Bounds for the Stanley Depth of Some Square Free Monomial Ideals

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Dedicated to

My loving Parents
for their Love,
Endless support &
Encouragement.

Abstract

In this thesis we review the paper [11], in which some different bounds related to the Stanley depth of monomial ideal $I \subset S$ are given. For example Stanley depth of monomial ideal denoted by I is less than or equal to the Stanley depth of any prime ideal Ω associated to S/I. Furthermore, if the associated prime ideals of S/I are generated by disjoint sets of variables, then Stanley's conjecture holds for I and S/I.

We find some classes of monomial ideals whose Stanley depth is equal to its lower bound. Furthermore, we also show that Stanley's conjecture holds for these classes of monomial ideals. For these ideals we have also shown that $\operatorname{sdepth}(I) \geq \operatorname{sdepth}(S/I) + 1$. If I is the intersection of four prime ideals that is $I = \bigcap\limits_{j=1}^{4} \Omega_j$ and $\Omega_k \subset \sum\limits_{1=j\neq k}^{4} \Omega_j$ for all $j \neq k$, then we have good bound for Stanley depth of I and S/I. Also if I is the intersection of five prime ideals that is $I = \bigcap\limits_{j=1}^{5} \Omega_j$ and $\Omega_i \subset \sum\limits_{1=j\neq i}^{5} \Omega_j$ for all $i \neq j$, then we have a good bound for Stanley depth of I.

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Sobia Ashraf

Introduction

In [20], Stanley introduced the idea of what is now called the Stanley depth of a $\mathbb{Z}^n$-graded module over a commutative ring. This is combintorial invariant which has some common properties with the homological invariant depth. Stanley conjectured that $\operatorname{depth} M \leq \operatorname{sdepth} M$, but it still remains largely open. Herzog, Vladoiu and Zheng considered the Stanley depth of monomial ideals in [8] and showed that the Stanley depth of a monomial ideal can be computed by partitioning a finite poset associated to the ideal into intervals. The difficulty of computing Stanley depths is one of the main obstacles for verifying the Stanley's conjecture. It is still practically very difficult to find the Stanley depth for modules even for monomial ideals, if the method of [8] is applied. In this thesis we try to find some good bounds for the Stanley depth of some square free monomial ideals.

This thesis consists of four chapters. First chapter is devoted to basic definitions and results that are fundamental to the development later in this project. We review the background on abstract algebra that we need in the study of commutative algebra.

Second chapter summarizes the essentials of the theory of Stanley depth and Stanley decomposition. Here we recall the principle results related to the Stanley depth of some multigraded S-modules. Let $S = K[x_1, \cdots, x_n]$ denotes the polynomial ring in n unknowns over a field K and a finitely generated $\mathbb{Z}^n$-graded S-module is denoted by M. It is well known that for a field K, the Stanley depth of the maximal ideal $(x_1, \cdots, x_n) \subset K[x_1, \cdots, x_n]$ is exactly $\lceil \frac{n}{2} \rceil$ by [2]. In this chapter, some important results about the depth and dimension of monomial ideals and module are given. For general monomial ideals result $\operatorname{sdepth}_{S[x_{n+1}]}(I, x_{n+1}) \leq \operatorname{sdepth}_S I + 1$ given by [10] is also mentioned in this chapter.

In third chapter, we review paper [11]. Let Ω denotes an associated prime ideal of S/I. We already know that $\operatorname{depth}_S S/I \leq \operatorname{depth}_S S/\Omega = \dim S/\Omega$ and $\operatorname{depth}_S I \leq \operatorname{depth}_S \Omega$. In [1], it was proved by Apel that $\operatorname{sdepth}_S S/I \leq \dim\ S/\Omega$

and Theorem 3.1.1 shows that $\operatorname{sdepth}_S I \leq \operatorname{sdepth}_S \Omega$. We denote the minimal monomial generators of I by $G(I)$ and let s denotes the cardinality of $G(I)$. Corollaries 3.1.3 and 3.1.4 says that if there exist an associated prime ideal Ω of S/I with $\operatorname{ht} \Omega = s$ then $\operatorname{sdepth}_S I = n - \lfloor \frac{s}{2} \rfloor$.

Let $\operatorname{Ass} S/I = \{\Omega_1, \cdots, \Omega_r\}$, if $\Omega_j \not\subset \sum\limits_{j \neq k}^{r} \Omega_k$, where $1 \leq j \leq r$ and I is square free ideal. Then by [18], we have $\operatorname{sdepth}_S(I) \geq \operatorname{depth}_S(I)$. Suppose that $G(\Omega_j) \cap G(\Omega_k) = \emptyset$ for all $j \neq k$. In particular, the result mentioned above holds in this situation as it was proved in [15]. Corollary 3.1.7 says that previous result holds even if I is not square free. Furthermore, Theorem 3.2.1 says that $\operatorname{sdepth}_S S/I \geq \operatorname{depth}_S S/I$. Hence in this situation Stanley conjecture holds for I and S/I.

It is very difficult to compute the Stanley depth even we adopt the method mentioned in [8]. So in this chapter, the detail of some results which gives tight bounds are mentioned. If $r = 3$, $a = \operatorname{ht} \Omega_1 \leq b = \operatorname{ht} \Omega_2 \leq c = \operatorname{ht} \Omega_3$ then an upper bound for $\operatorname{sdepth}_S S/I$ is given by $b + \lceil \frac{c}{2} \rceil$ except possibly in the case when $a = b \neq c$, $b = a + 1$ and a is odd when $a < c$ (see Proposition 3.2.4 and Corollary 3.2.3). Lemma 3.2.5 says that $\operatorname{sdepth}_S S/I \geq \min\{a + b, a + \lceil \frac{c}{2} \rceil, \lceil \frac{a}{2} \rceil + \lceil \frac{c}{2} \rceil\}$. Some good upper bounds for the $\operatorname{sdepth}_S I$, when $r = 3$ but $G((\Omega_j))_j$ are not necessarily disjoint, are mentioned in section 3. Bounds given by Lemma 3.3.5, Theorem 3.3.7 and Proposition 3.3.8 are not good but very strong in certain cases as in example 3.3.1, 3.3.2 and 3.3.3. Some good results of Stanley depth are stated in Corollaries 3.1.3, 3.1.4 and 3.2.6.

In fourth chapter, we find some classes of monomial ideals whose Stanley depth is equal to its lower bound. Further, we have shown that Stanley's conjecture holds for these classes. For these ideals Corollary 4.1.4 shows that $\operatorname{sdepth}_S(I) \geq \operatorname{sdepth}_S(S/I)$ $+1$. Let Ω be a prime ideal, $I = \bigcap\limits_{j=1}^{4} \Omega_j$ and $\Omega_k \subset \sum\limits_{1=j \neq k}^{4} \Omega_j$ for all $j \neq k$, Lemma 4.2.1 says that I is a square free monomial ideal and all generators of I are of degree two. Lemma 4.2.2 and 4.2.3 gives good bound for I and S/I respectively. Also if

$I = \bigcap\limits_{j=1}^{5} \Omega_j$ and $\Omega_i \subset \sum\limits_{1=j\neq i}^{5} \Omega_j$ for all $i \neq j$ then Lemma 4.2.4 says that I is a square free monomial ideal and all generators of I are of degree three. Lemma 4.2.5 says that $3 \leq \operatorname{sdepth}(I) \leq \left\lfloor \frac{n+3}{2} \right\rfloor$.

Contents

Chapter 1

Preliminaries

This chapter is devoted to a brief introduction of basic concepts of abstract algebra and commutative algebra. These concepts will be helpful in the results of next chapters.

1.1 Rings, fields and modules

Definition 1.1.1. Let R be a non empty set with two binary operations, called *addition* and *multiplication* and denoted by " + " and ".", respectively. Then the algebraic system $\langle R, +, .\rangle$ is called a ring, if

$\Re_1$. $\langle R, +\rangle$ is an abelian group.

$\Re_2$. Multiplication is associative, i.e $\langle R, .\rangle$ is a semigroup.

$\Re_3$. For all $x, y, z \in R$, *left distributive law*, $x.(y + z) = (x.y) + (x.z)$ and the *right distributive law* $(x + y).z = (x.z) + (y.z)$ hold.

Example 1.1.1. $\langle \mathbb{Z}, +, .\rangle$, $\langle \mathbb{Q}, +, .\rangle$, $\langle \mathbb{R}, +, .\rangle$, and $\langle \mathbb{C}, +, .\rangle$, are rings.

Example 1.1.2. For any non-empty set X, the power set $P(X)$ is a ring, where addition and multiplication is defined as follows:
for all $A, B \in P(X)$,

(i) $A + B = (A \backslash B) \cup (B \backslash A)$,

(ii) $A \cdot B = A \cap B$.

Example 1.1.3. Let $A = \{g \mid g : \mathbb{R} \longrightarrow \mathbb{R}\}$. We define pointwise addition and multiplication on A as:
for all $g, h \in A$, and $r \in \mathbb{R}$

(i) $(g + h)(r) = g(r) + h(r)$,

(ii) $(gh)(r) = g(r)h(r)$.

Definition 1.1.2. A ring R is called a *commutative ring*, if the multiplication in R is commutative. A ring R is said to be *ring with identity*, if R has an identity for multiplication.

Definition 1.1.3. Let R be a ring with identity. An element $a \in R$ is called a *unit* if it has an inverse in R with respect to multiplication.

Definition 1.1.4. Let R be a ring with identity. R is called a *division ring* if every non-zero element of R is a unit. A commutative division ring is called a *field*.

Example 1.1.4. In $\mathbb{Z}$, 3 has no multiplicative inverse, so 3 is not a unit in $\mathbb{Z}$ hence $\mathbb{Z}$ is not a field. 1 and -1 are only units in $\mathbb{Z}$. However, $\mathbb{Q}$ and $\mathbb{R}$ are fields.

Proposition 1.1.1 ([6])**.** *Let R be a ring. Then*

(i) $0.r = r.0 = 0$ *for all* $r \in R$.

(ii) $(-r_1)r_2 = r_1(-r_2) = -(r_1 r_2)$, *for all* $r_1, r_2 \in R$ *(recall* $-r_1$ *is the additive inverse of* r_1.

(iii) $(-r_1)(-r_2) = r_1 r_2$ *for all* $r_1, r_2 \in R$.

(iv) *If* R *has an identity* 1, *then the identity is unique and* $-r = (-1)r$.

Definition 1.1.5. If x and y are two non-zero elements of a ring R such that $xy = 0$, then x and y are called *zero divisors*.

Theorem 1.1.2 ([6])**.** *In the ring* $\mathbb{Z}_n$, *the zero divisors are precisely those non-zero elements that are not relatively prime to* n.

Corollary 1.1.3 ([6])**.** *If* p *is a prime, then* $\mathbb{Z}_p$ *has no zero divisors.*

Definition 1.1.6. A commutative ring with identity $1 \neq 0$ without zero divisors is called an *integral domain*.

Theorem 1.1.4 ([6])**.** *Any finite integral domain is a field.*

Corollary 1.1.5 ([6])**.** *If* p *is a prime, then* $\mathbb{Z}_p$ *is a field.*

Definition 1.1.7. The polynomial ring $R[x]$ in the variable x with coefficients from R is the set of all polynomials $b_n x^n + b_{n-1} x^{n-1} + \cdots + b_1 x + b_0$ with $n \geq 0$ and each $b_j \in R$. If $b_n \neq 0$ then the polynomial is of degree n, $b_n x^n$ is the leading term, and b_n is the leading coefficient. The *polynomial ring* in the variables $x_1, x_2, \cdots, x_n$ with coefficients in R, denoted by $R[x_1, x_2, \cdots, x_n]$, is defined inductively by

$$R[x_1, x_2, \cdots, x_n] = R[x_1, x_2, \cdots, x_{n-1}][x_n].$$

This definition means that we can consider polynomials in n variables with coefficients in R simply as polynomials in one variable (say x_n) but now with coefficients that are themselves polynomials in $n-1$ variables.

Definition 1.1.8. A non-empty subset of a ring R is called a *subring* of R, if it is a ring itself with respect to the same binary operations as R.

Example 1.1.5. Following are examples of subrings:

(1) $\mathbb{Z}$ is a subring of $\mathbb{Q}$.

(2) $T = \{\overline{0}, \overline{2}, \overline{4}, \overline{6}, \overline{8}, \overline{10}\}$ is a subring of $\mathbb{Z}_{12}$ and $S = \{\overline{0}, \overline{4}, \overline{8}\}$ is a subring of T.

(3) $A = \{\overline{0}, \overline{3}\}$ is a subring of $\mathbb{Z}_6$.

Remark 1.1.1. Let S be a subring of a ring R. Then it is possible that:

(1) R has an identity but S does not has an identity.

(2) R does not has identity but S has an identity.

(3) R and S have same identities i.e. $1_R = 1_S$.

(4) R and S have different identities i.e. $1_R \neq 1_S$.

Example 1.1.6. (1) $2\mathbb{Z}$ is a subring of $\mathbb{Z}$. The ring $\mathbb{Z}$ has the identity 1 but $2\mathbb{Z}$ has no identity.

(2) S of Example 1.1.5(2) has an identity which is $\overline{4}$, but T has no identity.

(3) $\mathbb{Q}$ and $\mathbb{Z}$ both have identities which are same.

(4) $\mathbb{Z}_6$ and A of Example 1.1.5(3) both have identities $\overline{1}$ and $\overline{3}$ respectively and $\overline{1} \neq \overline{3}$.

Proposition 1.1.6 ([6])**.** *A non-empty subset S of a ring R is a subring of R if and only if $x - y \in S$ and $xy \in S$ whenever $x, y \in S$.*

Definition 1.1.9. A non-empty set I of a ring R is called an *ideal*, if:

(1) $(I, +)$ is a subgroup of $(R, +)$.

(2) for all $a \in I$, $r \in R$,

  (i) $ra \subset I$,

(ii) $ar \subset I$.

Remark 1.1.2. (1) Every ideal of a ring R is a subring of R.

(2) A subring in general is not an ideal. For instance $\mathbb{Z}$ is a subring of $\mathbb{Q}$ but not an ideal of $\mathbb{Q}$. Because $3 \in \mathbb{Z}$ and $\frac{2}{5} \in \mathbb{Q}$ but $(3)(\frac{2}{5}) = \frac{6}{5} \notin \mathbb{Z}$.

Proposition 1.1.7 ([6]). *A non-empty subset I of a ring R is an ideal in R if and only if for all $a, b \in I$ and $r \in R, a - b \in I, ar \in I, ra \in I$.*

Theorem 1.1.8 ([6]). *Let R be a ring with identity and I is an ideal in R. If I contains a unit element of R, then $I = R$.*

Corollary 1.1.9 ([6]). *The only ideal of a field F are $\{0_F\}$ and F.*

From now onwards, R is a commutative ring with unity.

Definition 1.1.10. Let $I \neq R$ is an ideal in R, then I is called *prime ideal* if $xy \in I$ then either $x \in I$ or $y \in I$ where $x, y \in R$.

Example 1.1.7. We know that $\mathbb{Z} \times \{0\}$ is an ideal in a commutative ring $\mathbb{Z} \times \mathbb{Z}$. Let $(w, x)(y, z) \in \mathbb{Z} \times \{0\}$ then $xz = 0 \in \mathbb{Z}$. It implies that either $x = 0$ or $z = 0$. So $(w, x) \in \mathbb{Z} \times \{0\}$ or $(y, z) \in \mathbb{Z} \times \{0\}$. Hence $\mathbb{Z} \times \{0\}$ is a prime ideal of $\mathbb{Z} \times \mathbb{Z}$.

Definition 1.1.11. An ideal m of a ring R is called *maximal ideal*, if $m \neq R$ and m is not properly contained in any other ideal I of R.

Definition 1.1.12. Let I be an ideal in R. I is called *principal ideal* if it is generated by a single element and it is written as

$$I = \langle x \rangle, \text{ where } x \in R.$$

Definition 1.1.13. An ideal I in a ring R is called *primary ideal* if it satisfies the following conditions

(i) $I \neq R$

(ii) For all $x, y \in R$ if $xy \in I$, then either $x \in I$ or there exist $r \geq 1$ such that $y^r \in I$.

Remark 1.1.3. (1) prime ideals are primary.

(2) Let P is an ideal whose radical is a maximal ideal, then P is a primary ideal.

Definition 1.1.14. Let A be any subset of R. (A) is said to be *the ideal generated* by A if it is the smallest ideal of R containing A. If A is a finite set then (A) is called a *finitely generated ideal.*

Definition 1.1.15. If every ideal of R is finitely generated, then R is called *Noetherian ring.*

Theorem 1.1.10 ([6])**.** *Every ideal in the polynomial ring $S = K[x_1, \cdots, x_n]$ over a field K is finitely generated.*

Definition 1.1.16. A ring that has a unique maximal ideal is called a *local ring.*

Definition 1.1.17. Let R and S be rings. A map $\psi : R \longrightarrow S$ is called *ring homomorphism* if it satisfying:

(1) $\psi(r_1 + r_2) = \psi(r_1) + \psi(r_2)$ for all $r_1, r_2 \in R$

(2) $\psi(r_1 r_2) = \psi(r_1)\psi(r_2)$ for all $r_1, r_2 \in R$.

The *kernel* of the ring homomorphism ψ, denoted as ker ψ, is the set

$$\ker \psi = \{r \in R : \psi(r) = 0_S\}.$$

A bijective ring homomorphism is called an *isomorphism.*

Theorem 1.1.11 ([6])**.** (First Isomorphism Theorem for Rings)*. If $\psi : R \longrightarrow S$ is a homomorphism of rings, then the kernel of ψ is an ideal of R, the image of ψ is a subring of S and $R/\ker\psi \cong \psi(R)$.*

Theorem 1.1.12 ([6])**.** (Second Isomorphism Theorem for Rings). *Let R_1 be a subring and let I be an ideal of R. $R_1 + I = \{r + x : r \in R_1,\ x \in I\}$ is a subring of R, $R_1 \cap I$ is an ideal of R_1 and $(R_1 + I)/I \cong R_1/(R_1 \cap I)$.*

Theorem 1.1.13 ([6])**.** (Third Isomorphism Theorem for Rings). *Let I and J be ideals of R with $I \subseteq J$. Then J/I is an ideal of R/I and $(R/I)/(J/I) \cong R/J$.*

Definition 1.1.18. Let R be a ring. Then a non-empty set M is called an R-module or a module over R if it satisfies the following conditions:

(1) (M,+) is an abelian group,

(2) $R \times M \longrightarrow R$ (the image of (r, m) is denoted by rm) such that for all $r_1, r_2 \in R$ and $m, n \in M$,

  (i) $(r_1 + r_2)m = r_1m + r_2m$,

  (ii) $(r_1r_2)m = r_1(r_2m)$,

  (iii) $r_1(m + n) = r_1m + r_1n$,

  (iv) $1m = m$ for all $m \in M$.

If R is a field F, then the axioms of an R-module coincide with the axioms of a vector space over F, it depicts that *modules over F and vector spaces over F are the same*.

Example 1.1.8. Let R be any ring. Then $M = R$ is an R-module, where action of a ring element on a module element is just the usual multiplication in the ring R. In particular, every field can be considered as a (1-dimensional) vector space over itself.

Example 1.1.9. Let $R = \mathbb{Z}$, let A be any additive abelian group (finite or infinite). Make A into a $\mathbb{Z}$-module as follows: for any $n \in \mathbb{Z}$ and $a \in A$, define

$$na = \begin{cases} a + a + \cdots + a \; (n \,\text{times}) & \text{if } n > 0 \\ 0 & \text{if } n = 0 \\ -a - a - \cdots - a \; (-n \,\text{times}) & \text{if } n < 0 \end{cases}$$

(here 0 is the additive identity of the group A). Above definition of an action of the integers on A makes A into a $\mathbb{Z}$-module, and the module axioms show that this is the only possible action of $\mathbb{Z}$ on A making it a (unital) $\mathbb{Z}$-module. Thus every abelian group is a $\mathbb{Z}$-module. Conversely, if M is any $\mathbb{Z}$-module, a fortiori M is an abelian group, so *$\mathbb{Z}$-modules are the same as abelian groups.* Moreover it is consequently from the definition that *$\mathbb{Z}$-submodules are the same as subgroups.*

Definition 1.1.19. Let R be a ring and M be an R-module. A subgroup N of M for which $rn \in N$, for all $r \in R$ and $n \in N$ is called an *R-submodule* of M.

Remark 1.1.4. Submodules of M are subsets of M which are themselves modules under the restricted operations.

Proposition 1.1.14 ([6])**.** *Let M_1 be a non-empty subset of the R-module M. Then M_1 is a submodule of M if and only if for all $x, y \in M_1, r \in R, x - y \in M_1$ and $xr \in M_1$.*

Definition 1.1.20. Let R be a ring and let M and N be R-modules. A map $\psi : M \longrightarrow N$ is an *R-module homomorphism* if it satisfies the following conditions:

(i) $\psi(m_1 + m_2) = \psi(m_1) + \psi(m_2)$, for all $m_1, m_2 \in M$ and

(ii) $\psi(rm) = r\psi(m)$, for all $r \in R, m \in M$.

Definition 1.1.21. An R-module homomorphism is an *isomorphism* (of R-module) if it is bijective. If $\psi : M \longrightarrow N$ is an R-module isomorphism, then the modules M and N are said to be isomorphic and it is denoted as $M \cong N$.

Example 1.1.10. If R is a ring and $M = R$ is a module over itself, then R-module homomorphism (from R to itself) need not be a ring homomorphisms and ring homomorphisms need not be R-module homomorphisms. For example, when $R = \mathbb{Z}$, the $\mathbb{Z}$-module homomorphism $n \mapsto 2n$ is not a ring homomorphism because $m \mapsto 2m$, $n \mapsto 2n$ and $mn \mapsto 2nm \neq 4nm$. When $R = F[x]$, the ring homomorphism $\psi : g(x) \longrightarrow g(x^2)$ is not an $F[x]$-module homomorphism because $\psi(x.1) = \psi(x) = x^2$ but $x\psi(1) = x$ so $\psi(x.1) \neq x\psi(1)$.

Definition 1.1.22. If $\psi : M \longrightarrow N$ is an R-module homomorphism, then kernel of ψ is the set

$$\ker \psi = \{m \in M : \psi(m) = 0\}.$$

Image of ψ is the set

$$\psi(M) = \{n \in N : n = \psi(m)\}.$$

Theorem 1.1.15 ([6])**.** (First Isomorphism Theorem for Modules)*. Let M, N be R-modules and let $\psi : M \longrightarrow N$ be an R-module homomorphism. Then* $\ker \psi$ *is a submodule of M and $M/\ker \psi \cong \psi(M)$.*

Theorem 1.1.16 ([9])**.** (Second Isomorphism Theorem for Modules)*. Let M_1, M_2 be submodules of the R-module M, and let $M_1 + M_2 = \{m + n : m \in M_1, n \in M_2\}$. Then $M_1 + M_2$ and $M_1 \cap M_2$ are submodules of M and $(M_1 + M_2)/M_2 \cong M_1/(M_1 \cap M_2)$.*

Theorem 1.1.17 ([6])**.** (Third Isomorphism Theorem for Modules)*. Let M be an R-module, and let M_1 and M_2 be submodules of M with $M_1 \subseteq M_2$. Then $(M/M_1)/(M_2/M_1) \cong M/M_2$.*

Definition 1.1.23. An R-module M is said to be *finitely generated* if and only if there exist $b_1, b_2, \cdots, b_n$ in M such that for all $w \in M$, there exist $s_1, s_2, \cdots, s_n \in R$ with $w = s_1b_1 + s_2b_2 + \cdots + s_nb_n$. The set $\{b_1, \cdots, b_n\}$ is referred as generating set for M in this case.

Definition 1.1.24. An R-module M is said to be *Noetherian R-module* or to satisfy the *ascending chain condition on submodules* if every increasing chain of submodules stops, that is,

$$M_1 \subseteq M_2 \subseteq M_3 \subseteq \cdots$$

is an increasing chain of submodules of M, then there is a positive integer r such that $M_k = M_r$ for all $k \geq r$.

Definition 1.1.25. Let M be an R-module. A non-zero element $r \in R$ is said to be M-*regular* if for any $m \in M$, $rm = 0$ implies that $m = 0$. In other words r is not a zero divisor on M.

Definition 1.1.26. A sequence $r = (r_1, \cdots, r_n)$ of elements of R is called an M-regular sequence, if it satisfy the following conditions:

(1) r_j is $M/(r_1, \cdots, r_{j-1})M$-regular for any j;

(2) $M \neq (r)M$.

Such a sequence is also called an M-sequence.

Example 1.1.11. (1) If $S = K[x_1, \cdots, x_n]$ be a module over itself, then $(x_1, \cdots, x_n)$ is a regular sequence on S.

(2) If $I = (x_1x_2^2, x_1^2x_3, x_2^2x_4^2) \subset S = K[x_1, x_2, x_3, x_4]$, then $(x_1 - x_3, x_2 - x_4)$ is a regular sequence on S/I.

1.2 Monomial ideals

Definition 1.2.1. Let $S = K[x_1, \cdots, x_n]$ be the polynomial ring in the n indeterminates x_i over a field K. Let $\mathbb{Z}_+^n$ denotes the set of vectors $a = (a_1, \cdots, a_n)$ with each $a_j \geq 0$. Any product $x_1^{a_1} \cdots x_n^{a_n}$ with $a_j \in \mathbb{Z}_+ \cup \{0\}$ is called a *monomial*. A monomial $w = x_1^{a_1} \cdots x_n^{a_n}$ can be written in the form $w = x^a$ with $a = (a_1, \cdots, a_n) \in \mathbb{Z}_+^n$. Let $\mathcal{W}$ be the set of all monomials of S, then $\mathcal{W}$ is a K-basis of S. In other words,

$g \in S$ can be written as linear combination of monomials with coefficients from field K. Write

$$g = \sum_{w \in \mathcal{W}} a_w w \quad \text{with } a_w \in K.$$

Then the set

$$\text{supp}(g) = \{w \in \mathcal{W} : a_w \neq 0\}$$

is called *support* of g. And the set

$$\text{supp}(w) = \{x_i : x_i \mid w\}$$

is called the *support* of monomial w.

Definition 1.2.2. If an ideal $I \subset S$ is generated by set of monomials, then it is called a *monomial ideal*.

Proposition 1.2.1 ([9])**.** *Every monomial ideal $I \subset S$ has a unique minimal set of monomial generators.*

Usually $G(I)$ denotes the unique minimal set of monomial generators of the monomial ideal I.

Example 1.2.1. Let $S = K[x_1, \cdots, x_5]$ and $G(I) = \{x_1^2x_2, x_3^2x_4^3, x_5^4\}$, then ideal generated by $G(I)$ is called monomial ideal in S.

Definition 1.2.3. A monomial w is called *square free*, if $w = x_{j_1}x_{j_2}\cdots x_{j_r}$ for some $1 \leq j_1 < j_2 < \cdots j_r \leq n$.

Definition 1.2.4. If a monomial ideal $I \subset S$ is generated by square free monomials then I is called *square free monomial ideal*.

Example 1.2.2. Let $S = K[x_1, \cdots, x_4]$, then $w = x_1x_2x_4$ is square free monomial in S and $I = (x_1x_2x_3, x_2x_3x_4, x_1x_3x_4, x_1x_2x_3x_4)$ is square free monomial ideal in S.

Definition 1.2.5. If $J \subset S$ is an ideal, then $\sqrt{J} = \{x \in S : x^l \in J \text{ for some } l > 0\}$ is called the *radical of ideal* J.

Example 1.2.3. In Example 1.2.1, radical of I is (x_1x_2, x_3x_4, x_5).

Proposition 1.2.2 ([9])**.** *Let I and J be monomial ideals. Then $I \cap J$ is a monomial ideal, and its set of generators is defined as*

$$\{\mathrm{lcm}(u, w) : u \in G(I), w \in G(J)\}.$$

Definition 1.2.6. Let I, J are two ideals in S. The set

$$I : J = \{f \in S : fh \in I \text{ for all } h \in J\}$$

is an ideal, called the *colon ideal* of I with respect to J.

Proposition 1.2.3 ([9])**.** *Let I and J be monomial ideals. Then $I : J$ is a monomial ideal, and*

$$I : J = \bigcap_{w \in G(J)} I : (w).$$

Moreover, set of generators of $I : (w)$ is defined as $\{u/\gcd(u, w) : u \in G(I)\}$.

Corollary 1.2.4 ([9])**.** *A square free monomial ideal is an intersection of monomial prime ideals.*

Definition 1.2.7. Let M be an R-module, then a prime ideal Ω is called an associated prime ideal of M if there is an injective morphism of R-modules:

$$\xi : R/\Omega \hookrightarrow M$$

The set of associated primes of M is denoted by $\mathrm{Ass}_R(M)$.

Definition 1.2.8. If P is a primary ideal, then the prime ideal $\Omega = \mathrm{rad}\, P$ is called the *associated prime* of R/P (in short we say that Ω is associated prime of P) and P

is said to be Ω-primary. A representation of an ideal I as an intersection $I = \bigcap\limits_{j=1}^{s} P_j$, where each P_j is a primary ideal is known as *Primary decomposition* of I. Suppose that $\{\Omega_j\} = \text{Ass}(P_j)$. If none of the P_j can be omitted in this intersection and if $\Omega_j \neq \Omega_k$ for all $j \neq k$, then this type of primary decomposition is called *irredundant primary decomposition.* If $I = \bigcap\limits_{j=1}^{s} P_j$ is an irredundant primary decomposition of I, then the P_j is called the *Ω_j-primary components* of I, and we have $\text{Ass}(I) = \{\Omega_1, \cdots, \Omega_s\}$.

Definition 1.2.9. Let R be a Noetherian ring and M a finitely generated R-module. If for a prime ideal $\Omega \subset R$, there exists an element $y \in M$ such that $\Omega = \text{Ann}(y)$, then Ω is said to be an *associated prime ideal* of M.

Example 1.2.4. Let $I = (x_1x_2^2, x_2x_3^3, x_3x_4^3, x_1^2x_2x_3)$, then

$$I = (x_1, x_3) \cap (x_2, x_4^3) \cap (x_1, x_3^3, x_4^3) \cap (x_2^2, x_3) \cap (x_1^2, x_2^2, x_3^3, x_4^3),$$

is irredundant primary decomposition of I and $\text{Ass}(I) = \{(x_1, x_3), (x_2, x_4), (x_1, x_3, x_4), (x_2, x_3), (x_1, x_2, x_3, x_4)\}$.

Corollary 1.2.5 ([9])**.** *Let $I \subset S$ be a monomial ideal, and let $\Omega \in$ Ass(I). Then there exists a monomial w such that $\Omega = I : w$.*

Definition 1.2.10. A ring R is called *graded ring* or H-graded if $(H, +)$ is an abelian semi-group and R is the direct sum of additive subgroups:

$$R = \bigoplus_{m \in H} R_m \text{ (as a group)}$$

such that $R_lR_m \subset R_{l+m}$ for all $l, m \in H$. The elements of R_k are said to be homogeneous of degree k, and R_k is called the homogeneous component of R of degree k.

Definition 1.2.11. If R is an H-graded ring and M is an R-module with a decomposition $M = \bigoplus_{m \in H} M_m$ (as a group) such that $R_l M_m \subset M_{l+m}$ for all $l, m \in H$, then M is called an *H-graded module*.

Example 1.2.5. The polynomial ring $S = \mathbb{Z}[x_1, \cdots, x_n]$ in n unknowns over the ring $\mathbb{Z}$ is a $\mathbb{Z}$-graded ring. Here $S_0 = \mathbb{Z}$ and the homogeneous component of degree k is the subgroup of all $\mathbb{Z}$-linear combinations of monomial of degree k. Polynomial ring is also $\mathbb{Z}^n$-graded. Monomial ideals in a polynomial ring are $\mathbb{Z}^n$-graded modules.

Definition 1.2.12. Let R be a ring. The chain of prime ideals of length k is

$$\Omega_0 \subset \Omega_1 \subset \Omega_2 \subset \cdots \subset \Omega_k = \Omega. \tag{1.2.1}$$

The supremum of lengths of chains of prime ideals as mentioned in (1.2.1) is referred as *krull dimension* of R. $\dim R = \infty$, if there is no upper bound for such chains in R. *Height* of a prime ideal Ω is define as
$\mathrm{ht}(\Omega) = \max\{k :$ there exist a chain of prime ideals $\Omega_0 \subset \Omega_1 \subset \cdots \subset \Omega_k = \Omega\}$.
and
$\dim(R/\Omega) = \max\{k :$ there exist a chain of prime ideals $\Omega = \Omega_0 \subset \Omega_1 \subset \cdots \subset \Omega_k\}$. Height of any ideal I is defined as $\mathrm{ht}(I) = \min\{\mathrm{ht}(\Omega) : \Omega$ is prime ideal and $I \subset \Omega\}$. Generally, $\dim(R/I) + \mathrm{ht}(I) \leq \dim(R)$.

Definition 1.2.13. Let R be a ring and M be an R-module, then *annihilator* of module M is defined as:

$$\mathrm{Ann}_R(M) = \{r \in R : rM = 0\}.$$

And *Krull dimension* of M is defined as $\dim(M) = \dim(R/\mathrm{Ann}(M))$. We put $\dim M = -1$ if $M = 0$. Let M be a finitely generated module over the Noetherian ring R. Then $\mathrm{Ass}(M)$ is finite and

$$\dim(M) = \sup\{\dim(R/\Omega) : \Omega \in \mathrm{Ass}(M)\}.$$

Chapter 2

Stanley decomposition and the Stanley depth

This chapter includes the discussion about Stanley decomposition, Stanley depth and depth of $\mathbb{Z}^n$-graded S-modules, where S is the polynomial ring in n variables over a field. We also discuss about a conjecture of R. P. Stanley given in [20]. Some results related to Stanley depth and Stanley's conjecture that obtained in recent years are also discussed.

2.1 Stanley decomposition and the Stanley depth

Definition 2.1.1. Let $S = K[x_1, \cdots, x_n]$ is a polynomial ring in n unknowns with coefficients in the field K and M be a finitely generated $\mathbb{Z}^n$-graded S-module. Let $v \in M$ be a homogeneous element in M and Z a subset of the set of variables. The K-subspace of M, generated by all elements vw, where w is a monomial in $K[Z]$, is denoted as $vK[Z]$. If $vK[Z]$ is a free $K[Z]$-module then $\mathbb{Z}^n$-graded K-space $vK[Z] \subset M$ is called the *Stanley space* of dimension $|Z|$. A *Stanley decomposition* of M is a representation of the $\mathbb{Z}^n$-graded K-vector space M as a finite direct sum of Stanley spaces

$$\mathcal{D} : M = \bigoplus_{j=1}^{r} v_j K[Z_j].$$

The number

$$\operatorname{sdepth} \mathcal{D} = \min\{|Z_j| : j = 1, \cdots, r\}$$

is called the Stanley depth of decomposition $\mathcal{D}$ and the number

$$\operatorname{sdepth} M := \max\{\operatorname{sdepth} \mathcal{D} : \mathcal{D} \text{ is a Stanley decomposition of } M\}$$

is called the *Stanley depth* of M.

Now we discuss the Herzog method as described in [8]. By using this method, we can compute the Stanley depth of a square free monomial ideal I using posets. Suppose that set of minimal monomial generators of I is denoted by $G(I) = \{w_1, \cdots, w_k\}$. The characteristics poset of I with respect to $q = (1, \cdots, 1)$ see [8], represented by $\mathcal{P}_I^q$, where

$$\mathcal{P}_I^q = \{Z \subset [n] \mid Z \text{ contains } \operatorname{supp}(w_j) \text{ for some } j\},$$

where $\operatorname{supp}(w_j) = \{l : x_l \mid w_j\} \subseteq [n] := \{1, \cdots, n\}$. For each $X, Y \in \mathcal{P}_I^q$ with $X \subseteq Y$ and the interval $[X, Y]$ is defined as $\{W \in \mathcal{P}_I^q : X \subseteq W \subseteq Y\}$. Suppose that partition of $\mathcal{P}_I^q$ is defined as $\mathcal{P} : \mathcal{P}_I^q = \cup_{j=1}^{s} [W_j, Z_j]$, and for all j, let $w(j) \in \{0, 1\}^n$ be the n-tuple such that $\operatorname{supp}(x^{w(j)}) = W_j$. Then there is a Stanley decomposition $\mathcal{D}(\mathcal{P})$ of I

$$\mathcal{D}(\mathcal{P}) : I = \bigoplus_{j=1}^{s} x^{w(j)} K[\{x_l \mid l \in Z_j\}].$$

By [8], we have

$$\operatorname{sdepth}(I) = \max\{\operatorname{sdepth} \mathcal{D}(\mathcal{P}) \mid \mathcal{P} \text{ is a partition of } \mathcal{P}_I^q\}.$$

The following conjecture is due to Stanley [20]

$$\operatorname{depth} M \leq \operatorname{sdepth} M,$$

for all finitely generated $\mathbb{Z}^n$-graded S-modules M.

Example 2.1.1. Let $S = K[x_1, \cdots, x_5]$ and $I = (x_1x_2, x_3x_4, x_1x_3x_5)$ is a square free monomial ideal in S.

$$\begin{aligned}
\mathcal{D}_1 := {} & x_1x_2K[x_1, x_2] \oplus x_3x_4K[x_3, x_4] \oplus x_1x_3x_5K[x_1, x_2, x_3, x_4, x_5] \oplus x_1x_2x_3K[x_1, \\
& x_2, x_3] \oplus x_1x_2x_4K[x_1, x_2, x_4] \oplus x_1x_2x_5K[x_1, x_2, x_5] \oplus x_1x_3x_4K[x_1, x_3, x_4] \oplus \\
& x_2x_3x_4K[x_2, x_3, x_4] \oplus x_3x_4x_5K[x_3, x_4, x_5] \oplus x_1x_2x_3x_4K[x_1, x_2, x_3, x_4] \oplus x_1 \\
& x_2x_4x_5K[x_1x_2x_4x_5] \oplus x_2x_3x_4x_5K[x_2, x_3, x_4, x_5].
\end{aligned}$$

$$\begin{aligned}
\mathcal{D}_2 := {} & x_1x_2K[x_1, x_2, x_3] \oplus x_3x_4K[x_1, x_3, x_4] \oplus x_1x_2x_5K[x_1, x_2, x_3, x_5] \oplus x_2x_3x_4K \\
& [x_2, x_3, x_4] \oplus x_1x_3x_5K[x_1, x_3, x_4, x_5] \oplus x_3x_4x_5K[x_2, x_3, x_4, x_5] \oplus x_1x_2x_4K[x_1, \\
& x_2, x_3, x_4, x_5].
\end{aligned}$$

$$\begin{aligned}
\mathcal{D}_3 := {} & x_1x_2K[x_1, x_2, x_3, x_4] \oplus x_3x_4K[x_1, x_3, x_4, x_5] \oplus x_1x_2x_5K[x_1, x_2, x_4, x_5] \oplus x_1 \\
& x_2x_5K[x_1, x_2, x_3, x_5] \oplus x_2x_3x_4K[x_2, x_3, x_4, x_5] \oplus x_1x_2x_3x_4x_5K[x_1, x_2, x_3, \\
& x_4, x_5].
\end{aligned}$$

$\operatorname{sdepth}(I) \geq \max\{\operatorname{sdepth}(\mathcal{D}_1), \operatorname{sdepth}(\mathcal{D}_2), \operatorname{sdepth}(\mathcal{D}_3)\} = \max\{2, 3, 4\} = 4$. Since I is not principal ideal, so $\operatorname{sdepth}(I) = 4$.

2.2 Some known values and bounds for Stanley depth

Proposition 2.2.1. [16, Proposition 1.3]. *Let I ba a monomial ideal in S. Then* $\operatorname{sdepth}_S(I) \leq \operatorname{sdepth}_S(I : w)$ *for each monomial* $w \notin I$.

Theorem 2.2.2. [5, Theorem 1.1]. *Let $I \subset S$ be a monomial ideal which is not principal. Assume $I = wI'$, where $w \in S$ is a monomial and $I' = (I : w)$. Then:*

(1) $\operatorname{sdepth}(S/I') = \operatorname{sdepth}(S/I)$

(2) $\operatorname{sdepth}(I^{'}) = \operatorname{sdepth}(I)$

Theorem 2.2.3. [2, Theorem 2.2]. *Let $S = K[x_1, \cdots, x_n]$, where K is a field. If $\boldsymbol{m} = (x_1, \cdots, x_n)$ be the maximal ideal in S, then*

$$\operatorname{sdepth} \boldsymbol{m} = \left\lceil \frac{n}{2} \right\rceil.$$

Theorem 2.2.4. [14, Theorem 2.3]. *Let I be a monomial ideal of S satisfying $|G(I)| = m$. Then*

$$\max\left\{1, n - \left\lfloor \frac{m}{2} \right\rfloor\right\} \leq \operatorname{sdepth} I.$$

Proposition 2.2.5. [5, Proposition 1.2]. *For a monomial ideal $I \subset S$, we have* $\operatorname{sdepth}(S/I) \geq n - G(I)$.

Lemma 2.2.6. [14, Lemma 2.4]. *Let M be a $\mathbb{Z}^n$-graded S-module and suppose that M_1 and M_2 be its two submodules. Let $0 \longrightarrow M_1 \longrightarrow M \longrightarrow M_2 \longrightarrow 0$ be an exact sequence, then*

$$\operatorname{sdepth} M \geq \min\{\operatorname{sdepth} M_1, \operatorname{sdepth} M_2\}.$$

Theorem 2.2.7. [17, Corollary 2.4]. *Let P and $P^{'}$ be two irreducible monomial ideals of S such that P and $P^{'}$ have different associated prime ideals. Then*

$$\operatorname{sdepth}_S S/(P \cap P^{'}) = \max\left\{ \min\left\{ \dim(S/P^{'}), \left\lceil \frac{\dim(S/P) + \dim(S/(P + P^{'}))}{2} \right\rceil \right\}, \min\left\{ \dim(S/P), \left\lceil \frac{\dim(S/P^{'}) + \dim(S/(P + P^{'}))}{2} \right\rceil \right\} \right\}.$$

Definition 2.2.1. Let $(R, \boldsymbol{m}, K)$ be a local Noetherian ring and M a finitely generated R-module. The common length of all maximal M-sequences in $\boldsymbol{m}$ is referred as the *depth* of M and denoted by $\operatorname{depth}(M)$.

Lemma 2.2.8. [17, Lemma 1.2]. *For a monomial primary ideal P in $S = K[x_1, \cdots, x_n]$ we have,* $\operatorname{sdepth} S/P = \dim\ S/P = \operatorname{depth} S/P$.

Lemma 2.2.9. [17, Lemma 1.3]. *Let I and J be two monomial ideals in a polynomial ring $S = K[x_1, \cdots, x_n]$. Then*

$$\operatorname{sdepth}(S/(I \cap J)) \geq \max\{\min\{\operatorname{sdepth}(S/I), \operatorname{sdepth}(I/(I \cap J))\}, \min\{\operatorname{sdepth}(S/J), \operatorname{sdepth}(J/(I \cap J))\}\}.$$

Theorem 2.2.10. [17, Theorem 1.5]. *Let P and $P^{'}$ are the distinct irreducible monomial ideals in a polynomial ring S. Then*

$$\operatorname{sdepth}_S S/(P \cap P^{'}) \geq \max\Big\{\min\Big\{\dim(S/P^{'}), \Big\lceil\frac{\dim(S/P) + \dim(S/(P + P^{'}))}{2}\Big\rceil\Big\}, \min\Big\{\dim(S/P), \Big\lceil\frac{\dim(S/P^{'}) + \dim(S/(P + P^{'}))}{2}\Big\rceil\Big\}\Big\}.$$

Theorem 2.2.11. [10, Theorem 2.1]. *Let I and J be two monomials ideals of S satisfying $I \subset J$ and $\sqrt{I}$ and $\sqrt{J}$ be the radical ideals of I and J, respectively. Then*

$$\operatorname{sdepth}(J/I) \leq \operatorname{sdepth}(\sqrt{J}/\sqrt{I}).$$

Corollary 2.2.12. [10, Corollary 2.2]. *Let $\sqrt{I}$ be a radical of monomial ideal I in S. Then* $\operatorname{sdepth}(S/I) \leq \operatorname{sdepth}(S/\sqrt{I})$ *and* $\operatorname{sdepth}(I) \leq \operatorname{sdepth}(\sqrt{I})$.

Corollary 2.2.13. [10, Corollary 2.3]. *Suppose that I and J be two monomial ideals in S satisfying $I \subset J$, and let radical ideals of I and J are $\sqrt{I}$ and $\sqrt{J}$, respectively. If* $\operatorname{sdepth}(J/I) = \dim(J/I)$, *then* $\operatorname{sdepth}(\sqrt{J}/\sqrt{I}) = \dim(\sqrt{J}/\sqrt{I})$.

Lemma 2.2.14. [10, Lemma 2.5]. *Suppose that P and $P^{'}$ be two primary ideals having $\sqrt{P} = (x_1, \cdots, x_r)$ and $\sqrt{P^{'}} = (x_1, \cdots, x_n)$. Then*

$$\operatorname{sdepth}(P \cap P^{'}) \leq n - \Big\lfloor\frac{r}{2}\Big\rfloor.$$

Lemma 2.2.15. [10, Lemma 2.6]. *Let P and $P^{'}$ be two primary ideals such that $\sqrt{P} = (x_1)$ and $\sqrt{P^{'}} = (x_2, \cdots, x_n)$. Then*

$$\operatorname{sdepth}(P \cap P^{'}) \leq 1 + \Big\lceil\frac{n-1}{2}\Big\rceil.$$

Theorem 2.2.16. [10, Theorem 2.8]. *Let P and $P^{'}$ be two primary ideals such that $\sqrt{P} = (x_1, \cdots, x_r)$ and $\sqrt{P'} = (x_{r+1}, \cdots, x_n)$, where $r \geq 2$ and $n \geq 4$. Then*

$$\operatorname{sdepth}(P \cap P^{'}) \leq \frac{n+2}{2}.$$

Corollary 2.2.17. [10, Corollary 2.9]. *Let P and $P^{'}$ be the distinct irreducible monomial ideals with $\sqrt{P} = (x_1, \cdots, x_r)$ and $\sqrt{P'} = (x_{r+1}, \cdots, x_n)$, where n is odd. Then* $\operatorname{sdepth}(P \cap P^{'}) = \lceil \frac{n}{2} \rceil$.

Corollary 2.2.18. [10, Corollary 2.10]. *Let P and $P^{'}$ be two irreducible monomial ideals with $\sqrt{P} = (x_1, \cdots, x_r)$ and $\sqrt{P'} = (x_{r+1}, \cdots, x_n)$, where n is even. Then*

$$\operatorname{sdepth}(P \cap P^{'}) = \begin{cases} \frac{n}{2} \text{ or } \frac{n}{2}+1, & \text{if } r \text{ is even;} \\ \frac{n}{2}+1, & \text{otherwise.} \end{cases}$$

Lemma 2.2.19. [10, Lemma 2.11]. *Let I be a monomial ideal in polynomial ring S, and let $I^{'} = (I, x_{n+1}) \subset S^{'} = S[x_{n+1}]$. Then*

$$\operatorname{sdepth}_S(I) \leq \operatorname{sdepth}_{S'}(I^{'}) \leq \operatorname{sdepth}_S(I) + 1.$$

Proposition 2.2.20. [10, Proposition 2.13]. *Let P and $P^{'}$ be two primary monomial ideals such that $\sqrt{P} = (x_1, \cdots, x_v)$ and $\sqrt{P'} = (x_{u+1}, \cdots, x_n)$. Suppose that $1 < u \leq v < n$, $n \geq 4$. Then*

$$\operatorname{sdepth}(P \cap P^{'}) \leq \frac{n+v-u+2}{2}.$$

Lemma 2.2.21. [10, Lemma 2.15]. *Let P and $P^{'}$ be two primary monomial ideals satisfying $\sqrt{P} = (x_1, \cdots, x_{n-1})$ and $\sqrt{P'} = (x_2, \cdots, x_n)$. Then*

$$\operatorname{sdepth}(P \cap P^{'}) \leq n - \left\lfloor \frac{n-1}{2} \right\rfloor.$$

Proposition 2.2.22. [10, Proposition 2.16]. *Let P and $P^{'}$ be two primary monomial ideals satisfying $\sqrt{P} = (x_1, \cdots, x_v)$ and $\sqrt{P'} = (x_{u+1}, \cdots, x_n)$, with $1 \leq u \leq v \leq n$. Then*

$$\operatorname{sdepth}(P \cap P^{'}) \leq \min \left\{ n - \left\lfloor \frac{v}{2} \right\rfloor, n - \left\lfloor \frac{n-v}{2} \right\rfloor \right\}.$$

Theorem 2.2.23. [10, Theorem 2.19]. *Let P and $P^{'}$ be two primary monomial ideals satisfying $\sqrt{P} = (x_1, \cdots, x_t)$ and $\sqrt{P'} = (x_{s+1}, \cdots, x_q)$, with $1 < s \leq t < q \leq n, n \geq 4$. Then*

$$\operatorname{sdepth}(P \cap P^{'}) \leq \min\Big\{\frac{2n+t-q-s+2}{2}, n - \Big\lfloor\frac{t}{2}\Big\rfloor, n - \Big\lfloor\frac{q-t}{2}\Big\rfloor\Big\}.$$

Theorem 2.2.24. [4, Theorem 2.1]. *Let I be a complete intersection monomial ideal in S and $\sqrt{I}$ be its radical. Then*

$$\operatorname{sdepth}(I) = \operatorname{sdepth}(\sqrt{I}).$$

Theorem 2.2.25. [13, Theorem 2.2]. *Let $\sqrt{I}$ be a radical of a monomial ideal $I = \bigcap\limits_{j=1}^{r} P_j$ in S satisfying each P_j is irreducible and $G(\sqrt{P_j}) \cap G(\sqrt{P_k}) = \emptyset$ for all $j \neq k$, then*

$$\operatorname{sdepth}(I) = \operatorname{sdepth}(\sqrt{I}).$$

Theorem 2.2.26. [13, Theorem 2.14]. *Let I be a monomial ideal and let $\operatorname{Min}(S/I) = \{\Omega_1, \cdots, \Omega_r\}$ with $\sum\limits_{j=1}^{r} \Omega_j = \boldsymbol{m}$. Let $h_j := |G(\Omega_j) \setminus G(\sum\limits_{j \neq k}^{r} \Omega_k)|$, and $s := |\{h_j : h_j \neq 0\}|$. Suppose that $s \geq 1$. Then*

$$\operatorname{sdepth}(I) \leq \Big(2n + s - \sum_{j=1}^{r} \Omega_j\Big)\Big/2.$$

Theorem 2.2.27. [18, Theorem 2.3]. *Let $I = \bigcap\limits_{j=1}^{r} \Omega_j$ be a reduced intersection of monomial prime ideals in S. Suppose that $\Omega_j \not\subset \sum\limits_{1=k\neq j}^{r} \Omega_k$ for all $j \in [r]$. Then Stanley's Conjecture holds for I, that is*

$$\operatorname{sdepth}_S I \geq r = \operatorname{depth}_S I.$$

Theorem 2.2.28. [12, Theorem 2.3]. *Let $I = \bigcap\limits_{j=1}^{r} P_j$ be the irredundant representation of I as an intersection of primary monomial ideals. Let $\Omega_j := \sqrt{P_j}$. If $\Omega_j \not\subset \sum\limits_{1=j\neq k}^{r-1} \Omega_k$ for all $j \in [r]$ then $\operatorname{sdepth}(S/I) \geq \operatorname{depth}(S/I)$, that is the Stanley's conjecture holds for S/I.*

Chapter 3

Values and bounds for the Stanley depth

In this chapter, we give review of paper [11]. The different bounds for the Stanley depth of a monomial ideal I of a polynomial ring S over a field K are discussed. If the associated prime ideals of S/I are generated by disjoint sets of variables, then it is shown that Stanley's conjecture holds for I and S/I.

3.1 Upper bounds of the Stanley depth of monomial ideals

Theorem 3.1.1. [11, Theorem 2.1]. *Let $I \subset S$ be a monomial ideal such that* $\mathrm{Ass}(S/I) = \{\Omega_1, \cdots, \Omega_r\}$. *Then*

$$\mathrm{sdepth}(I) \leq \min\{\mathrm{sdepth}(\Omega_j), 1 \leq j \leq r\}.$$

Proof. Suppose that $\Omega_j \in \mathrm{Ass}(S/I)$, then Ω_j is a monomial prime ideal and there exists a monomial $v_j \notin I$ such that $I : v_j = \Omega_j$. By [16, Proposition 1.3] $\mathrm{sdepth}(I) \leq$

sdepth$(I : v_j) =$ sdepth(Ω_j). Hence we get

$$\text{sdepth}(I) \leq \min\{\text{sdepth}(\Omega_j), 1 \leq j \leq r\}.$$

□

Corollary 3.1.2. [11, Corollary 2.1]. *Let $I \subset S$ be a monomial ideal such that $\boldsymbol{m} \in \text{Ass}(S/I)$, then* sdepth$(I) \leq \lceil \frac{n}{2} \rceil$.

Proof. Theorem 2.2.3 says that sdepth $\boldsymbol{m} = \lceil \frac{n}{2} \rceil$ and by Theorem 3.1.1 sdepth$(I) \leq$ sdepth $\boldsymbol{m}$, hence we obtain sdepth$(I) \leq \lceil \frac{n}{2} \rceil$. □

Corollary 3.1.3. [11, Corollary 2.2]. *Let $I \subset S$ be a monomial ideal with $|G(I)| = m$, where m is even. Let there exists a prime ideal $\Omega \in \text{Ass}(S/I)$ with $\text{ht}(\Omega) = m$. Then*

$$\text{sdepth}_S(I) = n - \tfrac{m}{2}.$$

Proof. By [14, Theorem 2.3] we have sdepth$(I) \geq n - \frac{m}{2}$. By hypothesis, there exist a prime ideal $\Omega \in \text{Ass}(S/I)$ with $\text{ht}(\Omega) = m$. Since sdepth$(\Omega) = n - \lfloor \frac{\text{ht}\Omega}{2} \rfloor = n - \frac{m}{2}$ then Theorem 3.1.1 implies that sdepth$(I) \leq n - \frac{m}{2}$. Thus we get sdepth$_S(I) = n - \frac{m}{2}$. □

Corollary 3.1.4. [11, Corollary 2.3]. *Let $I \subset S$ be a monomial ideal with $|G(I)| = m$ (m is odd). Let there exists a prime ideal $\Omega \in \text{Ass}(S/I)$ with $\text{ht}(\Omega) \geq m - 1$. Then*

$$\text{sdepth}_S(I) = n - \lfloor \tfrac{m}{2} \rfloor.$$

Proof. By [14, Theorem 2.3] we have sdepth$(I) \geq n - \lfloor \frac{m}{2} \rfloor$.
Case 1 :
Suppose that there exist a prime ideal $\Omega \in \text{Ass}(S/I)$ with $\text{ht}(\Omega) = m - 1$, then sdepth$(\Omega) = n - \lfloor \frac{\text{ht}\Omega}{2} \rfloor = n - \lfloor \frac{m-1}{2} \rfloor = n - \lfloor \frac{m}{2} \rfloor$. So Theorem 3.1.1 implies that sdepth$(I) \leq n - \lfloor \frac{m}{2} \rfloor$.

Case 2 :

Suppose that there exist a prime ideal $\Omega \in \text{Ass}(S/I)$ with $\text{ht}(\Omega) = m$ then $\text{sdepth}(\Omega) = n - \lfloor \frac{\text{ht}\Omega}{2} \rfloor = n - \lfloor \frac{m}{2} \rfloor$. So by Theorem 3.1.1, we have $\text{sdepth}(I) \leq n - \lfloor \frac{m}{2} \rfloor$. Hence we get $\text{sdepth}_S(I) = n - \lfloor \frac{m}{2} \rfloor$. □

Lemma 3.1.5. [11, Lemma 2.1]. *Let* $S = K[x_1, \cdots, x_n]$ *and* $I \subset S^{'} = K[x_1, \cdots, x_m]$, $J \subset S^{''} = K[x_{m+1} \cdots, x_n]$ *be monomials ideals, where* $1 < m < n$. *Then*

$$\text{sdepth}_S(IS \cap JS) \geq \text{sdepth}_{S'}(I) + \text{sdepth}_{S''}(J).$$

Proposition 3.1.6. [11, Proposition 2.1]. *Suppose that* $I \subset S$ *be a monomial ideal and* $\text{Min}(S/I) = \{\Omega_1, \cdots, \Omega_r\}$. *If* $\Omega_i \not\subset \sum_{j \neq i}^{r} \Omega_j$ *for all* $i \in [r]$. *Then* $\text{depth}(I) \leq r$ *and* $\text{depth}(S/I) \leq r - 1$.

Proof. By [8, Lemma 3.6], it is sufficient to consider that $\sum_{j=1}^{r} \Omega_j = \boldsymbol{m}$. Since $\sqrt{I} = \bigcap_{j=1}^{r} \Omega_j$. By [18, Theorem 2.3] and [7, Theorem 2.6] we obtained $\text{depth}(I) \leq \text{depth}(\sqrt{I})$ $= r$ and accordingly $\text{depth}(S/I) \leq r - 1$. □

Corollary 3.1.7. [11, Corollary 2.4]. *Let* $I \subset S$ *be a monomial ideal such that* $I = \bigcap_{j=1}^{r} P_j$ *is a reduced primary decomposition of* I *and* $\Omega_j = \sqrt{P_j}$ *with* $G(\Omega_i) \cap G(\Omega_j) = \emptyset$ *for* $i \neq j$. *Then* $\text{sdepth}(I) \geq r$. *In particular, we can say that Stanley's conjecture holds for* I.

Proof. By [8, Lemma 3.6] it is sufficient to consider the case $\sum_{j=1}^{r} \Omega_j = \boldsymbol{m}$. If $\text{ht}(\Omega_j) = 1$ for all j then I is principal monomial ideal and it follows that $\text{sdepth}(I) = n \geq r$. Now suppose that I is not principal monomial ideal. Since for all j $\text{sdepth}(\Omega_j) \geq 1$ then by using Lemma 3.1.5 by recurrence we obtained $\text{sdepth}(I) \geq r$ and by Proposition 3.1.6 $\text{depth}(I) \leq r$. Hence we get $\text{sdepth}(I) \geq \text{depth}(I)$. □

3.2 Stanley depth of multigraded cyclic modules

Theorem 3.2.1. [11, Theorem 3.2]. *Let $I \subset S$ be a monomial ideal such that $I = \bigcap_{j=1}^{r} P_j$ is a reduced primary decomposition of I and $\Omega_j = \sqrt{P_j}$ with $G(\Omega_i) \cap G(\Omega_j) = \emptyset$ for $i \neq j$. Then* $\operatorname{sdepth}(S/I) \geq r-1$. *In particular, we can say that Stanley's conjecture holds for S/I.*

Proof. Using [8, Lemma 3.6] it is sufficient to consider that $\sum_{j=1}^{r} \Omega_j = (x_1, \cdots, x_n)$. Let $\sum_{j=1}^{r-1} \Omega_j = (x_1, \cdots, x_t)$ and $S^{'} = K[x_1, \cdots, x_t]$. Firstly apply induction on r. If $r = 1$ then I is maximal monomial ideal in S thus $\operatorname{sdepth}(S/I) = 0 = 1 - 1 = r - 1$. Hence result hold for $r = 1$. Fix $r > 1$ and apply induction on $n - t$. Suppose that l be the minimum positive integer such that $x_n^l \in P_r$. Now for some ideal $I_j \subset S^{''} = K[x_1, \cdots, x_{n-1}]$, we define I_j by $I \cap x_n^j S^{''} = x_n^j I_j$. Then

$$S/I = S^{''}/I_0 \oplus x_n(S^{''}/I_1) \oplus \cdots \oplus x_n^{l-1}(S^{''}/I_{l-1}) \oplus x_n^l(S^{''}/I_l)[x_n].$$

Let $J = P_1 \cap P_2 \cap \cdots \cap P_{r-1}$. If $n - t = 1$, then $I = P_1 \cap P_2 \cap \cdots P_{r-1} \cap (x_n)$. It implies that P_r is an (x_n)-primary and hence it is given by a power of x_n, that is $P_r = (x_n^k)$. By [5, Theorem 1.1], [8, Lemma 3.6] and by using induction on r, we have $\operatorname{sdepth} S/(J \cap x_n^k) = \operatorname{sdepth} S/(x_n^k J) = \operatorname{sdepth} S/J = \operatorname{sdepth}(S^{'}/J) + 1 \geq (r-1-1) + 1 = r-1$. If $n - t > 1$, then by induction we have $\operatorname{sdepth}_{S''}(S^{''}/I_j) \geq r-1$ for all $j < l$ and by using [8, Lemma 3.6] we have $\operatorname{sdepth}_{S''}(S^{''}/I_l) = \operatorname{sdepth}_{S'}(S^{'}/I_l) + n - 1 - t \geq r - 2 + n - t - 1 \geq r - 2$ because $t < n$. Then

$$\operatorname{sdepth}_S(S/I) \geq \min\{\{\operatorname{sdepth}_{S''}(S^{''}/I_j)\}_{j=0,1,\cdots,l-1}, 1 + \operatorname{sdepth}_{S''}(S^{''}/I_l)\}.$$

If minimum is $1 + \operatorname{sdepth}_{S''}(S^{''}/I_l)$, then we get $\operatorname{sdepth}_S(S/I) \geq r - 2 + 1 = r - 1$. If the minimum is $\operatorname{sdepth}_{S''}(S^{''}/I_j)$ for some $0 \leq j < l$, then again $\operatorname{sdepth}_S(S/I) \geq r - 1$. Now using Proposition 3.1.6, we obtained $\operatorname{depth}(S/I) \leq r - 1$ this yield $\operatorname{sdepth}(S/I) \geq \operatorname{depth}(S/I)$. □

The lower and upper bounds for the Stanley depth of S/I with unique irredundant presentation of $I = \Omega_1 \cap \Omega_2 \cap \Omega_3$ is given as the intersection of its minimal monomial prime ideals. By [8, Lemma 3.6] it is sufficient to consider that $\Omega_1 + \Omega_2 + \Omega_3 = \boldsymbol{m}$. Let $\mathcal{D} : S/I = \bigoplus_{j=1}^{m} v_j K[Z_j]$ be a Stanley decomposition. Then Z_j cannot have in the same time variables from all $G(\Omega_j)$, otherwise $v_j K[Z_j]$ will not be a free $K[Z_j]$-module.

Lemma 3.2.2. [11, Lemma 3.2]. *Let $\mathcal{D} : S/I = \bigoplus_{j=1}^{r} v_j K[Z_j]$ be a Stanley decomposition of S/I. Suppose that $v_1 = 1$ and $Z_1 \subset (G(\Omega_1) \cup G(\Omega_2)) \setminus G(\Omega_3)$. Then*

$$\operatorname{sdepth}(\mathcal{D}) \leq \max\{\dim(S/(\Omega_2 + \Omega_3)), \dim(S/(\Omega_1 + \Omega_3))\} + \left\lceil \frac{\operatorname{ht}(\Omega_3) - t}{2} \right\rceil,$$

where $t = |G(\Omega_1) \cap G(\Omega_2) \cap G(\Omega_3)|$.

Proof. Suppose that $Z := G(\Omega_3) \setminus (G(\Omega_1) \cap G(\Omega_2))$ and $\Psi : \Omega_3 \cap K[Z] \hookrightarrow S/I$ be the inclusion map given by $K[Z] \hookrightarrow S/I$. Then $\Omega_3 \cap K[Z] = \bigoplus_j \Psi^{-1}(v_j K[Z_j])$. If $\Psi^{-1}(v_j K[Z_j]) \neq 0$, then there exist $v_j f \in v_j K[Z_j]$ with $v_j f \in \Omega_3 \cap K[Z]$. Since all variables of Z_1 are in $\Omega_1 + \Omega_2$ and Ω_3 is the maximal ideal of $K[Z]$, therefore $v_j K[Z_j] \cap \Omega_3 \cap K[Z] \neq 0$ implies that $v_j \neq 1$ and so $v_j \in \Omega_3 \cap K[Z]$. Let $Z_j^{'} = Z_j \cap Z$ and $\Omega_3^{'} = \Omega_3 \cap K[Z]$. Then $\Psi^{-1}(v_j K[Z_j]) = v_j K[Z_j^{'}]$ and we obtained Stanley decomposition of $\Omega_3^{'}$ that is $\Omega_3^{'} = \bigoplus_{v_j \in \Omega_3^{'}} v_j K[Z_j^{'}]$. Since $|Z_j^{'}| \leq \operatorname{sdepth}(\Omega_3^{'}) = |Z| - \lfloor \frac{\operatorname{ht}(\Omega_3^{'})}{2} \rfloor = |Z| - \lfloor \frac{|Z|}{2} \rfloor = \lceil \frac{|Z|}{2} \rceil$. But either $Z_j \subset \Big(G(\Omega_3) \cup G(\Omega_1)\Big) \setminus G(\Omega_2)$ or $Z_j \subset \Big(G(\Omega_3) \cup G(\Omega_2)\Big) \setminus G(\Omega_1)$ because otherwise $v_j K[Z_j]$ will not be a free $K[Z_j]$ module. In the first case, we have $Z_j \subset \Big(G(\Omega_3) \setminus G(\Omega_2)\Big) \cup \Big(G(\Omega_1) \setminus \Big(G(\Omega_2) \cup G(\Omega_3)\Big)\Big)$ and we get $Z_j \subset Z_j^{'} \cup \Big(G(\Omega_1) \setminus \Big(G(\Omega_2) \cup G(\Omega_3)\Big)\Big)$. It follows $|Z_j| \leq |Z_j^{'}| + \Big|\Big(G(\Omega_1) \setminus \Big(G(\Omega_2) \cup G(\Omega_3)\Big)\Big)\Big| \leq \lceil \frac{|Z|}{2} \rceil + \dim(S/(\Omega_2 + \Omega_3))$. Now for the case $Z_j \subset \Big(G(\Omega_3) \cup G(\Omega_2)\Big) \setminus G(\Omega_1)$, we have $Z_j \subset \Big(G(\Omega_3) \setminus G(\Omega_1)\Big) \cup \Big(G(\Omega_2) \setminus \Big(G(\Omega_1) \cup G(\Omega_3)\Big)\Big)$ and we get $Z_j \subset Z_j^{'} \cup \Big(G(\Omega_2) \setminus \Big(G(\Omega_1) \cup G(\Omega_3)\Big)\Big)$. It follows $|Z_j| \leq$

$|Z_j'|+\Big|\Big(G(\Omega_2)\setminus\Big(G(\Omega_1)\cup G(\Omega_3)\Big)\Big)\Big| \leq \lceil\frac{|Z|}{2}\rceil+\dim(S/(\Omega_1+\Omega_3))$. As $|Z| = \mathrm{ht}(\Omega_3)-t$. Hence we get

$$\mathrm{sdepth}(\mathcal{D}) \leq \max\{\dim(S/(\Omega_2+\Omega_3)),\dim(S/(\Omega_1+\Omega_3))\} + \left\lceil\frac{\mathrm{ht}(\Omega_3)-t}{2}\right\rceil.$$

□

Corollary 3.2.3. [11, Corollary 3.5]. *Let $a \leq b \leq c$ be some positive integers with $a+b+c=n$, $\Omega_1 = (x_1,\cdots,x_a)$, $\Omega_2 = (x_{a+1},\cdots,x_{a+b})$, $\Omega_3 = (x_{a+b+1},\cdots,x_{a+b+c})$ prime ideals of $S = K[x_1,\cdots,x_n]$ and $I = \Omega_1\cap\Omega_2\cap\Omega_3$. Then*

$$\mathrm{sdepth}(S/I) \leq b + \left\lceil\frac{c}{2}\right\rceil.$$

Proof. Suppose that $\mathcal{D}: S/I = \bigoplus_{j=1}^{r} v_jK[Z_j]$ be a Stanley decomposition of S/I and $v_1 = 1$ then there arise three cases:

Case 1. If $Z_1 \subset (G(\Omega_1)\cup G(\Omega_2))\setminus G(\Omega_3)$, then by Lemma 3.2.4

$$\begin{aligned}\mathrm{sdepth}(\mathcal{D}) &\leq \max\{\dim(S/(\Omega_2+\Omega_3)),\dim(S/(\Omega_1+\Omega_3))\} + \left\lceil\tfrac{\mathrm{ht}\,(\Omega_3)}{2}\right\rceil \\ &= \max\{a,b\} + \lceil\tfrac{c}{2}\rceil = b + \lceil\tfrac{c}{2}\rceil.\end{aligned}$$

Case 2. If $Z_1 \subset (G(\Omega_1)\cup G(\Omega_3))\setminus G(\Omega_2)$, then by Lemma 3.2.4

$$\begin{aligned}\mathrm{sdepth}(\mathcal{D}) &\leq \max\{\dim(S/(\Omega_1+\Omega_2)),\dim(S/(\Omega_2+\Omega_3))\} + \left\lceil\tfrac{\mathrm{ht}(\Omega_2)}{2}\right\rceil \\ &= \max\{c,a\} + \lceil\tfrac{b}{2}\rceil = c + \lceil\tfrac{b}{2}\rceil.\end{aligned}$$

Case 3. If $Z_1 \subset (G(\Omega_2)\cup G(\Omega_3))\setminus G(\Omega_1)$, then again by Lemma 3.2.4 we get

$$\begin{aligned}\mathrm{sdepth}(\mathcal{D}) &\leq \max\{\dim(S/(\Omega_1+\Omega_2)),\dim(S/(\Omega_1+\Omega_3))\} + \left\lceil\tfrac{\mathrm{ht}(\Omega_1)}{2}\right\rceil \\ &= \max\{c,b\} + \lceil\tfrac{a}{2}\rceil = c + \lceil\tfrac{a}{2}\rceil.\end{aligned}$$

Above three cases imply that

$$\begin{aligned}\mathrm{sdepth}(\mathcal{D}) &\leq \max\{b+\lceil\tfrac{c}{2}\rceil,\ c+\lceil\tfrac{b}{2}\rceil,\ c+\lceil\tfrac{a}{2}\rceil\} \\ &= c+\lceil\tfrac{b}{2}\rceil.\end{aligned}$$

Hence we have

$$\mathrm{sdepth}(S/I) \leq \mathrm{sdepth}(\mathcal{D}) \leq c + \lceil\frac{b}{2}\rceil.$$

□

From following proposition, the above bound can be improved.

Proposition 3.2.4. [11, Proposition 3.2]. *Let $a \leq b \leq c$ be some positive integers with $a+b+c = n$, $\Omega_1 = (x_1, \cdots, x_a)$, $\Omega_2 = (x_{a+1}, \cdots, x_{a+b})$, $\Omega_3 = (x_{a+b+1}, \cdots, x_{a+b+c})$ prime ideals of $S = K[x_1, \cdots, x_n]$ and $I = \Omega_1 \cap \Omega_2 \cap \Omega_3$. Then*

$$\operatorname{sdepth}(S/I) \leq a + \min\left\{b, \left\lceil \frac{c}{2} \right\rceil\right\},$$

except in the case when $a = b \neq c, b = a+1$ and a is odd when $a < c$.

Proof. Suppose that $a = b = c$ it follows from Corollary 3.2.3 $\operatorname{sdepth}_S(S/I) \leq c + \left\lceil \frac{b}{2} \right\rceil = a + \min\{b, \left\lceil \frac{c}{2} \right\rceil\}$. Hence result holds in this case. Now suppose that $a \neq b \neq c$. Let $R' = K[x_2, \cdots, x_n]$ and $R'' = K[x_1, x_{a+1}, \cdots, x_n]$. If $a = 1$, then by using [5, Theorem 1.1(1)] and [17, Corollary 2.4], we have $\operatorname{sdepth}(S/I) = \operatorname{sdepth}(S/(I : x_1)) = 1 + \operatorname{sdepth}_{R'} R'/((\Omega_2 \cap \Omega_3) \cap R' = 1 + \max\{\min\{a+c-1, a-1+\lceil \frac{b}{2} \rceil\}, \min\{a+b-1, a-1+\lceil \frac{c}{2} \rceil\} = a-1+1+\max\{\min\{c, \lceil \frac{b}{2} \rceil\}, \min\{b, \lceil \frac{c}{2} \rceil\} = a + \max\{\lceil \frac{b}{2} \rceil, \min\{b, \lceil \frac{c}{2} \rceil\} = a + \min\{b, \lceil \frac{c}{2} \rceil\}$, so inequality holds in this case. Now assume that $\Omega_1 = (\Omega_1', x_2, \cdots, x_a)$, where $\Omega_1' = (x_1)$ and $J = \Omega_1' \cap \Omega_2 \cap \Omega_3$, that is $JS = I \cap (\Omega_1' S)$. Now assume the following exact sequence

$$0 \longrightarrow I/JS \longrightarrow S/JS \longrightarrow S/I \longrightarrow 0.$$

Using [19, Lemma 2.2] we get

$$\operatorname{sdepth}(S/JS) \geq \min\{\operatorname{sdepth}(I/JS), \operatorname{sdepth}(S/I)\}.$$

Since $I/JS \simeq I/(I \cap \Omega_1')S \simeq (I + \Omega_1')/\Omega_1' \simeq I \cap K[x_2, \cdots, x_n] = (x_2, \cdots, x_a) \cap \Omega_2 \cap \Omega_3$, using [17, Lemma 4.1] by recurrence we have $\operatorname{sdepth}(I/JS) \geq \lceil \frac{a-1}{2} \rceil + \lceil \frac{b}{2} \rceil + \lceil \frac{c}{2} \rceil$. Now $\operatorname{sdepth}(S/JS) = (a-1) + \operatorname{sdepth}(R''/(x_1) \cap \Omega_2 \cap \Omega_3) = (a-1) + \operatorname{sdepth}(R''/(\Omega_2 \cap \Omega_3) : x_1) = a-1+\operatorname{sdepth}(R''/(\Omega_2 \cap \Omega_3) \cap R'') = a-1+1+\min\{b, \lceil \frac{c}{2} \rceil\} = a + \min\{b, \lceil \frac{c}{2} \rceil\}$. It can be observed that $\lceil \frac{a-1}{2} \rceil + \lceil \frac{b}{2} \rceil + \lceil \frac{c}{2} \rceil > a + \min\{b, \lceil \frac{c}{2} \rceil\}$ for all cases except $b = a \neq c, b = a+1$ and a is odd. Therefore, with these exceptions we have sdepth$(I/JS) >$ sdepth(S/JS). It implies that sdepth$(S/I) \leq$ sdepth$(S/JS) = a + \min\{b, \lceil \frac{c}{2} \rceil\}$ except in the cases when $b = a \neq c, b = a + 1$ and a is odd when $a < c$. ☐

Remark 3.2.1. Suppose that $I \subset S$ be a monomial ideal and $\text{Min}(I) = \{\Omega_1, \Omega_2, \Omega_3\}$ satisfying $G(\Omega_j) \cap G(\Omega_k) = \emptyset$ for all $j \neq k$. From [10, Corollary 2.2] sdepth$(S/I) \leq$ sdepth$(S/\Omega_1 \cap \Omega_2 \cap \Omega_3)$ and the upper bounds in Proposition 3.2.4 (including exceptions mentioned in the proposition) and Corollary 3.2.3 are also upper bounds for the Stanley depth of S/I.

Lemma 3.2.5. [11, Lemma 3.3]. *Let $1 \leq a \leq b \leq c$ be some positive integers with $a + b + c = n$, $\Omega_1 = (x_1, \cdots, x_a)$, $\Omega_2 = (x_{a+1}, \cdots, x_{a+b})$, $\Omega_3 = (x_{a+b+1}, \cdots, x_{a+b+c})$ prime ideals of $S = K[x_1, \cdots, x_n]$ and $I = \Omega_1 \cap \Omega_2 \cap \Omega_3$. Then*

$$\text{sdepth}(S/I) \geq \min\left\{a + b,\ a + \left\lceil \frac{c}{2} \right\rceil,\ \left\lceil \frac{b}{2} \right\rceil + \left\lceil \frac{c}{2} \right\rceil\right\}.$$

Proof. We can write a K-linear space S/I in the direct sum of multi-graded modules:

$$S/I = S/\Omega_3 \oplus \Omega_3/(\Omega_2 \cap \Omega_3) \oplus (\Omega_3 \cap \Omega_2)/I$$

and we get

$$\text{sdepth}(S/I) \geq \min\{\text{sdepth}(S/\Omega_3), \text{sdepth}(\Omega_3/(\Omega_2 \cap \Omega_2)), \text{sdepth}((\Omega_2 \cap \Omega_3)/I)\}.$$

sdepth $S/\Omega_3 =$ sdepth $K[x_1, \cdots, x_{a+b}] = a + b$. By second isomorphism theorem $\Omega_3/(\Omega_2 \cap \Omega_3) \simeq (\Omega_2 + \Omega_3)/\Omega_2 \simeq \Omega_3 \cap K[x_1, \cdots, x_a, x_{a+b+1}, \cdots, x_n]$. Since sdepth$(\Omega_3 \cap K[x_1, \cdots, x_a, x_{a+b+1}, \cdots, x_n]) = n - b - \lfloor \frac{\text{ht}\Omega_3}{2} \rfloor = a + b + c - b - \lfloor \frac{c}{2} \rfloor = a + c - \lfloor \frac{c}{2} \rfloor = a + \lceil \frac{c}{2} \rceil$. It implies that sdepth$(\Omega_3/(\Omega_2 \cap \Omega_3)) = a + \lceil \frac{c}{2} \rceil$. Again by second isomorphism theorem $(\Omega_2 \cap \Omega_3)/I \simeq ((\Omega_2 \cap \Omega_3) + \Omega_1)/\Omega_1 \simeq (\Omega_2 \cap \Omega_3) \cap K[x_{a+1}, \cdots, x_n]$, by using [17, Lemma 4.1], sdepth$((\Omega_2 \cap \Omega_3) \cap K[x_{a+1}, \cdots, x_n]) \geq \lceil \frac{b}{2} \rceil + \lceil \frac{c}{2} \rceil$. It implies that sdepth$((\Omega_2 \cap \Omega_3)/I) \geq \lceil \frac{b}{2} \rceil + \lceil \frac{c}{2} \rceil$. Hence we get

$$\text{sdepth}(S/I) \geq \min\left\{a + b,\ a + \left\lceil \frac{c}{2} \right\rceil,\ \left\lceil \frac{b}{2} \right\rceil + \left\lceil \frac{c}{2} \right\rceil\right\}.$$

☐

Corollary 3.2.6. [11, Corollary 3.6]. *With the hypothesis from the Lemma 3.2.5, suppose that* $a \leq \lceil \frac{b}{2} \rceil$. *Then* $\operatorname{sdepth}(S/I) = a + \min\left\{b, \left\lceil \frac{c}{2} \right\rceil\right\}$.

Proof. By Lemma 3.2.5, we have $\operatorname{sdepth}(S/I) \geq \min\left\{a+b,\ a+\left\lceil \frac{c}{2} \right\rceil,\ \left\lceil \frac{b}{2} \right\rceil + \left\lceil \frac{c}{2} \right\rceil\right\} \geq \min\left\{a+b,\ a+\left\lceil \frac{c}{2} \right\rceil\right\} = a + \min\left\{b,\ \left\lceil \frac{c}{2} \right\rceil\right\}$. Also by Proposition 3.2.4 we have $\operatorname{sdepth}(S/I) \leq a+\min\left\{b,\ \left\lceil \frac{c}{2} \right\rceil\right\}$, it follows that $\operatorname{sdepth}(S/I) = a+\min\left\{b,\ \left\lceil \frac{c}{2} \right\rceil\right\}$. □

3.3 Upper bounds for intersection of three prime ideals

Theorem 3.1.1 gives an upper bound for the Stanley depth of any monomial ideal, but in general it is not a good bound. This section contains results related to upper bounds for the Stanley depth of ideals such that their minimal associated primes set having three prime ideals. These bounds are stronger than the bound that mentioned in Theorem 3.1.1. [10, Corollary 2.2] which says that it is fair enough to give an upper bound for the Stanley depth of intersection of three minimal prime ideal. Let $I = \Omega_1 \cap \Omega_2 \cap \Omega_3$, where Ω_1, Ω_2 and Ω_3 are monomial prime ideals of S. Assume that $\Omega_j \subset \Omega_k$ for all $j \neq k$ and $x_r \notin G(I)$ for all $1 \leq r \leq n$ (for upper bound, if $G(I)$ contains some variables then by applying [10, Lemma 2.11] by recurrence $G(I)$ can be reduce to the case when it contains no variable). Using [8, Lemma 3.6] it is enough to consider that $\Omega_1 + \Omega_2 + \Omega_3 = \boldsymbol{m}$. After renumbering the variables assume that $\Omega_1 = (x_1, \cdots, x_b)$, $\Omega_2 = (x_{a+1}, \cdots, x_d)$ and $\Omega_3 = (x_{c+1}, \cdots, x_n, x_1, \cdots, x_e)$, with $a \leq b$, $c \leq d$, $b > e \geq 0$. Consider the following cases:

(1) $e > 0$, $b \leq c$,

(2) $e = 0$, $a = b$, $c = d$,

(3) $e = 0$, $a \leq b$, $c \leq d$.

The purpose of considering the above cases is to find best possible upper bounds that depend on the behavior of these prime ideals.

Lemma 3.3.1. [11, Lemma 4.4]. *Let* $J \subset S^{'} = S[x_{n+1}]$ *be a monomial ideal,* x_{n+1} *being a new variable. If* $J \cap S \neq (0)$*, then* $\operatorname{sdepth}_S(J \cap S) \geq \operatorname{sdepth}_{S[x_{n+1}]} J - 1$.

Proof. Suppose that $\mathcal{D} : J = \bigoplus_{j=1}^{r} v_j K[Z_j]$ be a Stanley decomposition of J such that $\operatorname{sdepth}(J) = \operatorname{sdepth} \mathcal{D}$. Let w be a monomial and $\operatorname{supp}(w) := \{l : x_l | w\}$. We claim that

$$J \cap S = \bigoplus_{n+1 \notin \operatorname{supp}(v_j)} v_j K[Z_j \backslash \{x_{n+1}\}],$$

which is enough. Suppose that $u \in J \cap S$ be a monomial. Then there exist j such that $u \in v_j K[Z_j]$ and so $x_{n+1} \nmid v_j$ then $u \in v_j K[Z_j \backslash \{x_{n+1}\}]$. Hence " $\subset$ " holds and other inclusion is obvious. As $v_j K[Z_j \backslash \{x_{n+1}\}] \cap v_l K[Z_l \backslash \{x_{n+1}\}] \subset v_j K[Z_j] \cap v_l K[Z_l] = 0$ for $j \neq l$, hence proved. □

Corollary 3.3.2. [11, Corallary 4.7]. *Let* $I \subset S$ *be a monomial ideal of* S *and* $I^{'} = (I, x_{n+1}) \subset S^{'} = S[x_{n+1}]$. *Then* $\operatorname{sdepth}_{S'}(I^{'}) \leq \operatorname{sdepth}_S(I) + 1$.

Corollary 3.3.3. [11, Corollary 4.8]. *Let* $0 < e < b < n$ *be some positive integers and suppose that* $I^{'} = \Omega_1 \cap \Omega_2 \cap \Omega_3 \subset S^{'} = S[x_{n+1}]$, *where* Ω_j *are prime monomial ideals with* $\Omega_1 = (x_1, \cdots, x_b)$, $\Omega_2 = (x_{e+1}, \cdots, x_n)$ *and* $\Omega_3 = (x_{b+1}, \cdots, x_n, x_{n+1}, x_1, \cdots, x_e)$. *Then*

$$\operatorname{sdepth}_{S'}(I^{'}) \leq \operatorname{sdepth}(I) + 1,$$

where $I = I^{'} \cap S = (x_1, \cdots, x_b) \cap (x_{e+1}, \cdots, x_n) \cap (x_{b+1}, \cdots, x_n, x_1, \cdots, x_a)$.

Next assume the special kind of ideals that belong to the case (1) by taking $e < 0$, $e = a$ and $b = c$. Then $\Omega_1 = (x_1, \cdots, x_b)$, $\Omega_2 = (x_{e+1}, \cdots, x_n)$ and $\Omega_3 = (x_{b+1}, \cdots, x_n, x_1, \cdots, x_e)$ are prime ideals of S, where $0 < e < b$ and $I = \Omega_1 \cap \Omega_2 \cap \Omega_3$. Then the following results hold.

Proposition 3.3.4. *Let* $0 < e < b$ *and* $I = \Omega_1 \cap \Omega_2 \cap \Omega_3$*, where* $\Omega_1 = (x_1, \cdots, x_b)$*,* $\Omega_2 = (x_{e+1}, \cdots, x_n)$ *and* $\Omega_3 = (x_{b+1}, \cdots, x_n, x_1, \cdots, x_e)$*. Then*

$$I = (x_1 \cdots, x_e) \cap (x_{b+1}, \cdots, x_n) + (x_{e+1}, \cdots, x_b) \cap (x_{b+1}, \cdots, x_n, x_1, \cdots, x_e).$$

Proof.
$$\begin{aligned}\Omega_1 \cap \Omega_2 &= (x_1 \cdots, x_e, x_{e+1}, \cdots, x_b) \cap (x_{e+1}, \cdots, x_b, x_{b+1}, \cdots, x_n)\\ &= (x_1 \cdots, x_e) \cap (x_{b+1}, \cdots, x_n) + (x_{e+1}, \cdots, x_b)\\ \Omega_1 \cap \Omega_2 \cap \Omega_3 &= [(x_1 \cdots, x_e) \cap (x_{b+1}, \cdots, x_n) + (x_{e+1}, \cdots, x_b)] \cap (x_{b+1}, \cdots, x_n,\\ &\quad x_1, \cdots, x_e) = (x_1 \cdots, x_e) \cap (x_{b+1}, \cdots, x_n) + (x_{e+1}, \cdots, x_b) \cap\\ &\quad (x_{b+1}, \cdots, x_n, x_1, \cdots, x_e).\end{aligned}$$

Hence proved. ☐

Lemma 3.3.5. [11, Lemma 4.5]. $\text{sdepth}(I) \leq 2 + \dfrac{\binom{n}{3} - \binom{e}{3} - \binom{b-e}{3} - \binom{n-b}{3}}{\binom{n}{2} - \binom{e}{2} - \binom{b-e}{2} - \binom{n-b}{2}}$ *where* $\binom{s}{t} = 0$ *when* $s < t$.

Proof. It is clear from Proposition 3.3.4 that I is generated by square free monomials of degree 2. Suppose that $t := \text{sdepth}(I)$. $\mathcal{P} : \mathcal{P}_I^q = \bigcup_{j=1}^{r} [W_j, Z_j]$ is the partition of the poset $\mathcal{P}_I^q$ and this partition satisfy $\text{sdepth}(\mathcal{D}(\mathcal{P})) = t$, where $\mathcal{D}(\mathcal{P})$ is the Stanley decomposition of I with respect to the partition $\mathcal{P}$. Each interval $[W_j, Z_j]$ in partition $\mathcal{P}$ satisfying $|W_j| = 2$ has $|Z_j| \geq t$. Each such interval has $|Z_j| - |W_j|$ subsets of cardinality 3. As these intervals are disjoint, therefore by counting the number of subsets of cardinality 2 and 3 we get

$$\left[\binom{n}{2} - \binom{e}{2} - \binom{b-e}{2} - \binom{n-b}{2}\right](t-2) \leq \binom{n}{3} - \binom{e}{3} - \binom{b-e}{3} - \binom{n-b}{3}.$$

It follows that

$$\text{sdepth}(I) \leq 2 + \frac{\binom{n}{3} - \binom{e}{3} - \binom{b-e}{3} - \binom{n-b}{3}}{\binom{n}{2} - \binom{e}{2} - \binom{b-e}{2} - \binom{n-b}{2}}.$$

☐

Example 3.3.1. Let $I = (x_1, x_2, x_3, x_4, x_5) \cap (x_4, x_5, x_6, x_7, x_8) \cap (x_6, x_7, x_8, x_1, x_2, x_3)$ then $e = 3, b = 5$ and $n = 8$ by the above lemma, we get $\operatorname{sdepth}(I) \leq 4$.

Proposition 3.3.6. [11, Proposition 4.3]. *If $0 < e \leq a \leq b \leq c \leq d \leq n$ and $I = \Omega_1 \cap \Omega_2 \cap \Omega_3$, where $\Omega_1 = (x_1, \cdots, x_b)$, $\Omega_2 = (x_{a+1}, \cdots, x_d)$ and $\Omega_3 = (x_{c+1}, \cdots, x_n, x_1, \cdots, x_e)$. Let $g = a - e + c - b + n - d$, $n - g \geq 3$, then*

$$\operatorname{sdepth}(I) \leq 2 + g + \frac{\binom{n-g}{3} - \binom{e}{3} - \binom{b-a}{3} - \binom{d-c}{3}}{\binom{n-g}{2} - \binom{e}{2} - \binom{b-a}{2} - \binom{d-c}{2}}.$$

Proof. Suppose that $J = (x_1, \cdots, x_e, x_{a+1}, \cdots, x_b) \cap (x_{a+1}, \cdots, x_b, x_{c+1}, \cdots, x_d) \cap (x_{c+1}, \cdots, x_d, x_1, \cdots, x_e) \subset R = K[x_1, \cdots, x_e, x_{a+1}, \cdots, x_b, x_{c+1}, \cdots, x_d]$. Then by using Lemma 3.3.5 we have

$$\operatorname{sdepth}_R(J) = 2 + \frac{\binom{n-g}{3} - \binom{e}{3} - \binom{b-a}{3} - \binom{d-c}{3}}{\binom{n-g}{2} - \binom{e}{2} - \binom{b-a}{2} - \binom{d-c}{2}}.$$

Now applying Corollary 3.3.3 by recurrence, we get

$$\operatorname{sdepth}_S(I) \leq \operatorname{sdepth}_R(J) + g.$$

Hence proved. □

Theorem 3.3.7. [11, Theorem 4.3]. *Let $I = \Omega_1 \cap \Omega_2 \cap \Omega_3$, where Ω_1, Ω_2 and Ω_3 are prime monomial ideals of S satisfying $G(\Omega_j) \cap G(\Omega_k) = \emptyset$ for all $j \neq k$, $\Omega_1 + \Omega_2 + \Omega_3 = \boldsymbol{m}$ and $\operatorname{ht}(\Omega_j) = h_j$. Then*

$$\operatorname{sdepth}(I) \leq 3 + \frac{1}{h_1 h_2 h_3}\Big[\binom{n}{4} - \sum_{j=1}^{3}\binom{h_j}{4} - \sum_{j=1}^{3}\binom{h_j}{3}(n - h_j) - \sum_{j<k}\binom{h_j}{2}\binom{h_k}{2}\Big].$$

Proof. We can observe that I is generated by square free monomials of degree 3. Suppose that $t := \text{sdepth}(I)$. $\mathcal{P} : \mathcal{P}_I^q = \bigcup_{j=1}^{r} [W_j, Z_j]$ is the partition of the poset $\mathcal{P}_I^q$ and this partition satisfies $\text{sdepth}(\mathcal{D}(\mathcal{P})) = t$, where $\mathcal{D}(\mathcal{P})$ is the Stanley decomposition of I with respect to the partition $\mathcal{P}$. Since every interval $[W_j, Z_j]$ in partition $\mathcal{P}$ satisfying $|W_j| = 3$ has $|Z_j| \geq t$. Since $|G(I)| = h_1h_2h_3$, so there are $h_1h_2h_3$ such intervals. Each such interval has $|Z_j| - |W_j|$ subsets of cardinality 4. Since these intervals are disjoint, so by counting the number of subsets of cardinality 4, we have

$$h_1h_2h_3(t-3) \leq \left[\binom{n}{4} - \sum_{j=1}^{3}\binom{h_j}{4} - \sum_{j=1}^{3}\binom{h_j}{3}(n-h_j) - \sum_{j<k}\binom{h_j}{2}\binom{h_k}{2}\right].$$

Hence proved. □

Example 3.3.2. Let $I = (x_1^{b_1}, \cdots, x_7^{b_7}) \cap (x_8^{b_8}, \cdots, x_{14}^{b_{14}}) \cap (x_{15}^{b_{15}}, \cdots, x_{21}^{b_{21}})$ for some positive integer b_j. Then by Corollary 3.1.5, we have $\text{sdepth}(I) \geq \lceil \frac{7}{2} \rceil + \lceil \frac{7}{2} \rceil + \lceil \frac{7}{2} \rceil = 12$. Now by Theorem 3.3.7, we get $\text{sdepth}(I) \leq 12$ it follows that $\text{sdepth}(I) = 12$.

Proposition 3.3.8. [11, Proposition 4.4]. *Suppose that* $\Omega_1 = (x_1, \cdots, x_b)$, $\Omega_2 = (x_{a+1}, \cdots, x_d)$ *and* $\Omega_3 = (x_{c+1}, \cdots, x_n)$. *with* $a \leq b$, $c \leq d$, *and* $I = \Omega_1 \cap \Omega_2 \cap \Omega_3$. *Let*

$$k := \min\left\{\frac{2n+b-d-a+2}{2}, \frac{n+d+a-c+2}{2}, n-\left\lfloor\frac{b}{2}\right\rfloor, n-\left\lfloor\frac{d-a}{2}\right\rfloor, n-\left\lfloor\frac{n-c}{2}\right\rfloor, \right\}.$$

Then $\text{sdepth}(I) \leq k$, *if* $b \geq c$, *and*

$$\text{sdepth}(I) \leq \min\left\{k, \frac{n+c-b+2}{2}\right\}, \qquad if \quad b < c.$$

Proof. Suppose that $w = x_n$. Since $I = \Omega_1 \cap \Omega_2 \cap \Omega_3$, so we can observe that $w \notin I$ and $J := I : w = (x_1, \cdots, x_b) \cap (x_{a+1}, \cdots, x_d) \subset R = K[x_1, \cdots, x_d]$ that is $J = \Omega_1 \cap \Omega_2$. Then by [16, Proposition 1.3] $\text{sdepth}_S(I) \leq \text{sdepth}_S(J)$ and applying [8, Lemma 3.6] by recurrence $\text{sdepth}_S(J) = \text{sdepth}_R(J) + n - d$. Using [10, Proposition 2.13] $\text{sdepth}_R(J) = \frac{b+d-a+2}{2}$. Since $\text{sdepth}_R(\Omega_1) = d -$

$\lfloor\frac{\mathrm{ht}\Omega_1}{2}\rfloor = d - \lfloor\frac{b}{2}\rfloor$ and $\mathrm{sdepth}_R(\Omega_2) = d - \lfloor\frac{\mathrm{ht}\Omega_2}{2}\rfloor = d - \lfloor\frac{d-a}{2}\rfloor$. Now by Theorem 3.1.1, $\mathrm{sdepth}_R(J) \leq \min\{\frac{b+d-a+2}{2}, d-\lfloor\frac{b}{2}\rfloor, d-\lfloor\frac{d-a}{2}\rfloor\}$. This yield $\mathrm{sdepth}_S(I) \leq \mathrm{sdepth}_S(J) \leq \min\{\frac{b+d-a+2}{2}+n-d, d-\lfloor\frac{b}{2}\rfloor+n-d, d-\lfloor\frac{d-a}{2}\rfloor+n-d\} = \min\{n+\frac{b-d-a+2}{2}, n-\lfloor\frac{b}{2}\rfloor, n-\lfloor\frac{d-a}{2}\rfloor\}$. Now if we consider $w = x_1$, then again $w \notin I$ and $J^{'} = I : w = (x_{a+1}, \cdots, x_d) \cap (x_{c+1}, \cdots, x_n) \subset R^{'} = K[x_{a+1}, \cdots, x_{c+1}, \cdots, x_n]$, that is $J^{'} = \Omega_2 \cap \Omega_3$. By [16, Proposition 1.3] $\mathrm{sdepth}_S(I) \leq \mathrm{sdepth}_S(J^{'})$ and applying [8, Lemma 3.6] by recurrence $\mathrm{sdepth}_S(J^{'}) = \mathrm{sdepth}_{R'}(J^{'}) + a$. Since $\mathrm{sdepth}_{R'}(\Omega_2) = n-a-\lfloor\frac{\mathrm{ht}\Omega_2}{2}\rfloor = n-a-\lfloor\frac{d-a}{2}\rfloor$ and $\mathrm{sdepth}_{R'}(\Omega_3) = n-a-\lfloor\frac{\mathrm{ht}\Omega_3}{2}\rfloor = n-a-\lfloor\frac{n-c}{2}\rfloor$ using [10, Proposition 2.13] $\mathrm{sdepth}_{R'}(J') \leq \frac{n+d-a-c+2}{2}$. By Theorem 3.1.1 $\mathrm{sdepth}_{R'}(J^{'}) \leq \min\{\frac{n+d-a-c+2}{2}, n-a-\lfloor\frac{d-a}{2}\rfloor, n-a-\lfloor\frac{n-c}{2}\rfloor\}$. It follows $\mathrm{sdepth}_S(I) \leq \mathrm{sdepth}_S(J^{'}) \leq \min\{\frac{n+d-a-c+2}{2}+a, n-a-\lfloor\frac{d-a}{2}\rfloor+a, n-a-\lfloor\frac{n-c}{2}\rfloor+a\} = \min\{\frac{n+d+a-c+2}{2}, n-\lfloor\frac{d-a}{2}\rfloor, n-\lfloor\frac{n-c}{2}\rfloor\}$. Hence $\mathrm{sdepth}_S(I) \leq k$. If $b < c$, then take $w = x_{b+1}$ and we have $J^{''} = I : w = (x_1, \cdots, x_b) \cap (x_{c+1}, \cdots, x_n) \subset R^{''} = K[x_1, \cdots, x_b, x_{c+1}, \cdots, x_n]$ and by [10, Theorem 2.8] it follows $\mathrm{sdepth}_{R''}(J^{''}) \leq \frac{n-(c-b)+2}{2}$. Applying [8, Lemma 3.6] by recurrence, we get

$$\mathrm{sdepth}_S(I) = \mathrm{sdepth}_{R''} J^{''} + c - b \leq \frac{n-(c-b)+2}{2} + c - b = \frac{n+c-b+2}{2}.$$

□

Example 3.3.3. Let $I = (x_1, \cdots, x_7) \cap (x_6, \cdots, x_{12}) \cap (x_{10}, \cdots, x_{15})$, that is $n = 15, a = 5, b = 7, c = 9, d = 12$. Then $k := \min\{11, 12, 12, 12, 12\} = 11$ and by Proposition 3.3.8 $\mathrm{sdepth}(I) \leq 9$.

Chapter 4

Bounds for the Stanley depth of some square free monomial ideals

In this chapter, we show that Stanley conjecture holds for some classes of monomial ideal whose Stanley depth is equal to its lower bound. For these ideals, we show that $\text{sdepth}(I) \geq \text{sdepth}(S/I) + 1$. We also give some good bounds for monomial ideal I, where I is an intersection of four and five prime ideals.

4.1 Some classes of monomial ideals whose Stanley depth is equal to its lower bound

Let $I \subset S = K[x_1, \cdots, x_n]$ be a monomial ideal such that for all x_i there exist a monomial $v \in G(I)$ such that $x_i | v$ and define $l_i = \sup\{j : x_i^j \mid v, \text{ for some } v \in G(I)\}$.

Theorem 4.1.1. *Let $I \subset S = K[x_1, \cdots, x_n]$ be a monomial ideal with $|G(I)| = m$. If there exist $A := \{x_{k_1}, x_{k_2}, \cdots, x_{k_m}\} \subset \{x_1, \cdots, x_n\}$and for each $v \in G(I)$ there exist unique $x_{k_i} \in A$ such that $x_{k_i}^{l_{k_i}} \mid v$ then* $\text{sdepth}_S(I) = n - \lfloor \frac{m}{2} \rfloor$.

Proof. Suppose that $I \subset S = K[x_1, \cdots, x_n]$ be a monomial ideal with $|G(I)| = m$. After renumbering the variables, let $A := \{x_1, \cdots, x_m\} \subset \{x_1, \cdots, x_n\}$ and for each $v \in G(I)$ there exist unique $x_i \in A$ such that $x_i^{l_i} \mid v$. Then there exist $\Omega = (x_1, \cdots, x_m) \in \mathrm{Ass}(S/I)$ with $\mathrm{ht}(\Omega) = m$. Because there exist a monomial $u = x_1^{l_1-1}x_2^{l_2-1}\cdots x_m^{l_m-1}x_{m+1}^{l_{m+1}}\cdots x_n^{l_n}$ such that $I : u = \Omega$. Also $u \notin I$ because there does not exist $x_i \in A$ such that $x_i^{l_i} \mid u$. Then by [11, Corollary 2.2] and [11, Corollary 2.3] $\mathrm{sdepth}_S(I) = n - \lfloor \frac{m}{2} \rfloor$. □

Example 4.1.1. Let $I = (x_1x_2x_3, x_1x_3x_4, x_1x_3x_5, x_1x_3x_6)$ is a monomial ideal in $S = K[x_1, \cdots, x_6]$. Then using Theorem 4.1.1, we have $\mathrm{sdepth}(I) = 4$ and corresponding Stanley decomposition is

$$\begin{aligned} D : I \;=\; & x_1x_2x_3K[x_1,x_2,x_3,x_4] \oplus x_1x_3x_4K[x_1,x_3,x_4,x_5] \oplus x_1x_3x_5K[x_1,x_2,x_3,x_5] \\ & \oplus x_1x_3x_6K[x_1,x_2,x_3,x_6] \oplus x_1x_3x_5x_6K[x_1,x_3,x_4,x_5,x_6] \oplus x_1x_3x_4x_6K[x_1, \\ & x_2,x_3,x_4,x_6] \oplus x_1x_2x_3x_5x_6K[x_1,x_2,x_3,x_4,x_5,x_6]. \end{aligned}$$

Remark 4.1.1. Converse of Theorem 4.1.1 is not true. As shown in the following example.

Example 4.1.2. Let $I = (x_1, x_2x_3, x_3x_4, x_2x_4x_5) \subset S = K[x_1, \cdots, x_5]$, I does not satisfy the hypothesis of Theorem 4.1.1 and primary decomposition of I is $I = (x_1, x_2, x_3) \cap (x_1, x_2, x_4) \cap (x_1, x_3, x_4) \cap (x_1, x_3, x_5)$. But by using cocoa, $\mathrm{sdepth}(I) = 3$ and corresponding Stanley decomposition is

$$\begin{aligned} D : I \;=\; & x_1K[x_1,x_2,x_3,x_4,x_5] \oplus x_3x_4K[x_3,x_4,x_5] \oplus x_2x_3K[x_2,x_3,x_4,x_5] \oplus x_2x_4 \\ & K[x_2,x_4,x_5]. \end{aligned}$$

Theorem 4.1.2. *Let $I \subset S = K[x_1, \cdots, x_n]$ be a monomial ideal with $|G(I)| = m$ (m is odd). If there exist $A := \{x_{k_1}, x_{k_2}, \cdots, x_{k_{m-1}}\} \subset \{x_1, \cdots, x_n\}$ and $B := \{v_{t_1}, v_{t_2}, \cdots, v_{t_{m-1}}\} \subset G(I)$ and for each $v_{t_i} \in B$, there exist unique $x_{k_i} \in A$ such that $x_{k_i}^{l_{k_i}} \mid v_{t_i}$, moreover there exist $x_j \in \{x_1, \cdots, x_n\} \backslash A$ such that $x_j^{l_j} | w$ where $w \in G(I) \backslash B$ and $x_j^{l_j}$ divide only one $v_{t_i} \in B$ then* $\mathrm{sdepth}_S(I) = n - \lfloor \frac{m}{2} \rfloor$.

Proof. Let $I \subset S$ be a monomial ideal with $|G(I)| = m$ (m is odd). After renumbering the variables let $A := \{x_1, \cdots, x_{m-1}\} \subset \{x_1, \cdots, x_n\}$ and $B := \{v_1, \cdots, v_{m-1}\} \subset G(I)$. Suppose that for each $v_i \in B$ there exist unique $x_i \in A$ such that $x_i^{l_i} \mid v_i$ and for $w \in G(I) \backslash B$ there exist $x_j \in \{x_1, \cdots, x_n\} \backslash A$ such that $x_j^{l_j} | w$ and

$x_j^{l_j}|v_k$ where $v_k \in B$. Then $\Omega = (x_1, \cdots, x_{k-1}, x_{k+1}, \cdots, x_{m-1}, x_j) \in \text{Ass}(S/I)$ with $\text{ht}(\Omega) = m-1$. Because there exist a monomial
$u = x_1^{l_1-1}x_2^{l_2-1}\cdots x_{k-1}^{l_{k-1}-1}x_k^{l_k}x_{k+1}^{l_{k+1}-1}\cdots x_{m-1}^{l_{m-1}-1}x_m^{l_m}x_{m+1}^{l_{m+1}}\cdots x_j^{l_j-1}\cdots x_n^{l_n}$
such that $I : u = \Omega$. Also $u \notin I$, because there does not exist $x_i \in A$ such that $x_i^{l_i}|u$ for $i \in \{1, \cdots, m-1\}\backslash\{k\}$, $x^k|u$ but $x_j^{l_j} \nmid u$. By [11, Corollary 2.3], we have $\text{sdepth}_S(I) = n - \lfloor \frac{m}{2} \rfloor$. □

Example 4.1.3. Let $I = (x_1x_2x_3, x_1x_3x_4, x_1x_3x_5, x_1x_3x_6, x_2x_3x_7)$ is a monomial ideal in $S = K[x_1, \cdots, x_7]$. Then by Theorem 4.1.2 we have $\text{sdepth}(I) = 5$ and corresponding Stanley decomposition is

$$\begin{aligned} D : I \;=\; & x_1x_2x_3K[x_1,x_2,x_3,x_4,x_5] \oplus x_1x_3x_4K[x_1,x_3,x_4,x_5,x_6] \oplus x_1x_3x_5K[x_1,\\ & x_3,x_5,x_6,x_7] \oplus x_1x_3x_6K[x_1,x_2,x_3,x_6,x_7] \oplus x_2x_3x_7K[x_1,x_2,x_3,x_4,x_7] \oplus\\ & x_1x_3x_4x_7K[x_1,x_3,x_4,x_5,x_7] \oplus x_2x_3x_5x_7K[x_1,x_2,x_3,x_5,x_7] \oplus x_2x_3x_6x_7\\ & K[x_2,x_3,x_4,x_6,x_7] \oplus x_1x_2x_3x_5x_6K[x_1,x_2,x_3,x_5,x_6,x_7] \oplus x_1x_3x_4x_6x_7K\\ & [x_1,x_3,x_4,x_5,x_6,x_7] \oplus x_1x_2x_3x_4x_6K[x_1,x_2,x_3,x_4,x_5,x_6,x_7] \oplus x_2x_3x_4x_5\\ & x_6x_7K[x_2,x_3,x_4,x_5,x_6,x_7]. \end{aligned}$$

Remark 4.1.2. Converse of Theorem 4.1.2 is not true. As shown in the following example.

Example 4.1.4. Let $I = (x_1x_2, x_2x_3, x_3x_4, x_4x_5, x_1x_5x_6) \subset S = K[x_1, \cdots, x_6]$, I does not satisfy the hypothesis of Theorem 4.1.2 and primary decomposition of I is $I = (x_1, x_3, x_4) \cap (x_1, x_3, x_5) \cap (x_2, x_3, x_5) \cap (x_1, x_2, x_4) \cap (x_2, x_4, x_5) \cap (x_2, x_4, x_6)$. But by using Cocoa $\text{sdepth}(I) = 4$ and the corresponding Stanley decomposition is

$$\begin{aligned} D : I \;=\; & x_1x_2K[x_1,x_2,x_3,x_4] \oplus x_2x_3K[x_2,x_3,x_4,x_5] \oplus x_3x_4K[x_1,x_3,x_4,x_5] \oplus x_4x_5\\ & K[x_1,x_2,x_4,x_5] \oplus x_1x_2x_5K[x_1,x_2,x_3,x_5] \oplus x_1x_2x_6K[x_1,x_2,x_3,x_6] \oplus x_1x_5\\ & x_6K[x_1,x_2,x_5,x_6] \oplus x_2x_3x_6K[x_2,x_3,x_4,x_6] \oplus x_3x_4x_6K[x_1,x_3,x_4,x_6] \oplus x_4\\ & x_5x_6K[x_1,x_4,x_5,x_6] \oplus x_1x_2x_3x_4x_5x_6K[x_1,x_2,x_3,x_4,x_5,x_6]. \end{aligned}$$

Let $I \subset S = K[x_1, \cdots, x_n]$ be a monomial ideal with $|G(I)| = m$. Let $A = \{x_{k_1}, \cdots, x_{k_m}\} \subset \{x_1, \cdots, x_n\}$ and for each $v \in G(I)$ there exist unique $x_{k_i} \in A$ such that $x_{k_i}^{l_{k_i}} \mid v$. Then, we have the following results.

Corollary 4.1.3. $\operatorname{sdepth}_S(S/I) = n - m$.

Proof. By proof of Theorem 4.1.1, there exists a prime ideal $\Omega \in \operatorname{Ass}(S/I)$ with $\operatorname{ht}(\Omega) = m$. By Proposition 2.2.5, we have $\operatorname{sdepth}(S/I) \geq n - m$ and by [1] $\operatorname{sdepth}(S/I) \leq \min\{\dim(S/\Omega) : \Omega \in \operatorname{Ass}(S/I)\} = n - m$. It follows that

$$\operatorname{sdepth}_S(S/I) = n - m.$$

□

Corollary 4.1.4. $\operatorname{sdepth}_S(I) \geq \operatorname{sdepth}_S(S/I) + 1$.

Proof. Using Theorem 4.1.1, we have $\operatorname{sdepth}_S(I) = n - \lfloor \frac{m}{2} \rfloor$ and by Corollary 4.1.3 $\operatorname{sdepth}_S(S/I) = n - m$. Since $n - \lfloor \frac{m}{2} \rfloor \geq n - m + 1$ hence

$$\operatorname{sdepth}_S(I) \geq \operatorname{sdepth}_S(S/I) + 1.$$

□

Corollary 4.1.5. $\operatorname{depth}_S(S/I) \leq \operatorname{sdepth}_S(S/I)$, *that is Stanley's conjecture holds for* S/I.

Proof. Since $\operatorname{depth}_S(S/I) \leq n-m$, hence by Corollary 4.1.3 we have $\operatorname{depth}_S(S/I) \leq \operatorname{sdepth}_S(S/I)$. □

Corollary 4.1.6. $\operatorname{depth}_S(I) \leq \operatorname{sdepth}_S(I)$, *that is Stanley's conjecture holds for* I.

Proof. By Corollary 4.1.4 $\operatorname{sdepth}(I) \geq \operatorname{sdepth}_S(S/I) + 1 \geq 1 + \operatorname{depth}(S/I) = \operatorname{depth}(I)$. Therefore $\operatorname{sdepth}_S(I) \geq \operatorname{depth}_S(I)$. □

Proposition 4.1.7. *Let* $I' = (I, x_{n+1}, x_{n+2})$ *is a monomial ideal in* $S' = S[x_{n+1}, x_{n+2}]$ *then* $\operatorname{sdepth}(I') = \operatorname{sdepth}(I) + 1$.

Proof. By proof of Theorem 4.1.1, we have a prime ideal $\Omega \in \operatorname{Ass}(S/I)$ with $\operatorname{ht}(\Omega) = m$ then $\Omega' = (\Omega, x_{n+1}, x_{n+2})$ is a prime ideal in $\operatorname{Ass}(S'/I')$ and $\operatorname{ht}(\Omega') = m + 2$ so using Theorem 4.1.1 we have $\operatorname{sdepth}_{S'}(I') = n + 2 - \lfloor \frac{m+2}{2} \rfloor = n - \lfloor \frac{m}{2} \rfloor + 1$. It follows $\operatorname{sdepth}_{S'}(I') = \operatorname{sdepth}_S(I) + 1$. □

For the general monomial ideals result given by Lemma 4.1.7 is not proved.

4.2 Bounds of some square free monomial ideals

In this section, we have good bounds for the Stanley depth of I, where $I = \bigcap_{j=1}^{s} \Omega_j$ and $\Omega_k \subset \sum_{1=j\neq k}^{s} \Omega_j$ for $s = 4, 5$ but [13, Theorem 2.6] does not give any information about sdepth(I) for this type of ideals.

Lemma 4.2.1. *Let $0 < q < r < s < n$ and $I = \Omega_1 \cap \Omega_2 \cap \Omega_3 \cap \Omega_4$ where $\Omega_1 = (x_1 \cdots, x_r)$, $\Omega_2 = (x_{q+1}, \cdots, x_s)$, $\Omega_3 = (x_{r+1}, \cdots, x_n)$ and $\Omega_4 = (x_{s+1}, \cdots, x_n, x_1, \cdots, x_q)$. Then*

$$I = (x_1, \cdots, x_q) \cap (x_{r+1}, \cdots, x_s) + (x_{q+1}, \cdots, x_r) \cap (x_{s+1}, \cdots, x_n).$$

Proof. Suppose that I' is the intersection of prime ideals Ω_1 and Ω_2

$$\begin{aligned} I' = \Omega_1 \cap \Omega_2 &= (x_1 \cdots, x_q, x_{q+1}, \cdots, x_r) \cap (x_{q+1}, \cdots, x_r, x_{r+1}, \cdots, x_s) \\ &= (x_1 \cdots, x_q) \cap (x_{r+1}, \cdots, x_s) + (x_{q+1}, \cdots, x_r). \end{aligned}$$

Now I'' intersection of prime ideals Ω_3 and Ω_4

$$\begin{aligned} I'' = \Omega_3 \cap \Omega_4 &= (x_{r+1}, \cdots, x_s, x_{s+1}, \cdots, x_n) \cap (x_{s+1}, \cdots, x_n, x_1, \cdots, x_q) \\ &= (x_1, \cdots, x_q) \cap (x_{r+1}, \cdots, x_s) + (x_{s+1}, \cdots, x_n). \end{aligned}$$

Now taking the intersection of I' and I'', we get

$$\begin{aligned} \Omega_1 \cap \Omega_2 \cap \Omega_3 \cap \Omega_4 &= \Big\{(x_1 \cdots, x_q) \cap (x_{r+1}, \cdots, x_s) + (x_{q+1}, \cdots, x_r)\Big\} \cap \Big\{(x_1, \cdots \\ &\quad , x_q) \cap (x_{r+1}, \cdots, x_s) + (x_{s+1}, \cdots, x_n)\Big\} \\ &= (x_1 \cdots, x_q) \cap (x_{r+1}, \cdots, x_s) + (x_1 \cdots, x_q) \cap (x_{r+1}, \cdots, x_s) \\ &\quad \cap (x_{s+1}, \cdots, x_n) + (x_1 \cdots, x_q) \cap (x_{q+1}, \cdots, x_r) \cap (x_{r+1}, \cdots \\ &\quad , x_s) + (x_{q+1}, \cdots, x_r) \cap (x_{s+1}, \cdots, x_n). \end{aligned}$$

So

$$I = (x_1, \cdots, x_q) \cap (x_{r+1}, \cdots, x_s) + (x_{q+1}, \cdots, x_r) \cap (x_{s+1}, \cdots, x_n).$$

□

Proposition 4.2.2. *Let $0 < q < r < s < n$ and $I = \Omega_1 \cap \Omega_2 \cap \Omega_3 \cap \Omega_4$ where $\Omega_1 = (x_1 \cdots, x_r)$, $\Omega_2 = (x_{q+1}, \cdots, x_s)$, $\Omega_3 = (x_{r+1}, \cdots, x_n)$ and $\Omega_4 =$*

$(x_{s+1},\cdots,x_n,x_1,\cdots,x_q)$. *Then*

$$2 \leq \operatorname{sdepth}(I) \leq 2 + \left\lfloor \frac{d_3}{d_2} \right\rfloor,$$

where $d_2 = q(s-r)+(r-q)(n-s)$ *and*
$d_3 = d_2(n-2) - \left\{ q\binom{s-r}{2} + (s-r)\binom{q}{2} + (r-q)\binom{n-s}{2} + (n-s)\binom{r-q}{2} \right\}$.

Proof. Let $I = \Omega_1 \cap \Omega_2 \cap \Omega_3 \cap \Omega_4$ then by Lemma 4.2.1

$$I = (x_1,\cdots,x_q) \cap (x_{r+1},\cdots,x_s) + (x_{q+1},\cdots,x_r) \cap (x_{s+1},\cdots,x_n).$$

Number of monomials of degree 2 in I is $d_2 = q(s-r)+(r-q)(n-s)$. And the number of monomials of degree 3 is

$$d_3 = d_2(n-2) - \left\{ q\binom{s-r}{2} + (s-r)\binom{q}{2} + (r-q)\binom{n-s}{2} + (n-s)\binom{r-q}{2} \right\}.$$

By using [13, Lemma 2.4], we have $2 \leq \operatorname{sdepth}(I) \leq 2 + \left\lfloor \frac{d_3}{d_2} \right\rfloor$. □

Example 4.2.1. Let $I = (x_1,\cdots,x_6) \cap (x_5,\cdots,x_8) \cap (x_7,\cdots,x_{10}) \cap (x_9,x_{10},x_1,\cdots,x_4)$, that is $q=4, r=6, s=8, n=10$. Then by Proposition 4.2.2, we get $2 \leq \operatorname{sdepth}(I) \leq 8$, while [13, Theorem 2.14] gives $\operatorname{sdepth}(I) \leq 10$. It is noted that Proposition 4.2.2 gives better bound as compare to [13, Theorem 2.14].

Theorem 4.2.3. *Let* $0 < q < r < s < n$ *and* $I = \Omega_1 \cap \Omega_2 \cap \Omega_3 \cap \Omega_4$ *where* $\Omega_1 = (x_1 \cdots, x_r)$, $\Omega_2 = (x_{q+1},\cdots,x_s)$, $\Omega_3 = (x_{r+1},\cdots,x_n)$ *and* $\Omega_4 = (x_{s+1},\cdots,x_n,x_1,\cdots,x_q)$. *Then*

$$\operatorname{sdepth}(S/I) \geq \max\left\{ \min\left\{q, \left\lceil \frac{s-r}{2} \right\rceil\right\}, \min\left\{s-r, \left\lceil \frac{q}{2} \right\rceil\right\}\right\} + \max\left\{ \min\left\{n-s, \left\lceil \frac{r-q}{2} \right\rceil\right\}, \min\left\{r-q, \left\lceil \frac{n-s}{2} \right\rceil\right\}\right\}.$$

Proof. Let $I = \Omega_1 \cap \Omega_2 \cap \Omega_3 \cap \Omega_4$ then by Lemma 4.2.1

$$I = (x_1,\cdots,x_q) \cap (x_{r+1},\cdots,x_s) + (x_{q+1},\cdots,x_r) \cap (x_{s+1},\cdots,x_n).$$

Let $I^{'} = (x_1, \cdots, x_q) \cap (x_{r+1}, \cdots, x_s) \subset R^{'} = K[x_1, \cdots, x_q, x_{r+1}, \cdots, x_s]$ and $I^{''} = (x_{q+1}, \cdots, x_r) \cap (x_{s+1}, \cdots, x_n) \subset R^{''} = \{x_{q+1}, \cdots, x_r, x_{s+1}, \cdots, x_n\}$.
Then by [19, Theorem 3.1], we have

$$\text{sdepth}(S/I) \geq \text{sdepth}(R^{'}/I^{'}) + \text{sdepth}(R^{''}/I^{''}).$$

Let $Q_1^{'} = (x_1, \cdots, x_q)$ and $Q_2^{'} = (x_{r+1}, \cdots, x_s)$. Then $\dim(R^{'}/Q_1^{'}) = (s - r + q) - q$ $= s - r$, $\dim(R^{'}/Q_2^{'}) = (s - r + q) - (s - r) = q$ and $\dim(R^{'}/(Q_1^{'} + Q_2^{'})) = 0$. So

$$\min\Big\{ \dim(R^{'}/Q_2^{'}), \Big\lceil \frac{\dim(R^{'}/Q_1^{'}) + \dim(R^{'}/(Q_1^{'} + Q_2^{'}))}{2} \Big\rceil \Big\} = \min\Big\{q, \Big\lceil \frac{s-r}{2} \Big\rceil \Big\}$$

$$\min\Big\{ \dim(R^{'}/Q_1^{'}), \Big\lceil \frac{\dim(R^{'}/Q_2^{'}) + \dim(R^{'}/(Q_1^{'} + Q_2^{'}))}{2} \Big\rceil \Big\} = \min\Big\{s - r, \Big\lceil \frac{q}{2} \Big\rceil \Big\}.$$

By [17, Corollary 2.4], we get $\text{sdepth}(R^{'}/I^{'}) = \max\Big\{ \min\Big\{q, \Big\lceil \frac{s-r}{2} \Big\rceil \Big\}, \min\Big\{s - r, \Big\lceil \frac{q}{2} \Big\rceil \Big\}\Big\}$. Now suppose that $Q_1^{''} = (x_{q+1}, \cdots, x_r)$ and $Q_2^{''} = (x_{s+1}, \cdots, x_n)$, then $\dim(R^{''}/Q_1^{''}) = (n - s) + (r - q) - (r - q) = n - s, \dim(R^{''}/Q_2^{''}) = (n - s) + (r - q) - (n - s) = r - q$ and $\dim(R^{''}/(Q_1 + Q_2)) = 0$. So

$$\min\Big\{ \dim(R^{''}/Q_1^{''}), \Big\lceil \frac{\dim(R^{''}/Q_2^{''}) + \dim(R^{''}/(Q_1^{''} + Q_2^{''}))}{2} \Big\rceil \Big\} = \min\Big\{n-s, \Big\lceil \frac{r-q}{2} \Big\rceil \Big\},$$

$$\min\Big\{ \dim(R^{''}/Q_2^{''}), \Big\lceil \frac{\dim(R^{''}/Q_1^{''}) + \dim(R^{''}/(Q_1^{''} + Q_2^{''}))}{2} \Big\rceil \Big\} = \min\Big\{r-q, \Big\lceil \frac{n-s}{2} \Big\rceil \Big\}.$$

Now again by using [17, Corollary 2.4], we have

$$\text{sdepth}(R^{''}/I^{''}) = \max\Big\{ \min\Big\{n - s, \Big\lceil \frac{r-q}{2} \Big\rceil \Big\}, \min\Big\{r - q, \Big\lceil \frac{n-s}{2} \Big\rceil \Big\}\Big\}.$$

Hence we get

$$\begin{aligned} \text{sdepth}(S/I) \geq \max\Big\{ \min\Big\{q, \Big\lceil \frac{s-r}{2} \Big\rceil \Big\}, \min\Big\{s - r, \Big\lceil \frac{q}{2} \Big\rceil \Big\}\Big\} \\ + \max\Big\{ \min\Big\{n - s, \Big\lceil \frac{r-q}{2} \Big\rceil \Big\}, \min\Big\{r - q, \Big\lceil \frac{n-s}{2} \Big\rceil \Big\}\Big\}. \end{aligned}$$

□

Remark 4.2.1. If $0 < q = 3 < r = 5 < s = 7 < n = 10$ and $I = \Omega_1 \cap \Omega_2 \cap \Omega_3 \cap \Omega_4$ where $\Omega_1 = (x_1, \cdots, x_5)$, $\Omega_2 = (x_4, \cdots, x_7)$, $\Omega_3 = (x_6, \cdots, x_{10})$ and $\Omega_4 = (x_8, \cdots, x_{10}, x_1, \cdots, x_3)$ then $I = (x_1x_6, x_2x_6, x_3x_6, x_1x_7, x_2x_7, x_3x_7, x_4x_8, x_5x_8, x_4x_9, x_5x_9, x_4x_{10}, x_5x_{10})$. And $\max\{\min\{3,1\}, \min\{2,2\}\}+\max\{\min\{3,1\}, \min\{2,2\}\} = 2+2 = 4$. So by applying Theorem 4.2.3 $\operatorname{sdepth}(S/I) \geq 4$, but by using Proposition 2.2.5, we have $\operatorname{sdepth}(S/I) \geq n - m = 10 - 12 = -2$. It is noted that Theorem 4.2.3 gives better bound as compare to Proposition 2.2.5.

Lemma 4.2.4. *Let $0 < q < r < s < t < n$ and $I = \Omega_1 \cap \Omega_2 \cap \Omega_3 \cap \Omega_4 \cap \Omega_5$ where $\Omega_1 = (x_1 \cdots, x_r)$, $\Omega_2 = (x_{q+1}, \cdots, x_s)$, $\Omega_3 = (x_{r+1}, \cdots, x_t)$, $\Omega_4 = (x_{s+1}, \cdots, x_n)$ and $\Omega_5 = (x_{t+1}, \cdots, x_n, x_1, \cdots, x_q)$, then*

$$I = (x_1, \cdots, x_q) \cap (x_{r+1}, \cdots, x_s) \cap (x_{s+1}, \cdots, x_t) + (x_1, \cdots, x_q) \cap (x_{r+1}, \cdots, x_s) \cap (x_{t+1}, \cdots, x_n) + (x_1, \cdots, x_q) \cap (x_{q+1}, \cdots, x_r) \cap (x_{s+1}, \cdots, x_t) + (x_{q+1}, \cdots, x_r) \cap (x_{r+1}, \cdots, x_t) \cap (x_{t+1}, \cdots, x_n).$$

Proof. Firstly we take intersection of prime ideals Ω_1 and Ω_2

$$\begin{aligned}
\Omega_1 \cap \Omega_2 &= (x_1, \cdots, x_q, x_{q+1}, \cdots, x_r) \cap (x_{q+1}, \cdots, x_r, x_{r+1}, \cdots, x_s) \\
&= (x_1, \cdots, x_q) \cap (x_{r+1}, \cdots, x_s) + (x_{q+1}, \cdots, x_r).
\end{aligned}$$

Suppose that I' is the intersection of Ω_1, Ω_2 and Ω_3

$$\begin{aligned}
I' = \Omega_1 \cap \Omega_2 \cap \Omega_3 &= \{(x_1, \cdots, x_q) \cap (x_{r+1}, \cdots, x_s) + (x_{q+1}, \cdots, x_r)\} \cap (x_{r+1}, \cdots, x_s, x_{s+1}, \cdots, x_t) \\
&= (x_1, \cdots, x_q) \cap (x_{r+1}, \cdots, x_s) + (x_{q+1}, \cdots, x_r) \cap (x_{r+1}, \cdots, x_s, x_{s+1}, \cdots, x_t).
\end{aligned}$$

Let

$$\begin{aligned}
I'' = \Omega_4 \cap \Omega_5 &= (x_{s+1}, \cdots, x_t, x_{t+1}, \cdots x_n) \cap (x_{t+1}, \cdots, x_n, x_1, \cdots, x_q) \\
&= (x_1, \cdots, x_q) \cap (x_{s+1}, \cdots, x_t) + (x_{t+1}, \cdots, x_n).
\end{aligned}$$

Now by taking intersection of I' and I'' we have

$$\begin{aligned}\Omega_1 \cap \Omega_2 \cap \Omega_3 \cap \Omega_4 \cap \Omega_5 &= \{(x_1, \cdots, x_q) \cap (x_{r+1}, \cdots, x_s) + (x_{q+1}, \cdots, x_r) \cap (x_{r+1}, \\ &\quad \cdots, x_s, x_{s+1}, \cdots, x_t)\} \cap \{(x_1, \cdots, x_q) \cap (x_{s+1}, \cdots, x_t) \\ &\quad +(x_{t+1}, \cdots, x_n)\} \\ &= (x_1, \cdots, x_q) \cap (x_{r+1}, \cdots, x_s) \cap (x_{s+1}, \cdots, x_t) + (x_1, \cdots \\ &\quad , x_q) \cap (x_{r+1}, \cdots, x_s) \cap (x_{t+1}, \cdots, x_n) + (x_1, \cdots, x_q) \cap \\ &\quad (x_{q+1}, \cdots, x_r) \cap (x_{s+1}, \cdots, x_t) + (x_{q+1}, \cdots, x_r) \cap (x_{r+1} \\ &\quad , \cdots, x_t) \cap (x_{t+1}, \cdots, x_n).\end{aligned}$$

□

Theorem 4.2.5. *Let* $0 < q < r < s < t < n$ *and* $I = \Omega_1 \cap \Omega_2 \cap \Omega_3 \cap \Omega_4 \cap \Omega_5$ *where* $\Omega_1 = (x_1 \cdots, x_r)$, $\Omega_2 = (x_{q+1}, \cdots, x_s)$, $\Omega_3 = (x_{r+1}, \cdots, x_t)$, $\Omega_4 = (x_{s+1}, \cdots, x_n)$ *and* $\Omega_5 = (x_{t+1}, \cdots, x_n, x_1, \cdots, x_q)$, *then*

$$3 \leq \operatorname{sdepth}(I) \leq \left\lfloor \frac{n+3}{2} \right\rfloor.$$

Proof. Let $I = \Omega_1 \cap \Omega_2 \cap \Omega_3 \cap \Omega_4 \cap \Omega_5$ then by Lemma 4.2.4

$$\begin{aligned} I =& (x_1, \cdots, x_q) \cap (x_{r+1}, \cdots, x_s) \cap (x_{s+1}, \cdots, x_t) + (x_1, \cdots, x_q) \cap (x_{r+1}, \cdots, x_s) \cap \\ & (x_{t+1}, \cdots, x_n) + (x_1, \cdots, x_q) \cap (x_{q+1}, \cdots, x_r) \cap (x_{s+1}, \cdots, x_t) + (x_{q+1}, \cdots, x_r) \\ & \cap (x_{r+1}, \cdots, x_t) \cap (x_{t+1}, \cdots, x_n).\end{aligned}$$

Number of monomials of degree 3 in I is $d_3 = q(s-r)(t-s) + q(s-r)(n-t) + (r-q)(t-r)(n-t) + q(r-q)(t-s)$, after simplifying we get

$$d_3 = q(s-r)(n-s) + (r-q)(t-r)(n-t) + q(r-q)(t-s). \tag{4.2.1}$$

For our convenience, we denote different components of ideal I as $I^{'} = (x_1, \cdots, x_q) \cap (x_{r+1}, \cdots, x_s) \cap (x_{s+1}, \cdots, x_t)$, $I^{''} = (x_1, \cdots, x_q) \cap (x_{r+1}, \cdots, x_s) \cap (x_{t+1}, \cdots, x_n)$, $J^{'} = (x_{q+1}, \cdots, x_r) \cap (x_{r+1}, \cdots, x_t) \cap (x_{t+1}, \cdots, x_n)$ and $J^{''} = (x_1, \cdots, x_q) \cap (x_{q+1}, \cdots, x_r) \cap (x_{s+1}, \cdots, x_t)$.

Since the number of monomials of degree 4 in $I^{'}$ are

$$\frac{q(s-r)(t-s)}{2}\Big\{q+s-r+t-s-3\Big\}+q(s-r)(t-s)\Big\{n-(q+s-r+t-s)\Big\}$$
$$=q(s-r)(t-s)\Big\{\frac{q-r+t-k+2n-2q+2r-2t}{2}\Big\}$$
$$=\frac{q(s-r)(t-s)}{2}\Big\{2n+r-q-t-3\Big\}.$$

Hence we get the number of monomials of degree 4 in $I^{'}$ is

$$=\frac{q(s-r)(t-s)(n-t)}{2}+\frac{q(s-r)(t-s)(r-q)}{2}+\frac{q(s-r)(t-s)(n-3)}{2}. \quad (4.2.2)$$

And the number of monomials of degree 4 in $I^{''}$ is

$$\frac{q(s-r)(n-t)}{2}\Big\{n+q+s-r-t-3\Big\}+q(s-r)(n-t)\Big\{n-(q+s-r+n-t)\Big\}$$
$$=q(s-r)(n-t)\Big\{\frac{n+q+s-r-t-3-2s-2q+2r+2t}{2}\Big\}$$
$$=\frac{q(s-r)(n-t)}{2}\Big\{n+r+t-s-q-3\Big\}.$$

Thus we have the number of monomials of degree 4 in $I^{''}$ is

$$=\frac{q(s-r)(n-t)(t-s)}{2}+\frac{q(r-q)(s-r)(n-t)}{2}+\frac{q(s-r)(n-t)(n-3)}{2}. \quad (4.2.3)$$

Also the number of monomials of degree 4 in $J^{'}$ is

$$\frac{(r-q)(t-r)(n-t)}{2}\Big\{r-q+t-r+n-t-3\Big\}+(r-q)(t-r)(n-t)\Big\{n-(r-q$$
$$+t-r+n-t)\Big\}=\frac{(r-q)(t-r)(n-t)}{2}(n+q-3)$$
$$=\frac{q(r-q)(t-r)(n-t)}{2}+\frac{(r-q)(t-r)(n-t)(n-3)}{2}.$$

So the number of monomials of degree 4 in $J^{'}$ is

$$=\frac{q(r-q)(t-s)(n-t)}{2}+\frac{q(r-q)(s-r)(n-t)}{2}+\frac{(r-q)(t-r)(n-t)(n-3)}{2}. \quad (4.2.4)$$

Number of monomials of degree 4 in J'' is

$$\frac{q(r-q)(t-s)}{2}\Big\{q+r-q+t-s-3\Big\}+q(r-q)(t-s)\Big\{n-(q+r-q+t-s)\Big\}$$
$$=\frac{q(r-q)(t-s)}{2}\Big\{2n+s-t-r-3\Big\}$$
$$=\frac{q(r-q)(t-s)(s-r)}{2}+\frac{q(r-q)(t-s)(n-t)}{2}+\frac{q(r-q)(t-s)(n-3)}{2}. \quad (4.2.5)$$

Now combining (4.2.2), (4.2.3), (4.2.4) and (4.2.5) we get

$$q(s-r)(t-s)(n-t)+q(r-q)(t-s)(s-r)+q(r-q)(s-r)(n-t)+q(r-q)$$
$$(t-s)(n-t)\frac{q(s-r)(t-s)(n-3)}{2}+\frac{q(s-r)(n-t)(n-3)}{2}+(r-q)(t-r)(n-t)$$
$$\frac{(n-3)}{2}+\frac{q(r-q)(t-s)(n-3)}{2}$$
$$=q(s-r)(t-s)(n-t)+q(r-q)(t-s)(s-r)+q(r-q)(s-r)(n-t)+q(r-q)$$
$$(t-s)(n-t)+\frac{n-3}{2}\Big\{q(s-r)(n-s)+(r-q)(t-r)(n-t)+q(r-q)(t-s)\Big\}.$$

Number of common monomials of degree 4 in components I' and I'', I' and J'', I'' and J', J' and J'' are given as $q(s-r)(n-t)(t-s)$, $q(t-s)(r-q)(s-r)$, $q(r-q)(s-r)(n-t)$ and $q(r-q)(t-s)(n-t)$, respectively. So total number of monomials of degree 4 in I is

$$d_4=q(s-r)(t-s)(n-t)+q(r-q)(t-s)(s-r)+q(r-q)(s-r)(n-t)+q(r-q)(t-s)(n-t)+\tfrac{n-3}{2}\Big\{q(s-r)(n-s)+(r-q)(t-r)(n-t)+q(r-q)(t-s)\Big\}-\Big\{q(s-r)(n-t)(t-s)+q(t-s)(r-q)(s-r)+q(r-q)(s-r)(n-t)+q(r-q)(t-s)(n-t)\Big\}$$
$$=\tfrac{n-3}{2}\Big\{q(s-r)(n-s)+(r-q)(t-r)(n-t)+q(r-q)(t-s)\Big\}.$$

Then from above and by (4.2.1), we get

$$d_4=\frac{n-3}{2}d_3. \quad (4.2.6)$$

By using [13, Lemma 2.4] we have $3\leq \operatorname{sdepth}(I)\leq 3+\left\lfloor\frac{d_4}{d_3}\right\rfloor$. Hence by (4.2.6), we have $3\leq \operatorname{sdepth}(I)\leq 3+\left\lfloor\frac{n-3}{2}\right\rfloor=\left\lfloor 3+\frac{n-3}{2}\right\rfloor=\left\lfloor\frac{n+3}{2}\right\rfloor$. □

Example 4.2.2. Let $I = (x_1, \cdots, x_5) \cap (x_4, \cdots, x_6) \cap (x_6, \cdots, x_9) \cap (x_7, \cdots, x_{12}) \cap (x_{10}, \cdots, x_{12}, x_1, \cdots, x_3)$ that is $q = 3, r = 5, s = 6, t = 9$ and $n = 12$. Then by Theorem 4.2.5, we have $3 \leq \text{sdepth}(I) \leq 7$, while [13, Theorem 2.14] gives $\text{sdepth}(I) \leq 12$. It is noted that Theorem 4.2.5 gives better bound as compare to [13, Theorem 2.14].

Example 4.2.3. Let $I = (x_1, \cdots, x_8) \cap (x_6, \cdots, x_{12}) \cap (x_9, \cdots, x_{15}) \cap (x_{13}, \cdots, x_{20}) \cap (x_{16}, \cdots, x_{20}, x_1, \cdots, x_5)$ that is $q = 5, r = 8, s = 12, t = 15$ and $n = 20$. Then by Theorem 4.2.5, we have $3 \leq \text{sdepth}(I) \leq 11$, while [13, Theorem 2.14] gives $\text{sdepth}(I) \leq 20$. It is observed that Theorem 4.2.5 gives better bound as compare to [13, Theorem 2.14].

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