TWO DIMENSIONAL HARMONIC OSCILLATORS AND THEIR PHYSICAL INTERPRETATION

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Dedicated to

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and my all Teachers of life

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Abstract

In this dissertation I have reviewed some basic Lie and Noether theory for the real domain as well as for the case when there is complex domain. Then, we introduce the concept of the approximate symmetries for the perturbed complex differential equation in general and first order approximate symmetries of the perturbed complex ordinary differential equation (p-CODE) in particular. Also, we apply the same procedure to work out the first order approximate symmetries of some well known scalar second order p-CODEs. We also study the harmonic oscillator which have wide applications in population dynamics, classical mechanics, pattern recognition, quantum mechanics, fluid mechanics, and solid mechanics, etc. Furthermore, it is seen that the two dimensional coupled harmonic oscillators can be represented with the help of one complex ordinary differential equation (CODE) and hence we worked out the first integrals (invariants) corresponding to the coupled system and it was found that there was a mistake in the published work [20] which was corrected and then appeared in [21] on account of this work. At the end we dealt the same system for more choices of the phase angles between the oscillators and hence obtained the generalized form of the invariants.

Contents

Chapter 1

Symmetries and invariants for ODEs

The existence of problems in this universe is linked with the existence of human beings and it is, infact, the nature of human beings to inquire and raise questions about all those things which are unknown to them and hence solve those problems after giving proper attention to the nature of the problem [1]. Mathematics is considered as a study of problems but, indeed, it helps us to answer all those questions by which we can understand the laws of the Universe, and the nature of objects in it. Moreover, it provides logical ways to obtain the best solutions to certain problems. Thus, its existence comes from the beginning of everything. People often describe it either as "the language of precision", as we use it to avoid emotional content [2], in other words, for obtaining correct results from correct premises or as the science that deals with numbers, shapes, quantities, arrangements, and chances but it practically deals with every single object that exists in this universe from big stars to microscopic particles and their motion. Mathematics is merely like a game which has certain rules and we enjoy watching it and if we actually want to play then we need to know its rules. Most of the time people play it to learn more about applications but if there are no as such visible applications then playing the game can still be of great interest [3]. Therefore, to play it one should know its two main perspectives i.e. geometry and symbolic representation which give rise to two basic problems; that is when going from one world to the other and returning back is unique (or possible) or may not be unique (or impossible) [4].

To understand the laws of this Universe one needs to understand "symmetry". The word "symmetry" has its origin from the Greeks and it has three types of meaning: as a phenomenon is what on the basis of our knowledge we have learned, a concept which deals with all such phenomena, or an operation which makes the phenomenon possible [5]. A symmetry is a transformation that leaves an object to look apparently invariant or it is a function that actually preserves what we feel special about any object.

Generally speaking, symmetry is not only restricted to mean a transformation but from a bigger perspective it is a "criterion for beauty" [6]. It is quite interesting to note that people have not yet defined the type, order, and degree of the beauty but here we are only concerned with its mathematical aspect as it helps us to understand well the laws of the nature. Symmetry is a comprehensive concept that appears in our everyday lives, in science, in art and we feel it so often as if everything was made up of entire symmetrical objects. Symmetry appears in our neighborhoods, in buildings, roads, bridges, machinery, in the smallest particles of crystals, in the rhythm of songs, in movements of dance, in poetry, in flowers of the plant, in leaves of the trees, in the structure of animals' bodies, in games, and it is also a symmetry if some object is painted a different colour. The concept of symmetry is found in almost every modern science. In physics it appears in the variational principles, electric circuits, in the conservation laws, and in the structure of matter. In earth sciences and astronomy it turns up in the investigation of periodical events and even in the preparation of calendars. In crystallography, it arises in regular space filling, in spatial arrangements, and in crystal distortions. In chemistry, it is seen in molecules and macromolecules. In biology, it is present in the structures of DNA and RNA molecules, in protein, in every living creature, in evolution. In psychology, it is found in the regularities of development of the consciousness of any living creature, in personality tests. In music, it is heavily found in the rules of rhythm, in melody, and in the musical works. In literature, it is in rhyme, in meter, in the structure of every literary work. It is the keynote principle in the movements of dance. In fine art, one may find it in proportions, in perspective, and in harmony of proportion and colour. It is found in ethics, in determining virtuous behaviour, in moderation, in search for the perfect way. In logic and philosophy, its presence is in the form and order of thought. In economics, its active role is in balance, in simulation [5]. Beside all of the arts and sciences it also plays a vital role in the search of a fair or an ideal person.

Mathematically, one of the best examples of geometric symmetry is reflection. If we reflect some shape about the axis of symmetry, then the corresponding points of the shape and its reflected image are exactly at the same distance from the axis, but in opposite directions. The shape, size, colour, and the angles of the shape remain unchanged and only the orientation of the shape gets changed. The second example is rotation. If we rotate any 2-dimensional shape about an axis perpendicular to the plane of that shape and it overlaps the original, and if repeating the rotation n times it returns to the starting position we call it an n-fold symmetry. For example, an equilateral triangle has 3-fold symmetry, a square 4-fold, and so on. The internal characteristics like shape, size, colour, and angles of any shape in rotation always remain the same. Rotation requires at least 2 dimensions. The third example is translation. If we translate (linear change in position) any object with some fixed period in some direction then the object remains invariant. In order to understand the idea of symmetry of some geometrical object let us consider an example of a square. There are a total of eight geometrical operations which can bring a square into equivalence with itself that is the shape looks exactly the same before and after the operations. We say that a square has eight symmetries (see Fig 1.1). The operations are: two reflections along the lines drawn

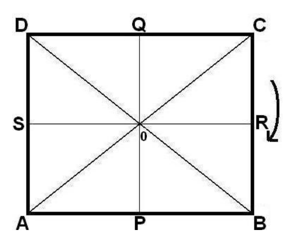

Figure 1.1: Symmetries of a square.
4 reflections along $\overline{PQ}$, $\overline{RS}$, $\overline{AC}$, and $\overline{BD}$,
4 rotations around 90, 180, 270, and 360 degrees

between the midpoints of opposite sides($\overline{PQ}$ and $\overline{RS}$), two reflections along the lines drawn between opposite vertices($\overline{AC}$ and $\overline{BD}$), and four planar rotations around 90, 180, 270, and 360 degrees. Further, the operation of leaving the square untouched can give us the idea of the neutral element (the identity) and the reverse operation as the inverse. These operations form a group of the symmetries of a square [5].

If we look back at the history then we find that it was Al-Khwarizmi $(780-850)$ who introduced algebra for the first time, Omar Khayyam $(1048-1131)$ gave connection between geometry and algebra. He also found the general geometric solution of the cubic equation, François Viéte $(1540-1603)$ in the late 1500s called algebra "the art of finding solutions to all problems". It was René Descartes $(1596-1650)$ who claimed that "all geometrical problems could be first quantified and then solved" and it was also his opinion that "analytic geometry" or "algebraic geometry" could be used to discover all of the science and the laws which govern the Universe. Pierre de Fermat (1601 or $1607/8-1665$) is given credit for the early developments that led to the infinitesimal calculus. Then, Isaac Newton $(1642-1727)$ and Gottfried Wilhelm von Leibnitz $(1646-1716)$ found that algebra is not sufficient to solve all of the scientific problems and invented calculus [7, 8].

The study of differential equations DEs (equations that contain the derivatives), began right after calculus [9]. In search for the most general methods of integrating DEs Isaac Newton classified first order (numerical value of the highest derivative appearing in a DE) DEs into three classes [7]

$$\begin{gathered} \tfrac{dy}{dx}=f(x), \\ \tfrac{dy}{dx}=f(x,y), \\ x\tfrac{\partial u}{\partial x}+y\tfrac{\partial u}{\partial y}=u(x,y). \end{gathered} \tag{1.0.1}$$

The first two classes involve ordinary derivative of one or more dependent variables with respect to only single independent variable and known as ordinary differential equations(ODEs). The last class involved the derivative (partial) of one dependent variable with respect to more than one independent variables as such equations are known today as partial differential equations(PDEs) [8].

There are many techniques to solve DEs but every method has its limitation and so no general method exists by which we can solve the linear or nonlinear DEs. To address this problem for finding the analytic solutions of DEs a Norwegian mathematician of the 19^{th} century, Sophus Lie $(1842 - 1899)$, proposed an entirely new approach for solving DEs in which he used continuous groups of transformations of the dependent and independent variables depending on continuous parameters known today as the Lie groups. He observed that DEs can remain invariant under continuous transformations and such transformations map any solution of the DEs to another solution which generalized various methods of the change of variables and led to a new branch of mathematics known as symmetry analysis for solving DEs. Lie used the symmetry approach to linearize (process of replacing an object by a simple linear object) certain DEs. He developed techniques one in which he used symmetries to solve the linear and nonlinear DEs and in which he used the group of Lie point symmetries to find the most general form of the nonlinear ODE. Also the symmetries have a strong relationship with the conservation laws (or the first integrals) which play a vital role in the reduction of order and in the study of the physical properties of geometry associated with the problem. The use of the complex variable in ODEs helps to provide new insights into the theory of DEs [10] since we split the complex variable into pair of real variables then the corresponding DE gives rise to a system of PDEs (if the independent variable is complex) and ODEs (if the independent variable is real). On the basis of this choice the system of two real second order ODEs can be represented by a single complex ODE (CODE). The examples of such systems include two oscillators, motion of two free particles, and generalized Lane-Emden can be described with the single CODE.

In the first chapter of this dissertation the basic mathematics to be used is introduced by reviewing some basic concepts of, geometry by explaining manifolds, transformations of Lie groups and their corresponding Lie algebras. Techniques are used to find transformations of some scalar ODEs (in which the coefficients are numbers). After which we state Noether's theorems and explain how its use in analysis helps us to obtain invariants corresponding to some physical problem. In chapter 2, we define complex manifolds, their need to understand certain phenomenons, transformations in complex domain or in other words complex Lie symmetry (CLS) analysis. At the end we discuss complex Noether approach and its double reduction and how the theory helps us to obtain invariants of the physical systems or conserved quantities. Chapter 3 deals with the treatment of perturbed complex ODEs (p-CODEs) by defining approximate transformations and their approximate algebras which helps us to understand the laws approximately. Finally, in chapter 4 complex harmonic oscillator is explained which infact give rise to a 2-dimensional system of coupled real harmonic oscillators and their invariants are worked out by using Noether's theorems. At the end we summarize our entire discussion and see what we have achieved.

1.1 Real manifolds

Manifolds arise in many physical situations and generally to deal with mechanics we need manifolds. Manifolds are, therefore, of interest in the study of Geometry, Topology and Analysis [11].

Definition 1.1.1 A real manifold M of dimension n is a set together with a collection of open subsets O_α satisfying the following properties [12].
(1) The set of O_α covers M.
(2) For each α there is a bijective map $\psi_\alpha : O_\alpha \longrightarrow U_\alpha$, where U_α is an open subset of $\mathbb{R}^n$.
(3) We can consider the map $\psi_\beta \circ \psi_\alpha^{-1}$ which takes the points in ψ_α to points in ψ_β only for the overlapping region i.e., where $O_\alpha \bigcap O_\beta \neq \phi$. (see Fig. 1.2)

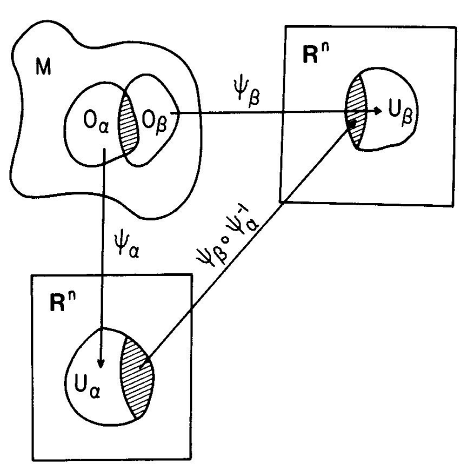

Figure 1.2: Manifold and two overlapping coordinate systems.

If all the maps (charts) ψ_α are homeomorphisms then we can define a topology on the manifold M. In the context of a topological space we can define a manifold as: "A manifold of dimension n is a separable, connected, Hausdorff space with a homeomorphism from each element of its open cover into $\mathbb{R}^n$". Thus we require that it does not have too many points in some place or the space has countable dense subset (i.e. it is separable), too few points or the space may not be divided into two or more disjoint sets (i.e. it is connected), and any two points can be enclosed in some disjoint regions or for any two elements of this space we have neighbourhoods which are disjoint (i.e. it is Hausdorff). In every small region one could assign coordinates so that to each point there would be a set of n numbers and to the set of n numbers there would be a specific point (i.e. there is a homeomorphism from its open cover to the set of n-tupples of numbers, $\mathbb{R}^n$) [6]. If the homeomorphic transformations are also differentiable then the manifold is said to be a *differentiable manifold.*

The basic example of a manifold is Euclidean space, and many of its properties carry over to manifolds. In addition, any smooth boundary of a subset of Euclidean space, like the straight line (non compact), circle and the sphere (compact), is a manifold.

1.2 Point transformations and Lie groups

To solve DEs by using symmetries one needs to define symmetry in the context of a continuum first, which requires some knowledge of transformations and their generators. For solving the DEs, one may simplify the equation by an appropriate transformation of the independent variable and the dependent variable,

$$\begin{aligned} \tilde{x} &= \tilde{x}(x, y), \\ \tilde{y} &= \tilde{y}(x, y), \end{aligned} \tag{1.2.1}$$

these are known as continuous point transformations that maps points (x, y) into points $(\tilde{x}, \tilde{y})$. To find the symmetries of DEs we need invertible point transformations that depend on (at least) one arbitrary parameter ε,

$$\tilde{x} = \tilde{x}(x, y; \varepsilon), \qquad \tilde{y} = \tilde{y}(x, y; \varepsilon). \tag{1.2.2}$$

In fact, repeated applications of these transformations yield a transformation of the same family and also the identity is contained. These group properties on a differentiable manifold ensure that the transformations (1.2.2) form a one-parameter Lie group of transformations with parameter ε [13]. Now to obtain the infinitesimal generator of such a group, we need to expand the transformations (1.2.2) by using a Taylor series about ε=0:

$$\begin{aligned} \tilde{x} &= x + \varepsilon\xi(x, y) + ... = x + \varepsilon\mathbf{X}x + ... = (e^{\varepsilon\mathbf{X}})x, \\ \tilde{y} &= y + \varepsilon\eta(x, y) + ... = y + \varepsilon\mathbf{X}y + ... = (e^{\varepsilon\mathbf{Y}})y, \end{aligned} \tag{1.2.3}$$

where the functions ξ and η are defined by the Lie equations,

$$\begin{aligned} \xi(x, y) &= \tfrac{\partial\tilde{x}}{\partial\varepsilon}\mid_{\varepsilon=0} = \mathbf{X}x \\ \eta(x, y) &= \tfrac{\partial\tilde{y}}{\partial\varepsilon}\mid_{\varepsilon=0} = \mathbf{X}y. \end{aligned} \tag{1.2.4}$$

Where $\mathbf{X}$ is called an infinitesimal generator of transformations and symmetry of DE, provided that it satisfies symmetry existence criteria, which we define later,

$$\mathbf{X} = \xi(x, y)\tfrac{\partial}{\partial x} + \eta(x, y)\tfrac{\partial}{\partial y}. \tag{1.2.5}$$

Example 1.2.1 Consider the one-paramter group of rotations in the X-Y plane

$$\begin{aligned} \tilde{x} &= x\cos\varepsilon - y\sin\varepsilon, \\ \tilde{y} &= x\sin\varepsilon + y\cos\varepsilon. \end{aligned} \tag{1.2.6}$$

Hence

$$\tfrac{\partial\tilde{x}}{\partial\varepsilon}\mid_{\varepsilon=0} = -y, \tfrac{\partial\tilde{y}}{\partial\varepsilon}\mid_{\varepsilon=0} = x, \tag{1.2.7}$$

so that the corresponding symmetry generator is given by

$$\mathbf{X} = -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}. \tag{1.2.8}$$

Example 1.2.2 Consider the one-parameter group of translations (shift of origin of X) we obtain

$$\tilde{x} = x + \varepsilon, \tilde{y} = y, \mathbf{X} = \frac{\partial}{\partial x}. \tag{1.2.9}$$

To apply a point transformation (1.2.2) to a scalar ODE

$$E\left(x, y, y', y'', ..., y^{(n)}\right) = 0, \tag{1.2.10}$$

we must know how to transform the derivatives $y', y'', ..., y^n$, that is, how to *extend* or *prolong* the point transformations to the derivatives of every order. This is done by defining

$$\begin{gathered} \tilde{y}' = \frac{d\tilde{y}'(x,y;\varepsilon)}{d\tilde{x}'(x,y;\varepsilon)} = \frac{\frac{\partial \tilde{y}}{\partial x} + y'\frac{\partial \tilde{y}}{\partial y}}{\frac{\partial \tilde{x}}{\partial x} + y'\frac{\partial \tilde{x}}{\partial x}} = \tilde{y}'\left(x, y, y'; \varepsilon\right), \\ \tilde{y}'' = \tilde{y}''\left(x, y, y', y''; \varepsilon\right), ... \end{gathered} \tag{1.2.11}$$

or equivalently,

$$\begin{aligned} \tilde{y}' &= y + \varepsilon\eta^{(1)}\left(x, y, y'\right) + O\left(\varepsilon^2\right) = y' + \varepsilon\mathbf{X}^{[n]}y' + O\left(\varepsilon^2\right), \\ \tilde{y}'' &= y + \varepsilon\eta^{(2)}\left(x, y, y', y''\right) + O\left(\varepsilon^2\right) = y'' + \varepsilon\mathbf{X}^{[n]}y'' + O\left(\varepsilon^2\right), \\ \tilde{y}^n &= y + \varepsilon\eta^{(n)}\left(x, y, y', ..., y^{(n)}\right) + O\left(\varepsilon^2\right) = y^{(n)} + \varepsilon\mathbf{X}^{[n]}y^{(n)} + O\left(\varepsilon^2\right), \end{aligned} \tag{1.2.12}$$

where the prolongation coefficients come out to be of the form

$$\begin{gathered} \eta^{(1)} = \frac{d\eta}{dx} - y'\frac{d\xi}{dx} = \eta_x + y'(\eta_y - \xi_x) - y'^2\xi_y, \\ \eta^{(n)} = \frac{d\eta^{(n-1)}}{dx} - y^{(n)}\frac{d\xi}{dx}, n \geq 2, \end{gathered} \tag{1.2.13}$$

where the operator $\frac{d}{dx}$ is defined as

$$\frac{d}{dx} = \frac{\partial}{\partial x} + y'\frac{\partial}{\partial y} + y''\frac{\partial}{\partial y'} + y'''\frac{\partial}{\partial y''} + ... \tag{1.2.14}$$

and the n^{th} order prolongation of the infinitesimal generator is

$$\mathbf{X}^{[n]} = \xi(x,y)\frac{\partial}{\partial x} + \eta(x,y)\frac{\partial}{\partial y} + \eta^{(1)}(x,y,y')\frac{\partial}{\partial y'} + ... \\ + \eta^{(n)}\left(x,y,y',...,y^{(n)}\right)\frac{\partial}{\partial y^{(n)}}. \tag{1.2.15}$$

1.3 The Lie algebra and its properties

Let $\mathbf{X}$ and $\mathbf{Y}$ be infinitesimal symmetry generators. Then, their commutator is defined by

$$[\mathbf{X},\mathbf{Y}] = \mathbf{XY} - \mathbf{YX}, \tag{1.3.1}$$

which itself is a symmetry generator and called the Lie product or Lie bracket of $\mathbf{X}$ and $\mathbf{Y}$. A Lie algebra, L, is a vector space over some field F equipped with the Lie product satisfying the following three properties

(i) **Bilinearity**

$[\alpha\mathbf{X}+\mathbf{Y},\mathbf{Z}] = \alpha[\mathbf{X},\mathbf{Z}] + [\mathbf{Y},\mathbf{Z}].$

$[\mathbf{X},\alpha\mathbf{Y}+\mathbf{Z}] = \alpha[\mathbf{X},\mathbf{Y}] + [\mathbf{X},\mathbf{Z}].(\forall\alpha\varepsilon F)$

(ii) **Anti-symmetry or skew-symmetry**

$[\mathbf{X},\mathbf{Y}] = -[\mathbf{Y},\mathbf{X}].$

(iii) **Jacobi identity**

$[[\mathbf{X},\mathbf{Y}],\mathbf{Z}] + [[\mathbf{Y},\mathbf{Z}],\mathbf{X}] + [[\mathbf{Z},\mathbf{X}],\mathbf{Y}] = 0.$

A Lie algebra is said to be real if F is the field of real numbers and complex if F is the field of complex numbers. If the generators of the Lie algebra are finite (say n) then it is known as a finite dimensional (n dimensional) Lie algebra otherwise it is known as an infinite dimensional Lie algebra. Also to every Lie group G, one can associate a unique Lie algebra, whose underlying vector space is the tangent space of G at the identity element, which completely captures the local structure of the group. Informally one can think of elements of the Lie algebra as elements of the group that are "infinitesimally close" to the identity. However, for a given Lie algebra there can be many groups with different topologies. Since a Lie algebra determines the local structure of the group, therefore two groups will be locally isomorphic iff their Lie algebras are isomorphic.

1.4 Lie point symmetries of ODEs

In order to find the Lie group of transformations one has to follow below mentioned criteria.

Theorem 1.4.1. *An n^{th} order ODE admits a group of symmetries with generator $\mathbf{X}^{[n]}$ if and only if*

$$\mathbf{X}^{[n]} E \mid_{E=0} = 0, \tag{1.4.1}$$

i.e. holds along the solutions of E=0.

Proof: If a DE admits a symmetry then (1.4.1) trivially holds. Conversely, since

$$E\left(\tilde{x}, \tilde{y}, \tilde{y}', \tilde{y}'', ..., \tilde{y}^{(n)}\right) = 0, \tag{1.4.2}$$

has to be valid for all values of ε, therefore, differentiating (1.4.2) we obtain

$$\begin{gathered} \tfrac{\partial}{\partial \varepsilon} E(\tilde{x}, \tilde{y}, \tilde{y}', \tilde{y}'', ..., \tilde{y}^{(n)}) \mid_{\varepsilon=0} = 0, \\ \tfrac{\partial E}{\partial \tilde{x}} \tfrac{\partial \tilde{x}}{\partial \varepsilon} + \tfrac{\partial E}{\partial \tilde{y}} \tfrac{\partial \tilde{y}}{\partial \varepsilon} + ... + \tfrac{\partial E}{\partial \tilde{y}^{(n)}} \tfrac{\partial \tilde{y}^{(n)}}{\partial \varepsilon} \mid_{\varepsilon=0} = 0. \end{gathered} \tag{1.4.3}$$

Using the definitions from (1.2.4), and

$$\frac{\partial E}{\partial \tilde{x}} \mid_{\varepsilon=0} = \frac{\partial E}{\partial x}, \;\; \frac{\partial E}{\partial \tilde{y}} \mid_{\varepsilon=0} = \frac{\partial E}{\partial y}, \;\; ..., \frac{\partial E}{\partial \tilde{y}^{(n)}} \mid_{\varepsilon=0} = \frac{\partial E}{\partial y^{(n)}},$$

(1.4.3) is equivalent to

$$\xi \frac{\partial E}{\partial x} + \eta \frac{\partial E}{\partial y} + \eta^{(1)} \frac{\partial E}{\partial y'} + ... + \eta^{(n)} \frac{\partial E}{\partial y^{(n)}} = 0,$$

which is equivalent to (1.4.1). This completes the proof.

Alternatively, the above theorem can be explicitly stated as follows.

Theorem 1.4.2. *An n^{th} order ODE admits a group of symmetries with generator* $\mathbf{X}$ *if and only if*

$$\mathbf{X}^{[n-1]}\omega = \eta^{n}_{|y^n=\omega}. \tag{1.4.4}$$

1.4.1 Lie symmetry conditions for second-order ODEs

To find the symmetry generators for second-order ODEs (or PDEs) [14] the method is the same as defined for the general case in the previous section but for the sake of understanding we are going to define it first for the particular case and then work out an example by using the same approach. First of all we are going to restate the previous theorem particularly for the second order ODEs.

Theorem 1.4.3. *Any 2^{nd} order ODE of the form $y'' = \omega(x, y, y')$ admits a group of symmetries with generator* $\mathbf{X}$ *if and only if*

$$\mathbf{X}^{[1]}\omega(x, y, y') = \eta^{2}_{|y''=\omega}, \tag{1.4.5}$$

where the expression for η^2 can be worked out by utilizing (1.2.13)

$$\begin{aligned}\eta^{(2)} = \eta_{xx} + y'(2\eta_{xy} - \xi_{xx}) + y'^2(\eta_{yy} - 2\xi_{xy}) - y'^3\xi_{yy}\\ +y''(\eta_y - 2\xi_x - 3y'\xi_y).\end{aligned} \tag{1.4.6}$$

In order to understand the above mentioned criteria we now apply the same procedure and give two examples.

Example 1.4.1

$$y'' = 0. \tag{1.4.7}$$

Here $\omega(x, y, y') = 0$, so by utilizing (1.4.5) we obtain

$$\eta^{(2)} = 0, \tag{1.4.8}$$

or equivalently,

$$\eta_{xx} + y'(2\eta_{xy} - \xi_{xx}) + y'^2(\eta_{yy} - 2\xi_{xy}) - y'^3\xi_{yy} = 0. \tag{1.4.9}$$

By comparing the coefficients of the powers of y', we get

$$\eta_{xx} = 0, \tag{1.4.10}$$

$$2\eta_{xy} - \xi_{xx} = 0, \tag{1.4.11}$$

$$\eta_{yy} - 2\xi_{xy} = 0, \tag{1.4.12}$$

$$\xi_{yy} = 0. \tag{1.4.13}$$

Solving above system for ξ and η, we get

$$\xi = (c_1x + c_2)y + c_3x^2 + c_7x + c_8, \tag{1.4.14}$$

$$\eta = c_1y^2 + (c_3x + c_4)y + c_5x + c_6, \tag{1.4.15}$$

which forms an 8-parameter Lie group of transformations

$$\begin{aligned} &\mathbf{X}_1 = \tfrac{\partial}{\partial x},\ \mathbf{X}_2 = \tfrac{\partial}{\partial y},\\ &\mathbf{X}_3 = x\tfrac{\partial}{\partial x},\ \mathbf{X}_4 = y\tfrac{\partial}{\partial y},\\ &\mathbf{X}_5 = x\tfrac{\partial}{\partial y},\ \mathbf{X}_6 = y\tfrac{\partial}{\partial x},\\ &\mathbf{X}_7 = x^2\tfrac{\partial}{\partial x} + xy\tfrac{\partial}{\partial y},\\ &\mathbf{X}_8 = xy\tfrac{\partial}{\partial x} + y^2\tfrac{\partial}{\partial y}. \end{aligned} \tag{1.4.16}$$

Example 1.4.2

$$y'' + y = 0, \tag{1.4.17}$$

with $\omega(x,y,y') = -y$, so by using (1.4.5) we obtain

$$\eta^{(2)} = -\eta, \tag{1.4.18}$$

or equivalently,

$$\begin{aligned} &\eta_{xx} + y'(2\eta_{xy} - \xi_{xx}) + y'^2(\eta_{yy} - 2\xi_{xy}) - y'^3\xi_{yy} \\ &+y(2\xi_x + 3y'\xi_y - \eta_y) = -\eta. \end{aligned} \tag{1.4.19}$$

By comparing the coefficients of y' and its powers, we get

$$\eta_{xx} + y(2\xi_x - \eta_y) + \eta = 0, \tag{1.4.20}$$

$$2\eta_{xy} - \xi_{xx} + 3y\xi_y = 0, \tag{1.4.21}$$

$$\eta_{yy} - 2\xi_{xy} = 0, \tag{1.4.22}$$

$$\xi_{yy} = 0. \tag{1.4.23}$$

The solution of the above system for ξ and η is

$$\xi = y(a\cos x + b\sin x) + e\sin 2x + f\cos 2x + g, \tag{1.4.24}$$

$$\begin{aligned} \eta = y^2(b\cos x - a\sin x) + y(h + e\cos 2x + f\sin 2x) + \\ c\cos x + d\sin x, \end{aligned} \tag{1.4.25}$$

where a, b, c, d, e, f, g, and h are some arbitrary constants.

Thus, ξ and η admits the 8-dimensional Lie algebra of point symmetry generators given by

$$
\begin{aligned}
&\mathbf{X}_1 = \tfrac{\partial}{\partial x},\ \mathbf{X}_2 = y\tfrac{\partial}{\partial y},\\
&\mathbf{X}_3 = \sin x\tfrac{\partial}{\partial y},\ \mathbf{X}_4 = \cos x\tfrac{\partial}{\partial y},\\
&\mathbf{X}_5 = \sin 2x\tfrac{\partial}{\partial x} + y\cos 2x\tfrac{\partial}{\partial y},\\
&\mathbf{X}_6 = \cos 2x\tfrac{\partial}{\partial x} - y\sin 2x\tfrac{\partial}{\partial y},\\
&\mathbf{X}_7 = y\cos x\tfrac{\partial}{\partial x} - y^2\sin x\tfrac{\partial}{\partial y},\\
&\mathbf{X}_8 = y\sin x\tfrac{\partial}{\partial x} + y^2\cos x\tfrac{\partial}{\partial y}.
\end{aligned}
\tag{1.4.26}
$$

1.5 Noether symmetries of ODEs

Conservation laws play an important role in understanding the physical properties of dynamical systems such as the law of conservation of energy and momentum, and to deal with such systems and their underlying DEs it is very important to search for the Noether symmetries and their corresponding first integrals because such symmetries reduces the order of equations by two as contrasted with those which reduced the order by one, i.e. Lie symmetries [15].

Definition 1.5.1 A vector field $\mathbf{X}$ is said to be a Noether point symmetry generator corresponding to a Lagrangian, $L(x, y, y')$, of an ODE if there exists a gauge function, $A(x, y)$, which satisfies

$$\mathbf{X}^{(1)}(L) + (\tfrac{d\xi}{dx})L = \tfrac{dA}{dx}, \tag{1.5.1}$$

where $\mathbf{X}^{(1)}$ is the first order prolongation of the symmetry generator $\mathbf{X}$

Theorem 1.5.1. *If* $\mathbf{X}$ *is a Noether point symmetry for a Lagrangian* $L(x, y, y')$ *of an ODE then*

$$I = \xi L + (\eta - y'\xi)L_{y'} - A, \tag{1.5.2}$$

is a first integral of the ODE corresponding to $\mathbf{X}$.

Theorem 1.5.2. *The first integral I corresponding to a Noether symmetry $\mathbf{X}$ satisfies the relation*

$$\mathbf{X}^{(1)}I = 0. \tag{1.5.3}$$

Theorem 1.5.3. *If for a Lagrangian $L(x, y, y')$ of an ODE there corresponds a Noether point symmetry, then that ODE is solvable by quadratures.*

In the subsequent chapters these results will be applied to get Noether symmetries and their corresponding first integrals.

Chapter 2

Symmetries and invariants for CODEs

In this chapter we explore CLS analysis [16] by using complex variables that encode the information of two real variables and a complex structure associated with each real variable. A CLS of a CODE gives rise to two real Lie symmetries for the systems; of DEs; PDEs if the independent variable involved is complex and ODEs if the variable is real. We review some basic definitions and give some examples and then at the end we generalize the Noether theorem for CLS and then state some Noether-like theorems.

2.1 Complex manifolds and CODEs

In order to deal with the theory of CODEs we first need to understand complex manifolds. We now defne a complex manifold as a geometric space that is locally isomorphic to an open subset of $\mathbb{C}^n$.

Definition 2.1.1 A complex manifold $\mathbb{M}_{\mathbb{C}}$ is a manifold whose coordinate charts are open subsets of $\mathbb{C}^n$ and the transition maps between the charts are holomorphic functions [11].

Alternatively, we can define a complex manifold as "A manifold $\mathbb{M}_{\mathbb{C}}$ that has a complex structure on it" [16]. It incorporates the properties like separability, connectedness, it is Hausdorff etc. A complex structure entails that the Cauchy Riemann equations (CREs) are satisfied, or in other words $\mathbb{M}_{\mathbb{C}}$ is totally filled with some open sets U_j, j=1,2, ..., n and holomorphic (complex analytic) functions $f_j : U_j \longrightarrow \mathbb{C}^n$ in the form of open charts (U_j, f_j), i.e. $\bigcup \mathrm{U}_j = \mathbb{M}_\mathbb{C}$. A holomorphic function f is one that is complex differentiable at every point and its open neighborhood, i.e. all first order partial derivatives of f are continuous and satisfy the CREs at every point in that neighborhood. A complex manifold of dimension n corresponds to a real manifold of dimension $2n$. However, the real and complex manifolds possess different properties as holomorphic functions are more rigid than smooth functions. In order to understand the difference between the both we can take the example that every smooth manifold can be embedded as a smooth submanifold of $\mathbb{R}^n$ whereas this is not generally true for a complex manifold i.e. no general holomorphic embedding into $\mathbb{C}^n$ exists. It is quite interesting to see that in any of the compact connected complex manifold $\mathbb{M}_\mathbb{C}$ every holomorphic function is locally constant. This means if we had a holomorphic embedding of $\mathbb{M}_\mathbb{C}$ into $\mathbb{C}^n$, then the coordinate functions of $\mathbb{C}^n$ would get restricted to the nonconstant holomorphic functions on $\mathbb{M}_\mathbb{C}$, which contradicts the well known property of the compactness except for the case when $\mathbb{M}_\mathbb{C}$ is just a complex point in $\mathbb{C}^n$. Those complex manifolds which can be embedded in $\mathbb{C}^n$ are known as "Stein manifolds". It is difficult to give a complete classification for complex manifolds due to their difference in nature when compared with the real manifolds. For example, a topological manifold of dimension other than four possess at most finitely many smooth structures whereas a topological manifold supporting a complex structure supports uncountably many complex structures and hence we obtain one major difference between the both. Two dimensional manifolds equipped with a complex structure such as a Riemann surface are an important example of this phenomenon. The set of complex structures on a given orientable surface itself form a complex algebraic variety called a moduli space and to discuss about its structure many people are working on it.

Therefore, we can particularly say that the complex manifolds are smooth and canonically oriented means not just orientable as a biholomorphic map to a subset of $\mathbb{C}^n$ gives an orientation because the biholomorphic maps are angle and orientation-preserving maps.

We now consider a general CODE

$$u^n(z) = \omega\left(z, u(z), u'(z), u''(z), ..., u^{(n-1)}(z)\right), \tag{2.1.1}$$

where ω is a holomorphic function and the derivative is with respect to z.

For understanding let us first consider a general first-order CODE of the form

$$u'(z) = \omega\left(z, u(z)\right). \tag{2.1.2}$$

If we substitute

$$\begin{aligned} z = x + \iota y, u(z) = f(x, y) + \iota g(x, y), \\ \omega(z, u(z)) = \text{G(x, y, f, g)} + \iota H(x, y, f, g), \\ u'(z) = \tfrac{1}{2}(f_x + g_y) + \tfrac{\iota}{2}(g_x - f_y) = h + \iota l, \end{aligned} \tag{2.1.3}$$

then we obtain the following system of PDEs

$$\begin{aligned} f_x + g_y = 2G(x, y, f, g), g_x - f_y = 2H(x, y, f, g), \\ f_x = g_y, f_y = -g_x, \end{aligned} \tag{2.1.4}$$

where u is analytic function and CREs are satisfied. Now let us consider one example

Example 2.1.1 Complexified Ricatti equation

$$u'(z) + u^2 = 0. \tag{2.1.5}$$

We obtain the following system of PDEs

$$\begin{aligned} f_x + g_y = 2(g^2 - f^2), g_x - f_y = -4fg, \\ f_x = g_y, f_y = -g_x, \end{aligned} \tag{2.1.6}$$

which has the solution

$$f = \tfrac{x}{x^2+y^2}, g = -\tfrac{y}{x^2+y^2}. \tag{2.1.7}$$

Also notice that the above Ricatti equation has the solution

$$u(z) = \tfrac{1}{z}, \tag{2.1.8}$$

and (2.1.8) can be obtained from (2.1.5).

Let us now consider a general second-order CODE of the form

$$u''(z) = \omega\left(z, u(z), u'(z)\right), \tag{2.1.9}$$

where ω is a holomorphic function and

$$\begin{aligned} \omega\left(z, u(z), u'(z)\right) &= G(x, y, f, g, h, l) + \iota H(x, y, f, g, h, l), \\ u'(z) &= \tfrac{1}{2}(f_x + g_y) + \tfrac{\iota}{2}(g_x - f_y) = h + \iota l. \end{aligned} \tag{2.1.10}$$

We obtain the following system of PDEs corresponding to CODE

$$\begin{aligned} f_{xx} - f_{yy} + 2g_{xy} &= 4G(x, y, f, g, h, l), \\ g_{xx} - g_{yy} - 2f_{xy} &= 4H(x, y, f, g, h, l), \\ f_x = g_y, f_y &= -g_x. \end{aligned} \tag{2.1.11}$$

Lets consider one example based on the above notation.

Example 2.1.2 For the complexified oscillator equation

$$u''(z) + u(z) = 0, \tag{2.1.12}$$

we get the following system of PDEs

$$\begin{aligned} f_{xx} - f_{yy} + 2g_{xy} &= -4f, \\ g_{xx} - g_{yy} - 2f_{xy} &= -4g, \\ f_x = g_y, f_y &= -g_x. \end{aligned} \tag{2.1.13}$$

By solving this system, we get

$$\begin{aligned} f &= \alpha_1 \cos x \cosh y + \alpha_2 \sin x \sinh y + \beta_1 \sin x \cosh y - \beta_2 \cos x \sinh y, \\ g &= \alpha_2 \cos x \cosh y - \alpha_1 \sin x \sinh y + \beta_1 \cos x \sinh y + \beta_2 \sin x \cosh y. \end{aligned} \tag{2.1.14}$$

The above solution can also be obtained from the complex solution $u = \alpha \cos z + \beta \sin z$ of the CODE (2.1.12).

Furthermore, it should be noted that since second-order DEs have more applications therefore we restrict the application of the theory to this order. However, it can equally be applied to CODEs of any order.

2.2 Restricted CODEs

If we restrict the complex holomorphic function u to depend explicitly on a single real variable x instead of the complex variable z, then we obtain r-CODE. Consider the general r-CODE of the form

$$u'(x) = \omega\left(x, u(x)\right). \tag{2.2.1}$$

After substituting $u(x) = f(x) + \iota g(x)$ and by using the fact that

$$\omega(x, u(x)) = \omega_1(x, f, g) + \iota \omega_2(x, f, g), \tag{2.2.2}$$

we get

$$\begin{aligned} f' &= \omega_1(x, f, g), \\ g' &= \omega_2(x, f, g). \end{aligned} \tag{2.2.3}$$

Similarly, a second order r-CODE has the form,

$$u''(x) = \omega\left(x, u(x), u'(x)\right), \tag{2.2.4}$$

and we obtain the following system of ODEs

$$f'' = \omega_1(x, f, g, f', g'),$$
$$g'' = \omega_2(x, f, g, f', g'). \tag{2.2.5}$$

Here in this section our aim is to extract a subalgebra from the complex algebra. So, let us take an example to demonstrate this fact.

Example 2.2.1

$$u''(x) + u(x) = 0, \tag{2.2.6}$$

which gives us

$$f'' = -f,$$
$$g'' = -g, \tag{2.2.7}$$

since the general solution of above r-CODE takes the form

$$u(x) = \alpha \cos x + \beta \sin x, \tag{2.2.8}$$

where $\alpha = \alpha_1 + \iota\alpha_2$, and $\beta = \beta_1 + \iota\beta_2$.

Now, we can obtain the general solution of the above system of two ODEs by using the values of α and β in the general solution of the r-CODE,

$$f(x) = \alpha_1 \cos x + \beta_1 \sin x,$$
$$g(x) = \alpha_2 \cos x + \beta_2 \sin x. \tag{2.2.9}$$

2.3 CLSs, prolongation and symmetry condition

The Lie point transformations with complex parameter are defined by

$$\tilde{z} = z(z, u; \epsilon), \tilde{u} = u(z, u; \epsilon), \tag{2.3.1}$$

where $\epsilon = \epsilon_1 + \iota\epsilon_2$, using Taylor's theorem we obtain

$$\xi\left(z,u\right) = \tfrac{\partial \tilde{z}}{\partial \varepsilon} \mid_{\varepsilon=0}, \eta\left(z,u\right) = \tfrac{\partial \tilde{u}}{\partial \varepsilon} \mid_{\varepsilon=0} . \tag{2.3.2}$$

Then we can define

$$\mathbf{Z} = \xi\left(z,u\right) \tfrac{\partial}{\partial z} + \eta\left(z,u\right) \tfrac{\partial}{\partial u}, \tag{2.3.3}$$

as the complex Lie symmetry (CLS) generator.

Definition 2.3.1 Let G be a complex Lie group that acts on a complex manifold $\mathbb{M}_{\mathbb{C}}$. Also a system of m-complex equations in n-complex variables takes the form

$$U_\nu(z^\mu) = 0, (\nu = 1, 2, ..., m, \mu = 1, 2, ..., n), \tag{2.3.4}$$

where U_ν are complex analytic functions. Then a CLS maps complex solutions of (2.3.4) into complex solutions. Next we give the prolongation of the transformations.

Consider an n^{th} order CODE of the form

$$\mathbf{E}\left(z, u(z), u'(z), u''(z), ..., u^{(n)}(z)\right) = 0. \tag{2.3.5}$$

By rearranging we get

$$u^{(n)} = \omega\left(z, u(z), u'(z), u''(z), ..., u^{(n-1)}(z)\right) = 0, \tag{2.3.6}$$

then the prolonged symmetry generator in the complex domain takes the form

$$\mathbf{Z} = \xi\left(z,u\right) \tfrac{\partial}{\partial z} + \eta\left(z,u\right) \tfrac{\partial}{\partial u} + \eta^{(n)}\left(z,u\right) \tfrac{\partial}{\partial u^{(n)}}, \tag{2.3.7}$$

where n can be any positive integer and prolongation coefficients

$$\eta^{(n)} = \tfrac{d\eta^{(n-1)}}{dz} - u^n(z)\tfrac{d\eta}{dz}. \tag{2.3.8}$$

The symmetry condition is also the same as it was in the case of real domain

$$\mathbf{Z}\omega = \eta^{(n)}_{|u^n=\omega}. \tag{2.3.9}$$

One can notice that a complex line in $\mathbb{C} \cong \mathbb{R}^2$ has 16 and $\mathbb{C}^2 \cong \mathbb{R}^4$ has 30 real symmetries whereas a real line in $\mathbb{R}^2$ has 15 and in $\mathbb{R}^4$ has 35 real symmetries which shows that in higher dimensions a real line has more real symmetries than a complex line and it is due to the fact that a single complex symmetry couples two real symmetries and it is one factor due to which complex symmetries have more importance than real.

2.4 Noether symmetries and first integrals

To discuss the two types of variational problems; first in which we are given a Lagrangian and we have to find the DE by solving Euler-Lagrange equations and the other in which the DE is known and we have to find or construct the Lagrangian which, perhaps, is a difficult problem for which there may be no solution and it is known as the "inverse problem". To deal with such problems let us consider a second-order CODE

$$u'' = \omega(z, u, u'), \tag{2.4.1}$$

where ω is some complex analytic function. In order to discuss the existence of the above CODE one obtains the real Lagrangian(r-Lagrangian) from the variational principle then complexifies that Lagrangian for the purpose of complex symmetry analysis (CSA). Since the new pair of terms does not satisfy the EL equation so we split the EL equation into two EL-like equations for the pair of two real variables and obtain two r-Lagrangians which give rise to the coupled system of DEs. The complex Lagrangian(c-Lagrangian) $L(z, u, u')$ [16, 17] satisfies the EL equation

$$\frac{\partial L}{\partial u} - \frac{d}{dz}(\frac{\partial L}{\partial u'}) = 0. \tag{2.4.2}$$

Definition 2.4.1 Z is said to be a complex Noether point symmetry corresponding to a c-Lagrangian $L(z, u, u')$ of (2.4.1) if there exists a complex guage function $A(z, u)$ such that

$$\begin{gathered} \mathbf{Z}L + (\tfrac{d\xi}{dz})L = \tfrac{dA}{dz}, \\ \mathbf{Z} = \xi\tfrac{\partial}{\partial z} + \eta\tfrac{\partial}{\partial u} + \eta^{(1)}\tfrac{\partial}{\partial u'}, \qquad \tfrac{d}{dz} = \tfrac{\partial}{\partial z} + u'\tfrac{\partial}{\partial u} + ... \end{gathered} \tag{2.4.3}$$

Next we state Noether theorems for the complex domain.

Theorem 2.4.1 *If* $\mathbf{Z}$ *is a Noether point symmetry corresponding to a c-Lagrangian* $L(z,u,u')$ *of (2.4.1) then*

$$I = \xi L + (\eta - u'\xi)L_{u'} - A, \tag{2.4.4}$$

is a complex first integral of (2.4.1) associated with $\mathbf{Z}$.

Theorem 2.4.2 *The first integral* I *associated with the complex Noether point symmetry* $\mathbf{Z}$ *satisfies the relation*

$$\mathbf{Z}I = 0. \tag{2.4.5}$$

Theorem 2.4.3 *If for a c-Lagrangian* $L(z,u,u')$ *of* (2.4.1) *there corresponds a complex Noether point symmetry, then* (2.4.1) *is solvable by quadratures.*

Next we give one example on NSs and the first integral

Example 2.4.1 The complexified free-particle equation

$$u''(z) = 0, \tag{2.4.6}$$

which admits the c-Lagrangian

$$L = \frac{1}{2}u'^{\,2}. \tag{2.4.7}$$

The CODE (2.4.6) has 5 NSs for the c-Lagrangian (2.4.7)

$$\mathbf{Z}_1 = \tfrac{\partial}{\partial z}, \quad \mathbf{Z}_2 = \tfrac{\partial}{\partial u}, \quad \mathbf{Z}_3 = 2z\tfrac{\partial}{\partial z} + u\tfrac{\partial}{\partial u}, \quad \mathbf{Z}_4 = z^2\tfrac{\partial}{\partial z} + zu\tfrac{\partial}{\partial u}, \quad \mathbf{Z}_5 = z\tfrac{\partial}{\partial u}. \tag{2.4.8}$$

Now to reduce the order of (2.4.6) by two with any of above symmetries. For example, if we apply NS $\mathbf{Z}_2$ on (2.4.6) we obtain

$$\frac{\partial}{\partial u}(u'') = 0, \tag{2.4.9}$$

then the first integral comes out to be

$$I = u' = a. \tag{2.4.10}$$

Notice that $\mathbf{Z}_2$ is also a symmetry of the c-Lagrangian (2.4.7). The above equation gives the solution

$$u = az + b, \tag{2.4.11}$$

where a and b are some complex constants.

On the other hand if we use NS $\mathbf{Z}_3$ we obtain the following first integral

$$I = u\ u' - zu'^2 = c. \tag{2.4.12}$$

Since (2.3.9) is satisfied by $\mathbf{Z}_3$ therefore it is symmetry of the above equation. If we introduce

$$w = uz^{-\frac{1}{2}}, \tag{2.4.13}$$

then the above first integral reduces to

$$w^2 - 4z^2w'^2 = d. \tag{2.4.14}$$

Thus we can say that one NS reduces the second order CODE into quadratures and now the solution can be found in new coordinates (z, w)

$$w(z) = \alpha z^{\frac{1}{2}} + \beta z^{-\frac{1}{2}}, \tag{2.4.15}$$

where α and β are some complex constants.

2.5 Noether-like operators and theorems

To define Noether-like operators let us begin by splitting the complex Lagrangian into the pair of two real Lagrangians (r-Lagrangians)

$$L = L_1 + iL_2. \tag{2.5.1}$$

The system of EL-like equations corresponding to (2.4.2) is

$$\frac{\partial L_1}{\partial f} + \frac{\partial L_2}{\partial g} - d_x(\frac{\partial L_1}{\partial h} + \frac{\partial L_2}{\partial l}) - d_y(\frac{\partial L_2}{\partial h} - \frac{\partial L_1}{\partial l}) = 0, \tag{2.5.2}$$

$$\frac{\partial L_2}{\partial f} - \frac{\partial L_1}{\partial g} - d_x(\frac{\partial L_2}{\partial h} - \frac{\partial L_1}{\partial l}) + d_y(\frac{\partial L_1}{\partial h} + \frac{\partial L_2}{\partial l}) = 0. \tag{2.5.3}$$

where

$$d_x = \partial_x + f_x\partial_f + g_x\partial_g + h_x\partial_h + l_x\partial_l, \tag{2.5.4}$$

$$d_y = \partial_y + f_y\partial_f + g_y\partial_g + h_y\partial_h + l_y\partial_l. \tag{2.5.5}$$

The system (2.5.2), (2.5.3) couples the two r-Lagrangians L_1 and L_2 which satisfy only the coupled system and not the EL equations so they are not Lagrangians when treated separately. Now we use the definition of Noether symmetry and split it into two real pairs and hence we get another system which has the form

$$2\mathbf{X}L_1 - 2\mathbf{Y}L_2 + (d_x\xi_1 + d_y\xi_2)L_1 - (d_x\xi_2 - d_y\xi_1)L_2 = d_xA_1 + d_yA_2, \tag{2.5.6}$$

$$2\mathbf{X}L_2 + 2\mathbf{Y}L_1 + (d_x\xi_1 + d_y\xi_2)L_2 + (d_x\xi_2 - d_y\xi_1)L_1 = d_xA_2 - d_yA_1. \tag{2.5.7}$$

The first integral (2.4.4) results in two real conserved quantities

$$I_1 = \xi_1L_1 - \xi_2L_2 + (\eta_1 - h\xi_1 - l\xi_2)(\partial_hL_1 + \partial_lL_2) - (\eta_2 - h\xi_2 - l\xi_1)(\partial_hL_2 - \partial_lL_1) - A_1, \tag{2.5.8}$$

$$I_2 = \xi_1L_2 + \xi_2L_1 + (\eta_1 - h\xi_1 - l\xi_2)(\partial_hL_2 - \partial_lL_1) + (\eta_2 - h\xi_2 - l\xi_1)(\partial_hL_1 + \partial_lL_2) - A_2. \tag{2.5.9}$$

The two first integrals satisfy the coupled equations

$$\mathbf{X}I_1 - \mathbf{Y}I_2 = 0, \tag{2.5.10}$$

$$\mathbf{X}I_2 + \mathbf{Y}I_1 = 0, \tag{2.5.11}$$

where

$$\mathbf{Z} = \mathbf{X} + i\mathbf{Y}. \tag{2.5.12}$$

In chapter 4, we give a detailed discussion of these theorems and how they are used to get the first integrals of the corresponding DEs.

Chapter 3

Approximate symmetries of p-CODEs

In this chapter we introduce the mathematical formulation of an approach for solving p-CODEs [18] and then apply it to solve a few well known physical models involving second order p-CODEs [1]. Complex DEs arise in many situations. However, in some cases the DEs describing a model may contain parameters that are known approximately. Since every DE corresponds to some geometry, so to look for whether its geometry possess symmetries one may generally consider manifolds to work on. In general a complex manifold may not possess an exact symmetry but may approximately do so. It would be of great interest to look at the approximate symmetries of the complex manifold. Then form an approximate complex Lie algebra.

The basics of the approximate symmetries were developed by Baikov et al. [19]. These authors showed that the main part of Lie's theory can be used in an approximate calculus taking into account the smallness of the critical parameter in the theory. The new theory maintains the essential features of the standard Lie theory. Here we provide a concise introduction to the theory of approximate transformation groups and approximate symmetries of DEs with a small parameter.

3.1 Mathematical formulation

In this section the method of determining the approximate symmetries for the p-CODEs is defined in detail. However, the theory has been generalized to PDEs.
Consider the general form of the perturbed complex differential equation E

$$E \equiv E_0 + \varepsilon E_1 + \varepsilon^2 E_2 + ... + \varepsilon^n E_n = 0, \tag{3.1.1}$$

where $E_0, E_1, E_2, ..., E_n$ are the exact, first order, second order, and so on so forth up to n^{th} order approximate parts of the complex DE E, and ε is some complex parameter for which the infinitesimal complex symmetry generator is

$$\mathbf{X} = \mathbf{X}_0 + \varepsilon \mathbf{X}_1 + \varepsilon^2 \mathbf{X}_2 + ... + \varepsilon^n \mathbf{X}_n, \tag{3.1.2}$$

where

$$\begin{aligned} \mathbf{X}_0 = \xi_0(z,u)\tfrac{\partial}{\partial z} + \eta_0(z,u)\tfrac{\partial}{\partial u} + \eta_0^1(z,u,u')\tfrac{\partial}{\partial u'} + \eta_0^2(z,u,u',u'')\tfrac{\partial}{\partial u''} + ... \\ +\eta_0^{(n)}(z,u,u',...,u^{(n)})\tfrac{\partial}{\partial u^{(n)}}, \end{aligned} \tag{3.1.3}$$

is the exact complex symmetry generator corresponding to the unperturbed equation E_0 and

$$\begin{aligned} \mathbf{X}_i = \xi_i(z,u)\tfrac{\partial}{\partial z} + \eta_i(z,u)\tfrac{\partial}{\partial u} + \eta_i^1(z,u,u')\tfrac{\partial}{\partial u'} + \eta_i^2(z,u,u',u'')\tfrac{\partial}{\partial u''} + ... \\ +\eta_i^{(n)}(z,u,u',...,u^{(n)})\tfrac{\partial}{\partial u^{(n)}}, \end{aligned} \tag{3.1.4}$$

is the corresponding i^{th}-order approximate symmetry generator for every $i \epsilon N$. Now we define the criteria for the existence of n^{th}-order approximate symmetry of E. If

$$\begin{aligned} \mathbf{X}E := [\mathbf{X}_0 + \varepsilon \mathbf{X}_1 + \varepsilon^2 \mathbf{X}_2 + ... + \varepsilon^n \mathbf{X}_n](E_0 + \varepsilon E_1 + \varepsilon^2 E_2 \\ +... + \varepsilon^n E_n)]_{E=E_0+\varepsilon E_1+...+\varepsilon^n E_n} = O(\varepsilon^{n+1}), \end{aligned} \tag{3.1.5}$$

then (3.1.2) is the n^{th}-order approximate symmetry of E.

Now for calculational simplicity we restrict the index to 1 and consider only the first order approximate symmetries and second order ODE , however, the procedure may be worked out for any finite index n and for any order ODE or even for any PDE [18]. Thus, if

$$\mathbf{X} = \mathbf{X}_0 + \varepsilon \mathbf{X}^1, \tag{3.1.6}$$

where

$$\mathbf{X}_0 = \xi_0 \frac{\partial}{\partial z} + \eta_0 \frac{\partial}{\partial u} \text{ and } \mathbf{X}^1 = \xi_1 \frac{\partial}{\partial z} + \eta_1 \frac{\partial}{\partial u}, \tag{3.1.7}$$

is the first order approximate symmetry generator of the p-CODE

$$E_o + \varepsilon E_1 = 0, \tag{3.1.8}$$

then (3.1.8) will be approximately invariant under the group of transformations with the generator (3.1.6) iff $\mathbf{X}E \mid_{E=0} = O(\varepsilon^2)$, or equivalently

$$\left[\mathbf{X}_0 E_0 + \varepsilon \left(\mathbf{X}_0 E_1 + \mathbf{X}^1 E_0\right)\right] \mid_{E=0} = O(\varepsilon^2). \tag{3.1.9}$$

Alternatively, If $\mathbf{X}_0$ is a CLS generator of a DE

$$E_0 = 0, \tag{3.1.10}$$

then an approximate symmetry of the perturbed DE (3.1.8) is obtained by solving for $\mathbf{X}^1$ in

$$\mathbf{X}^1(E_0)|_{E_0=0} + H = 0, \tag{3.1.11}$$

where

$$H = \frac{1}{\varepsilon} \mathbf{X}_0 (E_0 + \varepsilon E_1)|_{E_0 + \varepsilon E_1 = 0}. \tag{3.1.12}$$

Here H is called an auxiliary function. In the next section we apply the same procedure and work out the approximate symmetries.

3.2 Application to complexified oscillator equation

Consider the perturbed complexified oscillator equation

$$u'' + u = \varepsilon u. \tag{3.2.1}$$

The exact symmetries of

$$u'' + u = 0, \tag{3.2.2}$$

are

$$\begin{aligned}
&\mathbf{X}_1=\tfrac{\partial}{\partial z},\quad \mathbf{X}_2=u\tfrac{\partial}{\partial u},\quad \mathbf{X}_3=\sin z\tfrac{\partial}{\partial u},\\
&\mathbf{X}_4=\cos z\tfrac{\partial}{\partial u},\quad \mathbf{X}_5=\sin 2z\tfrac{\partial}{\partial z}+u\cos 2z\tfrac{\partial}{\partial u},\\
&\mathbf{X}_6=\cos 2z\tfrac{\partial}{\partial z}-u\sin 2z\tfrac{\partial}{\partial u},\quad \mathbf{X}_7=u\cos z\tfrac{\partial}{\partial z}-u^2\sin z\tfrac{\partial}{\partial u},\\
&\mathbf{X}_8=u\sin z\tfrac{\partial}{\partial z}+u^2\cos z\tfrac{\partial}{\partial u},
\end{aligned}\tag{3.2.3}$$

using the criteria as defined in (3.1.11) and (3.1.12)

$$\begin{aligned}
&\mathbf{X}^1(E_o)|_{E_o=0}+H=0,\\
&H=\tfrac{1}{\varepsilon}\mathbf{X}_0(E_0+\varepsilon E_1)|_{E_0+\varepsilon E_1=0},
\end{aligned}\tag{3.2.4}$$

then the eight-parameter exact symmetry generator corresponding to (3.2.3) is

$$\begin{aligned}
\mathbf{X}_0=&(c_1+c_5\sin 2z+c_6\cos 2z+c_7u\cos z+c_8u\sin z)\tfrac{\partial}{\partial z}+\\
&(c_2u+c_3\sin z+c_4\cos z+c_5u\cos 2z-c_6u\sin 2z-\\
&c_7u^2\sin z+c_8u^2\cos z)\tfrac{\partial}{\partial u}.
\end{aligned}\tag{3.2.5}$$

Using (3.2.4) we obtain

$$\begin{aligned}
H=&(c_7u^2-c_3-3c_8uu')\sin z-(c_4+c_8u^2+3c_7uu')\cos z+\\
&4c_6u\sin 2z-4c_5u\cos 2z.
\end{aligned}\tag{3.2.6}$$

Substituting (3.2.6) in (3.2.4) we get

$$\begin{aligned}
&u'(2\eta_{zu}-\xi_{zz}+3u\xi_u-3uc_7\cos z-3uc_8\sin z)+u'^2(\eta_{uu}-2\xi_{zu})-u'^3\xi_{uu}\\
&+\eta+\eta_{zz}-u\eta_u+2u\xi_z+4c_6u\sin 2z-4c_5u\cos 2z+(c_7u^2-c_3)\sin z\\
&-(c_8u^2+c_4)\cos z=0.
\end{aligned}\tag{3.2.7}$$

By equating the powers of u' we get

$$2\eta_{zu}-\xi_{zz}+3u\xi_u-3uc_7\cos z-3uc_8\sin z=0,\tag{3.2.8}$$

$$\eta_{uu}-2\xi_{zu}=0,\tag{3.2.9}$$

$$\xi_{uu}=0,\tag{3.2.10}$$

$$\eta + \eta_{zz} - u\eta_u + 2u\xi_z + 4c_6 u \sin 2z - 4c_5 u \cos 2z +$$
$$(c_7 u^2 - c_3)\sin z - (c_8 u^2 + c_4)\cos z = 0. \tag{3.2.11}$$

Solving the above system for ξ and η we get only six first order complex approximate symmetry generators corresponding to (3.2.1)

$$\begin{aligned}
&\mathbf{X}^3 = \sin z\frac{\partial}{\partial u} + \varepsilon(\tfrac{\sin z}{2} - \tfrac{z\cos z}{2})\frac{\partial}{\partial u}, \mathbf{X}^4 = \cos z\frac{\partial}{\partial u} + \varepsilon\tfrac{z\sin z}{2}\frac{\partial}{\partial u},\\
&\mathbf{X}^5 = \sin 2z\frac{\partial}{\partial z} + u\cos 2z\frac{\partial}{\partial u} + \varepsilon[(\tfrac{\sin 2z}{2} - z\cos 2z)\frac{\partial}{\partial z} + (zu\sin 2z)\frac{\partial}{\partial u}],\\
&\mathbf{X}^6 = \cos 2z\frac{\partial}{\partial z} - u\sin 2z\frac{\partial}{\partial u} + \varepsilon[(z\sin 2z + \cos 2z)\frac{\partial}{\partial z} + (zu\cos 2z - \tfrac{u\sin 2z}{2})\frac{\partial}{\partial u}],\\
&\mathbf{X}^7 = u\cos z\frac{\partial}{\partial z} - u^2\sin z\frac{\partial}{\partial u} + \varepsilon[\tfrac{zu\sin z}{2}\frac{\partial}{\partial z} + \tfrac{u^2}{2}(z\cos z + \sin z)\frac{\partial}{\partial u}],\\
&\mathbf{X}^8 = u\sin z\frac{\partial}{\partial z} + u^2\cos z\frac{\partial}{\partial u} + \varepsilon[\tfrac{u}{2}(\sin z - z\cos z)\frac{\partial}{\partial z} + \tfrac{u^2}{2}z\sin z\frac{\partial}{\partial u}],
\end{aligned} \tag{3.2.12}$$

where $\mathbf{X}^i = \mathbf{X}_0 + \varepsilon\mathbf{X}^1$.

3.3 Application to the complexified damped harmonic oscillator equation

Let us now consider the complexified damped harmonic oscillator equation

$$u'' + u + 2\varepsilon u' = 0. \tag{3.3.1}$$

Following the same procedure we get three first order approximate symmetries of p-CODE.

$$\begin{aligned}
&\mathbf{X}^3 = \sin z\frac{\partial}{\partial u} - \varepsilon z\sin z\frac{\partial}{\partial u},\\
&\mathbf{X}^4 = \cos z\frac{\partial}{\partial u} - \varepsilon z\sin z\frac{\partial}{\partial u},\\
&\mathbf{X}^7 = u\cos z\frac{\partial}{\partial z} - u^2\sin z\frac{\partial}{\partial u} + \varepsilon[\big(-\tfrac{5}{3}zu\cos z + 4\sin z - 4z\cos z)\big)\frac{\partial}{\partial z} +\\
&\big(-\tfrac{5}{3}u^2(\cos z - z\sin z) + 2u^2\cos z + 2zu\sin z\big)\frac{\partial}{\partial u}].
\end{aligned} \tag{3.3.2}$$

The existence of first (or even of higher) order complex approximate symmetries help us to obtain an analytic solution of the perturbed DEs in general. This approach can be very useful in obtaining the approximately conserved quantities (invariants).

Chapter 4

Complex harmonic oscillator and its physical interpretation

The mechanism for the coupled harmonic oscillator involves systems of two second order ODEs, which have wide applications in the general theory of relativity, in nonlinear oscillations, in population dynamics, in the classical theory of mechanics, in pattern recognition, in quantum mechanics, in fluid mechanics, in solid mechanics, etc.

In order to obtain a deeper understanding of the system of two DEs and its physical properties we search for its solution and sometimes the system looks quite complicated. So if we are able to find another system in which one system is much easier to solve than the other, then surely we have discovered something significant about the more difficult looking system. We often use transformations to obtain another equation (DE or its system) and since every analytic method has its limitation so we prefer to use symmetries i.e. Lie, Noether, etc. which form a group of continuous point transformations. When we talk about the system of two DEs, then beside linearity and nonlinearity one may consider the system to be coupled or uncoupled: If there are two (or more) equations such that both can be solved independently without the help of the other equation and solution can be found then such system is called an "uncoupled system". If both equations can only be solved simultaneously and it is not possible to solve either of the equations independently, the systems are called "coupled systems".

Let us consider the dynamical equation of two particles interacting with each other and can be represented by system of two second-order ODEs of the form

$$x'' = G(t, x, y, x', y'),$$
$$y'' = H(t, x, y, x', y'), \tag{4.0.1}$$

in which a time dependent oscillator equation appears in several physical situations e.g. motion of charged particle in the magnetic field and in the study of plasmas can be regarded to deal with such an equation whereas in the theory of radiation it is with variable frequency and in small oscillations of the pendulum whose length changes uniformly. The complex extension of such an oscillator and complex symmetry analysis will help us in working out invariants of the dynamical equations of the two interacting charged particles in magnetic fields.

In the next section I review some work on the time-independent harmonic oscillator which admits Noether symmetries and corresponding to some complex Lagrangian, work out its two first integrals to see where the energy is stored and during this analysis pointed out some errors in [20] which are given in an erratum [21]. At the end I give a generalization of the same work and obtain a general expression for the two first integrals in which it can be seen how the change of phase angle affects the invariants thus obtained. Finally, I summarize the entire discussion.

4.1 Mathematical formulation

Consider a system of two coupled time-independent harmonic oscillators oscillating with different frequencies in different phases

$$x'' = -\alpha x + \beta y,$$
$$y'' = -\beta x - \alpha y, \tag{4.1.1}$$

where α and β have units of inverse time squared. Here, it can be observed that there is no exchange of energy for $\beta = 0$ and if we choose α to be some multiple of β this corresponds to selecting some angle between the principal axis and the original coordinates. Choose the phase angle to be $\frac{\pi}{4}$ radians, i.e. $\alpha = -\beta$ in the above system for definiteness. The system admits Noether symmetry $\mathbf{X} = \frac{\partial}{\partial t}$, which is also a Noether-like symmetry. Notice further that the system is relative to the complex Lagrangian

$$L = \tfrac{1}{2}z'^2 - \tfrac{1}{2}\lambda z^2, \tag{4.1.2}$$

where $\lambda = \alpha + \iota\beta$, using (2.4.3) we get $A = constant$, and CODE corresponding to this system where $\alpha = -\beta$ takes the form

$$z'' = \beta z - \iota\beta z, \tag{4.1.3}$$

and if we choose $k = \beta - \iota\beta$ then CODE becomes

$$z'' = kz. \tag{4.1.4}$$

Now by utilizing (2.4.4), (2.4.5), (2.5.8), and (2.5.9) we obtain the two invariants

$$\begin{aligned} I_1 &= \tfrac{1}{2}(x'^2 - \beta x^2) - \tfrac{1}{2}(y'^2 - \beta y^2) - \beta xy, \\ I_2 &= x'y' + \tfrac{\beta}{2}(x^2 - y^2) - \beta xy. \end{aligned} \tag{4.1.5}$$

Where $L_1 = \frac{1}{2}(x'^2 - y'^2 - \lambda x^2 + \lambda y^2), L_2 = x'y' - xy$ are the two r-Lagrangians. Also, the errors in [20] have been corrected and appeared in [21].

Notice that the integral I_1 corresponds to the total energy of the system, the last term of I_1 corresponds to the exchange of the energy between the two oscillators and the integral I_2 corresponds to the exchanged kinetic and potential energies between the two oscillators that remains conserved. This analysis explains the fact that the energy is not stored in either oscillator but it is stored in the field between them.

Furthermore, if we change the phase angle that is for $\alpha = -\beta$, $\alpha = -\frac{\beta}{2}$, $\alpha = -\frac{\beta}{3}$, $\alpha = -\frac{\beta}{4}$, or generally to $\alpha = -\frac{\beta}{n}$, for $n \neq 0$ then we get the following two integrals

$$\begin{aligned} I_1 &= \tfrac{1}{2}(y'^2 - x'^2) + \tfrac{\beta}{2n}(x^2 - y^2) + \beta xy, \\ I_2 &= \tfrac{\beta}{2}(y^2 - x^2) + \tfrac{\beta}{n}xy - x'y', \end{aligned} \tag{4.1.6}$$

and for $\alpha = \beta$, $\alpha = \frac{\beta}{2}$, $\alpha = \frac{\beta}{3}$, and $\alpha = \frac{\beta}{4}$ or generally to $\alpha = \frac{\beta}{n}$, for $n \neq 0$ we get the following two integrals

$$\begin{aligned} I_1 &= \tfrac{1}{2}(y'^2 - x'^2) + \tfrac{\beta}{2n}(y^2 - x^2) + \beta xy, \\ I_2 &= \tfrac{\beta}{2}(y^2 - x^2) - \tfrac{\beta}{n}xy - x'y'. \end{aligned} \tag{4.1.7}$$

This shows that if we change the phase angle then we get the different invariants [1]. And by this analysis we are able to convert the single complex oscillator into a pair of coupled oscillators and by using both approaches we can find the conservation laws.

4.2 Summary

The first chapter of this dissertation contains the concept of real manifolds, one parameter point transformations, symmetry in general and mathematical symmetry in particular and the theory of Lie groups and Lie algebras, the criteria for finding the exact symmetries of DEs and application of such analysis by solving few DEs. Later we define the Noether symmetries and corresponding first integrals for the ODEs.

The second chapter is begun by defining the manifolds having complex structure on it or in other words "complex manifolds", then we complexify the DEs and define the theory of CODEs, r-CODEs, their point transformations, CLSs, Noether theorems and Noether-like operators and how one can obtain the real invariants corresponding to both approaches.

Chapter 3 contains the mathematical treatment of perturbed DEs and particularly p-CODEs in which the parameter involved is complex and explains how such an analysis helps us to find first-order approximate symmetries of second-order p-CODEs. We are currently engaged in applying it to the complexified oscillator equation. Approximate symmetries have been worked out which can help us to calculate the approximate invariants and to understand the physical systems in greater detail. The work is still in progress [1].

In the end the theory of two dimensional coupled harmonic oscillators is considered, which can be represented with the help of one CODE and we worked out the first integrals (invariants) corresponding to the coupled system and it was found that there was a mistake in the published work [20] which was corrected [21] on account of this work. At the end we dealt with the same system for more choices of the α and β (or for other choices of the phase angles between the oscillators) and obtained the generalized form of the invariants. Also, It was noticed that a change in the phase angle between the oscillators yields the different invariants.

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