f}_r &= 4\mathfrak{f}_{r-1} - \mathfrak{f}_{r-2} - \mathfrak{d}_{r-1} + \mathfrak{b}_{r-1}.
\end{aligned} \tag{4.6}$$

Since, we have

$$\bar{x}_r = \mathfrak{a}_r\bar{x}_1 - \mathfrak{b}_r\bar{x}_0 - \mathfrak{c}_r\bar{y}_1 + \mathfrak{d}_r\bar{y}_0 - \mathfrak{e}_r\bar{z}_1 + \mathfrak{f}_r\bar{z}_0.$$

Putting $r = 0$ and comparing the coefficients, we get $\mathfrak{a}_0 = 0, \mathfrak{b}_0 = -1, \mathfrak{c}_0 = 0, \mathfrak{d}_0 = 0, \mathfrak{e}_0 = 0, \mathfrak{f}_0 = 0$, and for $r = 1$ we get $\mathfrak{a}_1 = 1, \mathfrak{b}_1 = 0, \mathfrak{c}_1 = 0, \mathfrak{d}_1 = 0, \mathfrak{e}_1 = 0, \mathfrak{e}_1 = 0$.

For $r = 2$, substituting all the above values in equation 4.6:

$$\mathfrak{a}_2 = 4\mathfrak{a}_1 - \mathfrak{a}_0 + \mathfrak{e}_1 + \mathfrak{c}_1,$$

$$\mathfrak{b}_2 = 4\mathfrak{b}_1 - \mathfrak{b}_0 + \mathfrak{f}_1 + \mathfrak{d}_1,$$

$$\mathfrak{c}_2 = 4\mathfrak{c}_1 - \mathfrak{c}_0 + \mathfrak{a}_1 - \mathfrak{e}_1,$$

$$\mathfrak{d}_2 = 4\mathfrak{d}_1 - \mathfrak{d}_0 + \mathfrak{b}_1 - \mathfrak{f}_1,$$

$$\mathfrak{e}_2 = 4\mathfrak{e}_1 - \mathfrak{e}_0 - \mathfrak{c}_1 + \mathfrak{a}_1,$$

$$\mathfrak{f}_2 = 4\mathfrak{f}_1 - \mathfrak{f}_0 - \mathfrak{d}_1 + \mathfrak{b}_1.$$

we get $\mathfrak{a}_2 = 4, \mathfrak{b}_2 = 1, \mathfrak{c}_2 = 1, \mathfrak{d}_2 = 0, \mathfrak{e}_2 = 1, \mathfrak{f}_2 = 0$. Similarly, for $r = 3$ we have $\mathfrak{a}_3 = 17, \mathfrak{b}_3 = 4, \mathfrak{c}_3 = 7, \mathfrak{d}_3 = 1, \mathfrak{e}_3 = 7, \mathfrak{f}_3 = 1$ and so on. The final result shows

$$\begin{cases} \mathfrak{a}_r = & 4\mathfrak{a}_{r-1} - \mathfrak{a}_{r-2} + \mathfrak{e}_{r-1} + \mathfrak{c}_{r-1}, \\ \mathfrak{b}_r = & 4\mathfrak{b}_{r-1} - \mathfrak{b}_{r-2} + \mathfrak{f}_{r-1} + \mathfrak{d}_{r-1}, \\ \mathfrak{c}_r = & 4\mathfrak{c}_{r-1} - \mathfrak{c}_{r-2} + \mathfrak{a}_{r-1} - \mathfrak{e}_{r-1}, \\ \mathfrak{b}_r = & \mathfrak{a}_{r-1}, \\ \mathfrak{d}_r = & \mathfrak{c}_{r-1}, \\ \mathfrak{e}_r = & \mathfrak{c}_r, \\ \mathfrak{f}_r = & \mathfrak{d}_r. \end{cases}$$

with the initial values $\mathfrak{a}_2 = 4, \mathfrak{a}_3 = 17, \mathfrak{b}_2 = 1, \mathfrak{c}_2 = 1$ and $\mathfrak{c}_3 = 7$ and concluded that $\mathfrak{b}_r, \mathfrak{d}_r, \mathfrak{e}_r$, can be written in terms of $\mathfrak{a}_r, \mathfrak{c}_r$ and $\mathfrak{f}_r$ can be written in terms of $\mathfrak{d}_r$.

Lemma 4.1.1.1. For all $r \geq 2$, we have $\mathfrak{a}_r - 2\mathfrak{c}_r = r$ and $\mathfrak{c}_r = 5\mathfrak{c}_r - \mathfrak{c}_{r-2} + r - 1$.

Proof. Since

$$\mathfrak{a}_r = 4\mathfrak{a}_{r-1} - \mathfrak{a}_{r-2} + \mathfrak{e}_{r-1} + \mathfrak{c}_{r-1},$$

$$\mathfrak{c}_r = 4\mathfrak{c}_{r-1} - \mathfrak{c}_{r-2} + \mathfrak{a}_{r-1} - \mathfrak{e}_{r-1}.$$

Then we have

$$\begin{aligned}\mathfrak{a}_r - 2\mathfrak{c}_r =& 4\mathfrak{a}_{r-1} - \mathfrak{a}_{r-2} + \mathfrak{e}_{r-1} + \mathfrak{c}_{r-1} - 2(4\mathfrak{c}_{r-1} - \mathfrak{c}_{r-2} + \mathfrak{a}_{r-1} - \mathfrak{e}_{r-1}),\\ =& 4\mathfrak{a}_{r-1} - \mathfrak{a}_{r-2} + \mathfrak{e}_{r-1} + \mathfrak{c}_{r-1} - 8\mathfrak{c}_{r-1} + 2\mathfrak{c}_{r-2} - 2\mathfrak{a}_{r-1} + 2\mathfrak{e}_{r-1},\\ =& 2\mathfrak{a}_{r-1} - \mathfrak{a}_{r-2} + 3\mathfrak{e}_{r-1} - 7\mathfrak{c}_{r-1} + 2\mathfrak{c}_{r-2}.\end{aligned}$$

Since $\mathfrak{e}_{r-1} = \mathfrak{c}_{r-1}$,

$$\begin{aligned}\mathfrak{a}_r - 2\mathfrak{c}_r =& 2\mathfrak{a}_{r-1} - \mathfrak{a}_{r-2} + 3\mathfrak{c}_{r-1} - 7\mathfrak{c}_{r-1} + 2\mathfrak{c}_{r-2}.\\ =& 2\mathfrak{a}_{r-1} - \mathfrak{a}_{r-2} - 4\mathfrak{c}_{r-1} + 2\mathfrak{c}_{r-2},\\ =& 2(\mathfrak{a}_{r-1} - 2\mathfrak{c}_{r-1}) - (\mathfrak{a}_{r-2} - 2\mathfrak{c}_{r-2}).\end{aligned}$$

Using initial values of $\mathfrak{a}_2$ and $\mathfrak{c}_2$ so we have $\mathfrak{a}_2 - 2\mathfrak{c}_2 = 4 - 2 = 2 \implies \mathfrak{a}_r - 2\mathfrak{c}_r = r$; hence, we get

$$\mathfrak{a}_r - 2\mathfrak{c}_r = 2(r-1) - (r-2) = r.$$

Hence proved.

Now consider,

$$\mathfrak{c}_r = 4\mathfrak{c}_{r-1} - \mathfrak{c}_{r-2} + \mathfrak{a}_{r-1} - \mathfrak{e}_{r-1},$$

as $\mathfrak{e}_{r-1} = \mathfrak{c}_{r-1}$

$$\mathfrak{c}_r = 4\mathfrak{c}_{r-1} - \mathfrak{c}_{r-2} + \mathfrak{a}_{r-1} - \mathfrak{c}_{r-1},$$

Since $\mathfrak{a}_r - 2\mathfrak{c}_r = r$ then $\mathfrak{a}_{r-1} - 2\mathfrak{c}_{r-1} = r - 1$. Thus we get

$$\mathfrak{c}_r = 4\mathfrak{c}_{r-1} - \mathfrak{c}_{r-2} + \mathfrak{a}_{r-1} - 2\mathfrak{c}_{r-1} + \mathfrak{c}_{r-1},$$
$$\implies \mathfrak{c}_r = 5\mathfrak{c}_{r-1} - \mathfrak{c}_{r-2} + r - 1.$$

Using $\mathfrak{c}_{-1} = \mathfrak{c}_0 = 0$ and $\mathfrak{c}_1 = 0$ for $n \geq 2$, the sequence $\mathfrak{c}_n$ can be extended.
For all $r \geq 0$, define $\mathfrak{J}_r = \mathfrak{c}_r - \mathfrak{c}_{r-1}$. Then for $r \geq 2$, we have $\mathfrak{J}_r = 5\mathfrak{J}_{r-1} - \mathfrak{J}_{r-2} + 1$, such that $\mathfrak{J}_0 = \mathfrak{J}_1 = 0$. Also $3\mathfrak{c}_n + n = \mathfrak{J}_{n+1} - \mathfrak{J}_n$

$$\begin{aligned}
\mathfrak{c}_r &= 5\mathfrak{c}_{r-1} - \mathfrak{c}_{r-2} + r - 1,\\
&= \mathfrak{c}_{r-1} - \mathfrak{c}_{r-2} + 4\mathfrak{c}_{r-1} + r - 1\\
&= \mathfrak{J}_{r-1} + \mathfrak{c}_{r-1} + (3\mathfrak{c}_{r-1} + r - 1),\\
&= \mathfrak{J}_{r-1} + \mathfrak{c}_{r-1} + \mathfrak{J}_r - \mathfrak{J}_{r-1},\\
&\implies \mathfrak{J}_r = \mathfrak{J}_r.
\end{aligned}$$

□

Theorem 4.1.1. Let $n \geq 3$. The relation matrix between the generators $\bar{x}_0$, $\bar{x}_1$, $\bar{y}_0$, $\bar{y}_1$, $\bar{z}_0$, $\bar{z}_1$ of the cokernel $\mathbb{Z}^{3n}/ImL(K_3 \times C_n)$ is equivalent to

$$\begin{pmatrix} n & 0 & 0 & 0 & 0 & 0 \\ \mathfrak{c}_n & \mathfrak{I}_n & \mathfrak{I}_{n+1} & 0 & 0 & 0 \\ \mathfrak{c}_{n+1} & \mathfrak{I}_{n+1} & \mathfrak{I}_{n+2}-1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \mathfrak{I}_{n+1}-\mathfrak{I}_n & \mathfrak{I}_n-\mathfrak{I}_{n-1}+1 & 0 \\ 0 & 0 & 0 & 3\mathfrak{I}_{n+1} & 3\mathfrak{I}_n & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} A_n & 0 \\ 0 & 0 \end{pmatrix}$$

. Alternatively, for $n \geq 3$, we have $S(K_3 \times C_n) \cong \mathbb{Z}^5/A_n\mathbb{Z}^5$.

Proof. Since,

$$\begin{aligned} \bar{x}_r =& 4\bar{x}_{r-1} - \bar{x}_{r-2} - \bar{y}_{r-1} - \bar{z}_{r-1}, \\ \bar{x}_r =& \mathfrak{a}_r\bar{x}_1 - \mathfrak{b}_r\bar{x}_0 - \mathfrak{c}_r\bar{y}_1 + \mathfrak{d}_r\bar{y}_0 - \mathfrak{e}_r\bar{z}_1 + \mathfrak{f}_r\bar{z}_0, \end{aligned}$$

Replacing r by n in above equations, we get

$$\bar{x}_n = 4\bar{x}_{n-1} - \bar{x}_{n-2} - \bar{y}_{n-1} - \bar{z}_{n-1}, \tag{4.7}$$

$$\bar{x}_n = \mathfrak{a}_n\bar{x}_1 - \mathfrak{b}_n\bar{x}_0 - \mathfrak{c}_n\bar{y}_1 + \mathfrak{d}_n\bar{y}_0 - \mathfrak{e}_n\bar{z}_1 + \mathfrak{f}_n\bar{z}_0, \tag{4.8}$$

$$\text{Also } \bar{x}_0 = 4\bar{x}_{n-1} - \bar{x}_{n-2} - \bar{y}_{n-1} - \bar{z}_{n-1}. \tag{4.9}$$

Using equation 4.7 and substituting all the values in the above equation

$$\begin{aligned} \mathfrak{a}_n\bar{x}_1 - \mathfrak{b}_n\bar{x}_0 - \mathfrak{c}_n\bar{y}_1 + \mathfrak{d}_n\bar{y}_0 - \mathfrak{e}_n\bar{z}_1 + \mathfrak{f}_n\bar{z}_0 &= 4\bar{x}_{n-1} - \bar{x}_{n-2} - \bar{y}_{n-1} - \bar{z}_{n-1} \\ &= 4(\mathfrak{a}_{n-1}\bar{x}_1 - \mathfrak{b}_{n-1}\bar{x}_0 - \mathfrak{c}_{n-1}\bar{y}_1 + \mathfrak{d}_{n-1}\bar{y}_0 - \mathfrak{e}_{n-1}\bar{z}_1 + \mathfrak{f}_{n-1}\bar{z}_0) \\ &- (\mathfrak{a}_{n-2}\bar{x}_1 - \mathfrak{b}_{n-2}\bar{x}_0 - \mathfrak{c}_{n-2}\bar{y}_1 + \mathfrak{d}_{n-2}\bar{y}_0 - \mathfrak{e}_{n-2}\bar{z}_1 + \mathfrak{f}_{n-2}\bar{z}_0) \\ &- (\mathfrak{a}_{n-1}\bar{y}_1 - \mathfrak{b}_{n-1}\bar{y}_0 - \mathfrak{c}_{n-1}\bar{z}_1 + \mathfrak{d}_{n-1}\bar{z}_0 - \mathfrak{e}_{n-1}\bar{x}_1, +\mathfrak{f}_{n-1}\bar{x}_0) \\ &- (\mathfrak{a}_{n-1}\bar{z}_1 - \mathfrak{b}_{n-1}\bar{z}_0 - \mathfrak{c}_{n-1}\bar{x}_1 + \mathfrak{d}_{n-1}\bar{x}_0 - \mathfrak{e}_{n-1}\bar{y}_1 + \mathfrak{f}_{n-1}\bar{y}_0) \end{aligned}$$

Comparing the coeffients,

$$\begin{aligned}
\mathfrak{a}_n &= 4\mathfrak{a}_{n-1} - \mathfrak{a}_{n-2} + \mathfrak{e}_{n-1} + \mathfrak{c}_{n-1},\\
\mathfrak{b}_n &= 4\mathfrak{b}_{n-1} - \mathfrak{b}_{n-2} + \mathfrak{f}_{n-1} - \mathfrak{d}_{n-1},\\
\mathfrak{c}_n &= 4\mathfrak{c}_{n-1} - \mathfrak{c}_{n-2} + \mathfrak{a}_{n-1} - \mathfrak{e}_{n-1},\\
\mathfrak{d}_n &= 4\mathfrak{d}_{n-1} - \mathfrak{d}_{n-2} + \mathfrak{b}_{n-1} - \mathfrak{f}_{n-1},\\
\mathfrak{e}_n &= 4\mathfrak{e}_{n-1} - \mathfrak{e}_{n-2} - \mathfrak{c}_{n-1} + \mathfrak{a}_{n-1},\\
\mathfrak{f}_n &= 4\mathfrak{f}_{n-1} - \mathfrak{f}_{n-2} - \mathfrak{d}_{n-1} + \mathfrak{b}_{n-1}.
\end{aligned}$$

Consider equation 4.8

$$\bar{x}_0 = 4\bar{x}_{n-1} - \bar{x}_{n-2} - \bar{y}_{n-1} - \bar{z}_{n-1},$$

By using equation 4.7 substituting values of $\bar{x}_{n-1}, \bar{x}_{n-2}, \bar{y}_{n-1}, \bar{z}_{n-1}$

$$\begin{aligned}
\bar{x}_0 =& 4(\mathfrak{a}_{n-1}\bar{x}_1 - \mathfrak{b}_{n-1}\bar{x}_0 - \mathfrak{c}_{n-1}\bar{y}_1 + \mathfrak{d}_{n-1}\bar{y}_0 - \mathfrak{e}_{n-1}\bar{z}_1 + \mathfrak{f}_{n-1}\bar{z}_0)\\
&- (\mathfrak{a}_{n-2}\bar{x}_1 - \mathfrak{b}_{n-2}\bar{x}_0 - \mathfrak{c}_{n-2}\bar{y}_1 + \mathfrak{d}_{n-2}\bar{y}_0 - \mathfrak{e}_{n-2}\bar{z}_1 + \mathfrak{f}_{n-2}\bar{z}_0)\\
&- (\mathfrak{a}_{n-1}\bar{y}_1 - \mathfrak{b}_{n-1}\bar{y}_0 - \mathfrak{c}_{n-1}\bar{z}_1 + \mathfrak{d}_{n-1}\bar{z}_0 - \mathfrak{e}_{n-1}\bar{x}_1, +\mathfrak{f}_{n-1}\bar{x}_0)\\
&- (\mathfrak{a}_{n-1}\bar{z}_1 - \mathfrak{b}_{n-1}\bar{z}_0 - \mathfrak{c}_{n-1}\bar{x}_1 + \mathfrak{d}_{n-1}\bar{x}_0 - \mathfrak{e}_{n-1}\bar{y}_1 + \mathfrak{f}_{n-1}\bar{y}_0)\\
&= (4\mathfrak{a}_{n-1} - \mathfrak{a}_{n-2} + \mathfrak{e}_{n-1} + \mathfrak{c}_{n-1})\bar{x}_1 - (4\mathfrak{b}_{n-1} - \mathfrak{b}_{n-2} + \mathfrak{f}_{n-1}\\
&- \mathfrak{d}_{n-1})\bar{x}_0 - (4\mathfrak{c}_{n-1} - \mathfrak{c}_{n-2} + \mathfrak{a}_{n-1} - \mathfrak{e}_{n-1})\bar{y}_1 + (4\mathfrak{d}_{n-1} -\\
&\mathfrak{d}_{n-2} + \mathfrak{b}_{n-1} - \mathfrak{f}_{n-1})\bar{y}_0 - (4\mathfrak{e}_{n-1} - \mathfrak{e}_{n-2} - \mathfrak{c}_{n-1} + \mathfrak{a}_{n-1})\\
&\bar{z}_1 + (4\mathfrak{f}_{n-1} - \mathfrak{f}_{n-2} - \mathfrak{d}_{n-1} + \mathfrak{b}_{n-1})\bar{z}_0.
\end{aligned}$$

Thus,

$$\bar{x}_0 = \mathfrak{a}_n \bar{x}_1 - \mathfrak{b}_n \bar{x}_0 - \mathfrak{c}_n \bar{y}_1 + \mathfrak{d}_n \bar{y}_0 - \mathfrak{e}_n \bar{z}_1 + \mathfrak{f}_n \bar{z}_0.$$

Similarly, for $\bar{y}_0, \bar{z}_0, \bar{x}_1, \bar{y}_1, \bar{z}_1$. Thus we deduced a new system of relations.

$$\begin{aligned}
\bar{x}_0 &= \mathfrak{a}_n \bar{x}_1 - \mathfrak{b}_n \bar{x}_0 - \mathfrak{c}_n \bar{y}_1 + \mathfrak{d}_n \bar{y}_0 - \mathfrak{e}_n \bar{z}_1 + \mathfrak{f}_n \bar{z}_0, \\
\bar{y}_0 &= \mathfrak{a}_n \bar{y}_1 - \mathfrak{b}_n \bar{y}_0 - \mathfrak{c}_n \bar{z}_1 + \mathfrak{d}_n \bar{z}_0 - \mathfrak{e}_n \bar{x}_1 + \mathfrak{f}_n \bar{x}_0, \\
\bar{z}_0 &= \mathfrak{a}_n \bar{z}_1 - \mathfrak{b}_n \bar{z}_0 - \mathfrak{c}_n \bar{x}_1 + \mathfrak{d}_n \bar{x}_0 - \mathfrak{e}_n \bar{y}_1 + \mathfrak{f}_n \bar{y}_0, \\
\bar{x}_{n+1} &= \mathfrak{a}_{n+1} \bar{x}_1 - \mathfrak{b}_{n+1} \bar{x}_0 - \mathfrak{c}_{n+1} \bar{y}_1 + \mathfrak{d}_{n+1} \bar{y}_0 - \mathfrak{e}_{n+1} \bar{z}_1 + \mathfrak{f}_{n+1} \bar{z}_0, \\
\bar{y}_{n+1} &= \mathfrak{a}_{n+1} \bar{y}_1 - \mathfrak{b}_{n+1} \bar{y}_0 - \mathfrak{c}_{n+1} \bar{z}_1 + \mathfrak{d}_{n+1} \bar{z}_0 - \mathfrak{e}_{n+1} \bar{x}_1 + \mathfrak{f}_{n+1} \bar{x}_0, \\
\bar{z}_{n+1} &= \mathfrak{a}_{n+1} \bar{z}_1 - \mathfrak{b}_{n+1} \bar{z}_0 - \mathfrak{c}_{n+1} \bar{x}_1 + \mathfrak{d}_{n+1} \bar{x}_0 - \mathfrak{e}_{n+1} \bar{y}_1 + \mathfrak{f}_{n+1} \bar{y}_0.
\end{aligned}$$

For $\bar{x}_0, \bar{y}_0, \bar{z}_0, \bar{x}_1, \bar{y}_1, \bar{z}_1$, the relation matrix can be given as:

$$B_n = \begin{pmatrix}
\mathfrak{a}_n & -(\mathfrak{b}_n + 1) & -\mathfrak{c}_n & \mathfrak{d}_n & -\mathfrak{e}_n & \mathfrak{f}_n \\
-\mathfrak{e}_n & \mathfrak{f}_n & \mathfrak{a}_n & -(\mathfrak{b}_{n+1}) & -\mathfrak{c}_n & \mathfrak{d}_n \\
-\mathfrak{c}_n & \mathfrak{d}_n & -\mathfrak{e}_n & \mathfrak{f}_n & \mathfrak{a}_n & -(\mathfrak{b}_n + 1) \\
\mathfrak{a}_{n+1} - 1 & -\mathfrak{b}_{n+1} & -\mathfrak{c}_{n+1} & \mathfrak{d}_{n+1} & -\mathfrak{e}_{n+1} & \mathfrak{f}_{n+1} \\
-\mathfrak{e}_{n+1} & \mathfrak{f}_{n+1} & \mathfrak{a}_{n+1} - 1 & -\mathfrak{b}_{n+1} & -\mathfrak{c}_{n+1} & \mathfrak{d}_{n+1} \\
-\mathfrak{c}_n & \mathfrak{d}_{n+1} & -\mathfrak{e}_n & \mathfrak{f}_{n+2} & \mathfrak{a}_{n+1} - 1 & -\mathfrak{b}_{n+1}
\end{pmatrix}$$

Using identities of Lemma 4.1.1.1 we get

$$B_n = \begin{pmatrix} 2\mathfrak{c}_n + n & -(2\mathfrak{c}_{n-1} + n) & -\mathfrak{c}_n & \mathfrak{c}_{n-1} & -\mathfrak{c}_n & \mathfrak{c}_{n-1} \\ -\mathfrak{c}_n & \mathfrak{c}_{n-1} & 2\mathfrak{c}_n + n & -(2\mathfrak{c}_{n-1} + n) & -\mathfrak{c}_n & \mathfrak{c}_{n-1} \\ -\mathfrak{c}_n & \mathfrak{c}_{n-1} & -\mathfrak{c}_n & \mathfrak{c}_{n-1} & 2\mathfrak{c}_n + n & -(2\mathfrak{c}_{n-1} + n) \\ 2\mathfrak{c}_{n+1} + n & -(2\mathfrak{c}_n + n) & -\mathfrak{c}_{n+1} & \mathfrak{c}_n & -\mathfrak{c}_{n+1} & \mathfrak{c}_n \\ -\mathfrak{c}_{n+1} & \mathfrak{c}_n & 2\mathfrak{c}_{n+1} + \mathfrak{c}_n & -(2\mathfrak{c}_n + n) & -\mathfrak{c}_{n+1} & \mathfrak{c}_n \\ -\mathfrak{c}_{n+1} & \mathfrak{c}_n & -\mathfrak{c}_{n+1} & \mathfrak{c}_n & 2\mathfrak{c}_{n+1} + n & -(2\mathfrak{c}_n + n) \end{pmatrix}$$

Adding all columns to the last column,

$$B_n = \begin{pmatrix} 2\mathfrak{c}_n + n & -(2\mathfrak{c}_{n-1} + n) & -\mathfrak{c}_n & \mathfrak{c}_{n-1} & -\mathfrak{c}_n & 0 \\ -\mathfrak{c}_n & \mathfrak{c}_{n-1} & 2\mathfrak{c}_n + n & -(2\mathfrak{c}_{n-1} + n) & -\mathfrak{c}_n & 0 \\ -\mathfrak{c}_n & \mathfrak{c}_{n-1} & -\mathfrak{c}_n & \mathfrak{c}_{n-1} & 2\mathfrak{c}_n + n & 0 \\ 2\mathfrak{c}_{n+1} + n & -(2\mathfrak{c}_n + n) & -\mathfrak{c}_{n+1} & \mathfrak{c}_n & -\mathfrak{c}_{n+1} & 0 \\ -\mathfrak{c}_{n+1} & \mathfrak{c}_n & 2\mathfrak{c}_{n+1} + \mathfrak{c}_n & -(2\mathfrak{c}_n + n) & -\mathfrak{c}_{n+1} & 0 \\ -\mathfrak{c}_{n+1} & \mathfrak{c}_n & -\mathfrak{c}_{n+1} & \mathfrak{c}_n & 2\mathfrak{c}_{n+1} + n & 0 \end{pmatrix}$$

Add row 2 times 2 to row 1, subtract row 2 from row 3, add 2 times row 5 to row 4 and subtract row 6 from row 5.

$$B_n \sim \begin{pmatrix} n & -n & 3\mathfrak{c}_n + 2n & -\mathfrak{c}_{n-1} - 2n & -3\mathfrak{c}_n & 0 \\ -\mathfrak{c}_n & \mathfrak{c}_{n-1} & 2\mathfrak{c}_n + n & -(2\mathfrak{c}_{n-1} + n) & -\mathfrak{c}_n & 0 \\ 0 & 0 & -3\mathfrak{c}_n - n & 3\mathfrak{c}_{n-1} & 3\mathfrak{c}_n + n & 0 \\ n & -n & 3\mathfrak{c}_{n+1} + 2n & -3\mathfrak{c}_n - 2n & -3\mathfrak{c}_n + 1 & 0 \\ 0 & 0 & 3\mathfrak{c}_{n+1} + n & -3\mathfrak{c}_n - n & -3\mathfrak{c}_n - n & 0 \\ -\mathfrak{c}_{n+1} & \mathfrak{c}_n & -\mathfrak{c}_{n+1} & \mathfrak{c}_n & 2\mathfrak{c}_{n+1} + n & 0 \end{pmatrix}$$

Subtract column 1 from column 3, add column 1 to column 4 and add 2 times column 1 to column 5.

$$B_n \sim \begin{pmatrix} n & -n & 3\mathfrak{c}_n + n & -3\mathfrak{c}_{n-1} - n & -3\mathfrak{c}_n + 2n & 0 \\ -\mathfrak{c}_n & \mathfrak{c}_{n-1} & 3\mathfrak{c}_n + n & -(3\mathfrak{c}_{n-1} + n) & -3\mathfrak{c}_n & 0 \\ 0 & 0 & -3\mathfrak{c}_n - n & 3\mathfrak{c}_{n-1} + n & 3\mathfrak{c}_n + n & 0 \\ n & -n & 3\mathfrak{c}_{n+1} + n & -3\mathfrak{c}_n - n & -3\mathfrak{c}_{n+1} + 2n & 0 \\ 0 & 0 & 3\mathfrak{c}_{n+1} + n & -3\mathfrak{c}_n - n & -3\mathfrak{c}_n - n & 0 \\ -\mathfrak{c}_{n+1} & \mathfrak{c}_n & 0 & 0 & n & 0 \end{pmatrix}$$

Multiply 0 to row 1, interchang row 3 and row 6, then interchange row 6 and row 4.

$$B_n \sim \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ -\mathfrak{c}_n & \mathfrak{c}_{n-1} & 3\mathfrak{c}_n + n & -(3\mathfrak{c}_{n-1} + n) & -3\mathfrak{c}_n & 0 \\ -\mathfrak{c}_{n+1} & \mathfrak{c}_n & 0 & 0 & n & 0 \\ 0 & 0 & -3\mathfrak{c}_n - n & 3\mathfrak{c}_{n-1} + n & 3\mathfrak{c}_n + n & 0 \\ 0 & 0 & 3\mathfrak{c}_{n+1} + n & -3\mathfrak{c}_n - n & -3\mathfrak{c}_n - n & 0 \\ n & -n & 3\mathfrak{c}_{n+1} + n & -3\mathfrak{c}_n - n & -3\mathfrak{c}_{n+1} + 2n & 0 \end{pmatrix}$$

Add row 4 to row 2 and subtract row 5 from row 6.

$$B_n \sim \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ -\mathfrak{c}_n & \mathfrak{c}_{n-1} & 0 & 0 & n & 0 \\ -\mathfrak{c}_{n+1} & \mathfrak{c}_n & 0 & 0 & n & 0 \\ 0 & 0 & -3\mathfrak{c}_n - n & 3\mathfrak{c}_{n-1} + n & 3\mathfrak{c}_n + n & 0 \\ 0 & 0 & 3\mathfrak{c}_{n+1} + n & -3\mathfrak{c}_n - n & -3\mathfrak{c}_{n+1} - n & 0 \\ n & -n & 0 & 0 & 3n & 0 \end{pmatrix}$$

Interchange column 4 and column 5, then interchange row 4 and row 6, and finally interchange row 5 and row 6.

$$B_n \sim \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ -\mathfrak{c}_n & \mathfrak{c}_{n-1} & 0 & n & 0 & 0 \\ -\mathfrak{c}_{n+1} & \mathfrak{c}_n & 0 & n & 0 & 0 \\ n & -n & 0 & 3n & 0 & 0 \\ 0 & 0 & -3\mathfrak{c}_n - n & 3\mathfrak{c}_n + n & 3\mathfrak{c}_{n-1} + n & 0 \\ 0 & 0 & 3\mathfrak{c}_{n+1} + n & -3\mathfrak{c}_{n+1} - n & -3\mathfrak{c}_n - n & 0 \end{pmatrix}$$

Add row 4 to row 2 and take - times row 6.

$$B_n \sim \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ -\mathfrak{c}_n & \mathfrak{c}_{n-1} & n & n & 0 & 0 \\ -\mathfrak{c}_{n+1} & \mathfrak{c}_n & n & n & 0 & 0 \\ n & -n & 3n & 3n & 0 & 0 \\ 0 & 0 & 0 & 3\mathfrak{c}_n + n & 3\mathfrak{c}_{n-1} + n & 0 \\ 0 & 0 & 0 & 3\mathfrak{c}_{n+1} + n & 3\mathfrak{c}_n + n & 0 \end{pmatrix}$$

Interchange $R_2 \iff R_4, R_3 \iff R_4$ and take - times column 1.

$$B_n \sim \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ -n & -n & 3n & 3n & 0 & 0 \\ \mathfrak{c}_n & \mathfrak{c}_{n-1} & n & n & 0 & 0 \\ \mathfrak{c}_{n+1} & \mathfrak{c}_n & n & n & 0 & 0 \\ 0 & 0 & 0 & 3\mathfrak{c}_n + n & 3\mathfrak{c}_{n-1} + n & 0 \\ 0 & 0 & 0 & 3\mathfrak{c}_{n+1} + n & 3\mathfrak{c}_n + n & 0 \end{pmatrix}$$

Subtract column 3 from column 4:

$$B_n \sim \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ -n & -n & 3n & 0 & 0 & 0 \\ \mathfrak{c}_n & \mathfrak{c}_{n-1} & n & 0 & 0 & 0 \\ \mathfrak{c}_{n+1} & \mathfrak{c}_n & n & 0 & 0 & 0 \\ 0 & 0 & 0 & 3\mathfrak{c}_n + n & 3\mathfrak{c}_{n-1} + n & 0 \\ 0 & 0 & 0 & 3\mathfrak{c}_{n+1} + n & 3\mathfrak{c}_n + n & 0 \end{pmatrix}$$

Add 3 times column 1 to column 3 and take - times column 2:

$$B_n \sim \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ -n & n & 0 & 0 & 0 & 0 \\ \mathfrak{c}_n & -\mathfrak{c}_{n-1} & 3\mathfrak{c}_n + n & 0 & 0 & 0 \\ \mathfrak{c}_{n+1} & -\mathfrak{c}_n & 3\mathfrak{c}_{n+1} + n & 0 & 0 & 0 \\ 0 & 0 & 0 & 3\mathfrak{c}_n + n & 3\mathfrak{c}_{n-1} + n & 0 \\ 0 & 0 & 0 & 3\mathfrak{c}_{n+1} + n & 3\mathfrak{c}_n + n & 0 \end{pmatrix}$$

Add column 1 to column 2 and take -times row 2:

$$B_n \sim \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ n & 0 & 0 & 0 & 0 & 0 \\ \mathfrak{c}_n & \mathfrak{c}_n - \mathfrak{c}_{n-1} & 3\mathfrak{c}_n + n & 0 & 0 & 0 \\ \mathfrak{c}_{n+1} & \mathfrak{c}_{n+1} - \mathfrak{c}_n & 3\mathfrak{c}_{n+1} + n & 0 & 0 & 0 \\ 0 & 0 & 0 & 3\mathfrak{c}_n + n & 3\mathfrak{c}_{n-1} + n & 0 \\ 0 & 0 & 0 & 3\mathfrak{c}_{n+1} + n & 3\mathfrak{c}_n + n & 0 \end{pmatrix}$$

Using identities of Lemma 4.1.1.1:

$$B_n \sim \begin{pmatrix} n & 0 & 0 & 0 & 0 & 0 \\ \mathfrak{c}_n & \mathfrak{I}_n & \mathfrak{I}_{n+1} & 0 & 0 & 0 \\ \mathfrak{c}_{n+1} & \mathfrak{I}_{n+1} & \mathfrak{I}_{n+2}-1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \mathfrak{I}_{n+1}-\mathfrak{I}_n & \mathfrak{I}_n-\mathfrak{I}_{n-1}+1 & 0 \\ 0 & 0 & 0 & 3\mathfrak{I}_{n+1} & 3\mathfrak{I}_n & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} A_n & 0 \\ 0 & 0 \end{pmatrix}.$$

. □

4.1.2 Two Sequences Related to the Number of Spanning Trees of $S(K_3 \times C_n)$

In this section, the SNF of matrix A_n is computed in the pursuance to adjudge the group structure $S(K_3 \times C_n)$. Consider

$$A_n = \begin{pmatrix} n & 0 & 0 & 0 & 0 \\ \mathfrak{c}_n & \mathfrak{I}_n & \mathfrak{I}_{n+1} & 0 & 0 \\ \mathfrak{c}_{n+1} & \mathfrak{I}_{n+1} & \mathfrak{I}_{n+2}-1 & 0 & 0 \\ 0 & 0 & 0 & \mathfrak{I}_{n+1}-\mathfrak{I}_n & \mathfrak{I}_n-\mathfrak{I}_{n-1}+1 \\ 0 & 0 & 0 & 3\mathfrak{I}_{n+1} & 3\mathfrak{I}_n \end{pmatrix}$$

$$Let\ P_n = \begin{pmatrix} \mathfrak{I}_{n+1}-\mathfrak{I}_n & \mathfrak{I}_n-\mathfrak{I}_{n-1}+1 \\ 3\mathfrak{I}_{n+1} & 3\mathfrak{I}_n \end{pmatrix},\ Q_n = \begin{pmatrix} n & 0 & 0 \\ \mathfrak{c}_n & \mathfrak{I}_n & \mathfrak{I}_{n+1} \\ \mathfrak{c}_{n+1} & \mathfrak{I}_{n+1} & \mathfrak{I}_{n+2}-1 \end{pmatrix}.$$

Then $A_n = \begin{pmatrix} Q_n & 0 \\ 0 & P_n \end{pmatrix} = Q_n \bigoplus P_n$. It is thus, clear that for each n we have $\mathfrak{J}_n\mathfrak{J}_{n+2} + \mathfrak{J}_{n+1} = \mathfrak{J}_{n+1}^2$. Hence, det$P_n$ can be deduced as

$$\begin{aligned} detP_n &= 3\mathfrak{J}_n(\mathfrak{J}_{n+1} - \mathfrak{J}_n) - 3\mathfrak{J}_{n+1}(\mathfrak{J}_n - \mathfrak{J}_{n-1} + 1) \\ &= 3\mathfrak{J}_n\mathfrak{J}_{n+1} - 3\mathfrak{J}_n^2 - 3\mathfrak{J}_{n+1}\mathfrak{J}_n - 3\mathfrak{J}_{n+1}\mathfrak{J}_{n-1} - 3\mathfrak{J}_{n+1} \\ &= -3\mathfrak{J}_n^2 + 3\mathfrak{J}_{n+1}\mathfrak{J}_{n-1} - 3\mathfrak{J}_{n+1}. \end{aligned}$$

As

$$\begin{aligned} &\mathfrak{J}_n\mathfrak{J}_{n+2} + \mathfrak{J}_{n+1} = \mathfrak{J}_{n+1}^2 \\ &\mathfrak{J}_{n-1}\mathfrak{J}_{n+1} + \mathfrak{J}_n = \mathfrak{J}_n^2 \\ &\mathfrak{J}_n^2 + \mathfrak{J}_{n-1}\mathfrak{J}_{n+1} = -\mathfrak{J}_n. \end{aligned}$$

$$\implies detP_n = -3(\mathfrak{J}_n + \mathfrak{J}_{n+1}).$$

Similarly,

$$\begin{aligned} detQ_n &= n(\mathfrak{J}_n(\mathfrak{J}_{n+2} - 1) - \mathfrak{J}_{n+1}^2) \\ &= n(\mathfrak{J}_n\mathfrak{J}_{n+2} - \mathfrak{J}_n - \mathfrak{J}_{n+1}^2) \\ &= n(-\mathfrak{J}_n - \mathfrak{J}_{n+1}) \\ detQ_n &= -n(\mathfrak{J}_n - \mathfrak{J}_{n+1}). \end{aligned}$$

Thus, the number of spanning trees of $K_3 \times C_n$ is $detA_n$=detQ_n·det$P_n = 3n(\mathfrak{J}_n + \mathfrak{J}_{n+1})^2$.

To quantify the SNF of the matrices P_n and Q_n; hence A_n, some divisibility properties about $\mathfrak{J}_n$ are required. For proving Lemmas, let $w_n = \mathfrak{J}_n + \mathfrak{J}_{n+1}$.

Lemma 4.1.2.1. For $n = 2i+1$ odd, $\widehat{w}_n = \mathfrak{J}_{2i+1} + \mathfrak{J}_{2i+2} = \check{\mu}_i^2$, where $\check{\mu}_i$ is given as;

$$\begin{cases} \check{\mu}_i = 5\check{\mu}_{i-1} - \check{\mu}_{i-2}, \\ \check{\mu}_0 = 1, \\ \check{\mu}_1 = 6. \end{cases}$$

For $n = 2i$ even, $\widehat{w}_n = \mathfrak{J}_{2i} + \mathfrak{J}_{2i+1} = 7\check{\vartheta}_i^2$, where $\check{\vartheta}_i$ is given as

$$\begin{cases} \check{\vartheta}_i = 5\check{\vartheta}_{i-1} - \check{\vartheta}_{i-2}, \\ \check{\vartheta}_0 = 0, \\ \check{\vartheta}_1 = 1. \end{cases}$$

Proof. The above results can be proved by induction. Since we know $\widehat{w} = \mathfrak{J}_n + \mathfrak{J}_{n+1} = \check{\mu}_i^2$, so dealing with he odd case first.
For $n = 2i+1, \widehat{w}_n = \mathfrak{J}_i + \mathfrak{J}_{i+1} = \check{\mu}_i^2$ becomes $\widehat{w}_{2i+1} = \mathfrak{J}_{2i+1} + \mathfrak{J}_{2i+2} = \check{\mu}_i^2$. Considering different values for i, we have
For $i = 0,\ \ \widehat{w}_1 = \mathfrak{J}_1 + \mathfrak{J}_2 \implies \mathfrak{c}_1 - \mathfrak{c}_0 + c - \mathfrak{c}_2 = 1 = \widehat{w}_1$
For $i = 1,\ \ \widehat{w}_3 = \mathfrak{J}_3 + \mathfrak{J}_4 \implies \mathfrak{c}_3 - \mathfrak{c}_2 + \mathfrak{c}_4 - \mathfrak{c}_3 = 36 = \widehat{w}_3$
For $i = 2,\ \ \widehat{w}_5 = \mathfrak{J}_5 + \mathfrak{J}_6 \implies \mathfrak{c}_5 - \mathfrak{c}_4 + \mathfrak{c}_6 - \mathfrak{c}_5 = 841 = \widehat{w}_5$
For $i = 3,\ \ \widehat{w}_7 = \mathfrak{J}_7 + \mathfrak{J}_8 \implies \mathfrak{c}_7 - \mathfrak{c}_6 + \mathfrak{c}_8 - \mathfrak{c}_7 = 1932 = \widehat{w}_7$
Clearly,
$\widehat{w}_1 = (1)^2 = \check{\mu}_0^2 \implies \check{\mu}_0 = 1$
$\widehat{w}_3 = (6)^2 = \check{\mu}_1^2 \implies \check{\mu}_1 = 6$
$\widehat{w}_5 = (29)^2 = \check{\mu}_2^2 \implies \check{\mu}_2 = 29$
$\widehat{w}_7 = (139)^2 = \check{\mu}_3^2 \implies \check{\mu}_3 = 139$
As a result, generalized form is deduced as $\widehat{w}_n = \check{\mu}_i^2$. Also, $\check{\mu}_2 = 5\check{\mu}_1 - \check{\mu}_0$,
$\check{\mu}_3 = 5\check{\mu}_2 - \check{\mu}_1 \implies \check{\mu}_i = 5\check{\mu}_{i-1} - \check{\mu}_{i-2}$.

Thus,

$$\begin{cases} \check{\mu}_i = 5\check{\mu}_{i-1} - \check{\mu}_{i-2}, \\ \check{\mu}_0 = 1, \\ \check{\mu}_1 = 6. \end{cases}$$

Now consider the even case:

For $n = 2i, \widehat{w}_n = \mathfrak{J}_i + \mathfrak{J}_{i+1} = 7\check{\vartheta}_i^2$ becomes $\widehat{w}_{2i} = \mathfrak{J}_{2i} + \mathfrak{J}_{2i+1} = 7\check{\vartheta}_i^2$.

For $i = 0,\ \widehat{w}_0 = \mathfrak{J}_0 + \mathfrak{J}_1 \implies \mathfrak{c}_0 - \mathfrak{c}_{-1} + \mathfrak{c}_1 - \mathfrak{c}_0 = 0 = \widehat{w}_0$

For $i = 1,\ \widehat{w}_2 = \mathfrak{J}_2 + \mathfrak{J}_3 \implies \mathfrak{c}_2 - \mathfrak{c}_1 + \mathfrak{c}_3 - \mathfrak{c}_2 = 7 = \widehat{w}_2$

For $i = 2,\ \widehat{w}_4 = \mathfrak{J}_4 + \mathfrak{J}_5 \implies \mathfrak{c}_4 - \mathfrak{c}_3 + \mathfrak{c}_5 - \mathfrak{c}_4 = 175 = \widehat{w}_4$

For $i = 3,\ \widehat{w}_6 = \mathfrak{J}_6 + \mathfrak{J}_7 \implies \mathfrak{c}_6 - \mathfrak{c}_5 + \mathfrak{c}_7 - \mathfrak{c}_6 = 4032 = \widehat{w}_6$

Clearly,

$\widehat{w}_0 = 0 = 7(0)^2 = \check{\vartheta}_0^2 = 7(\check{\vartheta}_0)^2$

$\widehat{w}_2 = 7 = 7(1)^2 = \check{\vartheta}_1^2 = 7(\check{\vartheta}_1)^2$

$\widehat{w}_4 = 175 = 7(5)^2 = \check{\vartheta}_2^2 = 7(\check{\vartheta}_2)^2$

$\widehat{w}_6 = 4032 = 7(24)^2 = \check{\vartheta}_3^2 = 7(\check{\vartheta}_3)^2$

As a result, generalized form is deduced as $\widehat{w}_n = 7\check{\vartheta}_i^2$. It can be further noted that, $\check{\vartheta}_4 = 5\check{\vartheta}_3 - \check{\vartheta}_2, \check{\vartheta}_3 = 5\check{\vartheta}_2 - \check{\mu}_1 \implies \check{\vartheta}_i = 5\check{\vartheta}_{i-1} - \check{\vartheta}_{i-2}$. It can be generalized as $\check{\mu}_i = \check{\vartheta}_i + \check{\vartheta}_{i+1}$.

Thus in general,

$$\begin{cases} \check{\vartheta}_i = 5\check{\vartheta}_{i-1} - \check{\vartheta}_{i-2}, \\ \check{\vartheta}_0 = 0, \\ \check{\vartheta}_1 = 1. \end{cases}$$

Now,

$$\begin{aligned}
\breve{\mu}_{i-1}^2\breve{\mu}_i^2 &= \widehat{w}_{2(i-1)+1}\widehat{w}_{2i+1} \\
&= (\mathfrak{J}_{2(i-1)+1} + \mathfrak{J}_{2(i-1)+2})(\mathfrak{J}_{2i+1} + \mathfrak{J}_{2i+2}) \\
&= (\mathfrak{J}_{2i-1} + \mathfrak{J}_{2i})(\mathfrak{J}_{2i+1} + \mathfrak{J}_{2i+2}) \\
&= \mathfrak{J}_{2i-1}\mathfrak{J}_{2i+1} + \mathfrak{J}_{2i}\mathfrak{J}_{2i+1} + \mathfrak{J}_{2i-1}\mathfrak{J}_{2i+2} + \mathfrak{J}_{2i}\mathfrak{J}_{2i+2}.
\end{aligned}$$

Since,

$$\begin{aligned}
\mathfrak{J}_n\mathfrak{J}_{n+2} + \mathfrak{J}_{n+1} &= \mathfrak{J}_{n+1}^2 \\
\mathfrak{J}_{2i-1}\mathfrak{J}_{2i+1} &= \mathfrak{J}_{2i}^2 - \mathfrak{J}_{2i} \\
\mathfrak{J}_{2i}\mathfrak{J}_{2i+2} &= \mathfrak{J}_{2i+1}^2 - \mathfrak{J}_{2i+1} \\
\mathfrak{J}_m &= 5\mathfrak{J}_{m-1} - \mathfrak{J}_{m-2} + 1 \\
\mathfrak{J}_{2i+1} &= 5\mathfrak{J}_{2i} - \mathfrak{J}_{2i-1} + 1 \\
\mathfrak{J}_{2i+2} &= 5\mathfrak{J}_{2i+1} - \mathfrak{J}_{2i} + 1.
\end{aligned}$$

Using the above values, we get:

$$\begin{aligned}
\breve{\mu}_{i-1}^2\breve{\mu}_i^2 &= \mathfrak{J}_{2i}\mathfrak{J}_{2i+1} + \mathfrak{J}_{2i-1}\mathfrak{J}_{2i+2} + \mathfrak{J}_{2i}^2 - \mathfrak{J}_{2i} + \mathfrak{J}_{2i+1}^2 - \mathfrak{J}_{2i+1} \\
&= \mathfrak{J}_{2i}\mathfrak{J}_{2i+1} + \mathfrak{J}_{2i-1}\mathfrak{J}_{2i+2} + \mathfrak{J}_{2i}^2 + \mathfrak{J}_{2i+1}^2 - (\mathfrak{J}_{2i+1} + \mathfrak{J}_{2i}) \\
&= \mathfrak{J}_{2i}\mathfrak{J}_{2i+1} + \mathfrak{J}_{2i-1}(5\mathfrak{J}_{2i+1} - \mathfrak{J}_{2i} + 1) + \mathfrak{J}_{2i}^2 + \mathfrak{J}_{2i+1}^2 - (\mathfrak{J}_{2i+1} + \mathfrak{J}_{2i}) \\
&= \mathfrak{J}_{2i}\mathfrak{J}_{2i+1} + 5\mathfrak{J}_{2i-1}\mathfrak{J}_{2i+1} - \mathfrak{J}_{2i-1}\mathfrak{J}_{2i} + \mathfrak{J}_{2i-1} + \mathfrak{J}_{2i}^2 + \mathfrak{J}_{2i+1}^2 - (\mathfrak{J}_{2i+1} + \mathfrak{J}_{2i}) \\
&= \mathfrak{J}_{2i}\mathfrak{J}_{2i+1} + 5\mathfrak{J}_{2i-1}\mathfrak{J}_{2i+1} - \mathfrak{J}_{2i-1}\mathfrak{J}_{2i} + \mathfrak{J}_{2i}^2 + \mathfrak{J}_{2i+1}^2 - (\mathfrak{J}_{2i+1} + \mathfrak{J}_{2i} - \mathfrak{J}_{2i-1}) \\
&= \mathfrak{J}_{2i}\mathfrak{J}_{2i+1} - (\mathfrak{J}_{2i+1} + \mathfrak{J}_{2i} - \mathfrak{J}_{2i-1}) + \mathfrak{J}_{2i+1}\mathfrak{J}_{2i} + \mathfrak{J}_{2i}^2 + \mathfrak{J}_{2i+1}^2 + 5(\mathfrak{J}_{2i}^2 - \mathfrak{J}_{2i}) \\
&= \mathfrak{J}_{2i}\mathfrak{J}_{2i+1} - (\mathfrak{J}_{2i+1} + \mathfrak{J}_{2i} - \mathfrak{J}_{2i-1}) + \mathfrak{J}_{2i+1}\mathfrak{J}_{2i} + 6\mathfrak{J}_{2i}^2 + \mathfrak{J}_{2i+1}^2 - 5\mathfrak{J}_{2i} \\
&= \mathfrak{J}_{2i}\mathfrak{J}_{2i+1} - (\mathfrak{J}_{2i+1} - \mathfrak{J}_{2i-1}) + \mathfrak{J}_{2i+1}\mathfrak{J}_{2i} + 6\mathfrak{J}_{2i}^2 + \mathfrak{J}_{2i+1}^2 - 6\mathfrak{J}_{2i}
\end{aligned}$$

As $\mathfrak{J}_n = 5\mathfrak{J}_{n-1} - \mathfrak{J}_{n-2} + 1 \implies \mathfrak{J}_{2i+1} = 5\mathfrak{J}_{2i} - \mathfrak{J}_{2i-1} + 1 \implies \mathfrak{J}_{2i-1} = 5\mathfrak{J}_{2i} - \mathfrak{J}_{2i+1} + 1.$ Putting the value in above equation, we get:

$$\begin{aligned}
\breve{\mu}_{i-1}^2\breve{\mu}_i^2 &= \mathfrak{J}_{2i}\mathfrak{J}_{2i+1} - \mathfrak{J}_{2i-1}\mathfrak{J}_{2i} - \mathfrak{J}_{2i+1} + 5\mathfrak{J}_{2i} - \mathfrak{J}_{2i+1} + 1 + 6\mathfrak{J}_{2i}^2 + \mathfrak{J}_{2i+1}^2 - 6\mathfrak{J}_{2i}\\
&= \mathfrak{J}_{2i}\mathfrak{J}_{2i+1} - \mathfrak{J}_{2i}(5\mathfrak{J}_{2i} - \mathfrak{J}_{2i+1} + 1) - \mathfrak{J}_{2i+1} + 5\mathfrak{J}_{2i} - \mathfrak{J}_{2i+1} + 1 + 6\mathfrak{J}_{2i}^2 + \mathfrak{J}_{2i+1}^2 - 6\mathfrak{J}_{2i}\\
&= \mathfrak{J}_{2i}\mathfrak{J}_{2i+1} - 5\mathfrak{J}_{2i}^2 - \mathfrak{J}_{2i}\mathfrak{J}_{2i+1} - \mathfrak{J}_{2i} + 5\mathfrak{J}_{2i} - \mathfrak{J}_{2i+1} + 1 + 6\mathfrak{J}_{2i}^2 + \mathfrak{J}_{2i+1}^2 - 6\mathfrak{J}_{2i}\\
&= 2\mathfrak{J}_{2i}\mathfrak{J}_{2i+1} + \mathfrak{J}_{2i}^2 + \mathfrak{J}_{2i+1}^2 - 2\mathfrak{J}_{2i} - 2\mathfrak{J}_{2i+1} + 1\\
&= (\mathfrak{J}_{2i} + \mathfrak{J}_{2i+1})^2 - 2(\mathfrak{J}_{2i} - \mathfrak{J}_{2i+1}) + 1\\
&= (\widehat{w}_{2i})^2 - 2\widehat{w}_{2i} + 1\\
\breve{\mu}_{i-1}^2\breve{\mu}_i^2 &= (\widehat{w}_{2i} - 1)^2\\
\breve{\mu}_{i-1}\breve{\mu}_i &= \widehat{w}_{2i} - 1.
\end{aligned}$$

Similarly, it can be easily shown that $7\check{\vartheta}_{2i-1}\check{\vartheta}_i = \widehat{w}_{2i+1} - 1.$ □

Proposition 4.1.1. For each $i, n \geq 1$ we have
$\check{\vartheta}_{i+n} = \check{\vartheta}_{i+1}\check{\vartheta}_n - \check{\vartheta}_i\check{\vartheta}_{n-1}$ and $p_{i+n} = \check{\vartheta}_{i+1}\breve{\mu}_n - \check{\vartheta}_i\breve{\mu}_{n-1}$.

Proof. From above proposition, the following identities are deduced that will be used in Lemma 4.1.2.4

$$\begin{aligned}
\check{\vartheta}_{3n} &= \check{\vartheta}_n(3\check{\vartheta}_{n-1}^2 - 15\check{\vartheta}_n\check{\vartheta}_{n-1} + 24\check{\vartheta}_n^2),\\
\check{\vartheta}_{3n-1} &= \check{\vartheta}_{n-1}^3 - 3\check{\vartheta}_n^2\check{\vartheta}_{n-1} + 5\check{\vartheta}_n^3,\\
\check{\vartheta}_{3n+1} &= 115\check{\vartheta}_n - \check{\vartheta}_{n-1}^3 + 15\check{\vartheta}_n\check{\vartheta}_{n-1}^2 - 92\check{\vartheta}_n^2\check{\vartheta}_{n-1}.
\end{aligned}$$

Moreover, if we extend the sequence $\check{\vartheta}_n$ by $Q_{-n} = -\check{\vartheta}_n$ then we have □

Lemma 4.1.2.2. For all integers i, n and j, we have

$$\check{\vartheta}_{in+j} = \sum_{k=0}^{1}(-1)^{i-k}\binom{i}{j}\check{\vartheta}_n^k\check{\vartheta}_{n-1}^{i-k}\check{\vartheta}_{j+k}.$$

proof 4.1.1. Let γ be the root of $y^2 - 5y + 1$. Then $\gamma^n = \gamma\check{\vartheta}_n - \check{\vartheta}_{n-1}$ is verified by induction on n.
For $n = 1$,

$$\gamma = \gamma\check{\vartheta}_1 - \check{\vartheta}_0$$
$$\gamma = \gamma.$$

For $n = 2$,

$$\gamma^2 = \gamma\check{\vartheta}_2 - \check{\vartheta}_1$$
$$\gamma^2 = 5\gamma - 1$$
$$\gamma^2 - 5\gamma + 1.$$

For $n = 3$,

$$\gamma^3 = \gamma\check{\vartheta}_3 - \check{\vartheta}_2$$
$$\gamma^3 = 24\gamma - 5$$
$$\gamma.\gamma^2 = 24\gamma - 5$$
$$\gamma(5\gamma - 1) = 24\gamma - 5$$
$$5\gamma^2 - \gamma = 24\gamma - 5$$
$$\gamma^2 - 5\gamma + 1 = 0.$$

Suppose the equality is true for $n = k$:

$$\gamma^k = \gamma\check{\vartheta}_k - \check{\vartheta}_{k-1}. \tag{4.10}$$

For $n = k + 1$,

$$\gamma^{k+1} = \gamma\check{\vartheta}_{k+1} - \check{\vartheta}_k$$
$$\gamma^k.\gamma = \gamma\check{\vartheta}_{k+1} - \check{\vartheta}_k$$
$$\gamma(\gamma\check{\vartheta}_k - \check{\vartheta}_{k-1}) = \gamma\check{\vartheta}_{k+1} - \check{\vartheta}_k$$
$$\gamma^2\check{\vartheta}_k - \gamma\check{\vartheta}_{k-1} = \gamma\check{\vartheta}_{k+1} - \check{\vartheta}_k.$$

Multiplying 4.10 by γ, we get

$$\gamma.\gamma^k = \gamma^2\check{\vartheta}_k - \gamma\check{\vartheta}_{k-1}$$
$$\gamma^{k+1} = \gamma^2\check{\vartheta}_k - \gamma\check{\vartheta}_{k-1}$$
$$\Longrightarrow \gamma^{k+1} = \gamma\check{\vartheta}_{k+1} - \check{\vartheta}_k.$$

Hence $\gamma^n = \gamma\check{\vartheta}_n - \check{\vartheta}_{n-1}$ holds by induction. Now consider:

$$\begin{aligned}
\gamma^{in+j} &= (\gamma^n)^i\gamma^j \\
&= \gamma^j(\gamma\check{\vartheta}_n - \check{\vartheta}_{n-1})^i && \text{Binomial expansion} \\
&= (\gamma\check{\vartheta}_j - \check{\vartheta}_{j-1})\left[\sum_{k=0}^{1}\binom{i}{k}\gamma^k\check{\vartheta}_n^k(-1)^{i-k}\check{\vartheta}_{n-1}^{i-k}\right] \\
&= (\gamma\check{\vartheta}_j - \check{\vartheta}_{j-1})\left[\sum_{k=0}^{1}\binom{i}{k}(\gamma\check{\vartheta}_k - \check{\vartheta}_{k-1})\check{\vartheta}_n^k(-1)^{i-k}\check{\vartheta}_{n-1}^{i-k}\right] \\
&= \sum_{k=0}^{1}(-1)^{i-k}\binom{i}{k}\check{\vartheta}_n^k\check{\vartheta}_{n-1}^{i-k}(\gamma\check{\vartheta}_{j+k} - \check{\vartheta}_{j+k-1}),
\end{aligned}$$

$\gamma^{in+j} = \gamma\check{\vartheta}_{in+j} - \check{\vartheta}_{in+j-1}$ and the irrationality ofγ, Lemma 4.1.2.2 holds.

Lemma 4.1.2.3. For $a \mid b$, $\check{\vartheta}_a$ divides $\check{\vartheta}_b$. Moreover, if $a \mid b$ implies $\det(A_a)$ divides $\det(A_b)$..

Proof. Let $b = ac$, then from Lemma 4.1.2.2

$$\check{\vartheta}_b = \check{\vartheta}_{ac} = \sum_{i=0}(-1)^{c-i}\binom{c}{i}\check{\vartheta}_a^i\check{\vartheta}_{a-1}^{c-i}\check{\vartheta}_i.$$

i.e. if $a \mid b$ then $\check{\vartheta}_a \mid \check{\vartheta}_b$. First consider if we put $n = i+1$, then Proposition 4.1.1 becomes $\check{\vartheta}_{2i+1} = \check{\vartheta}_{i+1}^2 - \check{\vartheta}_i^2$.

Now suppose if $2a+1$ divides $2b+1$, then one can show that $\check{\mu}_a$ divides $\check{\mu}_b$. By definition of divisibility, if $2a+1$ divides $2b+1$ then $(2b+1) = (2a+1)(2c+1)$. Thus $b = 2ac + a + c$.

The statement $\check{\mu}_a \mid \check{\mu}_b = \mu_{2ac+a+c}$ can be proved by using induction on c. It is true if $c = 0$, since then $b = a \implies \check{\mu}_a = \check{\mu}_b$ and $\check{\mu}_a \mid \check{\mu}_a$.

For $c = c+1$, $b = 2a(c+1) + a + (c+1)$, then $\check{\mu}_{2a(c+1)+a+(c+)} = \check{\mu}_{(2a+1)+(2ac+a+c)}$. Using property $p_{i+n} = \check{\vartheta}_{i+1}\check{\mu}_n - \check{\vartheta}_i\check{\mu}_{n-1}$, we get;

$$\check{\mu}_{(2a+1)+(2ac+a+c)} = \check{\vartheta}_{2a+2}\check{\mu}_{2ac+a+c} - \check{\vartheta}_{2a+1}\check{\mu}_{2ac+a+c-1}.$$

The first term $\check{\vartheta}_{2a+2}\check{\mu}_{2ac+a+c}$ is multiple of $\check{\mu}_a$; since by inductive hypothesis $\check{\mu}_a \mid \check{\mu}_{2ac+a+c}$, so it is divisible by $\check{\mu}_a$. Also by Proposition 4.1.1, $\check{\vartheta}_{2a+1} = \check{\vartheta}_{a+1}^2 - \check{\vartheta}_a^2 = (\check{\vartheta}_{2a+1} - \check{\vartheta}_a)(\check{\vartheta}_{2a+1} + \check{\vartheta}_a)$ which gives $\check{\vartheta}_{2a+1} = \check{\mu}_a(\check{\vartheta}_{a+1} - \check{\vartheta}_a)$, (since as $\check{\mu}_a = \check{\vartheta}_{a+1} + \check{\vartheta}_a$) that is also a multiple of $\check{\mu}_a$. Hence second term is also divisible by $\check{\mu}_a$.

Now if $2a+1$ divides $2b$, then $\check{\mu}_a$ divides $\check{\vartheta}_b$. Let $2b = (2a+1)2c$ by definition of divisibility, then $b = (2a+1)c$. We need to show that $\check{\mu}_a \mid \check{\vartheta}_b = \check{\vartheta}_{(2a+1)c}$. Previously, $\check{\mu}_a$ divides $\check{\vartheta}_{2a+1}$ and $\check{\vartheta}_{2a+1}$ divides $\check{\vartheta}_{(2a+1)c}$ then by transitivity of divisibilty $\check{\mu}_a$ divides $\check{\vartheta}_{(2a+1)c}$. □

Lemma 4.1.2.4. For $\mathcal{Q} \geq 1$, we have $\check{\vartheta}_{3^{\mathcal{Q}}} = 2.3^{\mathcal{Q}} \mod 3^{\mathcal{Q}+1}$, $\check{\vartheta}_{3^{\mathcal{Q}}-1} = 2.3^{\mathcal{Q}} - 1 \mod 3^{\mathcal{Q}+1}$, $\check{\vartheta}_{3^{\mathcal{Q}}+1} = 2.3^{\mathcal{Q}} + 1 \mod 3^{\mathcal{Q}+1}$, $\check{\vartheta}_{2\cdot 3^{\mathcal{Q}}} \equiv 3^{\mathcal{Q}} \mod 3^{\mathcal{Q}+1}$.

Proof. The identities $\check{\vartheta}_{3^{\mathcal{Q}}} = 2.3^{\mathcal{Q}} \mod 3^{\mathcal{Q}+1}$, $\check{\vartheta}_{3^{\mathcal{Q}}-1} = 2.3^{\mathcal{Q}} - 1 \mod 3^{3\mathcal{Q}+1}$ can be proved by induction on $\mathcal{Q}$. Thus for $\mathcal{Q} = 1$ we have, $\check{\vartheta}_3 = 5\check{\vartheta}_2 - \check{\vartheta}_1 = 24 \equiv 6 \mod 9$

that can be written as $\check{\vartheta}_3 = 2.3 \mod 3^{1+1}$ and $\check{\vartheta}_2 = 5\check{\vartheta}_1 - \check{\vartheta}_0 = 5 \equiv 5 \mod 9$, that can be written as $\check{\vartheta}_2 = 2.3^1 - 1 \mod 3^{1+1}$, which is true. Let the result also holds for $\mathcal{Q} \geq 1$ that is $\check{\vartheta}_{3\mathcal{Q}} = 2.3^{\mathcal{Q}} + x3^{\mathcal{Q}+1}$, $\check{\vartheta}_{3\mathcal{Q}-1} = 2.3^{\mathcal{Q}} - 1 + y3^{3\mathcal{Q}+1}$ for some integers x and y.

$$\begin{aligned}
\check{\vartheta}_{3\mathcal{Q}}^3 &= (2.3^{\mathcal{Q}} + x.3^{\mathcal{Q}+1})^3 \\
&= (2.3^{\mathcal{Q}})^3 + 3(2.3^{\mathcal{Q}})^2(x.3^{\mathcal{Q}+1}) + 3(2.3^{\mathcal{Q}})(x.3^{\mathcal{Q}+1})^2 + (x.3^{\mathcal{Q}+1})^3 \\
&= (2^3.3^{2\mathcal{Q}-2})3^{\mathcal{Q}+2} + (x.2^2.3^{2\mathcal{Q}})3^{\mathcal{Q}+2} + (x^2.2.3^{2\mathcal{Q}+1})3^{\mathcal{Q}+2} + (x^3.3^{2\mathcal{Q}+1})3^{\mathcal{Q}+2} \\
&\equiv 0 \mod 3^{\mathcal{Q}+2}.
\end{aligned}$$

$$\begin{aligned}
3\check{\vartheta}_{3\mathcal{Q}}\check{\vartheta}_{3\mathcal{Q}-1}^2 &= 3(2.3^{\mathcal{Q}} + x.3^{\mathcal{Q}+1})(2.3^{\mathcal{Q}} - 1 + y.3^{\mathcal{Q}+1})^2 \\
&= 3(2 \cdot 3^{\mathcal{Q}} + x.3^{\mathcal{Q}+1})[(2 \cdot 3^{\mathcal{Q}})^2 + (-1)^2 + (y \cdot 3^{\mathcal{Q}+1})^2 + 2(2 \cdot 3^{\mathcal{Q}}(-1) + y \cdot 3^{\mathcal{Q}+1}(-1) + (y \cdot 3^{\mathcal{Q}+1}) \\
&\cdot 2 \cdot 3^{\mathcal{Q}}] \\
&= 3[(2 \cdot 3u)^3 + x \cdot 3^{\mathcal{Q}+1}(2 \cdot 3^{\mathcal{Q}})^2 + 2 \cdot 3^{\mathcal{Q}} + x \cdot 3^{\mathcal{Q}+1} + 2 \cdot 3^{\mathcal{Q}}(y \cdot 3^{\mathcal{Q}+1})^2 + x \cdot y^2(3^{\mathcal{Q}+1})^3 + 2^2 \cdot 3^{\mathcal{Q}} \\
&(-2 \cdot 3^{\mathcal{Q}} - y \cdot 3^{\mathcal{Q}+1} + (y \cdot 3^{\mathcal{Q}+1}) \cdot 2 \cdot 3^{\mathcal{Q}}) + x \cdot 2 \cdot 3^{\mathcal{Q}+1}(-2 \cdot 3^{\mathcal{Q}} - y \cdot 3^{\mathcal{Q}+1} + (y \cdot 3^{\mathcal{Q}+1}) \cdot 2 \cdot 3^{\mathcal{Q}})] \\
&= (2 \cdot 3^{2\mathcal{Q}-1})3^{\mathcal{Q}+2} + x \cdot (2 \cdot 3^{\mathcal{Q}})^2 3^{\mathcal{Q}+2} + 2 \cdot 3^{\mathcal{Q}+1} + x \cdot 3^{\mathcal{Q}+2} + (y.3^{2\mathcal{Q}+1})3^{\mathcal{Q}+2} + (x \cdot y^2 \cdot 3^{2u+2})3^{\mathcal{Q}+2} \\
&+ 2^2(-2 \cdot 3^{\mathcal{Q}-1} - y \cdot 3^{\mathcal{Q}} + y.2.3^{2\mathcal{Q}})3^{\mathcal{Q}+2} + x \cdot 2(-2 \cdot 3^{\mathcal{Q}} - y \cdot 3^{\mathcal{Q}+1} + y \cdot 2 \cdot 3^{2\mathcal{Q}+1})3^{\mathcal{Q}+2} \\
&\equiv 2 \cdot 3^{\mathcal{Q}+1} \mod 3^{\mathcal{Q}+2}.
\end{aligned}$$

$$
\begin{aligned}
\check{\vartheta}_{3^{\mathcal{Q}}}^2 \check{\vartheta}_{3^{\mathcal{Q}}-1} &= 3(2 \cdot 3^{\mathcal{Q}} + x \cdot 3^{\mathcal{Q}+1})^2 (2 \cdot 3^{\mathcal{Q}} - 1 + y \cdot 3^{\mathcal{Q}+1}) \\
&= ((2 \cdot 3^{\mathcal{Q}})^2 + 2(2 \cdot 3^{\mathcal{Q}})(x \cdot 3^{\mathcal{Q}+1}) + (x \cdot 3^{\mathcal{Q}+1})^2)(2 \cdot 3^{\mathcal{Q}} - 1 + y \cdot 3^{\mathcal{Q}+1}) \\
&= (2 \cdot 3^{\mathcal{Q}})^3 - (2 \cdot 3^{\mathcal{Q}})^2 + (2 \cdot 3^{\mathcal{Q}})^2 (y \cdot 3^{\mathcal{Q}+1}) + 2(2 \cdot 3^{\mathcal{Q}})^2 (x \cdot 3^{\mathcal{Q}+1}) - 2(2 \cdot 3^{\mathcal{Q}})(x \cdot 3^{\mathcal{Q}+1}) + x \cdot y \cdot 2(2 \\
&+ (2 \cdot 3^{\mathcal{Q}})(x \cdot 3^{\mathcal{Q}+1})^2 - (x \cdot 3^{\mathcal{Q}+1})^2 + x^2 \cdot y (3^{\mathcal{Q}+1})^3 \\
&= (2^3 \cdot 3^{2\mathcal{Q}-2}) 3^{\mathcal{Q}+2} - (2^2 \cdot 3^{\mathcal{Q}-2}) 3^{\mathcal{Q}+2} + (y \cdot 2^2 \cdot 3^{2\mathcal{Q}-1}) 3^{\mathcal{Q}+2} + (x \cdot 2^3 \cdot 3^{2\mathcal{Q}-1}) 3^{\mathcal{Q}+2} - (a \cdot 2^2 \cdot 3^{\mathcal{Q}-1}) 3 \\
&+ (x \cdot y \cdot 2^2 \cdot 3^{2\mathcal{Q}}) 3^{\mathcal{Q}+2} + (\cdot \cdot 2 \cdot 3^{2\mathcal{Q}}) 3^{\mathcal{Q}+2} - (x^2 \cdot 3^{\mathcal{Q}}) 3^{\mathcal{Q}+2} + (x^2 \cdot y \cdot 3^{2\mathcal{Q}+1}) 3^{\mathcal{Q}+2} \\
&\equiv 0 \mod 3^{\mathcal{Q}+2}.
\end{aligned}
$$

The identities from proposition,

$$
\begin{aligned}
\check{\vartheta}_{3^{\mathcal{Q}+1}} &= \check{\vartheta}_{3 \cdot 3^{\mathcal{Q}}} = \check{\vartheta}_{3^{\mathcal{Q}}} (3\check{\vartheta}_{3^{\mathcal{Q}}-1}^2 - 15 \check{\vartheta}_{3^{\mathcal{Q}}} \check{\vartheta}_{3^{\mathcal{Q}}-1} + 24 \check{\vartheta}_{3^{\mathcal{Q}}}^2), \\
\check{\vartheta}_{3^{\mathcal{Q}+1}-1} &= \check{\vartheta}_{3^{\mathcal{Q}} . 3^{\mathcal{Q}}-1} = \check{\vartheta}_{3^{\mathcal{Q}}-1}^3 - 3 \check{\vartheta}_{3^{\mathcal{Q}}}^2 \check{\vartheta}_{3^{\mathcal{Q}}-1} + 5 \check{\vartheta}_{3^{\mathcal{Q}}}^3.
\end{aligned}
$$

can be used to prove the remaining identities. Thus we have,

$$
\begin{aligned}
\check{\vartheta}_{3^{\mathcal{Q}+1}} = \check{\vartheta}_{3.3^{\mathcal{Q}}} &= \check{\vartheta}_{3^{\mathcal{Q}}} 3 \check{\vartheta}_{3^{\mathcal{Q}}-1}^2 - 15 \check{\vartheta}_{3^{\mathcal{Q}}}^2 \check{\vartheta}_{3^{\mathcal{Q}}-1} + 24 \check{\vartheta}_{3^{\mathcal{Q}}}^3 \\
&= \check{\vartheta}_{3^{\mathcal{Q}}} 3 \check{\vartheta}_{3^{\mathcal{Q}}-1}^2 \mod 3^{\mathcal{Q}+2} - 15 \check{\vartheta}_{3^{\mathcal{Q}}}^2 \check{\vartheta}_{3^{\mathcal{Q}}-1} \mod 3^{\mathcal{Q}+2} + 24 \check{\vartheta}_{3^{\mathcal{Q}}}^3 \mod 3^{\mathcal{Q}+2} \\
&\equiv \check{\vartheta}_{3^{\mathcal{Q}}} 3 \check{\vartheta}_{3^{\mathcal{Q}}-1}^2 \mod 3^{\mathcal{Q}+2} - 0 \mod 3^{\mathcal{Q}+2} + 0 \mod 3^{\mathcal{Q}+2} \\
&\equiv \check{\vartheta}_{3^{\mathcal{Q}}} 3 \check{\vartheta}_{3^{\mathcal{Q}}-1}^2 \mod 3^{\mathcal{Q}+2} \\
&\equiv 2 \cdot 3^{\mathcal{Q}+1} \mod 3^{\mathcal{Q}+2}.
\end{aligned}
$$

Also,

$$\begin{aligned}
\check{\vartheta}_{3^{\mathcal{Q}+1}-1} &= \check{\vartheta}^3_{3^{\mathcal{Q}}-1} - 3\check{\vartheta}^2_{3^{\mathcal{Q}}}\check{\vartheta}_{3^{\mathcal{Q}}-1} + 5\check{\vartheta}^3_{3^{\mathcal{Q}}} \\
&= \check{\vartheta}^3_{3^{\mathcal{Q}}-1} \mod 3^{\mathcal{Q}+2} - 3\check{\vartheta}^2_{3^{\mathcal{Q}}}\check{\vartheta}_{3^{\mathcal{Q}}-1} \mod 3^{\mathcal{Q}+2} + 5\check{\vartheta}^3_{3^{\mathcal{Q}}} \mod 3^{\mathcal{Q}+2} \\
&\equiv \check{\vartheta}^3_{3^{\mathcal{Q}}-1} \mod 3^{\mathcal{Q}+2} - 0 \mod 3^{\mathcal{Q}+2} + 0 \mod 3^{\mathcal{Q}+2} \\
&\equiv (2\cdot 3^{\mathcal{Q}} - 1 + y\cdot 3^{\mathcal{Q}+1})^3 \mod 3^{\mathcal{Q}+1} \\
&\equiv [(2\cdot 3^{\mathcal{Q}})^3 + 3(2\cdot 3^{\mathcal{Q}})^2(-1) + 3(2\cdot 3^{\mathcal{Q}})(-1)^2 + 6(2\cdot 3^{\mathcal{Q}})(-1)(y\cdot 3^{\mathcal{Q}+1}) + 3(2\cdot 3^{\mathcal{Q}})^2(y\cdot 3^{\mathcal{Q}+1}) + (-1 \\
&+ 3(-1)^2(y\cdot 3^{\mathcal{Q}+1}) + 3(-1)(y\cdot 3^{\mathcal{Q}+1})^2 + (y\cdot 3^{\mathcal{Q}+1})^3] \mod 3^{\mathcal{Q}+2} \\
&\equiv 3(2\cdot 3^{\mathcal{Q}}) - 1 \mod 3^{\mathcal{Q}+2} \\
&\equiv 2\cdot 3^{\mathcal{Q}+1} - 1 \mod 3^{\mathcal{Q}+2}.
\end{aligned}$$

Similarly,

$$\begin{aligned}
\check{\vartheta}_{3^{\mathcal{Q}+1}+1} = \check{\vartheta}_{3.3^{\mathcal{Q}}+1} &= 115\check{\vartheta}^3_{3^{\mathcal{Q}}} - \check{\vartheta}^3_{3^{\mathcal{Q}}-1} + 15\check{\vartheta}_{3^{\mathcal{Q}}}\check{\vartheta}^2_{3^{\mathcal{Q}}-1} - 92\check{\vartheta}^2_{3^{\mathcal{Q}}}\check{\vartheta}_{3^{\mathcal{Q}}-1} \\
&= 115\check{\vartheta}^3_{3^{\mathcal{Q}}} \mod 3^{\mathcal{Q}+2} - \check{\vartheta}^3_{3^{\mathcal{Q}}-1} \mod 3^{\mathcal{Q}+2} + 15\check{\vartheta}_{3^{\mathcal{Q}}}\check{\vartheta}^2_{3^{\mathcal{Q}}-1} \mod 3^{\mathcal{Q}+2} - 92\check{\vartheta}^2_{3^{\mathcal{Q}}}\check{\vartheta}_{3^{\mathcal{Q}}-1} \\
&\quad \mod 3^{\mathcal{Q}+2} \\
&\equiv 0 \mod 3^{\mathcal{Q}+2} - \check{\vartheta}^3_{3^{\mathcal{Q}}-1} \mod 3^{\mathcal{Q}+2} + 5\cdot 3\check{\vartheta}_{3^{\mathcal{Q}}}q^2_{3^{\mathcal{Q}}-1} \mod 3^{\mathcal{Q}+2} - 0 \mod 3^{\mathcal{Q}+2} \\
&\equiv -2\cdot 3^{\mathcal{Q}+1} - 1 \mod 3^{\mathcal{Q}+2} + 5\cdot 2\cdot 3^{\mathcal{Q}+1} \mod 3^{\mathcal{Q}+2} \\
&\equiv 8\cdot 3^{\mathcal{Q}+1} \mod 3^{\mathcal{Q}+2} + 1 \mod 3^{\mathcal{Q}+2} \\
&\equiv (6+2)3^{\mathcal{Q}+1} \mod 3^{\mathcal{Q}+2} + 1 \mod 3^{\mathcal{Q}+2} \\
&\equiv 2\cdot 3^{\mathcal{Q}+2} \mod 3^{\mathcal{Q}+2} + 2\cdot 3^{\mathcal{Q}+1} \mod 3^{\mathcal{Q}+2} + 1 \mod 3^{\mathcal{Q}+2} \\
&\equiv 2\cdot 3^{\mathcal{Q}+1} + 1 \mod 3^{\mathcal{Q}+2}.
\end{aligned}$$

also,

$$
\begin{aligned}
\check{\vartheta}_{2\cdot 3^{\mathcal{Q}}} = \check{\vartheta}_{3^{\mathcal{Q}}+3^{\mathcal{Q}}} &= \check{\vartheta}_{3^{\mathcal{Q}}}(\check{\vartheta}_{3^{\mathcal{Q}}+1} - \check{\vartheta}_{3^{\mathcal{Q}}-1}) \\
&= (2\cdot 3^{\mathcal{Q}} + x\cdot 3^{\mathcal{Q}+1})(2\cdot 3^{\mathcal{Q}} + 1 + z\cdot 3^{\mathcal{Q}+1} - 2\cdot 3^{\mathcal{Q}} + 1 - y\cdot 3^{\mathcal{Q}+1}) \\
&= 2^2\cdot 3^{\mathcal{Q}} + 2\cdot 3^{\mathcal{Q}}(z-y)3^{\mathcal{Q}+1} + x\cdot 2\cdot 3^{\mathcal{Q}+1} + x(z-y)\cdot(3^{\mathcal{Q}+1})^2 \\
&= 2^2\cdot 3^{\mathcal{Q}} + (2\cdot 3^{\mathcal{Q}}(z-y))3^{\mathcal{Q}+1} + (x\cdot 2)3^{\mathcal{Q}+1} + (x(z-y)\cdot 3^{\mathcal{Q}+1})3^{\mathcal{Q}+1} \\
&\equiv 2^2\cdot 3^{\mathcal{Q}} \mod 3^{\mathcal{Q}+1} \\
&\equiv (3+1)3^{\mathcal{Q}} \mod 3^{\mathcal{Q}+1} \\
&\equiv 3^{\mathcal{Q}+1} \mod 3^{\mathcal{Q}+1} + 3^{\mathcal{Q}} \mod 3^{\mathcal{Q}+1} \\
&\equiv 3^{\mathcal{Q}} \mod 3^{\mathcal{Q}+1}.
\end{aligned}
$$

Let $pk(h)$ be the period of the sequence $\check{\vartheta}_n$ modulo h. Then we have some quick results:

$\check{\vartheta}_n \equiv \check{\vartheta}_{n+s\cdot pk(h)} \mod h$,

$\check{\vartheta}_{pk(h)} \equiv 0 \mod h$. □

Lemma 4.1.2.5. If $h_1|h_2$, then $pk(h_1)|pk(h_2)$.

Proof. Let $\pi = pk(h_2)$ be the period of sequence $\check{\vartheta}_i$ modulo h_2. Since $h_1|h_2$, to prove $pk(h_1)|pk(h_2)$, it is required to show that $\check{\vartheta}_n(\text{mod } h_1)$ repeats in blocks of length π. It is possible by showing $\check{\vartheta}_n \equiv q_{n+\pi} \mod h_1$ for each n. Clearly, $\check{\vartheta}_n \equiv q_{n+\pi} \mod h_2$. Then for some $0 \le s \le h_2$, it follows, $\check{\vartheta}_n = s + h_2 x$ and $\check{\vartheta}_n = s + h_2 y$.

Now, since $h_1|h_2$, by definition of divisibility $h_2 = h_1 t$. By substitution, $\check{\vartheta}_n = s + h_i t x$ and $\check{\vartheta}_n = s + h_1 t y$. As $h_2 \ge h_1$, for some $0 \le s' \le h_1$, we can have $s = s' + h_1\alpha$. Finally by substituting value of s we get $\check{\vartheta}_n = s' + h_1(\alpha + tx)$ and $\check{\vartheta}_n = s' + h_1(\alpha + ty)$. This implies $\check{\vartheta}_n = q_{n+\pi}(\mod h_1)$(i.e. $\check{\vartheta}_n$ repeats in blocks of length π); hence, the required result. □

Lemma 4.1.2.6. Let $\mathcal{Q} \geq 1$. If $3^{\mathcal{Q}}|n$.then$3^{\mathcal{Q}}|\check{\vartheta}_n$, and if $7^{\mathcal{Q}}|n$ then $7^{\mathcal{Q}}|\check{\vartheta}_n$. If $3^{\mathcal{Q}}|2i+1$, then $3^{\mathcal{Q}}|\check{\mu}_i$.

Proof. In order to have explicit formula for $\check{\vartheta}_n$ and $\check{\mu}_n$, the recurrence relations are needed to be solved, respectively:

The solution to recurrence relation $\check{\vartheta}_n = 5\check{\vartheta}_{n-1} - \check{\vartheta}_{n-2}$ with $\check{\vartheta}_0 = 0$ and $\check{\vartheta}_1 = 1$ can be find as under.

The characteristic equation for this recurrence is $r^2 - 5r + 1 = 0$, which has roots $r_1 = \frac{5-\sqrt{21}}{2}$ and $r_2 = \frac{5+\sqrt{21}}{2}$.Thus $\check{\vartheta}_n = \alpha(\frac{5-\sqrt{21}}{2})^n + \beta(\frac{5+\sqrt{21}}{2})^n$. The constants can be solved by using initial conditions. Thus, we have:

$$\begin{aligned}\check{\vartheta}_n &= \frac{1}{\sqrt{21}}(\frac{5-\sqrt{21}}{2})^n - \frac{1}{\sqrt{21}}(\frac{5+\sqrt{21}}{2})^n \\ &= \frac{1}{2^n}\sum_{1\leq 2k+1\leq n}\binom{n}{2k+1}5^{n-2k-1}21^k\end{aligned}$$

Hence ,if $3^{\mathcal{Q}}|n$, then $3^{\mathcal{Q}}|\check{\vartheta}_n$, and if $7^{\mathcal{Q}}|n$ then $7^{\mathcal{Q}}|\check{\vartheta}_n$. For example, for $\mathcal{Q} = 1$ and $n = 3$, $3|3$ and $3|\check{\vartheta}_3$ that is 24. Similarly, for $n = 7$, $7|7$ and $7|q_7 = 12649$.

Now the solution to recurrence relation $\check{\mu}_i = 5\check{\mu}_{i-1} - \check{\mu}_{i-2}$ with $\check{\mu}_0 = 1$ and $\check{\mu}_1 = 6$ can be given as:

The characteristic equation is $r^2 - 5r + 1 = 0$, which has roots $r_1 = \frac{5-\sqrt{21}}{2}$ and $r_2 = \frac{5+\sqrt{21}}{2}$.Thus $\check{\mu}_i = \alpha(\frac{5-\sqrt{21}}{2})^i + \beta(\frac{5+\sqrt{21}}{2})^i$. We can find the constants from the initial values we know. Thus,

$$\begin{aligned}
\check{\mu}_i &= (\frac{3+\sqrt{21}}{6})(\frac{5+\sqrt{21}}{2})^i + (\frac{3-\sqrt{21}}{6})(\frac{5-\sqrt{21}}{2})^i \\
&= (\frac{3+\sqrt{21}}{6})(\frac{\sqrt{3}+\sqrt{7}}{2})^{2i} + (\frac{3-\sqrt{21}}{6})(\frac{\sqrt{3}-\sqrt{7}}{2})^{2i} \\
&= (\frac{\sqrt{3}+\sqrt{7}}{2\sqrt{3}})(\frac{\sqrt{3}+\sqrt{7}}{2})^{2i} + (\frac{\sqrt{3}-\sqrt{7}}{2\sqrt{3}})(\frac{\sqrt{3}-\sqrt{7}}{2})^{2i} \\
&= \frac{1}{\sqrt{3}}(\frac{\sqrt{3}-\sqrt{7}}{2})^{2i+1} + \frac{1}{\sqrt{3}}(\frac{\sqrt{3}+\sqrt{7}}{2})^{2i+1} \\
&= \frac{1}{2^{2i}}\sum_{k=0}^{i}\binom{2i+1}{2k+1}7^{i-k}3^k.
\end{aligned}$$

So, if $3^{\mathcal{Q}}|2i+1$ then $3^{\mathcal{Q}}|\check{\mu}_i$. □

Corollary 4.1.1. Let $i \geq 1$. Then $(2i, 7\check{\vartheta}_i) = (2i, \check{\vartheta}_i)$.

Lemma 4.1.2.7. $pk(3^{\mathcal{Q}}) = 3^{\mathcal{Q}}$

Proof. From Lemma 4.1.2.4 we have $\check{\vartheta}_{3^{\mathcal{Q}}} \equiv 0 \pmod{3^{\mathcal{Q}}}$ and $\check{\vartheta}_{3^{\mathcal{Q}}+1} \equiv 1(\text{mod}3^{\mathcal{Q}})$ for all u. Using Lemma 4.1.2.4, we have $pk(3^{\mathcal{Q}}) = 3^{\mathcal{Q}}|3^{\mathcal{Q}}$ for all u.

Using induction on u to show $3^{\mathcal{Q}}|pk(3^{\mathcal{Q}})$. This is true for $u = 1$ since $\check{\vartheta}_{pk(3)} \equiv 0($ mod 3). By induction hypothesis suppose it is true for u i.e. $3^{\mathcal{Q}}|pk(3^{\mathcal{Q}})$. Now for $u = u + 1$, as we know that $pk(3^{\mathcal{Q}})|pk(3^{\mathcal{Q}+1})$; also, $3^{\mathcal{Q}}|pk(3^{\mathcal{Q}})$ (by induction hypothesis) this implies $3^{\mathcal{Q}}|pk(3^{\mathcal{Q}+1})$. From Lemma 4.1.2.6, it follows $pk(3^{\mathcal{Q}+1})|3^{\mathcal{Q}+1}$. This shows that $pk(3^{\mathcal{Q}+1})$ is either $3^{\mathcal{Q}}$ or $3^{\mathcal{Q}+1}$. But the facts from Lemma4.1.2.4, proved that $\check{\vartheta}_{3^{\mathcal{Q}}} = 2.3^{\mathcal{Q}} \mod 3^{\mathcal{Q}+1} \not\equiv 0 \mod 3^{\mathcal{Q}+1}$ hence clearly, $3^{\mathcal{Q}+1}|3^{\mathcal{Q}+1}$ which implies $pk(3^{\mathcal{Q}+1}) = 3^{\mathcal{Q}+1}$; hence the required proof. □

Lemma 4.1.2.8. If $3^{\mathcal{Q}}|n$ and $3^{\mathcal{Q}+1} \nmid n$. Then $3^{\mathcal{Q}-1}|\frac{\check{\vartheta}_n - n}{3}$ and $3^{\mathcal{Q}} \nmid \frac{\check{\vartheta}_n - n}{3}$.

Proof. By definition of divisibility if $3^{\mathcal{Q}}$ divides n then $n = 3i$ also 3 must not divide i, then we can have $i = 3b + a$, where a can either be 1 or 2. S ubstituting value of

i, we get;

$n = 3^{\mathcal{Q}}(3b + a) \implies n = a \cdot 3^{\mathcal{Q}} + b \cdot 3^{\mathcal{Q}+1} \equiv a \cdot 3^{\mathcal{Q}} \mod 3^{\mathcal{Q}+1}$.

$$a \cdot 3^{\mathcal{Q}} \mod 3^{\mathcal{Q}+1} = \begin{cases} 3^{\mathcal{Q}} \mod 3^{\mathcal{Q}+1}, & a = 1, \\ 2 \cdot 3^{\mathcal{Q}} \mod 3^{\mathcal{Q}+1}, & a = 2. \end{cases}$$

Also, $\check{\vartheta}_n = \check{\vartheta}_{a.3^{\mathcal{Q}}+b.3^{\mathcal{Q}+1}} \equiv \check{\vartheta}_{a.3^{\mathcal{Q}}} \mod 3^{\mathcal{Q}+1}$

$$\check{\vartheta}_{a\cdot 3^{\mathcal{Q}}} \mod 3^{\mathcal{Q}+1} = \begin{cases} 2 \cdot 3^{\mathcal{Q}} \mod 3^{\mathcal{Q}+1}, & a = 1, \\ 3^{\mathcal{Q}} \mod 3^{\mathcal{Q}+1}, & a = 2. \end{cases}$$

(Since when $a = 2$, then we get $2 \cdot 2 \cdot 3^{\mathcal{Q}} \mod 3^{\mathcal{Q}+1} = (3 + 1)3^{\mathcal{Q}} \mod 3^{\mathcal{Q}+1}$, upon solving we finally have $3^{\mathcal{Q}} \mod 3^{\mathcal{Q}+1}$). From the above properties, we get:

$$\check{\vartheta}_n - n = \begin{cases} 3^{\mathcal{Q}} \mod 3^{\mathcal{Q}+1}, & a = 1, \\ 2 \cdot 3^{\mathcal{Q}} \mod 3^{\mathcal{Q}+1}, & a = 2. \end{cases}$$

(When $a = 2$ we get $-3^{\mathcal{Q}} \mod 3^{\mathcal{Q}+1} = (-3 + 2)3^{\mathcal{Q}} \mod 3^{\mathcal{Q}+1}$, by property of mod finally we are left with $2 \cdot 3^{\mathcal{Q}} \mod 3^{\mathcal{Q}+1}$).

Hence proved that $3^{\mathcal{Q}-1} | \frac{\check{\vartheta}_n - n}{3}$ and $3^{\mathcal{Q}} \nmid \frac{\check{\vartheta}_n - n}{3}$. □

Lemma 4.1.2.9. For all $n \geq 2$, we have $(\check{\mu}_n, \check{\mu}_{n-1}) = (\check{\vartheta}_n, \check{\vartheta}_{n-1}) = 1$ and $(\check{\mu}_n - \check{\mu}_{n-1}, \check{\mu}_{n+1} - \check{\mu}_n) = (\check{\vartheta}_n - \check{\vartheta}_{n-1}, \check{\vartheta}_{n+1} - \check{\vartheta}_n) = 1$

We list some other relationships linking $\check{\mu}_n$, $\check{\vartheta}_n$ and $\mathfrak{J}_n$

- $\mathfrak{J}_{n+1} - \mathfrak{J}_n = \check{\vartheta}_n$.
- $\mathfrak{J}_{2i+1} = \check{\mu}_i \check{\vartheta}_i$.
- $\mathfrak{J}_{2i+2} = \check{\mu}_i \check{\vartheta}_{i+1}$.

- $(\mathfrak{J}_{2i}, \mathfrak{J}_{2i+1}) = \check{\vartheta}_i, (\mathfrak{J}_{2i+1}, \mathfrak{J}_{2i+2}) = \check{\mu}_i$.

4.1.3 The Smith Normal Form (SNF) A_n

With the help of SNF of P_n and Q_n, the SNF of A_n can be computed.

4.1.4 Computation of the SNF of P_n

Since $P_n = \begin{pmatrix} \mathfrak{J}_{n+1} - \mathfrak{J}_n & \mathfrak{J}_n - \mathfrak{J}_{n-1} + 1 \\ 3\mathfrak{J}_{n+1} & 3\mathfrak{J}_n \end{pmatrix}$. As $\mathfrak{J}_{n+1} - \mathfrak{J}_n = \check{\vartheta}_n$ If $n = 2i+1$, then we have:

$$\mathfrak{J}_{2i+2} - \mathfrak{J}_{2i+1} = \check{\vartheta}_{2i+1} = \check{\mu}_i(\check{\vartheta}_{i+1} - \check{\vartheta}_i)$$

,

$$\begin{aligned}
\mathfrak{J}_n - \mathfrak{J}_{n-1} + 1 &= \mathfrak{J}_{2i+1} - (\mathfrak{J}_{2i} - 1) \\
&= \mathfrak{J}_{2i+1} - 5\mathfrak{J}_{2i+1} + \mathfrak{J}_{2i+2} \\
&= \mathfrak{J}_{2i+2} - 4\mathfrak{J}_{2i+1} \\
&= \check{\mu}_i\check{\vartheta}_{i+1} - 4\check{\mu}_i\check{\vartheta}_i \\
&= \check{\mu}_i(\check{\vartheta}_{i+1} + 4\check{\vartheta}_i) \\
&= \check{\mu}_i(\check{\vartheta}_{i+1} - 5\check{\vartheta}_i + \check{\vartheta}_i) \\
&= \check{\mu}_i(\check{\vartheta}_i - \check{\vartheta}_{i-1}).
\end{aligned}$$

Thus, the GCD of all entries of P_n for $n = 2i+1$ is

$$\begin{aligned}
(\mathfrak{J}_{n+1} - \mathfrak{J}_n, \mathfrak{J}_n - \mathfrak{J}_{n-1} + 1, 3\mathfrak{J}_{n+1}, 3\mathfrak{J}_n) &= (\mathfrak{J}_{2i+2} - \mathfrak{J}_{2i+1}, \mathfrak{J}_{2i+1} - \mathfrak{J}_{2i} + 1, 3\mathfrak{J}_{2i+2}, 3\mathfrak{J}_{2i+1}) \\
&= (\check{\mu}_i(\check{\vartheta}_{i+1} - \check{\vartheta}_i), \check{\mu}_i(\check{\vartheta}_i - \check{\vartheta}_{i-1}), 3\check{\mu}_i\check{\vartheta}_i, 3\check{\mu}_i\check{\vartheta}_{i+1}) \\
&= \check{\mu}_i(\check{\vartheta}_{i+1} - \check{\vartheta}_i, \check{\vartheta}_i - \check{\vartheta}_{i-1}, 3\check{\vartheta}_i, 3\check{\vartheta}_{i+1}) \\
&= \check{\mu}_i.
\end{aligned}$$

Since $(\check{\vartheta}_n - \check{\vartheta}_{n-1}, \check{\vartheta}_{n+1} - \check{\vartheta}_n) = 1$ and $(\check{\vartheta}_n, \check{\vartheta}_{n-1}) = 1$, now consider

$$\begin{aligned}
\begin{vmatrix} \mathfrak{J}_{n+1} - \mathfrak{J}_n & \mathfrak{J}_n - \mathfrak{J}_{n-1} + 1 \\ 3\mathfrak{J}_{n+1} & 3\mathfrak{J}_n \end{vmatrix} =& 3\mathfrak{J}_n(\mathfrak{J}_{n+1} - \mathfrak{J}_n) - 3\mathfrak{J}_{n+1}(\mathfrak{J}_n - \mathfrak{J}_{n-1} + 1) \\
=& 3\mathfrak{J}_n\mathfrak{J}_{n+1} - 3\mathfrak{J}_n^2 - 3\mathfrak{J}_{n+1}\mathfrak{J}_{n-1} - 3\mathfrak{J}_{n+1} \\
=& -3\mathfrak{J}_n^2 + 3\mathfrak{J}_{n+1}\mathfrak{J}_{n-1} - 3\mathfrak{J}_{n+1} \\
=& -3(\mathfrak{J}_n + \mathfrak{J}_{n+1}).
\end{aligned}$$

When $n = 2i + 1$;

$$\begin{aligned}
-3(\mathfrak{J}_n + \mathfrak{J}_{n+1}) =& -3(\mathfrak{J}_{2i+1} - \mathfrak{J}_{2i+2}) \\
=& -3(\check{\mu}_i^2).
\end{aligned}$$

Thus we can have $\begin{pmatrix} \mathfrak{J}_{n+1} - \mathfrak{J}_n & \mathfrak{J}_n - \mathfrak{J}_{n-1} + 1 \\ 3\mathfrak{J}_{n+1} & 3\mathfrak{J}_n \end{pmatrix} = \begin{pmatrix} \mathfrak{J}_{2i+2} - \mathfrak{J}_{2i+1} & \mathfrak{J}_{2i+1} - \mathfrak{J}_{2i} + 1 \\ 3\mathfrak{J}_{2i+2} & 3\mathfrak{J}_{2i+1} \end{pmatrix}$
$\sim \begin{pmatrix} \check{\mu}_i & 0 \\ 0 & 3\check{\mu}_i \end{pmatrix}$
Now if $n = 2i$, then $\mathfrak{J}_{n+1} - \mathfrak{J}_n = q_{2i} = \mathfrak{J}_{2i+1} - \mathfrak{J}_{2i} = \check{\vartheta}_i(\check{\mu}_i - \check{\mu}_{i-1})$ (since $\mathfrak{J}_{2(i-1)+2} = \check{\mu}_{i-1}\check{\vartheta}_i$),

$$\begin{aligned}
\mathfrak{J}_n - \mathfrak{J}_{n-1} + 1 &= \mathfrak{J}_{2i} - \mathfrak{J}_{2i-1} + 1 \\
&= \mathfrak{J}_{2i} - 5\mathfrak{J}_{2i} + \mathfrak{J}_{2i+1} \\
&= \mathfrak{J}_{2i+1} - 4\mathfrak{J}_{2i} \\
&= \check{\mu}_i\check{\vartheta}_i - 4\check{\mu}_{i-1}\check{\vartheta}_i \\
&= \check{\vartheta}_i(\check{\mu}_i + 4\check{\mu}_{i-1}) \\
&= \check{\vartheta}_i(\check{\mu}_i - 5\check{\mu}_{i-1} + \check{\mu}_{i-1}) \\
&= \check{\vartheta}_i(\check{\mu}_{i-1} - \check{\mu}_{i-2}).
\end{aligned}$$

Thus, the GCD of all items of P_n for $n = 2i$ is

$$\begin{aligned}(\mathfrak{J}_{n+1} - \mathfrak{J}_n, \mathfrak{J}_n - \mathfrak{J}_{n-1} + 1, 3\mathfrak{J}_{n+1}, 3\mathfrak{J}_n) &= (\mathfrak{J}_{2i+1} - \mathfrak{J}_{2i}, \mathfrak{J}_{2i} - \mathfrak{J}_{2i-1} + 1, 3\mathfrak{J}_{2i+1}, 3\mathfrak{J}_{2i}) \\ &= (\check{\vartheta}_i(\check{\mu}_i - \check{\vartheta}_{i-1}), \check{\vartheta}_i(\check{\mu}_{i-1} - \check{\mu}_{i-2}), 3\check{\mu}_i\check{\vartheta}_i, 3\check{\mu}_{i-1}\check{\vartheta}_i) \\ &= \check{\vartheta}_i(\check{\mu}_i - \check{\vartheta}_{i-1}, \check{\mu}_{i-1} - \check{\mu}_{i-2}, 3\check{\mu}_i, 3\check{\mu}_{i-1}) \\ &= \check{\vartheta}_i.\end{aligned}$$

Now,

$$\begin{aligned}\begin{vmatrix} \mathfrak{J}_{n+1} - \mathfrak{J}_n & \mathfrak{J}_n - \mathfrak{J}_{n-1} + 1 \\ 3\mathfrak{J}_{n+1} & 3\mathfrak{J}_n \end{vmatrix} =& 3\mathfrak{J}_n(\mathfrak{J}_{n+1} - \mathfrak{J}_n) - 3\mathfrak{J}_{n+1}(\mathfrak{J}_n - \mathfrak{J}_{n-1} + 1) \\ =& 3\mathfrak{J}_n\mathfrak{J}_{n+1} - 3\mathfrak{J}_n^2 - 3\mathfrak{J}_{n+1}\mathfrak{J}_{n-1} - 3\mathfrak{J}_{n+1} \\ =& -3\mathfrak{J}_n^2 + 3\mathfrak{J}_{n+1}\mathfrak{J}_{n-1} - 3\mathfrak{J}_{n+1} \\ =& -3(\mathfrak{J}_n + \mathfrak{J}_{n+1}).\end{aligned}$$

When $n = 2i$, then

$$\begin{aligned}-3(\mathfrak{J}_n + \mathfrak{J}_{n+1}) =& -3(\mathfrak{J}_{2i} - \mathfrak{J}_{2i+1}) \\ =& -3(7\check{\vartheta}_i^2) \\ =& -21\check{\vartheta}_i^2.\end{aligned}$$

Thus, we can have $\begin{pmatrix} \mathfrak{J}_{n+1} - \mathfrak{J}_n & \mathfrak{J}_n - \mathfrak{J}_{n-1} + 1 \\ 3\mathfrak{J}_{n+1} & 3\mathfrak{J}_n \end{pmatrix} = \begin{pmatrix} \mathfrak{J}_{2i+1} - \mathfrak{J}_{2i} & \mathfrak{J}_{2i} - \mathfrak{J}_{2i-1} + 1 \\ 3\mathfrak{J}_{2i+1} & 3\mathfrak{J}_{2i} \end{pmatrix}$
$\sim \begin{pmatrix} \check{\vartheta}_i & 0 \\ 0 & 21\check{\vartheta}_i \end{pmatrix}$

4.1.5 Computation of the Smith Normal Form of Q_n

The SNF of the matrix $Q_n = \begin{pmatrix} n & 0 & 0 \\ \mathfrak{c}_n & \mathfrak{J}_n & \mathfrak{J}_{n+1} \\ \mathfrak{c}_{n+1} & \mathfrak{J}_{n+1} & \mathfrak{J}_{n+2}-1 \end{pmatrix}$ with determinant $n(\mathfrak{J}_n(\mathfrak{J}_{n+2}-1)-\mathfrak{J}_{n+1}^2) = n(-\mathfrak{J}_n - \mathfrak{J}_{n+1}) = -n(\mathfrak{J}_n - \mathfrak{J}_{n+1})$ can be computed by calculating the GCD of all $i \times i$ minor determinants of Q_n.

Computation of H_{11}

H_{11} is the GCD of all 1×1 minor determinants of Q_n. The non negative integer entries of H_{11} are

$$H_{11} = (0, n, \mathfrak{c}_n, \mathfrak{c}_{n+1}, \mathfrak{J}_n, \mathfrak{J}_{n+1}, \mathfrak{J}_{n+2}-1).$$

Since we have $3\mathfrak{c}_n + n = \mathfrak{J}_{n+1} - \mathfrak{J}_n$, by some computation and $\mathfrak{J}_n = \mathfrak{c}_n - \mathfrak{c}_{n-1}$ for all $n \geq 0$. Thus $\mathfrak{c}_n = \frac{\mathfrak{J}_{n+1}-\mathfrak{J}_n-n}{3}$ and $\mathfrak{c}_{n+1} = \mathfrak{J}_{n+1} + \mathfrak{c}_n$. This follows,

$$H_{11} = (n, \mathfrak{J}_n, \mathfrak{J}_{n+1}, \tfrac{\mathfrak{J}_{n+1}-\mathfrak{J}_n-n}{3}).$$

If $n = 2i+1$ by relationship of $\mathfrak{J}_n$ and $\breve{\mu}_n$ we have $(\mathfrak{J}_{2i+1}, \mathfrak{J}_{2i+2}) = \breve{\mu}_i$. This implies $H_{11} = (2i+1, \breve{\mu}_i, \frac{\mathfrak{J}_{2i+2}-\mathfrak{J}_{2i+1}-2i-1}{3})$. Also $\mathfrak{J}_{2i+2} = \breve{\mu}_i\check{\vartheta}_{i+1}$ and $\mathfrak{J}_{2i+1} = \breve{\mu}_i\check{\vartheta}_i$. By substituting, we get:

$$H_{11} = (2i+1, \breve{\mu}_i, \tfrac{\breve{\mu}_i(\check{\vartheta}_{i+1}-\check{\vartheta}_i-2i-1)}{3}).$$

Now for $n = 2i+1$ if $3 \nmid n$, then by Lemma 4.1.2.8, $3 | \frac{\check{\vartheta}_n - n}{3}$ then $H_{11} = (2i+1, \breve{\mu}_i, \frac{\breve{\mu}_i(\check{\vartheta}_{i+1}-\check{\vartheta}_i-2i-1)}{3}) = (2i+1, \breve{\mu}_i)$.

For $n = 2i+1$ if $3|n$ then by Lemma 4.1.2.8, $3 \nmid \frac{\check{\vartheta}_n - n}{3}$ then $H_{11} = (2i+1, \breve{\mu}_i, \frac{\breve{\mu}_i(\check{\vartheta}_{i+1}-\check{\vartheta}_i-2i-1)}{3}) = (n, \breve{\mu}_i, \frac{\check{\vartheta}_n - n)}{3}) = \frac{(2i+1, \breve{\mu}_i)}{3}$.

Consider the case when $n = 2i$, then H_{11} becomes

$$H_{11} = (2i, \mathfrak{J}_{2i}, \mathfrak{J}_{2i+1}, \tfrac{\mathfrak{J}_{2i+1}-\mathfrak{J}_{2i}-2i}{3}).$$

By relationship $(\mathfrak{J}_{2i}, \mathfrak{J}_{2i+1}) = \check{\vartheta}_i$ and$\mathfrak{J}_{2i+1} = \breve{\mu}_i\check{\vartheta}_i$, $\mathfrak{J}_{2i} = \check{\vartheta}_i\breve{\mu}_{i-1}$. This implies $H_{11} =$

$(2i, \check{\vartheta}_i, \frac{\check{\vartheta}_i(\check{\mu}_i-\check{\mu}_{i-1})-2i}{3})$.

Similarly, for $n = 2i$ if $3 \nmid n$ then $3|\frac{\check{\vartheta}_n-n}{3}$; hence, $H_{11} = (2i, \check{\vartheta}_i, \frac{\check{\vartheta}_i(\check{\mu}_i-\check{\mu}_{i-1})-2i}{3}) = (2i, \check{\vartheta}_i, \frac{\check{\vartheta}_n-n}{3}) = (2i, \check{\vartheta}_i)$. On the same lines, if $3|n$ then $3 \nmid \frac{\check{\vartheta}_n-n}{3}$; hence $H_{11} = (2i, \check{\vartheta}_i, \frac{\check{\vartheta}_i(\check{\mu}_i-\check{\mu}_{i-1})-2i}{3}) = (2i, \check{\vartheta}_i, \frac{\check{\vartheta}_n-n}{3}) = \frac{(2i,\check{\vartheta}_i)}{3}$.

Computation of H_{22}

In order to compute H_{22} the entities involved in computation of H_{11} must be kept under consideration. $H_{11}H_{22}$ is the GCD of all 2×2 minor determinants. Thus, for $Q_n = \begin{pmatrix} n & 0 & 0 \\ \mathfrak{c}_n & \mathfrak{J}_n & \mathfrak{J}_{n+1} \\ \mathfrak{c}_{n+1} & \mathfrak{J}_{n+1} & \mathfrak{J}_{n+2}-1 \end{pmatrix}$. All 2×2 minor determinants are:

$$\begin{aligned} Q_{11} &= \begin{vmatrix} \mathfrak{J}_n & \mathfrak{J}_{n+1} \\ \mathfrak{J}_{n+1} & \mathfrak{J}_{n+2}-1 \end{vmatrix} = \mathfrak{J}_n(\mathfrak{J}_{n+2}-1) - \mathfrak{J}_{n+1}^2 \\ &= \mathfrak{J}_n\mathfrak{J}_{n+2} - \mathfrak{J}_{n+1}^2 - \mathfrak{J}_n \\ &= -(\mathfrak{J}_{n+1} + \mathfrak{J}_n). \end{aligned}$$

$$\begin{aligned} Q_{12} &= \begin{vmatrix} \mathfrak{c}_n & \mathfrak{J}_{n+1} \\ \mathfrak{c}_{n+1} & \mathfrak{J}_{n+2}-1 \end{vmatrix} = \mathfrak{c}_n(\mathfrak{J}_{n+2}-1)) - \mathfrak{J}_{n+1}\mathfrak{c}_{n+1}) \\ &= \mathfrak{c}_n(\mathfrak{J}_{n+2}-1) - \mathfrak{J}_{n+1}(\mathfrak{c}_n + \mathfrak{J}_{n+1}) \\ &= \mathfrak{c}_n(\mathfrak{J}_{n+2} - \mathfrak{J}_{n+1} - 1) - \mathfrak{J}_{n+1}^2 \\ &= \mathfrak{c}_n(5\mathfrak{J}_{n+1} - \mathfrak{J}_n - \mathfrak{J}_{n+1} + 1 - 1) - \mathfrak{J}_{n+1}^2 \\ &= \mathfrak{c}_n(4\mathfrak{J}_{n+1} - \mathfrak{J}_n) - \mathfrak{J}_{n+1}^2. \end{aligned}$$

$$\begin{aligned} Q_{13} &= \begin{vmatrix} \mathfrak{c}_n & \mathfrak{J}_n \\ \mathfrak{c}_{n+1} & \mathfrak{J}_{n+1} \end{vmatrix} = \mathfrak{c}_n\mathfrak{J}_{n+1} - \mathfrak{J}_n\mathfrak{c}_{n+1} \\ &= \mathfrak{c}_n\mathfrak{J}_{n+1} - \mathfrak{J}_n(\mathfrak{c}_n + \mathfrak{J}_{n+1}) \\ &= \mathfrak{c}_n(\mathfrak{J}_{n+1} - \mathfrak{J}_n) - \mathfrak{J}_n\mathfrak{J}_{n+1}. \end{aligned}$$

$$Q_{21} = \begin{vmatrix} 0 & 0 \\ \mathfrak{c}_{n+1} & \mathfrak{J}_{n+2}-1 \end{vmatrix} = 0.$$

$$Q_{22} = \begin{vmatrix} n & 0 \\ \mathfrak{c}_{n+1} & \mathfrak{J}_{n+2}-1 \end{vmatrix} = n(\mathfrak{J}_{n+2}-1).$$

$$Q_{23} = \begin{vmatrix} n & 0 \\ \mathfrak{c}_{n+1} & \mathfrak{J}_{n+1} \end{vmatrix} = n\mathfrak{J}_{n+1}.$$

$$Q_{31} = \begin{vmatrix} 0 & 0 \\ \mathfrak{J}_{n} & \mathfrak{J}_{n+1} \end{vmatrix} = 0.$$

$$Q_{32} = \begin{vmatrix} n & 0 \\ \mathfrak{c}_{n} & \mathfrak{J}_{n+1} \end{vmatrix} = n\mathfrak{J}_{n+1}.$$

$$Q_{33} = \begin{vmatrix} n & 0 \\ \mathfrak{c}_{n} & \mathfrak{J}_{n} \end{vmatrix} = n\mathfrak{J}_{n}.$$

Thus

$$H_{11}H_{22} = \left(\begin{vmatrix} \mathfrak{J}_n & \mathfrak{J}_{n+1} \\ \mathfrak{J}_{n+1} & \mathfrak{J}_{n+2}-1 \end{vmatrix}, \begin{vmatrix} \mathfrak{c}_n & \mathfrak{J}_{n+1} \\ \mathfrak{c}_{n+1} & \mathfrak{J}_{n+2}-1 \end{vmatrix}, \begin{vmatrix} \mathfrak{c}_n & \mathfrak{J}_{n} \\ \mathfrak{c}_{n+1} & \mathfrak{J}_{n+1} \end{vmatrix}, \begin{vmatrix} 0 & 0 \\ \mathfrak{c}_{n+1} & \mathfrak{J}_{n+2}-1 \end{vmatrix}, \begin{vmatrix} n & 0 \\ \mathfrak{c}_{n+1} & \mathfrak{J}_{n+2}-1 \end{vmatrix}, \begin{vmatrix} n & 0 \\ \mathfrak{c}_{n+1} & \mathfrak{J}_{n+1} \end{vmatrix}, \begin{vmatrix} n & 0 \\ \mathfrak{c}_{n} & \mathfrak{J}_{n+1} \end{vmatrix}, \begin{vmatrix} n & 0 \\ \mathfrak{c}_{n} & \mathfrak{J}_{n+1} \end{vmatrix}, \begin{vmatrix} n & 0 \\ \mathfrak{c}_{n} & \mathfrak{J}_{n} \end{vmatrix} \right).$$

$$H_{11}H_{22} = \big(0, n\mathfrak{J}_n, n\mathfrak{J}_{n+1}, n(\mathfrak{J}_{n+2}-1), \mathfrak{J}_n + \mathfrak{J}_{n+1}, \mathfrak{c}_n(4\mathfrak{J}_{n+1} - \mathfrak{J}_n) - \mathfrak{J}_{n+1}^2, \mathfrak{c}_n(\mathfrak{J}_{n+1} - \mathfrak{J}_n) - \mathfrak{J}_n\mathfrak{J}_{n+1} \big),$$

For $n = 2i + 1$, then
$(\mathfrak{I}_n, \mathfrak{I}_{n+1}) = (\mathfrak{I}_{2i+1}, \mathfrak{I}_{2i+2}) = \breve{\mu}_i$ and $\mathfrak{I}_n + \mathfrak{I}_{n+1} = \mathfrak{I}_{2i+1} + \mathfrak{I}_{2i+2} = \breve{\mu}_i^2$ also $\mathfrak{c}_n = \frac{\check{\vartheta}_n - n}{3}$

$$\begin{aligned}
\mathfrak{c}_n(4\mathfrak{I}_{n+1} - \mathfrak{I}_n) - \mathfrak{I}_{n+1}^2 &= \frac{\check{\vartheta}_n - n}{3}(4\mathfrak{I}_{2i+2} - \mathfrak{I}_{2i+1}) - \mathfrak{I}_{2i+2}^2 \\
&= \frac{\check{\vartheta}_n - n}{3}(4\breve{\mu}_i\check{\vartheta}_{i+1} - \breve{\mu}_i\vartheta_i) - \breve{\mu}_i^2 q_{i+1}^{\hat{2}} \\
&= \frac{\check{\vartheta}_n - n}{3}\breve{\mu}_i(4\check{\vartheta}_{i+1} - \check{\vartheta}_i) - \breve{\mu}_i^2 q_{i+1}^{\hat{2}} \\
&= \frac{\check{\vartheta}_n - n}{3}\breve{\mu}_i(5\check{\vartheta}_{i+1} - \check{\vartheta}_i\check{\vartheta}_{i+1}) - \breve{\mu}_i^2 q_{i+1}^{\hat{2}} \\
&= \frac{\check{\vartheta}_n - n}{3}\breve{\mu}_i(\check{\vartheta}_{i+2} - \check{\vartheta}_{i+1}) - \breve{\mu}^2 i\check{\vartheta}_{i+1}^2.
\end{aligned}$$

$$\begin{aligned}
\mathfrak{c}_n(\mathfrak{I}_{n+1} - \mathfrak{I}_n) - \mathfrak{I}_n\mathfrak{I}_{n+1} &= \frac{\check{\vartheta}_n - n}{3}(\mathfrak{I}_{2i+2} - \mathfrak{I}_{2i+1}) - \mathfrak{I}_{2i+1}\mathfrak{I}_{2i+2} \\
&= \frac{\check{\vartheta}_n - n}{3}(\breve{\mu}_i\check{\vartheta}_{i+1} - \breve{\mu}_i\check{\vartheta}_i) - \breve{\mu}_i\check{\vartheta}_i\breve{\mu}_i\check{\vartheta}_{i+1} \\
&= \frac{\check{\vartheta}_n - n}{3}\breve{\mu}_i(\check{\vartheta}_{i+1} - \check{\vartheta}_i) - \breve{\mu}_i^2\check{\vartheta}_i\check{\vartheta}_{i+1},
\end{aligned}$$

$$\begin{aligned}
H_{11}H_{22} &= \Big(\ n\breve{\mu}_i, \breve{\mu}_i^2, \frac{\check{\vartheta}_n - n}{3}\breve{\mu}_i(\check{\vartheta}_{i+2} - \check{\vartheta}_{i+1}) - \breve{\mu}_i^2\check{\vartheta}_{i+1}^2, \frac{\check{\vartheta}_n - n}{3}\breve{\mu}_i(\check{\vartheta}_{i+1} - \check{\vartheta}_i) - \breve{\mu}_i^2\check{\vartheta}_i\check{\vartheta}_{i+1}\Big) \\
&= \breve{\mu}_i\Big(\ n, \breve{\mu}_i, \frac{\check{\vartheta}_n - n}{3}(\check{\vartheta}_{i+2} - \check{\vartheta}_{i+1}), \frac{\check{\vartheta}_n - n}{3}(\check{\vartheta}_{i+1} - \check{\vartheta}_i)\Big) \\
&= \breve{\mu}_i\Big(\ n, \breve{\mu}_i, \frac{\check{\vartheta}_n - n}{3}(\check{\vartheta}_{i+2} - \check{\vartheta}_{i+1}, \check{\vartheta}_{i+1} - \check{\vartheta}_i)\Big) \\
&= \breve{\mu}_i\Big(\ n, \breve{\mu}_i, \frac{\check{\vartheta}_n - n}{3}(1)\Big) \\
&= \breve{\mu}_i\Big(\ n, \breve{\mu}_i, \frac{\check{\vartheta}_n - n}{3}\Big).
\end{aligned}$$

Thus $H_{22} = \breve{\mu}_i$ since, $H_{22} = \frac{\breve{\mu}_i\big(\ n, \breve{\mu}_i, \frac{\check{\vartheta}_n - n}{3}\big)}{H_{11}}$.
Now for $n = 2i$:
$H_{11}H_{22} = \big(\ 0, n\mathfrak{I}_n, n\mathfrak{I}_{n+1}, n(\mathfrak{I}_{n+2}-1), \mathfrak{I}_n+\mathfrak{I}_{n+1}, \mathfrak{c}_n(4\mathfrak{I}_{n+1}-\mathfrak{I}_n)-\mathfrak{I}_{n+1}^2, \mathfrak{c}_n(\mathfrak{I}_{n+1}-\mathfrak{I}_n)-\mathfrak{I}_n\mathfrak{I}_{n+1}\big)$, becomes, $(\mathfrak{I}_n, \mathfrak{I}_{n+1}) = (\mathfrak{I}_{2i}, \mathfrak{I}_{2i+1}) = \check{\vartheta}_i$ and $\mathfrak{I}_n + \mathfrak{I}_{n+1} = \mathfrak{I}_{2i} + \mathfrak{I}_{2i+1} = 7\check{\vartheta}_i^2$.

Also $\mathfrak{c}_n = \frac{\check{\vartheta}_n - n}{3}$.

$$\begin{aligned}
\mathfrak{c}_n(\mathfrak{J}_{n+2} - \mathfrak{J}_{n+1} - 1) - \mathfrak{J}_{n+1}^2 &= \frac{\check{\vartheta}_n - n}{3}(4\mathfrak{J}_{2i+1} - \mathfrak{J}_{2i}) - \mathfrak{J}_{2i+1}^2 \\
&= \frac{\check{\vartheta}_n - n}{3}(4\check{\mu}_i\check{\vartheta}_i - \check{\mu}_{i-1}\check{\vartheta}_i) - \check{\mu}_i^2\check{\vartheta}_i^2 \\
&= \frac{\check{\vartheta}_n - n}{3}\check{\vartheta}_i(4\check{\mu}_i - \check{\mu}_{i-1}) - \check{\mu}_i^2\check{\vartheta}_i^2 \\
&= \frac{\check{\vartheta}_n - n}{3}\check{\mu}_i(5\check{\mu}_i - \check{\mu}_{i-1} - \check{\mu}_i) - \check{\mu}_i^2\check{\vartheta}_i^2 \\
&= \frac{\check{\vartheta}_n - n}{3}\check{\vartheta}_i(\check{\mu}_{i+1} - \check{\mu}_i) - p^2 i\check{\vartheta}_i^2.
\end{aligned}$$

$$\begin{aligned}
\mathfrak{c}_n(\mathfrak{J}_{n+1} - \mathfrak{J}_n) - \mathfrak{J}_n\mathfrak{J}_{n+1} &= \frac{\check{\vartheta}_n - n}{3}(\mathfrak{J}_{2i+1} - \mathfrak{J}_{2i}) - \mathfrak{J}_{2i}\mathfrak{J}_{2i+1} \\
&= \frac{\check{\vartheta}_n - n}{3}(\check{\mu}_i\check{\vartheta}_i - \check{\mu}_{i-1}\check{\vartheta}_i) - \check{\mu}_{i-1}\check{\mu}_i\check{\vartheta}_i \\
&= \frac{\check{\vartheta}_n - n}{3}\check{\vartheta}_i(\check{\mu}_i - \check{\mu}_{i-1}) - \check{\mu}_i\check{\mu}_{i-1}\check{\vartheta}_i^2.
\end{aligned}$$

$$\begin{aligned}
H_{11}H_{22} &= \big(\ n\check{\vartheta}_i, 7\check{\vartheta}_i^2, \frac{\check{\vartheta}_n - n}{3}\check{\vartheta}_i(\check{\mu}_{i+1} - \check{\mu}_i) - p_i^2\check{\vartheta}_i^2, \frac{\check{\vartheta}_n - n}{3}\check{\vartheta}_i(\check{\mu}_i - \check{\mu}_{i-1}) - \check{\mu}_i\check{\mu}_{i-1}\check{\vartheta}_i^2\big) \\
&= \check{\vartheta}_i\big(\ n, 7\check{\vartheta}_i, \frac{\check{\vartheta}_n - n}{3}(\check{\mu}_{i+1} - \check{\mu}_i) - p_i^2\check{\vartheta}_i, \frac{\check{\vartheta}_n - n}{3}(\check{\mu}_i - \check{\mu}_{i-1}) - \check{\mu}_i\check{\mu}_{i-1}\check{\vartheta}_i\big) \\
&= \check{\vartheta}_i\big(\ n, \check{\vartheta}_i, \frac{\check{\vartheta}_n - n}{3}(\check{\mu}_{i+1} - \check{\mu}_i) - \check{\mu}_i^2\check{\vartheta}_i, \frac{\check{\vartheta}_n - n}{3}(\check{\mu}_i - \check{\mu}_{i-1}) - \check{\mu}_i\check{\mu}_{i-1}\check{\vartheta}_i\big) \\
&= \check{\vartheta}_i\big(\ n, \check{\vartheta}_i, \frac{\check{\vartheta}_n - n}{3}(\check{\mu}_{i+1} - \check{\mu}_i, \check{\mu}_i - \check{\mu}_{i-1})\big) \\
&= \check{\vartheta}_i\big(\ n, \check{\vartheta}_i, \frac{\check{\vartheta}_n - n}{3}(1)\big) \\
&= \check{\vartheta}_i\big(\ n, \check{\vartheta}_i, \frac{\check{\vartheta}_n - n}{3}\big).
\end{aligned}$$

Here, for $n = 2i$ $H_{22} = \check{\vartheta}_i$, since $H_{22} = \frac{\check{\vartheta}_i\big(\ n, \check{\vartheta}_i,, \frac{\check{\vartheta}_n - n}{3}\big)}{H_{11}}$

Computation of H_{33}

Since $H_{33} = \frac{|det(Q_n)|}{H_{11}H_{22}}$. Then:

$$detQ_n = \begin{vmatrix} n & 0 & 0 \\ \mathfrak{c}_n & \mathfrak{J}_n & \mathfrak{J}_{n+1} \\ \mathfrak{c}_{n+1} & \mathfrak{J}_{n+1} & \mathfrak{J}_{n+2} - 1 \end{vmatrix}$$

$$\begin{aligned} &= n(\mathfrak{J}_n(\mathfrak{J}_{n+2} - 1) - \mathfrak{J}_{n+1}^2) \\ &= n(\mathfrak{J}_n\mathfrak{J}_{n+2} - \mathfrak{J}_n - \mathfrak{J}_{n+1}^2) \\ &= n(-\mathfrak{J}_n - \mathfrak{J}_{n+1}) \\ &= -n(\mathfrak{J}_n - \mathfrak{J}_{n+1}). \end{aligned}$$

This implies $H_{33} = \frac{n(\mathfrak{J}_n - \mathfrak{J}_{n+1})}{H_{11}H_{22}}$. For $n = 2i+1$, $H_{33} = \frac{n(\mathfrak{J}_{2i+1} - \mathfrak{J}_{2i+2})}{H_{11}H_{22}} = \frac{n\breve{\mu}_i}{H_{11}H_{22}}$ and for $n = 2i$, $\underline{H_{33} = \frac{n(\mathfrak{J}_{2i} - \mathfrak{J}_{2i+1})}{H_{11}H_{22}} = \frac{7n\breve{\vartheta}_i}{H_{11}H_{22}}}$.
Now, the SNF of matrix Q_n is

- For $n = 2i+1$ and $3 \nmid n$
$$\begin{pmatrix} H_{11} & 0 & 0 \\ 0 & H_{22} & 0 \\ 0 & 0 & H_{33} \end{pmatrix} = \begin{pmatrix} (n, \breve{\mu}_i) & 0 & 0 \\ 0 & \breve{\mu}_i & 0 \\ 0 & 0 & \frac{n\breve{\mu}_i}{(n,\breve{\mu}_i)} \end{pmatrix}.$$
- For $n = 2i+1$ and $3 \mid n$
$$\begin{pmatrix} H_{11} & 0 & 0 \\ 0 & H_{22} & 0 \\ 0 & 0 & H_{33} \end{pmatrix} = \begin{pmatrix} \frac{(n,\breve{\mu}_i)}{3} & 0 & 0 \\ 0 & \breve{\mu}_i & 0 \\ 0 & 0 & \frac{3n\breve{\mu}_i}{(n,\breve{\mu}_i)} \end{pmatrix}.$$
- For $n = 2i$ and $3 \nmid n$
$$\begin{pmatrix} H_{11} & 0 & 0 \\ 0 & H_{22} & 0 \\ 0 & 0 & H_{33} \end{pmatrix} = \begin{pmatrix} (n, \breve{\vartheta}_i) & 0 & 0 \\ 0 & \breve{\vartheta}_i & 0 \\ 0 & 0 & \frac{7n\breve{\vartheta}_i}{(n,\breve{\vartheta}_i)} \end{pmatrix}.$$
- For $n = 2i$ and $3 \mid n$
$$\begin{pmatrix} H_{11} & 0 & 0 \\ 0 & H_{22} & 0 \\ 0 & 0 & H_{33} \end{pmatrix} = \begin{pmatrix} \frac{(n,\breve{\vartheta}_i)}{3} & 0 & 0 \\ 0 & \breve{\vartheta}_i & 0 \\ 0 & 0 & \frac{21n\breve{\vartheta}_i}{(n,\breve{\vartheta}_i)} \end{pmatrix}.$$

Note that we can calculate the Smith normal of $A_n = \begin{pmatrix} Q_n & 0 \\ 0 & P_n \end{pmatrix}$ with the help of SNF calculated for Q_n and P_n.

Theorem 4.1.2. For $n = 2i+1$(odd) and $3 \nmid n$ $S(K_3 \times C_n) = \mathbb{Z}_{(n,\check{\mu}_i)} \oplus (\mathbb{Z}_{\check{\mu}_i})^3 \oplus \mathbb{Z}_{\frac{n\check{\mu}_i}{(n,\check{\mu}_i)}}$ and when $3 \mid n$ then, $S(K_3 \times C_n) = \mathbb{Z}_{\frac{(n,\check{\mu}_i)}{3}} \oplus (\mathbb{Z}_{\check{\mu}_i})^2 \oplus (\mathbb{Z}_{3\check{\mu}_i}) \oplus \mathbb{Z}_{\frac{3n\check{\mu}_i}{(n,\check{\mu}_i)}}$.
For $n = 2i$(even) and $3 \nmid n$ then $S(K_3 \times C_n) = \mathbb{Z}_{(n,\check{\vartheta}_i)} \oplus (\mathbb{Z}_{\check{\vartheta}_i})^2 \oplus \mathbb{Z}_{7\check{\vartheta}_i} \oplus \mathbb{Z}_{\frac{7n\check{\vartheta}_i}{(n,\check{\vartheta}_i)}}$.
Similarly, when $3 \mid n$ $S(K_3 \times C_n) = \mathbb{Z}_{\frac{(n,\check{\vartheta}_i)}{3}} \oplus (\mathbb{Z}_{\check{\vartheta}_i})^2 \oplus (\mathbb{Z}_{21\check{\vartheta}_i}) \oplus \mathbb{Z}_{\frac{21n\check{\vartheta}_i}{(n,\check{\vartheta}_i)}}$,

Proof.

- For$n = 2i + 1$(odd) and $3 \nmid n$

$$\begin{pmatrix} Q_n & 0 \\ 0 & P_n \end{pmatrix} = \begin{pmatrix} (n,\check{\mu}_i) & 0 & 0 & 0 & 0 \\ 0 & \check{\mu}_i & 0 & 0 & 0 \\ 0 & 0 & \frac{n\check{\mu}_i}{(n,\check{\mu}_i)} & & \\ 0 & 0 & 0 & \check{\mu}_i & 0 \\ 0 & 0 & 0 & 0 & 3\check{\mu}_i \end{pmatrix}.$$

$S(K_3 \times C_n) = \mathbb{Z}_{(n,\check{\mu}_i)} \oplus (\mathbb{Z}_{\check{\mu}_i})^3 \oplus \mathbb{Z}_{\frac{n\check{\mu}_i}{(n,\check{\mu}_i)}}$

- For $n = 2i + 1$(odd) and $3 \mid n$.

$$\begin{pmatrix} Q_n & 0 \\ 0 & P_n \end{pmatrix} = \begin{pmatrix} \frac{(n,\check{\mu}_i)}{3} & 0 & 0 & 0 & 0 \\ 0 & \check{\mu}_i & 0 & 0 & 0 \\ 0 & 0 & \frac{3n\check{\mu}_i}{(n,\check{\mu}_i)} & 0 & 0 \\ 0 & 0 & 0 & \check{\mu}_i & 0 \\ 0 & 0 & 0 & 0 & 3\check{\mu}_i \end{pmatrix}.$$

$S(K_3 \times C_n) = \mathbb{Z}_{\frac{(n,\check{\mu}_i)}{3}} \oplus (\mathbb{Z}_{\check{\mu}_i})^2 \oplus (\mathbb{Z}_{3\check{\mu}_i}) \oplus \mathbb{Z}_{\frac{3n\check{\mu}_i}{(n,\check{\mu}_i)}}$.

- For n=2i(even) and $3 \nmid n$

$$\begin{pmatrix} Q_n & 0 \\ 0 & P_n \end{pmatrix} = \begin{pmatrix} (n,\check{\vartheta}_i) & 0 & 0 & 0 & 0 \\ 0 & \check{\vartheta}_i & 0 & 0 & 0 \\ 0 & 0 & \frac{7n\check{\vartheta}_i}{(n,\check{\vartheta}_i)} & 0 & 0 \\ 0 & 0 & 0 & \check{\vartheta}_i & 0 \\ 0 & 0 & 0 & 0 & 21\check{\vartheta}_i \end{pmatrix}.$$

$S(K_3 \times C_n) = \mathbb{Z}_{(n,\check{\vartheta}_i)} \oplus (\mathbb{Z}_{\check{\vartheta}_i})^2 \oplus \mathbb{Z}_{21\check{\vartheta}_i} \oplus \mathbb{Z}_{\frac{7n\check{\vartheta}_i}{(n,\check{\vartheta}_i)}}$.

- For $n = 2i$(even) and $3 \mid n$

$$\begin{pmatrix} Q_n & 0 \\ 0 & P_n \end{pmatrix} = \begin{pmatrix} \frac{(n,\check{\vartheta}_i)}{3} & 0 & 0 & 0 & 0 \\ 0 & \check{\vartheta}_i & 0 & 0 & 0 \\ 0 & 0 & \frac{21n\check{\vartheta}_i}{(n,\check{\vartheta}_i)} & 0 & 0 \\ 0 & 0 & 0 & \check{\vartheta}_i & 0 \\ 0 & 0 & 0 & 0 & 21\check{\vartheta}_i \end{pmatrix}$$

$S(K_3 \times C_n) = \mathbb{Z}_{\frac{(n,\check{\vartheta}_i)}{3}} \oplus (\mathbb{Z}_{\check{\vartheta}_i})^2 \oplus (\mathbb{Z}_{21\check{\vartheta}_i}) \oplus \mathbb{Z}_{\frac{21n\check{\vartheta}_i}{(n,\check{\vartheta}_i)}}$.

□

Bibliography

[1] A. A. Jarai, Sandpile Models, *arXiv:1401.0354v2(2014)*.

[2] B. Bauslag, B. Chandler, Schaum's Outline of Theory and Problems of Group Theory, (1968).

[3] D. Dhar, The Abelian Sandpile and Related Models. *ArXiv:cond-mat/9808047(1998)*.

[4] D. Dhar, Self-organized Critical State of Sandpile Automaton Models. Phys. Rev. Lett.64 (1990) 1613—1616.

[5] D. S. Dummit, R. M. Foote, Abstract Algebra, 3rd Edition (June 2003).

[6] J. A. Bondy and U. S. R Murty(2010), Graph Theory, *Mathematics Subject classification* ,(2000) 1-99.

[7] J. S. Milne, Group Theory, www.jmilne.org/math/(2013) 1–20.

[8] K. Ruohonen, Graph Theory, (Translation by Janne Tamminen, Kung-Chung Lee and Robert Piché) (2013).

[9] N. Deo, Graph Theory with Applications to Engineering and Computer Science, *Prentice Hall of India*, (2004), 41–43.

[10] P. Bak, C. Tang, K. Wiesenfled, Self-organized criticality: an explanation of I/f noise, Phys. Rev. Lett. 59, (1987), 381–384.

[11] R. Cori and D. Rossin, On the Sandpile Group of a Graph, European J. Combin. 21 (2000) 447–459.

[12] Scott M. Lalonde, Notes on Abstract Algebra,(2013).

[13] W. Chen, T. Schedler, Concrete and Abstract Structure of the Sandpile Group for Thick Trees with Loops. *arXiv:math/0701381v1(2007)*.

[14] Y. Hou, C. Woo, P. Chen, On the Sandpile Group of C_n^2, *Linear Algebra Appl.* 418 (2006), 457–467.

[15] Y. Hou, T. Lei and C. Woo, On the Sandpile Group of $K_3 \times C_n$, *Linear Algebra Appl.* 418 (2008), 1886–1898.