Abstract

This dissertation deals with necessary and sufficient conditions for the existence of axes of symmetry and planes of symmetry. We discuss matrix representation of an elasticity tensor belonging to a trigonal, a tetragonal or a hexagonal material.

The planes of symmetry of an anisotropic elastic material (if they exist) can be found by the Cowin-Mehrabadi theorem [8], and the modified Cowin-Mehrabadi theorem proved by Ting [23]. Using the Cowin-Mehrabadi formalism Ahmad [2] proved the result that an anisotropic material possesses a plane of symmetry if and only if the matrix associated with the material commutes with the matrix representing the elasticity tensor. A necessary and sufficient condition to determine an axis of symmetry of an anisotropic elastic material is given by Ahmad [3].

We review the Cowin-Mehrabadi theorem for an axis of symmetry and develop a straightforward way to find the matrix representation for a trigonal, a tetragonal or a hexagonal material.

Contents

Chapter 1

Introduction

Symmetry in nature underlies one of the most fundamental concepts of beauty. It connects balance, order, and thus, to some, a type of divine principle. This is the property of an object that enables it to undergo certain manipulations, called symmetry operations, such that its new state is indistinguishable from its original state. Inversion through a point, reflection through a plane and rotation about an axis are examples of symmetry operations.

The symmetry axes of an object are lines about which it can be rotated through some angle which brings the object to a new orientation which appears identical to its starting position. The symmetry planes of an object are imaginary mirrors in which it can be reflected while appearing unchanged. When a mirror is placed on a line of symmetry of a two-dimensional shape and looked at from either side, the shapes look identical. In other words, each half of the shape is a mirror image of the other half. In a similar way, when a plane cuts a 3-D shape in two so that each half is a mirror image of the other half, the plane is called a plane of symmetry

Elastic materials can be divided into eight classes on the basis of their symmetries. Let $\mathbf{a}, \mathbf{b}, \mathbf{c}$ denote the sides of a unit cell of a crystal and let α, β, γ denote the angles between these vectors so that β is the angle between $\mathbf{b}$ and $\mathbf{c}$ etc. For an isotropic material every line is an axis of symmetry and every plane is a plane of symmetry. The crystal structure of the rest of the seven crystal systems for a triclinic, a monoclinic, an orthorhombic, a trigonal, a tetragonal, a hexagonal and a cubic system is as follows [26],

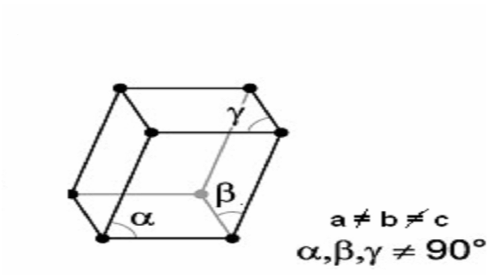

Figure 1.1: Triclinic

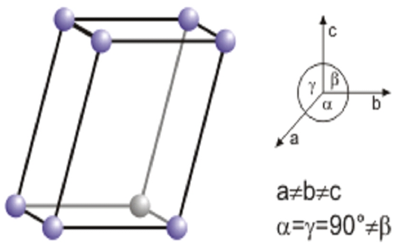

Figure 1.2: Monoclinic

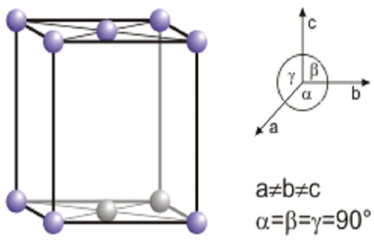

Figure 1.3: Orthorhombic

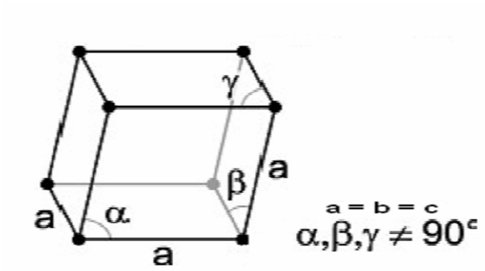

Figure 1.4: Trigonal

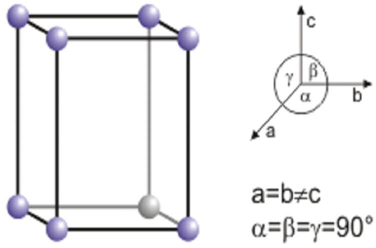

Figure 1.5: Tetragonal

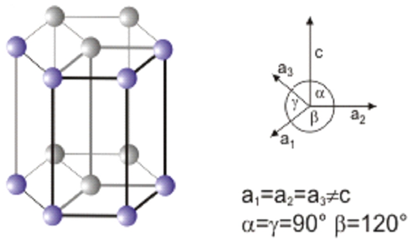

Figure 1.6: Hexagonal

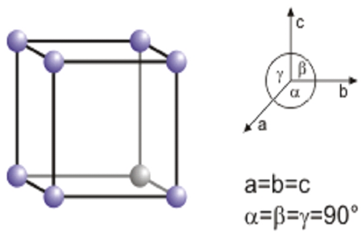

Figure 1.7: Cubic

Sr.no.	Crystal systems	No. of axes of symmetry	No. of planes of symmetry	No. of ind. components
1	Triclinic	no axis of symmetry	no plane of symmetry	21
2	Monoclinic	one 2-fold axis of rotation	one symmetry plane	13
3	Orthorhombic	3 mutually perpendicular 2-fold rotation axes	three coordinate planes are symmetry planes	9
4	Trigonal	one 3-fold rotation	three symmetry planes	7
5	Tetragonal	one 4-fold rotation	five symmetry planes	7
6	Hexagonal	one 6-fold rotation	any plane containing x_3 axis	5
7	Cubic	four independent 3-fold rotation axes	nine planes of symmetry	3
8	Isotropic	any axis is an axis of symmetry	any plane is a symmetry plane	2

Table 1.1: Description of Crystal Systems.

The generalized Hooke's law

$$T_{ij} = C_{ijks}E_{ks}, \tag{1.1}$$

is the constitutive relation for linear anisotropic elasticity, where summation over repeated indices is taken and free indices take the value $1, 2, 3$, and T_{ij} are the components of the stress tensor, E_{ij} are the components of the strain tensor and C_{ijks} are the components of the fourth-rank elasticity tensor. Since the strain and the stress tensors are both symmetric, C_{ijks} must have the following symmetries

$$C_{ijks} = C_{jiks} = C_{ijsk}.$$

The elasticity tensor has $3^4 = 81$ components, but due to the above symmetries the number of independent components reduces to 36. Voigt(1910) introduced a two index notation as follows

$$11 \leftrightarrow 1, 22 \leftrightarrow 2, 33 \leftrightarrow 3, 32 = 23 \leftrightarrow 4, 31 = 13 \leftrightarrow 5, 21 = 12 \leftrightarrow 6.$$

This means $C_{1111} = c_{11},\ C_{1223} = c_{64},\ C_{1331} = c_{55}$ etc.

Now equation (1.1) can be written as

$$\begin{bmatrix} T_{11} \\ T_{22} \\ T_{33} \\ T_{23} \\ T_{31} \\ T_{12} \end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} & c_{13} & c_{14} & c_{15} & c_{16} \\ c_{21} & c_{22} & c_{23} & c_{24} & c_{25} & c_{26} \\ c_{31} & c_{32} & c_{33} & c_{34} & c_{35} & c_{36} \\ c_{41} & c_{42} & c_{43} & c_{44} & c_{45} & c_{46} \\ c_{51} & c_{52} & c_{53} & c_{54} & c_{55} & c_{56} \\ c_{61} & c_{62} & c_{63} & c_{64} & c_{65} & c_{66} \end{bmatrix} \begin{bmatrix} E_{11} \\ E_{22} \\ E_{33} \\ 2E_{23} \\ 2E_{31} \\ 2E_{12} \end{bmatrix} \tag{1.2}$$

$$\text{or} \quad \mathbf{T} = \mathbf{CE}.$$

The number of independent constants are further reduced from the consideration of strain energy due to the thermodynamics requirement. Ting [23] gave this condition in the following manner. The strain energy W per unit volume of a material is

$$W = \int_0^{\epsilon_{pq}} \sigma_{ij} d\epsilon_{ij} = \int_0^{\epsilon_{pq}} C_{ijks}\epsilon_{ks} d\epsilon_{ij}. \tag{1.3}$$

This integral must be independent of the path traversed by ϵ_{ij} from 0 to ϵ_{pq}. We consider a loading from 0 to ϵ_{pq} and unloading from ϵ_{pq} to 0. By taking the difference of these two paths of the $W's$ we obtain energy. By successive repetition of this process we gain infinite energy, but physically this is not possible for real materials. Therefore the integral in (1.3) is independent of the path taken by ϵ_{ij} and W is the function of final strain ϵ_{pq} only. Therefore

$$dW = C_{ijks}\epsilon_{ks} d\epsilon_{ij}.$$

But

$$dW = \frac{\partial W}{\partial \epsilon_{ij}} d\epsilon_{ij}.$$

Thus

$$\frac{\partial W}{\partial \epsilon_{ij}} d\epsilon_{ij} = C_{ijks}\epsilon_{ks} d\epsilon_{ij}.$$

Since $d\epsilon_{ij}$ is arbitrary, so we get

$$\frac{\partial W}{\partial \epsilon_{ij}} = C_{ijks}\epsilon_{ks}.$$

Differentiating it with respect to ϵ_{ks}, we get

$$\frac{\partial^2 W}{\partial \epsilon_{ks} \partial \epsilon_{ij}} = C_{ijks}. \tag{1.4}$$

Similarly

$$\frac{\partial^2 W}{\partial \epsilon_{ij} \partial \epsilon_{ks}} = C_{ksij}. \tag{1.5}$$

As we know from calculus $\frac{\partial^2 W}{\partial \epsilon_{ks} \partial \epsilon_{ij}} = \frac{\partial^2 W}{\partial \epsilon_{ij} \partial \epsilon_{ks}}$, so from (1.4) and (1.5), we have

$$C_{ijks} = C_{ksij}$$

which is the required condition for the integral in (1.3) to be independent of the loading path.

Thus because of $C_{ijks} = C_{ksij}$, the number of independent elastic constants are reduced from 36 to 21, and we have a representation of (1.2) as follows

$$\begin{bmatrix} T_{11} \\ T_{22} \\ T_{33} \\ T_{23} \\ T_{31} \\ T_{12} \end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} & c_{13} & c_{14} & c_{15} & c_{16} \\ c_{12} & c_{22} & c_{23} & c_{24} & c_{25} & c_{26} \\ c_{13} & c_{23} & c_{33} & c_{34} & c_{35} & c_{36} \\ c_{14} & c_{24} & c_{34} & c_{44} & c_{45} & c_{46} \\ c_{15} & c_{25} & c_{35} & c_{45} & c_{55} & c_{56} \\ c_{16} & c_{26} & c_{36} & c_{46} & c_{56} & c_{66} \end{bmatrix} \begin{bmatrix} E_{11} \\ E_{22} \\ E_{33} \\ 2E_{23} \\ 2E_{31} \\ 2E_{12} \end{bmatrix}.$$

A line, A_n is called an n-fold axis of symmetry of an elastic material if it leaves the material invariant when we rotate it about A_n through an angle $2\pi/n$.

If the material is invariant with respect to inversion through some point then that point is called a centre of symmetry. If the material is invariant with respect to reflection in some plane then that plane is called a plane of symmetry.

If an axis of symmetry A_n exists and we choose x_3-axis parallel to A_n, then the rotation matrix about this axis for rotation through an angle θ is

$$(a_{ij}) = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{1.6}$$

The invariance of a material with respect to a rotation through θ, about x_3-axis gives

$$C_{ijkl} = a_{ip} a_{jq} a_{kr} a_{ls} C_{pqrs}, \tag{1.7}$$

In some standard text books ([14] and [15]) rotation of a rigid body about an axis $\mathbf{p}$ by an arbitrary angle θ is described by a tensor $\mathbf{Q}$ (a three dimensional rotation matrix),

$$\mathbf{Q} = \mathbf{I} + \sin\theta \mathbf{P} + (1 - \cos\theta)\, \mathbf{P}^2 = e^{\boldsymbol{\theta}\mathbf{P}}, \tag{1.8}$$

where $\mathbf{P}$ is the three dimensional skew symmetric tensor whose components are $P_{ij} = -\varepsilon_{ijk}p_k$, and $\mathbf{p}$ is a unit vector represented by $\mathbf{p} = (p_{1,}p_2, p_3)^T$. Beatty [6] presented a simple derivation of (1.8). He found a vector representation of rigid body rotation. Beatty [5] gave simple proof of some fundamental theorems on the basis of kinematics of finite rigid body displacements by using basic vector and matrix methods. He proved displacement theorems like Euler's theorem, the parallel axis theorem, Chasles's theorem and other equivalent displacement theorems.

Let $\mathbf{n}$ be a unit normal to a plane of symmetry of an anisotropic material and let $\mathbf{m}$ be any vector lying in the symmetry plane. Cowin and Mehrabadi [8] proved the following theorem.

Theorem 1 *[8] A set of necessary and sufficient conditions for a unit vector* $\mathbf{n}$ *to be a normal to a plane of symmetry is that it should be a common eigenvector of the tensors* $U_{ij} = C_{ijkk}$, $V_{ij} = C_{ikjk}$, $Q_{ik}(\mathbf{n}) = C_{ijks}n_jn_s$, $Q_{ik}(\mathbf{m}) = C_{ijks}m_jm_s$, *where summation over the repeated indices is understood and free indices take the value* $1, 2, 3$.

In 1989 Cowin and Norris modified the above conditions and showed that it is necessary and sufficient that $\mathbf{n}$ be a common eigenvector of $Q_{ik}(\mathbf{n})$ and $Q_{ik}(\mathbf{m})$. Ting [23] found other alternative necessary and sufficient conditions. Ting [24] gave simplified versions of the theorem 1.

For a linear anisotropic elastic material the number of elastic symmetries are eight [22]. Further Forte and Vianello [13] and Chadwick et al. [10] proved that there are exactly eight dissimilar classes of symmetry planes of an elasticity tensor. Each of these class can be taken as initiator of an associated group that exemplify one of the traditional symmetry class. Bona, Bucataru and Slawinski [7] proved that there are eight subgroups of the orthogonal group $O(3)$ that determine all symmetry classes of an elasticity tensor. They proved a necessary and sufficient condition for finding the symmetry classes of associated elasticity tensor and they provide a method to determine the natural coordinate system for each symmetry class. Ting [24] derived this result by applying one more direct approach using generalized Cowin-Mehrabadi theorems.

Theorem 1 deals with a normal to a plane of symmetry $\mathbf{n}$. Such type of results are available in literature but similar results involving axes of symmetry $\mathbf{p}$ do not exist. Ahmad [3] showed that any tensor of rank two associated with the material, which is independent of $\mathbf{n}$, must have $\mathbf{n}$ as an eigenvector and he also showed that an axis of symmetry satisfies first three conditions of theorem 1 when $\mathbf{n}$ is replaced by $\mathbf{p}$ and all four conditions when $\mathbf{m}$ is replaced by $\mathbf{p}$ for monoclinic, orthotropic and hexagonal materials, i.e.

Theorem *A necessary condition for a vector p to be an axis of symmetry is that it is a common eigenvector of U, V and $Q(\mathbf{p})$ as defined*

$$\begin{aligned} U_{ij} &= C_{ijkk}, \\ V_{ij} &= C_{ikjk}, \\ Q_{ik}(\mathbf{p}) &= C_{ijks}p_j p_s. \end{aligned}$$

Chadwick, Vianello and Cowin [10] proved the result that elastic symmetries can be found either by normals to symmetry planes or by axes of rotational symmetry and the transformation of reflection through a plane or a rotation about an axis belongs to the symmetry group of the material.

The matrix representation of triclinic, monoclinic, orthorhombic, cubic and of isotropic systems are very straightforward and can be obtained in a simple manner as can be seen in standard text-books, but the matrix representation of trigonal, tetragonal and of hexagonal systems, is not so simple. Some standard text-books discuss the problem only partially and leave unanswered some questions related to matrix representation, especially of the trigonal system ([18] and [19]). Hearmon [16] found transformation equations involving components $c_{11}, c_{12}, c_{22}, c_{16}, c_{26}$ and c_{66} then equated *"only the rational terms"* in the equations for c_{16} and c_{26} to conclude that these components vanish. Such an argument could be justified only if the components of the elasticity tensor were rational.

Dieulesant and Royer [11] tackled this problem by diagonalizing the rotation matrix (1.6) followed by a rotation of the coordinate axes where the transformed axes are now complex. It can be shown that an axis of symmetry implies the existence of a certain number of planes of symmetry [11]. Furthermore, Ting [23] has dealt with the trigonal, tetragonal and hexagonal symmetries by choosing two symmetry planes and considering invariance with respect to reflections in them.

In chapter 3, we discuss the necessary and sufficient condition for the existence of a plane of symmetry and its application by an alternative criterion to determine the orientation of a normal to a symmetry plane formulated by Ahmad [2]. In this chapter we give numerical examples of seven crystal system for finding symmetry planes of these crystal systems.

In chapter 4, we first review Cowin-Mehrabadi theorems for an axis of symmetry and discuss how we can determine an axis of symmetry for an unknown elastic material. Here we develop a new and easier technique to find the matrix representation for a trigonal, a tetragonal or a hexagonal material.

Chapter 2

Preliminaries

2.1 Introduction

The word symmetry is generally used with two meanings in our everyday language. The first meaning is well-balanced, well-proportioned or harmonious. Beauty is bound up with symmetry. The second meaning is a precise and well-defined concept of balance or "patterned self-similarity" that can be demonstrated or proved according to the rules of a formal system: by geometry, through physics or otherwise.

This chapter is devoted to some basic definitions and some basic mathematical preliminaries. It is concerned about the definition of symmetry and types of symmetry operation. Here we examine constants that characterize the elastic properties of crystals. These constants are the components of a tensor of rank 4. Also the relationship between the stresses and the strains in an anisotropic elastic material and some contracted notations are presented in this chapter.

2.2 What is symmetry?

"Symmetry" is a fundamental scientific concept, which alongside with the "harmony" concept has a relation practically to all forms of nature, science and art.

The work of outstanding mathematician Hermann Weyl (9 Nov. 1885 – 8 Dec. 1955) who was a German mathematician, is highly appreciated in respect to the role of symmetry in modern science. According to him

"Symmetry, as though is wide or narrow we did not perceive this

> word, there is the idea, with the help of which a man attempted to explain and to create the order, beauty and perfection" [25].

Marcus Vitruvius Pollio (C. 90-20 B.C.) was a Roman writer, architect and engineer, he has been called the world's first known engineer.

Vitruvius defines symmetry as

> "Symmetry results from proportion . . . Proportion is the commensuration of the various constituent parts with the whole" [25].

What is "symmetry"? When we look in the mirror we can see in it our reflection; this is an example of the "mirror" symmetry. The mirror reflection is an example of the so-called "orthogonal" transformation varying orientation. A modern poet Anna Wickhan (1884–1947) said about symmetry as

> "God, Thou great symmetry
> From whence my sorrows spring,
> For all the frittered days
> That I have spent in shapeless ways
> Gives me one perfect thing. " [25].

Morton Hamermesh [17] defines the symmetry and its types as follows:

The symmetry of a body is described by giving the set of all those transformations which preserves the distances between all pair of points of the body and bring the body into coincidence with itself. Any such transformation is called *symmetry transformation*.

He defines types of symmetry as follows.

The distance preserving transformations can all be built up from three fundamental types:

- *Rotation through a definite angle about some axis.*
- *Mirror reflection in a plane.*
- *Parallel displacement (translation)*

The symmetry element *translation* can occur only if the body is infinite in extent (e.g., an infinite crystal lattice).

For a body of finite extension, only first two symmetry types are possible. In fact, all transformations of symmetry group of a finite body must leave at least one point of the body fixed. In other words, all axes of rotation and all planes of reflection must intersect in (at least) one point.

Let us suppose that the body is brought into coincidence with itself if we rotate it through an angle $\Psi = 2\pi/n$ (n integral) about a certain axis. Such an axis is said to be an *n-fold rotation axis*. The symmetry axes of an object are lines about which it can be rotated through some angle which brings the object to a new orientation which appears identical to its starting position [17].

If a figure or object looks the same after being reflected in a mirror, then it has *reflectional symmetry*. The line where we must put the mirror for this to work is called the line of symmetry. In 2D there is an axis of symmetry, and in 3D there is a plane of symmetry.

Something may have more than one line of symmetry; in fact a circle has infinitely many lines of symmetry passing through its centre. Reflection in a plane is the replacement of each point on one side of a line by the point symmetrically placed on the other side of the line and in space reflection is the replacement of each point on one side of a plane by the symmetric point on the other side of the plane.

An object or figure which is indistinguishable from its transformed image is called mirror symmetric.

An image has *translational symmetry* if it can be divided by straight lines into a sequence of identical figures. Translational symmetry results from moving a figure a certain distance in a certain direction.

Translational symmetry leaves an object invariant under a discrete or continuous group of translations $T_a(p) = p + a$.

2.3 Some basic definitions and concepts

The following definition and concepts will be used throughout this dissertation.

In physics, *elasticity* is the physical property of a material when it returns to its original shape after the stress (e.g. external forces) under which it deforms is removed. The relative amount of deformation is called the strain.

By definition, a medium is said to be *elastic* if it returns to its initial state after the external forces are removed. This returns to its initial state due to

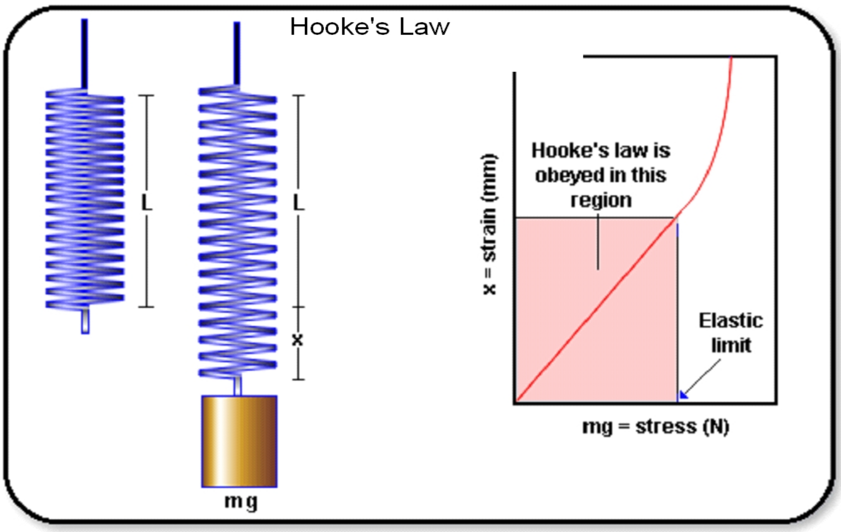

Figure 2.1: Classical model of linear elasticity.

internal stress. The elastic regime is characterized by a linear relationship between the stress and the strain, called the linear elasticity. The classic example is a metal spring. This idea was first stated by Robert Hooke in 1675 as a Latin anagram "ceiiinosssttuu" whose solution he published in 1678 as "Ut tensio, sic vis" which means "As the extension, so the force."

This linear relationship is called *Hooke's law*. The classic model of linear elasticity is the perfect spring. Although the general proportionality constant between the stress and the strain in three dimensions is a fourth rank tensor, when considering simple situations of higher symmetry such as a rod in one dimensional loading, the relationship may often be reduced to applications of Hooke's law.

The Generalized Hooke's law is defined as

$$T_{ij} = C_{ijks} E_{ks} \tag{2.1}$$

where T_{ij} is the stress tensor, E_{ij} is the strain tensor and C_{ijks} is the fourth rank elasticity tensor which is the coefficient of linearity. Because most materials are elastic only under relatively small deformations, several assumptions are used to linearize the theory. Most importantly, higher order terms are generally discarded based on the small deformation assumption. In certain special cases, such as when considering a rubbery material, these assumptions may not be

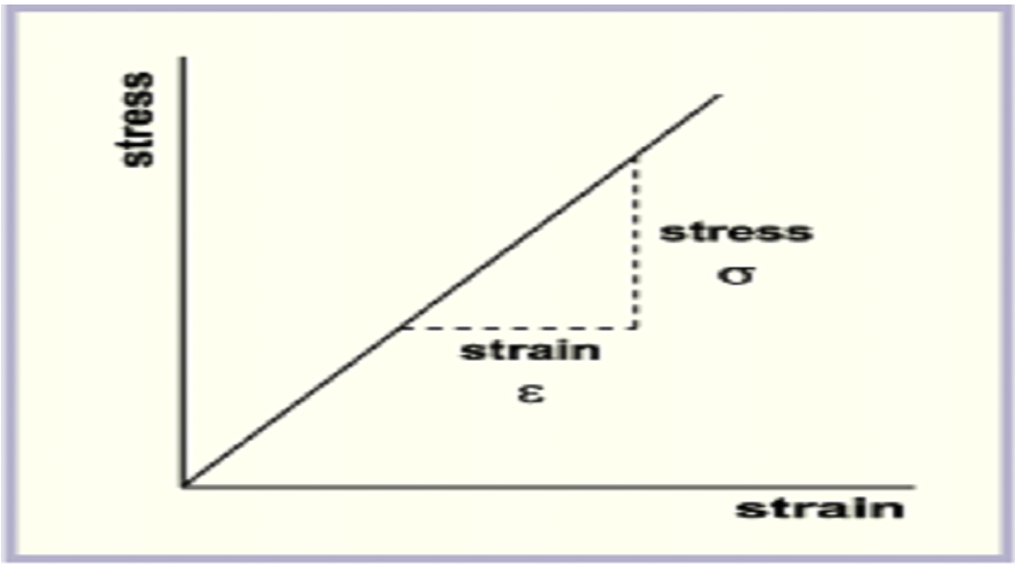

Figure 2.2: Linear relationship between stress and strain.

permissible. However, in general, elasticity refers to the linearized theory of the continuum stresses and strains.

2.4 Elastic stiffness

The coefficients C_{ijks}, that describe the most general linear relationship between the two second rank tensors T_{ij} and $E_{ks,}$ are the component of a fourth rank tensor called the *elastic stiffness* tensor

The *elastic stiffness* tensor is a fourth rank tensor and it has $3^4 = 81$ components. But since T_{ij} and E_{ks} are symmetric tensors, so the elastic constants remain unchanged under a permutation of i and j, k or s:

$$C_{ijks} = C_{jiks} \qquad ; \qquad C_{ijks} = C_{ijsk}. \tag{2.2}$$

Hooke's Law (2.1) can be written in terms of displacement:

$$T_{ij} = \frac{1}{2} C_{ijks} \frac{\partial u_k}{\partial x_s} + \frac{1}{2} C_{ijks} \frac{\partial u_s}{\partial x_k}.$$

Since $C_{ijks} = C_{ijsk}$, so both terms in the above equation are equal, or

$$T_{ij} = C_{ijks} \frac{\partial u_s}{\partial x_k} \tag{2.3}$$

[23].

The *elastic stiffness* tensor satisfies these symmetry conditions as in (2.2)

$$C_{ijks} = C_{jiks}, \tag{2.4}$$

$$C_{ijks} = C_{ijsk}, \tag{2.5}$$

$$C_{ijks} = C_{ksij}. \tag{2.6}$$

Using (2.6), (2.4) and (2.6) in that order we have

$$C_{ijks} = C_{ksij} = C_{skij} = C_{ijsk}, \tag{2.7}$$

which proves (2.5). Therefore the three equations (2.4), (2.5) and (2.6) are written as

$$C_{ijks} = C_{jiks} = C_{ijsk} = C_{ksij}. \tag{2.8}$$

From the above symmetries, we are left with 21 independent elastic constants instead of 81.

2.5 Contracted notations

Introducing the Voigt contracted notation

ij(or ks)	$\leftrightarrow$	α(or β)
11	$\leftrightarrow$	1
22	$\leftrightarrow$	2
33	$\leftrightarrow$	3
23 or 32	$\leftrightarrow$	4
31 or 13	$\leftrightarrow$	5
12 or 21	$\leftrightarrow$	6

The transformation of this table can be written in the following form

$$\alpha = \left\{ \begin{array}{ll} i & \text{if } i = j \\ 9 - i - j & \text{if } i \neq j \end{array} \right\}, \beta = \left\{ \begin{array}{ll} k & \text{if } k = s \\ 9 - k - s & \text{if } k \neq s \end{array} \right\} \tag{2.9}$$

[23].

2.6 Notations for generalized Hooke's Law

Mehrabadi and Cowin [20] defined the contracted notation as follows:

The Generalized Hooke's Law

$$T_{ij} = C_{ijks}E_{ks} \tag{2.10}$$

is the most general linear relation between the stress tensor (T_{ij}) and the strain tensor (E_{ij}). Fourth rank elasticity tensor (C_{ijks}) is the coefficient of linearity. These are the symmetries of elasticity tensor C_{ijks}.

$$C_{ijks} = C_{jiks}, \quad C_{ijks} = C_{ijsk}, \quad C_{ijks} = C_{ksij} \tag{2.11}$$

which follows respectively from the symmetry of the stress tensor, the symmetry of the strain tensor, and the strain energy due to thermodynamic requirement that the strain energy is independent of loading path. The number of independent components of a fourth-rank in three dimensions is 81, but the restrictions (2.11) reduces the number of independent components of C_{ijks} to 21 [20].

2.6.1 Three notations for generalized Hooke's Law

The purpose of this section is to present different notations for the Generalized Hooke's Law (2.10). An effort is made to present this material in a format that allows for a comparison of the notations. These notations are called the *fourth-rank tensor* notation, the *Voigt notation* and the *second-rank tensor* notation. The *fourth-rank tensor* notation employs a fourth-rank Cartesian tensor in three dimensions. The *Voigt* notation is a non-tensorial notation that employs a 6×6 matrix. The *second-rank tensor* notation employs a second-rank Cartesian tensor in six dimensions.

Generalized Hooke's Law can be represented in the *fourth-rank tensor* notation which is represented in Cartesian-index notation so Hooke's law (2.10) in matrix notation is

$$0000000 \begin{bmatrix} T_{11} \\ T_{22} \\ T_{33} \\ T_{23} \\ T_{13} \\ T_{12} \end{bmatrix} = \begin{bmatrix} C_{1111} & C_{1122} & C_{1133} & C_{1123} & C_{1113} & C_{1112} \\ C_{2211} & C_{2222} & C_{2233} & C_{2223} & C_{2213} & C_{2212} \\ C_{3311} & C_{3322} & C_{3333} & C_{3323} & C_{3313} & C_{3312} \\ C_{2311} & C_{2322} & C_{2333} & C_{2323} & C_{2313} & C_{2312} \\ C_{1311} & C_{1322} & C_{1333} & C_{1323} & C_{1313} & C_{1312} \\ C_{1211} & C_{1222} & C_{1233} & C_{1223} & C_{1213} & C_{1212} \end{bmatrix} \begin{bmatrix} E_{11} \\ E_{22} \\ E_{33} \\ 2E_{23} \\ 2E_{13} \\ 2E_{12} \end{bmatrix}. \tag{2.12}$$

The representation (2.12) is obtained from (2.10) by use of the index notation conventions of the free and summation indices. The factor of 2 that pre multiplies the shearing strains is significant because of the notational difficulties it causes.

The *Voigt* notation for the generalized Hooke's Law represents (2.12) in the form

$$\begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \sigma_4 \\ \sigma_5 \\ \sigma_6 \end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} & c_{13} & c_{14} & c_{15} & c_{16} \\ c_{21} & c_{22} & c_{23} & c_{24} & c_{25} & c_{26} \\ c_{31} & c_{32} & c_{33} & c_{34} & c_{35} & c_{36} \\ c_{41} & c_{42} & c_{43} & c_{44} & c_{45} & c_{46} \\ c_{51} & c_{52} & c_{53} & c_{54} & c_{55} & c_{56} \\ c_{61} & c_{62} & c_{63} & c_{64} & c_{65} & c_{66} \end{bmatrix} \begin{bmatrix} \epsilon_1 \\ \epsilon_2 \\ \epsilon_3 \\ \epsilon_4 \\ \epsilon_5 \\ \epsilon_6 \end{bmatrix}. \tag{2.13}$$

The *Voigt* notation is important as it has become the standard in anisotropic elasticity [20].

If we rewrite (2.12) with factors of 2 and its square root as multipliers of various terms, then we get the six dimensional second-rank *tensor* notation.

$$\begin{bmatrix} T_{11} \\ T_{22} \\ T_{33} \\ \sqrt{2}T_{23} \\ \sqrt{2}T_{13} \\ \sqrt{2}T_{12} \end{bmatrix} = \begin{bmatrix} C_{1111} & C_{1122} & C_{1133} & \sqrt{2}C_{1123} & \sqrt{2}C_{1113} & \sqrt{2}C_{1112} \\ C_{2211} & C_{2222} & C_{2233} & \sqrt{2}C_{2223} & \sqrt{2}C_{2213} & \sqrt{2}C_{2212} \\ C_{3311} & C_{3322} & C_{3333} & \sqrt{2}C_{3323} & \sqrt{2}C_{3313} & \sqrt{2}C_{3312} \\ \sqrt{2}C_{2311} & \sqrt{2}C_{2322} & \sqrt{2}C_{2333} & 2C_{2323} & 2C_{2313} & 2C_{2312} \\ \sqrt{2}C_{1311} & \sqrt{2}C_{1322} & \sqrt{2}C_{1333} & 2C_{1323} & 2C_{1313} & 2C_{1312} \\ \sqrt{2}C_{1211} & \sqrt{2}C_{1222} & \sqrt{2}C_{1233} & 2C_{1223} & 2C_{1213} & 2C_{1212} \end{bmatrix} \begin{bmatrix} E_{11} \\ E_{22} \\ E_{33} \\ \sqrt{2}E_{23} \\ \sqrt{2}E_{13} \\ \sqrt{2}E_{12} \end{bmatrix}. \tag{2.14}$$

The difference between (2.13) and (2.14) is that (2.14) is a tensor equation in six dimensions. This was formulated by Mehrabadi-Cowin [20] and its details are given in the next chapter.

It is easy to verify that (2.14) is equivalent to (2.12) by performing the indicated matrix multiplications. Based on (2.14) the second-rank tensor notation for Hooke's law is written in the form

$$\begin{bmatrix} \hat{T}_1 \\ \hat{T}_2 \\ \hat{T}_3 \\ \hat{T}_4 \\ \hat{T}_5 \\ \hat{T}_6 \end{bmatrix} = \begin{bmatrix} \hat{c}_{11} & \hat{c}_{12} & \hat{c}_{13} & \hat{c}_{14} & \hat{c}_{15} & \hat{c}_{16} \\ \hat{c}_{21} & \hat{c}_{22} & \hat{c}_{23} & \hat{c}_{24} & \hat{c}_{25} & \hat{c}_{26} \\ \hat{c}_{31} & \hat{c}_{32} & \hat{c}_{33} & \hat{c}_{34} & \hat{c}_{35} & \hat{c}_{36} \\ \hat{c}_{41} & \hat{c}_{42} & \hat{c}_{43} & \hat{c}_{44} & \hat{c}_{45} & \hat{c}_{46} \\ \hat{c}_{51} & \hat{c}_{52} & \hat{c}_{53} & \hat{c}_{54} & \hat{c}_{55} & \hat{c}_{56} \\ \hat{c}_{61} & \hat{c}_{62} & \hat{c}_{63} & \hat{c}_{64} & \hat{c}_{65} & \hat{c}_{66} \end{bmatrix} \begin{bmatrix} \hat{E}_1 \\ \hat{E}_2 \\ \hat{E}_3 \\ \hat{E}_4 \\ \hat{E}_5 \\ \hat{E}_6 \end{bmatrix}. \qquad (2.15)$$

2.7 Isotropy

A continuum is called *isotropic* if its properties are same in all directions. This will happen if the tensor C_{ijks} is an isotropic tensor. Most metals (steel, aluminum) are isotropic materials. They respond in the same way in all directions.

An *isotropic tensor* is define as a tensor whose components remain invariant under all transformations of the reference frame. $\lambda\delta_{ij}$ is a second-rank isotropic tensor, where λ is a scalar. $\delta_{ij}\delta_{ks}, \delta_{ik}\delta_{js}, \delta_{is}\delta_{jk}$ are all isotropic tensors of rank four.

In *isotropic medium* properties of a material are identical in all directions. Elasticity tensor C_{ijks} must be a linear combination of the above three tensors

$$C_{ijks} = \lambda\delta_{ij}\delta_{ks} + \mu_1\delta_{ik}\delta_{js} + \mu_2\delta_{is}\delta_{jk},$$

where λ, μ_1, μ_2 are constants.

As

$$C_{ijks} = C_{jiks}$$

we have

$$\mu_1 = \mu_2 = \mu.$$

Thus

$$\begin{aligned} C_{ijks} &= \lambda\delta_{ij}\delta_{ks} + \mu(\delta_{ik}\delta_{js} + \delta_{is}\delta_{js}) \\ c_{11} &= C_{1111} = \lambda(1) + \mu(1+1) \end{aligned}$$

or

$$c_{11} = \lambda + 2\mu.$$

Similarly

$$\begin{aligned} c_{22} &= \lambda + 2\mu, \\ c_{33} &= \lambda + 2\mu, \\ c_{12} &= C_{1122} = \lambda(1) + \mu(0) \end{aligned}$$

which implies

$$c_{12} = \lambda,$$

and

$$\begin{aligned} c_{13} &= \lambda = c_{23}, \\ c_{14} &= C_{1123} = \lambda(0) + \mu(0) \end{aligned}$$

which implies

$$c_{14} = 0.$$

Similarly

$$\begin{aligned} c_{15} &= 0 = c_{16} = c_{24} = c_{25} = c_{26} = c_{34} = c_{35} = c_{36} = c_{45} = c_{46} = c_{56}, \\ c_{44} &= C_{2323} = \lambda(0) + \mu(1+0). \end{aligned}$$

which implies $c_{44} = \mu$, and $c_{55} = \mu = c_{66}$.

Thus isotropic medium is characterized by the 6×6 matrix representation

$$c_{\alpha\beta} = \begin{bmatrix} \lambda + 2\mu & \lambda & \lambda & 0 & 0 & 0 \\ \lambda & \lambda + 2\mu & \lambda & 0 & 0 & 0 \\ \lambda & \lambda & \lambda + 2\mu & 0 & 0 & 0 \\ 0 & 0 & 0 & \mu & 0 & 0 \\ 0 & 0 & 0 & 0 & \mu & 0 \\ 0 & 0 & 0 & 0 & 0 & \mu \end{bmatrix}.$$

For an isotropic medium there are two independent parameters in terms of which all 81 components can be expressed.

2.8 Anisotropic material

Anisotropy is the property of being directionally dependent, as opposed to isotropy, which means homogeneity in all directions. In an anisotropic material, properties of a material depend on the direction; for example, wood. In a piece of wood, we can see lines going in one direction; this direction is referred to as "with the grain". The wood is stronger "with the grain" than "against the grain". Strength is a property of the wood and this property depends on the direction; thus wood is anisotropic.

2.9 Eigenvalues and eigenvectors

Eigenvalues and eigenvectors are defined for an even order tensor. A unit vector **a** is an eigenvector of a second rank tensor **T** if and only if there exists a scalar α such that

$$\mathbf{Ta} = \alpha\mathbf{a} \quad \text{or} \quad T_{ij}a_j = \alpha a_i. \tag{2.16}$$

The scalar α is called the eigenvalue corresponding to the eigenvector **a**. The necessary and sufficient condition for α to be an eigenvalue of **T** is that

$$|\mathbf{T} - \alpha\mathbf{I}| = 0. \tag{2.17}$$

The L.H.S of (2.16) is a cubic polynomial in α. Thus (2.16) has either three real roots or one real and two complex conjugate roots. For symmetric **T**, all three roots of (2.16) are real. When **T** is symmetric and positive definite (i.e. $T_{ij}b_ib_j > 0$ for every $b \neq 0$), then all the three roots of (2.16) are positive.

Chapter 3

Planes of symmetry of an anisotropic elastic material

3.1 Introduction

This chapter is basically concerned with identification of planes of symmetry of an anisotropic elastic material. We review Cowin Mehrabadi theorems for planes of symmetry. Also we discuss second-rank Cartesian tensor formulation of Hooke's law in six-dimensions and the construction of a six dimensional rotation matrix from a given rotation matrix in three dimension. Ting [24] has discussed generalized Cowin Mehrabadi theorems and he has given the technique of adding a symmetry plane at each stage that leads to a new elastic symmetry. It takes a minimum of three and maximum of five symmetry planes to reduce a triclinic material to an anisotropic elastic material.

In this chapter we find a necessary and sufficient condition for the existence of a plane of symmetry. Ahmad [2] has formulated an alternative criterion to determine the orientation of a normal to a symmetry plane. He defined a reflection matrix $\hat{\mathbf{N}}$ which depends only on $\mathbf{n}$ and proved that $\mathbf{n}$ is normal to plane of symmetry if and only if $\hat{\mathbf{N}}$ commutes with $\hat{\mathbf{c}}$, the elasticity tensor. Here we give examples of seven crystal systems for finding symmetry planes of these crystal systems.

3.2 Material symmetry

The 6×6 matrix $\mathbf{c}$ contains 21 independent elastic constants, the number of these independent elastic constants can be reduced when the material possesses a certain material symmetry. C_{ijks} are components of the fourth-rank elastic stiffness tensor.

The elastic stiffness C'_{ijks} referred to the x'_i coordinator system are

$$C'_{ijks} = \Omega_{ip}\Omega_{jq}\Omega_{kr}\Omega_{st}C_{pqrt}. \tag{3.1}$$

When $C'_{ijks} = C_{ijks}$ the above equation become

$$C_{ijks} = \Omega_{ip}\Omega_{jq}\Omega_{kr}\Omega_{st}C_{pqrt}. \tag{3.2}$$

If (3.2) holds, a material is said to possess a symmetry with respect to Ω.

An anisotropic material possesses a *symmetry of central inversion* if (3.2) is satisfied by

$$\Omega = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} = -\mathbf{I}, \tag{3.3}$$

for any C_{ijks}. All anisotropic elastic materials have the symmetry of central inversion because any tensor of even rank will satisfy (3.2) with respect to Ω given by (3.3).

The transformation represented by

$$\Omega^{(r)}(\theta) = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{3.4}$$

represents a rotation about x_3-axis through an angle θ. If a tensor C_{ijks} satisfies (3.2) with respect to $\Omega^{(r)}(\theta)$ then the material possesses a *rotational symmetry* with respect to a rotation through an angle θ about the x_3-axis.

An orthogonal transformation Ω is a *reflection* if

$$\Omega = \mathbf{I} - 2\mathbf{n}\mathbf{n}^T \text{ or } \Omega_{ij} = \delta_{ij} - 2n_in_j, \tag{3.5}$$

where $\mathbf{n}$ is a unit vector normal to the reflection plane. If $\mathbf{m}$ is any vector in the plane,

$$\boldsymbol{\Omega}\mathbf{n} = -\,\mathbf{n} \text{ or } \Omega_{ij}n_j = -n_i, \tag{3.6}$$

$$\mathbf{\Omega m} = \mathbf{m} \text{ or } \Omega_{ij} m_j = m_i. \tag{3.7}$$

Thus a vector normal to the reflection plane reverses its direction after the transformation while a vector on the reflection plane remain unchanged. When (3.2) is satisfied by the Ω of (3.5), the material is said to possess a *symmetry plane.*

For example, let

$$\mathbf{n} = \begin{bmatrix} \cos\theta \\ \sin\theta \\ 0 \end{bmatrix} \tag{3.8}$$

be the normal to a symmetry plane containing x_3-axis. The Ω of (3.5), takes the form

$$\Omega(\theta) = \begin{bmatrix} -\cos 2\theta & -\sin 2\theta & 0 \\ -\sin 2\theta & \cos 2\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad -\pi/2 < \theta \le \pi/2. \tag{3.9}$$

At $\theta = 0$,

$$\Omega(0) = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{3.10}$$

represents a reflection about the plane $x_1 = 0$. When (3.2) is satisfied by (3.10), the material has a symmetry plane at $x_1 = 0$ [23].

Below we discuss some properties of the reflection matrix. These results are given by Ting [23].

Theorem 2 *[23] If an anisotropic elastic material possesses a material symmetry with the orthogonal matrix Ω, it possesses the material symmetry with $\Omega^T = \Omega^{-1}$.*

Proof. Consider the Eq.(3.2)

$$C_{ijks} = \Omega_{ip}\Omega_{jq}\Omega_{kr}\Omega_{st}C_{pqrt},$$

multiply both sides of (3.2) by $\Omega_{ia}\Omega_{jb}\Omega_{kc}\Omega_{sd}$, we get

$$\begin{aligned} \Omega_{ia}\Omega_{jb}\Omega_{kc}\Omega_{sd}C_{ijks} &= \Omega_{ia}\Omega_{jb}\Omega_{kc}\Omega_{sd}\Omega_{ip}\Omega_{jq}\Omega_{kr}\Omega_{st}C_{pqrt} \\ &= (\Omega_{ip}\Omega_{ia})(\Omega_{jq}\Omega_{jb})(\Omega_{kr}\Omega_{kc})(\Omega_{st}\Omega_{sd})C_{pqrt}. \end{aligned}$$

Using orthogonality of Ω and definition of Kronecker delta, we have

$$\begin{aligned}\Omega_{ia}\Omega_{jb}\Omega_{kc}\Omega_{sd}C_{ijks} &= \delta_{pa}\delta_{qb}\delta_{rc}\delta_{td}C_{pqrt}\\ &= C_{abcd},\end{aligned}$$

or

$$C_{abcd} = \Omega_{ia}\Omega_{jb}\Omega_{kc}\Omega_{sd}C_{ijks}. \tag{3.11}$$

This implies that the material is symmetric with respect to the orthogonal matrix $\Omega^T = \Omega^{-1}$. ■

Theorem 3 *[23] If an anisotropic elastic material possesses a material symmetry with respect to Ω' and Ω'', it possesses a symmetry with $\Omega = \Omega'\Omega''$.*

Proof. Let the material possess a symmetry with Ω' and Ω'', i.e. ,

$$C_{ijks} = \Omega'_{ia}\Omega'_{jb}\Omega'_{kc}\Omega'_{sd}C_{abcd} \tag{3.12}$$

and

$$C_{abcd} = \Omega''_{ap}\Omega''_{bq}\Omega''_{cr}\Omega''_{dt}C_{pqrt}. \tag{3.13}$$

Insertion of (3.13) in (3.12) leads to

$$\begin{aligned}C_{ijks} &= \Omega'_{ia}\Omega'_{jb}\Omega'_{kc}\Omega'_{sd}\Omega''_{ap}\Omega''_{bq}\Omega''_{cr}\Omega''_{dt}C_{pqrt}\\ &= (\Omega'_{ia}\Omega''_{ap})(\Omega'_{jb}\Omega''_{bq})(\Omega'_{kc}\Omega''_{cr})(\Omega'_{sd}\Omega''_{dt})C_{pqrt}\end{aligned}$$

which is of the form

$$C_{ijks} = \Omega_{ip}\Omega_{jq}\Omega_{kr}\Omega_{st}C_{pqrt},$$

when $\Omega = \Omega'\Omega''$.

Hence the material possesses a symmetry with $\Omega = \Omega'\Omega''$. ■

These two theorems are useful in determining the structure of C_{ijks} when the material possesses symmetry planes [23].

3.3 Cowin-Mehrabadi theorem

If we have an anisotropic elastic material which possesses an axis of symmetry or a plane of symmetry and we choose coordinate axes so as to align one of the axes along an axis of symmetry or normal to a plane of symmetry then symmetry forces some components of elasticity tensor to become zero and elasticity tensor

C takes a simple form and we have the knowledge about the number and location of symmetry planes of a given material. But when we come across an unknown material, we choose an arbitrary set of coordinate axes and determine the elastic constants of the material with respect to this system. The problem of how to locate symmetry planes, if they exist, of the given matrix C, in an arbitrary Cartesian coordinate system, was showed by Cowin and Mehrabadi who proved the following theorem.

Theorem 4 *[8] A set of necessary and sufficient conditions for a unit vector* $\mathbf{n}$ *to be a normal to a plane of symmetry is that it should be a common eigenvector of the following tensors*

$$\begin{aligned} U_{ij} &= C_{ijkk}, \\ V_{ij} &= C_{ikjk}, \\ Q_{ik}(\mathbf{n}) &= C_{ijks}n_j n_s, \\ Q_{ik}(\mathbf{m}) &= C_{ijks}m_j m_s, \end{aligned}$$

where $\mathbf{m}$ *is any vector perpendicular to* $\mathbf{n}$ *.*

Proof. First we show that the conditions are necessary. To show that $\mathbf{n}$ is a common eigenvector of above four tensors we have to prove the following equations

$$C_{ijkk}n_j = (C_{pqtt}n_p n_q)\, n_i, \tag{3.14}$$

$$C_{isks}n_k = (C_{pqrq}n_p n_r)\, n_i, \tag{3.15}$$

$$C_{ijks}n_j n_s n_k = (C_{pqrt}n_p n_q n_r n_t)\, n_i, \tag{3.16}$$

$$C_{ijks}m_j m_s n_k = (C_{pqrt}n_p m_q n_r m_t)\, n_i, \tag{3.17}$$

To prove (3.14) consider (3.2) and multiply both sides by n_j, we have

$$\begin{aligned} C_{ijks}n_j &= \Omega_{ip}\Omega_{jq}\Omega_{kr}\Omega_{st}C_{pqrt}n_j, \\ &= (\Omega_{jq}n_j)\Omega_{ip}\Omega_{kr}\Omega_{st}C_{pqrt}. \end{aligned}$$

Using (3.6)

$$\begin{aligned} C_{ijks}n_j &= (-n_q)\Omega_{ip}\Omega_{kr}\Omega_{st}C_{pqrt}, \\ &= -\Omega_{ip}\Omega_{kr}\Omega_{st}C_{pqrt}n_q. \end{aligned}$$

Let $s = k$

$$C_{ijkk}n_j = -\Omega_{ip}\Omega_{kr}\Omega_{kt}C_{pqrt}n_q.$$

Since Ω is symmetric and orthogonal, so we have

$$\begin{aligned}\Omega_{kr}\Omega_{kt} &= \Omega_{rk}\Omega_{kt},\\ &= \delta_{rt}.\end{aligned}$$

Therefore

$$\begin{aligned}C_{ijkk}n_j &= -\Omega_{ip}\delta_{rt}C_{pqrt}n_q,\\ &= -\Omega_{ip}C_{pqrr}n_q,\\ &= -\left(\delta_{ip} - 2n_i n_p\right)C_{pqrr}n_q,\\ &= -\delta_{ip}C_{pqrr}n_q + 2C_{pqrr}n_i n_p n_q,\\ &= -C_{iqrr}n_q + 2C_{pqrr}n_i n_p n_q.\end{aligned}$$

Here q and r are dummy indices so change q to j and r to k

$$\begin{aligned}C_{ijkk}n_j &= -C_{ijkk}n_j + 2C_{pqrr}n_i n_p n_q,\\ &\implies C_{ijkk}n_j = C_{pqrr}n_i n_p n_q,\end{aligned}$$

or

$$C_{ijkk}n_j = (C_{pqrr}n_p n_q)n_i,$$

which is the required result (3.14).

To prove (3.15) consider (3.2) and multiply both sides by n_k, we have

$$\begin{aligned}C_{ijks}n_k &= \Omega_{ip}\Omega_{jq}\Omega_{kr}\Omega_{st}C_{pqrt}n_k\\ &= (\Omega_{kr}n_k)\Omega_{ip}\Omega_{jq}\Omega_{st}C_{pqrt}.\end{aligned}$$

By using (3.6)

$$\begin{aligned}C_{ijks}n_k &= (-n_r)\Omega_{ip}\Omega_{jq}\Omega_{st}C_{pqrt}\\ &= -\Omega_{ip}\Omega_{jq}\Omega_{st}C_{pqrt}n_r.\end{aligned}$$

Let $j = s$

$$C_{isks}n_k = -n_r\Omega_{ip}\Omega_{sq}\Omega_{st}C_{pqrt}.$$

Since Ω is symmetric and orthogonal, so we have

$$\begin{aligned}\Omega_{sq}\Omega_{st} &= \Omega_{qs}\Omega_{st},\\ &= \delta_{qt}.\end{aligned}$$

Therefore

$$\begin{aligned}
C_{isks}n_k &= -\Omega_{ip}\delta_{qt}C_{pqrt}n_r, \\
&= -\Omega_{ip}C_{pqrq}n_r, \\
&= -\left(\delta_{ip} - 2n_in_p\right)C_{pqrq}n_r, \\
&= -\delta_{ip}C_{pqrq}n_r + 2C_{pqrq}n_in_pn_r, \\
&= -C_{iqrq}n_r + 2C_{pqrq}n_in_pn_r.
\end{aligned}$$

Here q and r are dummy indices so change q to s and r to k so

$$\begin{aligned}
C_{isks}n_k &= -C_{isks}n_k + 2C_{pqrq}n_in_pn_r, \\
&\Longrightarrow \; C_{isks}n_k = C_{pqrq}n_in_pn_r,
\end{aligned}$$

or

$$C_{isks}n_k = (C_{pqrq}n_pn_r)n_i,$$

is the required result (3.15).

To prove (3.16) consider (3.2) and multiply both sides by n_j, n_k, n_s, we have

$$\begin{aligned}
C_{ijks}n_jn_kn_s &= \Omega_{ip}\Omega_{jq}\Omega_{kr}\Omega_{st}C_{pqrt}n_jn_kn_s, \\
&= (\Omega_{jq}n_j)(\Omega_{kr}n_k)(\Omega_{st}n_s)\Omega_{ip}C_{pqrt}, \\
&= \left(-n_q\right)\left(-n_r\right)\left(-n_t\right)\Omega_{ip}C_{pqrt}, \\
&= -\Omega_{ip}C_{pqrt}n_qn_rn_t, \\
&= -\left(\delta_{ip} - 2n_in_p\right)C_{pqrt}n_qn_rn_t, \\
&= -\delta_{ip}C_{pqrt}n_qn_rn_t + 2n_in_pC_{pqrt}n_qn_rn_t, \\
&= -C_{iqrt}n_qn_rn_t + 2C_{pqrt}n_qn_rn_tn_pn_i.
\end{aligned}$$

In the first term of last equation q, r, and t are dummy indices so we change q to j, r to k, and s to t

$$\begin{aligned}
C_{ijks}n_jn_kn_s &= -C_{ijks}n_jn_kn_s + 2C_{pqrt}n_qn_rn_tn_pn_i \\
&\Longrightarrow \; C_{ijks}n_jn_kn_s = C_{pqrt}n_qn_rn_tn_pn_i
\end{aligned}$$

or

$$C_{ijks}n_jn_kn_s = (C_{pqrt}n_pn_qn_rn_t)n_i$$

which is the required result (3.16).

To prove (3.17) consider (3.2) and multiply both sides by m_j, n_k, m_s , we have

$$\begin{aligned} C_{ijks}m_jn_km_s &= \Omega_{ip}\Omega_{jq}\Omega_{kr}\Omega_{st}C_{pqrt}m_jn_km_s, \\ &= (\Omega_{jq}m_j)(\Omega_{kr}n_k)(\Omega_{st}m_s)\Omega_{ip}C_{pqrt}, \\ &= (m_q)\,(-n_r)\,(m_t)\,\Omega_{ip}C_{pqrt}, \\ &= -\Omega_{ip}C_{pqrt}m_qn_rm_t, \\ &= -\left(\delta_{ip}-2n_in_p\right)C_{pqrt}m_qn_rm_t, \\ &= -\delta_{ip}C_{pqrt}m_qn_rm_t+2n_in_pC_{pqrt}m_qn_rm_t, \\ &= -C_{iqrt}m_qn_rm_t+2C_{pqrt}m_qn_rm_tn_pn_i. \end{aligned}$$

In the first term of last equation q, r,and t are dummy indices so we change q to j, r to k, and s to t

$$\begin{aligned} C_{ijks}m_jm_kn_s &= -C_{ijks}m_jn_km_s+2C_{pqrt}m_qn_rm_tn_pn_i \\ &\Longrightarrow\ C_{ijks}m_jn_km_s = C_{pqrt}m_qn_rm_tn_pn_i \end{aligned}$$

or

$$C_{ijks}m_jn_km_s = (C_{pqrt}n_pm_qn_rm_t)n_i,$$

which is the required result (3.17).

Now we prove that the conditions are sufficient. For this we suppose that $x_1 = 0$ is a plane of symmetry so that

$$n_i = \delta_{i1}, \tag{3.18}$$

$$m_i = \delta_{i2}\cos\theta + \delta_{i3}\sin\theta, \tag{3.19}$$

where θ is an arbitrary constant. Also the equations (3.14), (3.15), (3.16) and (3.17) are satisfied. Use (3.18) into (3.14), (3.15) and (3.16) respectively. We have

$$\begin{aligned} C_{ijkk}\delta_{j1} &= \left(C_{pqtt}\delta_{p1}\delta_{q1}\right)\delta_{i1} \\ &\Longrightarrow\ C_{i1kk} = C_{11tt}\delta_{i1}. \end{aligned}$$

For $i = 1$, Eqs. (3.14) to (3.17) are trivial.

Now for $i = 2, 3$ we have

$$C_{21kk} = 0,\ C_{31kk} = 0$$

or

$$C_{2111} + C_{2122} + C_{2133} = 0,\ C_{3111} + C_{3122} + C_{3133} = 0$$

or

$$c_{16} + c_{26} + c_{36} = 0,\ \ c_{15} + c_{25} + c_{35} = 0 \tag{3.20}$$

and

$$\begin{aligned} C_{isks}\delta_{k1} &= C_{pqrq}\delta_{p1}\delta_{r1}\delta_{i1} \\ &\Longrightarrow C_{is1s} = C_{1q1q}\delta_{i1}. \end{aligned}$$

Now for $i = 2, 3$ we have

$$C_{2s1s} = 0\ C_{3s1s} = 0$$

or

$$C_{2111} + C_{2212} + C_{2313} = 0,\ \ C_{3111} + C_{3212} + C_{3313} = 0$$

or

$$c_{16} + c_{26} + c_{45} = 0,\ c_{15} + c_{46} + c_{35} = 0 \tag{3.21}$$

and

$$\begin{aligned} C_{ijks}\delta_{j1}\delta_{s1}\delta_{k1} &= C_{pqrt}\delta_{p1}\delta_{q1}\delta_{r1}\delta_{t1}\delta_{i1} \\ &\Longrightarrow C_{i111} = C_{1111}\delta_{i1}. \end{aligned}$$

Now for $i = 2, 3$ we have

$$C_{2111} = 0,\ C_{3111} = 0$$

or

$$c_{16} = 0,\ c_{15} = 0. \tag{3.22}$$

Now put (3.19) in (3.17), we have

$$\begin{aligned}
& C_{ijks}\left(\delta_{j2}\cos\theta + \delta_{j3}\sin\theta\right)\left(\delta_{s2}\cos\theta + \delta_{s3}\sin\theta\right)\delta_{k1} \\
= \ & C_{pqrt}\delta_{p1}\left(\delta_{q2}\cos\theta + \delta_{q3}\sin\theta\right)\delta_{r1}\left(\delta_{t2}\cos\theta + \delta_{t3}\sin\theta\right)\delta_{i1} \\
\Longrightarrow \ & C_{ij1s}\left(\delta_{j2}\cos\theta + \delta_{j3}\sin\theta\right)\left(\delta_{s2}\cos\theta + \delta_{s3}\sin\theta\right) \\
= \ & C_{1q1t}\left(\delta_{q2}\cos\theta + \delta_{q3}\sin\theta\right)\left(\delta_{t2}\cos\theta + \delta_{t3}\sin\theta\right)\delta_{i1} \\
\Longrightarrow \ & C_{i21s}\cos\theta\left(\delta_{s2}\cos\theta + \delta_{s3}\sin\theta\right) + C_{i31s}\sin\theta\left(\delta_{s2}\cos\theta + \delta_{s3}\sin\theta\right) \\
= \ & C_{121t}\cos\theta\left(\delta_{t2}\cos\theta + \delta_{t3}\sin\theta\right)\delta_{i1} + C_{131t}\sin\theta\left(\delta_{t2}\cos\theta + \delta_{t3}\sin\theta\right)\delta_{i1} \\
\Longrightarrow \ & C_{i212}\cos^2\theta + C_{i213}\cos\theta\sin\theta + C_{i312}\sin\theta\cos\theta + C_{i313}\sin^2\theta \\
= \ & \left[C_{1212}\cos^2\theta + C_{1213}\cos\theta\sin\theta + C_{1312}\sin\theta\cos\theta + C_{1313}\sin^2\theta\right]\delta_{i1}.
\end{aligned}$$

Now for $\theta = 0, \pi/2$, and for arbitrary θ respectively, we have

$$\begin{aligned} C_{i212} &= C_{1212}\delta_{i1}, \\ C_{i313} &= C_{1313}\delta_{i1}, \\ (C_{i213} + C_{i312})\sin\theta\cos\theta &= (C_{1213} + C_{1312})\cos\theta\sin\theta\delta_{i1} \end{aligned}$$

for $\theta \neq 0, \pi/2$, above equation becomes

$$C_{i213} + C_{i312} = (C_{1213} + C_{1312})\delta_{i1}.$$

For $i = 1$, above mentioned three equations are satisfied so for $i = 2, 3$ in above three equations respectively we have

$$\begin{aligned} C_{2212} &= 0 = C_{3212} \\ &\Longrightarrow c_{26} = 0 = c_{46} = 0, \end{aligned}$$

and

$$\begin{aligned} C_{2313} &= 0 = C_{3313} \\ &\Longrightarrow c_{45} = 0 = c_{35} = 0, \end{aligned}$$

and

$$\begin{aligned} C_{2213} + C_{2312} &= 0 = C_{3213} + C_{3312} \\ &\Longrightarrow c_{25} + c_{46} = 0 = c_{45} + c_{36}. \end{aligned}$$

Thus we have

$$c_{26} = c_{46} = c_{45} = c_{35} = c_{25} = c_{36} = 0. \tag{3.23}$$

Thus we notice that (3.20)-(3.23) are specialization of (3.14)-(3.17) respectively only if unit normal $\mathbf{n}$ is along x_1-axis, hence $\mathbf{n}$ is normal to a symmetry plane. If $x_1 = 0$ is a plane of symmetry we have

$$c_{15} = c_{16} = c_{25} = c_{26} = c_{35} = c_{36} = c_{45} = c_{46} = 0. \tag{3.24}$$

which leads to one symmetry plane ■

3.3.1 Modified Cowin-Mehrabadi theorem

Cowin showed that (3.16) and (3.17) are necessary and sufficient conditions. In fact (3.14) and (3.17), or (3.15) and (3.17), are also necessary and sufficient conditions. Therefore Ting [23] stated a modified Cowin-Mehrabadi theorem as follows

Theorem 5 *[23] An anisotropic elastic material with given elastic compliances C_{ijks} has a plane of symmetry if and only if* $\mathbf{n}$ *is an eigenvector of* (i) $\mathbf{Q}(\mathbf{n})$ *and* $\mathbf{Q}(m)$*, or* (ii) $\mathbf{U}$ *and* $\mathbf{Q}(m)$*, or* (iii) $\mathbf{V}$ *and* $\mathbf{Q}(m)$*. The vector* $\mathbf{n}$ *is normal to the plane of symmetry while* $\mathbf{m}$ *is any vector on the plane of symmetry.*

But this theorem is not suitable for determining $\mathbf{n}$ because the matrix $\mathbf{Q}(m)$ depends on $\mathbf{m}$ which, depends on $\mathbf{n}$. Therefore following theorem is more useful for determining $\mathbf{n}$.

Theorem 6 *[23] An anisotropic elastic material has a plane of symmetry if and only if normal* $\mathbf{n}$ *to the plane of symmetry is a common eigenvector of* U *and* V*, and satisfies*

$$C_{ijks}m_in_jn_kn_s = 0, \tag{3.25}$$

$$C_{ijks}m_im_jm_kn_s = 0, \tag{3.26}$$

for any two independent vectors $m^{(\alpha)}$ $(\alpha = 1, 2)$ *on the plane of symmetry that don't form an angle a multiple of* $\pi/3$.

Proof. Suppose that a normal $\mathbf{n}$ is along x_1-axis so

$$n_i = \delta_{i1} \tag{3.27}$$

and two $\mathbf{m}'s$ are

$$m_i^{(1)} = \delta_{i2}\cos\theta + \delta_{i3}\sin\theta, \tag{3.28}$$

$$m_i^{(2)} = \delta_{i2}\cos\Psi + \delta_{i3}\sin\Psi, \tag{3.29}$$

where θ and Ψ are arbitrary constants with condition $\theta - \Psi \neq \frac{1}{3}k\pi$, k integer. Substitute (3.27), (3.28) and (3.29) in (3.25). We have

$$C_{ijks}\left(\delta_{i2}\cos\theta + \delta_{i3}\sin\theta\right)\delta_{j1}\delta_{k1}\delta_{s1} = 0,$$

or

$$C_{i111}\left(\delta_{i2}\cos\theta + \delta_{i3}\sin\theta\right) = 0,$$

or

$$C_{2111}\cos\theta + C_{3111}\sin\theta = 0,$$

or

$$c_{16}\cos\theta + c_{15}\sin\theta = 0,$$

and similarly

$$c_{16}\cos\Psi + c_{15}\sin\Psi = 0.$$

Now determinant for $[c_{15}, c_{16}]$ of

$$\begin{aligned} c_{16}\cos\theta + c_{15}\sin\theta &= 0 \\ c_{16}\cos\Psi + c_{15}\sin\Psi &= 0 \end{aligned}$$

is $\sin(\theta - \Psi)$, so for $\theta - \Psi \neq \frac{1}{3}k\pi$, k integer $\sin(\theta - \Psi) \neq 0$ which implies

$$c_{15} = c_{16} = 0. \tag{3.30}$$

When $n_i = \delta_{i1}$ is a common eigenvector of $\mathbf{U}$ and $\mathbf{V}$, so from equations (3.20), (3.21) and (3.30) we have

$$c_{36} = c_{45} = -c_{26},\ c_{25} = c_{46} = -c_{35}. \tag{3.31}$$

Now substitute (3.27), (3.28) and (3.29) in (3.26), we have

$$\begin{aligned} C_{ijks}\left(\delta_{i2}\cos\theta + \delta_{i3}\sin\theta\right)\left(\delta_{j2}\cos\theta + \delta_{j3}\sin\theta\right)\left(\delta_{k2}\cos\theta + \delta_{k3}\sin\theta\right)\delta_{s1} &= 0, \\ \left(C_{2jk1}\cos\theta + C_{3jk1}\sin\theta\right)\left(\delta_{j2}\cos\theta + \delta_{j3}\sin\theta\right)\left(\delta_{k2}\cos\theta + \delta_{k3}\sin\theta\right) &= 0, \\ \left(C_{22k1}\cos^2\theta + C_{23k1}\cos\theta\sin\theta + C_{32k1}\sin\theta\cos\theta + C_{33k1}\sin^2\theta\right)\left(\delta_{k2}\cos\theta + \delta_{k3}\sin\theta\right) &= 0, \end{aligned}$$

$$C_{2221}\cos^3\theta + C_{2231}\cos^2\theta\sin\theta + (C_{2321} + C_{3221})\cos^2\theta\sin\theta + (C_{2331} + C_{3231})\cos\theta\sin^2\theta + C_{3321}\sin^2\theta\cos\theta + C_{3331}\sin^3\theta = 0,$$

or

$$c_{26}\cos^3\theta + c_{25}\cos^2\theta\sin\theta + 2c_{46}\cos^2\theta\sin\theta + 2c_{45}\cos\theta\sin^2\theta + c_{36}\sin^2\theta\cos\theta + c_{35}\sin^3\theta = 0.$$

Using (3.31) and after simplification we get

$$\cos\theta\left(2\cos 2\theta - 1\right)c_{26} - \sin\theta\left(2\cos 2\theta + 1\right)c_{35} = 0.$$

Similarly

$$\cos\Psi\left(2\cos 2\Psi - 1\right)c_{26} - \sin\Psi\left(2\cos 2\Psi + 1\right)c_{35} = 0.$$

Now their determinant for $[c_{26}, c_{35}]$ is $\sin(\theta-\Psi)\left[4\cos^2(\theta-\Psi)-1\right]$, so for $\theta-\Psi\neq\frac{1}{3}k\pi$, k integer $\sin(\theta-\Psi)\left[4\cos^2(\theta-\Psi)-1\right]\neq 0 \Longrightarrow$

$$c_{26}=c_{35}=0. \tag{3.32}$$

Hence from equations (3.30), (3.31) and (3.32), we get

$$c_{15}=c_{16}=c_{25}=c_{26}=c_{35}=c_{36}=c_{45}=c_{46}=0$$

which proves that material has a symmetry plane.

Conversely suppose that material has a symmetry plane. To show that $\mathbf{n}$ is a common eigenvector of $\mathbf{U}$ and $\mathbf{V}$ we have to prove (3.14) and (3.15) which we have proved in theorem 4 already. Next we have to prove Eqs.(3.25) and (3.26). To prove (3.25) consider (3.2) and multiply both sides by m_i, n_j, n_k, n_s, we have

$$\begin{aligned} C_{ijks}m_in_jn_kn_s &= \Omega_{ip}\Omega_{jq}\Omega_{kr}\Omega_{st}C_{pqrt}m_in_jn_kn_s, \\ &= (\Omega_{ip}m_i)(\Omega_{jq}n_j)(\Omega_{kr}n_k)(\Omega_{st}n_s)C_{pqrt}, \end{aligned}$$

by using (3.6) and (3.7), we have

$$\begin{aligned} C_{ijks}m_in_jn_kn_s &= (m_p)(-n_q)(-n_r)(-n_t)C_{pqrt}, \\ &= -m_pn_qn_rn_tC_{pqrt}, \end{aligned}$$

here all indices are dummy so change p, q, r, t to i, j, k, s, so we have

$$\begin{aligned} C_{ijks}m_in_jn_kn_s &= -C_{ijks}m_in_jn_kn_s \\ &\Longrightarrow C_{ijks}m_in_jn_kn_s = 0 \end{aligned}$$

Now to prove (3.26) consider (3.2) and multiply both sides by m_i, m_j, m_k, n_s, we have

$$\begin{aligned} C_{ijks}m_im_jm_kn_s &= \Omega_{ip}\Omega_{jq}\Omega_{kr}\Omega_{st}C_{pqrt}m_im_jm_kn_s, \\ &= (\Omega_{ip}m_i)(\Omega_{jq}m_j)(\Omega_{kr}m_k)(\Omega_{st}n_s)C_{pqrt}, \end{aligned}$$

by using (3.6) and (3.7), we have

$$\begin{aligned} C_{ijks}m_im_jm_kn_s &= (m_p)(m_q)(m_r)(-n_t)C_{pqrt}, \\ &= -m_pm_qm_rn_tC_{pqrt}, \end{aligned}$$

again here all indices are dummy so change p, q, r, t to i, j, k, s, so we have

$$\begin{aligned} C_{ijks}m_im_jm_kn_s &= -C_{ijks}m_im_jm_kn_s \\ &\Longrightarrow C_{ijks}m_im_jm_kn_s = 0. \end{aligned}$$

Which is the required result. ■

3.3.2 Generalized Cowin-Mehrabadi theorems

In this section Cowin-Mehrabadi theorem (theorem 4) is generalized. Ting [24] showed that there is no need for $\mathbf{n}$ to satisfy (3.17) for all $\mathbf{m}$. It suffices to satisfy (3.17) for any one $\mathbf{m}$, when $\mathbf{n}$ is an eigenvector of $\mathbf{U}$, $\mathbf{V}$, and $\mathbf{Q}(\mathbf{n})$. If $\mathbf{n}$ is an eigenvector of two of three matrices $\mathbf{U}$, $\mathbf{V}$, and $\mathbf{Q}(\mathbf{n})$, it suffices to satisfy (3.17) for any two distinct $\mathbf{m}$, when $\mathbf{n}$ is an eigenvector of any one of three matrices $\mathbf{U}$, $\mathbf{V}$, and $\mathbf{Q}(\mathbf{n})$, it suffices to satisfy (3.17) for any three distinct $\mathbf{m}$. In particular (3.16) and (3.17) with any three distinct $\mathbf{m}$ are necessary and sufficient conditions for $\mathbf{n}$ to be normal to a symmetry plane. Therefore Ting [24] stated the following theorems.

Theorem 7 *[24] A necessary and sufficient condition for* $\mathbf{n}$ *to be normal to a symmetry plane is that* $\mathbf{n}$ *be an eigenvector of* $\mathbf{U}$*,* $\mathbf{V}$*,* $\mathbf{Q}(\mathbf{n})$ *and* $\mathbf{Q}(\mathbf{m})$ *for any one* $\mathbf{m}$ *.*

Theorem 8 *[24] A necessary and sufficient condition for* $\mathbf{n}$ *to be normal to a symmetry plane is that* $\mathbf{n}$ *be an eigenvector of* $\mathbf{U}$*,* $\mathbf{V}$*, and* $\mathbf{Q}(\mathbf{m})$ *for any two distinct* $\mathbf{m}$ *.*

3.4 Mehrabadi-Cowin formalism

In this section a tensorial equivalence is developed between three-dimensional tensor and six-dimensional tensor that is presented by Mehrabadi and Cowin [20].

For this tensor formulation, first we introduce Cartesian basis in three-dimensions to construct Cartesian basis in six-dimensions, the Cartesian base vectors in three-dimensions are denoted by $\mathbf{e}_i$, $i = 1, 2, 3$ and the Cartesian base vectors in six-dimensions are denoted by $\hat{\mathbf{e}}_\alpha$, $\alpha = 1, 2, ...6$. The relationship between these two bases is as follows

$$\left\{ \begin{array}{c} \hat{\mathbf{e}}_1 = \mathbf{e}_1 \otimes \mathbf{e}_1 \\ \hat{\mathbf{e}}_2 = \mathbf{e}_2 \otimes \mathbf{e}_2 \\ \hat{\mathbf{e}}_3 = \mathbf{e}_3 \otimes \mathbf{e}_3 \\ \hat{\mathbf{e}}_4 = \frac{1}{\sqrt{2}}(\mathbf{e}_2 \otimes \mathbf{e}_3 + \mathbf{e}_3 \otimes \mathbf{e}_2) \\ \hat{\mathbf{e}}_5 = \frac{1}{\sqrt{2}}(\mathbf{e}_1 \otimes \mathbf{e}_3 + \mathbf{e}_3 \otimes \mathbf{e}_1) \\ \hat{\mathbf{e}}_6 = \frac{1}{\sqrt{2}}(\mathbf{e}_1 \otimes \mathbf{e}_2 + \mathbf{e}_2 \otimes \mathbf{e}_1) \end{array} \right\}, \tag{3.33}$$

where $\otimes$ indicate tensor product.

The six-dimensional base vectors $\hat{e}_1, \hat{e}_{2,}...\hat{e}_6$ are either as vectors in the six-dimensional space $(1,0,0,0,0,0)$, $(0,1,0,0,0,0)$, $(0,0,1,0,0,0)$, $(0,0,0,1,0,0)$, $(0,0,0,0,1,0)$, $(0,0,0,0,0,1)$, or as a special second-rank tensor in three-dimensions

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, ..., 1/\sqrt{2}\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{3.34}$$

relative to these two bases, the components of stress are expressed as

$$T_{ij}\mathbf{e}_i\otimes\mathbf{e}_j = \hat{T}_\alpha\hat{\mathbf{e}}_\alpha. \tag{3.35}$$

Now using (3.33) and summation convention over repeated indices, we prove (3.35) as follows

$$\begin{aligned} T_{ij}\mathbf{e}_i\otimes\mathbf{e}_j &= T_{1j}\mathbf{e}_1\otimes\mathbf{e}_j + T_{2j}\mathbf{e}_2\otimes\mathbf{e}_j + T_{3j}\mathbf{e}_3\otimes\mathbf{e}_j \\ &= T_{11}\mathbf{e}_1\otimes\mathbf{e}_1 + T_{12}\mathbf{e}_1\otimes\mathbf{e}_2 + T_{13}\mathbf{e}_1\otimes\mathbf{e}_3 \\ &\quad +T_{21}\mathbf{e}_2\otimes\mathbf{e}_1 + T_{22}\mathbf{e}_2\otimes\mathbf{e}_2 + T_{23}\mathbf{e}_2\otimes\mathbf{e}_3 \\ &\quad +T_{31}\mathbf{e}_3\otimes\mathbf{e}_1 + T_{32}\mathbf{e}_3\otimes\mathbf{e}_2 + T_{33}\mathbf{e}_3\otimes\mathbf{e}_3. \end{aligned}$$

As T is symmetric, so we have

$$\begin{aligned} T_{ij}e_i \otimes e_j &= T_{11}\mathbf{e}_1\otimes\mathbf{e}_1 + T_{22}\mathbf{e}_2\otimes\mathbf{e}_2 \\ &\quad +T_{33}\mathbf{e}_3\otimes\mathbf{e}_3 + T_{12}(\mathbf{e}_1\otimes\mathbf{e}_2+\mathbf{e}_2\otimes\mathbf{e}_1) \\ &\quad +T_{13}(\mathbf{e}_1\otimes\mathbf{e}_3+\mathbf{e}_3\otimes\mathbf{e}_1) + T_{23}\left(\mathbf{e}_2\otimes\mathbf{e}_3+\mathbf{e}_3\otimes\mathbf{e}_2\right) \\ &= T_{11}\hat{\mathbf{e}}_1 + T_{22}\hat{\mathbf{e}}_2 + T_{33}\hat{\mathbf{e}}_3 + T_{23}\sqrt{2}\hat{\mathbf{e}}_4 + T_{13}\sqrt{2}\hat{\mathbf{e}}_5 + T_{12}\sqrt{2}\hat{\mathbf{e}}_6 \\ &= \hat{T}_1\hat{\mathbf{e}}_1 + \hat{T}_2\hat{\mathbf{e}}_2 + \hat{T}_3\hat{\mathbf{e}}_3 + \hat{T}_4\hat{\mathbf{e}}_4 + \hat{T}_5\hat{\mathbf{e}}_5 + \hat{T}_6\hat{\mathbf{e}}_6 \end{aligned}$$

$$T_{ij}\mathbf{e}_i\otimes\mathbf{e}_j = \hat{T}_\alpha\hat{\mathbf{e}}_\alpha,\ \alpha = 1,2,...,6. \tag{3.36}$$

Now with respect to the three-dimensional and six-dimensional bases, the component of a tensor of elastic constants is expressed as

$$C_{ijkm}\mathbf{e}_i\otimes\mathbf{e}_j\otimes\mathbf{e}_k\otimes\mathbf{e}_m = \hat{c}_{\alpha\beta}\hat{\mathbf{e}}_\alpha\otimes\hat{\mathbf{e}}_\beta. \tag{3.37}$$

Let the stress and the strain tensor be denoted by T_{ij} and E_{ij} respectively in Cartesian basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$, they are related through generalized Hooke's Law

$$T_{ij} = C_{ijkl}E_{kl}.$$

Thus concisely

$$\hat{e}_{\alpha(i,j)} = 2^{-\frac{1}{2-\delta_{ij}}} \left(\mathbf{e}_i \otimes \mathbf{e}_j + \mathbf{e}_j \otimes \mathbf{e}_i\right) \tag{3.38}$$

where $\alpha(i,j) = i\delta_{ij} + (9 - i - j)$. Define $\hat{T}_{\alpha(i,j)} = 2^{\frac{1}{2-\delta_{ij}}} T_{ij}$ and $\hat{E}_{\alpha(i,j)} = 2^{\frac{1}{2-\delta_{ij}}} E_{ij}$ where $i, j = 1, 2, 3$ and $\alpha = 1, 2, ..., 6$. With respect to the basis $\{\hat{\mathbf{e}}_\alpha, \alpha = 1, 2, ..., 6\}$, Eq.(3.37) becomes

$$\hat{T}_\alpha = \hat{c}_{\alpha\beta} \hat{E}_\beta, \quad \alpha, \beta = 1, 2, ..., 6 \tag{3.39}$$

where $\hat{c}_{\alpha\beta}$ has the matrix representation

$$\hat{c}_{\alpha\beta} = \begin{bmatrix} c_{11} & c_{12} & c_{13} & \sqrt{2}c_{14} & \sqrt{2}c_{15} & \sqrt{2}c_{16} \\ c_{12} & c_{22} & c_{23} & \sqrt{2}c_{24} & \sqrt{2}c_{25} & \sqrt{2}c_{26} \\ c_{13} & c_{23} & c_{33} & \sqrt{2}c_{34} & \sqrt{2}c_{35} & \sqrt{2}c_{36} \\ \sqrt{2}c_{14} & \sqrt{2}c_{24} & \sqrt{2}c_{34} & 2c_{44} & 2c_{45} & 2c_{46} \\ \sqrt{2}c_{15} & \sqrt{2}c_{25} & \sqrt{2}c_{35} & 2c_{45} & 2c_{55} & 2c_{56} \\ \sqrt{2}c_{16} & \sqrt{2}c_{26} & \sqrt{2}c_{36} & 2c_{46} & 2c_{56} & 2c_{66} \end{bmatrix} \tag{3.40}$$

[20].

3.4.1 Construction of $\hat{Q}$ from given Q

Mehrabadi and Cowin [20] constructed $\hat{Q}$ (a six dimensional rotation matrix) from given Q (a three dimensional rotation matrix) as follows. Let $e'_i\,(i = 1, 2, 3)$ denote second three-dimensional Cartesian basis and $\hat{e}'_\alpha\,(\alpha = 1, 2, ..., 6)$ denote second six-dimensional Cartesian basis. By the transformation law, the basis e_i and e'_i are related by

$$\mathbf{e}'_i = Q_{ij}\mathbf{e}_j \tag{3.41}$$

where Q is an orthogonal three-dimensional tensor. Similarly the basis $\hat{e}_\alpha$ and $\hat{e}'_\alpha$ are related by

$$\hat{\mathbf{e}}'_\alpha = \hat{Q}_{\alpha\beta}\hat{\mathbf{e}}_\beta. \tag{3.42}$$

where $\hat{Q}$ is an orthogonal six-dimensional tensor.

If we are given Q, we can construct $\hat{Q}$ as follows:

First we note that from (3.41)

$$\mathbf{e}'_i \otimes \mathbf{e}'_j = Q_{ik}\mathbf{e}_k \otimes Q_{jm}\mathbf{e}_m.$$

$$\mathbf{e}'_i \otimes \mathbf{e}'_j = Q_{ik}Q_{jm}\mathbf{e}_k \otimes \mathbf{e}_m. \tag{3.43}$$

interchange i, j in (3.43)

$$\mathbf{e}'_j \otimes \mathbf{e}'_i = Q_{jk} Q_{im} \mathbf{e}_k \otimes \mathbf{e}_m.$$

and adding these two equations we have

$$\left(\mathbf{e}'_i \otimes \mathbf{e}'_j + \mathbf{e}'_j \otimes \mathbf{e}'_i\right) = \left(Q_{ik} Q_{jm} + Q_{im} Q_{jk}\right) \mathbf{e}_k \otimes \mathbf{e}_m.$$

which can be written as

$$\frac{1}{2}\left(\mathbf{e}'_i \otimes \mathbf{e}'_j + \mathbf{e}'_j \otimes \mathbf{e}'_i\right) = 1/2 \left(Q_{ik} Q_{jm} + Q_{im} Q_{jk}\right) \mathbf{e}_k \otimes \mathbf{e}_m. \tag{3.44}$$

Now consider (3.44) and using (3.33) and then using (3.42), we have

$$\frac{1}{2}\left(Q_{ik} Q_{jm} + Q_{im} Q_{jk}\right) \mathbf{e}_i \otimes \mathbf{e}_j \otimes \mathbf{e}_k \otimes \mathbf{e}_m = \hat{Q}_{\alpha\beta} \hat{\mathbf{e}}_\alpha \otimes \hat{\mathbf{e}}_\beta \tag{3.45}$$

from this formula, we can construct the following matrix

$$\hat{Q} = \begin{bmatrix} \hat{Q}_{11} & \hat{Q}_{12} & \hat{Q}_{13} & \hat{Q}_{14} & \hat{Q}_{15} & \hat{Q}_{16} \\ \hat{Q}_{21} & \hat{Q}_{22} & \hat{Q}_{23} & \hat{Q}_{24} & \hat{Q}_{25} & \hat{Q}_{26} \\ \hat{Q}_{31} & \hat{Q}_{32} & \hat{Q}_{33} & \hat{Q}_{34} & \hat{Q}_{35} & \hat{Q}_{36} \\ \hat{Q}_{41} & \hat{Q}_{42} & \hat{Q}_{43} & \hat{Q}_{44} & \hat{Q}_{45} & \hat{Q}_{46} \\ \hat{Q}_{51} & \hat{Q}_{52} & \hat{Q}_{53} & \hat{Q}_{54} & \hat{Q}_{55} & \hat{Q}_{56} \\ \hat{Q}_{61} & \hat{Q}_{62} & \hat{Q}_{63} & \hat{Q}_{64} & \hat{Q}_{65} & \hat{Q}_{66} \end{bmatrix}, \tag{3.46}$$

where

$$\begin{aligned}
\hat{Q}_{11} &= Q_{11}^2, \\
\hat{Q}_{12} &= Q_{12}^2, \ \hat{Q}_{21} = Q_{21}^2, \\
\hat{Q}_{13} &= Q_{13}^2, \ \hat{Q}_{31} = Q_{31}^2, \\
\hat{Q}_{14} &= \sqrt{2} Q_{12} Q_{13}, \ \hat{Q}_{41} = \sqrt{2} Q_{21} Q_{31}, \\
\hat{Q}_{15} &= \sqrt{2} Q_{11} Q_{13}, \ \hat{Q}_{51} = \sqrt{2} Q_{11} Q_{31}, \\
\hat{Q}_{16} &= \sqrt{2} Q_{11} Q_{12}, \ \hat{Q}_{61} = \sqrt{2} Q_{11} Q_{21}, \\
\hat{Q}_{22} &= Q_{22}^2, \\
\hat{Q}_{23} &= Q_{23}^2, \ \hat{Q}_{32} = Q_{32}^2, \\
\hat{Q}_{24} &= \sqrt{2} Q_{22} Q_{23}, \ \hat{Q}_{42} = \sqrt{2} Q_{22} Q_{32}, \\
\hat{Q}_{25} &= \sqrt{2} Q_{21} Q_{23}, \ \hat{Q}_{52} = \sqrt{2} Q_{12} Q_{32}, \\
\hat{Q}_{26} &= \sqrt{2} Q_{22} Q_{21}, \ \hat{Q}_{62} = \sqrt{2} Q_{12} Q_{22}, \\
\hat{Q}_{33} &= Q_{33}^2, \\
\hat{Q}_{34} &= \sqrt{2} Q_{33} Q_{32}, \ \hat{Q}_{43} = \sqrt{2} Q_{23} Q_{33}, \\
\hat{Q}_{35} &= \sqrt{2} Q_{33} Q_{31}, \ \hat{Q}_{53} = \sqrt{2} Q_{13} Q_{33},
\end{aligned}$$

$$\begin{aligned}
\hat{Q}_{36} &= \sqrt{2}Q_{31}Q_{32},\ \hat{Q}_{63} = \sqrt{2}Q_{13}Q_{23},\\
\hat{Q}_{44} &= Q_{22}Q_{33} + Q_{23}Q_{32},\\
\hat{Q}_{45} &= Q_{21}Q_{33} + Q_{31}Q_{23},\ \hat{Q}_{54} = Q_{12}Q_{33} + Q_{32}Q_{13},\\
\hat{Q}_{46} &= Q_{21}Q_{32} + Q_{31}Q_{22},\ \hat{Q}_{64} = Q_{12}Q_{23} + Q_{22}Q_{13},\\
\hat{Q}_{55} &= Q_{11}Q_{33} + Q_{13}Q_{31},\\
\hat{Q}_{56} &= Q_{11}Q_{32} + Q_{31}Q_{12},\ \hat{Q}_{65} = Q_{11}Q_{23} + Q_{21}Q_{13},\\
\hat{Q}_{66} &= Q_{11}Q_{22} + Q_{21}Q_{12},
\end{aligned}$$

(3.46) express the relationship between components of Q and $\hat{Q}$. It should be noted that

$$\text{orthogonality of } Q \Longrightarrow \text{orthogonality of } \hat{Q}$$

i.e.

$$QQ^T = Q^TQ = \mathbf{I} \Longrightarrow \hat{Q}\hat{Q}^T = \hat{Q}^T\hat{Q} = \mathbf{I} \tag{3.47}$$

[20].

Eq.(3.47) can be proved as follows, consider

$$Q = \begin{bmatrix} Q_{11} & Q_{12} & Q_{13} \\ Q_{21} & Q_{22} & Q_{23} \\ Q_{31} & Q_{32} & Q_{33} \end{bmatrix} \text{ and } Q^T = \begin{bmatrix} Q_{11} & Q_{21} & Q_{31} \\ Q_{12} & Q_{22} & Q_{32} \\ Q_{13} & Q_{23} & Q_{33} \end{bmatrix},$$

so

$$\begin{aligned}
&QQ^T\\
&= \begin{bmatrix} Q_{11}^2 + Q_{12}^2 + Q_{13}^2 & Q_{11}Q_{21} + Q_{12}Q_{22} + Q_{13}Q_{23} & Q_{11}Q_{31} + Q_{12}Q_{32} + Q_{13}Q_{33} \\ Q_{11}Q_{21} + Q_{12}Q_{22} + Q_{13}Q_{23} & Q_{21}^2 + Q_{22}^2 + Q_{23}^2 & Q_{21}Q_{31} + Q_{22}Q_{32} + Q_{33}Q_{32} \\ Q_{11}Q_{31} + Q_{12}Q_{32} + Q_{13}Q_{33} & Q_{21}Q_{31} + Q_{22}Q_{32} + Q_{33}Q_{32} & Q_{31}^2 + Q_{32}^2 + Q_{33}^2 \end{bmatrix}
\end{aligned}$$

so L.H.S of (3.47) implies i.e. $QQ^T = Q^TQ = \mathbf{I} \Longrightarrow$

$$Q_{11}^2 + Q_{12}^2 + Q_{13}^2 = 1 \tag{3.48}$$

$$Q_{21}^2 + Q_{22}^2 + Q_{23}^2 = 1 \tag{3.49}$$

$$Q_{31}^2 + Q_{32}^2 + Q_{33}^2 = 1 \tag{3.50}$$

$$Q_{11}Q_{21} + Q_{12}Q_{22} + Q_{13}Q_{23} = 0 \tag{3.51}$$

$$Q_{11}Q_{31} + Q_{12}Q_{32} + Q_{13}Q_{33} = 0 \tag{3.52}$$

$$Q_{21}Q_{31} + Q_{22}Q_{32} + Q_{33}Q_{32} = 0. \tag{3.53}$$

Now consider (3.46) and its transpose is

$$\hat{Q}^T = \begin{bmatrix} \hat{Q}_{11} & \hat{Q}_{21} & \hat{Q}_{31} & \hat{Q}_{41} & \hat{Q}_{51} & \hat{Q}_{61} \\ \hat{Q}_{12} & \hat{Q}_{22} & \hat{Q}_{32} & \hat{Q}_{42} & \hat{Q}_{52} & \hat{Q}_{62} \\ \hat{Q}_{13} & \hat{Q}_{23} & \hat{Q}_{33} & \hat{Q}_{43} & \hat{Q}_{53} & \hat{Q}_{63} \\ \hat{Q}_{14} & \hat{Q}_{24} & \hat{Q}_{34} & \hat{Q}_{44} & \hat{Q}_{54} & \hat{Q}_{64} \\ \hat{Q}_{15} & \hat{Q}_{25} & \hat{Q}_{35} & \hat{Q}_{45} & \hat{Q}_{55} & \hat{Q}_{65} \\ \hat{Q}_{16} & \hat{Q}_{26} & \hat{Q}_{36} & \hat{Q}_{46} & \hat{Q}_{56} & \hat{Q}_{66} \end{bmatrix}, \tag{3.54}$$

where all entries of $\hat{Q}^T$ is defined above, so

$$\begin{aligned} \hat{Q}\hat{Q}^T &= \mathbf{I} \Longrightarrow \\ \left(\hat{Q}\hat{Q}^T\right)_{11} &= 1,\ \left(\hat{Q}\hat{Q}^T\right)_{22} = 1,\ \left(\hat{Q}\hat{Q}^T\right)_{33} = 1, \\ \left(\hat{Q}\hat{Q}^T\right)_{44} &= 1,\ \left(\hat{Q}\hat{Q}^T\right)_{55} = 1,\ \left(\hat{Q}\hat{Q}^T\right)_{66} = 1 \\ \left(\hat{Q}\hat{Q}^T\right)_{12} &= \left(\hat{Q}\hat{Q}^T\right)_{21} = 0,\ \left(\hat{Q}\hat{Q}^T\right)_{13} = \left(\hat{Q}\hat{Q}^T\right)_{31} = 0, \\ \left(\hat{Q}\hat{Q}^T\right)_{23} &= \left(\hat{Q}\hat{Q}^T\right)_{32} = 0, \end{aligned}$$

consider

$$\begin{aligned} \left(\hat{Q}\hat{Q}^T\right)_{11} &= 1 \Longrightarrow Q_{11}^4 + Q_{12}^4 + Q_{13}^4 + 2Q_{11}^2Q_{12}^2 + 2Q_{13}^2Q_{12}^2 + 2Q_{11}^2Q_{13}^2 = 1 \\ &\Longrightarrow \left(Q_{11}^2 + Q_{12}^2 + Q_{13}^2\right)^2 = 1 \end{aligned}$$

which is satisfied because of (3.48), also consider

$$\begin{aligned} \left(\hat{Q}\hat{Q}^T\right)_{21} &= \left(\hat{Q}\hat{Q}^T\right)_{12} = 0 \\ &\Longrightarrow Q_{11}^2Q_{21}^2 + Q_{12}^2Q_{22}^2 + Q_{13}^2Q_{23}^2 + 2Q_{12}Q_{13}Q_{22}Q_{23} + 2Q_{11}Q_{13}Q_{21}Q_{23} + 2Q_{11}Q_{12}Q_{22}Q \\ &\Longrightarrow \left(Q_{11}Q_{21} + Q_{12}Q_{22} + Q_{13}Q_{23}\right)^2 = 0 \end{aligned}$$

which is satisfied because of (3.51), similarly it can be shown for others equations and we have proved (3.47).

3.5 Existence of a plane of symmetry

Consider a plane $\mathbf{P}$ with normal $\mathbf{n}$ and the second-rank tensor Ω in three-dimensions as defined above in (3.5) i.e.

$$\Omega = \mathbf{I} - 2\mathbf{n}\mathbf{n}^T \text{ or } \Omega_{ij} = \delta_{ij} - 2n_in_j,$$

and let $\mathbf{m}$ be the unit vector in the plane $\mathbf{P}$. As we defined above in (3.6) and (3.7) i.e.

$$\mathbf{\Omega n} = -\,\mathbf{n} \text{ or } \Omega_{ij}n_i = -n_j,$$

$$\mathbf{\Omega m} = \mathbf{m} \text{ or } \Omega_{ij}m_i = m_j.$$

(3.5) shows that orthogonal transformation Ω is a reflection in the plane $\mathbf{P}$. As $\mathbf{P}$ is plane of symmetry so C must be invariant under this transformation

$$C_{ijks} = \Omega_{ip}\Omega_{jq}\Omega_{kr}\Omega_{st}C_{pqrt}.$$

The matrix form of (3.5) is

$$\Omega(n) = \begin{bmatrix} 1-2n_1^2 & -2n_1n_2 & -2n_1n_3 \\ -2n_1n_2 & 1-2n_2^2 & -2n_2n_3 \\ -2n_1n_3 & -2n_2n_3 & 1-2n_3^2 \end{bmatrix}. \tag{3.55}$$

Now the transformation matrix $\hat{N}(n)$ corresponding to the reflection in the plane $\mathbf{P}$ will be obtain by just replacing Q_{ij} by Ω_{ij} $(i,j = 1,2,3)$ in $\hat{Q}$ (3.46).i.e.

$$\hat{N}(n) = \begin{bmatrix} \hat{N}_{11}(n) & \hat{N}_{12}(n) & \hat{N}_{13}(n) & \hat{N}_{14}(n) & \hat{N}_{15}(n) & \hat{N}_{16}(n) \\ \hat{N}_{12}(n) & \hat{N}_{22}(n) & \hat{N}_{23}(n) & \hat{N}_{24}(n) & \hat{N}_{25}(n) & \hat{N}_{26}(n) \\ \hat{N}_{13}(n) & \hat{N}_{23}(n) & \hat{N}_{33}(n) & \hat{N}_{34}(n) & \hat{N}_{35}(n) & \hat{N}_{36}(n) \\ \hat{N}_{14}(n) & \hat{N}_{24}(n) & \hat{N}_{34}(n) & \hat{N}_{44}(n) & \hat{N}_{45}(n) & \hat{N}_{46}(n) \\ \hat{N}_{15}(n) & \hat{N}_{25}(n) & \hat{N}_{35}(n) & \hat{N}_{45}(n) & \hat{N}_{55}(n) & \hat{N}_{56}(n) \\ \hat{N}_{16}(n) & \hat{N}_{26}(n) & \hat{N}_{36}(n) & \hat{N}_{46}(n) & \hat{N}_{56}(n) & \hat{N}_{66}(n) \end{bmatrix}. \tag{3.56}$$

Where

$$\begin{aligned}
\hat{N}_{11}(n) &= \Omega_{11}^2, \\
\hat{N}_{12}(n) &= \hat{N}_{21}(n) = \Omega_{12}^2, \\
\hat{N}_{13}(n) &= \hat{N}_{31}(n) = \Omega_{13}^2, \\
\hat{N}_{14}(n) &= \hat{N}_{41}(n) = \sqrt{2}\Omega_{12}\Omega_{13}, \\
\hat{N}_{15}(n) &= \hat{N}_{51}(n) = \sqrt{2}\Omega_{11}\Omega_{13}, \\
\hat{N}_{16}(n) &= \hat{N}_{61}(n) = \sqrt{2}\Omega_{11}\Omega_{12}, \\
\hat{N}_{22}(n) &= \Omega_{22}^2, \\
\hat{N}_{23}(n) &= \hat{N}_{32}(n) = \Omega_{23}^2, \\
\hat{N}_{24}(n) &= \hat{N}_{42}(n) = \sqrt{2}\Omega_{22}\Omega_{23}, \\
\hat{N}_{25}(n) &= \hat{N}_{52}(n) = \sqrt{2}\Omega_{21}\Omega_{23}, \\
\hat{N}_{26}(n) &= \hat{N}_{62}(n) = \sqrt{2}\Omega_{22}\Omega_{21}.
\end{aligned}$$

$$\begin{aligned}
\hat{N}_{33}(n) &= \Omega_{33}^2, \\
\hat{N}_{34}(n) &= \hat{N}_{43}(n) = \sqrt{2}\Omega_{33}\Omega_{32}, \\
\hat{N}_{35}(n) &= \hat{N}_{53}(n) = \sqrt{2}\Omega_{33}\Omega_{31}, \\
\hat{N}_{36}(n) &= \hat{N}_{63}(n) = \sqrt{2}\Omega_{31}\Omega_{32}, \\
\hat{N}_{44}(n) &= \Omega_{22}\Omega_{33} + \Omega_{23}\Omega_{32}, \\
\hat{N}_{45}(n) &= \hat{N}_{54}(n) = \Omega_{21}\Omega_{33} + \Omega_{23}\Omega_{31}, \\
\hat{N}_{46}(n) &= \hat{N}_{64}(n) = \Omega_{21}\Omega_{32} + \Omega_{31}\Omega_{32}, \\
\hat{N}_{55}(n) &= \Omega_{11}\Omega_{33} + \Omega_{13}\Omega_{31}, \\
\hat{N}_{56}(n) &= \hat{N}_{65}(n) = \Omega_{11}\Omega_{32} + \Omega_{31}\Omega_{12}, \\
\hat{N}_{66}(n) &= \Omega_{11}\Omega_{22} + \Omega_{12}\Omega_{21},
\end{aligned}$$

where

$$\begin{aligned}
\Omega_{11} &= 1 - 2n_1^2, \\
\Omega_{12} &= \Omega_{21} = -2n_1n_2, \\
\Omega_{22} &= 1 - 2n_2^2, \\
\Omega_{23} &= \Omega_{32} = -2n_2n_3, \\
\Omega_{13} &= \Omega_{31} = -2n_1n_3, \\
\Omega_{33} &= 1 - 2n_3^2.
\end{aligned}$$

Ahmad [2] gave the following theorem and its proof.

Theorem 9 *[2] An anisotropic material has a plane of symmetry with normal n if and only if*

$$\hat{N}\hat{c} = \hat{c}\hat{N}. \tag{3.57}$$

Proof. Since Ω is symmetric so $\hat{N}$ is symmetric. Also the orthogonality of Ω implies orthogonality of $\hat{N}$ i.e.

$$\hat{N}\hat{N}^T = \hat{N}^2 = \mathbf{I} \tag{3.58}$$

where $\mathbf{I}$ is the unit matrix in six dimensions. Thus condition of invariance under reflection in the plane P now becomes

$$\hat{N}\hat{c}\hat{N}^T = \hat{c} \tag{3.59}$$

since $\hat{N}^T = \hat{N}^{-1}$, so the above condition becomes

$$\hat{N}\hat{c} = \hat{c}\hat{N}.$$

Conversely suppose that condition (3.57) is satisfied for an anisotropic material i.e.

$$\hat{N}\hat{c} = \hat{c}\hat{N},$$

since $\hat{N}$ is orthogonal so$\hat{N}^T = \hat{N}^{-1}$, so

$$\begin{aligned} \hat{N}\hat{c} &= \hat{c}\hat{N}, \\ &\Longrightarrow \hat{N}\hat{c}\hat{N}^T = \hat{c} \end{aligned}$$

which is the condition of invariance under reflection in the plane P, hence an anisotropic material has a plane of symmetry with normal $\mathbf{n}$. ■

Therefore *a material possess a plane of symmetry if and only if the matrix associated with the normal to the plane commutes with the matrix representing the elasticity tensor.* However in order to apply this necessary and sufficient condition, we need some means to find an $\mathbf{n}$ for being a normal to a possible plane of symmetry. Here a common eigenvector of the tensors U and V comes to our help.

3.6 Application of theorem 9

Ahmad [2] developed a technique of adding symmetry planes to crystal systems, he constructed numerical example of adding symmetry planes to a triclinic system and in this way he regained the results of [24]. So here are given some numerical examples of this technique as follows.

3.6.1 Adding symmetry planes to a triclinic system

Consider a crystal whose elasticity tensor is given by (3.40). Without any loss of generality we take a normal to plane of symmetry along x_1-axis and we add a plane of symmetry with normal along x_1-axis. This means

$$n_1 = 1, \; n_2 = 0 = n_3$$

so (3.55) becomes

$$\Omega\left(n\right)=\begin{bmatrix}-1&0&0\\0&-1&0\\0&0&-1\end{bmatrix},$$

and $\hat{N}_1$ becomes

$$\hat{N}_1=\begin{bmatrix}1&0&0&0&0&0\\0&1&0&0&0&0\\0&0&1&0&0&0\\0&0&0&1&0&0\\0&0&0&0&-1&0\\0&0&0&0&0&-1\end{bmatrix}.$$

The condition (3.57) becomes $\hat{N}_1\hat{c}=\hat{c}\hat{N}_1$, where $\hat{c}$ is given by (3.40) and from this condition we have

$$\hat{c}_{\alpha\beta}=\begin{bmatrix}0&0&0&0&2\sqrt{2}c_{15}&2\sqrt{2}c_{16}\\0&0&0&0&2\sqrt{2}c_{26}&2\sqrt{2}c_{27}\\0&0&0&0&2\sqrt{2}c_{35}&2\sqrt{2}c_{36}\\0&0&0&0&4c_{45}&4c_{46}\\-2\sqrt{2}c_{15}&-2\sqrt{2}c_{26}&-2\sqrt{2}c_{35}&-4c_{45}&0&0\\-2\sqrt{2}c_{16}&-2\sqrt{2}c_{27}&-2\sqrt{2}c_{36}&-4c_{46}&0&0\end{bmatrix},$$

hence by adding a single plane of symmetry a triclinic system reduces to a monoclinic system. Next add another plane of symmetry and assume that its normal lies in x_1x_2 plane if the angle between the two normals is taken to be $\pi/3$ then direction cosines of second normal are

$$n_1=-1/2,\ n_2=\sqrt{3/2},\ n_3=0,$$

so (3.55) becomes

$$\Omega\left(n\right)=\begin{bmatrix}1/2&\sqrt{3/2}&0\\\sqrt{3/2}&-1/2&0\\0&0&1\end{bmatrix},$$

and $\hat{N}_2$ becomes

$$\hat{N}_2 = \begin{bmatrix} \frac{1}{4} & \frac{3}{4} & 0 & 0 & 0 & \frac{\sqrt{\frac{3}{2}}}{2} \\ \frac{3}{4} & \frac{1}{4} & 0 & 0 & 0 & -\frac{\sqrt{\frac{3}{2}}}{2} \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{1}{2} & \frac{\sqrt{3}}{2} & 0 \\ 0 & 0 & 0 & \frac{\sqrt{3}}{2} & \frac{1}{2} & 0 \\ \frac{\sqrt{\frac{3}{2}}}{2} & -\frac{\sqrt{\frac{3}{2}}}{2} & 0 & 0 & 0 & \frac{1}{2} \end{bmatrix}.$$

Let $P = \hat{N}_2\hat{c} - \hat{c}\hat{N}_2$ then elements of P are

$$\begin{aligned}
P_{11} &= 0,\ P_{12} = -\frac{3}{4}\left(c_{11} - c_{22}\right),\ P_{13} = -\frac{3}{4}\left(c_{13} - c_{23}\right),\ P_{14} = \frac{3}{2\sqrt{2}}\left(c_{14} + c_{24}\right), \\
P_{15} &= \frac{3}{\sqrt{2}}\left(-c_{14} + c_{56}\right),\ P_{16} = \frac{\sqrt{3}}{2\sqrt{2}}\left(-c_{11} + c_{12} + 2c_{66}\right),\ P_{22} = 0,\ P_{23} = -P_{13}, \\
P_{24} &= P_{14},\ P_{25} = \sqrt{\frac{3}{2}}\left(c_{24} + c_{56}\right),\ P_{26} = -\frac{\sqrt{3}}{2\sqrt{2}}\left(-c_{22} + c_{12} + 2c_{66}\right),\ P_{33} = 0, \\
P_{34} &= \frac{3c_{34}}{\sqrt{2}},\ P_{35} = -\sqrt{\frac{3}{2}}c_{34},\ P_{36} = \frac{1}{2}\sqrt{\frac{3}{2}}\left(-c_{13} + c_{23}\right),\ P_{44} = 0, \\
P_{45} &= \sqrt{3}\left(-c_{44} + c_{55}\right),\ P_{46} = \frac{\sqrt{3}}{2}\left(-c_{14} + c_{24} + 2c_{56}\right),\ P_{55} = 0,\ P_{56} = 0,\ P_{66} = 0.
\end{aligned}$$

Equating each of the above element to zero we have following results

$$c_{11} = c_{22},\ c_{13} = c_{23},\ c_{14} = -c_{24} = c_{56},\ c_{34} = 0,\ c_{44} = c_{55},\ c_{66} = \frac{(c_{11} - c_{12})}{2}$$

hence we have obtained the elasticity tensor for trigonal material i.e.

$$\hat{c}_{\alpha\beta} = \begin{bmatrix} c_{11} & c_{12} & c_{13} & \sqrt{2}c_{14} & 0 & 0 \\ c_{12} & c_{11} & c_{13} & -\sqrt{2}c_{14} & 0 & 0 \\ c_{13} & c_{13} & c_{33} & 0 & 0 & 0 \\ \sqrt{2}c_{14} & -\sqrt{2}c_{14} & 0 & 2c_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & 2c_{44} & 2c_{14} \\ 0 & 0 & 0 & 0 & 2c_{14} & c_{11} - c_{12} \end{bmatrix}.$$

Again we add another plane of symmetry and assume that its normal lies in x_1x_2 plane if the angle between the two normals is taken to be $\pi/4$ then direction cosines of second normal are

$$n_1 = 1/\sqrt{2},\ n_2 = 1/\sqrt{2},\ n_3 = 0,$$

so (3.55) becomes

$$\Omega(n) = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

and $\hat{N}_2$ becomes

$$\hat{N}_2 = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \end{bmatrix}.$$

Let $P = \hat{N}_2\hat{c} - \hat{c}\hat{N}_2$ then elements of P are

$$\begin{aligned}
P_{11} &= 0,\ P_{12} = -(c_{11} - c_{22}),\ P_{13} = -(c_{13} - c_{23}),\ P_{14} = \sqrt{2}c_{24}, \\
P_{15} &= -\sqrt{2}c_{14},\ P_{16} = 0,\ P_{22} = 0,\ P_{23} = -P_{13},\ P_{24} = -P_{15}, \\
P_{25} &= -P_{14},\ P_{26} = 0,\ P_{33} = 0,\ P_{34} = \sqrt{2}c_{34},\ P_{35} = -P_{34}, \\
P_{36} &= 0,\ P_{44} = 0,\ P_{45} = 2(-c_{44} + c_{55}),\ P_{46} = 2c_{56}, \\
P_{55} &= 0,\ P_{56} = 2c_{56},\ P_{66} = 0.
\end{aligned}$$

Equating each of the above element to zero we have following results

$$c_{11} = c_{22},\ c_{13} = c_{23},\ c_{14} = c_{24} = c_{34} = c_{56} = 0,\ c_{34} = 0,\ c_{44} = c_{55}$$

hence we have obtained the elasticity tensor for tetragonal material i.e.

$$\hat{c}_{\alpha\beta} = \begin{bmatrix} c_{11} & c_{12} & c_{13} & 0 & 0 & 0 \\ c_{12} & c_{11} & c_{13} & 0 & 0 & 0 \\ c_{13} & c_{13} & c_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & c_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & c_{44} & 0 \\ 0 & 0 & 0 & 0 & 0 & c_{66} \end{bmatrix}.$$

3.6.2 Numerical examples for identification of planes of symmetry

Now we illustrate some application of above theorem by considering numerical examples and finding all planes of symmetry for the material concerned.

Example of Triclinic material:

Let the tensor C be represented by the matrix c in the Voigt notation where the components are in units of $10^{10}N/m^2$.

$$c = \begin{bmatrix} -3.2612 & 1.1235 & 6.5897 & -4.2095 & 2.7351 & 10.7112 \\ 1.1235 & 3.8787 & 8.2231 & 11.9217 & 9.2135 & 4.8251 \\ 6.5897 & 8.2231 & 12.1125 & -7.1352 & -3.9567 & 6.8253 \\ -4.2095 & 11.9217 & -7.1352 & 12.2069 & 15.1515 & -8.6527 \\ 2.7351 & 9.2135 & -3.9567 & 15.1515 & 16.8898 & -9.3737 \\ 10.7112 & 4.8251 & 6.8253 & -8.6527 & -9.3737 & 12.8593 \end{bmatrix}, \quad (3.60)$$

matrices representing the tensors U, V are

$$\begin{aligned} U_{ij} &= C_{ijkk}, \\ V_{ij} &= C_{ikjk}, \end{aligned}$$

in this example these matrices become

$$U = \begin{bmatrix} 4.452 & 22.3616 & 7.9919 \\ 22.3616 & -3.2209 & 0.577 \\ 7.9919 & 0.577 & 10.4791 \end{bmatrix}, \; V = \begin{bmatrix} 26.4879 & 30.6878 & -9.8743 \\ 30.6878 & 28.9449 & -4.5872 \\ -9.8743 & -4.5872 & 41.2092 \end{bmatrix}.$$

U has eigenvalues $26.0793, -22.759, 8.38996$ and eigenvectors

$$u_1 = \begin{bmatrix} -0.727119 \\ -0.562675 \\ -0.393311 \end{bmatrix}, \; u_2 = \begin{bmatrix} 0.6532 \\ -0.74337 \\ -0.144154 \end{bmatrix}, \; u_3 = \begin{bmatrix} -0.211251 \\ -0.361728 \\ 0.908034 \end{bmatrix}.$$

V has eigenvalues $63.1698, 36.8305, -3.35826$ and eigenvectors

$$v_1 = \begin{bmatrix} -0.645687 \\ -0.635665 \\ 0.423105 \end{bmatrix}, \; v_2 = \begin{bmatrix} 0.229124 \\ 0.367276 \\ 0.901449 \end{bmatrix}, \; v_3 = \begin{bmatrix} 0.728416 \\ -0.678997 \\ 0.0914996 \end{bmatrix},$$

here U and V have no common eigenvector so no symmetry plane exists, hence the matrix (3.60) represents a *triclinic material.*

Example of Monoclinic material:

Let the tensor C be represented by the matrix c in the Voigt notation where the components are in units of $10^{10}N/m^2$.

$$c = \begin{bmatrix} 4.85081 & -0.788167 & 4.47176 & 2.17137 & -0.889572 & -4.48786 \\ -0.788167 & 7.58602 & 3.74454 & 5.63785 & -1.6478 & 3.49424 \\ 4.47176 & 3.74454 & 7.2829 & 4.01409 & -1.30445 & 0.26418 \\ 2.17137 & 5.63785 & 4.01409 & 8.8909 & -1.06086 & -1.14552 \\ -0.889572 & -1.6478 & -1.30445 & -1.06086 & 5.9706 & 0.625526 \\ -4.48786 & 3.49424 & 0.26418 & -1.14552 & 0.625526 & 6.09683 \end{bmatrix}, \tag{3.61}$$

from (3.40) we have Mehrabadi-Cowin matrix $\hat{c}$ corresponding to the above matrix is as follows

$$\hat{c} = \begin{bmatrix} 4.85081 & -0.788167 & 4.47176 & 3.07078 & -1.25804 & -6.34679 \\ -0.788167 & 7.58602 & 3.74454 & 7.97312 & -2.33034 & 4.9416 \\ 4.47176 & 3.74454 & 7.2829 & 5.67678 & -1.84477 & 0.373607 \\ 3.07078 & 7.97312 & 5.67678 & 17.7818 & -1.12172 & -2.29104 \\ -1.25804 & -2.33034 & -1.84477 & -1.12172 & 11.9412 & 1.25105 \\ -6.34679 & 4.9416 & 0.373607 & -2.29104 & 1.25105 & 12.1937 \end{bmatrix}, \tag{3.62}$$

matrices U and V become

$$U = \begin{bmatrix} 8.5344 & -0.72944 & -3.84182 \\ -0.72944 & 10.5424 & 11.8233 \\ -3.84182 & 11.8233 & 15.4992 \end{bmatrix}, \quad V = \begin{bmatrix} 16.9182 & -2.05448 & -3.33954 \\ -2.05448 & 22.5738 & 10.2775 \\ -3.33954 & 10.2775 & 22.1444 \end{bmatrix}.$$

U has eigenvalues $25.7931, 8.29739, 0.485472$ and eigenvectors

$$u_1 = \begin{bmatrix} -0.19706 \\ 0.606554 \\ 0.770226 \end{bmatrix}, \quad u_2 = \begin{bmatrix} 0.95105 \\ 0.30903 \\ 0 \end{bmatrix}, \quad u_3 = \begin{bmatrix} 0.238028 \\ -0.732524 \\ 0.637771 \end{bmatrix}.$$

V has eigenvalues $33.5132, 16.2507, 11.8725$ and eigenvectors

$$v_1 = \begin{bmatrix} 0.223984 \\ -0.689329 \\ 0.688953 \end{bmatrix}, \quad v_2 = \begin{bmatrix} 0.95105 \\ 0.30904 \\ 0 \end{bmatrix}, \quad v_3 = \begin{bmatrix} 0.212927 \\ -0.655224 \\ 0.724806 \end{bmatrix}.$$

The common eigenvector of U and V is

$$n = \begin{bmatrix} 0.95105 \\ 0.30903 \\ 0 \end{bmatrix}.$$

So from (3.56) we have $\hat{N}(n)$ as follows

$$\hat{N}(n) = \begin{bmatrix} 0.654475 & 0.345523 & 0 & 0 & 0 & 0.672512 \\ 0.345523 & 0.654476 & 0 & 0 & 0 & -0.672513 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.808997 & -0.587812 & 0 \\ 0 & 0 & 0 & -0.587812 & -0.808996 & 0 \\ 0.672512 & -0.672513 & 0 & 0 & 0 & -0.3089 \end{bmatrix}, \tag{3.63}$$

multiplying (3.63) and (3.62), (3.62) and (3.63) we have

$$\hat{N}(n)\,\hat{c} = \begin{bmatrix} -1.36589 & 5.4286 & 4.47173 & 3.22389 & -0.787196 & 5.75403 \\ 5.4286 & 1.36925 & 3.74456 & 7.82 & -2.80118 & -7.15922 \\ 4.47173 & 3.74456 & 7.2829 & 5.67678 & -1.84477 & 0.373607 \\ 3.22389 & 7.82 & 5.67678 & 15.6326 & -8.73565 & -2.58883 \\ -0.787196 & -2.80118 & -1.84477 & -8.73565 & -8.41321 & 0.334607 \\ 5.75403 & -7.15922 & 0.373607 & -2.58883 & 0.334607 & -11.3589 \end{bmatrix},$$

$$\hat{c}\hat{N}(n) = \begin{bmatrix} -1.36589 & 5.4286 & 4.47173 & 3.22389 & -0.787196 & 5.75403 \\ 5.4286 & 1.36925 & 3.74456 & 7.82 & -2.80118 & -7.15922 \\ 4.47173 & 3.74456 & 7.2829 & 5.67678 & -1.84477 & 0.373607 \\ 3.22389 & 7.82 & 5.67678 & 15.6326 & -8.73565 & -2.58883 \\ -0.787196 & -2.80118 & -1.84477 & -8.73565 & -8.41321 & 0.334607 \\ 5.75403 & -7.15922 & 0.373607 & -2.58883 & 0.334607 & -11.3589 \end{bmatrix},$$

i.e.

$$\hat{N}(n)\,\hat{c} = \hat{c}\hat{N}(n).$$

Thus $\mathbf{n}$ is common eigenvector and normal to a plane of symmetry. There is a single plane of symmetry, hence the material represented by (3.61) must be a *monoclinic material.*

Example of Orthotropic material:

Let the tensor C be represented by the matrix c in the Voigt notation where the components are in units of $10^{10}N/m^2$.

$$c = \begin{bmatrix} 9.76968 & 1.19131 & 3.75164 & -0.930095 & -1.2095 & -3.72247 \\ 1.19131 & 6.6526 & 6.97086 & -1.08713 & 1.16198 & 4.55952 \\ 3.75164 & 6.97086 & 10.6253 & -0.356033 & 0.122621 & -0.605921 \\ -0.930095 & -1.08713 & -0.356033 & 1.07158 & 0.985289 & 1.679 \\ -1.2095 & 1.16198 & 0.122621 & 0.985289 & 3.11501 & 1.09189 \\ -3.72247 & 4.55952 & -0.605921 & 1.679 & 1.09189 & 6.12073 \end{bmatrix}, \tag{3.64}$$

from (3.40) we have Mehrabadi-Cowin matrix $\hat{c}$ corresponding to the above matrix is as follows

$$\hat{c} = \begin{bmatrix} 9.76968 & 1.19131 & 3.75164 & -1.31535 & -1.71049 & -5.26437 \\ 1.19131 & 6.6526 & 6.97086 & -1.53743 & 1.64329 & 6.44814 \\ 3.75164 & 6.97086 & 10.6253 & -0.503507 & 0.173412 & -0.856902 \\ -1.31535 & -1.53743 & -0.503507 & 2.14316 & 1.97058 & 3.358 \\ -1.71049 & 1.64329 & 0.173412 & 1.97058 & 6.23002 & 2.18378 \\ -5.26437 & 6.44814 & -0.856902 & 3.358 & 2.18378 & 12.2415 \end{bmatrix}, \tag{3.65}$$

matrices U and V become

$$U = \begin{bmatrix} 14.7126 & 0.231129 & 0.075101 \\ 0.231129 & 14.8148 & -2.237326 \\ -3.84182 & -2.237326 & 21.3478 \end{bmatrix}, \quad V = \begin{bmatrix} 19.0054 & 1.82234 & 0.592121 \\ 1.82234 & 13.8449 & -0.351273 \\ 0.592121 & -0.351273 & 14.8119 \end{bmatrix}.$$

U has eigenvalues $22.1189, 14.7916, 13.9647$ and eigenvectors

$$u_1 = \begin{bmatrix} 0 \\ -0.309017 \\ 0.951057 \end{bmatrix}, \quad u_2 = \begin{bmatrix} 0.95105 \\ 0.29389 \\ 0.095492 \end{bmatrix}, \quad u_3 = \begin{bmatrix} 0.30902 \\ -0.90450 \\ -0.29389 \end{bmatrix}.$$

V has eigenvalues $19.628, 14.926, 13.1082$ and eigenvectors

$$v_1 = \begin{bmatrix} 0 \\ -0.309017 \\ 0.951057 \end{bmatrix}, \quad v_2 = \begin{bmatrix} 0.95105 \\ 0.29389 \\ 0.095492 \end{bmatrix}, \quad v_3 = \begin{bmatrix} 0.30901 \\ -0.90450 \\ -0.29389 \end{bmatrix}.$$

Here

$$n^{(1)} = \begin{bmatrix} 0 \\ -0.309017 \\ 0.951057 \end{bmatrix}, \; n^{(2)} = \begin{bmatrix} 0.95105 \\ 0.29389 \\ 0.095492 \end{bmatrix}, \; n^{(3)} = \begin{bmatrix} 0.30901 \\ -0.90450 \\ -0.29389 \end{bmatrix},$$

are three common eigenvector.

So from (3.56) we have $\hat{N}\left(n^{(1)}\right)$ as follows

$$\hat{N}\left(n^{(1)}\right) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0.654498 & 0.345502 & 0.672503 & 0 & 0 \\ 0 & 0.345502 & 0.654499 & -0.672504 & 0 & 0 \\ 0 & 0.672503 & -0.672504 & -0.308997 & 0 & 0 \\ 0 & 0 & 0 & 0 & -0.809011 & 0.587794 \\ 0 & 0 & 0 & 0 & 0.587794 & 0.509011 \end{bmatrix}, \tag{3.66}$$

multiplying (3.66) and (3.65), (3.65) and (3.66) we have

$$\hat{N}\left(n^{(1)}\right)\hat{c} = \begin{bmatrix} 9.76968 & 1.19131 & 3.75164 & -1.31535 & -1.71049 & -5.26437 \\ 1.19131 & 5.72864 & 7.89483 & 0.261054 & 2.46067 & 6.105256 \\ 3.75164 & 7.89483 & 9.70136 & -2.302 & -0.643972 & -0.591329 \\ -1.31535 & 0.261054 & -2.302 & -1.35759 & 0.379546 & 3.87495 \\ -1.71049 & 2.46067 & -0.643972 & 0.379546 & -3.75661 & 5.42866 \\ -5.26437 & 6.105256 & -0.591329 & 3.87495 & 5.42866 & 11.1872 \end{bmatrix},$$

$$\hat{c}\hat{N}\left(n^{(1)}\right) = \begin{bmatrix} 9.76968 & 1.19131 & 3.75164 & -1.31535 & -1.71049 & -5.26437 \\ 1.19131 & 5.72864 & 7.89483 & 0.261054 & 2.46067 & 6.105256 \\ 3.75164 & 7.89483 & 9.70136 & -2.302 & -0.643972 & -0.591329 \\ -1.31535 & 0.261054 & -2.302 & -1.35759 & 0.379546 & 3.87495 \\ -1.71049 & 2.46067 & -0.643972 & 0.379546 & -3.75661 & 5.42866 \\ -5.26437 & 6.105256 & -0.591329 & 3.87495 & 5.42866 & 11.1872 \end{bmatrix},$$

i.e.

$$\hat{N}\left(n^{(1)}\right)\hat{c} = \hat{c}\hat{N}\left(n^{(1)}\right).$$

Similarly from (3.56) we have $\hat{N}\left(n^{(2)}\right)$ as follows

$$\hat{N}\left(n^{(2)}\right)=\begin{bmatrix} 0.654468 & 0.31249 & 0.0329914 & 0.143593 & 0.207807 & 0.639554 \\ 0.31249 & 0.684355 & 0.00315038 & -0.0656655 & 0.0443726 & -0.653994 \\ 0.0329914 & 0.00315038 & 0.963858 & -0.0779297 & -0.252186 & 0.0144177 \\ 0.143593 & -0.0656655 & -0.0779297 & 0.815321 & -0.538618 & -0.118883 \\ 0.207807 & 0.0443726 & -0.252186 & -0.538618 & -0.761247 & 0.146943 \\ 0.639554 & -0.653994 & 0.0144177 & -0.118883 & 0.146943 & -0.356755 \end{bmatrix}, \tag{3.67}$$

multiplying (3.67) and (3.65), (3.65) and (3.67) we have

$$\hat{N}\left(n^{(2)}\right)\hat{c}=\begin{bmatrix} 2.97881 & 7.33318 & 4.3999 & 1.50697 & 2.37402 & 7.30644 \\ 7.33318 & 0.9038 & 6.57753 & -3.71417 & -0.690508 & -5.36442 \\ 4.3999 & 6.57753 & 10.3702 & -1.1491 & -1.57732 & -1.61521 \\ 1.50697 & -3.71417 & -1.1491 & 0.238082 & -2.3756 & -1.00625 \\ 2.37402 & -0.690508 & -1.57732 & -2.3756 & -5.80935 & -2.26403 \\ 7.30644 & -5.36442 & -1.61521 & -1.00625 & -2.26403 & -12.0418 \end{bmatrix},$$

$$\hat{c}\hat{N}\left(n^{(2)}\right)=\begin{bmatrix} 2.97881 & 7.33318 & 4.3999 & 1.50697 & 2.37402 & 7.30644 \\ 7.33318 & 0.9038 & 6.57753 & -3.71417 & -0.690508 & -5.36442 \\ 4.3999 & 6.57753 & 10.3702 & -1.1491 & -1.57732 & -1.61521 \\ 1.50697 & -3.71417 & -1.1491 & 0.238082 & -2.3756 & -1.00625 \\ 2.37402 & -0.690508 & -1.57732 & -2.3756 & -5.80935 & -2.26403 \\ 7.30644 & -5.36442 & -1.61521 & -1.00625 & -2.26403 & -12.0418 \end{bmatrix},$$

i.e.

$$\hat{N}\left(n^{(2)}\right)\hat{c}=\hat{c}\hat{N}\left(n^{(2)}\right).$$

Similarly from (3.56) we have $\hat{N}\left(n^{(3)}\right)$ as follows

$$\hat{N}\left(n^{(3)}\right)=\begin{bmatrix} 0.654502 & 0.3125 & 0.0329916 & 0.143596 & 0.207813 & 0.639581 \\ 0.3125 & 0.404802 & 0.282649 & 0.478365 & -0.420304 & -0.502992 \\ 0.0329916 & 0.282649 & 0.684355 & -0.621984 & 0.212499 & -0.136565 \\ 0.143596 & 0.478365 & -0.621984 & -0.243686 & 0.365885 & -0.412764 \\ 0.207813 & -0.420304 & 0.212499 & 0.365885 & 0.702254 & -0.328572 \\ 0.639581 & -0.502992 & -0.136565 & -0.412764 & -0.328572 & -0.202227 \end{bmatrix}, \tag{3.68}$$

multiplying (3.68) and (3.65), (3.65) and (3.68) we have

$$\hat{N}\left(n^{(3)}\right)\hat{c}=\begin{bmatrix} 2.979 & 7.33347 & 4.40008 & 1.50702 & 2.37408 & 7.30668 \\ 7.33347 & 0.366073 & 7.1147 & -2.6678 & -2.59458 & -4.74598 \\ 4.40008 & 7.1147 & 9.8326 & -2.19537 & 0.326694 & -2.23388 \\ 1.50702 & -2.6678 & -2.19537 & -1.79847 & 1.3305 & -2.21053 \\ 2.37408 & -2.59458 & 0.326694 & 1.3305 & 3.36924 & -5.24628 \\ 7.30668 & -4.74598 & -2.23388 & -2.21053 & -5.24628 & -11.7025 \end{bmatrix},$$

$$\hat{c}\hat{N}\left(n^{(3)}\right)=\begin{bmatrix} 2.979 & 7.33347 & 4.40008 & 1.50702 & 2.37408 & 7.30668 \\ 7.33347 & 0.366073 & 7.1147 & -2.6678 & -2.59458 & -4.74598 \\ 4.40008 & 7.1147 & 9.8326 & -2.19537 & 0.326694 & -2.23388 \\ 1.50702 & -2.6678 & -2.19537 & -1.79847 & 1.3305 & -2.21053 \\ 2.37408 & -2.59458 & 0.326694 & 1.3305 & 3.36924 & -5.24628 \\ 7.30668 & -4.74598 & -2.23388 & -2.21053 & -5.24628 & -11.7025 \end{bmatrix},$$

i.e.

$$\hat{N}\left(n^{(3)}\right)\hat{c}=\hat{c}\hat{N}\left(n^{(3)}\right).$$

Hence the material has only three planes of symmetry which are mutually orthogonal. Therefore the material represented by (3.64) must be a *orthotropic material.*

Example of Trigonal material:

Let the tensor C be represented by the matrix c in the Voigt notation where the components are in units of $10^{10}N/m^2$.

$$c=\begin{bmatrix} 10.2182 & 1.59431 & 0.57910 & 0.100242 & 1.16443 & 0.884643 \\ 1.59431 & 8.72171 & 2.16381 & 0.753152 & -1.05575 & -1.56371 \\ 0.57910 & 2.16381 & 10.6856 & -0.402904 & 0.212931 & 0.8074449 \\ 0.100242 & 0.753152 & -0.402904 & 4.45907 & 0.954172 & -0.688198 \\ 1.16443 & -1.05575 & 0.212931 & 0.954172 & 2.77358 & 0.615089 \\ 0.884643 & -1.56371 & 0.8074449 & -0.688198 & 0.615089 & 2.70458 \end{bmatrix}, \tag{3.69}$$

from (3.40) we have Mehrabadi-Cowin matrix $\hat{c}$ corresponding to the above matrix is as follows

$$\hat{c} = \begin{bmatrix} 10.2182 & 1.59431 & 0.57910 & 0.141764 & 1.64675 & 1.25107 \\ 1.59431 & 8.72171 & 2.16381 & 1.06512 & -1.49306 & -2.21142 \\ 0.57910 & 2.16381 & 10.6856 & -0.569792 & 0.301129 & 1.14190 \\ 0.141764 & 1.06512 & -0.569792 & 8.91813 & 1.90834 & -1.37640 \\ 1.64675 & -1.49306 & 0.301129 & 1.90834 & 5.54716 & 1.23018 \\ 1.25107 & -2.21142 & 1.14190 & -1.37640 & 1.23018 & 5.40915 \end{bmatrix}, \tag{3.70}$$

matrices U and V become

$$U = \begin{bmatrix} 12.39165 & 0.128383 & 0.321609 \\ 0.128383 & 12.479831 & 0.450491 \\ 0.321609 & 0.450491 & 13.428516 \end{bmatrix}, \; V = \begin{bmatrix} 15.6964 & 0.275106 & 0.689163 \\ 0.275106 & 15.885351 & 0.965337 \\ 0.689163 & 0.965337 & 17.918248 \end{bmatrix}.$$

U has eigenvalues $13.7, 12.3, 12.3$ and eigenvectors

$$u_1 = \begin{bmatrix} 0.25586 \\ 0.35840 \\ 0.89782 \end{bmatrix}, \; u_2 = \begin{bmatrix} 0.124937 \\ -0.933207 \\ 0.336921 \end{bmatrix}, \; u_3 = \begin{bmatrix} 0.958607 \\ 0.0259676 \\ -0.283546 \end{bmatrix}.$$

V has eigenvalues $18.5, 15.5, 15.5$ and eigenvectors

$$v_1 = \begin{bmatrix} 0.25586 \\ 0.35840 \\ 0.89782 \end{bmatrix}, \; v_2 = \begin{bmatrix} 0.031394 \\ -0.931327 \\ 0.362829 \end{bmatrix}, \; v_3 = \begin{bmatrix} 0.966204 \\ -0.0646476 \\ -0.249542 \end{bmatrix},$$

Here

$$n = \begin{bmatrix} 0.25586 \\ 0.35840 \\ 0.89782 \end{bmatrix},$$

is a common eigenvector.

So from (3.56) we have $\hat{N}(n)$ as follows

$$\hat{N}(n) = \begin{bmatrix} 0.755276 & 0.033637 & 0.211087 & 0.119166 & -0.546675 & -0.225412 \\ 0.033637 & 0.552196 & 0.414167 & -0.676316 & 0.166921 & -0.192739 \\ 0.211087 & 0.414167 & 0.374747 & 0.557149 & 0.397754 & 0.418151 \\ 0.119166 & -0.676316 & 0.557149 & -0.040733 & 0.407951 & -0.22338 \\ -0.546675 & 0.166921 & 0.397754 & 0.407951 & -0.320926 & -0.475031 \\ -0.225412 & -0.192739 & 0.418151 & -0.22338 & -0.475031 & 0.679439 \end{bmatrix}, \tag{3.71}$$

multiplying (3.71) and (3.70), (3.70) and (3.71) we have

$$\hat{N}(n)\,\hat{c}=\begin{bmatrix} 6.69854 & 3.42273 & 2.27033 & 0.318005 & -1.92508 & -0.966364 \\ 4.40178 & 5.22254 & 5.85548 & -5.09071 & -1.24617 & -0.612438 \\ 4.29125 & 3.83451 & 5.30254 & 5.40973 & 3.62602 & 1.76033 \\ 0.848553 & -4.66157 & 4.45 & -0.298215 & 3.28418 & 1.63053 \\ -6.3384 & 3.38038 & 3.41277 & 3.55058 & -2.64547 & -4.14722 \\ -2.33231 & -2.16677 & 4.68061 & -4.3093 & -2.18306 & 4.01997 \end{bmatrix},$$

$$\hat{c}\hat{N}(n)=\begin{bmatrix} 6.69854 & 1.40178 & 4.29125 & 0.848553 & -6.3384 & -2.33231 \\ 3.42273 & 5.22254 & 3.83451 & -4.66157 & 3.38038 & -2.16677 \\ 2.27033 & 5.85548 & 5.30254 & 4.45 & 3.41277 & 4.68061 \\ 0.318005 & -5.09071 & 5.40973 & -0.298215 & 3.55058 & -4.3093 \\ -1.92508 & -1.24617 & 3.62602 & 3.28418 & -2.64547 & -2.18306 \\ -0.966364 & -0.612438 & 1.76033 & 1.63053 & -4.14722 & 4.01997 \end{bmatrix},$$

$$\hat{N}(n)\,\hat{c}\neq\hat{c}\hat{N}(n)\,.$$

To find symmetry planes we define a unit vector

$$u(t)=u_2\cos t+u_3\sin t,\quad 0\leq t\leq 2\pi, \tag{3.72}$$

we use the above vector to define a matrix $\hat{N}[u(t)]$. Let $W(t)=\hat{N}[u(t)]\,\hat{c}-\hat{c}\hat{N}[u(t)]$ denote the commutator of $\hat{N}$ and $\hat{c}$. If $W(t)$ vanishes for a value of t, say t_1 , then $u(t_1)$ will define a plane of symmetry for the material. Let

$$f(t)=W^T(t)\,W(t) \tag{3.73}$$

finding f with the help of a mathematical software is straightforward and a simple affair. Solution of $f(t)=0$ is easy. Here for trigonal material (3.72) becomes

$$u(t)=\begin{bmatrix} 0.124937 \\ -0.933207 \\ 0.336921 \end{bmatrix}\cos t+\begin{bmatrix} 0.958607 \\ 0.0259676 \\ -0.283546 \end{bmatrix}\sin t,\quad 0\leq t\leq 2\pi,$$

and (3.73) becomes

$$f(t)=106.589-105.001\cos 6t+1803274\sin 6t,$$

this implies that $f(t)$ has exactly six zeros in the interval $[0, 2\pi]$. These zeros are equally spaced with the spacing $\pi/3$ and are given as

$$\begin{aligned} t_1 &= 1.0184, \\ t_2 &= 2.06559, \\ t_3 &= 3.11279, \\ t_4 &= 4.15999, \\ t_5 &= 5.20719, \\ t_6 &= 6.25438, \end{aligned}$$

each of these zeros define a symmetry plane. The normal to these planes are found from (3.72). If we let $n^{(i)} = u(t_i)$,we find

$$\begin{aligned} n^{(1)} &= \begin{bmatrix} 0.881591 \\ -0.467575 \\ -0.064582 \end{bmatrix}, \; n^{(2)} = \begin{bmatrix} 0.784312 \\ 0.465987 \\ -0.409526 \end{bmatrix}, \; n^{(3)} = \begin{bmatrix} -0.0972786 \\ 0.933568 \\ -0.344947 \end{bmatrix}, \\ n^{(4)} &= -n^{(1)}, \; n^{(5)} = -n^{(2)}, \; n^{(6)} = -n^{(3)}. \end{aligned}$$

These are three planes of symmetry in addition to one, which is orthogonal to these planes and angle between successive plane is $\frac{\pi}{3}$. Thus the material represented by the matrix (3.69) must belong to a trigonal material.

Example of Tetragonal material:

Let the tensor C be represented by the matrix c in the Voigt notation where the components are in units of $10^{10} N/m^2$.

$$c = \begin{bmatrix} 7.0431 & 4.60528 & 6.38762 & -0.647475 & 2.22024 & 6.8332 \\ 4.60528 & 8.6977 & 4.5475 & -2.21305 & -2.00823 & -6.18069 \\ 6.38762 & 4.5475 & 5.3436 & 3.4315 & -0.212014 & -0.652512 \\ -0.647475 & -2.21305 & 3.4315 & 3.0706 & -0.652512 & -2.00823 \\ 2.22024 & -2.00823 & -0.212014 & -0.652512 & 5.20437 & 0.256274 \\ 6.8332 & -6.18069 & -0.652512 & -2.00823 & 0.256274 & 5.90983 \end{bmatrix}, \tag{3.74}$$

from (3.40) we have Mehrabadi-Cowin matrix $\hat{c}$ corresponding to the above matrix is as follows

$$\hat{c} = \begin{bmatrix} 7.0431 & 4.60528 & 6.38762 & -0.915668 & 3.13989 & 9.6636 \\ 4.60528 & 8.6977 & 4.5477 & -3.12973 & -2.84007 & -8.74082 \\ 6.38762 & 4.5477 & 5.3436 & 4.85287 & -0.299833 & -0.922791 \\ -0.915668 & -3.12973 & 4.85287 & 6.1412 & -1.30502 & -4.01646 \\ 3.13989 & -2.84007 & -0.299833 & -1.30502 & 10.4087 & 0.512548 \\ 9.6636 & -8.74082 & -0.922791 & -4.01646 & 0.512548 & 11.8197 \end{bmatrix}, \tag{3.75}$$

matrices U and V become

$$U = \begin{bmatrix} 18.036 & 0 & 0 \\ 0 & 17.8507 & 0.570975 \\ 0 & 0.570975 & 16.2789 \end{bmatrix}, \quad V = \begin{bmatrix} 18.1573 & 0 & 0 \\ 0 & 17.6781 & 1.47472 \\ 0 & 1.47472 & 13.6186 \end{bmatrix}.$$

U has eigenvalues $18.0362, 18.036, 16.0934$ and eigenvectors

$$u_1 = \begin{bmatrix} 0 \\ -0.309017 \\ 0.951057 \end{bmatrix}, \quad u_2 = \begin{bmatrix} 0.651464 \\ -0.721547 \\ -0.234446 \end{bmatrix}, \quad u_3 = \begin{bmatrix} 0.75868 \\ 0.619579 \\ 0.201312 \end{bmatrix}.$$

V has eigenvalues $18.15, 18.1573, 13.1394$ and eigenvectors

$$v_1 = \begin{bmatrix} 0 \\ -0.309017 \\ 0.951057 \end{bmatrix}, \quad v_2 = \begin{bmatrix} 0.826373 \\ -0.535561 \\ -0.174015 \end{bmatrix}, \quad v_3 = \begin{bmatrix} -0.563122 \\ -0.785928 \\ -0.255363 \end{bmatrix}.$$

Here

$$n = \begin{bmatrix} 0 \\ -0.309017 \\ 0.951057 \end{bmatrix},$$

is a common eigenvector.

So from (3.56) we have $\hat{N}(n)$ as follows

$$\hat{N}(n) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0.654508 & 0.345492 & 0.672499 & 0 & 0 \\ 0 & 0.345492 & 0.654511 & -0.6725 & 0 & 0 \\ 0 & 0.672499 & -0.6725 & -0.309018 & 0 & 0 \\ 0 & 0 & 0 & 0 & -0.809019 & 0.587786 \\ 0 & 0 & 0 & 0 & 0.587786 & 0.809017 \end{bmatrix}, \tag{3.76}$$

multiplying (3.76) and (3.75), (3.75) and (3.76) we have

$$\hat{N}(n)\,\hat{c} = \begin{bmatrix} 7.04312 & 4.60528 & 6.38764 & -0.915673 & 3.14063 & 9.66306 \\ 4.60528 & 5.15917 & 8.08625 & 3.75801 & -2.84006 & -8.74082 \\ 6.38764 & 8.08625 & 1.80508 & -2.03487 & -0.299833 & -0.922791 \\ -0.915673 & 3.75801 & -2.03487 & -7.26604 & -1.30503 & -4.01646 \\ 3.14063 & -2.84006 & -0.299833 & -1.30503 & -8.11957 & 6.53274 \\ 9.66306 & -8.74082 & -0.922791 & -4.01646 & 6.53274 & 9.86361 \end{bmatrix},$$

$$\hat{c}\hat{N}(n) = \begin{bmatrix} 7.04312 & 4.60528 & 6.38764 & -0.915673 & 3.14063 & 9.66306 \\ 4.60528 & 5.15917 & 8.08625 & 3.75801 & -2.84006 & -8.74082 \\ 6.38764 & 8.08625 & 1.80508 & -2.03487 & -0.299833 & -0.922791 \\ -0.915673 & 3.75801 & -2.03487 & -7.26604 & -1.30503 & -4.01646 \\ 3.14063 & -2.84006 & -0.299833 & -1.30503 & -8.11957 & 6.53274 \\ 9.66306 & -8.74082 & -0.922791 & -4.01646 & 6.53274 & 9.86361 \end{bmatrix},$$

i.e.

$$\hat{N}(n)\,\hat{c} = \hat{c}\hat{N}(n)\,.$$

(3.72) becomes

$$u(t) = \begin{bmatrix} 0.651464 \\ -0.721547 \\ -0.234446 \end{bmatrix} \cos t + \begin{bmatrix} 0.75868 \\ 0.619579 \\ 0.201312 \end{bmatrix} \sin t, \quad 0 \le t \le 2\pi, \qquad (3.77)$$

and (3.73) becomes

$$\begin{aligned} f(t) &= 1002.2 + 89.1547\cos 2t - 4.96305\cos 4t - 47.7259\cos 6t + 257.853\cos 8t \\ &\quad -75.4311\sin 2t - 0.808551\sin 4t + 102.397\sin 6t + 961.823\sin 8t, \end{aligned}$$

this implies that $f(t)$ has exactly eight zeros in the interval $[0, 2\pi]$. These zeros are equally spaced with the spacing $\pi/4$ and are given as

$$\begin{aligned} t_1 &= 35.8959, \\ t_2 &= 1.33839, \\ t_3 &= 10.7632, \\ t_4 &= 45162.2, \\ t_5 &= 4.47998, \\ t_6 &= 4.47998, \\ t_7 &= 12585.8, \\ t_8 &= 10.7632, \end{aligned}$$

each of these zeros define a symmetry plane. The normal to these planes are found from (3.72).If we let $n^{(i)} = u(t_i)$, we find

$$n^{(1)} = \begin{bmatrix} -0.888332 \\ -0.429817 \\ -0.1419 \end{bmatrix}, \; n^{(2)} = -n^{(1)}, \; n^{(3)} = -n^{(1)}, \; n^{(4)} = \begin{bmatrix} -0.586294 \\ -0.777429 \\ -0.250333 \end{bmatrix},$$

$$n^{(5)} = -n^{(1)}, \; n^{(6)} = n^{(1)}, \; n^{(7)} = \begin{bmatrix} 0.960653 \\ -0.289255 \\ -0.0858321 \end{bmatrix}, \; n^{(8)} = n^{(1)}.$$

These are three planes of symmetry in addition to n found above, and one, which is orthogonal to these planes and angle between successive plane is $\frac{\pi}{4}$, so total are five symmetry planes. Thus the material represented by the matrix (3.74) must belong to a tetragonal material.

Example of Hexagonal material:

Let the tensor C be represented by the matrix c in the Voigt notation where the components are in units of $10^{10} N/m^2$.

$$c = \begin{bmatrix} 13.9182 & 10.6092 & 16.1537 & -2.14454 & 5.37335 & -1.83576 \\ 10.6092 & 14.2834 & 15.95085 & -5.54564 & 1.94045 & -1.80189 \\ 16.1537 & 15.95085 & 21.6003 & 2.71558 & -2.58266 & 2.0213 \\ -2.14454 & -5.54564 & 2.71558 & 3.13989 & 2.46945 & 0.628666 \\ 5.37335 & 1.94045 & -2.58266 & 2.46945 & 3.38785 & -0.765265 \\ -1.83576 & -1.80189 & 2.0213 & 0.628666 & -0.765265 & 1.40933 \end{bmatrix}, \tag{3.78}$$

from (3.40) we have Mehrabadi-Cowin matrix $\hat{c}$ corresponding to the above matrix is as follows

$$\hat{c} = \begin{bmatrix} 13.9182 & 10.6092 & 16.1537 & -3.03284 & 7.59906 & -2.59616 \\ 10.6092 & 14.2834 & 15.95085 & -7.84272 & 2.74421 & -2.54826 \\ 16.1537 & 15.95085 & 21.6003 & 3.84041 & -3.65243 & 2.85855 \\ -3.03284 & -7.84272 & 3.84041 & 6.27978 & 4.9389 & 1.25733 \\ 7.59906 & 2.74421 & -3.65243 & 4.9389 & 6.7757 & -1.53053 \\ -2.59616 & -2.54826 & 2.85855 & 1.25733 & -1.53053 & 2.81866 \end{bmatrix}, \tag{3.79}$$

matrices U and V become

$$U=\begin{bmatrix} 40.6811 & -1.61635 & 4.37114 \\ -1.61635 & 40.8434 & -4.9746 \\ 4.37114 & -4.9746 & 53.7048 \end{bmatrix},\ V=\begin{bmatrix} 18.7154 & -1.1682 & 3.41936 \\ -1.1682 & 18.8326 & -3.59532 \\ 3.41936 & -3.59532 & 28.128 \end{bmatrix}.$$

U has eigenvalues $56.9416, 39.1439, 39.1439$ and eigenvectors

$$u_1=\begin{bmatrix} 0.293893 \\ -0.309017 \\ 0.904509 \end{bmatrix},\ u_2=\begin{bmatrix} 0.0697118 \\ -0.936849 \\ -0.342716 \end{bmatrix},\ u_3=\begin{bmatrix} 0.953293 \\ 0.163777 \\ -0.253791 \end{bmatrix}.$$

V has eigenvalues $30.4674, 17.6044, 17.6043$ and eigenvectors

$$v_1=\begin{bmatrix} 0.293893 \\ -0.309016 \\ 0.904508 \end{bmatrix},\ v_2=\begin{bmatrix} 0.935058 \\ -0.103296 \\ -0.33911 \end{bmatrix},\ v_3=\begin{bmatrix} 0.198223 \\ 0.94543 \\ 0.25859 \end{bmatrix}.$$

Here

$$n=\begin{bmatrix} 0.293893 \\ -0.309016 \\ 0.904508 \end{bmatrix},$$

is a common eigenvector.

So from (3.56) we have $\hat{N}(n)$ as follows

$$\hat{N}(n)=\begin{bmatrix} 0.684349 & 0.0329914 & 0.282659 & -0.136567 & -0.621993 & 0.212498 \\ 0.0329914 & 0.65451 & 0.312498 & 0.639583 & 0.143595 & 0.207813 \\ 0.282659 & 0.312498 & 0.404839 & -0.503013 & 0.478396 & -0.42031 \\ -0.136567 & 0.639583 & -0.503013 & -0.202256 & -0.412773 & -0.328583 \\ -0.621993 & 0.143595 & 0.478396 & -0.412773 & -0.243697 & 0.36588 \\ 0.212498 & 0.207813 & -0.42031 & -0.328583 & 0.36588 & 0.702255 \end{bmatrix}, \tag{3.80}$$

multiplying (3.80) and (3.79), (3.79) and (3.80)) we have

$$\hat{N}(n)\,\hat{c}=\begin{bmatrix} 9.57686 & 11.063 & 20.0413 & -4.91113 & -0.955614 & 0.326473 \\ 11.063 & 9.53167 & 20.2488 & 0.953848 & 4.71917 & 0.309922 \\ 20.0413 & 20.2488 & 13.4147 & -3.07788 & 2.92729 & -2.92227 \\ -4.91113 & 0.953848 & -3.07788 & -10.2556 & -0.738256 & -3.26187 \\ -0.955614 & 4.71917 & 2.92729 & -0.738256 & -10.3297 & 3.50168 \\ 0.326473 & 0.309922 & -2.92227 & -3.26187 & 3.50168 & -1.27643 \end{bmatrix},$$

$$\hat{c}\hat{N}(n) = \begin{bmatrix} 9.57686 & 11.063 & 20.0413 & -4.91113 & -0.955614 & 0.326473 \\ 11.063 & 9.53167 & 20.2488 & 0.953848 & 4.71917 & 0.309922 \\ 20.0413 & 20.2488 & 13.4147 & -3.07788 & 2.92729 & -2.92227 \\ -4.91113 & 0.953848 & -3.07788 & -10.2556 & -0.738256 & -3.26187 \\ -0.955614 & 4.71917 & 2.92729 & -0.738256 & -10.3297 & 3.50168 \\ 0.326473 & 0.309922 & -2.92227 & -3.26187 & 3.50168 & -1.27643 \end{bmatrix},$$

$$\hat{N}(n)\,\hat{c} = \hat{c}\hat{N}(n).$$

(3.72) becomes

$$u(t) = \begin{bmatrix} 0.935058 \\ -0.103296 \\ -0.33911 \end{bmatrix} \cos t + \begin{bmatrix} 0.198223 \\ 0.94543 \\ 0.25859 \end{bmatrix} \sin t, \quad 0 \le t \le 2\pi,$$

and (3.73) becomes

$$f(t) = 0, \tag{3.81}$$

from (3.81) it is clear that any plane that contain the x_3-axis and x_3-axis is the plane of symmetry. Thus the material represented by the matrix (3.78) must belong to a hexagonal material.

Example of Cubic material:

Let the tensor C be represented by the matrix c in the Voigt notation where the components are in units of $10^{10}N/m^2$.

$$c = \begin{bmatrix} 26.4265 & -7.229 & -10.4429 & -1.16753 & 1.68967 & -5.20027 \\ -7.229 & 23.1763 & -7.19278 & -4.41183 & -1.52832 & 4.70369 \\ -10.4429 & -7.19278 & 26.3903 & 5.57938 & -0.161349 & 0.496582 \\ -1.16753 & -4.41183 & 5.57938 & 12.7417 & 0.496582 & -1.52832 \\ 1.68967 & -1.52832 & -0.161349 & 0.496582 & 9.49156 & -1.16755 \\ -5.20027 & 4.70369 & 0.496582 & -1.52832 & -1.16755 & 12.7055 \end{bmatrix}, \tag{3.82}$$

from (3.40) we have Mehrabadi-Cowin matrix $\hat{c}$ corresponding to the above matrix is as follows

$$\hat{c} = \begin{bmatrix} 26.4265 & -7.229 & -10.4429 & -1.65114 & 2.38955 & -7.35429 \\ -7.229 & 23.1763 & -7.19278 & -6.23927 & -2.16137 & 6.65202 \\ -10.4429 & -7.19278 & 26.3903 & 7.89043 & -0.228182 & 0.702273 \\ -1.65114 & -6.23927 & 7.89043 & 25.4834 & 0.993164 & -3.05664 \\ 2.38955 & -2.16137 & -0.228182 & 0.993164 & 18.9831 & -2.3351 \\ -7.35429 & 6.65202 & 0.702273 & -3.05664 & -2.3351 & 25.411 \end{bmatrix}, \tag{3.83}$$

matrices U and V become

$$U = \begin{bmatrix} 8.7546 & 0 & 0 \\ 0 & 8.7546 & 0 \\ 0 & 0 & 8.7546 \end{bmatrix}, \ V = \begin{bmatrix} 48.6236 & 0 & 0 \\ 0 & 48.6236 & 0 \\ 0 & 0 & 48.6236 \end{bmatrix}.$$

U has eigenvalues $8.7547, 8.7547, 8.7547$ and eigenvectors

$$u_1 = \begin{bmatrix} 0.0568194 \\ 0.189721 \\ 0.980193 \end{bmatrix}, \ u_2 = \begin{bmatrix} 0.99816 \\ 0.0100094 \\ -0.0597983 \end{bmatrix}, \ u_3 = \begin{bmatrix} 0.0211561 \\ -0.981787 \\ 0.188803 \end{bmatrix}.$$

V has eigenvalues $48.6236, 48.6236, 48.6236$ and eigenvectors

$$v_1 = \begin{bmatrix} 0.718394 \\ 0.023541 \\ 0.695238 \end{bmatrix}, \ v_2 = \begin{bmatrix} -0.69484 \\ -0.0235406 \\ 0.718779 \end{bmatrix}, \ v_3 = \begin{bmatrix} -0.0332871 \\ 0.999446 \\ 0.00055417 \end{bmatrix}.$$

Now here we see that U has same three eigenvalues and same for V, so for this case it is very difficult to find symmetry planes by using theorem 9. So to find the symmetry planes of cubic material, we define an eigenvalue λ such that

$$C_{ijks} n_i n_j n_k n_s = \lambda,$$

so eigenvector is

$$C_{pjks} n_j n_k n_s = \lambda n_p,$$

thus equation becomes

$$C_{pjks} n_j n_k n_s = \left(C_{ijks} n_i n_j n_k n_s\right) n_p, \tag{3.84}$$

where summation over repeated indices is taken and free indices take the value $1, 2, 3$. With the aid of eq(3.84) we determine n let us suppose that n is normal to a plane of symmetry of C, then the components n_1, n_2, n_3 must satisfies 3 equations implied by the above condition such that

$$n = \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix} \text{ where } n_1^2 + n_2^2 + n_3^2 = 1 \text{ so } n = \begin{bmatrix} n_1 \\ n_2 \\ \sqrt{1 - n_1^2 - n_2^2} \end{bmatrix}.$$

Now from eq(3.84) we get three equations and solving simultaneously these equations numerically we get roots (n_1, n_2) of equation as we solve numerically for n_1 and n_2 some of the roots will not appear in the solution set because the method fail to converge to a root (n_1, n_2). Here we have following nine roots by solving eq(3.84) for $p = 2, 3$ and which satisfy eq(3.84) for $p = 1$, $(0.951056, \ -0.293893)$, $(0.672499, \ 0.0106943)$, $(0.218509, \ 0.858092)$, $(0.45399, \ -0.847398)$, $(0, \ 0.309016)$, $(-0.218509, \ -0.421077)$, $(-0.672499, \ 0.42632)$, $(-0.309017, \ -0.904508)$, $(-0.891007, \ -0.431771)$.

Here are nine unit vectors as follows

$$\begin{aligned}
n^{(1)} &= \begin{bmatrix} 0.951056 \\ -0.293893 \\ 0.0954955 \end{bmatrix}, \ n^{(2)} = \begin{bmatrix} 0.672499 \\ 0.0106943 \\ 0.740021 \end{bmatrix}, \ n^{(3)} = \begin{bmatrix} 0.218508 \\ 0.858092 \\ 0.464685 \end{bmatrix}, \\
n^{(4)} &= \begin{bmatrix} 0.45399 \\ -0.847398 \\ 0.275336 \end{bmatrix}, \ n^{(5)} = \begin{bmatrix} 0 \\ 0.309008 \\ 0.951059 \end{bmatrix}, \ n^{(6)} = \begin{bmatrix} -0.218509 \\ -0.421077 \\ 0.880311 \end{bmatrix}, \\
n^{(7)} &= \begin{bmatrix} -0.672499 \\ 0.42632 \\ 0.604976 \end{bmatrix}, \ n^{(8)} = \begin{bmatrix} -0.309017 \\ -0.904508 \\ 0.293894 \end{bmatrix}, \ n^{(9)} = \begin{bmatrix} -0.891007 \\ -0.431771 \\ 0.140287 \end{bmatrix}.
\end{aligned}$$

Now we apply condition (3.57) of theorem 9, as follows for finding normals to symmetry planes. From (3.56) we have $\hat{N}\left(n^{(1)}\right)$ as follows

$$\hat{N}\left(n^{(1)}\right) = \begin{bmatrix}
0.654505 & 0.3125 & 0.0329942 & -0.143602 & 0.207822 & -0.639583 \\
0.3125 & 0.684349 & 0.00315068 & 0.0656683 & 0.0443754 & 0.654002 \\
0.0329942 & 0.00315068 & 0.963855 & 0.0779333 & -0.252197 & -0.014419 \\
-0.143602 & 0.0656683 & 0.0779333 & 0.815316 & 0.538626 & -0.118887 \\
0.207822 & 0.0443754 & -0.252197 & 0.538626 & -0.761265 & -0.146952 \\
-0.639583 & 0.654002 & -0.014419 & -0.118887 & -0.146952 & -0.35676
\end{bmatrix}, \tag{3.85}$$

multiplying (3.85) and (3.83), (3.83) and (3.85) we have

$$\hat{N}\left(n^{(1)}\right)\hat{c}=\begin{bmatrix} 20.13 & -1.53386 & -9.84161 & -4.2682 & 6.17698 & -19.0103 \\ -1.53386 & 17.4237 & -7.13532 & -5.04248 & -1.35267 & 18.5708 \\ -9.84161 & -7.13532 & 25.7316 & 9.3107 & -4.82431 & 0.439488 \\ -4.2682 & -5.04248 & 9.3107 & 22.1177 & 10.8093 & -5.22326 \\ 6.17698 & -1.35267 & -4.82431 & 10.8093 & -13.1148 & -5.01327 \\ -19.0103 & 18.5708 & 0.439488 & -5.22326 & -5.01327 & 0.684894 \end{bmatrix},$$

$$\hat{c}\hat{N}\left(n^{(1)}\right)=\begin{bmatrix} 20.13 & -1.53386 & -9.84161 & -4.2682 & 6.17698 & -19.0103 \\ -1.53386 & 17.4237 & -7.13532 & -5.04248 & -1.35267 & 18.5708 \\ -9.84161 & -7.13532 & 25.7316 & 9.3107 & -4.82431 & 0.439488 \\ -4.2682 & -5.04248 & 9.3107 & 22.1177 & 10.8093 & -5.22326 \\ 6.17698 & -1.35267 & -4.82431 & 10.8093 & -13.1148 & -5.01327 \\ -19.0103 & 18.5708 & 0.439488 & -5.22326 & -5.01327 & 0.684894 \end{bmatrix}.$$

i.e.

$$\hat{N}\left(n^{(1)}\right)\hat{c}=\hat{c}\hat{N}\left(n^{(1)}\right).$$

Similarly from (3.56) we have $\hat{N}\left(n^{(2)}\right)$ as follows

$$\hat{N}\left(n^{(2)}\right)=\begin{bmatrix} 0.009118 & 0.0002069 & 0.990675 & 0.0202467 & -0.134412 & -0.001943 \\ 0.0002069 & 0.999543 & 0.0002505 & -0.022379 & 0.0003219 & -0.020337 \\ 0.990675 & 0.0002505 & 0.009074 & 0.0021323 & 0.13409 & 0.022279 \\ 0.0202467 & -0.022379 & 0.0021323 & -0.0949891 & 0.017124 & -0.994871 \\ -0.134412 & 0.0003219 & 0.13409 & 0.0171243 & 0.98157 & 0.012805 \\ -0.001943 & -0.020337 & 0.022279 & -0.994871 & 0.012805 & 0.095675 \end{bmatrix}, \tag{3.86}$$

multiplying (3.86) and (3.83), (3.83) and (3.86) we have

$$\hat{N}\left(n^{(2)}\right)\hat{c}=\begin{bmatrix} -10.4464 & -7.03556 & 26.2365 & 8.18887 & -2.73172 & 0.83265 \\ -7.03556 & 23.1661 & -7.37599 & -6.74257 & -2.12857 & 6.19852 \\ 26.2365 & -7.37599 & -10.1059 & -1.4463 & 4.86029 & -7.03119 \\ 8.18887 & -6.74257 & -1.4463 & 0.760273 & 2.65009 & -25.3265 \\ -2.73172 & -2.12857 & 4.86029 & 2.65009 & 18.268 & -0.934187 \\ 0.83265 & 6.19852 & -7.03119 & -25.3265 & -0.934187 & 5.33699 \end{bmatrix},$$

$$\hat{c}\hat{N}\left(n^{(2)}\right)=\begin{bmatrix} -10.4464 & -7.03556 & 26.2365 & 8.18887 & -2.73172 & 0.83265 \\ -7.03556 & 23.1661 & -7.37599 & -6.74257 & -2.12857 & 6.19852 \\ 26.2365 & -7.37599 & -10.1059 & -1.4463 & 4.86029 & -7.03119 \\ 8.18887 & -6.74257 & -1.4463 & 0.760273 & 2.65009 & -25.3265 \\ -2.73172 & -2.12857 & 4.86029 & 2.65009 & 18.268 & -0.934187 \\ 0.83265 & 6.19852 & -7.03119 & -25.3265 & -0.934187 & 5.33699 \end{bmatrix},$$

i.e.

$$\hat{N}\left(n^{(2)}\right)\hat{c}=\hat{c}\hat{N}\left(n^{(2)}\right).$$

Similarly condition (3.57) of theorem 9 is satisfied for $n^{(3)}$, $n^{(4)}$, $n^{(5)}$, $n^{(6)}$, $n^{(7)}$, $n^{(8)}$, and $n^{(9)}$. Hence by theorem 9 these nine unit vectors are normals to symmetry planes, thus the material represented by the matrix (3.82) must belong to a cubical material.

Now we verify from these normals that material is cubic as follows:

From these nine normals to symmetry plane, we see that $n^{(1)}$, $n^{(5)}$ and $n^{(8)}$ are mutually perpendicular and also $n^{(1)} \times n^{(5)} = n^{(8)}$, and the triplets $n^{(1)}$, $n^{(5)}$ and $n^{(8)}$ is such a normal which generate cubic material as follows:

We choose such coordinate axis such that $n^{(1)}$, $n^{(5)}$ and $n^{(8)}$ are three coordinate planes, from these three unit vectors we generate three-dimensional rotation matrix Q, such that

$$Q=\begin{bmatrix} n_1^{(1)} & n_2^{(1)} & n_3^{(1)} \\ n_1^{(5)} & n_2^{(5)} & n_3^{(5)} \\ n_1^{(8)} & n_2^{(8)} & n_3^{(8)} \end{bmatrix},$$

$$Q=\begin{bmatrix} 0.951056 & -0.293893 & 0.0954955 \\ 0 & 0.309008 & 0.951059 \\ -0.309017 & -0.904508 & 0.293894 \end{bmatrix}. \tag{3.87}$$

By using eq (3.46) we formulate six-dimensional rotation matrix $\hat{Q}$, such that

$$\hat{Q}=\begin{bmatrix} 0.904508 & 0.0863731 & 0.00911939 & -0.0396906 & 0.128441 & -0.395285 \\ 0 & 0.0954859 & 0.904513 & 0.415616 & 0 & 0 \\ 0.0954915 & 0.818135 & 0.0863737 & -0.37594 & -0.128436 & 0.395284 \\ 0 & -0.395273 & 0.395288 & -0.769425 & -0.293893 & -0.0954887 \\ -0.415627 & 0.375938 & 0.0396907 & -0.17275 & 0.25 & -0.76942 \\ 0 & -0.128432 & 0.128441 & -0.250001 & 0.90451 & 0.293884 \end{bmatrix}. \tag{3.88}$$

Now from the transformation $\hat{Q}^T\hat{c}\hat{Q}$ or $\hat{c}'_{ij} = Q_{ip}Q_{jq}\hat{c}_{pq}$, where summation over repeated indices is taken and free indices take the value $1, 2, 3$, by using equations (3.83) and (3.88), we have

$$\hat{c}' = \begin{bmatrix} 30.3992 & -10.8223 & -10.8223 & 0 & 0 & 0 \\ -10.8223 & 30.3992 & -10.8223 & 0 & 0 & 0 \\ -10.8223 & -10.8223 & 30.3992 & 0 & 0 & 0 \\ 0 & 0 & 0 & 18.2243 & 0 & 0 \\ 0 & 0 & 0 & 0 & 18.2243 & 0 \\ 0 & 0 & 0 & 0 & 0 & 18.2243 \end{bmatrix}, \quad (3.89)$$

equation (3.89) shows that material is cubic as it have only 3 independent components.

3.7 Revised Cowin-Mehrabadi theorem for a plane of symmetry

Ahmad [2] stated revised *Cowin-Mehrabadi theorem for a plane of symmetry,* which is as follows

Theorem 10 *[2] A necessary condition for a unit vector* **n** *to be normal to a plane of symmetry is that the matrix Ω commutes with every symmetric tensor of rank two associated with the material.*

Proof. Let **T** be symmetric tensor of rank two associated with the material. Since **n** is normal to plane of symmetry, **T** must be invariant with respect to the transformation (3.55) i.e.

$$\begin{aligned} T_{ij} &= \Omega_{ip}\Omega_{jq}T_{pq} \\ &\Longrightarrow \mathbf{T\Omega} = \mathbf{\Omega T}. \end{aligned} \quad (3.93)$$

■

Theorem 11 *[2] A necessary condition for a unit vector* **n** *to be normal to a plane of symmetry is that it is an eigenvector of any symmetric tensor of rank two associated with the material.*

Proof. Since $\mathbf{n}$ is an eigenvector of Ω and this eigenvector belongs to non degenerate eigenvalue -1, so from (3.93) $\mathbf{n}$ must also eigenvector of $\mathbf{T}$, by using the result " If $\mathbf{L}$ and $\mathbf{K}$ commute and if one of the operator has an eigenvalue of finite multiplicity, both operators have common eigenvector."[[12], pg 63-64]. ■

Chapter 4

Axis of symmetry of an anisotropic elastic material

4.1 Introduction

In this chapter we quote some basic useful definitions concerned about rotation of a rigid body and Euler's theorem ([5], [14]). Ahmad [3] has given Cowin-Mehrabadi theorem for an axis of symmetry. We discuss how to determine an axis of symmetry for an unknown elastic material. We find matrix representation of an elasticity tensor of an anisotropic elastic material possessing an axis of symmetry. A simple way is described here for finding a 6×6 matrix representation of the elasticity tensor belonging to a trigonal, a tetragonal or a hexagonal materials.

4.2 Some definitions

Here some elementary definitions concerning rotation of a rigid body are presented [5].

Let a rigid body undergo a general displacement in space relative to a spatial Cartesian frame X with origin at a point O. If X $\{X'\}$ denotes the position vector of a particle P in its intial {final} position from O, then the vector $d = X' - X$ represents the *finite displacement* of P.

A displacement in which every material point of the body undergoes the same displacement d is called a *parallel translation*.

If the spatial position vector of one point P is the same in every configuration of the body, then $d = 0$ for that point, and the displacement is called a *rotation about a fixed point.*

If the spatial position of every point that lies on some straight line L is unaltered, the displacement is called *rotation about a fixed line.* The line L is called the *axis of rotation.*

Let a rigid body undergo a general displacement in a space relative to a spatial Cartesian frame X. Consider another Cartesian frame x imbedded in the body and parallel to the frame X initially; the origin of the x frame is called the *base point* [5].

4.3 Fundamental equation of finite rotation

Let the particle of rigid body be rotated in a circular arc through a finite angle θ about a fixed line OA (axis of rotation) in a right hand sense with respect to a unit axial vector $\mathbf{e}_b$ directed from O to A positively as shown in figure (4.1). This axis of rotation passes through the centre of circle and it is perpendicular to the plane of circle.

Since the body is rigid and OA is fixed, the trajectory of P is a circle centered on the axis of rotation and in the plane of the circle, in this case the displacement of P is given by

$$\mathbf{d} = \mathbf{X}' - \mathbf{X} = \mathbf{TX}, \qquad d_i = x_i' - x_i = T_{ij}x_j. \tag{4.1}$$

From any point O on the axis of rotation let r is the initial position vector of any point P of the body, let $\mathbf{R}$ be the vector from centre of circle at O', so

$$\mathbf{R} = (\mathbf{r}.\mathbf{e}_b)\,\mathbf{e}_b$$

and let

$$\boldsymbol{\rho} = \mathbf{e}_b \times (\mathbf{r} \times \mathbf{e}_b)\,. \tag{4.2}$$

Let the unit vector directed from O' to P be $\mathbf{e}_p$, we may define a unit vector

$$\mathbf{e}_t = \mathbf{e}_b \times \mathbf{e}_p,$$

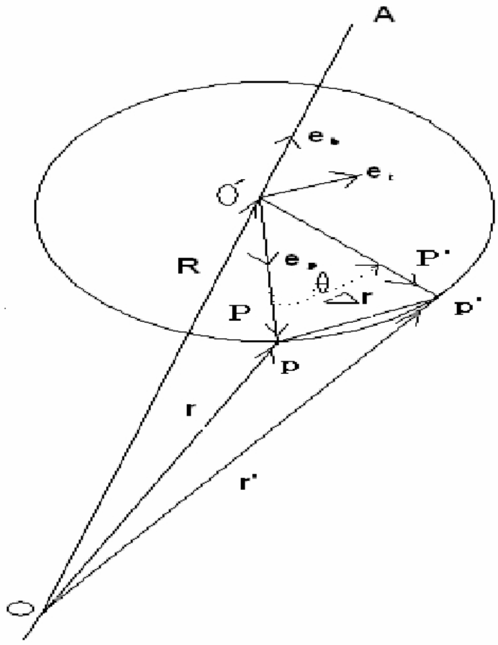

Figure 4.1: Finite rotation about a fixed line.

then we have

$$\begin{aligned}
\rho \mathbf{e}_t &= \boldsymbol{\rho}\left(\mathbf{e}_b \times \mathbf{e}_p\right), \\
-\rho \mathbf{e}_t &= \boldsymbol{\rho}\left(\mathbf{e}_p \times \mathbf{e}_b\right), \\
\mathbf{e}_b \times \left(-\rho \mathbf{e}_t\right) &= \mathbf{e}_b \times \rho\left(\mathbf{e}_p \times \mathbf{e}_b\right), \\
&= \rho\left(\mathbf{e}_b \times \left(\mathbf{e}_p \times \mathbf{e}_b\right)\right), \\
&= \rho[\mathbf{e}_p\left(\mathbf{e}_b . \mathbf{e}_b\right) - \mathbf{e}_b\left(\mathbf{e}_b . \mathbf{e}_p\right)], \\
&= \rho \mathbf{e}_p,
\end{aligned}$$

so we have

$$\boldsymbol{\rho} = \rho \mathbf{e}_p = \mathbf{e}_b \times \left(-\rho e_t\right), \qquad \rho = |\boldsymbol{\rho}|. \tag{4.3}$$

On comparing (4.2) and (4.3), we see that $\rho \mathbf{e}_t - \mathbf{e}_b \times \mathbf{r}$ is not parallel to $\mathbf{e}_b$, thus

$$\rho \mathbf{e}_t = \mathbf{e}_b \times \mathbf{r}, \tag{4.4}$$

and

$$\rho \mathbf{e}_p = \mathbf{e}_b \times \left(\mathbf{r} \times \mathbf{e}_b\right). \tag{4.5}$$

Let the final position of P be the vector $\mathbf{r}'$ and the radius vector of $\mathbf{r}'$ is $\boldsymbol{\rho}'$, then

$$\boldsymbol{\Delta}\mathbf{r} = \mathbf{r}' - \mathbf{r} = \boldsymbol{\rho}' - \boldsymbol{\rho}.$$

Since $|\boldsymbol{\rho}'| = |\boldsymbol{\rho}| = \rho$ (constant), thus from geometry we have

$$\boldsymbol{\rho}' = \rho\left(\cos\theta \mathbf{e}_p + \sin\theta \mathbf{e}_t\right), \tag{4.6}$$

$$\begin{aligned}
\boldsymbol{\rho}' - \boldsymbol{\rho} &= \rho\left(\cos\theta \mathbf{e}_p + \sin\theta \mathbf{e}_t\right) - \boldsymbol{\rho}, \\
\Delta r &= \rho \cos\theta \mathbf{e}_p + \rho \sin\theta \mathbf{e}_t - \rho \mathbf{e}_p, \\
\mathbf{r}' - \mathbf{r} &= \rho \sin\theta \mathbf{e}_t - \rho\left(1 - \cos\theta\right) \mathbf{e}_p, \\
\mathbf{r}' &= \mathbf{r} + \rho \sin\theta \mathbf{e}_t - \rho\left(1 - \cos\theta\right) \mathbf{e}_p,
\end{aligned}$$

putting values from (4.4) and (4.5), we have

$$\mathbf{r}' = \mathbf{r} + \sin\theta \mathbf{e}_b \times \mathbf{r} - \left(1 - \cos\theta\right) \mathbf{e}_b \times \left(\mathbf{r} \times \mathbf{e}_b\right). \tag{4.7}$$

Let the unit vectors directed along the axes of a rectangular Cartesian coordinate system centered at O, be i_1, i_2, i_3. If we write $\mathbf{r}' = x'_k i_k$, $\mathbf{r} = x_k i_k$, $\mathbf{e}_b = \alpha_k i_k$, then by (4.7) Cartesian components of $\mathbf{r}'$ are

$$x'_i = x_i + \sin\theta \epsilon_{ijk} \alpha_j x_k + \left(1 - \cos\theta\right)\left(\alpha_i \alpha_j - \delta_{ij}\right) x_j,$$

[6].

If $x\,\{x'\}$ denotes the initial {final} spatial position vector of P from O and T is a linear form defined by

$$\mathbf{T} = \boldsymbol{\alpha} \times [\quad] \sin\theta + (1 - \cos\theta)\, \boldsymbol{\alpha} \times (\boldsymbol{\alpha} \times [\quad]), \tag{4.8}$$

$$T_{ij} = \epsilon_{ijk}\alpha_k \sin\theta + (1 - \cos\theta)(\alpha_i\alpha_j - \delta_{ij}), \tag{4.9}$$

we call $\mathbf{T}$ the rotator. Here α_k are the direction cosines of $\boldsymbol{\alpha}$, ϵ_{ijk} is the usual permutation symbol, and δ_{ij} is the Kronecker delta.

We observe from (4.9) that the trace of $\mathbf{T}$ is related in a simple way to the angle of rotation i.e.

$$\begin{aligned} tr\mathbf{T} &= T_{kk} = T_{11} + T_{22} + T_{33}, \\ &= (1 - \cos\theta)\left(\alpha_1^2 - \delta_{11} + \alpha_2^2 - \delta_{22} + \alpha_3^2 - \delta_{33}\right), \\ &= (1 - \cos\theta)\left(\left(\alpha_1^2 + \alpha_2^2 + \alpha_3^2\right) - \left(\delta_{11} + \delta_{22} + \delta_{33}\right)\right), \\ &= (1 - \cos\theta)(1 - 3), \\ &= -2(1 - \cos\theta) = 2(\cos\theta - 1), \end{aligned}$$

i.e.

$$tr\mathbf{T} = 2(\cos\theta - 1), \tag{4.10}$$

moreover with the aid of (4.8), we have

$$\mathbf{T}\boldsymbol{\alpha} = 0, \qquad T_{ij}\alpha_j = 0. \tag{4.11}$$

Physically, (4.11) states that *points on the axis of rotation undergo no resultant displacement.*Also we notice from (4.1) that $\mathbf{X}'$ is given by

$$\mathbf{X}' = \mathbf{R}\mathbf{X}, \quad x_i' = R_{ij}x_j, \tag{4.12}$$

where the rotation matrix R is defined by

$$\mathbf{R} = \mathbf{T} + \mathbf{I}, \qquad R_{ij} = T_{ij} + \delta_{ij}, \tag{4.13}$$

and $\mathbf{I}$ is the unit matrix. Since the particle is always at the same distance from O, it follows from (4.12) that

$$\mathbf{R}\mathbf{R}^T = \mathbf{R}^T\mathbf{R} = \mathbf{I} \qquad \text{and} \qquad \det\mathbf{R} = \mathbf{I}, \tag{4.14}$$

which is the condition for the matrix R to be proper orthogonal. In terms of R (4.10) and (4.11) become

$$\mathbf{R}\boldsymbol{\alpha} = \boldsymbol{\alpha} \qquad tr\mathbf{R} = 1 + 2\cos\theta, \tag{4.15}$$

which may be used to determine the angle and the axis of rotation when $\mathbf{R}$ is given [5].

4.4 Euler's theorem

Euler's theorem is based on the rotation of a rigid body and it has many forms given in literature ([5], and [14]).

Goldstein [14] stated Euler's theorem in the following form.

Theorem 12 *[14] The general displacement of a rigid body with one point fixed is a rotation about some axis.*

The theorem means that for every such rotation it is always possible to find an axis through the fixed point oriented at a particular polar angle θ and ϕ such that a rotation by particular angle ψ about this axis duplicates the general rotation. Thus, three parameters (angles) characterize the general rotation [14].

Euler's theorem will be proven if it can be shown that there exists a vector $\mathbf{R}$ having the same components in both systems. Using matrix notation for the vector

$$\mathbf{R}' = \mathbf{A}\mathbf{R} = \mathbf{R}. \tag{4.16}$$

Eq (4.16) for the more general case becomes

$$\mathbf{R}' = \mathbf{A}\mathbf{R} = \boldsymbol{\lambda}\mathbf{R}, \tag{4.17}$$

where $\boldsymbol{\lambda}$ is some constant, which may be complex. The values of $\boldsymbol{\lambda}$ for which (4.17) is soluble are known as the characteristic values, or eigenvalues of the matrix. The problem of finding vectors that satisfy (4.17) is therefore called eigenvalue problem for the given matrix, and the vector solutions are the eigenvectors of $\mathbf{A}$. Goldstein (2002) restated Euler's theorem in the following language:

The real orthogonal matrix specifying the physical motion of a rigid body with one point fixed always has the eigenvalue $+1$.

The eigenvalue equation (4.17) may be written

$$(\mathbf{A} - \boldsymbol{\lambda}\mathbf{I})\,\mathbf{R} = \mathbf{0}.$$

Thus Euler's theorem can be proven directly by using orthogonality property of $\mathbf{A}^T$.

Proof. Consider the expression

$$(\mathbf{A} - \mathbf{I})\,\mathbf{A}^T = \mathbf{I} - \mathbf{A}^T.$$

On taking determinant on the both sides,

$$|\mathbf{A} - \mathbf{I}||\mathbf{A}^{\mathbf{T}}| = |\mathbf{I} - \mathbf{A}^{\mathbf{T}}|. \tag{4.18}$$

To describe the motion of a rigid body, the matrix $\mathbf{A}\,(t)$ must correspond to a proper rotation; therefore $|\mathbf{A}^{\mathbf{T}}| = |\mathbf{A}| = +\mathbf{1}$. Further, since in general $|\mathbf{A}^{\mathbf{T}}| = |\mathbf{A}|$ and $|\mathbf{A}| = \mathbf{1}$ so (4.18) become:

$$|\mathbf{A} - \mathbf{I}| = |\mathbf{I} - \mathbf{A}^{\mathbf{T}}|, \tag{4.19}$$

$$|\mathbf{A} - \mathbf{I}| = |\,(\mathbf{I} - \mathbf{A})^{\,\mathbf{T}}| = |\mathbf{I} - \mathbf{A}|.$$

Suppose $\mathbf{B}$ is some $n \times n$ matrix. Then

$$|-\mathbf{B}| = (-1)^n\,|\mathbf{B}|,$$

Since $\mathbf{A}$ is in three dimensional space ($n = 3$), (4.19) hold for any arbitrary proper rotation only if

$$|\mathbf{A} - \mathbf{I}| = (-\mathbf{1})^{\mathbf{3}}\,||\mathbf{A} - \mathbf{I}|,$$

$$\Longrightarrow \quad |\mathbf{A} - \mathbf{I}| = -|\mathbf{A} - \mathbf{I}|,$$

$$\Longrightarrow |\mathbf{A} - \mathbf{I}| = \mathbf{0}. \tag{4.20}$$

If

$$(\mathbf{A} - \boldsymbol{\lambda}\mathbf{I})\,\mathbf{R} = \mathbf{0},$$

is a homogeneous equation we can have a non trtrival solution only when determinant of the coefficients vanishes i.e.

$$|\mathbf{A} - \boldsymbol{\lambda}\mathbf{I}| = \mathbf{0}. \tag{4.21}$$

Comparing (4.20) with (4.21), we can see that one of the eigenvalues satisfying eq (4.21) must always be $\lambda = +1$, which is the desired result of Euler's theorem [14]. ■

4.5 Cowin-Mehrabadi theorem for an axis of symmetry

Ahmad [3] proved a useful theorem for an axis of symmetry, which we describe here briefly. As we have seen in this dissertation in section 3.3 of third chapter that the well known Cowin-Mehrabadi theorem deals with necessary and sufficient conditions for a normal $\mathbf{n}$ to a symmetry plane, the necessary condition requires that $\mathbf{n}$ be a common eigenvector of C_{ijkk} and C_{ikjk}. It is shown that a vector $\mathbf{p}$ parallel to an axis of symmetry must also satisfy this condition. This condition is generalized so that $\mathbf{n}$ and $\mathbf{p}$ must be eigenvectors of any tensor of rank two associated with the material. This condition is also sufficient for a vector to be an axis of symmetry.

A_n is said to be a n-fold *axis of symmetry*, if a crystal remain unchanged after rotation of an angle $2\pi/n$ about a vector $\mathbf{p}$. An axis of rotational symmetry is also a normal to a plane of symmetry, but it is not true for the trigonal material.

Euler's theorem characterizes the three dimensional rotation by an angle θ about a specific axis. Mehrabadi and Cowin [21] developed a formula from this theorem which is represented by the action of a three dimensional orthogonal tensor $\mathbf{Q}$ .i.e.

$$\mathbf{Q} = \mathbf{I} + \sin\theta\mathbf{P} + (1-\cos\theta)\,\mathbf{P}^2 = \mathbf{e}^{\theta\mathbf{P}}, \tag{4.22}$$

where $\mathbf{P}$ is a three dimensional skew-symmetric tensor which represents a unit vector $\mathbf{p} = (p_1, p_2, p_3)^T$

$$\mathbf{P} = \begin{bmatrix} 0 & -p_3 & p_2 \\ p_3 & 0 & -p_1 \\ -p_2 & p_1 & 0 \end{bmatrix} \quad \text{or} \quad P_{ij} = -\varepsilon_{ijk}p_k. \tag{4.23}$$

A unit vector $\mathbf{p}$ indicates that a specific axis of rotation and P_{ij} represents the components of $\mathbf{P}$, and $e^{\theta\mathbf{P}}$ is the exponential matrix function. Equation (4.22) is equivalent to (4.9) above.

It is worthwhile to note that $\mathbf{P}$ has the following important properties :

$$\mathbf{P} = -\mathbf{P}^T,\ \mathbf{Pp} = \mathbf{0},\ \mathbf{P}^3 = -\mathbf{P}. \tag{4.24}$$

$\mathbf{Q}$ is an orthogonal tensor i.e.

$$\mathbf{Q}\mathbf{Q}^T = \mathbf{I}. \tag{4.25}$$

Orthogonality of $\mathbf{Q}$ can be proven in the following way, as

$$\mathbf{Q} = \mathbf{I} + \sin\theta\mathbf{P} + (1-\cos\theta)\,\mathbf{P}^2,$$

and

$$\mathbf{Q}^T = \mathbf{I} + \sin\theta\mathbf{P}^T + (1-\cos\theta)\,\mathbf{P}^{2T},$$

so

$$\begin{aligned}\mathbf{Q}\mathbf{Q}^T &= \mathbf{I} + \sin\theta\mathbf{P}^T + (1-\cos\theta)\,\mathbf{P}^{2T} + \sin\theta\mathbf{P} + \sin^2\theta\mathbf{P}\mathbf{P}^T + \sin\theta\,(1-\cos\theta)\,\mathbf{P}\mathbf{P}^{2T}\\ &\quad + (1-\cos\theta)\,\mathbf{P}^2 + \sin\theta\,(1-\cos\theta)\,\mathbf{P}^T\mathbf{P}^2 + (1-\cos\theta)^2\,\mathbf{P}^{T2}\mathbf{P}^2,\end{aligned}$$

using (4.24), we get

$$\begin{aligned}\mathbf{Q}\mathbf{Q}^T &= \mathbf{I} - \sin\theta\mathbf{P} + (1-\cos\theta)\,\mathbf{P}^2 + \sin\theta\mathbf{P} - \sin^2\theta\mathbf{P}^2 + \sin\theta\,(1-\cos\theta)\,\mathbf{P}^3,\\ &\quad + (1-\cos\theta)\,\mathbf{P}^2 - \sin\theta\,(1-\cos\theta)\,\mathbf{P}^3 + (1-\cos\theta)^2\,\mathbf{P}^4,\\ &= \mathbf{I} + 2\,(1-\cos\theta)\,\mathbf{P}^2 - \sin^2\theta\mathbf{P}^2 - (1-\cos\theta)^2\,\mathbf{P}^2,\\ &= \mathbf{I} + \left(2 - 2\cos\theta - \sin^2\theta - 1 - \cos^2\theta + 2\cos\theta\right)\mathbf{P}^2,\\ &= \mathbf{I},\end{aligned}$$

Thus $\mathbf{Q}\mathbf{Q}^T = \mathbf{I}$, it has eigenvalues $1, e^{i\theta}, e^{-i\theta}$ [14]. The eigenvector corresponding to 1 is the vector $\mathbf{p}$ which is clear from second equation of (4.24). Now consider a tensor $\mathbf{T}$ of rank two associated with an elastic material, it may be $\mathbf{U} = \mathbf{C}_{ijkk}$, $\mathbf{V} = \mathbf{C}_{ikjk}$, $\mathbf{G} = \frac{1}{2}\epsilon_{ijk}\epsilon_{lmn}C_{iljm}$, $\mathbf{H} = \mathbf{C}_{ijkl}C_{ijkm}$, the dielectric tensor ϵ, the conductivity tensor k, $\mathbf{J} = \boldsymbol{\epsilon}_{ip}\boldsymbol{\epsilon}_{jp}$, $\mathbf{K} = \boldsymbol{\epsilon}_{ip}\mathbf{k}_{jp}$, $\mathbf{L} = \mathbf{C}_{ijkl}\epsilon_{kl}$ or some other tensor of rank two which can be associated with the anisotropic material under consideration. Among the above tensors $\mathbf{G}$ is

$$\mathbf{G} = \begin{bmatrix} c_{23}-c_{44} & c_{45}-c_{36} & c_{46}-c_{25}\\ c_{45}-c_{36} & c_{13}-c_{55} & c_{56}-c_{14}\\ c_{46}-c_{25} & c_{56}-c_{14} & c_{12}-c_{66}\end{bmatrix}. \tag{4.26}$$

For a tensor $\mathbf{T}$ of rank two associated with an elastic material, if $\mathbf{p}$ is an axis of symmetry for the material, the tensor $\mathbf{T}$ should be invariant with respect to the similarity transformation associated with $\mathbf{Q}$ i.e.

$$\mathbf{T} = \mathbf{Q}\mathbf{T}\mathbf{Q}^{-1} \tag{4.27}$$

Therefore $\mathbf{T}\mathbf{Q} = \mathbf{Q}\mathbf{T}$.

Ahmad [3] stated an important theorem in respect of axis of rotation which we review as follows.

Theorem 13 *[3] A necessary and sufficient condition for a vector* $\mathbf{p}$ *to be an axis of symmetry for a material is that any tensor of rank two, independent of* $\mathbf{p}$*, associated with that material has* $\mathbf{p}$ *as an eigenvector.*

Proof. Let $\mathbf{T}$ be a tensor. Since $\mathbf{T}$ commutes with $\mathbf{Q}$ which has $\mathbf{p}$ as a nondegenerate eigenvector corresponding to the eigenvalue 1, $\mathbf{p}$ must also be an eigenvector of $\mathbf{T}$ [Friedman (1966)]. ■

In component form (4.22)is written as

$$Q_{ij} = \delta_{ij} + \epsilon_{ijk} p_k \sin\theta + (1 - \cos\theta)\,(p_i p_j - \delta_{ij})\,,$$

further it can be written as

$$Q_{ij} = \delta_{ij} + \epsilon_{ijk} p_k \sin\theta + (1 - \cos\theta)\, p_i p_j - \delta_{ij} + \cos\theta \delta_{ij},$$

$$Q_{ij} = \delta_{ij}\cos\theta + (1 - \cos\theta)\, p_i p_j + \epsilon_{ijk} p_k \sin\theta. \tag{4.28}$$

[6].

As in chapter 3 of this dissertation, we have seen that $\mathbf{U}, \mathbf{V}$ or their linear combination, which are independent of $\mathbf{n}$, must have $\mathbf{n}$ as a common eigenvector. Ahmad [3] gave generalization of this fact and proved that: *"Any tensor of rank two, independent of n, associated with the material will have n as an eigenvector."*

4.6 How to determine an axis of symmetry

The 6×6 matrices C contain 21 independent elastic constants, the number of independent constants are reduced when the material possesses a certain material symmetry. Written as C_{ijks} the elastic stiffness are component of a fourth rank tensor. Under an orthogonal transformation

$$x' = Qx \quad \text{or} \quad x'_i = Q_{ij} x_j, \tag{4.29}$$

in which $\mathbf{Q}$ is an orthogonal matrix which satisfies the relation

$$Q_{ij} Q_{kj} = \delta_{ik} = Q_{ji} Q_{jk},$$

$$\mathbf{Q}\mathbf{Q}^T = \mathbf{I} = \mathbf{Q}^T\mathbf{Q}, \quad \det \mathbf{Q} = \mathbf{I}. \tag{4.30}$$

The elastic compliances C'_{ijks} referred to the x'_i coordinate system are

$$C'_{ijks} = Q_{ia}Q_{jb}Q_{kr}Q_{st}C_{abrt},$$

where summation over repeated indices is taken and free indices take the value $1, 2, 3$, where

$$C'_{ijks} = C_{ijks},$$

i.e

$$C_{ijks} = Q_{ia}Q_{jb}Q_{kr}Q_{st}C_{abrt}, \tag{4.31}$$

the material is said to possess a symmetry w.r.t $\mathbf{Q}$.

If $\mathbf{Q}$ is proper orthogonal, the transformation (4.29) represents a rigid body rotation about an axis. When (4.31) is satisfied by a proper orthogonal $\mathbf{Q}$, the material possesses rotational symmetry, where (4.30) is the condition for the matrix $\mathbf{Q}$ to be proper orthogonal.

With aid of Euler's theorem, three dimensional orthogonal tensor $\mathbf{Q}$, skew symmetric three dimensional tensor P, and the rotation matrix $\mathbf{Q}$ is may be defined as

$$Q_{ij} = T_{ij} + \delta_{ij},$$

where

$$T_{ij} = \epsilon_{ijk}p_k \sin\theta + (1 - \cos\theta)(p_i p_j - \delta_{ij}).$$

Ahmad [3] proved Theorem 4 when $\mathbf{n}$ and $\mathbf{m}$ are replaced by $\mathbf{p}$ in first three conditions of Theorem 4.

Theorem 14 *[3] A necessary condition for a vector* $\mathbf{p}$ *to be an axis of symmetry is that it is a common eigenvector of* $\mathbf{U}, \mathbf{V}$ *and* $\mathbf{W}(\mathbf{p})$ *as defined*

$$\begin{aligned} U_{ij} &= C_{ijkk}, \\ V_{ij} &= C_{ikjk}, \\ W_{ik}(\mathbf{p}) &= C_{ijkl}p_j p_l. \end{aligned}$$

Proof. An arbitrary vector is indeed a 1-fold axis of symmetry. It may or may not be an eigenvector of any tensor of rank two associated with the material, here we will not discuss this case. If a rotation through an angle θ leaves a material invariant, we must have

$$C_{ijkl} = Q_{ir}Q_{js}Q_{kt}Q_{lu}C_{rstu}. \tag{4.32}$$

Multiply boyh sides with $p_j p_k p_l$ and sum over repeated indices. Noting the fact $Q_{vw} p_v = p_w$, the above equation becomes

$$\begin{aligned} C_{ijkl} p_j p_k p_l &= Q_{ir} \left(Q_{js} p_j \right) \left(Q_{kt} p_k \right) \left(Q_{lu} p_l \right) C_{rstu}, \\ &= Q_{ir} p_s p_t p_u C_{rstu}. \end{aligned}$$

Substituting from (4.28) and rearranging the terms, we have

$$\begin{aligned} C_{ijkl} p_j p_k p_l &= Q_{ir} p_s p_t p_u C_{rstu}, \\ &= \left[\delta_{ir} \cos\theta + (1 - \cos\theta) p_i p_r + \epsilon_{irk} p_k \sin\theta \right] p_s p_t p_u C_{rstu}, \\ &= \delta_{ir} \cos\theta p_s p_t p_u C_{rstu} + (1 - \cos\theta) p_i p_r p_s p_t p_u C_{rstu} \\ &\quad - \sin\theta \epsilon_{irk} p_k p_s p_t p_u C_{rstu}, \end{aligned}$$

for $r = i$, we have

$$\begin{aligned} C_{ijkl} p_j p_k p_l &= \cos\theta p_s p_t p_u C_{istu} + (1 - \cos\theta) p_i p_r p_s p_t p_u C_{rstu} \\ &\quad - \sin\theta \epsilon_{irk} p_k p_s p_t p_u C_{rstu}, \end{aligned}$$

in the first term s, i, and u are dummy indices, so change them by j, k, and l, respectively, we have

$$\begin{aligned} C_{ijkl} p_j p_k p_l (1 - \cos\theta) &= (1 - \cos\theta) p_i p_r p_s p_t p_u C_{rstu} \\ &\quad - \sin\theta \epsilon_{irk} p_k p_s p_t p_u C_{rstu}, \end{aligned}$$

or

$$(1 - \cos\theta) \left(C_{ijkl} p_j p_k p_l - C_{rstu} p_r p_s p_t p_u p_i \right) + \sin\theta \epsilon_{irk} p_k p_s p_t p_u C_{rstu} = 0. \quad (4.33)$$

There are two possible ways of satisfying the above equation. First by requiring $\sin\theta = 0$ and $1 - \cos\theta = 0$. This leads to the case of $\theta = 2\pi$ i.e. a 1-fold rotation axis, which we have excluded here. Otherwise we must require that the vector $C_{ijkl} p_j p_k p_l$ be equal to $C_{rstu} p_r p_s p_t p_u p_i$. This implies that $\mathbf{p}$ is an eigenvector of $\mathbf{W}(\mathbf{p})$, where $W_{ik}(\mathbf{p}) = C_{ijkl} p_j p_l$. Now $C_{rstu} p_s p_t p_u = \lambda p_r$, where $\lambda = C_{rstu} p_r p_s p_t p_u$ and the second term on the right in the equation (4.33) vanishes since $\epsilon_{irk} p_k p_r = 0$. To prove that $\mathbf{p}$ is also an eigenvector of $\mathbf{U}$ and $\mathbf{V}$, the above argument may be repeated with the tensors C_{ijrr} and C_{irjr}. This proves the theorem. ■

4.7 Representation of the elasticity tensor of an anisotropic material possessing an axis of symmetry

As we have already discussed in first chapter of this dissertation that matrices representing triclinic, monoclinic, orthorhombic, cubic and isotropic materials are found in a straightforward manner, the same task for the trigonal, tetragonal and hexagonal materials requires some ingenuity. Some standard text-books discuss the problem only partially and leave unanswered some questions related to matrix representation, especially of the trigonal system [18], [19].

In general, it would be natural to first consider the simpler case of the effect of an axis of symmetry on a tensor of rank 2. Next, we shall show that much of the information related to the representation of the trigonal, tetragonal and hexagonal materials can be gained by simply reducing the problem to a consideration of tensors of rank 2.

4.7.1 Matrix representation of trigonal, tetragonal and hexagonal materials

Let us consider a material possessing an axis of symmetry A_n with $n = 3, 4$ or 6. We choose a rectangular coordinate system so that x_3-axis is aligned with the axis of symmetry. Let **B** be a tensor of rank 2 associated with the material. It is easily shown that, the matrix representation of **B** will be [11]

$$(\mathbf{B}) = \begin{bmatrix} b_{11} & 0 & 0 \\ 0 & b_{11} & 0 \\ 0 & 0 & b_{33} \end{bmatrix}. \tag{4.34}$$

Now define a tensor **G** such that $G_{kn} = \frac{1}{2}\, \epsilon_{ijk}\epsilon_{lmn}C_{iljm}$, where ϵ_{ijk} are the components of the alternating tensor. The matrix representation of **G** is

$$(\mathbf{G}) = \begin{pmatrix} c_{23} - c_{44} & c_{45} - c_{36} & c_{46} - c_{25} \\ c_{45} - c_{36} & c_{13} - c_{55} & c_{56} - c_{14} \\ c_{46} - c_{25} & c_{56} - c_{14} & c_{12} - c_{66} \end{pmatrix} \tag{4.35}$$

This tensor was used by Ahmad [1] in connection with quadratic invariants of the elasticity tensor. On comparison with (4.34), we find

$$c_{36} = c_{45},\ c_{25} = c_{46},\ c_{14} = c_{56},\ c_{23} - c_{44} = c_{13} - c_{55}. \tag{4.36}$$

Same results can be obtained by a consideration of $U_{ij} = C_{ijkk}$ and $V_{ij} = C_{ikjk}$. For example $U_{12} = 0 = V_{12}$ will give the first relation of (4.36).

Let $\boldsymbol{\tau}$ and $\boldsymbol{\sigma}$ be two tensors of rank two which are independent of each other and the elasticity tensor. They might, for example, be tensors based on measurement of physical data such as the permittivity tensor and the conductivity tensor. But for our purpose, *any* pair of hypothetical tensors will suffice. These tensors will exhibit the symmetry due to A_n and will have matrix representations of the form (4.34). Also the tensors $M_{ij} = \tau_{pq}C_{ijpq}$ and $N_{ij} = \sigma_{pq}C_{ijpq}$ will be similarly represented. Equating M_{12} and N_{12} to zero, we get

$$\tau_{11}(c_{16} + c_{26}) + \tau_{33}c_{36} = 0, \tag{4.37a}$$

and

$$\sigma_{11}(c_{16} + c_{26}) + \sigma_{33}c_{36} = 0. \tag{4.37b}$$

Since tensors $\boldsymbol{\tau}$ and $\boldsymbol{\sigma}$ are independent, the above equations imply

$$c_{36} = 0,\ c_{16} + c_{26} = 0. \tag{4.38}$$

Similarly $M_{13} = 0 = N_{13}$ and $M_{23} = 0 = N_{23}$ yield

$$c_{35} = 0,\ c_{15} + c_{25} = 0,\ c_{34} = 0,\ c_{14} + c_{24} = 0. \tag{4.39}$$

Equations $M_{11} = M_{22}$ and $N_{11} = N_{22}$ respectively give

$$\tau_{11}(c_{11} - c_{22}) + \tau_{33}(c_{13} - c_{23}) = 0,\ \sigma_{11}(c_{11} - c_{22}) + \sigma_{33}(c_{13} - c_{23}) = 0,$$

which imply

$$c_{11} = c_{22},\ c_{13} = c_{23} \tag{4.40}$$

The last equality of (4.40) compared with the last equality of (4.36) gives

$$c_{44} = c_{55}. \tag{4.41}$$

Conditions (4.36), (4.38)-(4.40) are common to all materials possessing an $n-$fold axis of symmetry with $n = 3, 4$ or 6. To proceed further, we have to consider them separately. The component $c_{11} = C_{1111}$ transforms, after a rotation about x_3-axis through θ, (see (3.2)), as

$$c_{11}\cos^4\theta + 4c_{16}\cos^3\theta\sin\theta + 2c_{12}\cos^2\theta\sin^2\theta +$$
$$4c_{66}\cos^2\theta\sin^2\theta + 4c_{26}\cos\theta\sin^3\theta + c_{22}\sin^4\theta.$$

Setting $c_{22} = c_{11}$ in the above expression, (see (4.40)), and equating it to c_{11}, we obtain

$$(c_{11} - c_{12} - 2c_{66})\sin^2\theta\cos^2\theta = 2\cos\theta\sin\theta(c_{16}\cos^2\theta + c_{26}\sin^2\theta). \tag{4.42}$$

A similar calculation with c_{16} gives

$$(c_{11} - c_{12} - 2c_{66})(\sin^3\theta\cos\theta - \cos^3\theta\sin\theta) = 8\sin^2\theta\cos^2\theta c_{16}. \tag{4.43}$$

For a tetragonal material, $\theta = \pi/2$, the above equations are identically satisfied and (4.42) and (4.43) impose no further conditions on the components of the elasticity tensor for this class. However an A_4 axis also behaves as an A_2 axis hence, in addition to (4.36), (4.38)-(4.41), we also have $0 = c_{14} = c_{15} = c_{24} = c_{25} = c_{46} = c_{56}$ i.e.

$$(\mathbf{c})_{\text{monoclinic}} = \begin{bmatrix} c_{11} & c_{12} & c_{13} & 0 & 0 & c_{16} \\ c_{12} & c_{22} & c_{23} & 0 & 0 & c_{26} \\ c_{13} & c_{23} & c_{33} & 0 & 0 & c_{36} \\ 0 & 0 & 0 & c_{44} & c_{45} & 0 \\ 0 & 0 & 0 & c_{45} & c_{55} & 0 \\ c_{16} & c_{26} & c_{36} & 0 & 0 & c_{66} \end{bmatrix} \tag{4.44}$$

Thus the matrix representation of the elasticity tensor for a tetragonal material is the following

$$(\mathbf{c})_{\text{tetragonal}} = \begin{bmatrix} c_{11} & c_{12} & c_{13} & 0 & 0 & c_{16} \\ c_{12} & c_{11} & c_{13} & 0 & 0 & -c_{16} \\ c_{13} & c_{13} & c_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & c_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & c_{44} & 0 \\ c_{16} & -c_{16} & 0 & 0 & 0 & c_{66} \end{bmatrix} \tag{4.45}$$

For a trigonal system, $\sin\theta\cos\theta \neq 0$. Using the condition $c_{16} = -c_{26}$, (see (4.38)), Eqs. (4.42) and (4.43) become

$$\begin{aligned} c_{11} - c_{12} - 2c_{66} &= 2\,(\cot\theta - \tan\theta)c_{16}, \\ (\tan\theta - \cot\theta)(c_{11} - c_{12} - 2c_{66}) &= 8c_{16}. \end{aligned}$$

The determinant of the above system fails to vanish if $\theta = 2\pi/3$. Hence, for a trigonal system

$$c_{16} = 0,\ c_{11} - c_{12} - 2c_{66} = 0. \tag{4.46}$$

The matrix representation of such a system follows

$$(\mathbf{c})_{\text{trigonal}} = \begin{bmatrix} c_{11} & c_{12} & c_{13} & c_{14} & c_{15} & 0 \\ c_{12} & c_{11} & c_{13} & -c_{14} & -c_{15} & 0 \\ c_{13} & c_{13} & c_{33} & 0 & 0 & 0 \\ c_{14} & -c_{14} & 0 & c_{44} & 0 & -c_{15} \\ c_{15} & -c_{15} & 0 & 0 & c_{44} & c_{14} \\ 0 & 0 & 0 & -c_{15} & c_{14} & (c_{11} - c_{12})/2 \end{bmatrix}. \tag{4.47}$$

Finally since an A_6 axis is simultaneously an A_2 axis as well as an A_3 axis, the matrix representing a hexagonal material should incorporate properties of both (4.44) and (4.47). Thus we get

$$(\mathbf{c})_{\text{hexagonal}} = \begin{bmatrix} c_{11} & c_{12} & c_{13} & 0 & 0 & 0 \\ c_{12} & c_{11} & c_{13} & 0 & 0 & 0 \\ c_{13} & c_{13} & c_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & c_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & c_{44} & 0 \\ 0 & 0 & 0 & 0 & 0 & (c_{11} - c_{12})/2 \end{bmatrix}. \tag{4.48}$$

The elasticity tensor for a hexagonal material has five independent components.

Chapter 5

Conclusion

Elastic materials can be divided into eight classes on the basis of their symmetries. We first investigate how to find planes of symmetry of an anisotropic elastic material and later find matrix representations for a trigonal, a tetragonal, or a hexagonal material.

We apply theorem 9 of chapter 3 to determine planes of symmetry. As a starting point we take a common eigenvector of the tensors **U** and **V**, which are defined as

$$U_{ij} = C_{ijkk},\ V_{ij} = C_{ikjk},$$

We consider an unknown material and investigate it for a triclinic, a monoclinic, an orthotropic, a trigonal, a tetragonal, a hexagonal, or a cubic material. We consider several examples in which numerical data is given in the form of 6×6 matrices and it is requried to find the class to which the material belongs. If there exists no symmetry plane, then material belongs to a *triclinic system*, if there exists one symmetry plane, then material belongs to a *monoclinic system,* if there exists three coordinates as a symmetry plane, then material belongs to an *orthorhombic material,* if there exists three symmetry planes, then material belongs to a *trigonal material,* if there exists five symmetry planes, then material belongs to a *tetragonal material,* if any plane containing x_3 axis is symmetry plane, then material belongs to a, in a *hexagonal material,* and if there exists nine symmetry planes, then material belongs to a *cubic material.*

The represention of a triclinic, a monoclinic, an orthorhombic, a cubic or an isotropic material is found in a straightforward manner, but the same task for a trigonal, a tetragonal or a hexagonal material is not so simple. We describe a simple method for finding a 6×6 matrix representation of the elasticity tensor belonging to these classes. Suitable tensors of rank two are formed and their symmetry properties lead to the vanishing of several components and certain relations are obtained among the components of the elasticity tensor belonging to these classes. This work should make the job of a student or a research worker easy to understand the structure of anisotropic materials. A paper based on this work is submitted for publication [4].

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