Compactified Non-Singular Coordinates and Foliation of Some Black Hole Spacetimes

by

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Abstract

The foliation of a manifold is its decomposition into disjoint pieces all having same dimension. The concept of a *moment of time* is ambiguous in general relativity and is replaced by a *time slice* or *three dimensional spacelike hypersurface.* So, in order to define a globally accepted time parameter, we slice up (foliate) the spacetime. In this thesis a complete foliation upto the inner horizon of the Carter-Penrose diagram of the Schwarzschild spacetime surrounded by quintessence (SHQ) with the hypersurfaces having constant mean extrinsic curvature is obtained. For this purpose, we first discussed the compactified Kruskal-Szekeres like coordinates for the SHQ geometry and have presented its maximally extended Carter-Penrose diagram.

In the last, we have obtained the compactified non-singular Kruskal-Szekeres like coordinates for the geometry outside the Reissner-Nordstrom black hole surrounded by quintessence (RNQ) for the case when it has three distinct real horizons.

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Introduction

Foliation of a space is splitting of the space into disjoint pieces whose union comprises the whole space. A sequence of one lower dimensional subspaces is generally known as hypersurfaces. Einstein unify space and time into spacetime and this unification led to commendable theories of special relativity and general relativity. These theories help out to understand the universe. In order to model behavior of the universe as a whole, we apply general relativity. To do this, we usually make some far-reaching assumptions. For example, the universe is isotropic or homogeneous etc. Evidently, matter distribution is highly irregular (non homogeneous) on small scales but on large scale it looks more and more uniform (homogeneous). So, we have good physical background to study simple cosmological models with the assumption of homogeneity and isotropy of the universe, which led to the *cosmological principle*, which states that, the universe looks the same at all times to all observers [1]. There is no ambiguity in the concept of a given moment of time in Newtonian theory while in Special relativity because of relativity of simultaneity there is some [2], but the concept become precise after the specification of an inertial frame. In case of General relativity unless the spacetime is flat there are no global inertial frames [3] and hence, the concept of a given moment of time is thoroughly vague. In general relativity the concept of a *moment of time* is replaced by a *time slice* or *three di-*

mensional spacelike hypersurface. So, in order to define a globally accepted time parameter, we slice up (foliate) the spacetime by a sequence of disjoint spacelike hypersurfaces. Hence, a particular time means a particular spacelike hypersurface. The hypersurfaces, $t =$ constant, may be constructed in a number of ways and there is no preferred way of slicing and consequently no preferred time. An unsuccessful attempt to obtain maximal slicing, foliation by hypersurfaces of zero mean extrinsic curvature, of the Schwarzschild spacetime was made by A. Lichnerowicz [4]. York in [5] defined a time parameter which is proportional to the mean extrinsic curvature; on the basis of which one may expect a foliation of a spacetime by constant mean extrinsic curvature (K) hypersurfaces. In [6] Brill *et.al* have provided a comprehensive discussion of foliation of static, spherically symmetric spacetimes by hypersurfaces of constant mean extrinsic curvature (named as K-surfaces) but, were unsuccessful to completely foliate the Schwarzschild spacetime as they had just used the non-negative values of K and had proposed that a use of full range of values might provide with a complete foliation. A complete foliation of the Schwarzschild geometry is provided in [7] and foliation of the Reissner-Nordstrom spacetime in [8]. The plan of the thesis is as follows:

In Chapter 1, the Schwarzschild black hole, its singularities, and non-singular compactified coordinates for the geometry are discussed in detail.

In Chapter 2, we have constructed the compactified Kruskal-Szekeres like coordinates for the geometry of the Schwarzschild black hole surrounded by quintessence (SHQ) and have presented its Carter-Penrose diagram.

In Chapter 3, K-surfaces, the procedure for the complete foliation of the Schwarzschild (SH) and the Reissner-Nordstrom (RN) spacetimes upto the inner horizon are reviewed. In the last section of this chapter, complete foliation of the Schwarzschild black hole surrounded by quintessence (SHQ) upto the inner horizon is presented.

In Chapter 4, compactified non-singular coordinates for the geometry of the Reissner-Nordstrom black hole surrounded by quintessence are constructed for the case when it has three distinct horizons.

Chapter 1

Black hole and its singularities

1.1 Black Hole

The thermonuclear reactions in a star generate pressure due to which star supports itself against gravity. These are not forever continuing reactions. When the nuclear fuel is exhausted the pressure reduces which break down its balance with gravity as a result of which the star starts contracting. In case the mass of the star is enough such that all outward acting forces get overcome by the inward acting force of gravity then the collapse continues. Due to which the volume of the star continues to decrease and consequently its density continues to increase, so that within a trapped surface escape velocity exceeds the velocity of light. The star is then said to become a *black hole* (this name was given by J.A. Wheeler in 1968). Now the question arises, what would be the geometry of space surrounding a black hole? Karl Schwarzschild, a German physicist, gave the description of the geometry of space surrounding a point mass and calculated the value of radial parameter, r, of the trapped surface, known as the *Schawarzschild radius*. In this way he discovered a black hole theoretically but had no idea of the physical existence of such an object. The main problem in observing black holes is that they can not be seen directly. In 1939, J.R Oppenheimer and H.Synder [9] explained that a sufficiently massive and cold star must collapse indefinitely.

Newtonian theory of gravity can not be employed to describe a black hole as it is restricted to the situation when the speeds are small (i.e not comparable to the speed of light, $c = 3 \times 10^8 m/s$) and gravity is week. It may be employed to describe the motion of a thrown ball or a car e.t.c. While special theory of relativity gives good approximation when employed to the objects having speed comparable to the speed of light but of very small masses. Hence, Einstein general theory of relativity (GR) and black holes are linked together.

After the formation of a black hole, there arises quite interesting questions like, what happens to the matter that makes a black hole? What lies at its center? The GR predicts that inside the black hole, matter keeps on collapsing till the time it has no volume but it still has mass i.e its density is infinite (as $\rho = m/v$). This is the theoretical end of the black hole and is known as the *essential singularity* of the black hole.

The first exact solution of the Einstein field equations (EFE) for gravitational fields that is spherically symmetric and static was obtained by Karl Schwarzschild in 1916 and is known as the *Schwarzschild black hole.* Reissner-Nordstrom [10, 11] obtained the metric describing the spacetime outside a static, spherically symmetric charged black hole, known as the *Reissner-Nordstrom black hole.* Roy Kerr [12] gave a solution of EFE for an axially symmetric rotating body. The solution describes a spinning blackhole, known as the *Kerr black hole.* Kerr and Newmann obtained a metric that describes a charged spinning black hole known as the *Kerr-Newmann* *black hole.*

In this chapter we talk about (in detail) the Schwarzschild black hole, its singularities and some coordinate systems adequate to cope with the metric describing the spacetime around the Schwarzschild black hole. At first, we review the derivations of the Einstein field equations.

1.2 The Einstein Field Equations

Geometry tells matter how to move, and matter tells geometry how to curve.

The basic idea of Einstein's gravitation theory lies in geometrizing the gravitational force [13], which maps all the properties of gravitational force and its influence upon physical processes to the properties of curved spacetime. We start our discussion by introducing the concept of covariant derivative.

1.2.1 Covariant Derivative

In case of flat space and inertial coordinates, the partial derivative operator, ∂_μ, is defined as a map given by

$$\partial_\mu : V_l^k \to V_{l+1}^k, \tag{1.2.1}$$

where V_l^k and V_{l+1}^k are tensor fields of rank $k+l$ and $k+l+1$ respectively. The map given in expression (1.2.1) satisfies the following properties:

1. Acts lineraly on its arguments.

2. Obeys Leibniz rule on tensor product.

The action of the partial derivative operator depends upon the choice of the coordinate system used. We want to define a covariant derivative operator denoted by ∇ to perform the same function as given in expression (1.2.1), but it should be independent of the coordinate system being used.

We begin our discussion by requiring that ∇ be a map satisfying properties (1)-(2) and along with that it should reduce to the partial derivative when applied on scalars i.e

$$\nabla_\nu \phi = \phi_{,\nu},$$

where ϕ is any scalar function. Let us consider the covariant derivative of a vector V^μ, for each direction ν. The covariant derivative ∇_ν is expressed as partial derivative ∂_ν plus a correction, $\Gamma^\rho_{\mu\sigma}$, where $\Gamma^\rho_{\mu\sigma}$ are known as the *connection coefficients* or the *Christofel symbols* and are given by

$$\Gamma^a_{bc} = \frac{1}{2} g^{al} \left(g_{bl,c} + g_{cl,b} - g_{bc,l} \right), \tag{1.2.2}$$

where g_{bc}, the metric tensor, gives the arclength of displacement vector dx^b

$$ds^2 = g_{bc} dx^b dx^c. \tag{1.2.3}$$

Also g^{bc} is the inverse matrix of g_{bc}, so that

$$g^{bc} g_{bd} = \delta^c_d, \tag{1.2.4}$$

and

$$g_{bc,d} = \frac{\partial(g_{bc})}{\partial x^d}.$$

So, finally we have

$$\nabla_\nu V^\mu = \partial_\nu V^\mu + \Gamma^\mu_{\nu\alpha} V^\alpha, \tag{1.2.5}$$

where repeated indices indicate summation (Einstein summation convention). The transformation laws for the connection coefficients are given by

$$\Gamma^{\acute{\mu}}_{\acute{\nu}\acute{\lambda}} = \frac{\partial x^\nu}{\partial x^{\acute{\nu}}} \frac{\partial x^\lambda}{\partial x^{\acute{\lambda}}} \frac{\partial x^{\acute{\mu}}}{\partial x^\mu} \Gamma^\mu_{\nu\lambda} + \frac{\partial x^\nu}{\partial x^{\acute{\nu}}} \frac{\partial x^\lambda}{\partial x^{\acute{\lambda}}} \frac{\partial^2 x^{\acute{\mu}}}{\partial x^\nu \partial x^\lambda}, \tag{1.2.6}$$

which is obviously not the way in which tensor transforms. So, the connection coefficients are non tensorial. It was purposely done, so, that the combination given by eq.(1.2.5) transforms like tensor.

In general

$$\begin{aligned} \nabla_\alpha T^{\nu_1\nu_2\ldots\nu_k}{}_{\mu_1\mu_2\ldots\mu_l} = T^{\nu_1\nu_2\ldots\nu_k}{}_{\mu_1\mu_2\ldots\mu_l,\alpha} + \Gamma^{\nu_1}_{\alpha\beta} T^{\beta\nu_2\ldots\nu_k}{}_{\mu_1\mu_2\ldots\mu_l} + \Gamma^{\nu_2}_{\alpha\beta} T^{\nu_1\beta\ldots\nu_k}{}_{\mu_1\mu_2\ldots\mu_l} \\ -\Gamma^\beta_{\mu_1\alpha} T^{\nu_1\nu_2\ldots\nu_k}{}_{\beta\mu_2\ldots\mu_l} - \Gamma^\beta_{\mu_2\alpha} T^{\nu_1\nu_2\ldots\nu_k}{}_{\mu_1\beta\ldots\mu_l}, \end{aligned} \tag{1.2.7}$$

eq.(1.2.7) is the general expression for the covariant derivative of a mixed tensor of rank $k + l$.

1.2.2 Parallel Transport

We can think of covariant derivative as somehow the measurement of rate of change of tensors. It quantifies the instantaneous rate of change of a tensor field in comparison of what the tensor would be if it were parallel transported i.e. a connection describes a particular way to keep a tensor constant along some specific path due to which we are able to compare nearby tensors (where, by comparing, we mean to add, to subtract, to take the dot product and so on).

In case of flat space, it is unnecessary to be careful that vectors are elements of tangent spaces which are defined at individual points. It is because moving of a vector from one point to another while keeping it constant make sense in flat space. Thus, after getting the vector from one point to another, we can perform the usual operations allowed in a vector space. The concept of moving a vector along a path while keeping it constant is known as *parallel transport*. The main difference between curved and flat space is that, in case of curved space, the parallel transportation of a vector from one point to another will be dependent on the path adopted between the two points. So, we can conclude that the parallel transport can be thought of as curved space generalization of the concept of *keeping the vector constant* as it moves along a path. Hence, for a given curve $x^\nu(\gamma)$ in flat space the consistency requirement of a tensor $V^{\alpha_1\alpha_2\ldots\alpha_k}{}_{\beta_1\beta_2\ldots\beta_l}$ along the curve is simply equivalent to the requirement that the components of tensor be constant i.e.

$$\frac{d}{d\gamma}V^{\alpha_1\alpha_2\ldots\alpha_k}{}_{\beta_1\beta_2\ldots\beta_l} = \frac{dx_\nu}{d\gamma}\frac{\partial}{\partial x^\nu}. \tag{1.2.8}$$

To make the result given by eq.(1.2.8) in tensorial form, we simply replace the partial derivative by covariant derivative. Hence, the direction covariant derivative is defined to be

$$\frac{D}{d\gamma} = \frac{dx^\nu}{d\gamma}\nabla_\nu. \tag{1.2.9}$$

It is a map which is defined only along the path and is between V_l^k to V_{l+1}^k tensor spaces. So, we can define the parallel transport of tensor $\underline{V}$ along the path $x^\nu(\gamma)$ to be equivalent to the requirement that its covariant derivative vanishes along the path i.e

$$\left(\frac{D}{d\gamma}T\right)^{\alpha_1\alpha_2\ldots\alpha_k}{}_{\beta_1\beta_2\ldots\beta_l} = \frac{dx^\lambda}{d\gamma}\nabla_\lambda T^{\alpha_1\alpha_2\ldots\alpha_k}{}_{\beta_1\beta_2\ldots\beta_l} = 0, \tag{1.2.10}$$

eq.(1.2.10) is known as the *equation of parallel transport*. In case of a vector $\underline{V} \in V_0^1$ it takes the form

$$\frac{d}{d\lambda}V^\nu + \Gamma^\nu_{\alpha\rho}\frac{dx^\alpha}{d\lambda}V^\rho = 0. \tag{1.2.11}$$

1.2.3 Geodesics

If we consider a triangle drawn on the surface of a curved sheet which is embedded in 3-dimensional flat space, then the interior angles of the triangle do not sum up to 180^o. It is due to the reason that on a curved space it is not possible to draw actual straight lines. However, on such a surface we can draw straight lines in the sense of shortest path between two points on the surface. Hence, we can say that a geodesic is a generalization of the notion of the straight line in Euclidean space to curved space.

A straight line is the path of the shortest distance between two points. This definition can be equivalently translated as a path that parallel transports its own tangent vector. Hence, we can say that a geodesic is a curve along which the tangent vector is parallel translated. Since, $\frac{dx^\nu}{d\lambda}$ is the tangent vector to the path $x^\nu(\lambda)$, so, above condition reads as

$$\frac{D}{d\lambda}\frac{dx^\nu}{d\lambda} = 0,$$

or

$$\frac{d^2x^\nu}{d\lambda^2} + \Gamma^\nu_{\rho\sigma}\frac{dx^\rho}{d\lambda}\frac{dx^\sigma}{d\lambda} = 0. \tag{1.2.12}$$

eq.(1.2.12) is known as the *geodesics equation.*

1.2.4 The Riemann Curvature Tensor

In a curved space, if a vector is displaced parallel from one point to another, then this parallel displacement is path dependent. Particularly, if we parallel displace a vector to itself along some closed contour, then it will not coincide with its original value upon returning to its initial point.

During the derivation of general expression for the change caused in a vector after its parallel displacement around any closed infinitesimal contour, we obtain the curvature tensor [14]. It measures the deviation of a given space from that of the Minkwoski space. Or, we can say that the non vanishing or vanishing of the curvature tensor is a criterion that whether a space is curved or flat. In the description and understanding of the geometry of curved space not only the curvature tensor but other tensors and scalars, which can be constructed from it are of great importance. They give a complete description of gravitation.

The mathematical apparatus required for curvature tensor is second covariant derivative of a covariant vector.

$$V^{a}_{\;;c;d} = V^{a}_{\;c,d} + \Gamma^{a}_{df} V^{f}_{\;,c} - \Gamma^{g}_{dc} V^{a}_{\;,g} + \Gamma^{a}_{bc,d} V^{b} + \Gamma^{a}_{df} \Gamma^{f}_{bc} V^{b}$$

$$-\Gamma^{h}_{db} \Gamma^{a}_{hc} V^{b} - \Gamma^{g}_{dc} \Gamma^{a}_{bg} V^{b} + \Gamma^{a}_{bc} V^{b}_{,d} + \Gamma^{a}_{bc} \Gamma^{b}_{dh} V^{h}, \qquad (1.2.13)$$

using eq.(1.2.13), we have

$$V^{a}_{\;;c;d} - V^{a}_{\;;d;c} = \left(\Gamma^{a}_{bd,c} - \Gamma^{a}_{bc,d} + \Gamma^{e}_{bd} \Gamma^{a}_{ce} - \Gamma^{e}_{bc} \Gamma^{a}_{de} \right) V^{b},$$

$$= R^{a}_{\;bcd} V^{b}.$$

The tensor

$$R^{a}_{\;bcd} = \Gamma^{a}_{bd,c} - \Gamma^{a}_{bc,d} + \Gamma^{e}_{bd} \Gamma^{a}_{ce} - \Gamma^{e}_{bc} \Gamma^{a}_{de}, \qquad (1.2.14)$$

is known as the *Riemann curvature tensor*. It has certain important symmetries. To study them it would be better if we consider the covariant Riemann curvature tensor i.e

$$R_{abcd} = g_{al} R^{l}_{\;bcd}, \qquad (1.2.15)$$

or equivalently

$$R_{abcd} = \frac{1}{2}\left(g_{bc,da} + g_{da,bc} - g_{bd,ac} - g_{ca,bd}\right) + g_{ef}\left(\Gamma^{f}_{da}\Gamma^{e}_{cb} - \Gamma^{f}_{ca}\Gamma^{e}_{db}\right), \qquad (1.2.16)$$

The following symmetry properties directly follows from eq.(1.2.16)

1. Skew symmetry in first two indices $R_{abcd} = -R_{bacd}$.

2. Skew symmetry in last two indices $R_{abcd} = -R_{abdc}$.

3. Block symmetry $R_{abcd} = R_{cdab}$.

Along with the above stated algebraic symmetries, the cyclic sum of any those components of R_{abcd} which are formed by permutations of any three fixed indices is zero. i.e.

$$R_{abcd} + R_{adbc} + R_{acdb} = 0. \qquad (1.2.17)$$

There are on maximum $\frac{1}{12}n^2(n^2-1)$ independent components of the Riemann curvature tensor, where n is the dimension of the manifold.

The contraction of the Riemann curvature tensor gives another tensor of rank two given as

$$R_{bd} = R^{c}_{bcd} = g^{ac}R_{abcd}, \qquad (1.2.18)$$

or

$$R_{bd} = \Gamma^{e}_{bd,e} - \Gamma^{e}_{be,d} + \Gamma^{e}_{bd}\Gamma^{f}_{fe} - \Gamma^{e}_{bf}\Gamma^{f}_{de}, \qquad (1.2.19)$$

R_{bd} is known as the *Ricci tensor.* It is straight forward to see that the Ricci tensor is symmetric i.e.

$$R_{bd} = R_{db}. \tag{1.2.20}$$

On taking the trace of the Ricci tensor, we get a scalar known as the *Ricci* or the *curvature scalar* given by

$$R = R^b{}_b = g^{bd} R_{bd}. \tag{1.2.21}$$

1.2.5 The Bianchi Identities

In addition to the algebraic symmetries, the Riemann curvature tensor satisfies some of the differential identities. They can be proved easily at a given point x^μ by using the geodesic coordinate system. In such a coordinate system at a given point Christoffel symbols (but not there derivatives) vanish.

So after differentiating R^a_{bcd} and substituting the Christoffel symbols equal to zero, we get at the point under consideration

$$R^a{}_{bcd;f} = \Gamma^a_{bd,cf} - \Gamma^a_{bc,df}. \tag{1.2.22}$$

After permuting the indices, c, d and f in eq.(1.2.22) (defined at a point) and adding the corresponding terms, we get

$$R^a{}_{bcd;f} + R^a{}_{bdf;c} + R^a{}_{bfc;d} = 0. \tag{1.2.23}$$

Since, each term of eq.(1.2.23) represents component of a tensor, so, this equation will hold in any coordinate system and at any given point. The above equation, eq.(1.2.23) is known as the *Bianchi Identity*.

On contracting indices c and a in eq.(1.2.23) and using the symmetry property of the Riemann curvature tensor, we have

$$R_{bd;f} - R_{bf;d} + R^{a}{}_{bdf;a} = 0.$$

After contracting indices b and d, we have

$$R^{d}{}_{f;d} - \frac{1}{2}\delta^{d}{}_{f} R_{;d} = 0,$$

or

$$\varepsilon^{d}{}_{f;d} = 0,$$

where

$$\varepsilon^{d}{}_{f} = R^{d}_{f} - \frac{1}{2}\delta^{d}_{f} R, \qquad (1.2.24)$$

or

$$\varepsilon_{df} = R_{df} - \frac{1}{2} R g_{df}, \qquad (1.2.25)$$

is known as the *Einstein tensor*.

The symmetry of the Einstein tensor follows from the symmetry of the metric tensor and the Ricci tensor i.e

$$\varepsilon_{df} = \varepsilon_{fd}. \tag{1.2.26}$$

1.2.6 Energy Momentum Tenor

General relativity deals with gravitational field and this field depends upon the spatial distribution of matter and on its temporal evolution. Hence, we require a mathematical description of the matter distribution in spacetime.

The action integral is

$$I_M = \frac{1}{2\kappa}\int L_M\sqrt{-g}d^4x, \tag{1.2.27}$$

where

L_m : Lagrangian density for all the fields with the exception of gravitational field.

κ : Constant.

From variational principle, we have

$$\delta I_M = 0. \tag{1.2.28}$$

Varying the action w.r.t. $g^{\mu\nu}$ and its derivative as $L_M = L_M[g^{\alpha\beta}, g^{\alpha\beta}_{,\gamma}]$, we get

$$\delta I_M = \int \delta(L_M\sqrt{-g})d^4x, \tag{1.2.29}$$

$$= \int \frac{\partial}{\partial g^{\alpha\beta}}(L_M\sqrt{-g})\delta g^{\alpha\beta}d^4x + \int \frac{\partial(L_M\sqrt{-g})}{\partial {g^{\alpha\beta}}_{,\gamma}}\delta g^{\alpha\beta}_{,\gamma}d^4x. \tag{1.2.30}$$

Since

$$\left[\frac{\partial L_M\sqrt{-g}}{\partial g^{\alpha\beta}{}_{,\gamma}}\delta g^{\alpha\beta}\right]_{,\gamma} = \left(\frac{\partial L_M\sqrt{-g}}{\partial g^{\alpha\beta}{}_{,\gamma}}\right)_{,\gamma}\delta g^{\alpha\beta} + \frac{\partial L_M\sqrt{-g}}{\partial g^{\alpha\beta}{}_{,\gamma}}\delta g^{\alpha\beta}_{,\gamma}, \tag{1.2.31}$$

from where we have

$$\frac{\partial L_M\sqrt{-g}}{\partial g^{\alpha\beta}{}_{,\gamma}}\delta g^{\alpha\beta}{}_{,\gamma} = \left[\frac{\partial L_M\sqrt{-g}}{\partial g^{\alpha\beta}{}_{,\gamma}}\delta g^{\alpha\beta}\right]_{,\gamma} - \left[\frac{\partial L_M\sqrt{-g}}{\partial g^{\alpha\beta}{}_{,\gamma}}\right]_{,\gamma}\delta g^{\alpha\beta}. \tag{1.2.32}$$

Using eq.(1.2.32) in eq.(1.2.30), we have

$$\delta I_M = \int \frac{\partial L_M\sqrt{-g}}{\partial g^{\alpha\beta}}\delta g^{\alpha\beta}d^4x + \int\left[\frac{\partial L_M\sqrt{-g}}{\partial g^{\alpha\beta}{}_{,\gamma}}\delta g^{\alpha\beta}\right]_{,\gamma}d^4x - \int\left[\frac{\partial L_M\sqrt{-g}}{\partial g^{\alpha\beta}{}_{,\gamma}}\right]_{,\gamma}\delta g^{\alpha\beta}d^4x.$$

by the Gauss divergence theorem, we can neglect the middle term, so, we have

$$\delta I_M = \int\left\{\frac{\partial L_M\sqrt{-g}}{\partial g^{\alpha\beta}}\delta g^{\alpha\beta} - \int\left[\frac{\partial L_M\sqrt{-g}}{\partial g^{\alpha\beta}{}_{,\gamma}}\right]_{,\gamma}\delta g^{\alpha\beta}\right\}d^4x.$$

After multiplying and dividing by $\frac{-2}{\sqrt{-g}}$, we get

$$\delta I_M = -\frac{1}{2}\int\frac{-2}{\sqrt{-g}}\left\{\frac{\partial L_M\sqrt{-g}}{\partial g^{\alpha\beta}} - \left[\frac{\partial L_M\sqrt{-g}}{\partial g^{\alpha\beta}{}_{,\gamma}}\right]_{,\gamma}\right\}\sqrt{-g}\delta g^{\alpha\beta}d^4x,$$

or

$$\delta I_M = -\frac{1}{2}\int T_{\alpha\beta}\sqrt{-g}\delta g^{\alpha\beta}d^4x, \tag{1.2.33}$$

where

$$T_{\alpha\beta} = \frac{-2}{\sqrt{-g}} \left\{ \frac{\partial L_M \sqrt{-g}}{\partial g^{\alpha\beta}} - \left[\frac{\partial L_M \sqrt{-g}}{\partial {g^{\alpha\beta}}_{,\gamma}} \right]_{,\gamma} \right\}. \tag{1.2.34}$$

We require that

$${T^{\alpha\beta}}_{;\beta} = 0.$$

The requirement of stress energy momentum tensor to be divergence free is equivalent to the requirement of the law of conservation of mass-energy and momentum.

1.2.7 The Einstein Field Equations (EFE)

The basis of Einstein gravitational theory lies in *geometrizing the gravitational force*, which means to map the influence of the gravitational force upon physical processes and the properties of the gravitational force onto that of a Riemannian space. In the context of general relativity, every thing that can produce a gravitational field is termed as *matter* (e.g. not only the electrons and atomic nuclei but also the electromagnetic field). Einstein postulated that in a given spacetime a particle will travel on the straightest path available to it and is known as its *geodesic*. This law should be a replacement for Newton's laws of motion. The basic idea behind the derivation of the Einstein field equations is that the extend of straightness of path depends upon the curvature of the spacetime, therefore, gravitation might also be expressed in terms of spacetime curvature. In Newtonian limits, the only source of the gravitational field is mass density and according to the special relativity there is no essential difference between energy and matter. Hence, from all the above stated arguments, we can conclude that in relativistic terms, curvature of spacetime must be related to the presence of matter-energy. The Einstein field equations describe this relation. Here we will review the derivation of the Einstein field equations as derived by Hilbert (1915) by using variational principle. The action, S, with which we have to deal with generally consist of a gravitational part, S_G, and a matter part, S_m, (apart from the action of other present forces, e.g, S_{em}: electromagnetic action).

Hence, we can write

$$S = S_G + S_m,$$

where

$$S_m = \int T_{\mu\nu} \; g^{\mu\nu} d\Omega.$$

Let the action vary due to the variations in $g^{\alpha\beta}$ and $\delta g^{\alpha\beta}$. Also, since the total action must remain invariant, so, we have

$$\delta S = \delta S_m + \delta S_G = 0,$$

$$\int T_{\mu\nu} \, \delta g^{\mu\nu} d\Omega + \delta S_G = 0. \tag{1.2.35}$$

Einstein and Hilbert independently proposed gravitational action as a purely geometric entity given as

$$S_G = -\lambda \int_\Omega \sqrt{|g|} R d\Omega, \tag{1.2.36}$$

where

R: Ricci scalar.

λ: Arbitrary constant.

Hence, the Lagrangian for gravity is

$$L_G[g^{\alpha\beta}, g^{\alpha\beta}_{,\gamma}] = \sqrt{|g|} R = \sqrt{|g|} g^{\alpha\beta} R_{\alpha\beta}, \tag{1.2.37}$$

$$\begin{aligned} L_G[g^{\alpha\beta}, g^{\alpha\beta}_{,\gamma}] &= \sqrt{|g|} g^{\alpha\beta} R_{\alpha\beta} \\ &= \sqrt{|g|} g^{\alpha\beta} \left[\Gamma^\gamma_{\alpha\beta,\gamma} - \Gamma^\gamma_{\alpha\gamma,\beta} + \Gamma^\gamma_{\pi\gamma} \Gamma^\pi_{\alpha\beta} - \Gamma^\gamma_{\pi\alpha} \Gamma^\pi_{\gamma\beta} \right]. \end{aligned}$$

In order to remove the terms involving higher derivatives, consider

$$2\left[\sqrt{|g|} g^{\alpha\beta} \Gamma^\gamma_{\alpha[\beta}\right]_{,\gamma]} = \sqrt{|g|} g^{\alpha\beta} \left[\Gamma^\gamma_{\alpha\beta,\gamma} - \Gamma^\gamma_{\alpha\gamma,\beta} \right] + 2\left[\sqrt{|g|} g^{\alpha\beta}_{[,\beta} \Gamma^\gamma_{\gamma]\alpha}\right], \tag{1.2.38}$$

or

$$\sqrt{|g|} g^{\alpha\beta} \left[\Gamma^\gamma_{\alpha\beta,\gamma} - \Gamma^\gamma_{\alpha\gamma,\beta} \right] = 2\left[\sqrt{|g|} g^{\alpha\beta} \Gamma^\gamma_{\alpha[\beta}\right]_{,\gamma]} + 2\left[\left(\sqrt{|g|} g^{\alpha\beta}\right)_{[,\beta} \Gamma^\gamma_{\gamma]\alpha}\right], \tag{1.2.39}$$

by using generalized Gauss divergence theorem the first term on the R.H.S. of eq.(1.2.1) can be neglected and hence, we have

$$\sqrt{|g|} g^{\alpha\beta} \left[\Gamma^\gamma_{\alpha\beta,\gamma} - \Gamma^\gamma_{\alpha\gamma,\beta} \right] = \left(\sqrt{|g|} g^{\alpha\beta}\right)_{,\beta} \Gamma^\rho_{\rho\mu} - \sqrt{|g|} g^{\alpha\beta}_{,\beta} \Gamma^\rho_{\beta\mu}. \tag{1.2.40}$$

Using eq.(1.2.40) in eq.(1.2.38), we have

$$L_G[g^{\alpha\beta}, g^{\alpha\beta}_{,\gamma}] = \left(\sqrt{|g|}g^{\alpha\beta}\right)_{,\beta}\Gamma^{\gamma}_{\gamma\alpha} - \left(\sqrt{|g|}g^{\alpha\beta}\right)_{,\gamma}\Gamma^{\gamma}_{\alpha\beta} + \sqrt{|g|}g^{\alpha\beta}\left[\Gamma^{\gamma}_{\gamma\pi}\Gamma^{\pi}_{\alpha\beta} - \Gamma^{\gamma}_{\pi\alpha}\Gamma^{\pi}_{\gamma\beta}\right]. \tag{1.2.41}$$

Eq.(1.2.41) can further be simplified by using the following identities

$$(\sqrt{|g|})_{,\gamma} = \sqrt{|g|}(\ln\sqrt{|g|})_{,\gamma} = \sqrt{|g|}\Gamma^{\pi}_{\pi\gamma}, \tag{1.2.42}$$

$$g^{\mu\nu}_{,\rho} = \frac{1}{2}\left\{\Gamma^{\mu}_{\pi\rho}g^{\nu\pi} + \Gamma^{\nu}_{\pi\rho}g^{\mu\pi}\right\}. \tag{1.2.43}$$

Using eqs.(1.2.42) and (1.2.43) in eq.(1.2.41) (after interchanging dummy indices), we get

$$L_G = -\sqrt{|g|}g^{\alpha\beta}\left[\Gamma^{\gamma}_{\alpha\pi}\Gamma^{\pi}_{\beta\gamma} - \Gamma^{\pi}_{\pi\gamma}\Gamma^{\gamma}_{\alpha\beta}\right]. \tag{1.2.44}$$

Now consider δS_G

$$\delta S_G = -\gamma\left[\int_{\Omega}(\delta\sqrt{|g|})Rd\Omega + \sqrt{|g|}\delta Rd\Omega\right]. \tag{1.2.45}$$

As

$$R = R_{\alpha\beta}g^{\alpha\beta},$$

and

$$\delta\sqrt{|g|} = -\frac{1}{2}\sqrt{|g|}g_{\alpha\beta}\delta g^{\alpha\beta},$$

$$\delta S_G = -\gamma\int_\Omega \left[-\frac{1}{2}g_{\alpha\beta}\delta g^{\alpha\beta}R + \sqrt{|g|}R_{\alpha\beta}\delta g^{\alpha\beta} + \sqrt{|g|}(\delta R_{\alpha\beta})g^{\alpha\beta}\right] d\Omega,$$

$$\delta S_G = -\gamma\int_\Omega \left(R_{\alpha\beta} - \frac{1}{2}Rg_{\alpha\beta}\right)\sqrt{|g|}\delta g^{\alpha\beta}d\Omega - \gamma\int_\Omega \sqrt{|g|}g^{\alpha\beta}\delta R_{\alpha\beta}d\Omega. \qquad (1.2.46)$$

On choosing the Riemann normal coordinates i.e.

$$\Gamma^\alpha_{\beta\gamma} \approx 0, \qquad \Gamma^\alpha_{\beta\gamma,\eta} \neq 0,$$

we get

$$g^{\alpha\beta}\delta R_{\alpha\beta} = g^{\alpha\beta}\delta\left[\Gamma^\gamma_{\alpha\beta,\gamma} - \Gamma^\gamma_{\alpha\gamma,\beta}\right] = 0. \qquad (1.2.47)$$

Using eq.(1.2.47) in eq.(1.2.46) we get

$$\delta S_G = -\gamma\int_\Omega \left(R_{\alpha\beta} - \frac{1}{2}Rg_{\alpha\beta}\right)\delta g^{\alpha\beta}d\Omega. \qquad (1.2.48)$$

Substituting eq.(1.2.48) in eq.(1.2.35) and choosing $\gamma = \frac{1}{\kappa}$, we get

$$0 = \int_\Omega \left(T_{\alpha\beta} - \frac{1}{\kappa}\varepsilon_{\alpha\beta}\right)\delta g^{\alpha\beta}d\Omega, \qquad (1.2.49)$$

where $\varepsilon_{\alpha\beta}$ is given by eq.(1.2.25). The integral given by eq.(1.2.49) vanishes only when the integrand is zero i.e.

$$T_{\alpha\beta} = \frac{1}{\kappa}\varepsilon_{\alpha\beta},$$

or

$$\kappa T_{\alpha\beta} = R_{\alpha\beta} - \frac{1}{2}Rg_{\alpha\beta}. \tag{1.2.50}$$

Above are known as the *Einstein field equations.*

1.3 The Schwarzschild Solution

Karl Schwarzschild found the first exact solution of the Einstein field equations in 1916, which on his name is known as the Schwarzschild solution. This solution describes the spacetime geometry outside a static and spherically symmetric matter distribution and is a solution of Einstein's vacuum field equations.

The general form of the metric in polar coordinates (known as the Schwarzschild coordinates) is

$$ds^2 = e^{2\rho(r)}c^2dt^2 - e^{2\mu(r)}dr^2 - e^{2\nu(r)}r^2d\theta^2 - e^{2\lambda(r)}r^2\sin^2\theta d\phi^2. \tag{1.3.1}$$

If we impose the condition of spherically symmetry then we get

$$\nu(r) = \lambda(r),$$

and eq.(1.3.1) gives

$$ds^2 = e^{2\rho(r)}c^2dt^2 - e^{2\mu(r)}dr^2 - e^{2\nu(r)}r^2d\theta^2 - e^{2\nu(r)}r^2\sin^2\theta d\phi^2. \tag{1.3.2}$$

Let $\overline{r} = re^{\nu(r)}$ which gives

$$d\overline{r} = e^{\nu}dr + e^{\nu}rd\nu,$$

or

$$d\overline{r} = e^{\nu}dr\left(1 + r\frac{d\nu}{dr}\right),$$

equivalently

$$dr = e^{-\nu}\left(1 + r\frac{d\nu}{dr}\right)^{-1}d\overline{r}. \tag{1.3.3}$$

using eq.(1.3.3) in eq.(1.3.2), we have

$$ds^2 = e^{2\rho(r)}c^2dt^2 - e^{2\mu(r)-2\nu(r)}\left(1 + r\frac{d\nu}{dr}\right)^{-2}d\overline{r}^2 - \overline{r}^2d\theta^2 - \overline{r}^2\sin^2\theta d\phi^2. \tag{1.3.4}$$

Let us define the following relabeling

$$\overline{r} \to r, \tag{1.3.5}$$

$$e^{2\mu(r)-2\nu(r)}\left(1 + r\frac{d\nu}{dr}\right)^{2} \to e^{2\mu}. \tag{1.3.6}$$

After using the transformations given in expressions (1.3.5) and (1.3.6) in eq.(1.3.4), we get

$$ds^2 = e^{2\rho(r)}c^2dt^2 - e^{2\mu(r)}dr^2 - r^2d\theta^2 - r^2\sin^2\theta d\phi^2. \tag{1.3.7}$$

We have not given a statement about geometry, but have chosen simply our radial coordinate to eradicate the factor $e^{2\nu(r)}$. We can write the metric given in eq.(1.3.7) as

$$ds^2 = e^{\sigma(r)}c^2dt^2 - e^{\gamma(r)}dr^2 - r^2d\theta^2 - r^2\sin^2\theta d\phi^2, \tag{1.3.8}$$

where we have used the following transformations

$$2\rho(r) \to \sigma(r),$$
$$2\mu(r) \to \gamma(r).$$

Using

$$d\Omega^2 = \sin^2\theta d\phi^2 + d\theta^2,$$

eq.(1.3.8) takes the form

$$ds^2 = e^{\sigma(r)}c^2dt^2 - e^{\gamma(r)}dr^2 - r^2d\Omega^2. \tag{1.3.9}$$

In this case the metric tensor and its inverse are given as

$$g_{ab} = \begin{bmatrix} e^{\sigma} & 0 & 0 & 0 \\ 0 & -e^{\gamma} & 0 & 0 \\ 0 & 0 & -r^2 & 0 \\ 0 & 0 & 0 & -r^2 \sin^2\theta \end{bmatrix},$$

$$g^{ab} = \begin{bmatrix} e^{-\sigma} & 0 & 0 & 0 \\ 0 & -e^{-\gamma} & 0 & 0 \\ 0 & 0 & -r^{-2} & 0 \\ 0 & 0 & 0 & \frac{-1}{r^2 \sin^2\theta} \end{bmatrix}.$$

Since, the EFE in vacuum are

$$R_{ab} = 0,$$

so, we get the following three non-vanishing independent equations

$$R_{00} : \sigma'' + \frac{1}{2}\sigma'\left(\sigma' - \gamma'\right) + \frac{2\sigma'}{r} = 0, \tag{1.3.10}$$

$$R_{11} = -\sigma'' - \frac{1}{2}\sigma'\left(\sigma' - \gamma'\right) + \frac{2\gamma'}{r} = 0, \tag{1.3.11}$$

$$R_{22} = 1 - e^{-\sigma} + \frac{1}{2}r\left(\gamma' - \sigma'\right)e^{-\sigma} = 0, \tag{1.3.12}$$

where $R_{33} = R_{22}\sin^2\theta$ and hence is not independent. Addition of eqs.(1.3.10) and (1.3.11) gives

$$\sigma' + \gamma' = 0,$$

or

$$\sigma + \gamma = constant. \tag{1.3.13}$$

We can choose the constant to be zero and can say that we have merged the constant into the units of measurement of time. Hence, we have

$$\sigma = -\gamma. \tag{1.3.14}$$

After using eq.(1.3.14) in eqs.(1.3.11) and (1.3.12), we get

$$e^{\sigma} = 1 + \frac{\alpha}{r}, \tag{1.3.15}$$

and

$$e^{\sigma} = e^{-\gamma}. \tag{1.3.16}$$

Using eqs.(1.3.15) and (1.3.16) in metric equation (1.3.10), we get

$$ds^2 = \left(1 + \frac{\beta}{r}\right)c^2dt^2 - \frac{dr^2}{\left(1 + \frac{\beta}{r}\right)} - r^2d\Omega^2, \tag{1.3.17}$$

to get the value of β, we use the geodesic equation given by eq.(1.2.12). The corresponding four geodesic equations are

$$c\frac{d^2t}{ds^2}+c\sigma'\left(\frac{dt}{ds}\right)\left(\frac{dr}{ds}\right)=0, \tag{1.3.18}$$

$$\frac{d^2r}{ds^2}+\frac{1}{2}c^2\sigma' e^{\sigma-\gamma}\left(\frac{dt}{ds}\right)^2+\frac{1}{2}\gamma'\left(\frac{dr}{ds}\right)^2-re^{-\gamma}\left(\frac{d\theta}{ds}\right)^2 -r\sin^2\theta e^{-\gamma}\left(\frac{d\phi}{ds}\right)^2=0, \tag{1.3.19}$$

$$\frac{d^2\theta}{ds^2}+\frac{2}{r}\left(\frac{dr}{ds}\right)\left(\frac{d\theta}{ds}\right)-\sin\theta cos\theta\left(\frac{d\phi}{ds}\right)^2=0, \tag{1.3.20}$$

$$\frac{d^2\phi}{ds^2}+\frac{2}{r}\left(\frac{dr}{ds}\right)\left(\frac{d\phi}{ds}\right)+2\cot\theta\left(\frac{d\theta}{ds}\right)\left(\frac{d\phi}{ds}\right)=0. \tag{1.3.21}$$

On choosing $\theta=\pi/2$ and hence $\dot{\theta}=0$ initially, eqs.(1.3.18), (1.3.19) and (1.3.21) take the following forms

$$\frac{d^2t}{ds^2}+\sigma'\left(\frac{dt}{ds}\right)\left(\frac{dr}{ds}\right)=0, \tag{1.3.22}$$

$$\frac{d^2r}{ds^2}+\frac{1}{2}c^2\sigma' e^{\sigma-\gamma}\left(\frac{dt}{ds}\right)^2+\frac{1}{2}\gamma'\left(\frac{dr}{ds}\right)^2-re^{-\gamma}\left(\frac{d\phi}{ds}\right)^2=0, \tag{1.3.23}$$

$$\frac{d^2\phi}{ds^2}+\frac{2}{r}\left(\frac{dr}{ds}\right)\left(\frac{d\phi}{ds}\right)=0. \tag{1.3.24}$$

Considering

$$\frac{d\sigma}{ds} = \frac{d\sigma}{dr}\frac{dr}{ds} = \sigma'\frac{dr}{ds},$$

hence, we can write eq.(1.3.22) as

$$\frac{d}{ds}\left(e^{\sigma}\frac{dt}{ds}\right) = 0,$$

or

$$\frac{dt}{ds} = \frac{k}{e^{\sigma}} = \frac{k}{\left(1 + \dfrac{\beta}{r}\right)}. \tag{1.3.25}$$

On requiring asymptotic flatness (sufficiently far from the gravitational source, the spacetime must be the Minkowski one i.e. mathematically saying, as $r \to \infty, \dot{t} \to 1/c$) we have from eq.(1.3.25)

$$c\frac{dt}{ds} = e^{-\sigma}, \tag{1.3.26}$$

eq.(1.3.24) after multiplication with r^2 gives

$$\frac{d}{ds}\left(r^2\frac{d\phi}{ds}\right) = 0,$$

or

$$r^2\frac{d\phi}{ds} = k_1. \tag{1.3.27}$$

Since, classically we know that

$$r^2 \frac{d\phi}{ds} = \frac{h}{c}, \tag{1.3.28}$$

where 'h' is Plank's constant whose value is $6.6260695710^{34} Js$ and 'c' is the speed of light. On comparing eqs.(1.3.27) and (1.3.28), we have

$$k_1 = h/c. \tag{1.3.29}$$

From eq.(1.3.10), we get

$$1 = e^{\sigma} \left(c \frac{dt}{ds} \right)^2 - e^{-\sigma} \left(\frac{dr}{ds} \right)^2 - r^2 \left(\frac{d\phi}{ds} \right)^2 . \tag{1.3.30}$$

After using eqs.(1.3.26) and (1.3.27) into eq.(1.3.30), we have

$$e^{-\sigma} \left(\frac{dr}{ds} \right)^2 = e^{-\sigma} - \frac{h^2/c^2}{r^2} - 1. \tag{1.3.31}$$

Using eqs.(1.3.26) and (1.3.30) and $(e^{\sigma})' = -\beta/r^2$ in eq.(1.3.23), we obtain

$$\frac{d^2r}{ds^2} - \frac{h^2/c^2}{r^3} = \frac{\beta/2}{r^2} + \frac{3}{2} \frac{\beta h^2/c^2}{r^4}. \tag{1.3.32}$$

From eq.(1.3.28)

$$d\phi = \frac{h/c}{r^2} ds,$$

or

$$\frac{d}{ds} = \frac{h/c}{r^2}\frac{d}{d\phi}. \tag{1.3.33}$$

Substituting $r = 1/U$ and using eq.(1.3.33), we have

$$\frac{d^2r}{ds^2} = -\frac{h^2}{c^2}U^2\frac{d^2U}{d\phi^2}. \tag{1.3.34}$$

Using eq.(1.3.34) in eq.(1.3.32), we get

$$\frac{d^2U}{d\phi^2} + U = \left(\frac{-c^2\beta/2}{h^2}\right) - \left(\frac{3\beta}{2}\right)U^2. \tag{1.3.35}$$

Since, the classical equation of motion of a particle in an external gravitational field is

$$\frac{d^2U}{d\phi^2} + U = \frac{Gm}{h^2}. \tag{1.3.36}$$

On comparing eqs.(1.3.35) and (1.3.36), we require that $\beta = -2Gm/c^2$ (since in the limit $r \to \infty$, U^2 is so small that we can neglect it).

The metric given by eq.(1.3.17) takes the form

$$ds^2 = \left(1 - \frac{2Gm}{c^2r}\right)dt^2 - \frac{dr^2}{\left(1 - \frac{2Gm}{c^2r}\right)} - r^2d\Omega^2, \tag{1.3.37}$$

which is known as the *Schwarzschild metric*. It is the most general static, spherically symmetric solution of EFE in a region of spacetime for which '$T_{\mu\nu}$' vanishes. It is known as the Birkhoff's theorem [1]. In case of use of gravitational units i.e

$c = G = 1$, eq.(1.3.37) takes the form

$$ds^2 = \left(1 - \frac{2m}{r}\right) dt^2 - \frac{dr^2}{\left(1 - \frac{2m}{r}\right)} - r^2 d\Omega^2. \quad (1.3.38)$$

1.4 Metric Singularities

If the metric coefficients become infinite at some point then the metric is said to have singularity there. This singularity can be of two types

1. Coordinate singularity.

2. Curvature singularity.

1.4.1 Coordinate Singularity

The singularity which arises merely due to a bad choice of coordinates (by bad choice of coordinates we mean that we have selected such coordinates which have a restricted domain of validity).

Let us consider the example of a 2-sphere of radius 'a', its metric (i.e the line element on its surface) is

$$ds^2 = a^2 d\theta^2 + a^2 \sin^2\theta d\phi^2. \quad (1.4.1)$$

After the coordinate transformation

$$a \sin\theta = \lambda, \quad (1.4.2)$$

the line element given in eq.(1.4.1) takes the form

$$ds^2 = \frac{d\lambda^2}{1-\lambda^2} + \lambda^2 d\phi^2, \tag{1.4.3}$$

which is singular at $\lambda = \pm 1$. This singularity is a coordinate singularity which arose due to a bad choice of coordinates.

1.4.2 Curvature Singularity

The singularity at which the manifold curvature diverges or equivalently, the singularity at which curvature invariants

$$\begin{aligned} I_1 = R = g^{cd}R_{cd}, \\ I_2 = R^{cd}_{ef}R^{ef}_{cd}, \\ I_3 = R^{cd}_{ef}R^{ef}_{gh}R^{gh}_{cd}, \\ \vdots \end{aligned}$$

become infinite is known as *curvature*, *essential* or *intrinsic* singularity.

Singularities of the Schwarzschild Black Hole (SH)

From eq.(1.3.38), it is clear that the SH metric coefficients become infinite at $r = 0$ and $r = 2m$. The question arises whether the singularities at these points are essential or coordinate. Here the curvature invariants at any point are given as [15]

$$I_1 = 0, \tag{1.4.4}$$

$$I_2 = \frac{48m^2}{r^4}, \tag{1.4.5}$$

$$I_3 = \frac{64m^3}{r^6}. \tag{1.4.6}$$

It is evident that the curvature invariants are regular at $r = 2m$ while I_2 and I_3 are infinite at $r = 0$. Also, since they are scalars so their value will remain the same in any coordinate system. Hence, we can say that the spacetime curvature is perfectly well behaved at $r = 2m$ and is singular at $r = 0$. Thus, the singularity at $r = 0$ is an intrinsic singularity while at $r = 2m$ is a coordinate singularity and can be removed by a better choice of coordinates.

To have a better understanding of the coordinate singularity, consider the radial motion of a photon falling freely towards the Schwarzschild radius. The radial geodesics of the photon are given as (obtained by using the Euler Lagrange equations)

$$\frac{dr}{d\tau} = \pm\sqrt{A^2 - \left(1 - \frac{2m}{r}\right)}, \tag{1.4.7}$$

where 'A' is a constant and 'τ' is the proper time of the observer who is at rest in the gravitational field. Using the fact that for null geodesics $g_{cd}\dot{x}^b\dot{x}^d = 0$, which is simply the first integral of the geodesic equation $\ddot{x}^c + \Gamma^c_{de}\dot{x}^d\dot{x}^e = 0$, we have

$$\frac{dt}{d\tau} = \frac{A}{\left(1 - \frac{2m}{r}\right)}. \tag{1.4.8}$$

Using eqs.(1.4.7) and (1.4.8), the coordinate time and proper time are given as

$$t = \int_0^t dt = \int_{r_0}^{2m} \frac{Adr}{\left(1 - \frac{2m}{r}\right)\sqrt{A^2 - \left(1 - \frac{2m}{r}\right)}}, \tag{1.4.9}$$

$$\tau = \int_{r_0}^{2m} \frac{dr}{\sqrt{A^2 - \left(1 - \frac{2m}{r}\right)}}. \tag{1.4.10}$$

It is clear from eqs.(1.4.9) and (1.4.10) that an infinitely long coordinate time is required to cover a finite distance, while proper time required for the same finite distance is finite. Hence, we can conclude that the freely falling photon will not notice any special thing at $r = 2m$, in fact the coordinates 't' and 'r' are unsuitable for the description of its motion. This problem was first resolved by Eddington in 1924 and was rediscovered by Finkelstein in 1958. The coordinates that they developed are known as the *Eddington-Finklestein coordinates*.

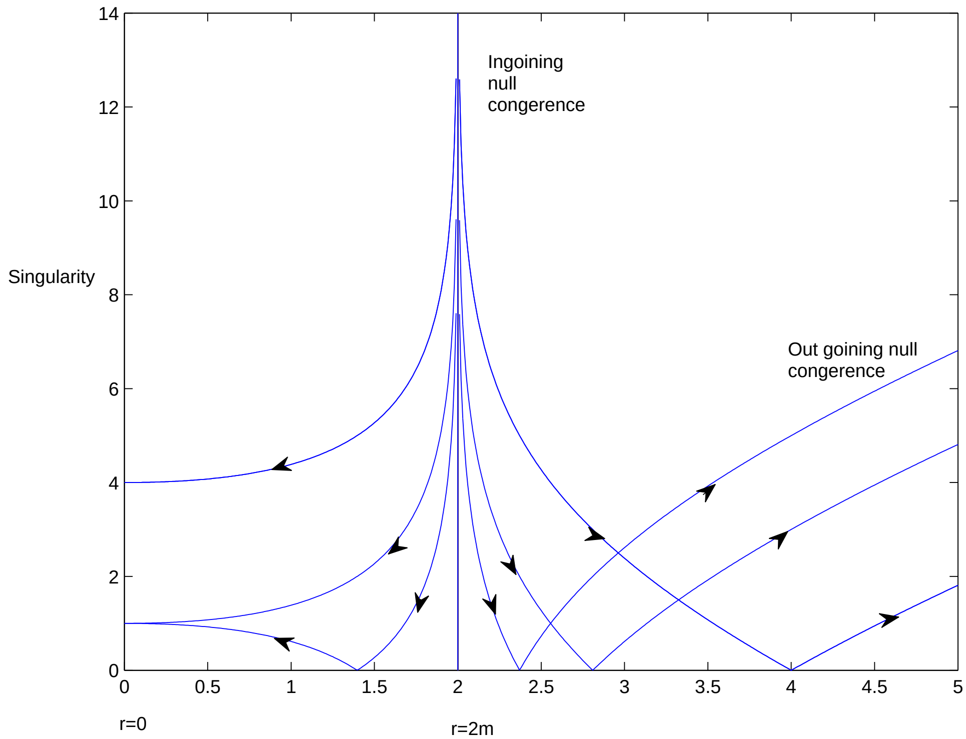

Figure 1.1: The figure shows the radial null geodesics of photons near the Schwarzschild black hole.

1.5 Eddington-Finkelstein Coordinates

In order to remove the coordinate singularity at $r = 2m$, Eddington [16] and Finkelstein [17] introduced a coordinate system (we will give the detail of it in Chapter no:2, here we will just state the coordinate transformations that they give for SH metric).

The retarded time of Eddington and Finkelstein is

$$u = t - r - 2m\ln(r - 2m). \tag{1.5.1}$$

Using eq.(1.5.1) in eq.(1.3.38), the line element given by eq.(1.3.38) transforms as

$$ds^2 = \left(1 - \frac{2m}{r}\right) du^2 + 2dudr - r^2 d\Omega^2. \tag{1.5.2}$$

Similarly, the advanced time of Eddington and Finkelstein is

$$v = t + r + 2m\ln(r - 2m), \tag{1.5.3}$$

in this case the line element given by eq.(1.3.38) transforms as

$$ds^2 = \left(1 - \frac{2m}{r}\right) dv^2 - 2dvdr - r^2 d\Omega^2. \tag{1.5.4}$$

Notice, that $r = 2m$ is no more a singularity of the metric (as clear from eq.(1.5.3) and eq.(1.6.1)). In comparison to the Schwarzschild coordinates, the Eddington-Finkelstein coordinates are well behaved at $r = 2m$, but they are not fully well behaved. Because, the advanced Eddington-Finkelstein coordinates (v, r, θ, ϕ) describe well the in fall of the particle through $r = 2m$, while the retarded Eddington-

Finkelstein coordinates (u, r, θ, ϕ) well describe the outward ejection of particle's from $r = 0$ through $r = 2m$. Hence, their usages are restricted.

The problem was resolved by Kruskal [18] and Szekeres [19] independently and they derived coordinates which are well behaved at $r = 2m$ and are known as the *Kruskal-Szekeres coordinates.*

1.6 Kruskal-Szekeres Coordinates

This coordinate system replaces

$$
\begin{aligned}
(t, r) &\to (v, u), \\
\theta &\to \theta, \\
\phi &\to \phi.
\end{aligned}
$$

These coordinate transformations divide the (v, u) plane into four parts labeled as I, II, III and IV.

$$
\begin{aligned}
&I : r \geq 2m, u \geq 0 \\
u &= \sqrt{\frac{r}{2m} - 1} \quad e^{r/4m} \quad \cosh(t/4m), \\
v &= \sqrt{\frac{r}{2m} - 1} \quad e^{r/4m} \quad \sinh(t/4m),
\end{aligned}
$$

$$II : r \leq 2m, v \geq 0$$

$$u = \sqrt{1 - \frac{r}{2m}} \quad e^{r/4m} \quad \sinh(t/4m),$$
$$v = \sqrt{1 - \frac{r}{2m}} \quad e^{r/4m} \quad \cosh(t/4m),$$

$$III : r \geq 2m, u \leq 0$$

$$u = -\sqrt{\frac{r}{2m} - 1} \quad e^{r/4m} \quad \cosh(t/4m),$$
$$v = -\sqrt{\frac{r}{2m} - 1} \quad e^{r/4m} \quad \sinh(t/4m),$$

$$IV : r \leq 2m, v \leq 0$$

$$u = -\sqrt{1 - \frac{r}{2m}} \quad e^{r/4m} \quad \sinh(t/4m),$$
$$v = -\sqrt{1 - \frac{r}{2m}} \quad e^{r/4m} \quad \cosh(t/4m).$$

In the Kruskal-Szekeres coordinate system the line element given by eq.(1.3.38) takes the form

$$ds^2 = \left(\frac{32m^3}{r}\right) \; e^{-r/2m}(dv^2 - du^2) - r^2 d\Omega^2. \tag{1.6.1}$$

The inverse transformations are

$$\left(\frac{r}{2m} - 1\right) \; e^{r/2m} = u^2 - v^2, \tag{1.6.2}$$

(in regions I, II, III and IV)

$$t = \begin{cases} 4m\tanh^{-1}\frac{v}{u} & : in \quad regions \quad I \quad and \quad III \\ 4m\tanh^{-1}\frac{u}{v} & : in \quad regions \quad II \quad and \quad IV. \end{cases}$$

The singularity $r = 0$ in new coordinates is located at

$$v^2 - u^2 = 1,$$

or at

$$v = \pm\sqrt{1+u^2}. \tag{1.6.3}$$

It is evident from eq.(1.6.3) that there are two singularities and both of them corresponds to $r = 0$ ($v = -\sqrt{1+u^2}$ corresponds to $r = 0$ in the past, while $v = \sqrt{1+u^2}$ corresponds to $r = 0$ in the future). See Figure.1.6.1.

Also, since the region $r \gg 2m$ corresponds to $u^2 \gg v^2$, which shows that there are actually two exterior regions i.e $u \gg |v|$ and $u \ll -|v|$, both correspond to $r \gg 2m$. Now, the question arises that is it possible to have one singularity and one exterior region when the spacetime is expressed in one coordinate system and have two of each in other? The answer is that, the Schwarzschild coordinates cover only a part of the spacetime manifold i.e they must be a local coordinate patch on the full manifold. Hence, two Schwarzschild coordinate patches i.e. I, II and III, IV are required to cover the whole Schwarzschild geometry, whereas on the other hand, a single Kruskal-Szekeres coordinate system is sufficient for the same purpose.

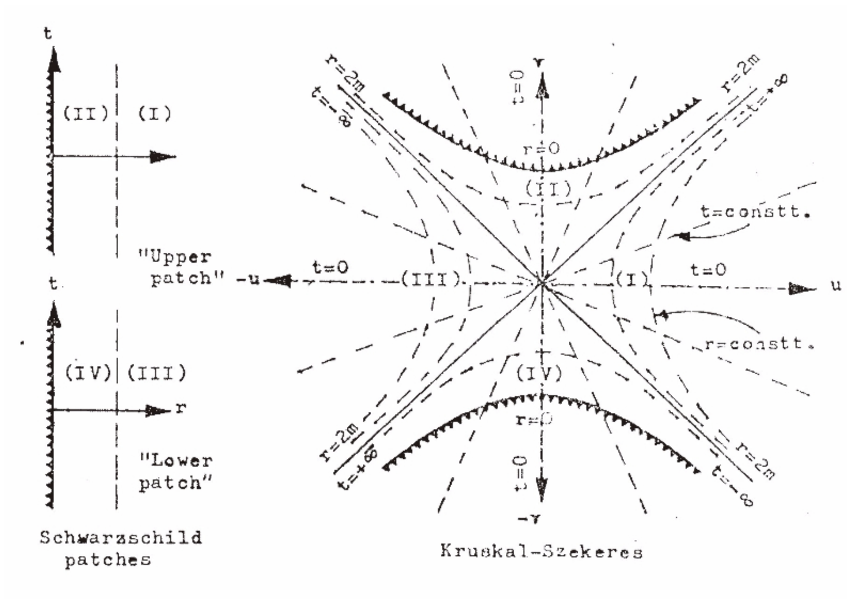

Figure 1.2: Transformation of SH geometry between the Schwarzschild, (t, r), and the Kruskal-Szekeres, (v, u), coordinates.

1.7 Compactified Kruskal-Szekeres Coordinates

These coordinates replace the v, u coordinates by ψ, ξ coordinates given as

$$\psi = \tan^{-1}(v+u) + \tan^{-1}(v-u), \tag{1.7.1}$$

$$\xi = \tan^{-1}(v+u) - \tan^{-1}(v-u). \tag{1.7.2}$$

The implicit relation between r, u, v, ψ and ξ is

$$\left(\frac{r}{2m} - 1\right) e^{r/2m} = u^2 - v^2 = -\tan\left(\frac{\psi+\xi}{2}\right) \tan\left(\frac{\psi-\xi}{2}\right). \tag{1.7.3}$$

1.8 Carter-Penrose Diagram

A two dimensional view of the spacetime was first employed by Eddington then Finkelstein and after that Kruskal for the Schwarzschild spacetime. The analogous techniques were employed by Brill and Graves [20] and Carter [21] for the RN solution. To study the asymptotic properties of spacetime near 'infinity', Penrose developed a mathematical technique. It is base on the use of '*conformal transformation*', which brings infinity to a finite parameter value and hence converts the asymptotic calculations to calculations at finite points. This technique also provides with rigorous definitions of several types of '*infinities*'. They are defined as

$$I^+ \equiv future\, timelike\, infinity.$$

The region where $t \to +\infty$ at finite radius, r.
(i.e, the region of spacetime towards which timelike lines extend).

$$I^- \equiv future\, timelike\, infinity.$$

The region where $t \to -\infty$ at finite radius, r.
(i.e, the region of spacetime from which timelike lines emerge).

$$I^0 \equiv spacelike \quad infinity.$$

The region where $r \to \infty$ at finite coordinate time, t.

(i.e, the region of spacetime towards which spacelike slices extend).

$$\mathscr{I}^+ \equiv future \quad null \quad infinity$$

The region where $t + r \to \infty$ while $t - r$ is finite

(i.e, the region of spacetime towards which outgoing null geodesics extend)

$$\mathscr{I}^- \equiv past \quad null \quad infinity$$

The region where $t - r \to -\infty$ while $t + r$ is finite

(i.e, the region of spacetime from which ingoing null geodesics emerge).

Chapter 2

Compactified non-singular coordinates for the Schwarzschild black hole surrounded by quintessence

In order to examine the geometrical and physical features of the Schwarzschild black hole surrounded by quintessence (SHQ), we need to construct the coordinates that can smoothly cross through the horizons. The technique for resolving the problem of unsatisfactory coordinates is to investigate and explore the spacetime with geodesics, which being coordinate independent will not be influenced by coordinate validity boundaries. In this regard, we have many options but we, like the Eddington-Finkelstein [16, 17] will use the worldlines of radially moving photons to probe.

The SHQ metric [22] is given as

$$ds^2 = \left(1 - \frac{2m}{r} - \alpha r\right) dt^2 - \frac{dr^2}{\left(1 - \frac{2m}{r} - \alpha r\right)} - r^2 \left(d\theta^2 + \sin^2\theta d\phi^2\right). \qquad (2.0.1)$$

The horizons are at

$$\begin{aligned} r_- = r_b &= \frac{1 - \sqrt{1 - 8\alpha m}}{2\alpha}, \\ r_+ = r_c &= \frac{1 + \sqrt{1 - 8\alpha m}}{2\alpha}, \end{aligned} \qquad (2.0.2)$$

where r_b and r_c are event and cosmological horizons respectively. The SHQ metric is regular in the following three regions

I :$r_+ < r < \infty$

II :$r_- < r < r_+$

III : $0 < r < r_-$

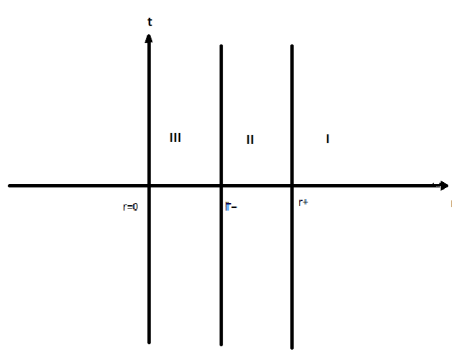

Figure 2.1: The SHQ geometry.

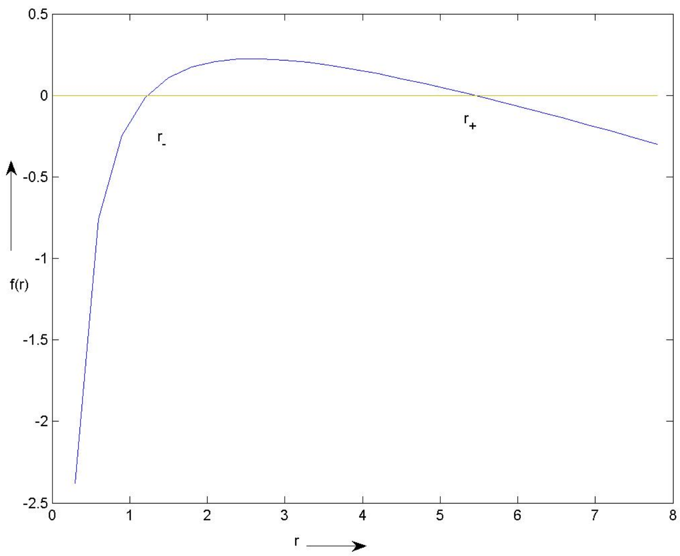

Figure 2.2: The figure depicts the relation between r and *f(r)* for $\alpha = 0.15$ and *m= 0.5*.

2.1 Radial Photon Worldlines in Schwarzschild coordinares

For radial photon worldlines $ds^2 = 0$, $\theta = constant$, $\phi = constant$, so, we have from eq.(2.0.1)

$$0 = f(r)dt^2 - \frac{1}{f(r)}dr^2,$$

or

$$dt = \pm\frac{1}{f(r)}dr, \qquad (2.1.1)$$

where

$$f(r) = 1 - \frac{2m}{r} - \alpha r, \tag{2.1.2}$$

which can be written as

$$f(r) = \frac{(r - r_+)(r - r_-)}{r}. \tag{2.1.3}$$

Eqs.(2.1.3) and (2.1.1) give us

$$dt = \pm \frac{r}{(r - r_+)(r - r_-)} dr. \tag{2.1.4}$$

Using partial fraction, we have

$$\frac{(r - r_+)(r - r_-)}{r} = \frac{1}{r_+ - r_-} \left\{ \frac{r_+}{r - r_+} - \frac{r_-}{r - r_-} \right\}. \tag{2.1.5}$$

Using eq.(2.1.5) in eq.(2.1.4) and integrating we get

$$t = \pm \frac{1}{r_+ - r_-} \left\{ r_+ \ln|r - r_+| - r_- \ln|r - r_-| + c_1 \right\}, \tag{2.1.6}$$

where c_1 is a constant. eq.(2.1.6) can be written as

$$t = \pm \left\{ \ln|\frac{r}{r_+} - 1|^{\frac{r_+}{r_+ - r_-}} - \ln|\frac{r}{r_-} - 1|^{\frac{r_-}{r_+ - r_-}} + constant \right\}, \tag{2.1.7}$$

where plus (minus) sign corresponds to the outgoing (incoming) photon.

2.2 Construction of Eddington-Finkelstein Like Coordinates

The idea here is to use the constants of integration appearing in eq.(2.1.7) as new coordinates, which we shall denote by p and q. Thus, the coordinate transformations are given by

$$p = t + \left\{ \ln |\frac{r}{r_+} - 1|^{\frac{r_+}{r_+ - r_-}} - \ln |\frac{r}{r_-} - 1|^{\frac{r_-}{r_+ - r_-}} \right\}, \tag{2.2.1}$$

$$q = t - \left\{ \ln |\frac{r}{r_+} - 1|^{\frac{r_+}{r_+ - r_-}} - \ln |\frac{r}{r_-} - 1|^{\frac{r_-}{r_+ - r_-}} \right\}. \tag{2.2.2}$$

2.3 Kruskal Like Coordinates

In 1961, Martin Kruskal suggested based on coordinate transformations given by Eddington-Finkelstein, the following coordinate transformations

$$\tilde{p} = \lambda \exp\left(\frac{p}{\beta}\right), \qquad \tilde{q} = -\lambda \exp\left(\frac{-q}{\beta}\right). \tag{2.3.1}$$

In order to get the usual form of the mertic, define new spacelike variable u and timelike variable v given by the following relations

$$v = \frac{\tilde{p} + \tilde{q}}{2}, \qquad u = \frac{\tilde{p} - \tilde{q}}{2}. \tag{2.3.2}$$

eq.(2.3.1) together with eq.(2.3.2) gives

$$v = \lambda \frac{e^{\frac{t}{\beta}} - e^{\frac{-t}{\beta}}}{2} \left|\frac{r}{r_+} - 1\right|^{\frac{r_+}{\beta(r_+ - r_-)}} \left|\frac{r}{r_-} - 1\right|^{\frac{-r_-}{\beta(r_+ - r_-)}},$$

$$u = \lambda \frac{e^{\frac{t}{\beta}} + e^{\frac{-t}{\beta}}}{2} \left|\frac{r}{r_+} - 1\right|^{\frac{r_+}{\beta(r_+ - r_-)}} \left|\frac{r}{r_-} - 1\right|^{\frac{-r_-}{\beta(r_+ - r_-)}},$$

which can be written as

$$v = \alpha \sinh\left(\frac{t}{\beta}\right) \left|\frac{r}{r_+} - 1\right|^{\frac{r_+}{\beta(r_+ - r_-)}} \left|\frac{r}{r_-} - 1\right|^{\frac{-r_-}{\beta(r_+ - r_-)}}, \tag{2.3.3}$$

$$u = \alpha \cosh\left(\frac{t}{\beta}\right) \left|\frac{r}{r_+} - 1\right|^{\frac{r_+}{\beta(r_+ - r_-)}} \left|\frac{r}{r_-} - 1\right|^{\frac{-r_-}{\beta(r_+ - r_-)}}. \tag{2.3.4}$$

The coordinates given by eqs.(2.3.3) and (2.3.4) are singular at $r = r_\pm$. In order to eliminate the singularity we need to cancel the factors containing $\left(1 - \frac{r}{r_-}\right)$ and $\left(1 - \frac{r}{r_+}\right)$, but in any way both can not be canceled simultaneously. So, atleast two patches are required to completely cover the SHQ geometry. Hence, to keep the coordinates regular at $r = r_+$, we take

$$\beta = \beta_+ = \frac{2r_+}{r_+ - r_-}, \qquad \lambda = \lambda_+, \tag{2.3.5}$$

and for coordinates to be regular at $r = r_-$, we take

$$\beta = \beta_- = \frac{-2r_-}{r_+ - r_-}, \qquad \lambda = \lambda_-, \tag{2.3.6}$$

these coordinates analogous to the generalized Kruskel-Szekeres coordinates for the two patches are

$$v_+ = \lambda_+ \sinh\left(\frac{t}{\beta_+}\right)|\frac{r}{r_+} - 1|^{\frac{r_+}{\beta_+(r_+ - r_-)}}|\frac{r}{r_-} - 1|^{\frac{-r_-}{\beta_+(r_+ - r_-)}}, \tag{2.3.7}$$

$$u_+ = \lambda_+ \cosh\left(\frac{t}{\beta_+}\right)|\frac{r}{r_+} - 1|^{\frac{r_+}{\beta_+(r_+ - r_-)}}|\frac{r}{r_-} - 1|^{\frac{-r_-}{\beta_+(r_+ - r_-)}}, \tag{2.3.8}$$

$$v_- = \lambda_- \sinh\left(\frac{t}{\beta_-}\right)|\frac{r}{r_+} - 1|^{\frac{r_+}{\beta_-(r_+ - r_-)}}|\frac{r}{r_-} - 1|^{\frac{-r_-}{\beta_-(r_+ - r_-)}}, \tag{2.3.9}$$

$$u_- = \lambda_- \cosh\left(\frac{t}{\beta_-}\right)|\frac{r}{r_+} - 1|^{\frac{r_+}{\beta_-(r_+ - r_-)}}|\frac{r}{r_-} - 1|^{\frac{-r_-}{\beta_-(r_+ - r_-)}}. \tag{2.3.10}$$

The values of $\lambda_\pm$ are chosen so that at a point r between r_+ and r_-, hypersurfaces get matched properly in (v_-, u_-) and (v_+, u_+) coordinates. The coordinates (v_-, u_-) are regular for the region in which $0 < r < r_+$. Similarly, the coordinates (v_+, u_+) are regular for $r_- < r < \infty$. For the overlapping region i.e $r_- < r < r_+$, v_+ and u_+ are interchanged. Same is true for the coordinates v_- and u_-. Also, $r = r_\pm$ corresponds to $v_+ = u_+ = 0$ and $v_- = u_- = 0$ respectively. The implicit relation betweens $u_\pm$, $v_\pm$ and r are

$$u_+^2 - v_+^2 = \lambda_+^2|\frac{r}{r_+} - 1|^{\frac{2r_+}{\beta_+(r_+ - r_-)}}|\frac{r}{r_-} - 1|^{\frac{-2r_-}{\beta_+(r_+ - r_-)}}, \tag{2.3.11}$$

and

$$u_-^2 - v_-^2 = \lambda_-^2 |\frac{r}{r_+} - 1|^{\frac{2r_+}{\beta_-(r_+-r_-)}} |\frac{r}{r_-} - 1|^{\frac{-2r_-}{\beta_-(r_+-r_-)}}. \tag{2.3.12}$$

2.4 Compactified Kruskal-Szekeres Like Coordinates

The coordinate transformations are given as

$$\begin{aligned}\psi_+ &= \tan^{-1}(v_+ + u_+) + \tan^{-1}(v_+ - u_+),\\ \xi_+ &= \tan^{-1}(v_+ + u_+) - \tan^{-1}(v_+ - u_+),\end{aligned} \tag{2.4.1}$$

and

$$\begin{aligned}\psi_- &= \tan^{-1}(v_- + u_-) + \tan^{-1}(v_- - u_-),\\ \xi_- &= \tan^{-1}(v_- + u_-) - \tan^{-1}(v_- - u_-).\end{aligned} \tag{2.4.2}$$

The implicit relations between these coordinates and r are

$$\begin{aligned}u_+^2 - v_+^2 = \lambda_+^2 |\frac{r}{r_+} - 1|^{\frac{2r_+}{\beta_+(r_+-r_-)}} |\frac{r}{r_-} - 1|^{\frac{-2r_-}{\beta_+(r_+-r_-)}} = \\ - \tan\left(\frac{\psi_+ + \xi_+}{2}\right) \tan\left(\frac{\psi_+ + \xi_+}{2}\right),\end{aligned} \tag{2.4.3}$$

and

$$u_-^2 - v_-^2 = \lambda_-^2 |\frac{r}{r_+} - 1|^{\frac{2r_+}{\beta_-(r_+-r_-)}} |\frac{r}{r_-} - 1|^{\frac{-2r_-}{\beta_-(r_+-r_-)}} = \tag{2.4.4}$$
$$-\tan\left(\frac{\psi_- + \xi_-}{2}\right)\tan\left(\frac{\psi_- + \xi_-}{2}\right).$$

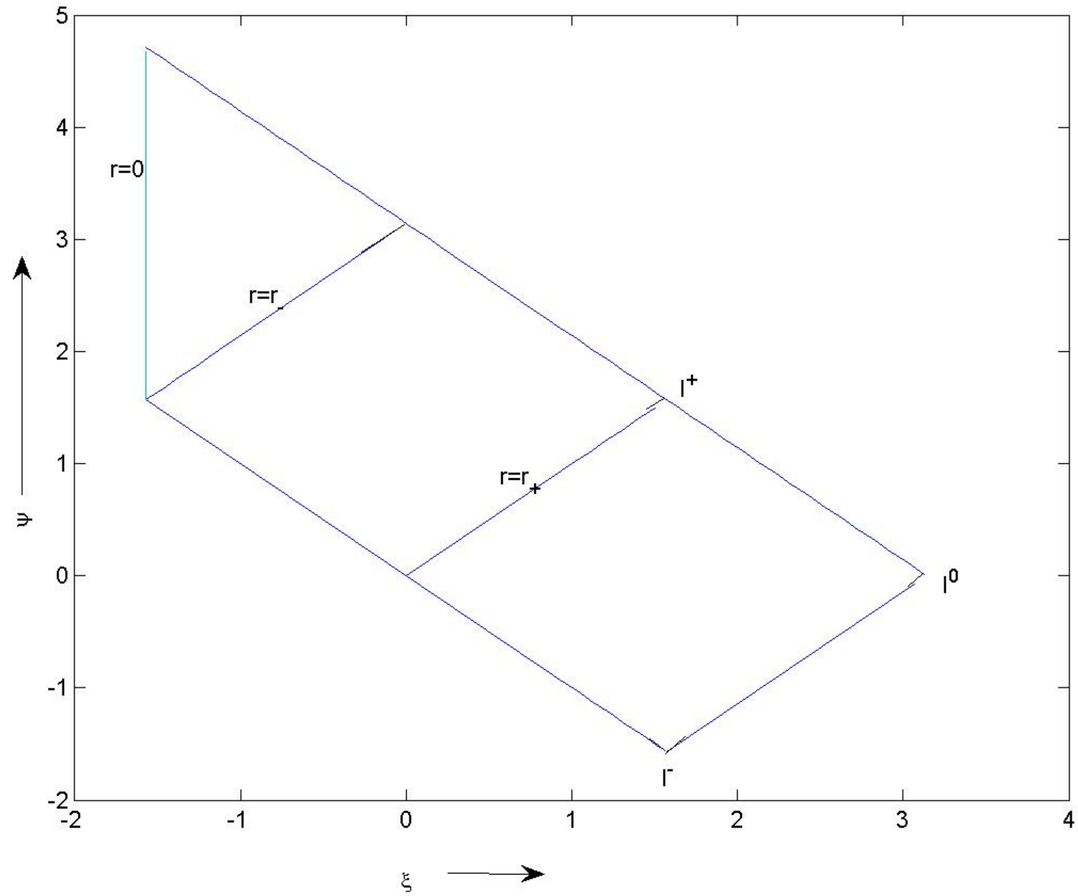

Figure 2.3: SHQ spacetime in $(\psi, \xi, \theta, \phi)$ coordinates. Note that the essential singularity here is timelike.

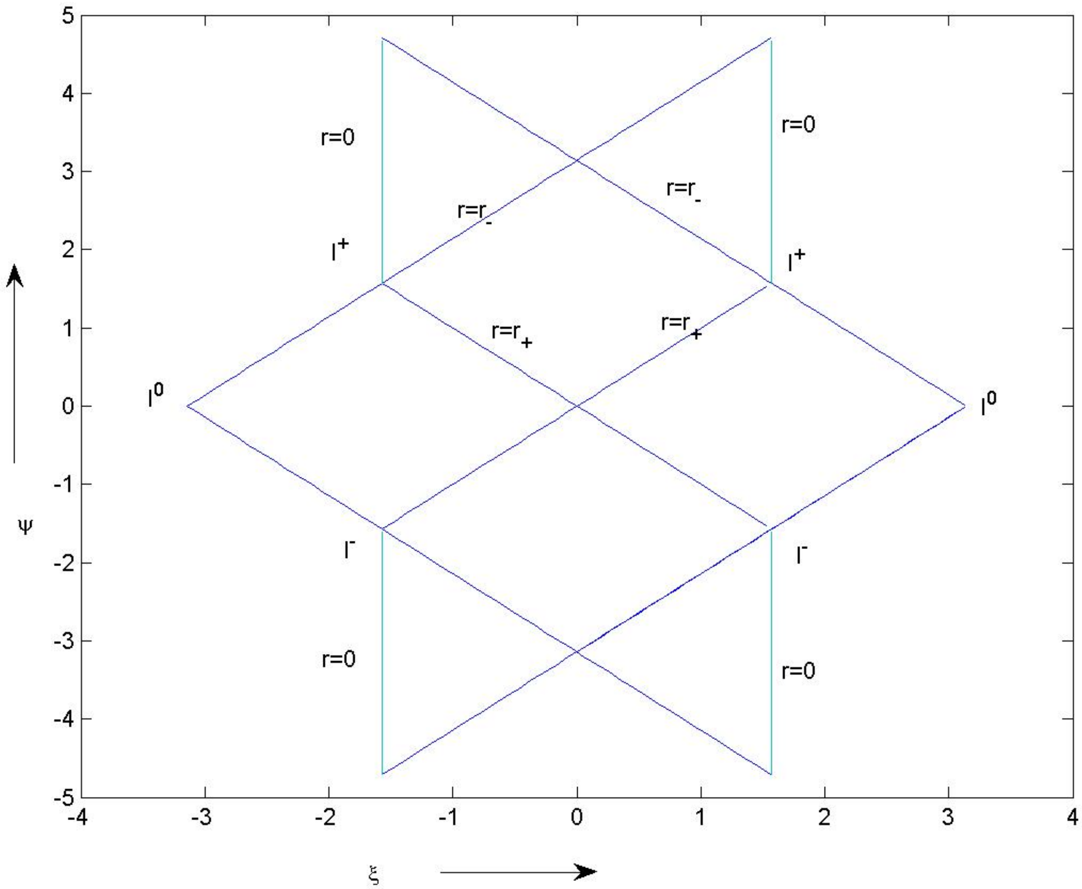

Figure 2.4: Maximal extension of the SHQ spacetime in $(\psi, \xi, \theta, \phi)$ coordinates. Note that as the essential singularity is timelike, so a timelike curve can enter the new universes by passing through the two coordinate singularities.

Chapter 3

Foliation of some black hole spacetimes by K-surfaces

One of the point of interest while studying the dynamics of a spacetime is hypersurfaces of constant mean extrinsic curvature. A comprehensive detail of foliation of Schwarzschild spacetime by hypersurfaces of constant mean extrinsic curvature has been given by Brill *et.al.* [6]. In this chapter, we first review a procedure based on variational principle as described in [6] and then review the foliation of SH and RN spacetimes explicitly [7, 8]. Finally, we apply the same technique to foliate SHQ spacetime.

3.1 K-Surface Foliation

In order to obtain the variational principle for K-surfaces Brill *et.al.* [6] generalized a well known extremum property that spheres in Euclidean space have least surface area for a fixed enclose volume and have constant mean curvature. Working on the same lines they extremized the three dimensional area, $A(S)$, of the hypersurface, S, by keeping constant the 4-volume, $V(S, S_1)$, enclosed by any fixed surface, S_1, and S. By using this constraint in variational principle Brill *et.al.* obtained

$$\delta I = 0, \tag{3.1.1}$$

with

$$I = A(s) + \mu V(S, S_1) = \int n^b d^3 S_b + \mu \int d^4 V, \tag{3.1.2}$$

where n^b represents a field of normal unit vectors to S_1 and S .

Since, the boundary of V is $S - S_1$, so, by using the divergence theorem integral I can be written as an integral over V

$$I = A(S) - A(S_1) + \mu V(S, S_1) + A(S_1), \tag{3.1.3}$$

or

$$I = \int_V [n^\lambda_{;\lambda} + \mu] d^4 V + A(S_1). \tag{3.1.4}$$

The variation of I can vanish for arbitrary variations of S only if the integrand vanishes everywhere on S. Hence

$$n^\lambda_{;\lambda} + \mu = 0, \tag{3.1.5}$$

which gives

$$K = \mu. \tag{3.1.6}$$

The above equation shows that the mean extrinsic curvature, K, is constant and is equal to the Lagrange multiplier. So, Brill *et.al.* have reached to the conclusion that if the three dimensional area, $A(S)$, of the hypersurface, S, is extremized, by keeping constant the 4-volume $V(S, S_1)$, enclosed by any fixed surface S_1 and S then the hypersurface, S, will have constant mean extrinsic curvature. On the basis of above stated result they obtained the differential equation satisfied by the K-surfaces whose detail is given in the next subsection.

3.1.1 *K*-Surface Equation for General Spacetimes

In this subsection we review the derivation of differential equation for K-Surfaces as derived by Azad [23]. Let us consider a general spacetime having metric of the form

$$\begin{aligned} ds^2 = B(r,\theta,\phi)dt^2 - C(r,\theta,\phi)dr^2 - D(r,\theta,\phi)d\theta^2 - E(r,\theta,\phi) \\ +2F(r,\theta,\phi)dtdr + 2G(r,\theta,\phi)dtd\theta + 2H(r,\theta,\phi)dtd\phi + 2I(r,\theta,\phi)drd\theta \\ +2J(r,\theta,\phi)drd\phi + 2L(r,\theta,\phi)d\theta d\phi. \end{aligned} \tag{3.1.7}$$

The spacetime 4-volume is

$$V = \int \sqrt{-g}dtdrd\theta d\phi, \tag{3.1.8}$$

here g is the determinant of the metric given by eq.(3.1.7). Let the foliating hypersurface be an implicit function of r, θ and ϕ given as

$$f(t, r, \theta, \phi) = 0, \tag{3.1.9}$$

then the 4-volume enclosed by an arbitrary hypersurface, t, and a fix hypersurface, S_1, (say $t = 0$) is

$$V(S_1, S) = \int \sqrt{-g(r, \theta, \phi)}t(r, \theta, \phi)dtdrd\theta d\phi. \tag{3.1.10}$$

From eq.(3.1.9), we have

$$df = 0,$$

which gives

$$dt = t_r dr + t_\theta d\theta + t_\phi d\phi, \tag{3.1.11}$$

with eq.(3.1.11) in (3.1.7), we get the following induced metric

$$ds^{*2} = -C^* dr^2 - D^* d\theta^2 - E^* d\phi^2 + 2I^* drd\theta + 2J^* drd\phi + 2L^* d\theta d\phi, \tag{3.1.12}$$

where C^*, D^*, E^*, I^*, J^*, L^* are

$$\begin{aligned} C^* &= C - 2Ft_r - B{t_r}^2, \\ D^* &= D - 2Gt_\theta - B{t_\theta}^2, \\ E^* &= E - 2Ht_\phi - B{t_\phi}^2, \\ I^* &= I - Ft_\theta - Gt_r - Bt_r t_\theta, \\ J^* &= J - Ft_\phi - Ht_r - Bt_r t_\phi, \\ L^* &= L - Gt_\phi - Ht_\theta - Bt_\theta t_\phi. \end{aligned} \tag{3.1.13}$$

Let the determinant of the metric given in eq.(3.1.12) is $detg_{ij}$, then the 3-area, $A(S)$, of the hypersurface, S, is

$$A(S) = \int \sqrt{-detg_{ab}} dr d\theta d\phi. \tag{3.1.14}$$

eq.(3.1.1) and (3.1.2) gives

$$\delta[A(S) + kV(S_1, S)] = 0, \tag{3.1.15}$$

with eq.(3.1.8), (3.1.14) in (3.1.15), we have

$$\delta\{\int (\sqrt{-detg_{ab}} + K\sqrt{-detg_{cd}}t) dr d\theta d\phi\} = 0, \tag{3.1.16}$$

which gives

$$\int \left[\delta(\sqrt{-detg_{ab}}) + K\sqrt{-detg_{cd}}\delta t\right] drd\theta d\phi = 0. \quad (3.1.17)$$

For the purpose of simplicity, let

$$-detg_{ab} = \Sigma^2, \quad (3.1.18)$$

which gives

$$\delta(\sqrt{-detg_{ab}}) = \delta\Sigma. \quad (3.1.19)$$

Since

$$\Sigma = \Sigma(t_r, t_\theta, t_\phi, r, \theta, \phi), \quad (3.1.20)$$

so, the variations of Σ with respect to 't' gives

$$\delta\Sigma = \frac{\partial\Sigma}{\partial t_r}\delta t_r + \frac{\partial\Sigma}{\partial t_\theta}\delta t_\theta + \frac{\partial\Sigma}{\partial t_\phi}\delta t_\phi. \quad (3.1.21)$$

Requiring the continuity

$$\delta t_r = \delta\frac{\partial t}{\partial r} = \frac{\partial}{\partial r}\delta t, \quad (3.1.22)$$

using eq.(3.1.22) in eq.(3.1.21), we get

$$\delta\Sigma = \left(\Sigma_{t_r}\frac{\partial}{\partial r} + \Sigma_{t_\theta}\frac{\partial}{\partial\theta} + \Sigma_{t_\phi}\frac{\partial}{\partial\phi}\right)\delta t. \quad (3.1.23)$$

Eqs.(3.1.19) and (3.1.23) together with eq.(3.1.16) gives us

$$\int \left(\Sigma_{t_r} \frac{\partial}{\partial r} + \Sigma_{t_\theta} \frac{\partial}{\partial \theta} + \Sigma_{t_\phi} \frac{\partial}{\partial \phi} + K\sqrt{-g} \right) \delta t dr d\theta d\phi = 0. \tag{3.1.24}$$

On integrating first, second and third terms of eq.(3.1.24) w.r.t. r, θ and ϕ respectively, we have

$$\int \left[-\left(\Sigma_{t_r}\right)_r - \left(\Sigma_{t_\theta}\right)_\theta - \left(\Sigma_{t_\phi}\right)_\phi + K\sqrt{-g} \right] \delta t dr d\theta d\phi. \tag{3.1.25}$$

The integral given in eq.(3.1.25) will vanish for arbitrary variations of 't' provided the integrand is zero. So, we get

$$\left(\Sigma_{t_r}\right)_r + \left(\Sigma_{t_\theta}\right)_\theta + \left(\Sigma_{t_\phi}\right)_\phi = K\sqrt{-g}. \tag{3.1.26}$$

The above equation, eq.(3.1.26) is the required equation, i.e. the equation satisfied by K-surfaces.

3.1.2 *K*-Surface Foliation Equation for Static Spherically Symmetric Spacetimes

The metric for static spherically symmetric spacetimes in Schwarzshild coordinates is

$$ds^2 = Bdt^2 - Cdr^2 - r^2(d\theta^2 + \sin^2\theta d\phi^2), \tag{3.1.27}$$

where B and C are functions of r only.

In this case using the variational principle under the same constraints as mentioned above, we have

$$\delta \int \left\{ \Sigma + K\sqrt{BC}t(r,\theta,\phi) \right\} r^2 \sin\theta dr d\theta d\phi = 0, \tag{3.1.28}$$

where

$$-det g_{ab} = \Sigma^2 = -Bt_r{}^2 - \left(\frac{BC}{r^2}\right)\left\{ t_\theta{}^2 + \left(\frac{t_\phi}{\sin\theta}\right)^2 \right\} + C. \tag{3.1.29}$$

Taking variations w.r.t 't' and using the continuity equation in eq.(3.1.29), we get

$$\delta\Sigma = -\frac{1}{\Sigma}\left\{ Bt_r \frac{\partial}{\partial r} + \frac{BC}{r^2}\left(t_\theta \frac{\partial}{\partial\theta} + \frac{t_\phi}{\sin^2\theta}\frac{\partial}{\partial\phi}\right)\right\}, \tag{3.1.30}$$

also

$$\delta\left\{ K\sqrt{B(r)C(r)}t \right\} = K\sqrt{B(r)C(r)}\delta t, \tag{3.1.31}$$

eqs.(3.1.30) and (3.1.31) together with eq.(3.1.28) give the following partial differential equation satisfied by K-surfaces

$$\left(\frac{B(r)t_r r^2 \sin\theta}{\Sigma}\right)_r + \left(\frac{B(r)C(r)t_\theta \sin\theta}{\Sigma}\right)_\theta + \left(\frac{B(r)C(r)t_\phi}{\Sigma\sin\theta}\right)_\phi = -Kr^2\sqrt{B(r)C(r)}\sin\theta. \tag{3.1.32}$$

In case of spherically symmetric hypersurfaces, we can let

$$t \equiv t(r). \tag{3.1.33}$$

Eq.(3.1.33) together with eq.(3.1.32) gives the ordinary differential equation satisfied by the K-surfaces

$$\frac{d}{dr}\left(\frac{B(r)t_r r^2}{\Sigma}\right) = -Kr^2\sqrt{B(r)C(r)}, \tag{3.1.34}$$

where

$$\Sigma^2 = C - Bt_r{}^2. \tag{3.1.35}$$

Eq.(3.1.34) can be written as

$$\frac{dt}{dr} = \sqrt{\frac{C(r)}{B(r)}}\frac{(H - J(r))}{\sqrt{B(r)r^4 + (H - J(r))^2}}, \tag{3.1.36}$$

where

$$J = K\int^r \sqrt{B(w)C(w)}w^2 dw, \tag{3.1.37}$$

and H is an integration constant.

3.2 K-Surface Foliation of the Schwarzschild Spacetime

In case of the Schwarzschild spacetime, we have

$$B = \frac{1}{C} = 1 - \frac{2m}{r}, \tag{3.2.1}$$

using eq.(3.2.1) in eq.(3.1.37), we get

$$J = \frac{Kr^3}{3}. \tag{3.2.2}$$

In this case the ordinary differential equation, eq.(3.1.36), reduces to

$$\frac{dt}{dr} = \frac{rE(r)}{(r-2m)A(r)}, \tag{3.2.3}$$

with

$$E(r) = H - \frac{Kr^3}{3}, \qquad A(r)^2 = E(r)^2 + r^3(r-2m). \tag{3.2.4}$$

Transforming eq.(3.2.3) in Kruskal-Szekeres coordinates, we have

$$\frac{du}{dr} = \frac{ur}{2r_s(r-r_s)} + \frac{v}{2r_s}\frac{dt}{dr}, \tag{3.2.5}$$

$$\frac{dv}{dr} = \frac{vr}{2r_s(r-r_s)} + \frac{u}{2r_s}\frac{dt}{dr}, \tag{3.2.6}$$

where $r_s = 2m$ is the Schwarzschild radius. Using eq.(3.2.3) in eqs.(3.2.5) and (3.2.6) we have

$$\frac{du}{dr} = \frac{r(Ev + Au)}{2r_s A(r-r_s)}, \tag{3.2.7}$$

and

$$\frac{dv}{dr} = \frac{r(Eu + Av)}{2r_s A(r - r_s)}. \tag{3.2.8}$$

So, the equation for K-surfaces in Kruskal-Szekeres coordinates takes the form

$$\frac{dv}{du} = \frac{Eu + Av}{Ev + Au}. \tag{3.2.9}$$

In order to display the foliating hypersurfaces, it is more feasible to foliate the Carter Penrose diagram of the spacetime. Therefore, converting eq.(3.2.9) in compactified Kruskal-Szekeres coordinates, we get the equations

$$\begin{aligned}
\frac{d\psi}{du} &= \frac{1}{1+(v+u)^2}\left(1+\frac{dv}{du}\right) - \frac{1}{1+(v-u)^2}\left(-1+\frac{du}{du}\right),\\
\frac{d\xi}{du} &= \frac{1}{1+(v+u)^2}\left(1+\frac{dv}{du}\right) + \frac{1}{1+(v-u)^2}\left(-1+\frac{du}{du}\right),
\end{aligned} \tag{3.2.10}$$

eq.(3.2.9) togather with eq.(3.2.10) yields

$$\frac{d\psi}{d\xi} = \frac{Av(u^2 - v^2 - 1) + Eu(-(u^2 - v^2) - 1)}{Au(-(u^2 - v^2) - 1) + Eu(u^2 - v^2 - 1)}. \tag{3.2.11}$$

Using eqs.(1.7.1), (1.7.2) and (1.7.3) in eq.(3.2.11) and simplifying we obtain K-surface equation in compactified Kruskal-szekeres coordinates as

$$\frac{d\psi}{d\xi} = \frac{A\sin\psi\cos\xi + E\sin\xi\cos\psi}{A\sin\xi\cos\psi + E\sin\psi\cos\xi}. \tag{3.2.12}$$

Brill *et.al.* numerically solved eq.(3.2.9) for $m = 0.5$ with the initial condition

$$\frac{d\psi}{d\xi}|_{u=0=\xi} = 0. \tag{3.2.13}$$

The above condition (3.2.13) further gives

$$H = \frac{Kr_i^3}{3} \pm \sqrt{r_i^3(2m - r_i)}. \tag{3.2.14}$$

The foliating hypersurfaces, their corresponding mean extrinsic curvature, initial value of r, and corresponding initial value of ψ are shown and given in Figure.3.1 and Table.3.1.

No	K	r_i	H	ψ_i
1	0	1	0	0
± 2	± 0.01	0.98	∓ 0.13406	± 0.45374
± 3	± 0.04	0.89	∓ 0.26907	± 0.95518
± 4	± 0.07	0.86	∓ 0.28356	± 1.0439
± 5	± 0.15	0.75	∓ 0.30366	± 1.2578
± 6	± 1.0	0.60	∓ 0.22193	± 1.41330

Table 3.1: The mean extrinsic curvatures, K, the initial value, r_i, of r, corresponding initial value, ψ_i, of ψ and the constant H are shown for eleven different $K-$surfaces. Here we have take $m = \frac{1}{2}$

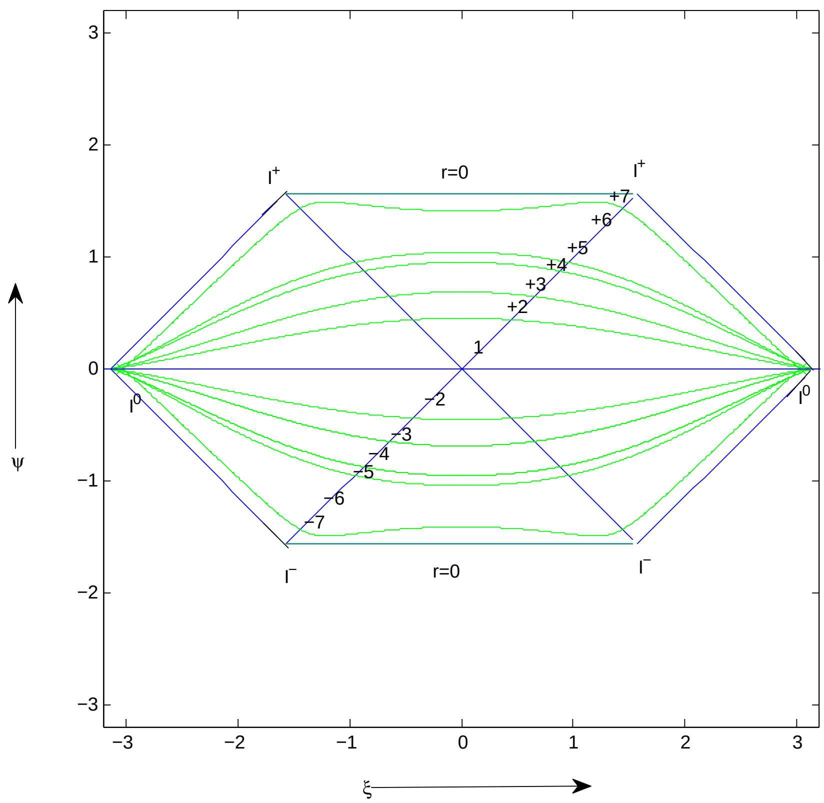

Figure 3.1: Eleven foliating hypersurfaces of SH spacetime for $K = 0$, ± 0.01, ± 0.04, ± 0.07, ± 0.15, ± 1 are shown.

3.3 K-Surface Foliation of RN Spacetime

The RN spacetime is static and spherically symmetric. In this case

$$B = \frac{1}{C} = 1 - \frac{2m}{r} + \frac{Q^2}{r^2}, \tag{3.3.1}$$

also

$$r_{\pm} = m \pm \sqrt{m^2 - Q^2}, \tag{3.3.2}$$

which are the coordinate singularities of the RN black hole. On following the same procedure as followed for SH spacetime one gets the same differential equation in KS and compactified KS coordinates as that for the SH case. Only 'A' and 'H' get altered and are given as

$$A^2 = E^2 + r^2(r - r_+)(r - r_-), \tag{3.3.3}$$

and

$$H = \frac{Kr_i^3}{3} \pm r_i m\sqrt{(r_i - r_+)(r_i - r_-)}, \tag{3.3.4}$$

where $r_{\pm}$ are given in eq.(3.3.2). Here E remains unaltered i.e

$$E = H - \frac{Kr^3}{3}.$$

Hypersurfaces are given for six values of $|K|$ with $Q/m = 0.8$

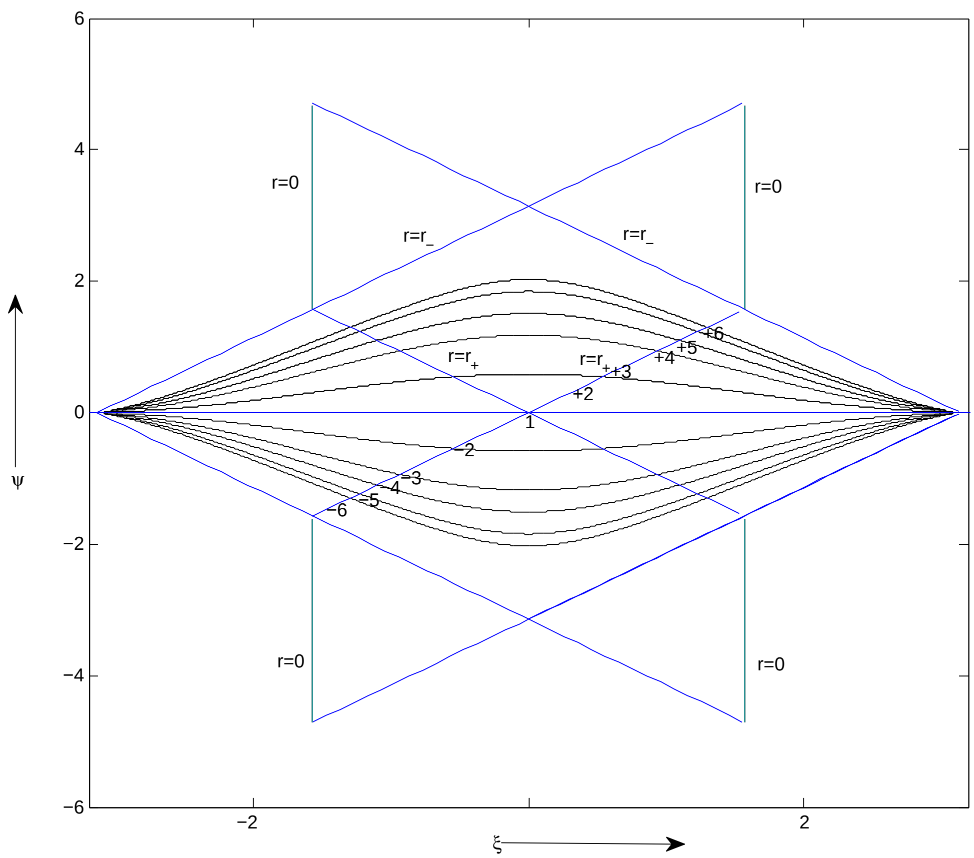

Figure 3.2: Eleven foliating hypersurfaces of RN spacetime for $K = 0$, ± 0.01, ± 0.02, ± 0.04, ± 0.06, ± 0.09 are shown.

No	K	r_i	H	ψ_i
1	0	0.8	0	0
± 2	± 0.01	0.79	∓ 0.0259	± 0.58231
± 3	± 0.02	0.798	∓ 0.0574	± 1.177658
± 4	± 0.04	0.78	∓ 0.0776	± 1.50958
± 5	± 0.06	0.76	∓ 0.1049	± 1.84436
± 6	± 0.09	0.74	∓ 0.1210	± 2.02577

Table 3.2: The mean extrinsic curvatures, K, the initial value, r_i, of r, corresponding initial value, ψ_i, of ψ and the constant H are shown for eleven different $K-$surfaces. Here we have taken $m = \frac{1}{2}$ and $Q = 0.4$.

3.4 *K*-Surface Foliation of the SHQ Spacetime

The SHQ spacetime is static and sphericaly symmetric given by eq.(2.0.1). On comparing eq.(2.0.1) with eq.(3.1.27), we have

$$B(r) = f(r) = \frac{1}{C(r)} = 1 - \frac{2m}{r} - \alpha r. \tag{3.4.1}$$

Following the same procedure as followed for the Schwarzschild spacetime, we obtain the same differential equations as given by eqs.(3.2.9) and (3.2.12) in generalized Kruskal-Szekeres coordinates i.e (v, u) and compactified generalized Kruskal-Szekeres coordinates (ψ, ξ). A in this case is changed to

$$A^2 = E^2 + r^3(r - r_+)(r - r_-), \tag{3.4.2}$$

while E remains unaltered and $r_\pm$ are given by eq.(2.0.2). We solve eq.(3.2.12) numerically with accuracy 1×10^{-4} for $m = 0.4$ and $\alpha = 0.12$. While solving eq.(3.2.12) we get a variety of hypersurfaces among them we like Qadir *et.al.* [8]

select those which satisfy

$$\frac{d\psi}{d\xi}|_{\xi=0} = 0. \tag{3.4.3}$$

The requirement given by eq.(3.4.3) gives $A = 0$ at $\xi = 0$, which yields a relationship between initial value, r_i, of r and H given as

$$H = \frac{Kr_i^3}{3} \pm \sqrt{r_i^3(r - r_+)(r - r_-)}, \tag{3.4.4}$$

here, obviously $r_i \leq r_+$, which is evident directly from the Carter-Penrose diagram of the SHQ geometry (as we have taken $\xi_i = 0$). Generally, the hypersurfaces so obtained do not approach I^0 in the Carter-Penrose diagram of the SHQ. To resolve this problem, we like Qadir *et.al.* [8] start changing the initially selected value, r_i, of r with small step sizes, so, that the point of intersection of the hypersurfaces and the line $\psi = 0$ extend towards I^0 (mathematically saying, $\psi(\pi) = 0$). Also, since the geometry of the SHQ spacetime is symmetric, so, it might be a fair requirement that for a possible complete foliation of the SHQ spacetime, we like Qadir *et.al.* [8] require the foliating hypersurfaces to be symmetric function of ψ and ξ.

For the numerical solution of the eq.(3.2.12), we like Qadir *et.al.* [8] select some value of mean extrinsic curvature, K, and some initial value, r_i, of r (where $r_i \leq r_+$). This yields two possible values of H, among which , we choose the one with the sign opposite to the sign of K (as the same sign value does not provide with a foliation). Then, we solve the differential equation given by eq.(3.2.12) numerically with the help of the implicit relation given by eq.(2.4.4). The foliating hypersurfaces are shown in Figures.3.3 and 3.4, for $m = 0.4$ and 1, where $\alpha = 0.12$. It is observed that increase in mass results in upward shifting of the hypersurfaces.

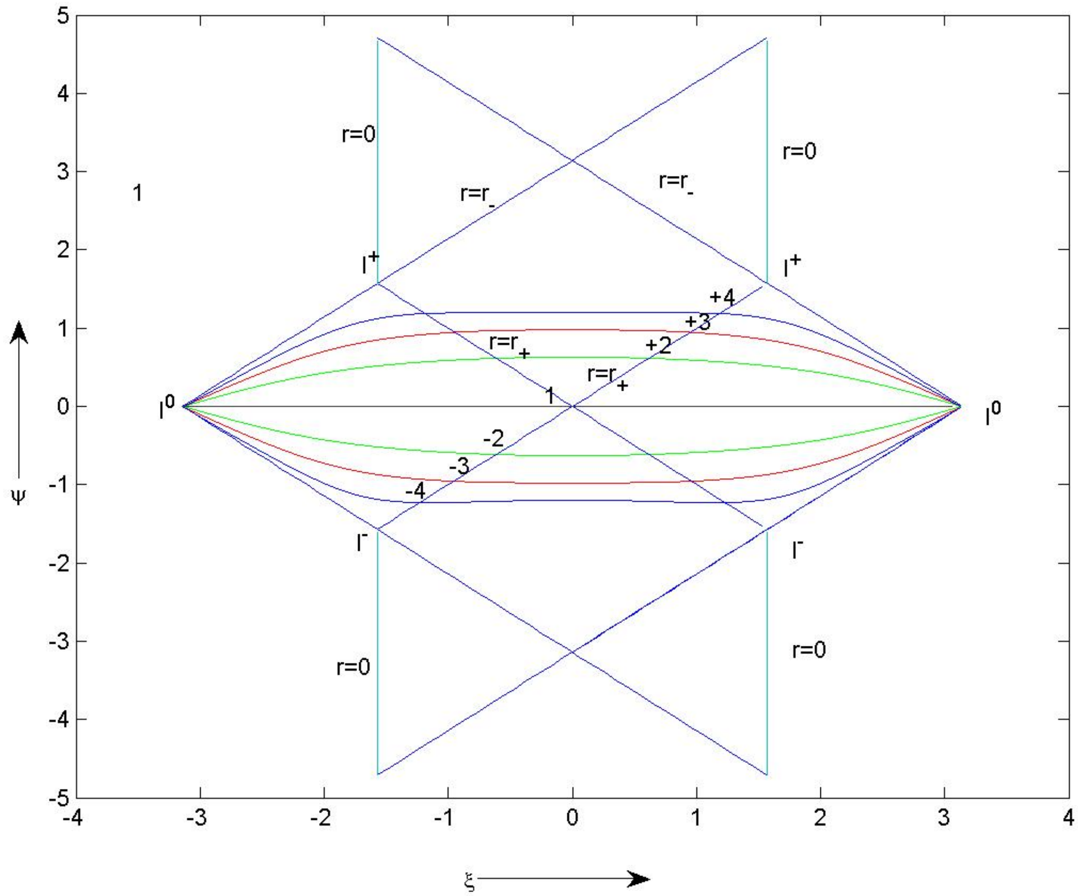

Figure 3.3: Foliation of the Carter-Penrose diagram of the SHQ geometry by K-surfaces, here $\alpha = 0.12$ and $m = 0.4$. Few spacelike hypersurfaces for $K = 0$, ± 0.00003, ± 0.001 and ± 0.015 are shown.

No	Colour	K	r_i	H	ψ_i
1	black	r_+	0	0	0
± 2	green	± 0.00003	7.2	∓ 23.6077	± 0.6257
± 3	red	± 0.001	6.8	∓ 34.2793	± 0.9786
± 4	blue	± 0.015	6.4	∓ 37.3670	± 1.1976

Table 3.3: The mean extrinsic curvatures, K, the initial value, r_i, of r, corresponding initial value, ψ_i, of ψ and the constant H are shown for seven different $K-$surfaces. Here, we have taken $m = 0.4$ and $\alpha = 0.12$.

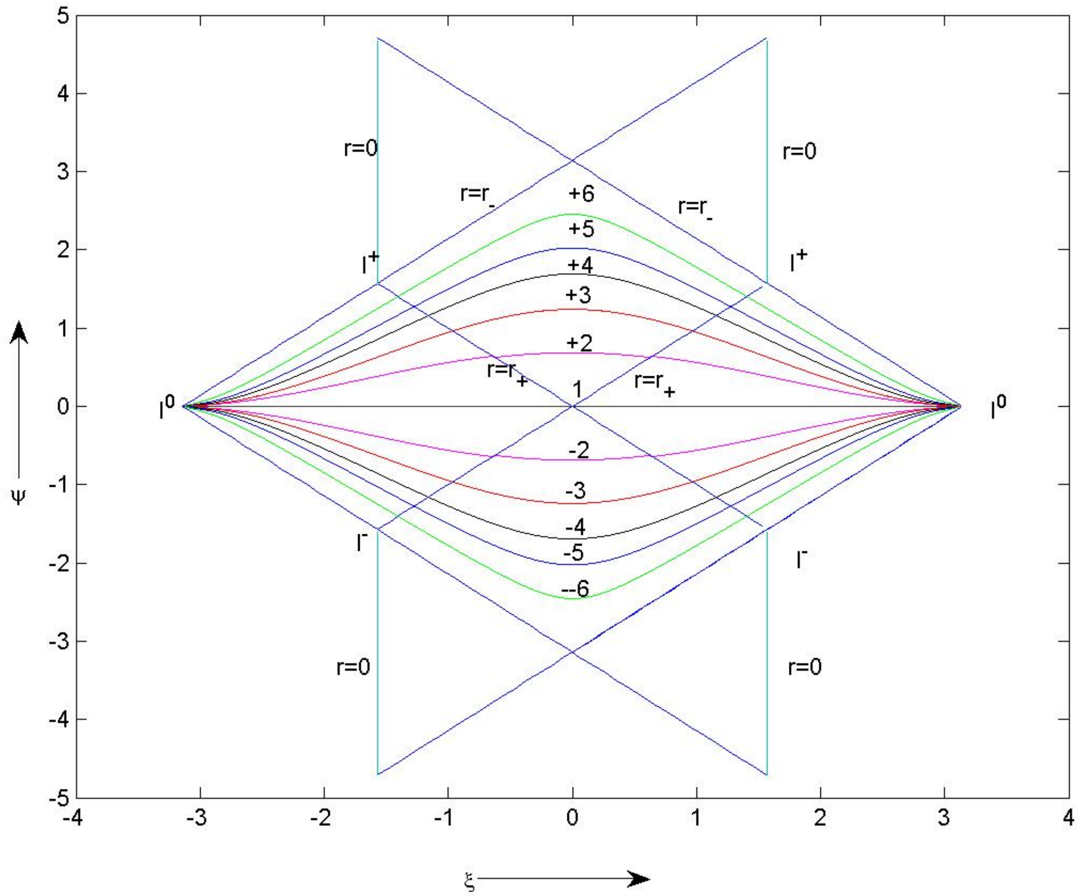

Figure 3.4: Foliation of the Carter-Penrose diagram of the SHQ geometry by K-surfaces, here $\alpha = 0.12$ and $m = 1$. Few spacelike hypersurfaces for $K = 0$, ± 0.00003, ± 0.005, ± 0.0009, ± 0.001 and ± 0.003 are shown.

No	Colour	K	r_i	H	ψ_i
1	black	r_+	0	0	0
± 2	magenta	± 0.00003	4.999	∓ 0.4549	± 0.6848
± 3	red	± 0.001	4.996	∓ 0.7028	± 1.2392
± 4	black	± 0.0009	4.990	∓ 1.3974	± 1.6920
± 5	blue	± 0.0015	4.980	∓ 1.9756	± 2.0245
± 6	green	± 0.003	4.940	∓ 3.2885	± 2.4543

Table 3.4: The mean extrinsic curvatures, K, the initial value, r_i, of r, corresponding initial value, ψ_i, of ψ and the constant H are shown for eleven different $K-$surfaces. Here, we have taken $m = 1$ and $\alpha = 0.12$.

Chapter 4

Compactified non-singular coordinates for the Reissner-Nordstrom black hole surrounded by quintessence

The metric describing the spacetime outside a Reissner-Nordstrom black hole surrounded by quintessence (RNQ) [24] is

$$ds^2 = f(r)dt^2 - \frac{1}{f(r)}dr^2 - r^2 d\Omega^2, \tag{4.0.1}$$

where

$$f(r) = 1 - \frac{2m}{r} + \frac{Q^2}{r^2} - \left(\frac{r_q}{r}\right)^{3\omega_q+1}, \tag{4.0.2}$$

where r_q is the dimensional normalization constant. Here we shall discuss the case when $\omega = \frac{-2}{3}$, with this value of ω, $f(r)$ given by eq.(4.0.2) becomes

$$f(r) = 1 - \frac{2m}{r} + \frac{Q^2}{r^2} - \frac{r}{r_q}, \tag{4.0.3}$$

which can be written as

$$f(r) = -\frac{(r^3 - r_q r^2 + 2m r_q r - Q^2 r_q)}{r^2 r_q},$$

which is cubic in $'r'$ and hence equation $f(r) = 0$ will have either

1. Two complex and one real root.

2. All the three roots to be real and equal.

3. All the three roots to be real among which two are equal.

4. All the three roots to be real and unequal.

Here we shall consider the case (4) and will call the three roots of $f(r) = 0$ as r_1, r_2 and r_3 where $r_3 < r_2 < r_1$. Hence apart from the essential singularity at $r = 0$, we have other three singularities at $r = r_1$, r_2 and r_3. So, $f(r)$ can be written as

$$f(r) = \frac{(r - r_1)(r - r_2)(r - r_3)}{r^2 r_q}. \tag{4.0.4}$$

The RNQ metric is regular in the following four regions

I :$r_1 < r < \infty$

II :$r_2 < r < r_1$

III : $r_3 < r < r_2$

IV : $0 < r < r_3$

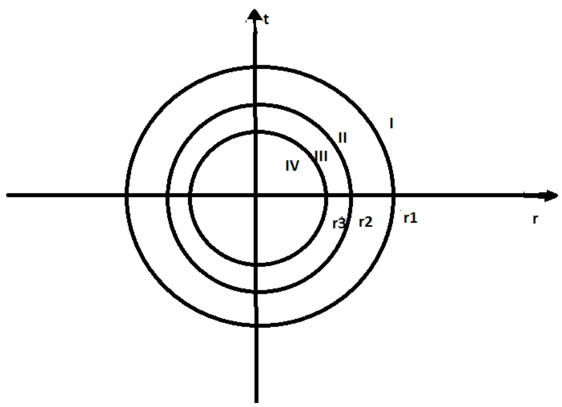

Figure 4.1: The RNQ geometry

4.1 Radial Photon Worldlines in Schwarzschild Coordinates

The radial photon world lines are given by

$$dt = \pm \frac{1}{f(r)} dr,$$

or

$$dt = \int \frac{r^2 r_q}{(r-r_1)(r-r_2)(r-r_3)} dr. \tag{4.1.1}$$

Using the partial fraction technique, we have

$$\frac{r^2 r_q}{(r-r_1)(r-r_2)(r-r_3)} = \frac{r_q r_1^2}{(r_1-r_2)(r_1-r_3)(r-r_1)} + \frac{r_q r_2^2}{(r_2-r_1)(r_2-r_3)(r-r_2)} + \frac{r_q r_3^2}{(r_3-r_1)(r_3-r_2)(r-r_3)}. \tag{4.1.2}$$

Using eq.(4.1.2) in eq.(4.1.1) and integrating, we get

$$t = \pm\eta \left\{ r_1^2(r_2-r_3)\ln|r-r_1| - r_1^2(r_2-r_3)\ln|r-r_1| + r_3^2(r_1-r_2)\ln|r-r_3| \right\}, \tag{4.1.3}$$

where

$$\eta = \frac{1}{(r_1-r_2)(r_1-r_3)(r_2-r_3)}. \tag{4.1.4}$$

Eq.(4.1.3) can be written as

$$t = \pm\eta \left\{ \ln|\frac{r}{r_1} - 1|^{r_1^2(r_2-r_3)} - \ln|\frac{r}{r_2} - 1|^{r_2^2(r_1-r_3)} + \ln|\frac{r}{r_3} - 1|^{r_3^2(r_1-r_2)} + c \right\}, \tag{4.1.5}$$

where c is a constant.

4.2 Eddington-Finkelstein Like Coordinates

The coordinate transformations like the case of SHQ as explained in chapter 2, are given as

$$p = t + \eta \left\{ \ln |\frac{r}{r_1} - 1|^{r_1^2(r_2-r_3)} - \ln |\frac{r}{r_2} - 1|^{r_2^2(r_1-r_3)} + \ln |\frac{r}{r_3} - 1|^{r_3^2(r_1-r_2)} \right\}, \quad (4.2.1)$$

and

$$q = t - \eta \left\{ \ln |\frac{r}{r_1} - 1|^{r_1^2(r_2-r_3)} - \ln |\frac{r}{r_2} - 1|^{r_2^2(r_1-r_3)} + \ln |\frac{r}{r_3} - 1|^{r_3^2(r_1-r_2)} \right\}. \quad (4.2.2)$$

4.3 Kruskal Like Coordinates

Using the coordinate transformations given by eq.(2.3.1) in chapter 2 for SHQ geometry, we get

$$\tilde{p} = \lambda e^{\frac{t}{\beta}} \left\{ |\frac{r}{r_1} - 1|^{\frac{r_1^2(r_2-r_3)\eta}{\beta}} |\frac{r}{r_2} - 1|^{-\frac{r_2^2(r_1-r_3)\eta}{\beta}} |\frac{r}{r_3} - 1|^{\frac{r_3^2(r_1-r_2)\eta}{\beta}} \right\}, \quad (4.3.1)$$

and

$$\tilde{q} = -\lambda e^{\frac{-t}{\beta}} \left\{ |\frac{r}{r_1} - 1|^{\frac{r_1^2(r_2-r_3)\eta}{\beta}} |\frac{r}{r_2} - 1|^{-\frac{r_2^2(r_1-r_3)\eta}{\beta}} |\frac{r}{r_3} - 1|^{\frac{r_3^2(r_1-r_2)\eta}{\beta}} \right\}. \quad (4.3.2)$$

Using eq.(4.1.4) in eqs.(4.3.1) and (4.3.2), we get

$$\tilde{p} = \lambda e^{\frac{t}{\beta}} \left\{ |\frac{r}{r_1} - 1|^{\frac{r_1^2}{\beta(r_1-r_2)(r_1-r_3)}} |\frac{r}{r_2} - 1|^{\frac{-r_2^2}{\beta(r_1-r_2)(r_2-r_3)}} |\frac{r}{r_3} - 1|^{\frac{r_3^2}{\beta(r_1-r_3)(r_2-r_3)}} \right\}, \quad (4.3.3)$$

and

$$\tilde{q} = -\lambda e^{\frac{-t}{\beta}} \left\{ |\frac{r}{r_1} - 1|^{\frac{r_1^2}{\beta(r_1-r_2)(r_1-r_3)}} |\frac{r}{r_2} - 1|^{\frac{-r_2^2}{\beta(r_1-r_2)(r_2-r_3)}} |\frac{r}{r_3} - 1|^{\frac{r_3^2}{\beta(r_1-r_3)(r_2-r_3)}} \right\}. \tag{4.3.4}$$

To get the usual form of metric, we define new variables v and u, which are timelike and spacelike respectively. They are given by eq.(2.3.2) and here take the form

$$v = \lambda \sinh\left(\frac{t}{\beta}\right) \left\{ |\frac{r}{r_1} - 1|^{\frac{r_1^2}{\beta(r_1-r_2)(r_1-r_3)}} |\frac{r}{r_2} - 1|^{\frac{-r_2^2}{\beta(r_1-r_2)(r_2-r_3)}} |\frac{r}{r_3} - 1|^{\frac{r_3^2}{\beta(r_1-r_3)(r_2-r_3)}} \right\}, \tag{4.3.5}$$

and

$$u = \lambda \cosh\left(\frac{t}{\beta}\right) \left\{ |\frac{r}{r_1} - 1|^{\frac{r_1^2}{\beta(r_1-r_2)(r_1-r_3)}} |\frac{r}{r_2} - 1|^{\frac{-r_2^2}{\beta(r_1-r_2)(r_2-r_3)}} |\frac{r}{r_3} - 1|^{\frac{r_3^2}{\beta(r_1-r_3)(r_2-r_3)}} \right\}. \tag{4.3.6}$$

The coordinates given by eqs.(4.3.5) and (4.3.6) are singular at $r = r_1$, r_2 and r_3, so, for the coordinates to be regular at $r = r_1$, we take

$$\beta = \beta_1 = \frac{2r_1^2}{(r_1 - r_2)(r_1 - r_3)}, \qquad \lambda = \lambda_1, \tag{4.3.7}$$

to be regular at $r = r_2$, we choose

$$\beta = \beta_2 = -\frac{2r_2^2}{(r_1 - r_2)(r_2 - r_3)}, \qquad \lambda = \lambda_2, \tag{4.3.8}$$

and for the regularity of the coordinates at $r = r_3$, we select

$$\beta = \beta_3 = \frac{2r_3^2}{(r_1 - r_3)(r_2 - r_3)}, \qquad \lambda = \lambda_3. \tag{4.3.9}$$

This shows that three patches are required to cover the whole RNQ geometry.

$$v_1 = \lambda_1 \sinh\left(\frac{t}{\beta_1}\right) \left\{ |\frac{r}{r_1} - 1|^{\frac{r_1^2}{\beta_1(r_1-r_2)(r_1-r_3)}} |\frac{r}{r_2} - 1|^{\frac{-r_2^2}{\beta_1(r_1-r_2)(r_2-r_3)}} |\frac{r}{r_3} - 1|^{\frac{r_3^2}{\beta_1(r_1-r_3)(r_2-r_3)}} \right\}, \tag{4.3.10}$$

$$u_1 = \lambda_1 \cosh\left(\frac{t}{\beta_1}\right) \left\{ |\frac{r}{r_1} - 1|^{\frac{r_1^2}{\beta_1(r_1-r_2)(r_1-r_3)}} |\frac{r}{r_2} - 1|^{\frac{-r_2^2}{\beta_1(r_1-r_2)(r_2-r_3)}} |\frac{r}{r_3} - 1|^{\frac{r_3^2}{\beta_1(r_1-r_3)(r_2-r_3)}} \right\}, \tag{4.3.11}$$

$$v_2 = \lambda_2 \sinh\left(\frac{t}{\beta_2}\right) \left\{ |\frac{r}{r_1} - 1|^{\frac{r_1^2}{\beta_2(r_1-r_2)(r_1-r_3)}} |\frac{r}{r_2} - 1|^{\frac{-r_2^2}{\beta_2(r_1-r_2)(r_2-r_3)}} |\frac{r}{r_3} - 1|^{\frac{r_3^2}{\beta_2(r_1-r_3)(r_2-r_3)}} \right\}, \tag{4.3.12}$$

$$u_2 = \lambda_2 \cosh\left(\frac{t}{\beta_2}\right) \left\{ |\frac{r}{r_1} - 1|^{\frac{r_1^2}{\beta_2(r_1-r_2)(r_1-r_3)}} |\frac{r}{r_2} - 1|^{\frac{-r_2^2}{\beta_2(r_1-r_2)(r_2-r_3)}} |\frac{r}{r_3} - 1|^{\frac{r_3^2}{\beta_2(r_1-r_3)(r_2-r_3)}} \right\}, \tag{4.3.13}$$

and

$$v_3 = \lambda_3 \sinh\left(\frac{t}{\beta_3}\right) \left\{ |\frac{r}{r_1} - 1|^{\frac{r_1^2}{\beta_3(r_1-r_2)(r_1-r_3)}} |\frac{r}{r_2} - 1|^{\frac{-r_2^2}{\beta_3(r_1-r_2)(r_2-r_3)}} |\frac{r}{r_3} - 1|^{\frac{r_3^2}{\beta_3(r_1-r_3)(r_2-r_3)}} \right\}, \tag{4.3.14}$$

$$u_3 = \lambda_3 \cosh\left(\frac{t}{\beta_3}\right)\left\{|\frac{r}{r_1} - 1|^{\frac{r_1^2}{\beta_3(r_1-r_2)(r_1-r_3)}}|\frac{r}{r_2} - 1|^{\frac{-r_2^2}{\beta_3(r_1-r_2)(r_2-r_3)}}|\frac{r}{r_3} - 1|^{\frac{r_3^2}{\beta_3(r_1-r_3)(r_2-r_3)}}\right\}. \tag{4.3.15}$$

The coordinates (v_1, u_1) are non-singular at $r = r_1$, so, they are valid for $r_2 < r < \infty$. Similarly, (v_2, u_2) are regular at $r = r_2$ and hence are valid for $r_3 < r < r_1$. On the same lines, since (v_3, u_3) are regular at $r = r_3$, so, they are used for $0 < r < r_2$. The implicit relations between these coordinates and r are

$$u_1^2 - v_1^2 = \lambda_1^2|\frac{r}{r_1} - 1|^{\frac{2r_1^2}{\beta_1(r_1-r_2)(r_1-r_3)}}|\frac{r}{r_2} - 1|^{\frac{-2r_2^2}{\beta_1(r_1-r_2)(r_2-r_3)}}|\frac{r}{r_3} - 1|^{\frac{2r_3^2}{\beta_1(r_1-r_3)(r_2-r_3)}}, \tag{4.3.16}$$

$$u_2^2 - v_2^2 = \lambda_2^2|\frac{r}{r_1} - 1|^{\frac{2r_1^2}{\beta_2(r_1-r_2)(r_1-r_3)}}|\frac{r}{r_2} - 1|^{\frac{-2r_2^2}{\beta_2(r_1-r_2)(r_2-r_3)}}|\frac{r}{r_3} - 1|^{\frac{2r_3^2}{\beta_2(r_1-r_3)(r_2-r_3)}}, \tag{4.3.17}$$

and

$$u_3^2 - v_3^2 = \lambda_3^2|\frac{r}{r_1} - 1|^{\frac{2r_1^2}{\beta_3(r_1-r_2)(r_1-r_3)}}|\frac{r}{r_2} - 1|^{\frac{-2r_2^2}{\beta_3(r_1-r_2)(r_2-r_3)}}|\frac{r}{r_3} - 1|^{\frac{2r_3^2}{\beta_3(r_1-r_3)(r_2-r_3)}}. \tag{4.3.18}$$

4.4 Compactified Kruskal-Szekeres Like Coordinates

The compactified Kruskal-Szekeres like coordinates for the RNQ metric are obtained using the same coordinate transformations as used for the SHQ case in chapter 2 and are given as

$$\psi_1 = \tan^{-1}(v_1+u_1) + \tan^{-1}(v_1-u_1),$$
$$\xi_1 = \tan^{-1}(v_1+u_1) - \tan^{-1}(v_1-u_1), \qquad (4.4.1)$$

$$\psi_2 = \tan^{-1}(v_2+u_2) + \tan^{-1}(v_2-u_2),$$
$$\xi_2 = \tan^{-1}(v_2+u_2) - \tan^{-1}(v_2-u_2), \qquad (4.4.2)$$

and

$$\psi_3 = \tan^{-1}(v_3+u_3) + \tan^{-1}(v_3-u_3),$$
$$\xi_3 = \tan^{-1}(v_3+u_3) - \tan^{-1}(v_3-u_3). \qquad (4.4.3)$$

The implicit relations between these coordinates, u_1, v_1, u_2, v_2, u_3 and v_3 and r are

$$u_1^2 - v_1^2 = \lambda_1^2 \left|\frac{r}{r_1} - 1\right|^{\frac{2r_1^2}{\beta_1(r_1-r_2)(r_1-r_3)}} \left|\frac{r}{r_2} - 1\right|^{\frac{-2r_2^2}{\beta_1(r_1-r_2)(r_2-r_3)}} \left|\frac{r}{r_3} - 1\right|^{\frac{2r_3^2}{\beta_1(r_1-r_3)(r_2-r_3)}}$$
$$= -\tan\left(\frac{\psi_1+\xi_1}{2}\right)\tan\left(\frac{\psi_1-\xi_1}{2}\right), \quad (4.4.4)$$

$$u_2^2 - v_2^2 = \lambda_2^2 \left|\frac{r}{r_1} - 1\right|^{\frac{2r_1^2}{\beta_2(r_1-r_2)(r_1-r_3)}} \left|\frac{r}{r_2} - 1\right|^{\frac{-2r_2^2}{\beta_2(r_1-r_2)(r_2-r_3)}} \left|\frac{r}{r_3} - 1\right|^{\frac{2r_3^2}{\beta_2(r_1-r_3)(r_2-r_3)}}$$
$$= -\tan\left(\frac{\psi_2+\xi_2}{2}\right)\tan\left(\frac{\psi_2-\xi_2}{2}\right), \quad (4.4.5)$$

and

$$u_3^2 - v_3^2 = \lambda_3^2 \left|\frac{r}{r_1} - 1\right|^{\frac{2r_1^2}{\beta_3(r_1-r_2)(r_1-r_3)}} \left|\frac{r}{r_2} - 1\right|^{\frac{-2r_2^2}{\beta_3(r_1-r_2)(r_2-r_3)}} \left|\frac{r}{r_3} - 1\right|^{\frac{2r_3^2}{\beta_3(r_1-r_3)(r_2-r_3)}}$$
$$= -\tan\left(\frac{\psi_3 + \xi_3}{2}\right)\tan\left(\frac{\psi_3 - \xi_3}{2}\right). \quad (4.4.6)$$

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