Finite Element Method for DEs

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Dedication

To my loving Parents, Siblings and Haroon.

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All praises and glories to **Almighty Allah** who answers my prayers and whispers to me "Never lose hope in Allah's mercy as with every Hardship, there is an Ease" and is actually **HIS** guidance and strength that enabled me to complete this work. I invoke peace for **Holy Prophet Hazrat Muhammad (PBUH)**, who is all the time a torch of Steering and braveness for the entire human race. I deeply pay my gratitude to my superviser Dr.Muhammad Asif Farooq whose guidance, continuos support and encouragement enabled me to successfully complete this thesis. I am very thankful to him for his time and help. I would like to thank Dr.Mujeeb-ur-Rehman for his help and suggestions. I would also like to thank Burhan for his help and moral support. Lastly, I would like to thank my family members for supporting and encouraging me to beleive in myself. This would not have been possible without them.

Once again, Thankyou to all those who helped me during my thesis.

Abstract

This thesis discusses solutions of ODEs and PDEs by FEM as this method has become a standard numerical technique for solving a lot of physical systems. We are comparing this method with other available methods and finding out that this is more general and powerful in terms of its applications. This method is endowed with three distinct features that account for its superiority over other numerical methods which include the representation of complex geometry into sub-domains, development of algebraic equations over each sub-domain and assembling all elements. We are applying this method on ODEs with constant coefficients and PDEs including Laplace, Burger and Heat equations by using MATLAB. The approach is to first introduce the method and its applications in different fields of science and engineering. In addition, Galerkin's method of weighted residuals is used for formulating and implementing this method. Initially for obtaining the approximate solutions, linear polynomials are assumed as trial functions. After that, shape functions are derived to obtain the estimated solutions. Finally, analytical and FEM solutions of the problems are presented in the form of tables. The study is worth investigating as it is showing that how much this numerical technique gives best approximations when compared with the exact solutions.

Contents

List of Abbreviations

FDM	Finite Difference Method
FEM	Finite Element Method
ODE	Ordinary Differential Equation
PDE	Partial Differential Equation

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Chapter 1

Introduction

1.1 Literature Review

Initially, Finite Difference Method was used for solving boundary value problems. In the field of finite element method (FEM), the most recent work started in the year 1940 with the work of Hrennikoff [1] and McHenry [2] who basically used 1-D elements in order to find stresses in continuous solids. Courant [3] through his published paper (although that was not known for many years) proposed the concept of variation form for the solution of stresses. After that, he came up with the idea of piecewise interpolation functions as a method for finding the approximate numerical solutions. Southwell [4] published a book based on finite difference method. That was actually the beginning of FEM that is from the concept of above mentioned numerical method. Levy [5] came up with the idea of flexibility (force) method. His work, [6] recommended that a new method named the displacement method could be a good option to use for the statistical analysis of aircraft structures. As it was easy to solve his equations by hand so his method became admired with the beginning of digital computer. Argyris and Kelsey [7, 8] made matrix structural analysis method by the use of energy principles which had a very important role in FEM because of the energy principles. Turner et al [9] first treated 2-D elements. He first obtained the stiffness matrices for two dimensional rectangular and triangular elements in plane stress and explained the process known as the direct stiffness method which is used to obtain the structured stiffness matrix. Early in 1950s, by the advancement of digital computers, Turner′s work was promoted for the progress of finite element stiffness equations expressed in matrix form. The term finite element was actually given by Clough [10] during plane stress analysis. Melosh [11] discovered a plane rectangular plate bending element stiffness matrix which was basically followed by the idea of curved shell bending elements stiffness matrix by Grafton and Strome [12]. The extended work on the method up to 3-D problems was done by Martin [13], Gallagher et al [14] and Melosh [15]. Argyris [16] extended this study on 3-D problems. Clough et al. [17] did work on axis symmetric solids. Zienkiewicz and Cheung [18] implemented this method on field problems like conduction of heat, irrotatinal flow of fluid etc. A large amount of work on nonlinear problems are available in Oden [19]. Early attempts made for modeling the transfer of heat problems on complex geometric shapes was found in Huebner [20]. Prior to that, a complete 3-D finite element model for heat conduction was given by Heuser [21]. Bakerin [22] applied finite element method to viscous fluid flow. All of these early works with rapid growth were continued upto the mid-1970s. Several articles were published and regular appearance of new applications found in the literature. Outstanding feedback and description of the method are available in some early texts by Finlayson [23], Desai [24], Baker [25], Fletcher [26], Reddy [27], Segerlind [28], Bickford [29], Zienkiewicz and Taylor [30], and Reddy [31].

Detailed discussion about the method is presented in Johnson [32], also Smith [33] who programmed the FEM. Furthermore, Owen and Hinton [34] gave a short dissertation on the progress of this method.

The basis of this method fundamentally lies with usual Rayleigh-Ritz and variational methods developed by Rayleigh [35] and Ritz [36]. These theories address the causes that why is the finite element approach best for type of problems where variational forms could be obtained. On the other hand, as FEM became interested to extend it to other types of problems, the classical theory became restricted and could not be applied to fluid related problems.

In early twentieth century, Galerkin [37] came up with a new method named Weighted Residuals. This method was actually found to present the theoretical basis for the problems in contrast to Rayleigh-Ritz method. Generally, this method of weighted residual was based on few steps which require; initially multiply the governing differential equation by some weight function, then the resulting product needs integration to vanish for whole domain. Galerkin method is basically a subset of the weighted residual method, since the choice of weight functions is not limited as in Galerkin method, the chosen weight functions are identical to the approximate function or trial function chosen for the approximation of corresponding problem.

Galerkin and Rayleigh-Ritz methods give the same results in a situation when obtaining a proper variational statement for the problem is possible and the same basis functions are used. More detailed information regarding the method is discussed in the earlier works by Portela and Chara [38], Chandrupatla and Belegundu [39], Liu and Quek [40], Hollig [41], Bohn and Garboczi [42], Hutton [43], Solin et _al. [44], Reddy [45], Becker [46], Ern and Guermond [47], Thompson [48], Gosz [49], Kattan [50], Moaveni [51], and more recently in Dow [52], and Bathe [53]. Most commonly, in using the FEM, the technique Galerkin method is used to find the approximate solutions to the governing problems. This is a very simple and rich method used to solve complicated problems. Once this basic concept is grasped, implementation of the FEM is very easy.

1.2 Background

Most of the physical and theoretical phenomena in the field of applied sciences and engineering can be expressed as differential equations. In general, it is not always possible to solve these equations using basic analytical methods for complex shapes. Number of natural phenomena are represented by equations that involve the rate where one variable changes with respect to the other variable(s) known as "derivative". The equation containing the derivative is called differential equation which can be further categorize into various types which assists in finding appropriate solution.

Differential equations are mainly characterized into two types of problems which are dependent on the type of conditions. One is the boundary value problem in which conditions are described at more than one point of the domain while the other type includes initial value problem that are described at only one point of the domain. BVPs are alternatively named as field problems and that field is actually the area of dependence of independent variables and usually represents the physical structure of the problem. We cannot get analytical solution for every type of differential equation which means we might face many practical situations where analytical solution does not exist so we need to devise a new method, numerical solution to solve the problem. We have numerous ways in order to solve a differential equation numerically. If the leading differential equation is of order one then we have different methods like Euler method, multiple Runge-Kutta methods, and multistep methods including Adam-Bash forth and Adam-Molten methods. Furthermore, we can transform the differential equation having higher order into a coupled first order differential equation system and after that solve the system using aforementioned methods. Not necessarily all physical problems are described by ODEs rather some from the field of science and engineering are coded in PDEs. So we have numerous techniques for finding the approximate solutions for partial differential equations. Some [54] of these are:

1. Finite Difference Method (FDM)

2. Finite Volume Method (FVM)

3. Finite Element Method (FEM)

1.2.1 Finite Difference Method (FDM)

Finite difference method (FDM) is the simplest numerical technique to understand. To solve the differential equation using FDM, first step is to discretize the whole domain into small intervals called mesh and label the grid points in the created mesh accordingly.

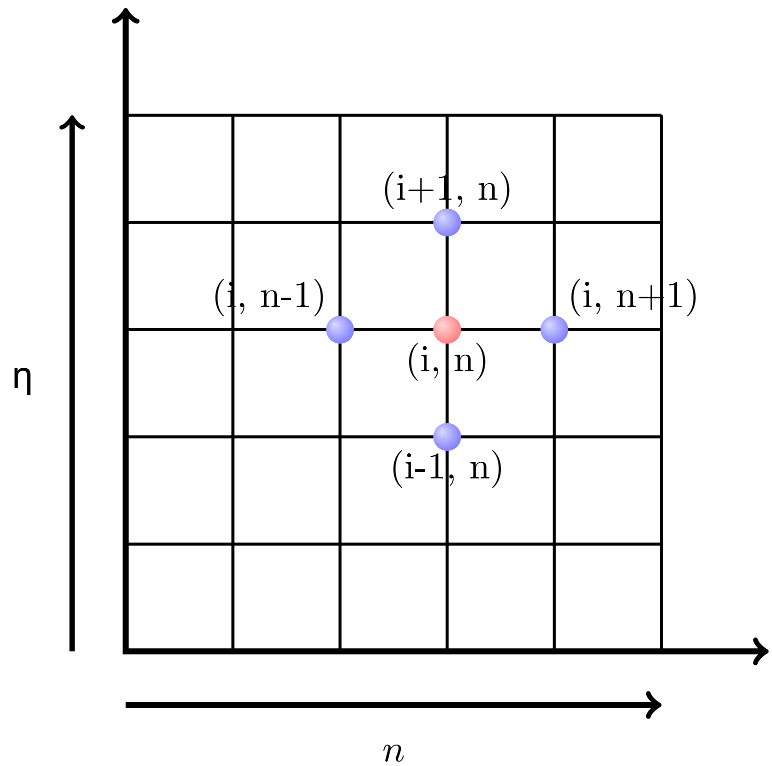

Figure 1.1: Finite difference method grid

After that, in the next step approximate the derivatives occurring in the leading differential equation by first order forward difference quotient, backward difference or second order central difference quotient. This process results in algebraic equation for the first grid point and so on. By repeating the same process for all interior grid points will lead us to a system of algebraic equations for which the grid point values of the nodal variables will remain unknown. After this, by putting in the boundary conditions we get the number of unknowns equal to the number of interior points and this system can be solved numerically using iterative methods like Jacobi Method, Gauss-Seidal Method etc.

1.2.2 Finite Element Method (FEM)

It is basically a mathematical technique which is used to find the approximate solution of differential and integral equations that occur in different fields of engineering and applied sciences. FEM was actually used to solve problems like stresses in complex aircraft structures, and then it was extended to the field of continuum mechanics which includes heat transfer and fluid flow.

Basic approach behind this technique is to replace the original problem which is

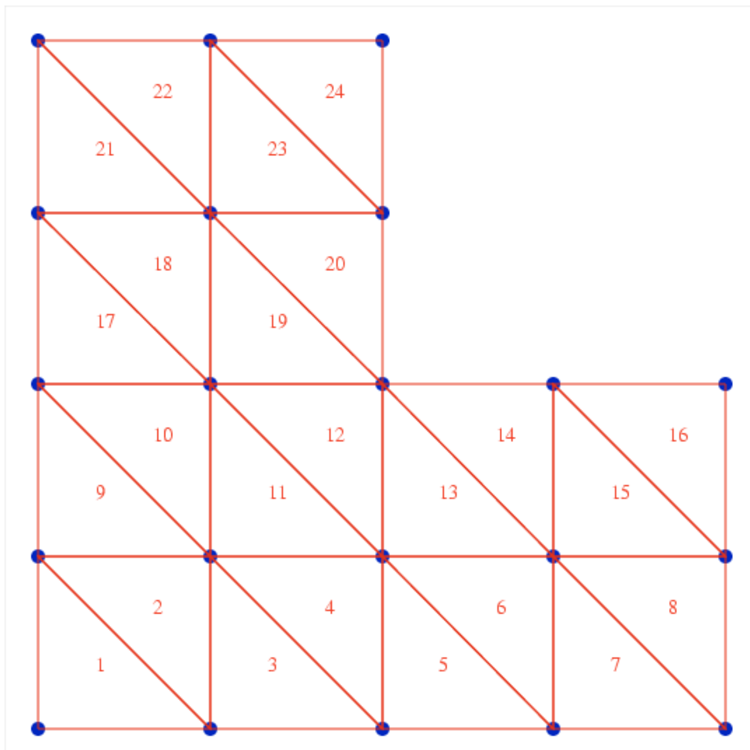

Figure 1.2: Finite element method grid

difficult with the simple one and by doing so we get an approximate solution instead of the exact solution. Further more in FEM, we can get better approximation by increasing the computational effort. In general, FEM is given preference over FDM because of the following reasons. First one is that FEM is comparatively more suitable for problems having complex geometrical domains. Another advantage is that writing the codes for FEM are easy in general. Basically, we have three different approaches available for the formulation of FEM which includes:

1. Direct method
2. Variational approach
3. Weighted residual method

Direct Method

This is an easiest approach to understand FEM. The benefit of this approach is that since it is a simple technique so we don't have to use difficult mathematics while formulating FEM via this method. But with this technique, we can only solve simple and easy problems.

Variational Approach

This approach is broadly used for solving those type of finite element equations for which variational statement is not complicated to form for certain problem. To use this approach, basic knowledge of calculus of variation is necessary. Drawback of this approach is that, we cannot find variational statement for every type of problem such as in the case of nonlinear problem. So this is not very useful approach for such type of problems. Another method named as Rayleigh-Ritz is also used to obtain approximate solutions which is based on this technique.

Weighted Residual Method

In this method, instead of depending on any variational principle we directly work on the governing differential equation. This method can be applied for both linear and nonlinear problems. This method has two basic steps. Firstly, according to the general behavior of the dependent variable, we suppose a trial function which should also satisfy the boundary conditions. Then, we substitute this assumed trial function into the governing differential equation. As this assumed solution is only an approximation to the exact solution, it will not satisfy the differential equation instead it will result into an error term generally called as residual. We need to vanish this residual over the whole domain. By this process, we will get a system of algebraic equations. Next step of this technique involves to get the solution of this system depending on the boundary conditions to get an estimated solution for the problem under consideration.

The major problem with the weighted residual method is that it is not an easy task to find an appropriate trial function to approximate the solution, because we might not have knowledge concerning the behavior of the solution since polynomial functions are designated as trial functions which may not be suitable for some cases, specifically, when the interval is large. So the low degree polynomials better reflect the behavior of the function when the interval is short. Hence, the weighted residual method is best to apply using polynomials of lower degree by subdividing the larger interval into smaller subintervals, that is, we can use the piecewise lower degree polynomials in smaller subintervals instead of going with polynomials of higher degree for the whole domain. This is actually the basic idea used in FEM.

Differences and Similarities in FDM and FEM

To differentiate between the finite element technique with finite difference technique, initial vital issue that differs in each strategies is that, in FEM, the variation of the field variable in physical domain is an essential part of the strategy that's relying upon the chosen interpolation functions, the variation of the field variable all over the finite element is taken into account to be the integral part for the formulation of the problem. Whereas in finite difference technique, this can not be true. Here, the field variable is computed at explicit points. The vital outcome of this difference is that derivatives are evaluated in FEM, however, FDM provides information solely on the variable itself. For structural issues, each strategies offer displacement solutions, FEM will be accustomed directly to calculate strain parts that's the first derivatives whereas to get a similar in FDM it needs additional concerns.

Now regarding the similarities between both strategies, the first one is that integration points within the finite difference technique are equivalent to the nodes in a finite element technique. Secondly, as if we select the smaller step size in FDM, we have a tendency to expect the convergence of solution to the exact solution that is analogous to the expected convergence in FEM in a sense that if we have a tendency to refine mesh of elements. In each cases, the refinement of step size and mesh both results in the reduction of mathematical model. Also in both cases, the governing problem reduces to an algebraic system of equations.

Most likely, the more expressive way to differentiate between both methods is to notice that the FDM models the differential equations of the problem and uses the numerical integration to find the solution at discrete points whereas the FEM models the whole domain of the problem and uses the known physical principles to build up the algebraic equations which describe the approximate solutions. Therefore, FDM models a differential equations whereas FEM models a physical problem. In some cases, combination of both methods is very helpful and able to find solutions for engineering problems mainly where dynamic effects are essential.

1.2.3 Finite Volume Method (FVM)

This finite volume method is a widely used mathematical approach. In this method, the partial differential equations which represent the conservation laws are transformed using differential volumes into system of distinct algebraic equations for finite volumes. Afterward, this system is used to calculate the values of the dependant variable for each element.

The finite volume method is in the form of integrals rather than differentials. In this method, domain is further divided into non-overlapping volumes known as finite volumes or cells. The conservation equations that are basically used to determine the

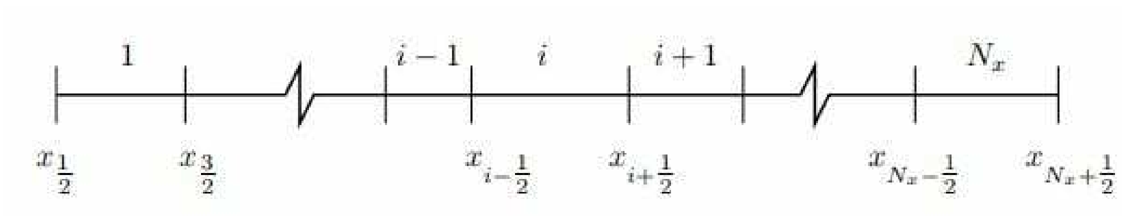

Figure 1.3: Finite volume method grid

flow variables in discrete points of the cells are defined for each finite volume. The integrals appearing in the expression are evaluated at computational nodes usually taken as centroids of finite volumes using interpolation functions. We use approximate integral formulas and interpolation techniques to get values of dependent variables at control surfaces. FVM is suitable for all type of complex geometries because it can accommodate any kind of grid.

Basic Steps in FVM

- Divide the domain into finite sized sub-domains known as finite volumes where each sub domain is represented by some grid points.
- Integrate the governing differential equation over each sub-domain by first converting volume into surface integral by using divergence theorem and then finding the flux through each face.
- Consider a suitable profile assumption for dependent variable in order to integrate.

1.3 Applications of FEM

1.3.1 Structural

- Structural areas [55] include thermal analysis of manufacturing parts including electronic devices and chips, pipes, valves, aircraft etc and stress analysis which includes frame analysis like pedestrian walk bridges, frames of residence buildings, and towers etc. Also stress attention problems usually related with holes, fillets or other types of changes in geometry of a body including automotive parts, medical devices, pressure vessels, aircraft and sports tools.

- Vibration analysis like in vibratory tools and buckling in columns, frames etc. Impact problems which include collision analysis of vehicles, projectile impact and bodies falling etc.

1.3.2 Non-Structural

- Nonstructural problems [55] include analysis of fluid flow which includes leakage all the way through porous media such as leakage of water through dams, cooling ponds and air in ventilation systems, air flow around racing cars, boats and surfboards etc.

- Also the movement of heat like in electronic devices which emit heat in computers microprocessor chip, engines, cooling fins in radiators.

- Division of electric potential for example in antennas and transistors etc.

- Investigation of surgical methods for example plastic surgery, reconstruction of jaw and many more.

These were just few applications of FEM in order to get the general idea. New areas of application in FEM are regularly appearing in the literature. This method has a very interesting contribution in medical field. Few years back, the medical society has come up with exciting choices of predictive, patient peculiar medicine. One perspective that is used in predictive machine is usage of medical imaging and data monitoring for the construction of a part of model for anatomy of individuals and physiology. This model is later utilized for the prediction of response of patients towards different treatments, for example, procedures that are used in surgery. As in finite difference methods, the finite elements are also utilized in scientific simulations like in tetrahedral mesh of a heart model on which a finite element method is used for simulating activities of the heart. Such type of models play a significant role for the development of cardiovascular surgical procedures and the design of artificial heart valves.

1.4 Ordinary Differential Equations

A lot of problems in the field of computational engineering [57] can be described in terms of differential equations. An ODE is defined to be a relation between a function $y(x)$ of a dependent variable x and its derivatives y', y'', y^n which can be expressed generally as:

$$\Psi(x, y, y', y'', , y^n) = 0.$$

A function $y = f(x)$ is the solution of the equation which satisfy the equation $\forall\ x$ in the specified domain Ω.

1.4.1 Types of Solutions

The solution of an ODE is not every time simple, specially for the nonlinear equations, it becomes quite complicated. Even for the case of linear equations, solutions are straightforward only for simple cases. Generally, solution falls into three categories:

- Closed form
- Integral form
- Series form

Solution of the type which can be described in the form of elementary functions and arbitrary constants is known as the closed form solution. Some specific type of equations for which closed form is not possible to form then we go towards series solution approach which we call a series form solution. Further, we have an integral form which is the only possible form when closed and series form is not possible. If all the above mentioned forms are impossible to apply than we need to approach the alternative techniques for finding the approximate and numerical solutions of the problems.

Linear differential equations with constant coefficients are widely used in the modeling of physical phenomena, for example, in the analysis of vibrating systems and electrical circuits. There are a lot of numerical methods (single and multistep) and most of them are based on the same Taylor series approximation discretization technique.

1.5 Partial Differential Equations

A partial differential equation [57] (PDE) is a relationship which contains one or more than one partial derivatives. In contrast with ODEs, PDEs are very complex especially in the case of nonlinear equations where the solution techniques are very difficult even the numerical simulations are not usually straight forward. In this chapter, we will discuss both linear and nonlinear PDEs namely Laplace, Poisson, Heat and Burgers equation. Like in ODE, the order n of the PDE is the highest n^{th} partial derivative in the equation. In general, PDE can be expressed as

$$\Psi(x, y, \cdots, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial^2 u}{\partial x^2}, \frac{\partial^2 u}{\partial y^2}, \frac{\partial^2 u}{\partial x \partial y}, \cdots) = 0 \tag{1.1}$$

where u corresponds to the dependent variable whereas $x, y, \cdots$ corresponds to the independent variables.

The linear partial differential equation having order one, can be written as

$$a(x, y)\frac{\partial u}{\partial x} + b(x, y)\frac{\partial u}{\partial y} = f(x, y) \tag{1.2}$$

which is for two independent variables and one dependent variable u.

To simplify the notations of PDEs , commonly used compact forms in literature are

$$u_x \equiv \frac{\partial u}{\partial x}, u_y \equiv \frac{\partial u}{\partial y}, u_{xx} \equiv \frac{\partial^2 u}{\partial x^2}, u_{yy} \equiv \frac{\partial^2 u}{\partial y^2}, u_{xy} \equiv \frac{\partial^2 u}{\partial x \partial y} \tag{1.3}$$

and therefore Eq. (1.2) can be written as

$$au_x + bu_y = f. \tag{1.4}$$

Chapter 2

FEM Solution of Scalar Second Order ODE

In this chapter, we will present the basic steps used in FEM. Furthermore, we will introduce the weighted residual technique used for FEM formulation. We will take an ordinary differential equation for a one-dimensional system and make its weak form and define shape functions for the implementation of FEM.

2.1 Basic Steps of FEM

Following are the main steps [54, 56]:

1. First of all, the domain of interest is subdivided into very small sub regions or subintervals which are non-overlapping called as finite elements. This process of subdivision is basically called as discretization. In this process, elements are joined at $x_1, x_2, \ldots, x_{n-1}$, here x_i are the interior nodes and we call $x_0 = a$, the left boundary point and $x_n = b$ the right boundary point and in general these both are the exterior nodes. The process of choosing the shape, size and number of elements is very important as we should have to choose them in such a way that the actual domain must be simulated as closely as possible so that we can get rid of extra computational efforts.

2. Next step is to assume trial function usually known as the interpolation polynomial for the differential equation which corresponds to the variation of dependent variable over elements. In general, accuracy of approximate solution is totally dependent on the chosen trial function.

3. After the assumption of trial or approximate function, we substitute that assumed function into the corresponding differential equation. As it is the approximate solution so it will not satisfy the differential equation identically. So, in order to satisfy the differential equation we get an error term which we call in FEM as residual.

4. Then we multiply this residual term with some weight function and integrate this product over the entire domain and set it equal to zero.

5. After that we decide the test function which is the weight function. We have several weighted residual methods available for the selection of weight function. More commonly, we use Galerkin approach of the weighted residual to evaluate the weight function.

6. In the next step, we construct the weak formulation of the problem by using integration by parts with appropriate functions to reduce the order of derivative of trial function which lead to weak formulation of differential equation.

7. For Galerkin method, weak formulation is very useful since here the test functions and trial functions are identical. If the leading differential equation is self adjoint operator (A second-order linear homogeneous differential equation is called self-adjoint if and only if it has the following form: $\frac{d}{dx}(h(x)y') + \Psi(x)y = 0 \quad \text{for} \quad a < x < b$, where $h(x) > 0$ on (a,b) and $\Psi(x), h'(x)$, are continuous functions on [a,b]) then we get symmetric matrix. Other than weak or strong formulation, the choice of selecting trial function is very important for accurate approximation of the solution. So far the best choice for choosing the trial function is piecewise linear functions in 1-D domain. Once we are done with weak formulation of the

leading differential equation, substitute the test and trial functions into it and evaluate the integral for each element.

8. Once we accomplish this the element wise evaluation of integral which will result in square matrices, the next step is to assemble the matrices that is the assembly process in which we gather all the matrices into one large matrix by adding the corresponding entries. It produces an algebraic system of equations which is the one equation for each element.

9. After this, we need to apply the boundary conditions for finding the nodal variables. Once the nodes are determined, solution can be obtained from corresponding nodes and trial functions within each element.

10. In general, this system of algebraic equations gives the approximate solution.

Selection of Elements

Selecting the certain type of elements and suitable trial functions are very important while using FEM. Trial function is usually referred to as interpolation polynomial. So the type of element you are going to use actually means the following points we have to consider:

1. Element's geometrical shape whether it is a line segment, triangle, or rectangle etc.

2. The number and type of nodes in each element whether it contains two or three nodes etc and are they interior or exterior. Exterior nodes are those which lie on the boundary of the element representing the point of connection between the bordering elements and interior nodes have no connection with neighboring elements.

3. Types of the nodal variable whether it may have single degree of freedom or several degree of freedom depending on the type of problem.

4. Type of the approximating function whether its polynomial or trigonometric function etc. Polynomials are used frequently in general because they are easy to handle.

2.2 Finite Element Formulation

In order to illustrate the FEM [58] we will solve the second order ordinary differential equation

$$a\frac{d^2u}{dx^2} + b\frac{du}{dx} + cu = f(x) \tag{2.1}$$

with the boundary conditions

$$u(0) = 0 \quad \& \quad u(L) = 0.$$

2.2.1 Weak Formulation

In case of complex domains, solving the strong form (governing differential equations) is not always efficient and there may not be smooth (classical) solutions to a particular problem. Moreover, while solving strong form directly it is not easy to incorporate boundary conditions. As on continuity, the requirement of field variables is much stronger hence weak formulation is given preference over strong formulation.
Weak formulation of Eq. (2.1) is :

$$\int_0^L w\left[a\frac{d^2u}{dx^2} + b\frac{du}{dx} + cu\right]dx = \int_0^L wf(x)dx. \tag{2.2}$$

By using Leibnitz's formula on the first term of the above expression i.e $aw\frac{d^2u}{dx^2}$ by letting $u = w$ and $dv = \frac{d^2u}{dx^2}$.

This implies $du = dw$ and $v = \frac{du}{dx}$

$$\int_0^L \left[-a\frac{dw}{dx}\frac{du}{dx} + bw\frac{du}{dx} + cwu\right]dx = \int_0^L wf(x)dx - \left[aw\frac{du}{dx}\right]_0^L. \tag{2.3}$$

2.3 Shape Functions

The chosen trial function corresponds to the accuracy of estimated solution. Also, it is observed that use of piecewise continuous functions as a test function is very advantageous because by increasing the number of sub-domains, representation of complex function in terms of sum of simple piecewise linear functions is possible.
Consider a finite element shown in Fig. 2.1
Here an element has two nodes, one at the left side and one at the right side. The

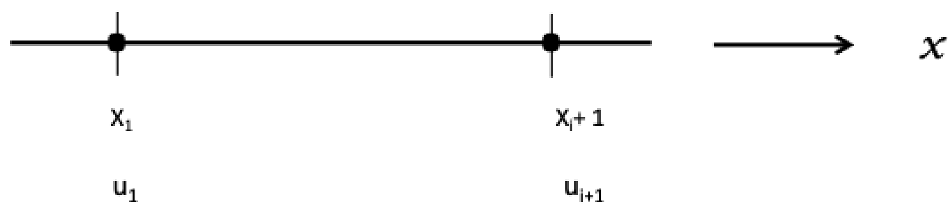

Figure 2.1: Two-node linear element

values are assigned at (x_i or x_{i+1}) and (u_i or u_{i+1}).
Assume the trial function as

$$u = b_1 x + b_2. \tag{2.4}$$

We replace b_1 and b_2 by u_i and u_{i+1} at $x = x_i$ and $x = x_{i+1}$ i.e.

$$u(x_i) = b_1 x_i + b_2 = u_i \tag{2.5}$$

$$u(x_{i+1}) = b_1 x_i + b_2 = u_{i+1} \tag{2.6}$$

By simultaneously solving for b_1 and b_2 we get

$$b_1 = \frac{u_{i+1} - u_i}{x_{i+1} - x_i} \; ; \; b_2 = \frac{u_i x_{i+1} - u_{i+1} x_i}{x_{i+1} - xi}$$

Putting the values of b_1 and b_2 into Eq. (2.4) , we get

$$u = \frac{u_{i+1} - u_i}{x_{i+1} - x_i} x + \frac{u_i x_{i+1} - u_{i+1} x_i}{x_{i+1} - xi}$$
$$h = x_{i+1} - x_i$$
$$u = \frac{u_i(x_{i+1} - x)}{h_i} + \frac{u_{i+1}(x - x_i)}{h_i} \tag{2.7}$$

$$u = H_1(x)u_i + H_2(x)u_{i+1} \tag{2.8}$$

where $H_1(x) = \frac{x_{i+1}-x}{h_i}$ and $H_2(x) = \frac{x-x_i}{h_i}$.

Eq. (2.7) is dependent on u_i and u_{i+1} and $H_1(x)$ and $H_2(x)$ are called linear shape functions.

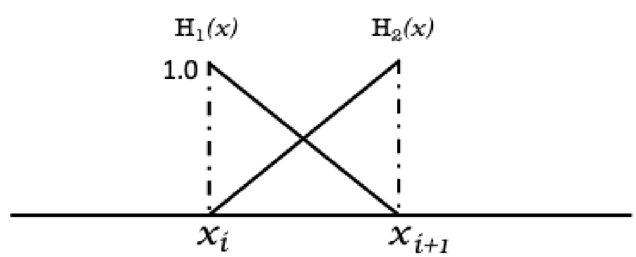

Figure 2.2: Linear shape function

Shape functions have two important characeristics:

(1) It corresponds to 1 at node i and zero otherwise.

$$H_1(x_i) = 1 \; ; \; H_1(x_{i+1}) = 0; \; H_2(x_i) = 0; \; H_2(x_{i+1}) = 1$$

(2) All shape functions add up to give unity.

$$\sum_{i=1}^{2} H_i(x) = 1 \tag{2.9}$$

Using the shape function $H_1(x)$ and $H_2(x)$ in Eq. (2.3) where $w = \begin{bmatrix} H_1 \\ H_2 \end{bmatrix}$ and $u = \begin{bmatrix} H_1 & H_2 \end{bmatrix}$. Also we have $H_1' = \frac{-1}{h_i}$ and $H_2' = \frac{1}{h_i}$.

Hence the left hand side of Eq. (2.3) becomes

$$\left[K^e\right] = \int_{x_i}^{x_{i+1}} \left(-a \begin{bmatrix} H_1' \\ H_2' \end{bmatrix} \begin{bmatrix} H_1' & H_2' \end{bmatrix} + b \begin{bmatrix} H_1' \\ H_2' \end{bmatrix} \begin{bmatrix} H_1' & H_2' \end{bmatrix} + c \begin{bmatrix} H_1' \\ H_2' \end{bmatrix} \begin{bmatrix} H_1 H_2 \end{bmatrix} \right) \mathrm{d}x. \tag{2.10}$$

After putting the shape functions and their derivatives in above equation and evaluation of integration gives

$$\left[K^e\right] = \frac{-a}{h_i} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} + \frac{b}{2} \begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix} + \frac{ch_i}{6} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}. \tag{2.11}$$

Now consider the right hand side of Eq. (2.3), here the element vector is:

$$\left[F^e\right] = \int_{x_i}^{x_{i+1}} \left(f(x) \begin{bmatrix} H_1 \\ H_2 \end{bmatrix} \right) \mathrm{d}x \tag{2.12}$$

if $f(x) = 1 \Longrightarrow$

$$F^e = \frac{h_i}{2} \begin{bmatrix} 1 \\ 1 \end{bmatrix}. \tag{2.13}$$

We have given a brief introduction of the method in the previous chapter, here we will illustrate this technique by alluding to an example.

Problem 2.1

Consider an ODE over the domain $0 < x < 1$ i.e.

$$a\frac{d^2u}{dx^2} + b\frac{du}{dx} + cu = f(x)$$

with boundary conditions

$$u(0) = 0 \;\; \& \;\; u(1) = 0.$$

Here a, b and c have the values of 1, -3 and 2 respectively. Also $f(x) = 1$.

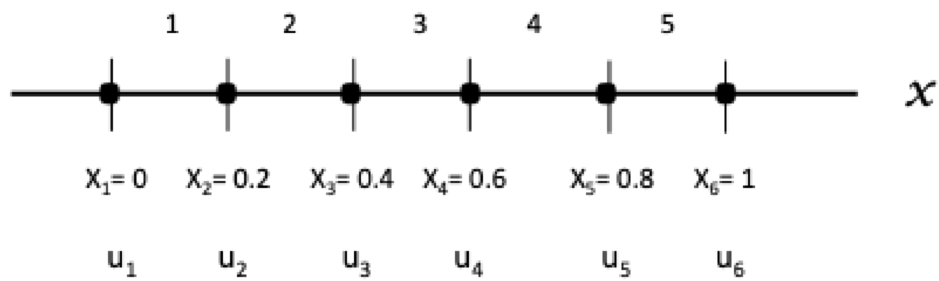

Figure 2.3: Mesh with five linear elements

By writing the MATLAB code using FEM [58], we have the following results as shown in the table and after that we have done with error analysis of exact solution and finite element solution.

Node#	FEM Sol	Exact Sol	Error
1	0.00000	0.00000	0.00000
2	-0.0621	-0.0610	0.00011
3	-0.1133	-0.1110	0.00023
4	-0.1388	-0.1355	0.00033
5	-0.1142	-0.1111	0.00031
6	0.00000	0.00000	0.00000

Table 2.1. Comparison of FEM solution and exact solution

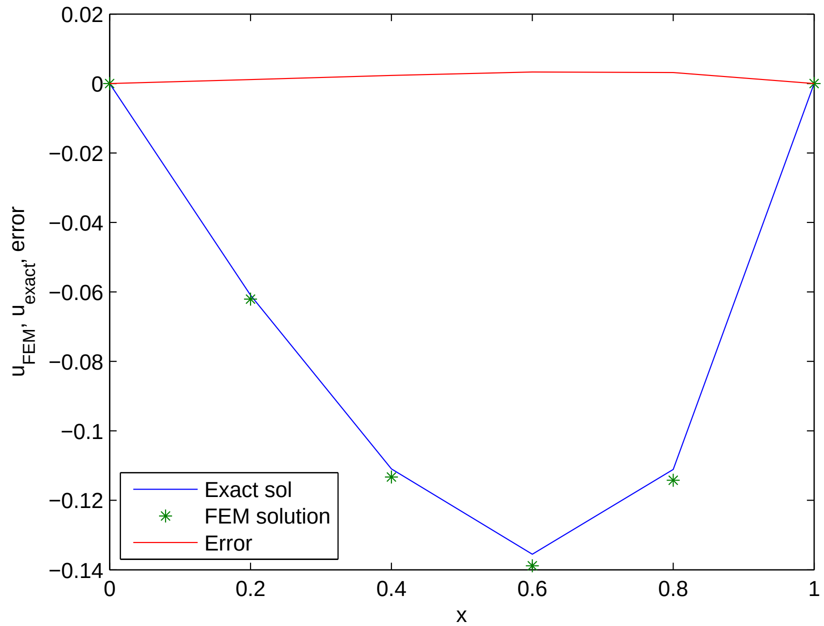

Figure 2.4: Exact and numerical solution with error for first order ODE

Problem 2.2

Consider second order ODE over the domain $0 < x < 1$ i.e.

$$a\frac{d^2u}{dx^2} + b\frac{du}{dx} + cu = f(x)$$

with boundary conditions

$$u(0) = 0 \quad \& \quad \frac{du(1)}{dx} = 1.$$

Here a, b and c have the values of 1, -3 and 2 respectively. Also $f(x) = 1$. Here, the left end corresponds to the essential boundary condition where as the right end corresponds to the natural boundary condition.

By writing the MATLAB code through FEM [58], we have the following results as mentioned below in the table and also the error plot is given in the figure.

Node#	FEM Sol	Exact Sol	Error
1	0.00000	0.00000	0.0000
2	-0.0588	-0.0578	0.0010
3	-0.1043	-0.1024	0.0019
4	-0.1203	-0.1180	0.0023
5	-0.0802	-0.0792	0.0010
6	0.0586	0.0546	-0.0041

Table 2.2. Comparison of FEM solution and exact solution

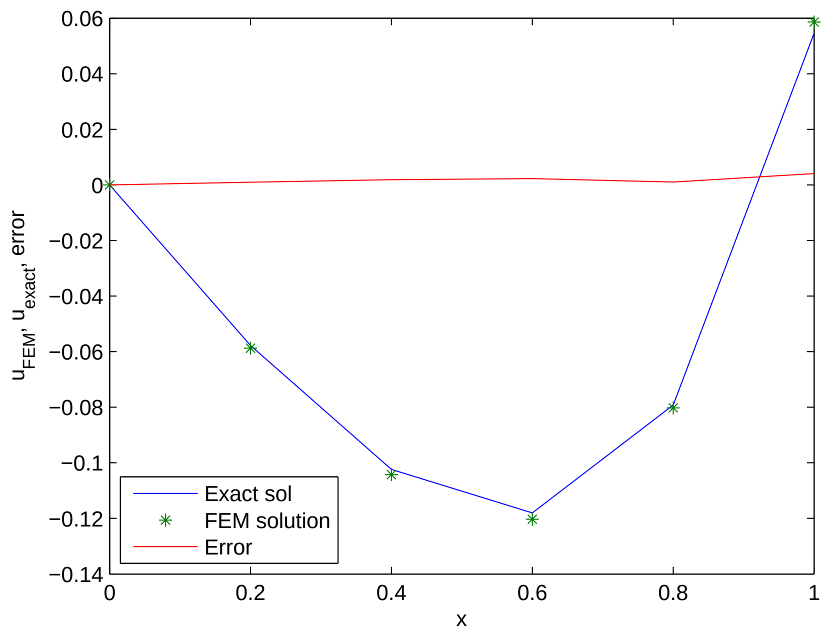

Figure 2.5: Exact and numerical solution with error for second order ODE

Chapter 3

FEM Solution of PDEs

In this chapter, we will discuss the finite element formulation for Poisson's equation for two types of elements that is the linear triangular elements and bilinear rectangular elements along with discussion on boundary integral used for computation of column vectors. In addition, we will discuss the general theory of FEM for 1-D and 2-D heat equation. Furthermore, PDE toolbox will be discussed for finding the solution of PDEs and also the FEM formulation for the case of non-linear PDEs.

3.1 Formulation for Poisson's Equation

To express different physical natures, Poisson's and Laplace's equations [58] are common field governing equations. These equations generally represent the heat conduction and flow of potential etc. Here we will apply the formulation of finite element on these equations. Hence, Laplace's equation is given as

$$\nabla^2 u = 0 \tag{3.1}$$

and Poisson's equation is given as

$$\nabla^2 u = g \tag{3.2}$$

where $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$ is the differential operator.
Since Poisson's equation is more general as compared to Laplace's equation, so we will apply the finite element formulation on it.

In terms of Cartesian coordinates, Poisson's equation is given as

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = g(x, y) \text{ in } \Omega \tag{3.3}$$

for the two-dimensional domain Ω. The relevant boundary conditions are defined as

$$u = \bar{u} \text{ on } \Gamma_e \tag{3.4}$$

and

$$\frac{\partial u}{\partial n} = \bar{q} \text{ on } \Gamma_n \tag{3.5}$$

where $\bar{u}$ and $\bar{q}$ represent the identified variable and flux boundary conditions, and n represents the unit normal vector. In addition, Γ_n and Γ_e represent the boundaries for natural and essential boundary conditions, respectively. For a well-posed bvp, we have

$$\Gamma_e \bigcup \Gamma_n = \Gamma \tag{3.6}$$

and

$$\Gamma_e \bigcap \Gamma_n = \emptyset \tag{3.7}$$

where $\bigcup$ is the sum and $\bigcap$ is the intersection, and Γ denotes the entire boundary of the respective domain Ω.

By integrating the weighted residual of the differential equation given in Eq. (3.3) and applying the boundary condition we get the following equation

$$I = \int_{\Omega} w \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} - g(x, y))d\Omega - \int_{\Gamma_e} w(\frac{\partial u}{\partial n} \right) d\Gamma \tag{3.8}$$

3.1.1 Weak Formulation

So as to develop the weak formulation of Eq. (3.8), we need to apply by parts integration to decrease the order of differentiation inside the integral.

First we will evaluate the first term of Eq. (3.8) that is

$$\int_\Omega w\frac{\partial^2 u}{\partial x^2}d\Omega. \tag{3.9}$$

Domain integral can be represented as below

$$\int_{y_1}^{y_2}\left(\int_{x_1}^{x_2} w\frac{\partial^2 u}{\partial x^2}dx\right)dy \tag{3.10}$$

y_1 and y_2 here represent the maximum and minimum values in the y-axis of the respective domain. By using Leibnitz's formula with respect to variable x, we get

$$-\int_{y_1}^{y_2}\int_{x_1}^{x_2}\frac{\partial w}{\partial x}\frac{\partial u}{\partial x}dxdy+\int_{y_1}^{y_2}\left[w\frac{\partial u}{\partial x}\right]_{x_1}^{x_2}dy \tag{3.11}$$

and again writing the above expression using the domain and boundary integrations, we get

$$-\int_\Omega\frac{\partial w}{\partial x}\frac{\partial u}{\partial x}d\Omega+\int_{\Gamma_2} w\frac{\partial u}{\partial x}n_x d\Gamma-\int_{\Gamma_1} w\frac{\partial u}{\partial x}n_x d\Gamma. \tag{3.12}$$

Here n_x represents the x-component of the unit normal vector. Now combining the both boundary integrals we get

$$-\int_\Omega\frac{\partial w}{\partial x}\frac{\partial u}{\partial x}d\Omega+\oint_\Gamma w\frac{\partial u}{\partial x}n_x d\Gamma. \tag{3.13}$$

In the similar manner, we can express second term of Eq. (3.8) as

$$-\int_\Omega\frac{\partial w}{\partial y}\frac{\partial u}{\partial y}d\Omega+\oint_\Gamma w\frac{\partial u}{\partial y}n_y d\Gamma \tag{3.14}$$

Adding the expressions (3.13) and (3.14) we get

$$\int_\Omega w\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right)d\Omega=-\int_\Omega\left(\frac{\partial w}{\partial x}\frac{\partial u}{\partial x}+\frac{\partial w}{\partial y}\frac{\partial u}{\partial y}\right)d\Omega+\oint w\left(\frac{\partial u}{\partial x}n_x+\frac{\partial u}{\partial y}n_y\right)d\Gamma \tag{3.15}$$

As the boundary integral can be expressed as

$$\frac{\partial u}{\partial n} = \frac{\partial u}{\partial x} n_x + \frac{\partial u}{\partial y} n_y. \tag{3.16}$$

Thus Eq. (3.15) becomes

$$\int_\Omega w \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) d\Omega = - \int_\Omega \left(\frac{\partial w}{\partial x} \frac{\partial u}{\partial x} + \frac{\partial w}{\partial y} \frac{\partial u}{\partial y} \right) d\Omega + \oint_\Gamma w \frac{\partial u}{\partial n} d_\Gamma \tag{3.17}$$

Above expression is known as Green's theorem.

Using Eq. (3.17) into Eq. (3.8) will result in the following equation

$$I = - \int_\Omega \left(\frac{\partial w}{\partial x} \frac{\partial u}{\partial x} + \frac{\partial w}{\partial y} \frac{\partial u}{\partial y} \right) d\Omega - \int_\Omega w g(x,y) d\Omega + \int_{\Gamma_n} w \frac{\partial u}{\partial n} d\Gamma \tag{3.18}$$

In the above expression, first term which is the volume integral will become a matrix term where as the second term which is volume integral and the third term which is line integral will turn out to be the vector terms.

3.1.2 Linear Triangular Elements

Domain descritization in Eq. (3.18) is done by using two-dimensional finite elements. One technique of discretization is by the use of triangular element having three nodes, which are linear triangular element (cf.3.1). The linear element in x and y is written as

$$u = a_1 + a_2 x + a_3 y \tag{3.19}$$

or

$$u = \begin{bmatrix} 1 & x & y \end{bmatrix} \begin{Bmatrix} a_1 \\ a_2 \\ a_3 \end{Bmatrix} \tag{3.20}$$

here a_i's are all constants which need to be determined. The interpolation function as in Eq. (3.19), corresponds to the nodal variables at three nodal points. Hence, putting in values of x and y at every nodal point is given as

$$\begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} = \begin{bmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{bmatrix} \begin{Bmatrix} a_1 \\ a_2 \\ a_3 \end{Bmatrix} \tag{3.21}$$

.

In above equation, x_i and y_i are the i^{th} coordinates where u_i are its nodal values. Inversion of the matrix and again writing the Eq. (3.21) gives

$$\begin{Bmatrix} a_1 \\ a_2 \\ a_3 \end{Bmatrix} = \frac{1}{2A} \begin{bmatrix} x_2y_3 - x_3y_2 & x_3y_1 - x_1y_3 & x_1y_2 - x_2y_1 \\ y_2 - y_3 & y_3 - y_1 & y_1 - y_2 \\ x_3 - x_2 & x_1 - x_3 & x_2 - x_1 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} \tag{3.22}$$

where

$$A = \frac{1}{2} det \begin{bmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{bmatrix} \tag{3.23}$$

Area of the linear triangular element equals the magnitude of matrix A. Although, when the node numbering of element is in the counter-clockwise direction, it will have a positive value and negative otherwise. For computation purpose, the nodal value must have the same direction as for the elements in the domain.

Substituting the Eq. (3.22) into Eq. (3.20) results in

$$u = H_1(x, y)u_1 + H_2(x, y)u_2 + H_3(x, y)u_3 \tag{3.24}$$

where $H_i(x, y)$ are known as the shape functions for the linear triangular element and

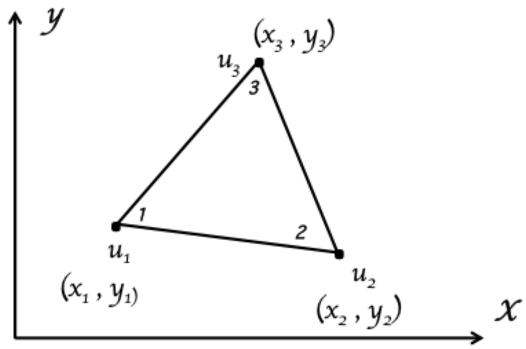

Figure 3.1: Linear triangular element

are given as:

$$H_1 = \frac{1}{2A}[(x_2y_3 - x_3y_2) + (y_2 - y_3)x + (x_3 - x_2)y] \tag{3.25}$$

$$H_2 = \frac{1}{2A}[(x_3y_1 - x_1y_3) + (y_3 - y_1)x + (x_1 - x_3)y] \tag{3.26}$$

$$H_3 = \frac{1}{2A}[(x_1y_2 - x_2y_1) + (y_1 - y_2)x + (x_2 - x_1)y] \tag{3.27}$$

These shape functions have some properties that are already mentioned in the previous chapter.

So our problem can be descritized by different elements as discussed below (cf.3.2).

Here the curved boundary which is the original boundary is estimated using linear piecewise boundary and the crude mesh (cf.3.2) is for the closer approximation of the original boundary while linear triangular elements have been used. In addition, another way is by using the finite elements of higher order which can easily fit in the curved boundary using higher order polynomial expressions.

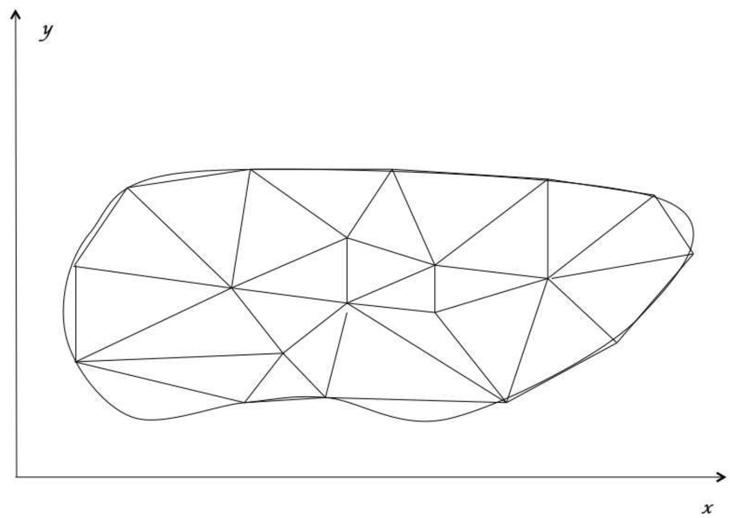

Figure 3.2: Finite element descritization

The element matrix for is computed below:

$$[K^e] = \int_{\Omega^e} \left(\frac{\partial w}{\partial x}\frac{\partial u}{\partial x} + \frac{\partial w}{\partial y}\frac{\partial u}{\partial y} \right) d\Omega = \int_{\Omega^e} \left(\begin{Bmatrix} \frac{\partial H_1}{\partial x} \\ \frac{\partial H_2}{\partial x} \\ \frac{\partial H_3}{\partial x} \end{Bmatrix} \right) \begin{Bmatrix} \frac{\partial H_1}{\partial x} & \frac{\partial H_2}{\partial x} & \frac{\partial H_3}{\partial x} \end{Bmatrix} + \left(\begin{Bmatrix} \frac{\partial H_1}{\partial y} \\ \frac{\partial H_2}{\partial y} \\ \frac{\partial H_3}{\partial y} \end{Bmatrix} \right) \begin{Bmatrix} \frac{\partial H_1}{\partial y} & \frac{\partial H_2}{\partial y} & \frac{\partial H_3}{\partial y} \end{Bmatrix} \quad (3.28)$$

where Ω^e denotes the domain of an element.

Now by substituting the shape function Eq. (3.25) through Eq. (3.27) into Eq. (3.28) and performing integration we get

$$[K^e] = \begin{bmatrix} k_{11} & k_{12} & k_{13} \\ k_{21} & k_{22} & k_{23} \\ k_{31} & k_{32} & k_{33} \end{bmatrix} \quad (3.29)$$

where

$$k_{11} = \frac{1}{4A}[(x_3 - x_2)^2 + (y_2 - y_3)^2], \tag{3.30}$$

$$k_{12} = \frac{1}{4A}[(x_3 - x_2)(x_1 - x_3) + (y_2 - y_3)(y_3 - y_1)], \tag{3.31}$$

$$k_{13} = \frac{1}{4A}[(x_3 - x_2)(x_2 - x_1) + (y_2 - y_3)(y_1 - y_2)], \tag{3.32}$$

$$k_{21} = k_{12}, \tag{3.33}$$

$$k_{22} = \frac{1}{4A}[(x_1 - x_3)^2 + (y_3 - y_1)^2], \tag{3.34}$$

$$k_{23} = \frac{1}{4A}[(x_1 - x_3)(x_2 - x_1) + (y_3 - y_1)(y_1 - y_2)], \tag{3.35}$$

$$k_{31} = k_{13}, \tag{3.36}$$

$$k_{32} = k_{23}, \tag{3.37}$$

$$k_{33} = \frac{1}{4A}[(x_2 - x_1)^2 + (y_1 - y_2)^2]. \tag{3.38}$$

The integrand in Eq. (3.28) is constant since $\frac{\partial H_i}{\partial x}$ and $\frac{\partial H_i}{\partial y}$ are constant for the linear triangular element so, as a result, the integration in Eq. (3.28) would become the integrand multiplied to the area of the element domain and we have the results as given in Eq. (3.29) through Eq. (3.38).

On the other hand, the term of the domain integral in Eq. (3.18) is evaluated as

$$\int_\Omega wg(x, y)\Omega. \tag{3.39}$$

As a result, this integration will become a column vector and computing the integral over every linear triangular element results in

$$\int_{\Omega^e} \begin{Bmatrix} H_1 \\ H_2 \\ H_3 \end{Bmatrix} g(x, y)\ d\Omega \tag{3.40}$$

3.1.3 Bilinear Rectangular Elements

The element for this is shown below (cf.3.3):

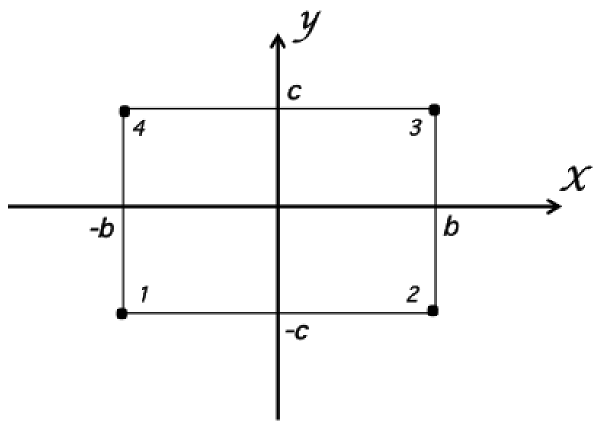

Figure 3.3: Bilinear element

Derivation of the shape functions can be done using the following interpolation function:

$$u = a_1 + a_2 x + a_3 y + a_4 xy. \tag{3.41}$$

As seen, the above function is linear in both x and y. Following the same method for deriving the shape functions as in the previous section yields

$$H_1 = \frac{1}{4bc}(b - x)(c - y) \tag{3.42}$$

$$H_2 = \frac{1}{4bc}(b + x)(c - y) \tag{3.43}$$

$$H_3 = \frac{1}{4bc}(b + x)(c + y) \tag{3.44}$$

$$H_4 = \frac{1}{4bc}(b - x)(c + y) \tag{3.45}$$

here 2b represent the length and 2c represent height of the element.

Shape functions in Eq. (3.42) through Eq. (3.45) can be obtained by taking the product of two classes of 1-D shape functions. Let the linear shape functions with nodes located at x=-b and x=b in the *x*-direction be given as

$$\Phi_1(x) = \frac{1}{2b}(b - x) \tag{3.46}$$

and

$$\Phi_2(x) = \frac{1}{2b}(b + x) \tag{3.47}$$

Also in the y-direction, the linear shape functions are given as

$$\Psi_1(x) = \frac{1}{2c}(c - y) \tag{3.48}$$

and

$$\Psi_2(x) = \frac{1}{2c}(c + y) \tag{3.49}$$

Taking the product of Eq. (3.46) and Eq. (3.47) and Eq. (3.48) and Eq. (3.49) would result in Eq. (3.42) through Eq. (3.45). Shape functions found by these products of equations are known as Lagrange shape functions.

The element matrix for a bilinear element as shown in Fig.3.3 is computed as:

$$[K^e] = \int_{\Omega^e} \left(\begin{Bmatrix} \frac{\partial H_1}{\partial x} \\ \frac{\partial H_2}{\partial x} \\ \frac{\partial H_3}{\partial x} \\ \frac{\partial H_4}{\partial x} \end{Bmatrix} \begin{bmatrix} \frac{\partial H_1}{\partial x} & \frac{\partial H_2}{\partial x} & \frac{\partial H_3}{\partial x} & \frac{\partial H_4}{\partial x} \end{bmatrix} + \begin{Bmatrix} \frac{\partial H_1}{\partial y} \\ \frac{\partial H_2}{\partial y} \\ \frac{\partial H_3}{\partial y} \\ \frac{\partial H_4}{\partial y} \end{Bmatrix} \begin{bmatrix} \frac{\partial H_1}{\partial y} & \frac{\partial H_2}{\partial y} & \frac{\partial H_3}{\partial y} & \frac{\partial H_4}{\partial y} \end{bmatrix} \right) d\Omega \tag{3.50}$$

where H_i is called the bilinear shape function. After putting the shape functions Eq. (3.42) through Eq. (3.45) and performing the integration over all terms will result in the bilinear rectangular element matrix given below

$$[K^e] = \begin{bmatrix} k_{11} & k_{12} & k_{13} & k_{14} \\ k_{12} & k_{22} & k_{23} & k_{24} \\ k_{13} & k_{23} & k_{33} & k_{34} \\ k_{14} & k_{24} & k_{34} & k_{44} \end{bmatrix} \tag{3.51}$$

in which

$$k_{11} = \frac{b^2 + c^2}{3bc}, \tag{3.52}$$

$$k_{12} = \frac{b^2 - 2c^2}{6bc}, \tag{3.53}$$

$$k_{13} = -\frac{b^2 + c^2}{6bc}, \tag{3.54}$$

$$k_{14} = \frac{c^2 - 2b^2}{6bc}, \tag{3.55}$$

$$k_{22} = k_{11}, \tag{3.56}$$

$$k_{23} = k_{14}, \tag{3.57}$$

$$k_{24} = k_{13}, \tag{3.58}$$

$$k_{33} = k_{11}, \tag{3.59}$$

$$k_{34} = k_{12}, \tag{3.60}$$

$$k_{44} = k_{11}. \tag{3.61}$$

Evaluation of the other domain integral yields

$$\int_{-b}^{b} \int_{-c}^{c} \begin{Bmatrix} H_1 \\ H_2 \\ H_3 \\ H_4 \end{Bmatrix} g(x,y) dy dx \tag{3.62}$$

which is same as Eq. (3.40).

3.1.4 Boundary Integral

The integral on the boundary as in Eq. (3.18) is given as

$$\int_{\Gamma_n} w \frac{\partial u}{\partial n} d\Gamma = \sum \int_{\Gamma^e} w \frac{\partial u}{\partial n} d\Gamma \tag{3.63}$$

Here we have two types of boundary conditions that are the natural boundary condition represented by n and element boundary condition represented by e.

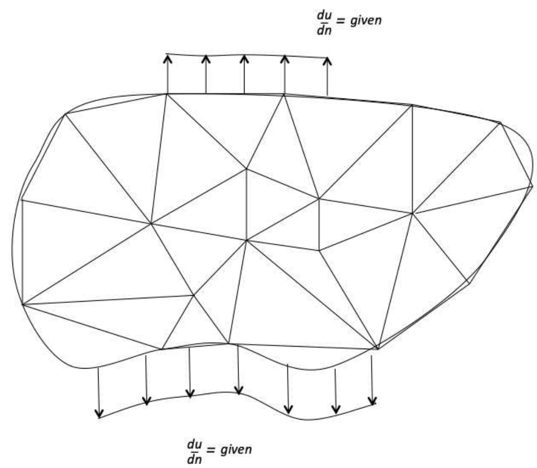

Figure 3.4: Elements at boundary

Here, the boundary of an element which is parallel to the x-axis is considered for simplicity, as shown below in Fig.3.5

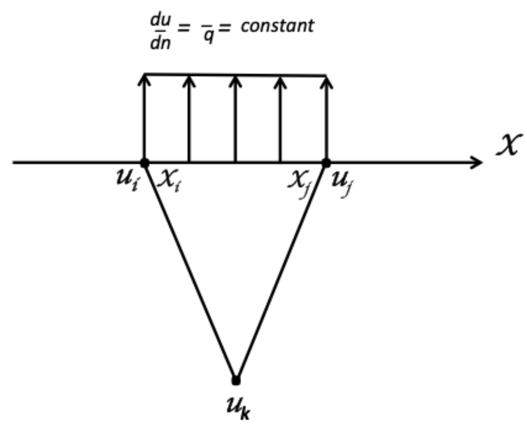

Figure 3.5: Triangular element subjected to constant flux

A positive constant flux is subjected to the element boundary i.e. the flux is in the outward direction which is supposed to be positive. Since for the domain's decritization we have used linear triangular elements hence the boundary of an element has two nodes as seen in Fig.3.5 and as a result, linear 1-D shape functions are used for interpolation of element boundary.

The mathematical term is

$$\int_{\Gamma^e} w\frac{\partial u}{\partial x}d\Gamma = \bar{q}\int_{x_i}^{x_j}\begin{Bmatrix}\frac{x_j-x}{x_j-x_i}\\ \frac{x-x_i}{x_j-x_i}\end{Bmatrix}dx = \frac{\bar{q}h_{ij}}{2}\begin{Bmatrix}1\\1\end{Bmatrix} \tag{3.64}$$

where $h_{ij}= x_j - x_i$ is the length of element boundary.

Hence this column vector is added to the corresponding nodes i and j and for the boundary of an element in the direction of y-axis or in any arbitrary direction (xy-axes), we get the results as far as h_{ij} is the length for element boundary.

Problem 1

Consider a Laplace equation

$$u_{xx} + u_{yy} = 0$$

for $(x, y) \in (0, 5) \times (0, 10)$ where the boundary conditions are given as

$$u = 0 \text{ at } y = 0 \text{ } and \text{ } x \in (0, 5)$$
$$u = 0 \text{ at } x = 0 \text{ } and \text{ } y \in (0, 10)$$
$$u(x, y) = 100 \sin(\pi x/10) \text{ at } y = 10 \text{ } and \text{ } x \in (0, 5)$$
$$u_x(x, y) = 0 \text{ at } x = 5 \text{ with } y \in (0, 10)$$

.

Fig.3.6 shows the finite element discretization and domain. Apply the finite element formulation for the above mentioned boundary conditions.

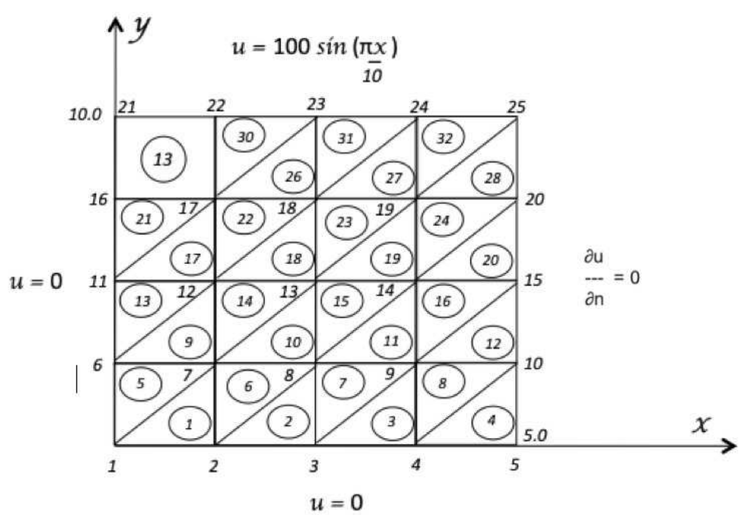

Figure 3.6: Mesh with linear triangular elements

By writing the MATLAB code using FEM [58], we have the results as shown in the Table 3.1

Dof#	FEM Sol	Exact Sol	Error
1	0.0000	0.0000	0.0000
2	0.0000	0.0000	0.0000
3	0.0000	0.0000	0.0000
4	0.0000	0.0000	0.0000
5	0.0000	0.0000	0.0000
6	0.0000	0.0000	0.0000
7	3.6896	2.8785	0.8112
8	6.5689	5.3187	1.2502
9	8.4046	6.9492	1.4553
10	9.0361	7.5218	1.5143
11	0.0000	0.0000	0.0000
12	10.6206	7.6257	2.9949
13	17.3122	14.0904	3.2218
14	21.6258	18.4100	3.2158
15	23.1241	19.9268	3.1973
16	0.0000	0.0000	0.0000
17	14.0437	17.3236	3.2799
18	37.5677	32.0099	5.5578
19	46.1080	41.8229	4.2851
20	49.1988	45.2688	3.9300
21	0.0000	0.0000	0.0000
22	38.2683	38.2683	0.0000
23	70.7101	70.7107	0.0006
24	92.3880	92.3880	0.0000
25	100.0000	100.0000	0.0000

Table 3.1. Comparison of FEM solution and exact solution

3.2 General Theory of FEM for Heat Equation

Problems of heat transfer [57] are commonly used in the field of engineering mostly with irregular geometry. Hence, FEM is very helpful in such cases.

3.2.1 Basic Formulation For 1-D Steady State Heat Conduction

The process of heat transfer is mainly governed through heat conduction equation or Poisson equation that is

$$\nabla.(k\nabla u) + Q = 0 \tag{3.65}$$

with boundary condition

$$u = \bar{u},\ x \ \epsilon\ \partial\Omega_E \tag{3.66}$$

which is of essential type and another boundary condition

$$k\frac{\partial u}{\partial n} - q = 0,\ x \ \epsilon\ \partial\Omega_N \tag{3.67}$$

which is of natural type.

Using the finite element formulation similar that is applying integration by parts and plugging $u_h = \sum_{j=1}^{M} u_j N_j(x)$ in the equation , we get

$$\sum_{j=1}^{M}[\int_{\Omega}(k\nabla N_i.\nabla N_j)d\Omega]u_j - \int_{\Omega} QN_i d\Omega - \int_{\partial\Omega_N} qN_i d\Gamma = 0 \tag{3.68}$$

We can write it in compact form as

$$\sum_{j=1}^{M} K_{ij}U_j = f_i,\ \mathbf{KU} = \mathbf{f}, \tag{3.69}$$

where $K = [K_{ij}]$,$(i, j = 1, 2, \cdots, M)$, $U^T = (u_1, u_2, \cdots, u_M)$, and $f^T = (f_1, f_2, \cdots, f_m)$. That is,

$$K_{ij} = \int_{\Omega} k \nabla N_i \nabla N_j d\Omega, \tag{3.70}$$

$$f_i = \int_{\Omega} Q N_i d\Omega + \int_{\partial\Omega_N} q N_i d\Gamma \tag{3.71}$$

Here, we consider 1-D steady-state heat conduction problem,

$$\text{u}''(\text{x})+\text{Q}(\text{x})=0$$

with following boundary conditions

$$u(0) = \beta \ , \ u'(1) = q$$

In a special case, where Q(x) = r exp(-x), the analytical solution have the form

$$u(x) = (\beta - r) + (re^{-}1 + q)x + re^{-}x \tag{3.72}$$

Hence, Eq. (3.71) then gets the following form

$$\sum_{j=1}^{M} (\int_0^1 N_i' N_j' dx) u_j = \int_0^1 Q N_i dx + q N_i(1). \tag{3.73}$$

Element By Element Assembly

The process of assembling the system of linear matrix is the common element-by-element formulation. In Eq. (3.69) and Eq. (3.71), the stiffness matrix K is the summation of the integral upon the entire solution domain, and the domain is further divided into m number of elements with each element situated on a sub-domain $\Omega_e (e = 1, 2, \cdots, m)$. The whole stiffness matrix is contributed by each element, and actually, its contribution is a pure number. Hence, we can done with assembly process of the stiffness matrix through an element-by-element approach. In addition, $K_{i,j} \neq 0$ iff nodes i and j belong to same elements. In the case of 1-D, $k_{i,j} \neq 0$ only for $j = i-1, i, i+1$. The shape functions N_j in finite element analysis are classically localized functions, hence the matrix K is more often sparse in nearly all cases.

We can write element-by-element formulation as

$$K_{i,j} = \sum_{e=1}^{m} K_{i,j}^{(e)}, K_{i,j}^{(e)} = \int_{\Omega_e} k \nabla N_i \nabla N_j d\Omega_e \tag{3.74}$$

and

$$f_i = \sum_{e=1}^{m} f_i^{(e)}, f_i^{(e)} = \int_{\Omega_e} Q N_i d\Omega_e + \int_{\partial\Omega_e} q N_i d\Gamma_e \tag{3.75}$$

Furthermore, as the simple number contributes to each element, hence we can integrate each element by using the local coordinates and local node numbers over an element.Then, the contribution of each part to the global system matrix K is solely assembled by direct addition to the corresponding global nodes or connected equations. This could be simply done by associate index matrix to trace the element contribution to the global system matrix.

The assembly of the global system matrix for the instance with four components and 5 nodes is shown below. For every element with i and j nodes, we have

$$N_i = 1 - \xi,\ N_j = \xi,\ \xi = \tfrac{x}{L},\ L = h_e,$$

$$K_{ij}^{(e)} = \left[\int_0^L k N_i' N_j' dx\right] = \frac{k}{h_e}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \tag{3.76}$$

$$f_i^{(e)} = \tfrac{Qh_e}{2}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

so that, for instance in elements 1 and 2 , these will reach all nodes
(with $h_i = x_i + 1 = x_i, i$, i=1,2,3,4),

$$K = \begin{pmatrix} k/h_1 & -k/h_1 & 0 & 0 & 0 \\ -k/h_1 & k/h_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix},\ f^{(1)} = Q/2 \begin{pmatrix} h_1 \\ h_2 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

so that global system matrix becomes

$$K = \begin{pmatrix} \frac{k}{h_1} & \frac{-k}{h_1} & 0 & 0 & 0 \\ \frac{-k}{h_1} & \frac{k}{h_1} + \frac{k}{h_2} & \frac{-k}{h_2} & 0 & 0 \\ 0 & \frac{-k}{h_2} & \frac{k}{h_2} + \frac{k}{h_3} & \frac{-k}{h_3} & 0 \\ 0 & 0 & \frac{-k}{h_3} & \frac{k}{h_3} + \frac{k}{h_4} & \frac{-k}{h_4} \\ 0 & 0 & 0 & \frac{-k}{h_4} & \frac{k}{h_4} \end{pmatrix},$$

$$U = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \end{pmatrix}, \quad f = \begin{pmatrix} Qh_1/2 \\ Q(h_1 + h_2)/2) \\ Q(h_2 + h_3)/2) \\ Q(h_3 + h_4)/2 \\ Qh_4/2 + q \end{pmatrix}$$

where the last row of f has already enclosed the natural boundary condition at $u'(1) = q$.

3.2.2 2-D Heat Transfer

More complicated elements and shape functions are used in higher dimensions. For simplicity, we will discuss 2-D steady state transfer

$$\frac{\partial}{\partial x}(k_x \frac{\partial u}{\partial x}) + \frac{\partial}{\partial y}(k_y \frac{\partial u}{\partial y}) + Q = 0 \tag{3.77}$$

where k_x and k_y are thermal conductivities on a triangular mesh, each element has three nodes i, j and m with three points (x_i, y_i), (x_j, y_j) and (x_m, y_m). The area of an element is

$$\Delta = \frac{1}{2} det \begin{vmatrix} 1 & x_1 & y_1 \\ 1 & x_j & y_j \\ 1 & x_m & y_m \end{vmatrix} \tag{3.78}$$

The shape functions are given as

$$N_i = \frac{(a_i + b_i x + c_i y)}{2\triangle}, \tag{3.79}$$

where $a_i = x_j y_m - x_m y_j$, $b_i = y_j - y_m = y_{jm}$, and $c_i = x_m - x_j = x_{mj}$ as well as their cyclic permutations of subscripts in the order of i, j, m. Therefore, the matrix K_{ij} becomes

$$K_{ij}^{(e)} = \int_{\Omega_e} \left(k_x \frac{\partial N_i}{\partial x} \frac{\partial N_j}{\partial x} + k_y \frac{\partial N_i}{\partial y} \frac{\partial N_j}{\partial y} \right) dxdy \tag{3.80}$$

This load matrix is simply

$$f^{(e)} = \frac{Q\triangle}{3} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \tag{3.81}$$

The index matrix for the global nodes or equation numbers are

$$D_{Index} = \begin{pmatrix} (i,i) & (i,j) & (i,m) \\ (j,i) & (j,j) & (j,m) \\ (m,i) & (m,j) & (m,m) \end{pmatrix} \tag{3.82}$$

This can be written as two separate matrices ID and JD for easily implementation

$$ID = \begin{pmatrix} i & i & i \\ j & j & j \\ m & m & m \end{pmatrix} JD = \begin{pmatrix} i & j & m \\ i & j & m \\ i & j & m \end{pmatrix} \tag{3.83}$$

$$ID = JD^T \tag{3.84}$$

So that the contribution of $K_{ij}^{(e)}$ to the global matrix K_{ij} is just

$$K_{[ID(I,J),JD(I,J)]} = K_{[ID(I,J),JD(I,J)]} + K_{(I,J)}(e), I, J = 1, 2, 3 \tag{3.85}$$

and

$$f(l) = f(l) + f^{(e)}(l)(l = i, j, m)\text{etc}$$

Problem 1: Heat Equation In 1-D

In this problem, we have solved the steady-state of the 1-D heat conduction equation given as

$$u''(x)+Q(x)=0$$

with following boundary conditions

$$u(0) = \beta \ , \ u'(1) = q$$

In a special case, where Q(x) = r exp(-x), the analytical solution have the form

$$u(x) = (\beta - r) + (re^{-}1 + q)x + re^{-}x \tag{3.86}$$

in MATLAB using FEM and we have the following results as shown below:

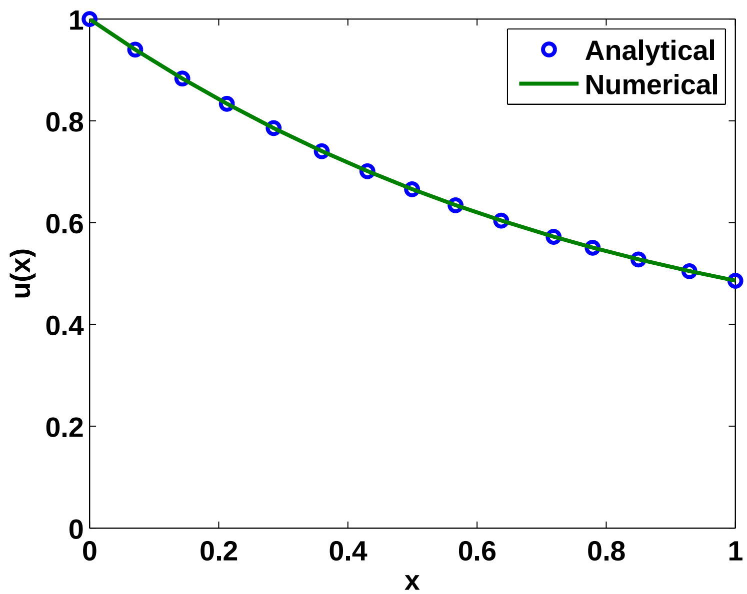

Figure 3.7: 1-D heat conduction(steady state)

Problem 2: Heat Equation in 2-D

Consider a 2-D steady state transfer

$$\frac{\partial}{\partial x}(k_x\frac{\partial u}{\partial x}) + \frac{\partial}{\partial y}(k_y\frac{\partial u}{\partial y}) + Q = 0 \tag{3.87}$$

where k_x and k_y are thermal conductivities and Q is the heat source on a triangular mesh, each element has three nodes i, j and m with three points (x_i, y_i), (x_j, y_j) and (x_m, y_m). Results are shown below:

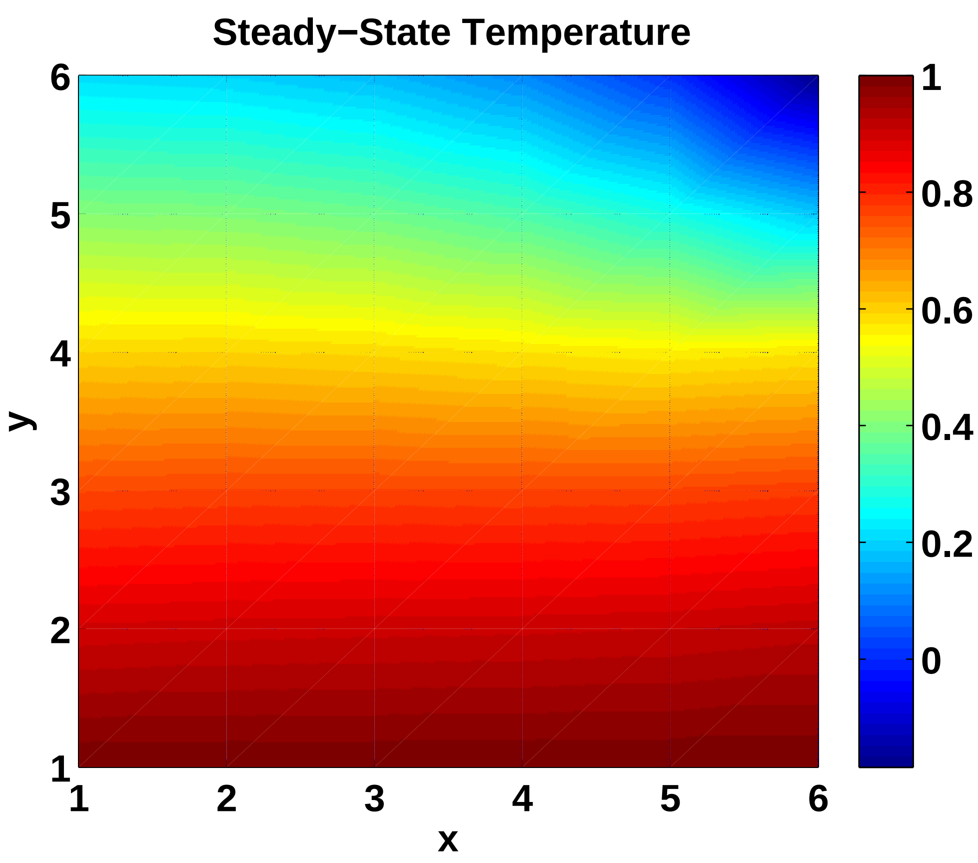

Figure 3.8: 2-D heat conduction on a rectangular domain

3.3 PDE Toolbox

3.3.1 Description

The Partial differential equation (PDE) toolbox is a powerful tool which provides us with a flexible environment to find the solution of PDEs in two space dimension and time. The process of discretization in PDE toolbox is done by using the finite element method (FEM). This is an alternate way to find the solution of PDEs as we do not need to write code by ourself instead this tool uses some built in functions for finding the solutions of PDEs. We just need to input some information into this toolbox regarding PDE that is geometry, boundary conditions of dirichlet, neumann or mix type, coefficients of PDE etc. Step by step procedure is mentioned below. In addition, toolbox has advantages as well as some disadvantages.

3.3.2 Basic steps to solve PDEs

- First step is to define a 2-D geometry.
- Secondly, we need to define the boundary conditions.
- Thirdly, we have to define the coefficients of a PDE.
- Than create a triangular mesh.
- Solve the PDE.
- At the end, plot the solution.

Advantages

- Easy to use.
- Easy to generate grid.
- Easy to handle complex geometry.
- Efficiency is higher.

- We can change the PDE and recompute the solution.
- Also we can change the mesh, boundary conditions and the geometry as well.

Disadvantages

- Cannot handle nonlinearity.
- Cannot handle 3-D properly.
- Cannot handle higher order equations.

Generally, PDE toolbox adds some necessary steps automatically if not defined that is:

- If we do not create a geometry, it will use the L-shaped geometry with the default boundary conditions.
- While in draw mode if we initialize the mesh than it will first decompose the geometry by using the current set formula and it will assign the default boundary conditions to the outer boundaries and after that it will generate a mesh.
- Before initializing if we refine the mesh than it will automatically intialize the mesh first.
- Without generating a mesh if we solve the PDE, it will initialize a mesh before solving it.
- If we choose to plot the solution, it will first check if the solution to the corresponding PDE is available, if not than it will first solve the corresponding PDE.
- If the coefficients are not specified while in generic scalar application mode, than it will solve the default PDE that is the Poisson equation.

The default settings of PDE depend on the application mode.

Problem 1

The problem here shows how we can solve a Poisson's Equation numerically using the assempde function in PDE toolbox.

Consider a PDE that is

$$-\Delta u = 1 \tag{3.88}$$

on the unit disk with zero-Dirichlet boundary conditions. The analytical solution is

$$u(x, y) = \frac{1-x^2-y^2}{4}$$

Since for many partial differential equations, the exact solution is not known so in this example, however, we have used the known solution that is exact to show that how the error decreases as we refine the mesh.

Generate initial mesh

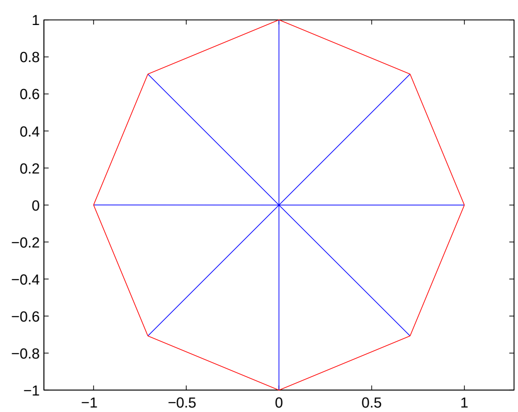

Figure 3.9: Initial mesh

Plot final mesh

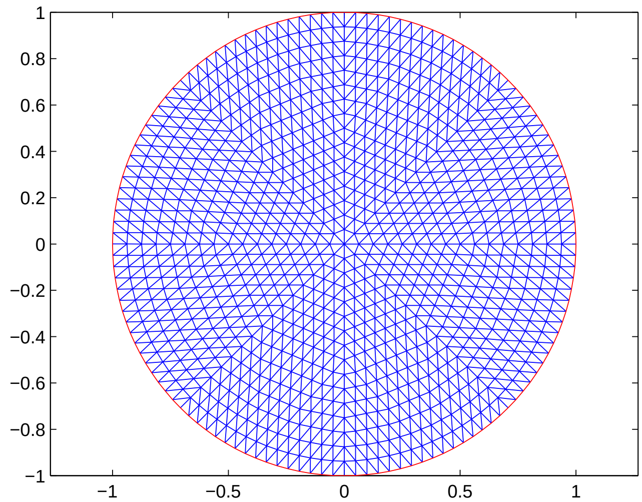

Figure 3.10: Final mesh

Plot error

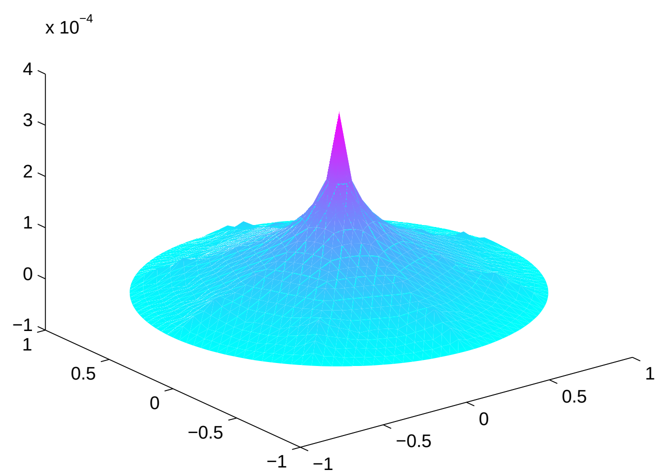

Figure 3.11: Error plot

Plot FEM solution

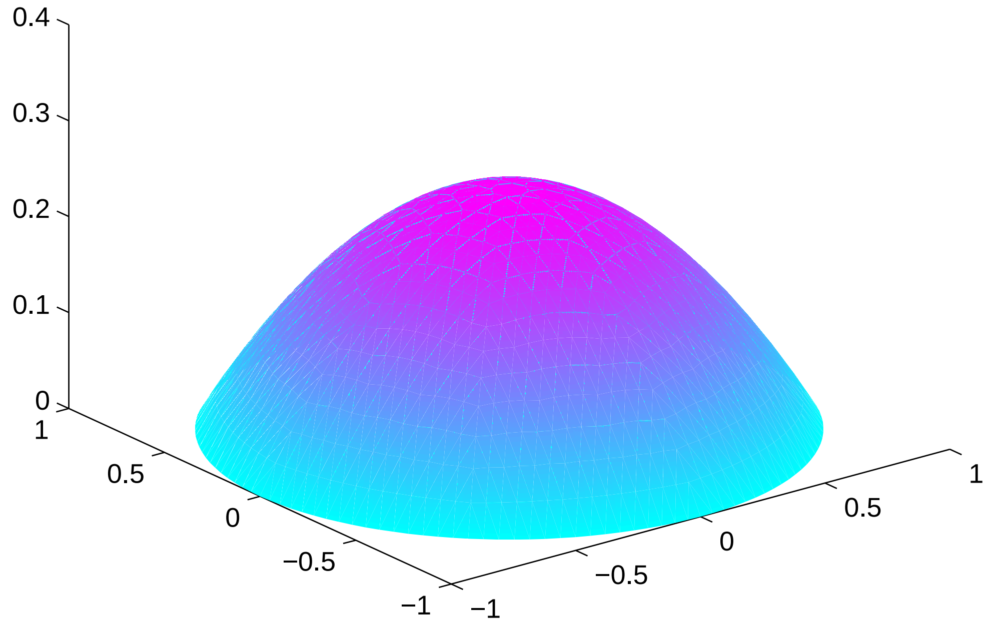

Figure 3.12: FEM solution

Problem 2

In this problem we have solved the heat equation with a source term using the parabolic function in the PDE Toolbox.

Consider the heat equation

$$\frac{\partial u}{\partial t} - \Delta u = 1 \tag{3.89}$$

Generate mesh

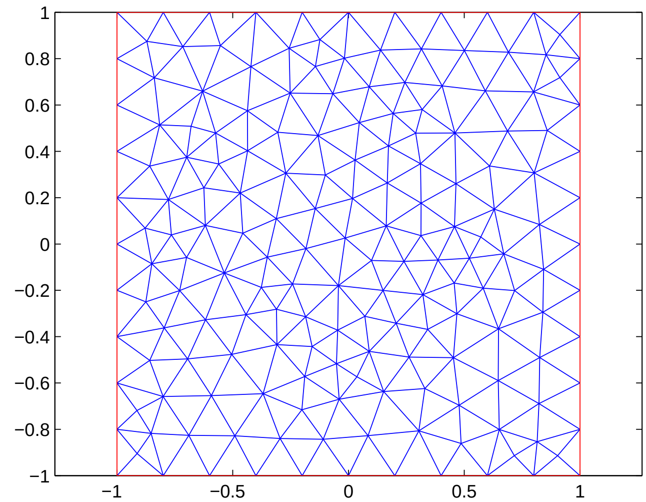

Figure 3.13: Generate mesh

Plot FEM solution

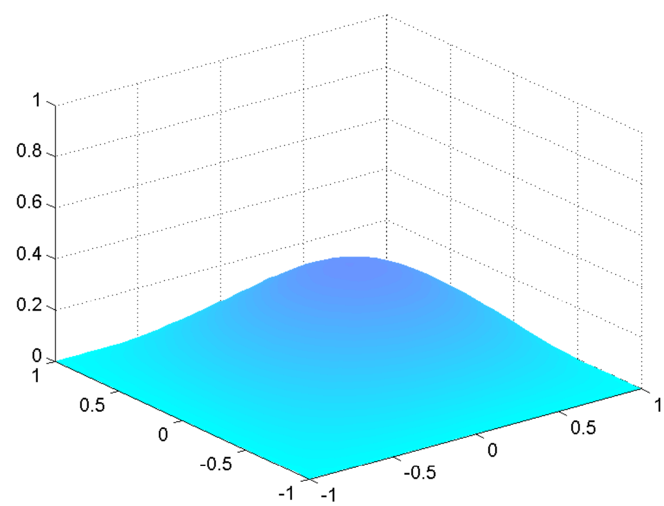

Figure 3.14: FEM solution

3.4 Nonlinear PDEs

3.4.1 Burgers Equation

In applied mathematics, nonlinear partial differential equations have a very important role. They are used for modeling and analyzing the physical problems. So the Burgers equation [59]

$$u_t + uu_x = \nu u_{xx} \tag{3.90}$$

which lies in the class of nonlinear PDEs is considered to be a center of interest for studying different physical phenomenon including shock waves, gas dynamics etc. Since this equation includes an advection term uu_x and viscosity term νu_{xx} so it had been used to test some numerical methods. In addition, this equation allow us to do the comparison for analyzing the quality of a numerical method applied to a nonlinear equation.

In this section, we are going to solve Burgers equation through FEM by writing the MATLAB code. Consider the Burger equation given below:

$$\frac{\partial x}{\partial t} = \mu\frac{\partial^2 x}{\partial z^2} - x\frac{\partial x}{\partial z} = \mu x_{zz} - xx_z \tag{3.91}$$

with $z\varepsilon[0,1]$ and t is non-negative. The relevant conditions are as:

$$x(z,t) = x_a(z,t)\ at\ t = 0 \tag{3.92}$$

$$x(z,t) = x_a(z,t)\ at\ z = 0 \tag{3.93}$$

$$x(z,t) = x_a(z,t)\ at\ z = 1 \tag{3.94}$$

where x_a is the analytical solution given as:

$$x_a(z,t) = \frac{0.1e^p + 0.5e^q + e^r}{e^p + e^q + e^r} \tag{3.95}$$

here

$$p = -(0.05/\mu)(z - 0.5 + 4.95t);$$
$$q = -(0.25/\mu)(z - 0.5 + 4.75t);$$
$$r = -(0.5/\mu)(z - 0.375);$$

The underlying method produce as following coupled ODEs:

$$M\frac{\partial \tilde{x}}{\partial t} = (\alpha_0 D_0 + \alpha_1 D_1 + \alpha_2 D_2)\tilde{x} + M\tilde{f}_{NL} + g \tag{3.96}$$

where

$$M\frac{d\tilde{x}}{dt} = \frac{\Delta z}{6}\begin{pmatrix} 2 & 1 & & & & & \\ 1 & 4 & 1 & & & & \\ & \cdot & \cdot & \cdot & & & \\ & & \cdot & \cdot & \cdot & & \\ & & & \cdot & \cdot & \cdot & \\ & & & & 1 & 4 & 1 \\ & & & & & 1 & 2 \end{pmatrix}\begin{pmatrix} X_1 \\ X_2 \\ \cdot \\ \cdot \\ \cdot \\ X_N - 1 \\ X_N \end{pmatrix} \tag{3.97}$$

$$D_0\tilde{x} = \frac{\Delta z}{6}\begin{pmatrix} 2 & 1 & & & & & \\ 1 & 4 & 1 & & & & \\ & & \cdot & \cdot & \cdot & & \\ & & & \cdot & \cdot & \cdot & \\ & & & & \cdot & \cdot & \cdot \\ & & & & 1 & 4 & 1 \\ & & & & & 1 & 2 \end{pmatrix}\begin{pmatrix} X_1 \\ X_2 \\ \cdot \\ \cdot \\ \cdot \\ X_N - 1 \\ X_N \end{pmatrix} \tag{3.98}$$

$$D_1\tilde{x} = \frac{1}{2}\begin{pmatrix} -1 & 1 & & & & & \\ -1 & 0 & 1 & & & & \\ & & \cdot & \cdot & \cdot & & \\ & & & \cdot & \cdot & \cdot & \\ & & & & \cdot & \cdot & \cdot \\ & & & & -1 & 0 & 1 \\ & & & & & -1 & 1 \end{pmatrix}\begin{pmatrix} X_1 \\ X_2 \\ \cdot \\ \cdot \\ \cdot \\ X_N - 1 \\ X_N \end{pmatrix} \tag{3.99}$$

$$D_2\tilde{x} = \frac{1}{\Delta z}\begin{pmatrix} 0 & 0 & & & & & \\ 1 & -2 & 1 & & & & \\ & & \cdot & \cdot & \cdot & & \\ & & & \cdot & \cdot & \cdot & \\ & & & & \cdot & \cdot & \cdot \\ & & & & 1 & -2 & 1 \\ & & & & & 0 & 0 \end{pmatrix}\begin{pmatrix} X_1 \\ X_2 \\ \cdot \\ \cdot \\ \cdot \\ X_N - 1 \\ X_N \end{pmatrix} \tag{3.100}$$

Details are mentioned in the book.

Now consider

$$M\frac{\partial \tilde{x}}{\partial t} = \mu D_2 \tilde{x} - \tilde{f}_{NL} + g \tag{3.101}$$

here $\tilde{f}_{NL}$ is the nonlinear term x_{zz}.

By writing the MATLAB code, we have the following result which shows the comparison between the analytical solution " $-$ " and numerical solution "." of Burgers equation by using the finite element code.

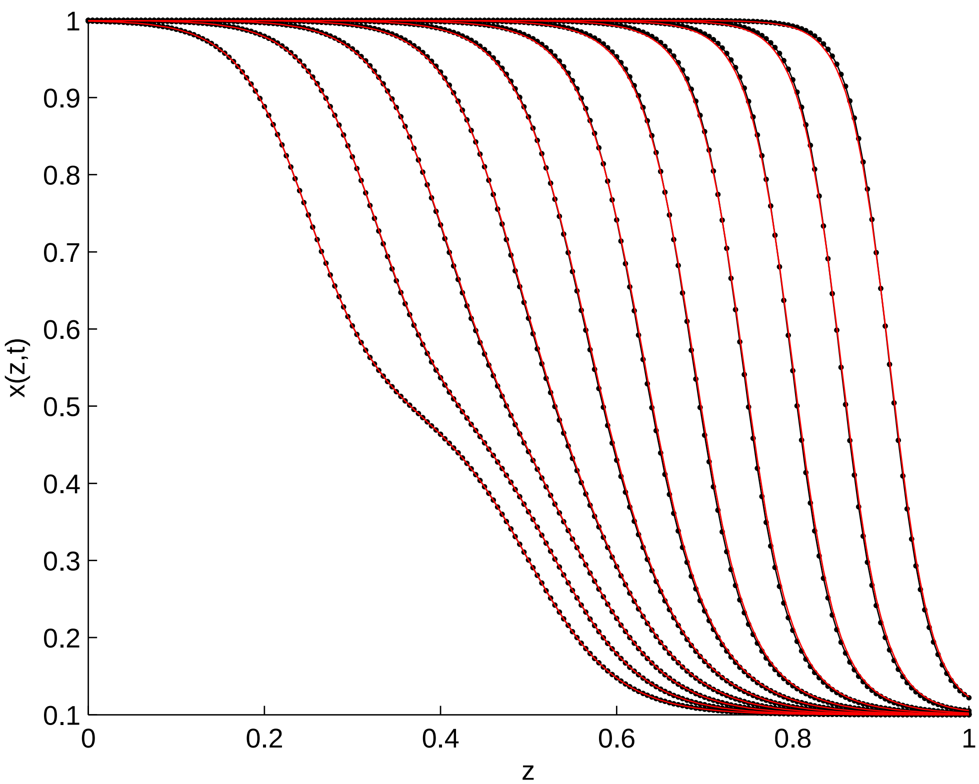

Figure 3.15: Comparison between the analytical and numerical solution

The figure shows that this numerical technique that is finite element method produces very satisfactory results. The discretization points as well as basis functions can be varied for different μ appear in Burgers equation.

Chapter 4

Conclusion

This work was intended as an attempt to motivate the implementation of finite element method for finding the numerical solutions to the problems. It is actually a generalization of the weighted residual method, which lies on the principle that the solution of a problem can be represented in a linear combination form including the unknown parameters along with suitably selected functions in the entire region of the problem. These unknown parameters are then determined in such a way that they should satisfy the differential equation in a weighted-integral sense. In addition, trial functions are assumed such that they have to satisfy the boundary conditions of the corresponding problem.

The traditional weighted-residual method suffer from one majorshort coming that is the construction of the approximation functions which is bound to satisfy the boundary conditions of the problem to be solved. Most of the problems are defined on a geometrically complex regions and therefore it becomes difficult to generate the approximation functions of the type which satisfy different types of boundary conditions on different parts of the boundary of that particular complex domain. However, if the domain can be represented as a collection of simple sub domains that permit construction of the approximation functions for any arbitrary but physically meaningful boundary conditions, then the traditional weighted residual method can be used to solve practical problems. So an idea behind the method is to analyze a respective domain as a collection of simple sub-domains known as the finite elements, for which we can generate the approximation functions necessary for the solution of differential equations by any weighted residual technique. The capability of representing the domains with irregular geometries along with the collection of finite elements makes this method an important practical tool for the solving the problems.

Galerkin finite element method (GFEM) has been used in this entry throughout for solving ODEs and PDEs. For a 1-D ordinary differential equation which has been solved in chapter two, weak formulation was constructed by a weighted residual technique and then by assuming the linear functions as the trial functions, we derived the shape functions which were later used along with weight functions into the governing ODE for finite element formulation. In chapter three, the method was used for formulating the Poisson and Laplace equation. For Laplace equation, first we have solved it with linear triangular elements and secondly theoretical results for bilinear rectangular elements are presented. The bilinear rectangular elements when compared with linear triangular elements will produce more accurate solutions. Also the steady state of 1-D heat conduction equation and 2-D heat transfer on a rectangular region are discussed along with graphs which shows both analytical and numerical solutions. In addition to the numerical technique used here, it is possible to apply FEM for solving the nonlinear problems; hence we have solved Burgers equation through finite element formulation. Therefore, this method is widely used for solving the practical engineering problems.

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