Relationship Between M$^\natural$-Convexity and Ultramodularity

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Dedication

To

HAZRAT ALI (R.A)

Abstract

In the theory of discrete convex analysis, problems related to discrete optimization are solved with the help of continuous optimization and combinatorial optimization. $\mathrm{M}^\natural$-convex functions play significant role in the theory of discrete optimization. On the other hand, ultramodular functions appears in several pure and applied research areas. In this thesis we investigate the relationship between two newly developed classes of functions known as ultramodular functions and $\mathrm{M}^\natural$-convex functions. We prove that the class of $\mathrm{M}^\natural$-convex functions is contained in the class of ultramodular functions on integer lattice. Moreover, we introduce a new class of functions, called the component wise ultramodular functions. We show that the class of component wise ultramodular functions is contained in the class of $\mathrm{M}^\natural$-convex functions, on integer lattice.

Introduction

In this thesis, as the name suggests, we will study the relationship between two recently developed classes of functions known as ultramodular functions and M$^\natural$-convex functions.

The class of ultramodular functions generalizes scalar convexity. These functions appear in several economic and statistical applications, where they provide extension of one-dimensional convexity. Ultramodular functions give stronger form of complementary than supermodularity. M$^\natural$-convex functions form the class of well behaved functions. They play very important role in the theory of discrete convex analysis. The theory of discrete convex analysis has broad applications in several areas, for instance, operation research, economics, mathematics etc. This thesis is organized as follows. In Chapter one we give introduction to the theory of Discrete Convex Analysis, developed by K. Murota. In this chapter we provide the notions and few results related to M$^\natural$-convexity in a compact way. Some characterizations of M$^\natural$-convexity will also be discussed.

In Chapter two, ultramodular functions will be discussed. Special class of ultramodular functions known as ultramodular aggregation functions will be given as well.

Finally, in Chapter three we present our main work. We give an equivalent definition of ultramodular functions. We develop a relationship between M$^\natural$-convex functions and ultramodular functions. We prove that the class of M$^\natural$-convex functions is contained in the class of ultramodular functions on integer lattice. Moreover, we introduce a new class of functions, called the component wise ultramodular functions. We show that the class of component wise ultramodular functions is contained in the class of M$^\natural$-convex functions, on integer lattice.

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Sumbul Azeem

Contents

Chapter 1

Discrete Convex Analysis

In this chapter we shall briefly portray history and development of discrete convex analysis. After this, we shall present the well known concept of convex sets and convex functions. Then, we present the concept of $\mathrm{M}^\natural$-convexity and some results associated with it. Finally, few characterizations related to $\mathrm{M}^\natural$-convex functions shall be discussed.

1.1 Introduction to Discrete Convex Analysis

K. Murota [13] developed and widen the theory of discrete convex analysis in the past few years. Formally speaking the fundamental purpose behind establishing discrete convex analysis is to provide a theoretical structure to solve combinatorial optimization problems. Ideas from discrete optimization and continuous optimization are merged together to solve discrete optimization problems. In this regard, the abstract structure of convex analysis is modified to discrete settings and mathematical consequences existing in matroid theory are made more general. Looking from the discrete area, discrete convex analysis is the study of discrete functions $f : \mathbf{Z}^n \to \mathbf{Z}$. These functions have the benefit of assured fine properties resembling to convexity. On the other hand, if one looks from the continuous perspective, the theory of discrete convex analysis can be categorized as a study of convex functions $f : \mathbf{R}^n \to \mathbf{R}$ having certain combinatorial properties. Thus,
Discrete convex analysis = Convex analysis + Matroid theory.

The theory of discrete convex analysis has its wide applications in the field of mathematics, operations research, economics and many more.

In 1935, H. Whitney [21] introduced the notions of matroids, along with the equivalence between exchange property of independent sets and submodular rank functions. In the recent theory of discrete convexity, this equivalence is the core of the conjugacy between M-convex functions and L-convex functions.

In 1990, Dress and Wenzel [3, 4] presented the notion of valuated matroids. By valuated matroid, we mean a nonseparable and nonlinear function on the set of bases of some matroid fulfilling certain exchange axiom. In 1996, Murota [14] extended the correspondence of valuated matroids to concave functions that led to the idea of M-concave and M-convex functions. M-convex functions have a common generalization of integral polymatroids and valuated matroids. In M-convexity, functions are defined on integer lattice points in terms of an exchange axiom. In early 1980s and early 1990s two independent developments related to convexity arguments for submodular functions and valuated matroids for M-convex functions were made. In 1998, Murota [15] combined these two classes into the theory of discrete convex analysis.

Later on class of $M^\natural$-convex functions were introduced by Murota and Shioura [16]. According to them $M^\natural$-convex functions are conceptually equivalent to M-convex functions. But in certain situations $M^\natural$-convex functions are more convenient. For instance, a convex function in one variable defined on integers is an $M^\natural$-convex function that is not M-convex.

1.2 Convex Sets and Convex Functions

Definition 1.2.1. Let N be a linear space. A subset S of N is said to be *convex set* if for any $x, y \in S$ and $0 \leq t \leq 1$

$$tx + (1-t)y \in S.$$

Here the set of points $z = tx + (1-t)y$ is called the line segment from x to y. The points x and y are known as the end points of line segment z. Thus, roughly speaking one can say that a set is convex if the line segment joining any two points of the given set lies within that set. A set which does not satisfy the stated criteria is called a *non convex set*. Few examples of convex sets include empty set, solid sphere, solid cube etc. Non convex sets are unfilled shapes or shapes having dent in them, for example, crescent. Following figure illustrates the concept of convex set and non convex set.

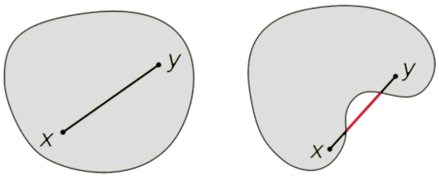

Figure 1.1: Convex Set and Non Convex Set

Definition 1.2.2. A function $f : S \subseteq N \to \mathbf{R}$ is called *convex function* if for any $x, y \in S$ and $0 \leq t \leq 1$, we have

$$tf(x) + (1-t)f(y) \geq f\big(tx + (1-t)y\big).$$

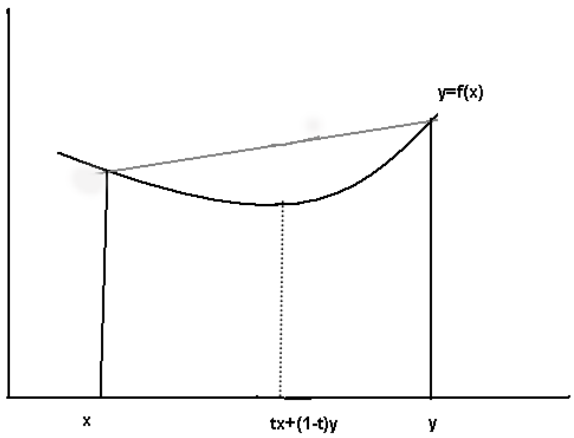

Figure 1.2: Graph of a convex function

The graph of a convex function lies on or below the line segment with the end points $\big(x, f(x)\big)$ and $\big(y, f(y)\big)$. Thus we can say that a function f is convex if and only if epigraph[1] of f is a convex set.

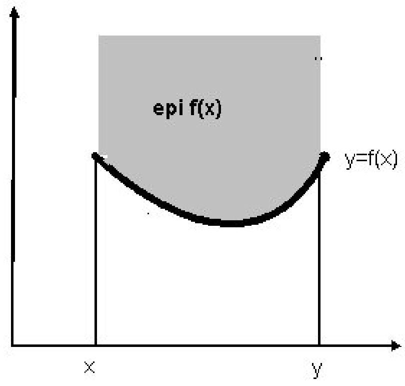

Figure 1.3: Epigraph of a function f

Convex functions play significant role in many branches of mathematics, as well as other fields of science and engineering. They are specially essential in the field of optimization problems. Convexity is simple concept which can be traced back to Archimedes around 250 B.C., in association with his famous approximation of the value of π by using circumscribed and inscribed regular polygons. He examined the significant fact that the perimeter of a convex figure is less than the perimeter of any other convex shape neighboring it. One of the consequences of convex functions is Jensen's inequality. Several important inequalities, for example, the Hölder's inequality and arithmetic-geometric mean inequality are consequences of Jensen's inequality.

The following definition gives the concept of convexity and mid point convexity in terms of real valued functions defined on an interval I.

Definition 1.2.3. Consider a subinterval I of $\mathbf{R}$ and let $f : I \to \mathbf{R}$ be a real valued function.

(i) The function f is *convex* if, for any $x, y \in I$ and for all t such that $t \in [0, 1]$

$$tf(x) + (1-t)f(y) \geq f\big(tx + (1-t)y\big). \tag{1.2.1}$$

[1]The set of points located on or above the graph of f.

(ii) The function f is said to be *mid point convex* or *Jensen convex* if, for any $x, y \in I$

$$f\left(\frac{x+y}{2}\right) \leq \frac{f(x)+f(y)}{2}. \tag{1.2.2}$$

Replacing $t = \frac{1}{2}$ in (1.2.1), we see that every convex function is also midpoint convex function. However the converse does not hold in general. The following remark, taken from E.P. Klement, M. Manzi, R. Mesiar [11] states certain conditions which are equivalent to the definition of convexity stated in (1.2.1).

Remark 1.2.1. Consider an interval I of $\mathbf{R}$ and let $f : I \to \mathbf{R}$ be a real valued function. Then:

(i) The function f is convex if and only if

$$f(x+h) - f(x) \geq f(y+h) - f(y) \tag{1.2.3}$$

holds for all $x, y \in I$ and for all $h > 0$ with $x > y$ and $x + h \in I$.

(ii) If f is a continuous function then f is convex if and only if it is Jensen convex.

(iii) If f is a monotone function then f is convex if and only if f is Jensen convex.

(iv) If f is a bounded function then f is convex if and only if f is Jensen convex.

Proof. (i). Suppose f is convex and let $x, y \in I$ and $h > 0$ such that $x > y$ and $x + h \in I$. Observe that $x + h > y + h > y$ and $x + h > x > y$. Then one can define $t, t' \in [0, 1]$ such that

$$y + h = t(x+h) + (1-t)y, \tag{1.2.4}$$

$$x = t'(x+h) + (1-t')y. \tag{1.2.5}$$

Adding (1.2.4) and (1.2.5) and comparing the coefficients of like terms we have $t + t' = 1$. From convexity of f we have

$$tf(x+h) + (1-t)f(y) \geq f(t(x+h) + (1-t)y).$$

Equation (1.2.4) implies

$$tf(x+h) + (1-t)f(y) \geq f(y+h). \tag{1.2.6}$$

Similarly,

$$t'f(x+h) + (1-t')f(y) \geq f(t'(x+h) + (1-t')y).$$

Equation (1.2.5) implies

$$t'f(x+h) + (1-t')f(y) \geq f(x). \tag{1.2.7}$$

Adding (1.2.6) and (1.2.7), we get

$$(t+t')f(x+h) + (1-t+1-t')f(y) \geq f(y+h) + f(x).$$

As stated earlier $t + t' = 1$ so we have

$$f(x+h) - f(x) \geq f(y+h) - f(y).$$

Which is as required. Converse can be proved in the similar manner.
(ii). The necessary condition can easily be obtained by putting $t = \frac{1}{2}$. For converse, suppose that f is not convex. Then there exists a subinterval $[a, b]$ of I such that the graph of $f_{[a,b]}$ that is, graph of f restricted to subinterval $[a, b]$ is not under the chord joining $(a, f(a))$ and $(b, f(b))$, that is, the function

$$\phi(x) = f(x) - \frac{f(b) - f(a)}{b - a}(x - a) - f(a), x \in [a, b]$$

satisfies

$$\gamma = \sup\{\phi(x) \mid x \in [a, b]\} > 0.$$

Note that ϕ is continuous and $\phi(a) = \phi(b) = 0$. Also a direct computation shows that ϕ is midpoint convex. Put $c = \inf\{x \in [a, b] \mid \phi(x) = \gamma\}$. Then necessarily $\phi(c) = \gamma$ and $c \in (a, b)$. For every $h > 0$ for which we have $c \pm h \in (a, b)$ we have $\phi(c - h) < \phi(c)$ and $\phi(c + h) < \phi(c)$. So that

$$\phi(c) > \frac{\phi(c-h) + \phi(c+h)}{2}$$

is in contradiction with the fact that ϕ is midpoint convex. Hence f is convex.

Proof of (iii) and (iv) can be done in a similar manner. □

We now present the following remark as a motivation for the exchange axiom of M-convex functions.

Remark 1.2.2. If $f : \mathbf{R}^n \to \mathbf{R} \cup \{+\infty\}$ is a convex function, then

$$f(x_1) + f(x_2) \geq f(x_1 - \delta(x_1 - x_2)) + f(x_2 + \delta(x_1 - x_2))$$

for every $\delta \in [0, 1]$.

Proof. Let $\delta \in [0, 1]$. Substituting $t = \delta$ and $t = 1 - \delta$ in (1.2.1), we get

$$\delta f(x_1) + (1 - \delta) f(x_2) \geq f\big(\delta x_1 + (1 - \delta) x_2\big)$$

and

$$(1 - \delta) f(x_1) + \delta f(x_2) \geq f\big((1 - \delta) x_1 + \delta x_2\big),$$

respectively. Adding these two inequalities, we get

$$f(x_1) + f(x_2) \geq f(x_1 - \delta(x_1 - x_2)) + f(x_2 + \delta(x_1 - x_2))$$

as required. □

1.3 M$^\natural$-Convex Sets

In this section we first review the concept of M-convex sets and then we will introduce the concept of M$^\natural$-convex sets as a projection of M-convex sets along an arbitrarily chosen coordinate axis. This section also contains some essential results about M$^\natural$-convex sets already present in the literature.

M-convex sets are defined in terms of exchange axiom. They have one-to-one correspondence with integer-valued submodular set functions.

Consider a nonempty finite set $V = \{1, 2, \ldots, n\}$. We denote the space of integer vectors indexed by the elements of V by $\mathbf{Z}^V$. If $\mid V \mid= n$ then $\mathbf{Z}^V$ may coincide with $\mathbf{Z}^n$. $\mathbf{R}^V$ denotes the real vector space indexed by the elements of V. If $\mid V \mid= n$ then $\mathbf{R}^V$ may coincide with $\mathbf{R}^n$. For a vector $z = (z(v) \in \mathbf{Z} \mid v \in V) \in \mathbf{Z}^V$, we define the *positive support* and the *negative support* of z by

$$\text{supp}^+(z) = \{v \in V \mid z(v) > 0\},$$

,

$$\mathrm{supp}^-(z) = \{v \in V \mid z(v) < 0\},$$

respectively.

The *characteristic vector* χ_S of $S \subseteq V$ is defined by

$$\chi_S(v) = \begin{cases} 1 & \text{if } v \in S \\ 0 & \text{if } v \in V \setminus S. \end{cases}$$

We shall denote the u^{th} unit vector by χ_u instead of $\chi_{\{u\}}$ for each $u \in V$. Thus $\chi_u = (0, 0, \ldots, \underbrace{1}_{u^{th} \text{ position}}, \ldots, 0, 0)$. For convenience, χ_0 is defined by the zero vector on V.

Definition 1.3.1 ([13])**.** A nonempty set $B \subseteq \mathbf{Z}^V$ is defined to be *M-convex set* if it satisfies the following *exchange axiom*:

(B-EXC[**Z**]) For $x_1, x_2 \in B$ and $u \in \mathrm{supp}^+(x_1 - x_2)$, there exists $v \in \mathrm{supp}^-(x_1 - x_2)$ such that

$$x_1 - \chi_u + \chi_v \in B \text{ and } x_2 + \chi_u - \chi_v \in B.$$

Here $\mathrm{supp}^+(x_1 - x_2)$ and $\mathrm{supp}^-(x_1 - x_2)$ are the positive support and the negative support of $x_1 - x_2$ respectively. χ_u is the characteristic vector of $u \in V$.

Base of matroid is an example of an M-convex set. The direct result of the exchange axiom (B-EXC[**Z**]) is that an M-convex set lies on hyperplane $\{x \in \mathbf{R}^V \mid x(V) = k\}$ for some $k \in \mathbf{Z}$ where,

$$x(X) = \sum_{v \in X} x(v) \quad (x \in \mathbf{R}^V, X \subseteq V).$$

Following proposition states this fact.

Proposition 1.3.1 ([13, Proposition 4.1])**.** *For any x_1, x_2 in an M-convex set B, we have $x_1(V) = x_2(V)$.*

Proof. We will prove this argument by mathematical induction on $\|x_1 - x_2\|_1$, where the norm $\|.\|_1$ is defined as

$$\|x\|_1 = \sum_{v \in V} |x(v)| \quad (x \in \mathbf{R}^V).$$

If $\|x_1 - x_2\|_1 = 0$ then
$$\sum_{v \in V} |x_1(v) - x_2(v)| = 0.$$
This implies that
$$|x_1(v) - x_2(v)| = 0 \quad \forall v \in V.$$
That is
$$x_1(V) = x_2(V).$$
We exclude the case $\|x_1 - x_2\|_1 = 1$ because of (B-EXC[$\mathbf{Z}$]). If $\|x_1 - x_2\|_1 \geq 2$, then by (B-EXC[$\mathbf{Z}$]) for $u \in \mathrm{supp}^+(x_1 - x_2)$, there exists $v \in \mathrm{supp}^-(x_1 - x_2)$ such that we have $x_2' = x_2 + \chi_u - \chi_v \in B$. Therefore, $x_2'(V) = x_2(V)$. Also $\|x_1 - x_2'\|_1 = \|x_1 - x_2\|_1 - 2$ so by inductive hypothesis $x_1(V) = x_2(V)$ for any $x_1, x_2 \in B$. □

There are number of conditions which are equivalent variants of (B-EXC[$\mathbf{Z}$]). For a set $B \subseteq \mathbf{Z}^V$ these conditions are:

(B-EXC$_+$[$\mathbf{Z}$]) For any $x_1, x_2 \in B$ and $u \in \mathrm{supp}^+(x_1 - x_2)$, there exists $v \in \mathrm{supp}^-(x_1 - x_2)$ such that $x_2 + \chi_u - \chi_v \in B$.

(B-EXC$_-$[$\mathbf{Z}$]) For any $x_1, x_2 \in B$ and $u \in \mathrm{supp}^+(x_1 - x_2)$, there exists $v \in \mathrm{supp}^-(x_1 - x_2)$ such that $x_1 - \chi_u + \chi_v \in B$.

(B-EXC$_W$[$\mathbf{Z}$]) For distinct $x_1, x_2 \in B$, there exists $u \in \mathrm{supp}^+(x_1 - x_2)$ and there exists $v \in \mathrm{supp}^-(x_1 - x_2)$ such that $x_1 - \chi_u + \chi_v \in B$ and $x_2 + \chi_u - \chi_v \in B$.

Now we define the notion of M$^\natural$-convex sets. We introduce a new element 0 not in V. Say $V' = \{0\} \cup V$. A set $X \subseteq \mathbf{Z}^V$ is called an M$^\natural$-convex set if $X = \{z \in \mathbf{Z}^V \mid (z_0, z) \in B, z_0 \in \mathbf{Z}\}$, for some M-convex set $B \subseteq \mathbf{Z}^{V \cup \{0\}}$. In terms of exchange axiom, we can define M$^\natural$-convex set as follows:

Definition 1.3.2 ([13]). A set $B \subseteq \mathbf{Z}^V$ is called M$^\natural$-convex if it satisfies the following exchange axiom:

(B$^\natural$-EXC[$\mathbf{Z}$]) For $x_1, x_2 \in B$ and $u \in \mathrm{supp}^+(x_1 - x_2)$, there exists $v \in \mathrm{supp}^-(x_1 - x_2) \cup \{0\}$ such that
$$x_1 - \chi_u + \chi_v \in B \text{ and } x_2 + \chi_u - \chi_v \in B.$$

1.4 M$^\natural$-Convex Functions

Murota [14] introduced the notion of M-convexity for real valued functions in integer variables. Later in 2000, Murota and Shioura [16] came up with the concept of M-convexity for real valued functions in real variables and called them Polyhedral M-convex functions. We define the *effective domain* of function $f : \mathbf{Z}^V \to \mathbf{R} \cup \{\pm\infty\}$ as

$$\mathrm{dom} f = \mathrm{dom}_{\mathbf{Z}} f = \{z \in \mathbf{Z}^V \mid -\infty < f(z) < +\infty\}.$$

Similarly the *effective domain* of function $f : \mathbf{R}^V \to \mathbf{R} \cup \{\pm\infty\}$ is defined as

$$\mathrm{dom} f = \mathrm{dom}_{\mathbf{R}} f = \{z \in \mathbf{R}^V \mid -\infty < f(z) < +\infty\}.$$

Definition 1.4.1. A function $f : \mathbf{Z}^V \to \mathbf{R} \cup \{+\infty\}$ with nonempty effective domain is called *M-convex function* whenever following exchange property is satisfied:

(M-EXC[$\mathbf{Z}$]) For $x_1, x_2 \in \mathrm{dom} f$ and $u \in \mathrm{supp}^+(x_1 - x_2)$, there exists $v \in \mathrm{supp}^-(x_1 - x_2)$ such that

$$f(x_1) + f(x_2) \geq f(x_1 - \chi_u + \chi_v) + f(x_2 + \chi_u - \chi_v). \tag{1.4.1}$$

The requirement of finiteness of right hand side of above inequality imposes the condition $x_1 - \chi_u + \chi_v \in \mathrm{dom} f$ and $x_2 + \chi_u - \chi_v \in \mathrm{dom} f$.

Let $V = \{1, 2, \ldots, n\}$ and let $f : \mathbf{Z}^V \to \mathbf{R} \cup \{+\infty\}$. The projection f^* of f along any arbitrarily chosen coordinate $v_0 \in V$ in $n-1$ variables is given as

$$f^*(z^*) = f(z_0, z^*) \quad for \quad z_0 = k - z^*(V^*)$$

where $V^* = V \setminus \{v_0\}$ and $(z_0, z^*) \in \mathbf{Z} \times \mathbf{Z}^{V^*}$. A function obtained by such a projection of an M-convex function along an arbitrarily chosen axis is called an M$^\natural$-convex function. Alternatively we can describe it as follows.

Introduce a new element 0 not in V. Say $V' = \{0\} \cup V$. A function

$$f : Z^V \to \mathbf{R} \cup \{+\infty\}$$

with nonempty dom$f$ is called M$^\natural$-convex function, if it can be written in terms of an M-convex function $f' : \mathbf{Z}^{V'} \to \mathbf{R} \cup \{+\infty\}$ as

$$f(z) = f'(z_0, z) \quad \text{with} \quad z_0 = -z(V); \quad z \in \mathbf{Z}^V.$$

Thus, an M$^\natural$-convex function is a function obtained as a projection of an M-convex function.

Conversely, an M$^\natural$-convex function $f : Z^V \to \mathbf{R} \cup \{+\infty\}$ determines the corresponding M-convex function $f' : \mathbf{Z}^{V'} \to \mathbf{R} \cup \{+\infty\}$ by

$$f'(z_0, z) = \begin{cases} f(z) & \text{if } z_0 = -z(V), \\ +\infty & \text{otherwise.} \end{cases}$$

upto a translation of dom$f$ in the direction of 0. Here $z_0 \in Z$ and $z \in \mathbf{Z}^V$. Now we define the M$^\natural$-convex function in terms of an exchange axiom.

Definition 1.4.2 ([12]). A function $f : \mathbf{Z}^V \to \mathbf{R} \cup \{+\infty\}$ is said to be M$^\natural$-convex if it satisfies the following condition:

(M$^\natural$-EXC) For $x_1, x_2 \in \text{dom} f$ and $u \in \text{supp}^+(x_1 - x_2)$, there exists $v \in \text{supp}^-(x_1 - x_2) \cup \{0\}$ such that

$$f(x_1) + f(x_2) \geq f(x_1 - \chi_u + \chi_v) + f(x_2 + \chi_u - \chi_v).$$

All those results which are valid for M-convex functions also hold for M$^\natural$-convex functions. However the class M$^\natural$-convex functions is larger than that of M-convex functions.

1.5 Few Results About M-Convex Functions and M$^\natural$-Convex Functions

In this section we present some important results related to M-convex functions and M$^\natural$-convex functions.

Proposition 1.5.1 ([13, Proposition 6.1]). (i) *The effective domain of an M-convex function is an M-convex set.*

(ii) *The effective domain of an M$^\natural$-convex function is an M$^\natural$-convex set.*

1.6 Few Characterizations of M$^\natural$-Convex Functions

There are numerous problems that involve allocation of a set of objects to a group of agents. For example job matching between firms and workers. Kelso and Crawford [8] introduced gross substitute conditions by generalizing the marriage model of Gale and Shapely [20]. M$^\natural$-convex functions have close relationship with gross substitutability. It is a basic property that guarantees existence of stable matching. [2]In this section we review some results on description of M$^\natural$-convex functions regarding substitutability. These results are discussed in detail by R. Farooq and A. Tamura in [6]. The definitions and results related to M$^\natural$-convex functions are already discussed in Section 1.4.

Consider a nonempty finite set $V = \{1, 2, \ldots, n\}$. Let $X \subseteq \mathbf{Z}^V$ and consider a function $f : \mathbf{Z}^V \to \mathbf{R} \cup \{+\infty\}$, define the set of minimizers of f on X as

$$\arg\min\{f(y) \mid y \in X\} = \{x \in X \mid \forall y \in X : f(y) \geq f(x)\}.$$

The function $f[p]$ at $y \in \mathbf{Z}^V$ is defined as

$$f[p](y) = f(y) + \langle p, y \rangle = f(y) + \sum_{u \in V} p(u)y(u).$$

Before proceeding further we state the generalized version of gross substitutability given for a function $f : \mathbf{Z}^V \to \mathbf{R} \cup \{+\infty\}$ in [17].

(M$^\natural$-GS$_\mathrm{W}$) For $p, q \in \mathbf{R}^V$ and $x_1 \in \arg\min f[p]$ with $p \leq q$ and nonempty $\arg\min f[q]$, there exists $x_2 \in \arg\min f[q]$ such that

$$x_2(v) \geq x_1(v) \text{ whenever } p(v) \geq q(v).$$

The following theorem shows that for set functions M$^\natural$-convexity and (M$^\natural$-GS$_\mathrm{W}$) are equivalent.

Theorem 1.6.1 ([17]). *Let $f : \mathbf{Z}^V \to \mathbf{R} \cup \{+\infty\}$ with nonempty effective domain, such that* $\mathrm{dom} f \subseteq \{0,1\}^V$. *Then f is M$^\natural$-convex if and only if f satisfies (M$^\natural$-GS$_\mathrm{W}$).*

[2]In matching a set of duo of players is formed from opposite teams such that each player appears at most once. Stable matching mean there exist no pair of players that prefer each other to their partners.

Now we analyze some properties as a variation of substitutability.

(SC^1) For $z_1, z_2 \in \mathbf{Z}^V$ with $z_1 \geq z_2$ and $\arg\min\{f(y) \mid y \leq z_2\}$ is nonempty, if $x_1 \in \arg\min\{f(y) \mid y \leq z_1\}$, then there exists x_2 such that

$$x_2 \in \arg\min\{f(y) \mid y \leq z_2\}, \quad x_2 \geq z_2 \wedge x_1.$$[3]

(SC^2) For $z_1, z_2 \in \mathbf{Z}^V$ with $z_1 \geq z_2$ and $\arg\min\{f(y) \mid y \leq z_2\}$ is nonempty, if $x_2 \in \arg\min\{f(y) \mid y \leq z_2\}$, then there exists x_1 such that

$$x_1 \in \arg\min\{f(y) \mid y \leq z_1\}, \quad x_2 \geq z_2 \wedge x_1.$$

Interpretation of these properties can be given as follows. Denote the set of indivisible goods [4] by V, the quantities $x(v)$ of v goods produced by a producer by $x \in \mathbf{Z}^V$ and the cost function of a producer by f. According to (SC^1) when the producible quota of each good remains constant or decreases, the producer requires a production so that the quantities of the goods whose quota remain the same do not reduce. According to (SC^2) when each quota increases or remains constant, the producer requires a production such that the quantities of goods failing to fill the original quotas either decreases or remains invariant.

(SC^1) and (SC^2) are independent properties, that is, it is not necessary that if f satisfies (SC^1) it must also satisfy (SC^2) and vise versa. Following examples illustrate this fact.

Example 1.6.1. Define $f : \{0,1\}^2 \to \mathbf{R} \cup \{+\infty\}$ as follows:

$$f(x) = \begin{cases} 2 & \text{if } x = (0,0) \\ 2 & \text{if } x = (1,0) \\ 2 & \text{if } x = (0,1) \\ 0 & \text{if } x = (1,1). \end{cases}$$

Here f satisfies (SC^1) but (SC^2) fails for $z_1 = (1,1)$ and $z_2 = (0,1)$.

[3] $z_2 \wedge x_1 = (\min\{z_2(i), x_1(i)\} \mid i \in \mathbf{Z}^+)$

[4] Those desirable goods which are sold only in integer quantities on the market for example computers, cars, ships etc.

Example 1.6.2. Define $f : \{0,1\}^2 \to \mathbf{R} \cup \{+\infty\}$ as follows:

$$f(x) = \begin{cases} 2 & \text{if } x = (0,0) \\ +\infty & \text{if } x = (0,1) \\ 1 & \text{if } x = (1,0) \\ 1 & \text{if } x = (1,1). \end{cases}$$

Here f satisfies (SC^2) but (SC^1) fails for $z_1 = (1,1)$ and $z_2 = (0,1)$.

Lemma 1.6.2 ([6, Lemma 3.1])**.** *An M$^\natural$-convex function with bounded effective domain satisfies* (SC^1) *and* (SC^2).

However the converse does not hold in general. In order to establish the converse relationship, two stronger properties are introduced by R. Farooq and A. Tamura [6] which are given below:

(SC$^1_\mathrm{G}$) For each $p \in \mathbf{R}^V$, (SC^1) is satisfied by $f[p]$.

(SC$^2_\mathrm{G}$) For each $p \in \mathbf{R}^V$, (SC^2) is satisfied by $f[p]$.

Lemma 1.6.3 ([6, Lemma 3.2])**.** *If $f : \mathbf{Z}^V \to \mathbf{R} \cup \{+\infty\}$ is an M$^\natural$-convex function with bounded effective domain then f satisfies* (SC$^1_\mathrm{G}$) *and* (SC$^2_\mathrm{G}$).

Proof. Since f is an M$^\natural$-convex function, therefore for each $p \in \mathbf{R}^V$, $f[p]$ is also M$^\natural$-convex function ([13]). So $f[p]$ satisfies (SC^1) and (SC^2). Consequently, f satisfies (SC$^1_\mathrm{G}$) and (SC$^2_\mathrm{G}$). □

The following lemma states the relationship between (SC$^1_\mathrm{G}$) and (SC$^2_\mathrm{G}$).

Lemma 1.6.4 ([6, Lemma 3.3])**.** *Let $f : \{0,1\}^V \to \mathbf{R} \cup \{+\infty\}$. Then f satisfies* (SC$^1_\mathrm{G}$) *if and only if it satisfies* (SC$^2_\mathrm{G}$).

Proof. Let $f : \{0,1\}^V \to \mathbf{R} \cup \{+\infty\}$. Suppose f satisfies (SC$^1_\mathrm{G}$). In order to show that f satisfies (SC$^2_\mathrm{G}$) it is enough to show that f satisfies (SC^2). For this we proceed as follows:

Let $z_1, z_2 \in \mathbf{Z}^V$ such that $z_1 \geq z_2$ and $\arg\min\{f(y) \mid y \leq z_2\} \neq \emptyset$. Further let $x_2 \in \arg\min\{f(y) \mid y \leq z_2\}$. We show that there exists $x_1 \in \arg\min\{f(y) \mid y \leq z_1\}$ such that $x_2 \geq z_2 \wedge x_1$. Define $q = \delta p \in \mathbf{R}^V$ where $\delta > 0$ is such that

$$\frac{\min\{|f(y) - f(x)| \mid f(y) \neq f(x), x, y \in \mathrm{dom} f\}}{\mid V \mid} \geq \delta, \tag{1.6.1}$$

and $p = \chi_V - 2x_2$. Suppose that for any $x, y \in \mathrm{dom}f$ we have $f(y) > f(x)$. Note that $\frac{\langle p, x-y\rangle}{|V|} \leq 1$. This is so because

$$\begin{aligned}
\langle p, x-y\rangle &= \sum_{v\in V} p(v)\big(x(v) - y(v)\big) \\
&\leq \sum_{v\in V} p(v) \\
&\leq 1 + 1 + \ldots + 1 = \mid V \mid .
\end{aligned}$$

Now using elementary properties of $\langle ., .\rangle$ we have

$$\begin{aligned}
f[q](x) &= f(x) + \langle q, x\rangle \\
&= f(x) + \langle q, x\rangle - \langle q, y\rangle + \langle q, y\rangle \\
&= f(x) + \langle \delta p, x\rangle - \langle \delta p, y\rangle + \langle q, y\rangle \\
&= f(x) + \delta\langle p, x\rangle - \delta\langle p, y\rangle + \langle q, y\rangle \\
&= f(x) + \delta\langle p, x-y\rangle + \langle q, y\rangle.
\end{aligned}$$

Thus we have

$$\begin{aligned}
f[q](x) &= f(x) + \delta\langle p, x-y\rangle + \langle q, y\rangle \\
&< f(x) + \frac{\min\{|f(y) - f(x)| \mid f(y) \neq f(x), x, y \in \mathrm{dom}f\}}{\mid V \mid}\langle p, x-y\rangle + \langle q, y\rangle \\
&\leq f(x) + \big(f(y) - f(x)\big)\frac{\langle p, x-y\rangle}{\mid V \mid} + \langle q, y\rangle \\
&\leq f(x) + \big(f(y) - f(x)\big) + \langle q, y\rangle \\
&= f(y) + \langle q, y\rangle \\
&= f[q](y),
\end{aligned}$$

where the first inequality holds because of definition (1.6.1) and the third inequality is valid because $\frac{\langle p, x-y\rangle}{|V|} \leq 1$. So for $x, y \in \mathrm{dom}f$ we have the following implication

$$f(y) > f(x) \Longrightarrow f[q](y) > f[q](x). \tag{1.6.2}$$

From this it follows that

$$\arg\min\{f[q](y) \mid y \leq z_1\} \subseteq \arg\min\{f(y) \mid y \leq z_1\}.$$

Since f satisfies (SC^1_G) therefore for all $x_1 \in \arg\min\{f[q](y) \mid y \leq z_1\}$ there exists x_2 such that $x_2 \in \arg\min\{f[q](y) \mid y \leq z_2\}$ with $x_2 \geq z_2 \wedge x_1$. Since $\arg\min\{f[q](y) \mid y \leq z_1\} \neq \emptyset$ therefore $\arg\min\{f(y) \mid y \leq z_1\} \neq \emptyset$. Thus ($\mathrm{SC}^2$) holds for f. From this it follows that f satisfies (SC^2_G).

For converse suppose f satisfies (SC^2_G). In order to show that f satisfies (SC^1_G) it is enough to show that f satisfies (SC^1). Proceeding in the same manner as earlier, let $z_1, z_2 \in \mathbf{Z}^V$ such that $z_1 \geq z_2$ and $\arg\min\{f(y) \mid y \leq z_1\} \neq \emptyset$. Further let $x_1 \in \arg\min\{f(y) \mid y \leq z_1\}$. We have to show that there exists $x_2 \in \arg\min\{f(y) \mid y \leq z_2\}$, where $x_2 \geq z_2 \wedge x_1$. Define $q = \delta p \in \mathbf{R}^V$ in the same way as above and we obtain (1.6.2). From this it follows that

$$\arg\min\{f[q](y) \mid y \leq z_2\} \subseteq \arg\min\{f(y) \mid y \leq z_2\}.$$

Since f satisfies (SC^2_G), therefore for all $x_2 \in \arg\min\{f[q](y) \mid y \leq z_2\}$, there exists x_1 such that $x_1 \in \arg\min\{f[q](y) \mid y \leq z_1\}$ with $x_2 \geq z_2 \wedge x_1$. Since $\arg\min\{f[q](y) \mid y \leq z_2\} \neq \emptyset$ therefore $\arg\min\{f(y) \mid y \leq z_2\} \neq \emptyset$. Thus ($\mathrm{SC}^1$) holds for f and hence it also satisfies (SC^1_G). This concludes the proof. □

The following lemma states that for set functions[5] converse of Lemma 1.6.3 holds.

Lemma 1.6.5 ([6, Lemma 3.4])**.** *Let $f : \{0,1\}^V \to \mathbf{R} \cup \{+\infty\}$. If f satisfies* (SC^1_G), *then f is $\mathrm{M}^\natural$-convex.*

Proof. Let $f : \{0,1\}^V \to \mathbf{R} \cup \{+\infty\}$ be a function satisfying (SC^1_G). We claim that f satisfies ($\mathrm{M}^\natural$-GS_W). It is enough to consider the case $q = p + t\chi_v$ for some $v \in V$ and $t > 0$. Let $x_1 \in \arg\min f[p]$. Now if $x_1 \in \arg\min f[q]$ then there is nothing to prove. So suppose that $x_1 \notin \arg\min f[q]$, then we have, $x_1(v) = 1$ and $x_2(v) = 0$ for all $x_2 \in \arg\min f[q]$. Take $z_1 = \chi_V$ and $z_2 = z_1 - \chi_v$ then we have the following relationship

$$\begin{aligned} \arg\min f[p] &= \arg\min\{f[p](x_2) \mid x_2 \leq z_1\}, \\ \arg\min f[q] &= \arg\min\{f[p](x_2) \mid x_2 \leq z_2\}. \end{aligned}$$

[5] A function $f : \mathbf{Z}^V \to \mathbf{R} \cup \{+\infty\}$ with $\emptyset \neq \mathrm{dom} f \subseteq \{0,1\}^V$. $\{0,1\}^V$ represents the vector space indexed by the elements of V. If $x \in \{0,1\}^V$ then $x = \{x(i) = 0 \text{ or } 1 \text{ for all } i \in V\}$.

Since (SC_G^1) hold, therefore for each $x_1 \in \arg\min f[p]$, there exists, $x_2 \in \arg\min f[q]$ such that $x_1 \wedge z_2 \leq x_2$. From this it follows that $x_2(v) \geq x_1(v)$ for all $v \in V$ with $p(v) = q(v)$. This shows that f satisfies ($\mathrm{M}^\natural$-GS_W). By theorem 1.6.1 f is $\mathrm{M}^\natural$-convex. This completes the proof. □

With the help of Lemmas 1.6.3- 1.6.5 the following relation was thus established between $\mathrm{M}^\natural$-convex functions and (SC_G^1), (SC_G^2).

Theorem 1.6.6 ([6, Theorem 3.1])**.** *For each $f : \{0,1\}^V \to \mathbf{R} \cup \{+\infty\}$, following conditions are equivalent:*

(i) *f satisfies* (SC_G^1).

(ii) *f satisfies* (SC_G^2).

(iii) *f is $M^\natural$-convex.*

Proof. (i) $\Longrightarrow$ (ii). This follows directly from Lemma (1.6.4).

(ii) $\Longrightarrow$ (iii). This implication is obtainable with the help of Lemmas (1.6.4) and (1.6.5).

(iii) $\Longrightarrow$ (i). The set $\{0,1\}^V$ is bounded, and from Lemma (1.6.3) we know that $\mathrm{M}^\natural$-convex function with bounded effective domain satisfies (SC_G^1).

□

The foundation of discrete convex analysis is laid down by Kazuo Murota. This field of mathematics developed gradually and took its present shape. The study of this theory is still expanding with the development of efficient algorithms. Convex functions play primary role in solving optimization problems. We had given the basic introduction of convex functions in our second section. Later on we moved to $\mathrm{M}^\natural$-convex sets and $\mathrm{M}^\natural$-convex functions. We wrapped up the chapter by giving some characterizations of $\mathrm{M}^\natural$-convex functions. We had seen the equivalence relationship between $\mathrm{M}^\natural$-convexity, (SC_G^1) and (SC_G^2) for $\mathrm{dom} f \subseteq \{0,1\}^V$, that is, when f is a set function. More general case of this, that is, when $\mathrm{dom} f$ is bounded is given by R. Farooq and A. Shioura [5].

Chapter 2

Ultramodular Functions

In the present chapter we shall review the definitions and results related to ultramodular functions, a class of functions that generalizes scalar convexity. Ultramodular functions have their wide applications in many branches of mathematics as well as economics. A special class of ultramodular functions known as ultramodular aggregation functions will also be discussed.

2.1 Introduction

The class of ultramodular functions generalizes convexity of one dimension. If a function $f : A \subseteq \mathbf{R}^n \to \mathbf{R}$ has increasing increments, that is,

$$f(y+h) - f(y) \leq f(x+h) - f(x) \tag{2.1.1}$$

for all $x, y \in A$ and $h \geq 0$ with $x \geq y$[1] and $x+h, y+h \in A$, then f is ultramodular. As seen in Remark (1.2.1) one-dimensional convex functions satisfy (2.1.1). However this does not remain valid for convex functions of several variables. Thus in this case convexity and ultramodularity are somewhat unrelated properties. Ultramodular functions occur unsurprisingly in some economic and statistical applications, providing extension of scalar convexity for some uses. First of all, we briefly describe the different areas, both applied and pure, from where these functions originate.

[1]Inequality between vectors means component wise inequality. Thus $x \geq y$ means $x = (x_i)_{i=1}^n \geq y = (y_i)_{i=1}^n$.

In mathematics, ultramodular functions were first introduced by Wright [23]. He defined them on the real line $\mathbf{R}$ and named them as "convex functions". Roberts and Varberg [19] used the terminology "Wright convexity" for these functions.

In economics, ultramodular functions are used for modelling complementarities. In the field of game theory, ultramodular functions came up in cooperative games of composition $f \circ P$ where f is the function defined on the range of vector measure P. In statistics, they play an essential role in modelling positive dependence between random vectors. In this area they are known as "directionally convex functions". This terminology, however, might not be confused with the functions defined in terms of properties of directional derivatives. Choquet [2] observed that definition (2.1.1) can be extended to functions defined on abstract domains with some algebraic structure. This significant realization makes it possible to deal with convex capacities, which have properties similar to (2.1.1). For the final point, the area of mathematics, where ultramodular functions appear is the Bernstein-Hausdorff theory of absolutely and completely monotonic functions on real line in Widder [22]. These are analytic functions represented by Laplace-Stieltjes integrals and having derivatives such that $(-1)f^j(x) \geq 0$ for all j or $f^j(x) \geq 0$ for all non negative integers j.

2.2 Preliminaries

In this section we give some definitions appearing in the coming sections. A nonempty set A together with binary relation $\preceq$ is called *partially order set* or in short *poset* if

(Reflexivity) for all $a \in A$, $a \preceq a$,

(Antisymmetry) for all $a, b \in A$ such that $a \preceq b$ and $b \preceq a$ we have $a = b$,

(Transitivity) for all $a, b, c \in A$ such that $a \preceq b$ and $b \preceq c$ we have $a \preceq c$,

hold. The relation $\preceq$ is called *ordering relation.*

Given a partially ordered set A, a function $f : A \to \mathbf{R}$ is *decreasing* if $x \geq y$ implies $f(x) \leq f(y)$ and is *increasing* if $x \geq y$ implies $f(x) \geq f(y)$. A function f is called *monotone* if it is either decreasing or increasing.

Let $P_1, P_2, \ldots, P_n$ be partially ordered sets (posets). Their *direct product* $P = \Pi_{i=1}^n P_i$ is again an partially ordered set with component wise order. For any $p \in P$, generally the canonical projection $\pi_i(p)$ is denoted by $p_i \in P_i$. Sometimes a point $p \in P$ is also represented as (p_i, p_{-i}), where p_{-i} is the projection of p onto $\Pi_{k \neq i} P_k$. A function $f : P \to \mathbf{R}$ is said to have *increasing differences* if, for each fixed k and any $p_k \leq q_k$ in P_k, the function $z \to f(q_k, z) - f(p_k, z)$ is increasing on $\Pi_{i \neq k} P_k$.

Let $f : P = \Pi_{i=1}^n P_i \to \mathbf{R}$ with $n \geq 2$. For each point p in P, and for each fixed i, we can define the *partial function* as

$$f_i(z_i, p) = f(p_1, p_2, \ldots, p_{i-1}, z_i, p_{i+1}, \ldots, p_n) \equiv f(z_i, p_{-i}).$$

Let $f : U \subseteq \Pi_{i=1}^n P_i \to \mathbf{R}$. A property is said to hold *separately* for f if, for each integer i, the function $f(p_i, p_{-i})$ of the single variable $p_i \in P_i$ enjoys such a property. The vector p_{-i} of all other variables are kept fixed.

Let $f : [0,1]^n \to [0,1]$. We define the *one-dimensional section* of f for each $x \in [0,1]^n$ and for each $i \in \{1, 2, \ldots, n\}$ by the function $f_{x,i} : [0,1] \to [0,1]$[2] given by $f_{x,i}(u) = f(y)$, where $y_i = u$ when $i = j$ and $y_j = x_j$ when $i \neq j$. Similarly *two-dimensional section* of f is defined by the function $f_{x,i,j} : [0,1]^2 \to [0,1]$ given by $f_{x,i,j}(u, v) = f(y)$ where $y_i = u$, $y_j = v$ and $y_k = x_k$ for $k \in \{1, 2, \ldots, n\} \setminus \{i, j\}$ for each $x \in [0,1]^n$ and for all $i, j \in \{1, 2, \ldots, n\}$ with $i \neq j$.

2.3 Ultramodular Sets and Functions

In this sub section we briefly discuss definitions and some elementary results related to ultramodular sets and ultramodular functions. These definitions and notations are taken from Marrinacci and Montrucchio [12].

Consider a collection of vectors $\{y, x, z, w\}$ of $\mathbf{R}^n$. This collection is called *test quadruple* if $y \leq x \leq w$ and $y + w = x + z$. Note that the vectors y, x, z, w need not to be distinct.

[2] $f_{x,i}$ means value of f at i^{th} component of x.

A set $A \subseteq \mathbf{R}^n$ is *ultramodular* if, given any triple $y, x, w \in A$ with $y \leq x \leq w$, we have $z \in A$ whenever $y + w = x + z$. A test quadruple $\{y, x, z, w\}$ is ultramodular if x and z are not comparable.

Proposition 2.3.1 ([12, Proposition 3.1])**.** *Let $A \subseteq \mathbf{R}^n$. If*

$$x, y \in A \text{ and } y \leq x \Longrightarrow [y, x] \subseteq A \tag{2.3.1}$$

then A is ultramodular. Converse holds whenever atleast one of the following conditions holds:

(i) *$int(A)$ is nonempty and A has a smallest and largest element.*

(ii) *A is open.*

Proof. Suppose that (2.3.1) is true and $w, x, y \in A$ such that $w \geq x \geq y$. Further, let $z \in \mathbf{R}^n$ such that $y + w = x + z$. We claim that $z \in A$. By given condition $[y, w] \subseteq A$. As $y \leq x$, so we have $z \leq w$. Likewise $x \leq w$ implies $y \leq z$. From this it follows that $z \in [y, w] \subseteq A$. Hence A is ultramodular. For converse we proceed as follows.

Let $C = [0, 1]^n$. Let A be any ultramodular set and $x, y \in A$. Also, suppose that $y + \varepsilon C \subseteq A$ for some $\varepsilon > 0$, and $y + \varepsilon C \leq x$. If $u \in C$ then, $\{y, y + \varepsilon u, x - \varepsilon u, x\}$ is a test quadruple under our supposition. So $y + \varepsilon C \leq x$ implies $x - \varepsilon C \subseteq A$. Similarly assume that $y - \varepsilon C \subseteq A$, and $x \leq y - \varepsilon C$. If $v \in C$ then, $\{x, y - \varepsilon v, y + \varepsilon v, x\}$ is a test quadruple. Therefore, $x + \varepsilon C \subseteq A$.

Take a and b as smallest and largest element of A. Then $A \subseteq [a, b]$. Further, as $int(A)$ is nonempty, so $a \ll b$, that is, a is very very less than b. Without loss of generality set $A \subseteq [0, 1]^n$ (by normalization), where 0 and 1 are in A. Fix an integer q and divide the cube $[0, 1]^n$ into q^n small cubes of size $1/q$. More precisely, set $A_1 = [0, 1/q], A_2 = [1/q, 2/q], \ldots, A_q = [(q-1)/q, 1]$, which are intervals on real line. We can decompose the cube $[0, 1]^n$ as

$$[0, 1]^n = \bigcup_i A^1_{i_1} \times A^2_{i_2}, \times \cdots \times A^n_{i_n},$$

$C_i = A^1_{i_1} \times A^2_{i_2}, \times \cdots \times A^n_{i_n}$ is a cube of size $1/q$ and the multi index $i = (i_1, i_2, \ldots, i_n)$ runs over $\{1, 2, \ldots, q\}^n$. All these cubes are indexed by i. As $int(A) \neq \emptyset$, there exists some integer q such that at least one cube of size $1/q$ is such that $C_i \subseteq A$. Exploiting the above mentioned property, the cubes $(1, 1, \ldots, 1)$ and $(q, q, \ldots, q)$ are contained in A. Consequently all the cubes $(q-1, q, \ldots, q), \ldots, (q, q, \ldots, q-1)$ are in A. Continuing in this way, we have all cubes C_i contained in A, so that $A = [0,1]^n$. The condition (2.3.1) clearly holds in $[0,1]^n$.

Now suppose that A is open. Let $x, y \in A$ with $y \leq x$. As A is open, there exist two points y_1 and x_1 in A such that $y_1 \leq y \leq x \leq x_1$ with $y_1 \ll x_1$. By (i), $[y, x] \subseteq [y_1, x_1] \subseteq A$. Thus we proved that if A is ultramodular then (2.3.1) holds provided any one of the conditions (i) or (ii) is satisfied. This completes the proof. □

A function $f : A \subseteq \mathbf{R}^n \to \mathbf{R}$ is *ultramodular* if for all test quadruples $\{y, x, z, w\}$ in A

$$f(y) + f(w) \geq f(x) + f(z). \tag{2.3.2}$$

An equivalent form of (2.3.2)

$$f(x+h) - f(x) \geq f(y+h) - f(y)$$

is obtained by setting $h = w - x = z - y$ in (2.3.2). Here $y \leq x$ and $h \geq 0$ with $x + h, y + h \in A$.

In the following proposition, some basic properties related to ultramodular functions are presented.

Proposition 2.3.2 ([12, Proposition 4.1])**.** *Consider two ultramodular functions $f, g : A \subseteq \mathbf{R}^n \to \mathbf{R}$ and let $\psi : B \subseteq \mathbf{R} \to \mathbf{R}$ be defined on an ultramodular set B containing the range of f. Then following results hold.*

(i) *If α and β are non-negative scalars, then the sum $\alpha f + \beta g$ is ultramodular.*

(ii) *If both f and g are increasing and non-negative functions then their product fg is ultramodular.*

(iii) *The composite function $\psi \circ f$ ultramodular if, in addition, f is monotone and ψ is increasing.*

Proof. While proving the theorem we shall make use of the notation $\Delta f(x)$ to represent the difference $f(x+h)-f(x)$ and $\Delta f(y)$ for $f(y+h)-f(y)$. Let $h \in \mathbf{R}^n$.
(i) Suppose $\{y, x, z, w\}$ is a test quadruple in A. Since f and g are ultramodular, we have

$$f(w)+f(y) \geq f(x)+f(z),$$
$$g(w)+g(y) \geq g(x)+g(z).$$

If we set $h = w - x = z - y$ then the above inequalities can be written as

$$f(x+h)-f(x) \geq f(y+h)-f(y),$$
$$g(x+h)-g(x) \geq g(y+h)-g(y).$$

Equivalently, we can write above inequality as

$$\Delta f(x)-\Delta f(y) \geq 0,$$
$$\Delta g(x)-\Delta g(y) \geq 0.$$

Multiplying above inequalities with α and β, respectively, then adding, we get

$$\alpha\Delta f(x)-\alpha\Delta f(y)+\beta\Delta g(x)-\beta\Delta g(y) \geq 0,$$

$$\Delta\alpha f(x)-\Delta\alpha f(y)+\Delta\beta g(x)-\Delta\beta g(y) \geq 0,$$

that is,

$$\Delta(\alpha f+\beta g)(x)-\Delta(\alpha f+\beta g)(y) \geq 0.$$

This shows that $\alpha f + \beta g$ is ultramodular.
(ii) Consider

$$\begin{aligned}\Delta(fg)(x)-\Delta(fg)(y) &= (fg)(x+h)-(fg)(x)-(fg)(y+h)+(fg)(y)\\ &= f(x+h)g(x+h)-f(x)g(x)-f(y+h)g(y+h)\\ &+ f(y)g(y).\end{aligned}$$

Adding and subtracting

$$g(x+h)f(x)+g(x+h)f(y+h)+g(x+h)f(y)+g(y+h)f(y)+g(y+h)f(x)+g(y)f(x)$$

in above equation and performing some elementary algebra, we get

$$\begin{aligned}\Delta(fg)(x) - \Delta(fg)(y) &= g(x+h)\big(f(x+h) - f(x) - f(y+h) + f(y)\big) \\ &+ \big(f(y+h) - f(y)\big)\big(g(x+h) - g(y+h)\big) \\ &+ \big(f(x) - f(y)\big)\big(g(y+h) - g(y)\big) \\ &+ f(x)\big(g(x+h) - g(x) - g(y+h) - g(y)\big) \\ &= g(x+h)\big(\Delta f(x) - \Delta f(y)\big) + \Delta f(y)\big(g(x+h) - g(y+h)\big) \\ &+ \Delta g(y)\big(f(x) - f(y)\big) + f(x)(\Delta g(x) - \Delta g(y)\big).\end{aligned}$$

Since f and g are non-negative and increasing functions, therefore, right hand side of above equation is non-negative. Hence

$$\Delta(fg)(x) - \Delta(fg)(y) \geq 0.$$

This shows that the product fg is ultramodular.
(iii) Consider a test quadruple $\{y, x, z, w\}$ in A. If f is increasing function, then $y \leq x \leq w$ implies $f(y) \leq f(x) \leq f(w)$. Since B contains the range of f, therefore, $f(y), f(x), f(w) \in B$. Hence by definition of ultramodular set, there exists $z^* \in B$ such that $\{f(y), f(x), z^*, f(w)\}$ is a test quadruple and $f(x) + z^* = f(w) + f(y)$. Moreover, $\{y, x, z, w\}$ is a test quadruple and f is ultramodular, therefore

$$f(w) + f(y) \geq f(x) + f(z),$$

that is,

$$f(x) + z^* \geq f(x) + f(z).$$

From this, we get $z^* \geq f(z)$. As ψ is ultramodular, we have

$$\psi(f(y)) + \psi(f(w)) \geq \psi(z^*) + \psi(f(x)).$$

Further, ψ is increasing function, therefore $\psi(z^*) \geq \psi(f(z))$ and hence

$$\psi(f(x)) + \psi(z^*) \geq \psi(f(x)) + \psi(f(z)).$$

Thus, we get

$$\psi(f(y)) + \psi(f(w)) \geq \psi(f(x)) + \psi(f(z)).$$

This shows that $\psi \circ f$ is ultramodular. Now if f is decreasing function then $y \leq x \leq w$ implies that $f(y) \geq f(x) \geq f(w)$, where $f(y), f(x), f(w) \in B$. Since B is ultramodular, there exists $z^* \in B$ such that $\{f(w), f(x), z^*, f(y)\}$ is a test quadruple with $f(x) + z^* = f(y) + f(w)$. By using similar arguments as before, we obtain

$$\psi(f(y)) + \psi(f(w)) \geq \psi(f(x)) + \psi(f(z)).$$

This shows that $\psi \circ f$ is ultramodular function, where f is decreasing. This concludes the proof. □

Proposition 2.3.3 ([12, Proposition 4.2])**.** *Let $U_1, U_2, \ldots, U_n$ be ultramodular sets and let $U = \Pi_{i=1}^n U_i$ be their direct product. A function $f : U \to \mathbf{R}$ is ultramodular if and only if it is separately ultramodular and has increasing differences on U.*

Proof. Suppose that f is ultramodular function defined on the direct product $U = \Pi_{i=1}^n U_i$ of ultramodular sets U_i, $i \in \{1, 2, \ldots, n\}$. We claim that f is separately ultramodular and has increasing differences. Fix an index i and consider

$$f_i(y_i, p) = f(p_1, p_2, \ldots, p_{i-1}, y_i, p_{i+1}, \ldots, p_n) \equiv f(y_i, p_{-i}), \qquad (2.3.3)$$

where $p \in U$. From equation (2.3.3) it is clear that the function $y \to f(y, p_{-i})$ is ultramodular on U_i. Hence we conclude that f is separately ultramodular. Now, we show that f has increasing differences. For fixed i, let $p_i, q_i \in U_i$ with $p_i \leq q_i$. Choose any two arbitrary values $y_{-i}, z_{-i} \in \Pi_{j \neq i} U_j$ with $y_{-i} \leq z_{-i}$. Denote $(y_1, y_2, \ldots, y_{i-1}, p_i, y_{i+1}, \ldots, y_n)$, $(y_1, y_2, \ldots, y_{i-1}, q_i, y_{i+1}, \ldots, y_n)$, $(z_1, z_2, \ldots, z_{i-1}, p_i, z_{i+1}, \ldots, z_n)$ and $(z_1, z_2, \ldots, z_{i-1}, q_i, z_{i+1}, \ldots, z_n)$ by (p_i, y_{-i}), (q_i, y_{-i}), (p_i, z_{-i}) and (q_i, z_{-i}), respectively. It can be easily seen that $(p_i, y_{-i}) + (q_i, z_{-i}) = (q_i, y_{-i}) + (p_i, z_{-i})$ and $(p_i, y_{-i}) \leq (q_i, y_{-i}) \leq (q_i, z_{-i})$. So $\{(p_i, y_{-i}), (q_i, y_{-i}), (p_i, z_{-i}), (q_i, z_{-i})\}$ is a test quadruple. This together with ultramodularity of f gives

$$f(q_i, z_{-i}) - f(p_i, z_{-i}) \geq f(q_i, y_{-i}) - f(p_i, y_{-i}).$$

From this it follows that f has increasing differences.

For converse, let $y, z \in U$ such that $y \leq z$ and let $h \geq 0$. Writing the telescopic expansion [3] of $f(y + h) - f(y)$ and using the property of increasing differences we

[3]Telescopic expansion means each term of the expansion cancels out the next term so that in the end we are left with only two terms.

have

$$
\begin{aligned}
f(y+h) - f(y) & \\
&= \sum_{i=1}^{n} f(y_1+h_1,\ldots,y_i+h_i,y_{i+1},\ldots,y_n) - f(y_1+h_1,\ldots,\\
&\quad y_{i-1}+h_{i-1},y_i,\ldots,y_n)\\
&\leq \sum_{i=1}^{n} f(z_1+h_1,\ldots,y_i+h_i,z_{i+1},\ldots,z_n) - f(z_1+h_1,\ldots,\\
&\quad z_{i-1}+h_{i-1},y_i,z_{i+1},\ldots,z_n)\\
&\leq \sum_{i=1}^{n} f(z_1+h_1,\ldots,z_i+h_i,z_{i+1},\ldots,z_n) - f(z_1+h_1,\ldots,\\
&\quad z_{i-1}+h_{i-1},z_i,z_{i+1},\ldots,z_n)\\
&= f(z+h) - f(z).
\end{aligned}
$$

The last inequality is valid because f is separately ultramodular. This completes the proof. □

2.4 Modular, Supermodular and Submodular Functions

In this section we give few definitions and results related to modular, supermodular, and submodular functions. The definitions and notations are taken mostly from E.P Klement, M. Manzi, R. Mesiar [11].

Definition 2.4.1. Let L be a nonempty set. For any $x, y \in L$ we define binary operations $\vee$ and $\wedge$ by $x \vee y = (\max\{x(i), y(i)\} \mid i \in \mathbf{Z}^+)$ and $x \wedge y = (\min\{x(i), y(i)\} \mid i \in \mathbf{Z}^+)$, respectively. A tuple $(L, \vee, \wedge)$ is called *lattice* if for any $x, y, z \in L$, following laws hold with respect to both $\vee$ and $\wedge$.

Idempotent law: $x \vee x = x, x \wedge x = x$.

Commutative law: $x \vee y = y \vee x, \quad x \wedge y = y \wedge x$.

Associative law: $x \vee (y \vee z) = (x \vee y) \vee z, \quad x \wedge (y \wedge z) = (x \wedge y) \wedge z$.

Absorption law: $x \wedge (x \vee y) = x, \quad x \vee (x \wedge y) = x.$

A lattice $(L, \vee, \wedge)$ is a *distributive lattice* if, in addition, the distributive law

$$\begin{aligned} x \wedge (y \vee z) &= (x \wedge y) \vee (x \wedge z), \\ x \vee (y \wedge z) &= (x \vee y) \wedge (x \vee z) \end{aligned}$$

holds true.

We may use L instead of $(L, \vee, \wedge)$ when there is no danger of confusion.

Definition 2.4.2. Let $(L, \vee, \wedge)$ be a lattice.

(i) A function $f : L \to \mathbf{R}$ is called *modular* if for all $x, y \in L$, we have

$$f(x) + f(y) = f(x \wedge y) + f(x \vee y).$$

(ii) A function $f : L \to \mathbf{R}$ is called *supermodular* if for all $x, y \in L$, we have

$$f(x) + f(y) \leq f(x \wedge y) + f(x \vee y).$$

(iii) A function $f : L \to \mathbf{R}$ is called *submodular* if for all $x, y \in L$, we have

$$f(x) + f(y) \geq f(x \wedge y) + f(x \vee y).$$

In the context of measures, i.e., when L is a σ-algebra[4] of subsets of universe, modular functions are named as valuations in Birkhoff [1]. So far the concept of modularity, supermodularity and submodularity is introduced for functions from a lattice L into $\mathbf{R}$. Now we take unit cube[5] $[0, 1]^n$ as L and focus our attention to n-ary aggregation functions.

[4]A collection $\mathscr{A}$ of subsets of a universal set X is called σ-algebra of sets if

(i) $A \in \mathscr{A}$ implies the complement of A i.e., $A^{\mathsf{C}} \in \mathscr{A}$.

(ii) $A_1, A_2, \ldots \in \mathscr{A}$ implies $\cup_{i=1}^{\infty} A_i \in \mathscr{A}$.

(iii) $\emptyset \in \mathscr{A}$ and by (i) $X \in \mathscr{A}$.

[5]Unit cube is a cube in n-dimensional space with sides having length 1 unit. $[0, 1]^n$ is also referred as unit cube. By $[0, 1]^n$ we mean the space of all n-tuples of real numbers having value with in the interval $[0, 1]$. Thus if $x \in [0, 1]^n$ then $x = \{x(i) \mid 0 \leq x(i) \leq 1 \forall i = 1, 2, \ldots, n\}$.

Definition 2.4.3. A non-decreasing function $A : [0,1]^n \to [0,1]$ satisfying $A(0,0,\ldots,0) = 0$ and $A(1,1,\ldots,1) = 1$ is called an *n-ary aggregation function.*

Following proposition gives characterization of modular functions.

Proposition 2.4.1 ([11, Proposition 2.2])**.** *An n-ary aggregation function $A : [0,1]^n \to [0,1]$ is modular if and only if there are non-decreasing functions $f_1, f_2, \ldots, f_n : [0,1] \to [0,1]$ with $\sum_{i=1}^n f_i(0) = 0$ and $\sum_{i=1}^n f_i(1) = 1$ such that, for all $(x_1, x_2, ..., x_n) \in [0,1]^n$,*

$$A(x_1, x_2, \ldots, x_n) = \sum_{i=1}^{n} f_i(x_i).$$

The following proposition gives reformulation of supermodular functions for the lattice $[0,1]^2$.

Proposition 2.4.2 ([11])**.** *For non-decreasing functions $f : [0,1]^2 \to [0,1]$, supermodularity can be reformulated as*

$$f(x_2, y_2) - f(x_2, y_1) - f(x_1, y_2) + f(x_1, y_1) \geq 0 \tag{2.4.1}$$

for all $x_1, x_2, y_1, y_2 \in [0,1]$ with $x_1 \leq x_2$ and $y_1 \leq y_2$.

Proof. Let $x, y \in [0,1]^2$ such that $x = (x_2, y_1)$ and $y = (x_1, y_2)$, where $x_1 \leq x_2$ and $y_1 \leq y_2$. Since f is supermodular, so

$$f(x \vee y) + f(x \wedge y) \geq f(x) + f(y).$$

As $x_1 \leq x_2$ and $y_1 \leq y_2$ therefore $x \vee y = (x_2, y_2)$ and $x \wedge y = (x_1, y_1)$. Thus above inequality becomes

$$f(x_2, y_2) + f(x_1, y_1) \geq f(x_2, y_1) + f(x_1, y_2).$$

Consequently

$$f(x_2, y_2) - f(x_2, y_1) - f(x_1, y_2) + f(x_1, y_1) \geq 0$$

as required. □

The following proposition describes the characteristic of supermodular functions.

Proposition 2.4.3 ([11, Proposition 2.3]). *An n-ary function* $f : [0,1]^n \to [0,1]$ *is supermodular if and only if any of its two-dimensional sections is supermodular.*

Nelsen [18] and Kimberling [9] called two-dimensional aggregation functions satisfying (2.4.1) as 2-increasing functions or functions with moderate growth.

Remark 2.4.1. If a binary aggregation function $A : [0,1]^2 \to [0,1]$ is supermodular, so is the function $B : [0,1]^2 \to [0,1]$ given by $B(x,y) = A(x,y) - A(x,0) - A(0,y)$.

Proof. Let $x_1, x_2 \in [0,1]^2$, where $x_1 = (x,y)$ and $x_2 = (x^*, y^*)$. We will discuss following cases:

Case 1: Let $x \le x^*$ and $y \le x^*$ then trivially

$$B(x_1 \vee x_2) + B(x_1 \wedge x_2) = B(x_1) + B(x_2).$$

Case 2: Let $x \ge x^*$ and $y \ge x^*$ then again B trivially satisfies the condition of supermodularity.

Case 3: Let $x \le x^*$ and $y^* \le y$. Since A is supermodular, therefore

$$A(x_1 \vee x_2) + A(x_1 \wedge x_2) \ge A(x_1) + A(x_2),$$

that is,

$$A(x^*, y) + A(x, y^*) \ge A(x,y) + A(x^*, y^*).$$

Now consider

$$\begin{aligned} B(x_1 \vee x_2) + B(x_1 \wedge x_2) &= B(x^*, y) + B(x, y^*) \\ &= A(x^*, y) - A(x^*, 0) - A(0, y) + A(x, y^*) \\ &- A(x, 0) + A(0, y^*) \\ &\ge A(x,y) + A(x^*, y^*) - A(x^*, 0) - A(0, y) \\ &- A(x, 0) + A(0, y^*) \\ &= B(x,y) + B(x^*, y^*) = B(x_1) + B(x_2). \end{aligned}$$

Thus we get

$$B(x_1 \vee x_2) + B(x_1 \wedge x_2) \ge B(x_1) + B(x_2).$$

Hence B is also supermodular.

Case 4: Let $x \geq x^*$ and $y^* \geq y$. This case can be proved by using similar arguments as of Case 3.

□

Generally, the composition of (super-)modular functions is not necessarily (super-)modular, as the following example illustrates.

Example 2.4.1. Consider the functions $A, B : [0,1]^2 \to [0,1]$ given by

$$A(x,y) = \sqrt[3]{x} \text{ and } B(x,y) = \frac{x+y}{3}.$$

Both functions are modular and hence supermodular. However the composite function $A(B,B) : [0,1]^2 \to [0,1]$ given by $A(B,B)(x,y) = \sqrt[3]{\frac{x+y}{3}}$ is not (super-)modular.

The following theorem shows that M$^\natural$-convex functions (discussed in chapter 1) are supermodular.

Theorem 2.4.4 ([13, Theorem 6.19])**.** *M$^\natural$-convex functions are supermodular on the integer lattice.*

The converse is however false. Following example illustrates this

Example 2.4.2. Define $f := \{0,1\}^3 \to \mathbf{R} \cup \{+\infty\}$ as follows

$$f(x) = \begin{cases} 2 & \text{if } x = (1,1,1) \\ 1 & \text{if } x = (1,1,0) \\ 1 & \text{if } x = (1,0,1) \\ 0 & \text{otherwise.} \end{cases}$$

It is easy to see that f is supermodular. However, it is not M$^\natural$-convex for $x = (0,1,1), y = (1,0,0)$ and $u = 2$.

2.5 Ultramodular Aggregation Functions

In this section we discuss ultramodular aggregation functions. Some constructions of ultramodular aggregation functions, particularly, related to composition of functions will be discussed.

Ultramodular functions are already discussed in Section (2.3). We can redefine an ultramodular function in context of n-ary aggregation function as follows.

Definition 2.5.1. An n-ary aggregation function $A : [0,1]^n \to [0,1]$ is ultramodular if

$$A(x+y+z) - A(x+y) \geq A(x+z) - A(x) \tag{2.5.1}$$

for all $x, y, z \in [0,1]^n$ with $x+y+z \in [0,1]^n$.

The following lemma establishes the relationship between ultramodular aggregation functions and supermodular aggregation functions.

Lemma 2.5.1 ([11])**.** *If an n-ary aggregation function $A : [0,1]^n \to [0,1]$ is ultramodular then it is supermodular.*

Proof. For any $x, y \in [0,1]^n$, put $u = y - x \wedge y$ and $v = x - x \wedge y$, Since $x + y = x \vee y + x \wedge y$. We have

$$x \vee y = x + y - x \wedge y = u + v + x \wedge y \in [0,1]^n.$$

From this it follows that

$$\begin{aligned} A(x \vee y) + A(x \wedge y) &= A(x \wedge y + u + v) + A(x \wedge y) \\ &\geq A(x \wedge y + u) + A(x \wedge y + v) \\ &= A(x) + A(y). \end{aligned}$$

Hence A is supermodular. □

In the following proposition we state the required condition for exact relationship between ultramodularity and supermodularity.

Proposition 2.5.2 ([11, Proposition 2.7])**.** *The necessary and sufficient condition for a function $f : [0,1]^n \to [0,1]$ to be ultramodular is that it is supermodular and each of its one-dimensional section is convex.*

Remark 2.5.1 ([11, Remark 2.9])**.** For an n-ary aggregation function $A : [0,1]^n \to [0,1]$ the following statements are equivalent:

(i) A is ultramodular.

(ii) Every two-dimensional section of A is ultramodular.

(iii) Each one-dimensional section of A is convex and any two-dimensional section of A is supermodular.

Proof. First we show that (i) implies (ii). Since A is ultramodular, by virtue of Lemma 2.5.1 it is supermodular. Consequently, each of its two-dimensional sections are supermodular. By Proposition 2.5.2, each of its two-dimensional sections are ultramodular. (ii) $\Rightarrow$ (iii) and (iii) $\Rightarrow$ (i) follows directly from Proposition 2.5.2. $\square$

2.5.1 Some Constructions Related to Ultramodular Aggregation Functions

We show that the composition of ultramodular aggregation function is again an ultramodular aggregation function. In this regard, we first present the following result.

Theorem 2.5.3 ([11, Theorem 3.1]). *For an n-ary aggregation function $A : [0,1]^n \to [0,1]$, following conditions are equivalent:*

(i) *A is ultramodular.*

(ii) *If $B_1, B_2, \ldots, B_n : [0,1]^k \to [0,1]$ are non-decreasing supermodular functions then the composition function $D : [0,1]^k \to [0,1]$ defined by*

$$D(x) = A(B_1(x), B_2(x), \ldots, B_n(x))$$

is a supermodular function for $k \geq 2$.

Proof. To prove (i) $\Rightarrow$ (ii), let $A : [0,1]^n \to [0,1]$ be an ultramodular aggregation function. Further, let $B_1, B_2, \ldots, B_n : [0,1]^k \to [0,1]$ be non-decreasing supermodular functions. Define the composite $D : [0,1]^k \to [0,1]$ by

$$D(x) = A(B_1(x), B_2(x), \ldots, B_n(x)).$$

Clearly, D is an aggregation function. Take $x, y \in [0,1]^k$ and define, for every $j \in \{1, 2, \ldots, n\}$,

$$\begin{aligned} a_j &= B_j(x) - B_j(x \wedge y), \\ b_j &= B_j(y) - B_j(x \vee y), \\ u &= (a_1, a_2, \ldots, a_n), \\ v &= (b_1, b_2, \ldots, b_n), \\ z &= (B_1(x \wedge y), B_2(x \wedge y), \ldots, B_n(x \wedge y)). \end{aligned}$$

Since B_j's are non-decreasing functions so, $u, v \in [0,1]^n$. Further, B_j's are supermodular, therefore

$$(B_1(x \vee y), B_2(x \vee y), \ldots, B_n(x \vee y)) \geq u + v + z. \tag{2.5.2}$$

This is so because for each fixed $j = 1, 2, \ldots, n$ we have

$$B_j(x \vee y) + B_j(x \wedge y) \geq B_j(x) + B_j(y).$$

This can be written as

$$B_j(x \vee y) \geq B_j(x) - B_j(x \wedge y) + B_j(y).$$

Adding and subtracting $B_j(x \wedge y)$, we obtain

$$B_j(x \vee y) \geq B_j(x) - B_j(x \wedge y) + B_j(y) - B_j(x \wedge y) + B_j(x \wedge y).$$

From this it follows that

$$B_j(x \vee y) \geq a_j + b_j + B_j(x \wedge y).$$

Now, since A is ultramodular and non-decreasing function, we have

$$\begin{aligned} D(x \vee y) &= A(B_1(x \vee y), B_2(x \vee y), \ldots, B_n(x \vee y)) \\ &\geq A(u + v + z) \\ &\geq A(z + u) + A(z + v) - A(z) \\ &= A\big(B_1(x), B_2(x), \ldots, B_n(x)\big) + A\big(B_1(y), B_2(y), \ldots, B_n(y)\big) \\ &- A\big(B_1(x \wedge y), B_2(x \wedge y), \ldots, B_n(x \wedge y)\big) \\ &= D(x) + D(y) - D(x \wedge y). \end{aligned}$$

where first inequality is due to (2.5.2) and the second inequality holds since A is ultramodular. Thus we have

$$D(x \vee y) + D(x \wedge y) \geq D(x) + D(y).$$

Conversely, suppose that (ii) holds. In order to show that A is ultramodular, we shall show that each of its one-dimensional section is convex. For this, with out loss of generality, consider a function $f : [0,1] \to [0,1]$ defined as $f(x^*) = A(x^*, u_2, u_3, \ldots, u_n)$, where $u_2, u_3, \ldots, u_n \in [0,1]$ are fixed. Also define $B_1, B_2, \ldots, B_n : [0,1]^k \to [0,1]$ by $B_1(x) = \frac{x_1+x_2}{2}$ and $B_j(x) = u_j$ for $j > 1$. For arbitrary $x_1, x_2 \in [0,1]$ if we put $x = (x_1, x_2, 0, 0, \ldots, 0)$ and $y = (x_2, x_1, 0, 0, \ldots, 0)$ then we have

$$D(x) = A\Big(B_1(x), B_2(x), \ldots, B_n(x)\Big) = A\left(\frac{x_1+x_2}{2}, 0, 0, \ldots, 0\right) = f\left(\frac{x_1+x_2}{2}\right).$$

Similarly,

$$D(y) = A\big(B_1(y), B_2(y), \ldots, B_n(y)\big) = A\left(\frac{x_2+x_1}{2}, 0, 0, \ldots, 0\right) = f\left(\frac{x_2+x_1}{2}\right).$$

Also we have $D(x \wedge y) = f(x_1 \wedge x_2)$ and $D(x \vee y) = f(x_1 \vee x_2)$. As $B_1, B_2, \ldots, B_n$ are non-decreasing supermodular functions, and by (ii) D is supermodular, we have

$$\begin{aligned} D(x \vee y) + D(x \wedge y) &\geq D(x) + D(y), \\ f(x_1 \vee x_2) + f(x_1 \wedge x_2) &\geq f\left(\frac{x_1+x_2}{2}\right) + f\left(\frac{x_1+x_2}{2}\right), \\ \frac{f(x_1) + f(x_2)}{2} &\geq f\left(\frac{x_1+x_2}{2}\right). \end{aligned}$$

It shows that f is Jensen convex function. Further, f is monotone thus by Remark 1.2.1 f is convex. Consequently A is ultramodular.

So far we have seen that A is ultramodular for $n = 1$. Now we claim that ultramodularity of A also holds for $n > 1$. For this we proceed as follows:

Consider a function $g : [0,1]^2 \to [0,1]$ defined as $g(x^*) = A(x^*, y^*, u_3, u_4, \ldots, u_n)$ where $u_3, u_4, \ldots, u_n \in [0,1]$ are fixed. Also define $B_1, B_2, \ldots, B_n : [0,1]^k \to [0,1]$ by $B_1(x) = x_1, B_2(x) = x_2$ and $B_j(x) = u_j$ for $j > 2$. With the help of similar arguments as above, we obtain

$$g(x_1 \vee x_2) + g(x_1 \wedge x_2) \geq g(x_1) + g(x_2).$$

This shows that g is supermodular. Hence, two-dimensional sections of A are supermodular. Using Proposition 2.4.3 and Proposition 2.5.2 we conclude that A is ultramodular. This completes the proof. □

Now in the following theorem we show that the class of ultramodular functions is closed under composition of functions.

Theorem 2.5.4 ([11, Theorem 3.2]). *Let $A : [0,1]^n \to [0,1]$ and $B_1, B_2, \ldots, B_n : [0,1]^k \to [0,1]$ be ultramodular aggregation functions. Then the composition $D : [0,1]^k \to [0,1]$ given by $D(x) = A\big(B_1(x), B_2(x), \ldots, B_n(x)\big)$ is also an ultramodular aggregation function. In other words, we can say that composition of ultramodular aggregation functions is again an ultramodular aggregation function.*

Proof. By virtue of Theorem 2.5.3, D is supermodular (the result of Theorem 2.5.3 is also true for $k = 1$). Thus it is sufficient to prove that each one-dimensional section of D is convex. Let $g : [0,1] \to [0,1]$ be a one-dimensional section of the composite function D. Then there exists one-dimensional sections $f_1, f_2, \ldots, f_n : [0,1] \to [0,1]$ of $B_1, B_2, \ldots, B_n$, respectively, such that $g(x^*) = A\Bigg(f_1(x^*), f_2(x^*), \ldots, f_n(x^*)\Bigg)$. Since $B_1, B_2, \ldots, B_n$ are ultramodular aggregation functions, therefore, by Proposition 2.5.2 $f_1, f_2, \ldots, f_n$ are convex. Clearly, g is non-decreasing. This together with its convexity is equivalent to the validity of Jensen inequality (by Remark 1.2.1). Now for all $x^*, a \in [0,1]$ with $x^* + 2a \leq 1$, we have

$$\frac{g(x^* + 2a) + g(x^*)}{2} \geq g\left(\frac{x^* + 2a + x^*}{2}\right).$$

Equivalently

$$g(x^* + 2a) - g(x^* + a) \geq g(x^* + a) - g(x^*). \tag{2.5.3}$$

Since $f_1, f_2, \ldots, f_n$ are convex, for each fixed $j \in \{1, 2, \ldots, n\}$, we have, $f_j(x^* + 2a) - f_j(x^* + a) \geq f_j(x^* + a) - f_j(x^*) \geq 0$. Put $a_j = f_j(x^* + a) - f_j(x^*)$ and $b_j = f_j(x^* + 2a) - f_j(x^* + a)$.

Since A is ultramodular and monotone, we have

$$\begin{aligned}
g(x^*+2a) &= A\big(f_1(x^*+2a), f_2(x^*+2a), \ldots, f_n(x^*+2a)\big)\\
&= A\big(f_1(x^*+2a) - f_1(x^*+a) + f_1(x^*+a) - f_1(x^*) + f_1(x^*), f_1(x^*+2a)\\
&- f_2(x^*+a) + f_2(x^*+a) - f_2(x^*) + f_2(x^*), \ldots, f_n(x^*+2a)\\
&- f_n(x^*+a) + f_n(x^*+a) - f_n(x^*) + f_n(x^*)\big)\\
&= A\big(a_1 + b_1 + f_1(x^*), a_2 + b_2 + f_2(x^*), \ldots, a_n + b_n + f_n(x^*)\big)\\
&\geq A\big(a_1 + f_1(x^*), a_2 + f_2(x^*), \ldots, a_n + f_n(x^*)\big) + A\big(b_1 + f_1(x^*), b_2 + f_2(x^*),\\
&\quad \ldots, b_n + f_n(x^*)\big) - A\big(f_1(x^*), f_2(x^*), \ldots, f_n(x^*)\big)\\
&\geq A\big(a_1 + f_1(x^*), a_2 + f_2(x^*), \ldots, a_n + f_n(x^*)\big) + A\big(a_1 + f_1(x^*), a_2 + f_2(x^*),\\
&\quad \ldots, a_n + f_n(x^*)\big) - A\big(f_1(x^*), f_2(x^*), \ldots, f_n(x^*)\big)\\
&= 2A\big(a_1 + f_1(x^*), a_2 + f_2(x^*), \ldots, a_n + f_n(x^*)\big)\\
&- A\big(f_1(x^*), f_2(x^*), \ldots, f_n(x^*)\big)\\
&= 2g(x^*+a) - g(x^*).
\end{aligned}$$

This proves (2.5.3). Thus each one-dimensional section of D is convex. Hence by Proposition 2.5.2, D is ultramodular aggregation function. This concludes the proof. □

In this chapter we discussed the structure and some properties of ultramodular functions. These functions play an important role in the areas of economics, statistics, pure and applied mathematics etc. Ultramodular aggregation functions and some constructions related to them were also given. These functions are important in the areas of supermodular measures and of bivariate copulas.

Chapter 3

Relationship between $M^\natural$-Convexity and Ultramodularity

As mentioned earlier in Chapter 1 that the motivation behind establishing discrete convex analysis is to set up the theoretical framework for solving problems related to discrete optimization with the aid of both combinatorial and continuous optimization. $M^\natural$-convex functions play significant role in the theory of discrete optimization. On the other hand, ultramodular functions appears in several pure and applied research areas. In this chapter we investigate the relationship between two newly developed classes of functions known as ultramodular functions and $M^\natural$-convex functions on integer lattice. We shall show that if a subset of an integer lattice is $M^\natural$-convex then it is ultramodular. Moreover, we prove that each $M^\natural$-convex function is an ultramodular function.

3.1 $M^\natural$-Convexity and Ultramodularity Revisited

In this section, we recall some basic definitions and results about $M^\natural$-convexity and ultramodularity. This will help us to construct relationship between $M^\natural$-convexity and ultramodularity.

Recall that a set $A \subseteq \mathbf{R}^n$ is *ultramodular* if given any elements $x, y, w \in A$ with $x \leq y \leq w$ the following holds:

$$x + w = y + z \text{ implies } z \in A.$$

A function $f : A \subseteq \mathbf{R}^n \to \mathbf{R}$ is said to be *ultramodular* if

$$f(x+h) - f(x) \geq f(y+h) - f(y) \tag{3.1.1}$$

for all $h \geq 0$ and $x \geq y$ such that $x + h, y + h \in A$.

The following lemma gives an equivalent condition for a function to be ultramodular.

Lemma 3.1.1. *A function $f : A \subseteq \mathbf{R}^n \to \mathbf{R}$ is ultramodular if and only if*

$$f(x) + f(y) \geq f(x-h) + f(y+h) \tag{3.1.2}$$

for all $x, y \in A$ and $h \geq 0$ with $x - h, y + h \in A$ and $x - h \geq y$.

Proof. Suppose that f is ultramodular. For any $x, y \in A$ and $h \geq 0$ such that $x - h, y + h \in A$ and $x - h \geq y$, we have

$$f(x) - f(x-h) \geq f(y+h) - f(y).$$

The above inequality can be written as

$$f(x) + f(y) \geq f(x-h) + f(y+h).$$

Conversely, suppose that (3.1.2) holds for all $x, y \in A$ and $h \geq 0$ with $x - h, y + h \in A$ and $x - h \geq y$. We show that f is ultramodular function. Let $x, y \in A$, $x \geq y$ and $h \geq 0$ with $x + h, y + h \in A$. Then from (3.1.2) we have

$$f(x+h) + f(y) \geq f(x) + (y+h),$$

The above inequality is equivalent to

$$f(x+h) - f(x) \geq f(y+h) - f(y).$$

Thus, f is ultramodular. □

Let V be a nonempty and finite set. Recall that a function $f : \mathbf{Z}^V \to \mathbf{R} \cup \{+\infty\}$ with dom$f \neq \emptyset$ is called *M$^\natural$-convex* if it satisfies the following condition:

(M$^\natural$-EXC) For all $x, y \in \text{dom}f$ and all $u \in \text{supp}^+(x-y)$, there exists $v \in \text{supp}^-(x-y) \cup \{0\}$ such that

$$f(x) + f(y) \geq f(x + \chi_v - \chi_u) + f(y - \chi_v + \chi_u).$$

A nonempty set $B \subseteq \mathbf{Z}^V$ is called *M$^\natural$-convex* if the following condition is satisfied:

(B$^\natural$-EXC) For all $x, y \in B$ and all $u \in \text{supp}^+(x-y)$, there exists $v \in \text{supp}^-(x-y) \cup \{0\}$ such that

$$x + \chi_v - \chi_u \in B \text{ and } y - \chi_v + \chi_u \in B.$$

We recall the following result about effective domain of an M$^\natural$-convex function.

Proposition 3.1.2 ([13, Proposition 6.1])**.** *The effective domain of an M$^\natural$-convex function is an M$^\natural$-convex set.*

3.2 M$^\natural$-Convex and Ultramodular Sets

In this section we construct a relationship between M$^\natural$-convex sets and Ultramodular sets on integer lattice. We show that each M$^\natural$-convex set is ultramodular. We give an example to show that an ultramodular set may not be an M$^\natural$-convex set. Infact, if $A \subseteq \mathbf{R}^n$ is a set such that $x \nleq y$ and $y \nleq x$ for each $x, y \in A$ then A is vacuously ultramodular.

Lemma 3.2.1. *If a set $A \subseteq \mathbf{Z}^V$ is M$^\natural$-convex then it is ultramodular.*

Proof. Assume that $A \subseteq \mathbf{Z}^V$ is an M$^\natural$-convex set. Let $x, y \in A$ with $y \leq x$ and $z \in [y, x]$. We show that $z \in A$. If $z = x$ or $z = y$ then obviously $z \in A$. Suppose that $z \neq x$. Then $\text{supp}^+(x-z) \neq \emptyset$ and let $u_1 \in \text{supp}^+(x-z)$, that is, $u_1 \in \text{supp}^+(x-y)$. Then (B$^\natural$-EXC) implies $x_1 = x - \chi_{u_1} \in A$. Clearly, $z \leq x_1$. If $z \neq x_1$ then take $u_2 \in \text{supp}^+(x_1 - z) \subseteq \text{supp}^+(x - z)$, that is, $u_2 \in \text{supp}^+(x_1 - y)$. Again by virtue of (B$^\natural$-EXC), $x_2 = x_1 - \chi_{u_2} \in A$. In other words, for $u_1, u_2 \in \mathrm{supp}^+(x-z)$ we have $x - \chi_{u_1} - \chi_{u_2} \in A$. Continuing this process, we get $x - \sum_{u \in \mathrm{supp}^+(x-z)} \big(x(u) - z(u)\big)\chi_u \in A$. But

$$z = x - \sum_{u \in \mathrm{supp}^+(x-z)} \big(x(u) - z(u)\big)\chi_u.$$

Thus $z \in A$. This shows that the $[y, x] \subseteq A$. By Proposition 2.3.1 A is ultramodular. □

The following example shows that the converse of the above lemma does not hold in general.

Example 3.2.1. Define a set $A \subseteq \mathbf{Z}^3$ by:

$$A = \{(0,0,0), (0,1,0), (1,0,1), (1,1,1)\}.$$

Clearly, A is ultramodular. However, it is not M$^\natural$-convex set for $x = (1,0,1), y = (0,1,0)$ and $u = 1$.

Corollary 3.2.2. *If a function $f : \mathbf{Z}^V \to \mathbf{R} \cup \{+\infty\}$ is M$^\natural$-convex then* dom$f$ *is an ultramodular set.*

Proof. By Proposition 3.1.2, dom$f$ is an M$^\natural$-convex set. Thus the assertion follows from Lemma 3.2.1. □

3.3 M$^\natural$-Convex and Ultramodular Functions

In this section, we give a relationship between M$^\natural$-convex functions and ultramodular functions on integer lattice. We prove that each M$^\natural$-convex function is ultramodular. We give an example to show that an ultramodular function may not be an M$^\natural$-convex function. Infact, if $f : A \subseteq \mathbf{R}^n \to \mathbf{R}$ is a function such that $x \not\leq y$ and $y \not\leq x$ for each $x, y \in A$ then f is vacuously ultramodular.

Lemma 3.3.1. *If a function $f : \mathbf{Z}^V \to \mathbf{R} \cup \{+\infty\}$ is M$^\natural$-convex then it is ultramodular on* dom$f$.

Proof. Let $x, y \in \mathrm{dom} f$ such that $x \geq y$ and $h \geq 0$ with $x+h, y+h \in \mathrm{dom} f$. Take $u_1 \in \mathrm{supp}^+(x-y)$. Then by Corollary 3.2.2, we have $x_1 = x - \chi_{u_1} \in \mathrm{dom} f$. Since $u_1 \in \mathrm{supp}^+(x+h-x_1)$ and f is M$^\natural$-convex, we have

$$f(x+h) + f(x_1) \geq f(x+h-\chi_{u_1}) + f(x).$$

This is equivalent to

$$f(x+h) - f(x) \geq f(x+h-\chi_{u_1}) - f(x_1). \tag{3.3.1}$$

Clearly $y \leq x_1$. If $x_1 = y$ then (3.3.1) implies that

$$f(x+h) - f(x) \geq f(y+h) - f(y),$$

that is, f is ultramodular. If $x_1 \neq y$ then we take $u_2 \in \mathrm{supp}^+(x_1-y) \subseteq \mathrm{supp}^+(x-y)$. Corollary 3.2.2 gives $x_2 = x_1 - \chi_{u_2} \in \mathrm{dom} f$ and (3.3.1) implies $x_1 + h \in \mathrm{dom} f$. As $u_2 \in \mathrm{supp}^+(x_1+h-x_2)$ and f is M$^\natural$-convex, we obtain

$$f(x_1+h) + f(x_2) \geq f(x_1+h-\chi_{u_2}) + f(x_1).$$

This is equivalent to

$$f(x_1+h) - f(x_1) \geq f(x_1+h-\chi_{u_2}) - f(x_2). \tag{3.3.2}$$

The inequalities (3.3.1) and (3.3.2) give

$$f(x+h) - f(x) \geq f(x+h-\chi_{u_1}-\chi_{u_2}) - f(x-\chi_{u_1}-\chi_{u_2}),$$

where $u_1, u_2 \in \mathrm{supp}^+(x-y)$. By applying this argument repeatedly, we get

$$f(x+h) - f(x) \geq f(x+h - \sum_{w \in \mathrm{supp}^+(x-y)} \lambda_w \chi_w) - f(x - \sum_{w \in \mathrm{supp}^+(x-y)} \lambda_w \chi_w), \tag{3.3.3}$$

where $\lambda_w = x(w) - y(w)$ for each $w \in \mathrm{supp}^+(x-y)$. But $x - \sum_{w \in \mathrm{supp}^+(x-y)} \lambda_w \chi_w = y$. Therefore (3.3.3) implies

$$f(x+h) - f(x) \geq f(y+h) - f(y).$$

Thus f is ultramodular. □

The following example shows that the converse of above lemma is not true, that is, an ultramodular function needs not to be an M$^\natural$-convex function.

Example 3.3.1. Define $f : \{0,1\}^3 \to \mathbf{R}$ as follows:

$$f(x) = \begin{cases} 2 & \text{if } x = (1,1,1) \\ 0 & \text{if } x = (0,1,0) \\ 1 & \text{otherwise.} \end{cases}$$

It is easy to see that f is ultramodular. However, it is not M$^\natural$-convex for $x = (1,0,1), y = (0,1,0)$ and $u = 1$.

3.4 Component-wise Ultramodular Functions

In this section, we define a new class of functions, called the component-wise ultramodular functions. This class of functions is contained in the class of ultramodular functions. Moreover, on integer lattice we will show that each component-wise ultramodular function is an M$^\natural$-convex function. We give an example to show that an M$^\natural$-convex function may not be a component-wise ultramodular function. Before defining component-wise ultramodular functions, we give few definitions.

Definition 3.4.1. A collection $\{x, y, z, w\}$ of vectors in $\mathbf{R}^n$ is said to be a component-wise test quadruple (denoted by CW-test quadruple) if it satisfies the following:

(i) $x + w = y + z$,

(ii) for each $i \in \{1, 2, \dots, n\}$, either $x(i) \le y(i) \le w(i)$ or $x(i) \ge y(i) \ge w(i)$.

Definition 3.4.2. A set $A \subseteq \mathbf{R}^n$ is *CW-ultramodular* if given any elements $x, y, w \in A$ with $x(i) \le y(i) \le w(i)$ or $x(i) \ge y(i) \ge w(i)$ for each $i \in \{1, 2, \dots, n\}$, the following holds:

$$x + w = y + z \text{ implies } z \in A. \tag{3.4.1}$$

In the following example, we show that an ultramodular set may not be a CW-ultramodular set.

Example 3.4.1. Define $A \subseteq \mathbf{R}^n$ by

$$A = \{(0,1,0), (1,0,0), (1,0,1)\}. \tag{3.4.2}$$

Then one can easily see that A is an ultramodular set. However, by setting $x = (0,1,0)$, $\quad y = (1,0,0)$, $\quad z = (0,1,1)$ and $w = (1,0,1)$, we see that $x(i) \leq y(i) \leq w(i)$ or $x(i) \geq y(i) \geq w(i)$ for each $i \in \{1,2,3\}$. Also, $x + w = y + z$ but $z \notin A$. Hence A is not a CW-ultramodular set.

Definition 3.4.3. A function $f : A \subseteq \mathbf{R}^n \longrightarrow \mathbf{R}$ is said to be component-wise ultramodular (denoted by CW-ultramodular) function if

$$f(x) + f(w) \geq f(y) + f(z)$$

holds for all CW-test quadruples $\{x, y, z, w\}$.

In definition 3.4.3, if we set $h(i) = z(i) - x(i) = w(i) - y(i)$ and define $h = (h(i) \mid i \in \{1, 2, \ldots, n\})$ then a CW-ultramodular function equivalently can be defined as follows:

A function $f : A \subseteq \mathbf{R}^n \longrightarrow \mathbf{R}$ is said to be a CW-ultramodular function if

$$f(x + h) - f(x) \geq f(y + h) - f(y)$$

holds for all $x, y \in A$ and $h \in \mathbf{R}^n$ such that $x \geq y$ and $x + h, y + h \in A$.

The following lemma gives the relationship between M$^\natural$-convex functions and CW-ultramodular functions.

Lemma 3.4.1. *If a function $f : \mathbf{Z}^V \to \mathbf{R}$ is CW-ultramodular, then it is M$^\natural$-convex.*

Proof. Let $x, y \in \mathbf{Z}^V$ and $u \in \mathrm{supp}^+(x - y)$. For each $w \in \mathrm{supp}^-(x - y)$, define $\lambda_w = y(w) - x(w)$. Take

$$h_1 = - \sum_{w \in \mathrm{supp}^-(x-y)} \lambda_w \chi_w$$

and set $x_1 = x - h_1$. Then $y \leq x_1$ and since f is CW-ultramodular, we have

$$f(y + h_1) - f(y) \leq f(x_1 + h_1) - f(x_1).$$

The above inequality is equivalent to

$$f(y+h_1)+f(x_1)\le f(x)+f(y). \tag{3.4.3}$$

Next, define

$$y_2=y+h_1,\quad x_2=x-\chi_u,\quad h_2=-(h_1-\chi_u).$$

Then $y_2\le x_2$ and the definition of CW-ultramodularity implies that

$$f(y_2+h_2)-f(y_2)\le f(x_2+h_2)-f(x_2).$$

The above inequality can further be written as:

$$f(y+\chi_u)+f(x-\chi_u)\le f(y+h_1)+f(x_1) \tag{3.4.4}$$

Combining (3.4.3) and (3.4.4), we get

$$f(x)+f(y)\ge f(x-\chi_u)+f(y+\chi_u)$$

which is (M$^\natural$-EXC) for $v=0$. Thus, f is M$^\natural$-convex function. □

The following example shows that the converse of the above lemma is not true.

Example 3.4.2. Define $f:\{0,1\}^3\to\mathbf{R}$ as follows:

$$f(x)=\begin{cases}2 & \text{if } x=(1,1,1)\\ 1 & \text{otherwise}\end{cases}$$

Consider the CW-test quadruple $\{x,y,z,w\}$ where

$$x=(0,1,1),\quad y=(1,1,1),\quad z=(0,0,0),\quad w=(1,0,0).$$

Then, f is not a CW-ultramodular function for $x=(0,1,1)$, $y=(1,1,1)$ and $h=(0,-1,-1)$. However, one can easily see that f is an M$^\natural$-convex function.

3.5 Concluding Remarks

Let $\mathcal{M}^\natural$, $\mathcal{UM}$ and $\mathcal{UM}_{\mathcal{CW}}$ denote the classes of M$^\natural$-convex functions, ultramodular functions and CW-ultramodular functions, respectively. Then from the above discussion, we conclude that on integer lattice, we have the following relation:

$$\mathcal{UM}_{\mathcal{CW}}\subset\mathcal{M}^\natural\subset\mathcal{UM}$$

We see that if $f : A \subseteq \mathbf{R}^n \to \mathbf{R}$ and any two elements of A are incomparable, that is, $x \not\leq y$ and $y \not\leq x$ for each $x, y \in A$ then f is vacuously ultramodular. This characteristic of ultramodular functions creates a gap between $\mathcal{M}^\natural$ and $\mathcal{UM}$.

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