Surface Gravity of Black Holes

by

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Supervised by

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Dean/Principal

Dedicated to

My Respected parents

for their never ending Love,

endless Support &

Encouragement.

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Knowledge is limited and words are bound to praise Almighty Allah for bestowing upon me strength and knowledge, to conclude this aspiration and to craft a useful contribution.

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Abstract

Surface gravity plays an important role in black hole thermodynamics. We have studied literature about the phenomenon of surface gravity. There are different formulae available in the literature to calculate the surface gravity of black hole spacetimes, however, it has not made explicit under what conditions these can be used. Generally, the Eddington-Finkelstein coordinates are used to find the surface gravity of black hole spacetimes, but the reason is not made clear. Using Killing vectors we have explicitly derive the formula to calculate the surface gravity for general spherically symmetric spacetimes, and also for general axially symmetric spacetime. Surface gravity of the Schwarzschild, the Reissner-Nordström and the Kerr black holes are calculated as examples.

Notations

In this thesis, signature convention used for the metric will be $(-, +, +, +)$ and the following list of acronyms and notations are used:

SR: Special Relativity

GR: General Relativity

$g_{\mu\nu}$: Metric Tensor

$T_{\mu\nu}$: Stress-Energy Tensor

$R_{\mu\nu}$: Ricci Tensor

R: Ricci Scalar

G: Gravitational Constant

c: Speed of Light

EFEs: Einstein field equations

κ: Surface Gravity

Contents

Preface

This thesis begins with a brief review of the GR in Chapter 1. Some basic terms of the Riemannian Geometry that are used in the analysis are defined in same chapter. Further, the formation of black holes, singularities and horizons are discussed.

In Chapter 2, the EFEs and some exact solutions of these equations are discussed. Further, geometric properties of the Schwarzschild, the Reissnerr-Nordström, the Kerr and the Kerr-Newmann black holes are discussed.

In Chapter 3, we study the Lie derivates and Killing vectors. Killing vectors of the schwarzschild metric and the Killing equations for Kerr metric are discussed.

In Chapter 4, we study the surface gravity of black holes. We have explicitly derived the formula for static, spherically symmetric spacetimes and also for axi-symmetric stationary spacetimes.

Thesis is concluded in chapter 5.

Chapter 1

Introduction

Physics is the science of measurement, among all the experimental sciences. Position plays an important role in all the measurements. In space (spacetime) position of an object is described by a coordinate system (CS). When we talk about CS we are actually labeling each point in space. We can assign different labels to same physical point. We can compare different CS through a set of equations which relate two different labels assign to same point. These relations are known as "coordinate transformations". In any CS the laws of physics remain invariant. The simplest CS is the cartesian CS, with three mutually perpendicular axes, known as x, y and z. However, it is not always the best choice. It is generally convenient to pick a CS that has the same symmetries as the situation being modeled. Subsequently, for physical problems having spherical symmetry, spherical polar coordinates are ones best choice. In spherical coordinates, a point is still marked by three numbers, i.e. the distance from the starting point and two angular coordinates. Coordinate transformation relates the label of a point in one CS to the label of same point in alternate CS.

In physics, the theory of relativity is thought to be one of the best accomplishments. The origins of GR can be traced to the conceptual revolution that followed Einstein's introduction of SR in 1905 [1]. It is a theory of gravity, so let us start by Newton's theory of gravity. There are two essential components: first an equation for the gravitational field as impacted by matter, and second an equation for the response of matter to this field. The ordinary Newtonian statement of these rules is in terms of forces between particles; the force between two objects of masses M and m is well known inverse square law [2]

$$\mathbf{F} = -\frac{GMm}{r^2}, \tag{1.1}$$

where M and m are not necessarily point masses, r is the distance between their centre of mass and G $(6.67 \times 10^{-8} dyn.\frac{cm^2}{g^2})$ is Newton's gravitational constant. The force F acts on a particle of mass m to give it an acceleration as indicated by Newton's second law

$$\mathbf{F} = ma. \tag{1.2}$$

Suppose that the mass M depends on time t then the above formula will become

$$\mathbf{F}(t) = -\frac{GM(t)m}{r^2}. \tag{1.3}$$

This implies that the force felt by the mass m at time t depends upon the value of the mass M at the same time t. There is no recompense for time delay, as SR would require. From our experience of advanced and retarded potentials in electrodynamics, one can say that SR would be satisfied if, in the above equation, $M(t)$ is modified to $M(t - \frac{r}{c})$. This would reflect the fact that the force felt by a small mass at time t depends on the value of the large mass at an earlier time $(t - \frac{r}{c})$; assuming, that the relevant gravitational 'information' travelled at the speed of light. However, this would then not be Newton's law. Newton's law, given by Eq.(1.3), which takes into account no time delay, implicitly suggests that the information that the mass M is changing travels with infinite velocity. Subsequent the effect of a changing M is felt at the same instant by the mass m. Since SR suggests that nothing can travel faster than the speed of light, Eqs.(1.1) and (1.3) are incompatible with it.

Confronted with such a dramatic situation – not to say crisis – the intuitive, and perfectly sensible, reaction of most physicists would be to attempt to 'tinker' with Newton's law; to change it slightly, in order to make it compatible with SR. And infact numerous such endeavors were made, however, none were successful. Einstein ultimately concluded that nothing less than a complete 'new look' at the problem of gravitation had to be taken. In spite of the fact that the Newton's equations are 'wrong', but they are an extremely good approximation to whatever 'correct' theory is discovered, so this theory ought to then give, as a first approximation, Newton's law. We have in no way finished with Newton!

Let $\mathbf{g} = \frac{F}{m}$, the gravitational field intensity. This is a parallel equation to $(\mathbf{E} = F/q)$ in electrostatics; the electric field E is the force per unit charge and the gravitational field g is the force per unit mass. Mass is the 'source' of gravitational field as the electric charge is the source of an electric field. Then Eq.(1.1) can be written as

$$\mathbf{g} = -\frac{GM}{r^2}, \tag{1.4}$$

the above equation gives an expression for the gravitational field intensity at a distance r from a mass M. This expression, on the other hand, is of a somewhat special form, since the right hand side is a gradient. We can write as

$$\mathbf{g} = -\nabla\phi,$$

where

$$\phi(r) = -\frac{GM}{r}. \tag{1.5}$$

Here the function $\phi(r)$ is the gravitational potential, a scalar field. Newton's theory is then depicted simply by one function. A mass, or a distribution of masses, gives rise to a scalar gravitational potential which completely determines the gravitational field. The potential ϕ in turn satisfies field equations. These are Laplace's and Poisson's equations, relevant to the cases where there is a vacuum, or a matter density ρ respectively:

$$(Laplace) \quad \nabla^2\phi = 0 \quad (vaccum), \tag{1.6}$$

$$(poisson) \quad \nabla^2\phi = 4\pi G\rho \quad (matter). \tag{1.7}$$

This completes our account for Newton's gravitational theory. The Newtonian gravitational force g acts instantaneously. The field g depends on r but not on t, therefore, it is incompatible with SR. Newtonian gravity can subsequently just be an approximation to a yet more fundamental theory.

The Newton's theory tells us to consider gravity as a force, and GR instructs us to consider it as curvature of spacetime. Newton's theory of gravity is an estimation to GR that works just when gravity is relatively weak, however, it breaks down when gravity is strong [3]. Einstein realized that he needed to build a theory that was capable of portraying the complicated nature of gravitation, and the universe, using his ideas of curved spacetime, however, he likewise needed to create this theory with the characteristic that at large distances from mass sources the universe can be clarified by Newtonian mechanics. Einstein says, mass-energy advises spacetime how to curve and curved spacetimes advises mass-energy how to move. In GR we study spacetime in terms of curvature. Various observational phenomena are effectively clarified by GR, for example, bending of light, precession of perihelion of Mercury and gravitational red shift [4]. GR reduces to the Newton theory under explicit circumstances when a particle is moving with slow velocity in a weak and static gravitational field. The two assumptions on which GR is actually based are

- Principle of equivalence

  It states that there are no physical affects which can distinguish whether a system is accelerated with a constant acceleration or whether it is under the influence of a homogenous gravitational field.

- Principle of general covariance

  It states that all frame of reference are physically equivalent.

1.0.1 The Metric Tensor

It is a tensor of second rank [5]. It defines the inner product between two vectors v_1 and v_2 i.e.

$$g(v_1, v_2) = v_1.v_2. \tag{1.8}$$

The contravariant and covariant components are given by [6]

$$g_{\alpha\beta} = g(e_\alpha, e_\beta) = e_\alpha.e_\beta, \tag{1.9}$$

and

$$g^{\alpha\beta} = g(e^\alpha, e^\beta) = e^\alpha . e^\beta. \tag{1.10}$$

The metric and the inverse of it are symmetric tensors and can be used for raising and lowering indices. The mixed component of metric tensor is

$$g^{\alpha\lambda} g_{\lambda\beta} = \delta^\alpha_\beta, \tag{1.11}$$

here δ^α_β is the kronecker delta and is defined as

$$\begin{aligned} \delta^\alpha_\beta &= 1 \qquad for \qquad \alpha = \beta, \\ &= 0 \qquad for \qquad \alpha \neq \beta. \end{aligned} \tag{1.12}$$

Line Element

The line element ds^2, which is the distance between two points x^α and $x^\alpha + dx^\alpha$ on a curve is a function of metric tensor given as

$$\begin{aligned} ds^2 =& g(u, u) d\lambda^2, \\ =& g_{\alpha\beta} u^\alpha u^\beta d\lambda^2, \\ =& g_{\alpha\beta} dx^\alpha dx^\beta, \end{aligned} \tag{1.13}$$

where $u^\alpha = \frac{dx^\alpha}{d\lambda}$, λ is an affine parameter. If the signature of the metric tensor $g_{\alpha\beta}$, is taken as $(-, +, +, +)$, then

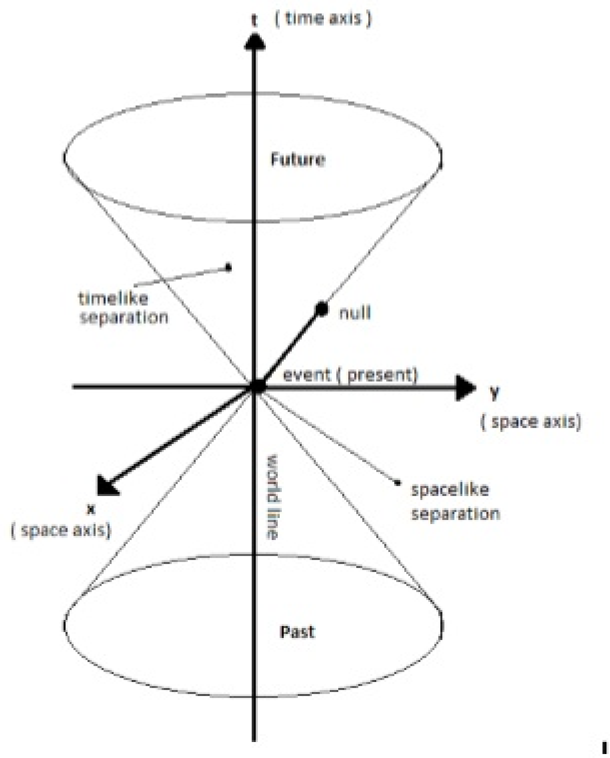
$$ds^2 < 0 \qquad represents \quad timelike \quad seperation, \tag{1.14}$$

$$ds^2 = 0 \qquad lightlike \quad or \quad null \quad seperation, \tag{1.15}$$

$$ds^2 > 0 \qquad spacelike \quad seperation. \tag{1.16}$$

In figure (1.1), the points which lie on the cone are **null separated**, the set of points which lies inside the future and past light cones are called **timelike**

separated and the set of points outside the light cone are called **spacelike separated** [5].



gure 1.1: A light cone, portrayed on a spacetime diagram.

Covariant Derivative

The tensorial character of a tensor can not be preserve by partial derivative, so we use an advance rule of differentiation called the covariant derivative. The covariant derivative of a vector (tensor of first rank) U_a is defined as

$$U_{a;b} = U_{a,b} - \Gamma^c_{ab} U_c, \tag{1.17}$$

where Γ is known as Christoffel symbol given as

$$\Gamma^{\alpha}_{\beta\gamma} = \frac{1}{2} g^{\alpha\lambda}(g_{\gamma\lambda,\beta} + g_{\beta\lambda,\gamma} - g_{\beta\gamma,\lambda}), \tag{1.18}$$

also the covariant derivative of a contravariant vector U^a is

$$U^a_{;b} = U^a, b + \Gamma^a_{cb} U^c. \tag{1.19}$$

The covariant derivative of a second rank tensor is described as

$$T_{\alpha\beta;\lambda} = T_{\alpha\beta,\lambda} - \Gamma^{\rho}_{\alpha\lambda} T_{\rho\beta} - \Gamma^{\rho}_{\beta\lambda} T_{\rho\alpha}. \tag{1.20}$$

The covariant derivative of metric tensor is always zero i.e.

$$g_{\alpha\beta;\lambda} = 0. \tag{1.21}$$

In general, we can construct a formula for covariant differentiation of any vector of t contravariant and s covariant components described as

$$\begin{aligned} T^{\alpha_1 ....\alpha_t}_{\beta_1 ....\beta_s;\gamma} &= T^{\alpha_1 ....\alpha_t}_{\beta_1 ....\beta_s,\gamma} + \Gamma^{\alpha_1}_{p\gamma} T^{p\alpha_2 ......\alpha_t}_{\beta_1\beta_2 ....\beta_s} + \Gamma^{\alpha_2}_{p\gamma} T^{\alpha_1 p ......\alpha_t}_{\beta_1\beta_2 ....\beta_s} + ... + \Gamma^{\alpha_t}_{p\gamma} T^{\alpha_1\alpha_2 ......p}_{\beta_1\beta_2 ....\beta_s} \\ &- \Gamma^{p}_{\beta_1\gamma} T^{\alpha_1\alpha_2 ......\alpha_t}_{p\beta_2 ....\beta_s} - \Gamma^{p}_{\beta_2\gamma} T^{\alpha_1\alpha_2 ......\alpha_t}_{\beta_1 p ....\beta_s} - ....\Gamma^{p}_{\beta_s\gamma} T^{\alpha_1\alpha_2 ......\alpha_t}_{\beta_1\beta_2 ....p} \end{aligned} \tag{1.22}$$

1.0.2 The Riemann Curvature Tensor

The Riemann curvature tensor is a tensor of rank four related to the curvature of spacetime and is given as, [7]

$$R^{\alpha}_{\beta\mu\nu} = \Gamma^{\alpha}_{\beta\nu,\mu} - \Gamma^{\alpha}_{\beta\mu,\nu} + \Gamma^{\alpha}_{\sigma\mu}\Gamma^{\sigma}_{\beta\nu} - \Gamma^{\alpha}_{\sigma\nu}\Gamma^{\sigma}_{\beta\mu}, \tag{1.23}$$

and

$$R_{\alpha\beta\mu\nu} = g_{\alpha\sigma} R^{\sigma}_{\beta\mu\nu}. \tag{1.24}$$

The Reimann tensor is anti-symmetric w.r.t first two and last two indices and it is symmetric if the first pair of indices is interchanged with the second pair of indices given as

$$R_{\alpha\beta\mu\nu} = -R_{\beta\alpha\mu\nu} = -R_{\alpha\beta\nu\mu} = R_{\mu\nu\alpha\beta}. \tag{1.25}$$

The Reimann tensor satisfies the two Bianchi identities which are as follow

$$R^{\alpha}_{\beta\mu\nu} + R^{\alpha}_{\nu\beta\mu} + R^{\alpha}_{\mu\nu\beta} = 0, \tag{1.26}$$

$$R_{\alpha\beta\mu\nu;\sigma} + R_{\alpha\beta\sigma\mu;\nu} + R_{\alpha\beta\nu\sigma;\mu} = 0. \tag{1.27}$$

The contraction of the curvature tensor gives us Ricci tensor which is symmetric tensor

$$R_{\beta\mu} = R^{\alpha}_{\beta\alpha\mu}, \tag{1.28}$$

or

$$R_{\beta\mu} = \Gamma^{\alpha}_{\beta\mu,\alpha} - \Gamma^{\alpha}_{\beta\alpha,\mu} + \Gamma^{\alpha}_{\alpha\nu}\Gamma^{\nu}_{\beta\mu} - \Gamma^{\alpha}_{\mu\nu}\Gamma^{\nu}_{\beta\alpha}, \tag{1.29}$$

and by further contraction one yields the Ricci scalar

$$R = g^{\beta\mu}R_{\beta\mu}. \tag{1.30}$$

The Stress-Energy momentum Tensor

The stress-energy momentum tensor is a tensor that describes the flux and density of momentum and energy in spacetime. It is the source of the gravitational field in EFEs, just as mass density is the source of gravitational field in Newtonian physics. It is denoted by $T_{\mu\nu}$. It is a second rank tensor and its components are displayed by square matrix of order 4 (in 4 dimesnsion) given as

$$\mathsf{T}^{\mu\nu} = \begin{pmatrix} T^{00} & T^{01} & T^{02} & T^{03} \\ T^{10} & T^{11} & T^{12} & T^{13} \\ T^{20} & T^{21} & T^{22} & T^{23} \\ T^{30} & T^{31} & T^{32} & T^{33} \end{pmatrix}. \tag{1.31}$$

Precession of Perihelion of Mercury

Mercury's orbit is shifting its position gradually over time, because of the curvature of space-time around the huge sun. The introduction of Mercury's orbit is found to precess in space after some time, this is normally called the precession of the perihelion, on the ground that it causes the position of perihelion to move. Only part of this can be represented for a perturbation in Newton's theory. There is an additional 43 seconds of arc per century in this precession that is anticipated by GR and observed to occur (a second of arc is 1/3600 of an angular degree). This impact is amazingly small, however, the measurement are extremely exact and can detect such little impacts exceptionally well .

Gravitational Red Shift

GR predicts that light originating from a strong gravitational field ought to have its wavelength shifted to its larger values. Point by point perceptions demonstrate such red shift and its magnitude is accurately given by Einstein's theory.

Bending of Light

Light bends because of an intrinsic curvature in spacetime surrounding a massive body [8]. According to GR, light coming from a star which is far away from the sun will be deviated due to the sun gravitational field by an amount which is inversely proportional to radial distance of the star from the sun. This amount is twice the amount that Einstein predicted in 1911 by using Newton's gravitational law.

1.0.3 Singularities

The place where curvature of the spacetime becomes infinite and also the known laws of physics no longer apply is known as a singularity. Singularities are of two types:

(i) essential singularity

(ii) coordinate singularity.

Coordinate singularity is due to bad choice of coordinates and can be removed by using appropriate coordinate transformations, but the essential singularity can not be removed by coordinate transformation. Such places are also known as missing pieces of spacetime. They disrupt the predictability of the spacetime and permit the uncontrollable information to enter into it. Essential singularities are of different kinds: point singularities, ring singularity and one dimensional lines.

If the singularity is of first kind (point singularity) then the black hole will be spherically symmetric like the Schwarzschild black hole (to be discussed in next chapter). The singularity will behave as a ring singularity if we consider rotating spherically symmetric black hole. Ring singularities also appear in case of cylindrically symmetric black hole. A point charge in space, will have spherical shape for its electric field, an electric current flowing throw the wire will produce cylindrically symmetric magnetic field and a spherically symmetric ball spinning about an axis not passing through its centre will move in a circle to form a ring.

Singularities can also be classified according to whether they are surrounded by event horizon or not. A naked singularity is that singularity which is not surrounded by an event horizon, therefore, does not represent a black hole. To identify singularities, the curvature invariants are constructed.

$$R_1 = g^{ab}R_{ab} = R, \tag{1.32}$$

$$R_2 = R^{ab}_{cd}R^{cd}_{ab}, \tag{1.33}$$

$$R_3 = R^{ab}_{cd}R^{cd}_{ef}R^{ef}_{ab}. \tag{1.34}$$

If the above defined curvature invariants R_1, R_2 and R_3 are all finite the singularity is coordinate, otherwise it is an essential singularity.

Gravitational collapse and black holes

In 1795 Laplace gave an idea of black holes first time when he discussed the classical body with escape velocity greater than the speed of light using Newton's theory of gravity, however, his idea did not attract much attention. In 1939, Julius Oppenheimer and Harland Synder [9] gave an idea that a sufficiently massive star must collapse indefinitely. In 1967, John Wheeler named such object a "black hole" [10] . GR anticipated the possibility of such dark objects from which even light would not have the capacity to escape.

A black hole is a region of spacetime exhibiting such strong gravitational effect that it precludes even light from escaping to infinity [11]. The gravity is so strong because matter has been squeezed into a tiny space. It is invisible but it pulls things in.

If a pressure gradient force is not strong enough, then a body can continue collapsing due to its own gravity. This phenomenon is known as "gravitational collapse [12]. A star is a self gravitating set of matter which is supported by its thermal pressure. This pressure is generated in stars by nuclear fusion reaction. Yet these reactions cannot continue forever and the star has no ability to maintain its equilibrium. If the star is massive enough, so that the inward force of gravity overcomes all outward acting forces, then the collapse continue. The volume of the star decreases and, therefore, the density increase. The escape velocity exceeds the speed of light with in a trapped surface which leads to the formation of black hole.

1.0.4 Horizons of Black Holes

Black holes are surrounded by singularities which are known as horizons. In literature there are different types of horizons depending upon properties. Some of them are discussed below.

Event horizon

It is a boundary in spacetime beyond which events can not effect an outside observer. Any object that approaches the horizon from the observer's side appears to slow down and never passes through the horizon. The light radiated from inside the horizon can never reach the observer. It is a place where light loses the capacity to escape from the black hole. It is sometimes characterized as the boundary within which black holes escape velocity is greater than the speed of light. Nothing can return once inside the event horizon not even light.

Apparent horizon

The apparent horizon dependably lies inside the event horizon and satisfies a stronger condition than that of event horizon. Within an apparent horizon , light is not moving away from the black hole while in event horizon light cannot escape to infinity. To comprehend difference between the event horizon and apparent horizon, consider a man going through a corridor filled with doors. He is attempting to go outward, yet they are closing in sequence. How many doors he passes through before he is stuck by a closed door? The door closest to him and is currently shut is analogous to apparent horizon. The door he actually reaches before he cannot travel any further is analogous to event horizon.

Killing Horizons

The Killing horizon is a null hypersurface characterized by the vanishing of the norm of a Killing vector field (both are named after Wilhelm Killing). κ is a geometrical quantity which is associated to a Killing horizon. The Killing horizon is said to be degenerate if the surface gravity vanishes. When we consider the flat spacetime Killing vectors are null all over the space time, so for this situation they do not characterized a hypersurface and we have no Killing horizon. In spacetime with time-translation symmetry the Killing horizon and event horizon are firmly related. We have following classification under specific conditions

- In a stationary ,asymptotically flat spacetime every event horizon is a Killing horizon for some killing vector field χ^{μ}.
- If the spacetime is static, x^{μ} is the Killing vector field $\chi^{\mu} = \frac{\partial}{\partial t}$ representing time translation symmetry.
- If the spacetime is stationary but not static, it is axisymmetric with a rotational Killing vector field $\xi^{\mu} = (\frac{\partial}{\partial_{\phi}})^{\mu}$ and k^{μ} is a linear combination

$\chi^\mu + \Omega_H \xi^\mu$ for some constant Ω_H, where $\chi^\mu = \frac{\partial}{\partial t}$.

Stationary Spacetime

A spacetime M is said to be stationary if there is a Killing vector ξ that is timelike in some region. The metric of a stationary spacetime can be written as

$$ds^2 = -e^{2U(x)}(dt + A(x))^2 + g_{\alpha\beta}(x)dx^\alpha dx^\beta, \tag{1.35}$$

where $x = (x^\alpha)$ represents the spatial coordinates. The Killing vector ξ can be written as $\xi = \partial_t$ in this coordinate system, hence

$$\xi_* = -e^{2U}(dt + A(x)). \tag{1.36}$$

The Kerr and the Kerr-Newmann spacetimes have offdiagonal term $dtd\phi$. The presence of this term shows that the Killing vector is not everywhere orthogonal to the spacelike surface. The geometry is no longer invariant under a change of $t \longrightarrow -t$. So these are stationary spacetimes.

Static Spacetime

A stationary spacetime M with the time translation Killing vector ξ is called static when the rotation of ξ vanishes. When a spacetime is static, from the rotation free condition, we can find a coordinate system locally in which the metric can be written as

$$ds^2 = -e^{2U(x)}dt^2 + g_{\alpha\beta}(x)dx^\alpha dx^\beta. \tag{1.37}$$

The Schwarzschild and the Reissner-Nordström spacetimes have timelike Killing vector which is everywhere orthogonal to the spacelike surface so they are static spacetimes.

Null Surface

A surface in spacetimes whose normal vector is everywhere null is called Null surface. The horizon of a black hole is a null surface.

Chapter 2

Exact Solutions of the Einstein Field Equation

2.1 The Einstein Field Equations

In 1915, Albert Einstein published a set of differential equation known as EFEs [13]. The EFEs are a set of second order non-linear partial differential equations which gives the relationship between matter and field and are independent of the choice of the coordinates. The EFEs basically relate the curvature of spacetime to radiations and matter. For deriving EFEs let us consider the variational principle in a local intertial frame. The variation is defined as

$$\delta I = \delta(I_m + I_G), \tag{2.1}$$

here I_G is the gravitational action integral and I_M is the energy matter action integral, defined as

$$I_G = \frac{1}{2k}\int_V \sqrt{-g}Rd^4x, \qquad I_M = \frac{1}{2k}\int_V \pounds_m\sqrt{-g}d^4x.$$

Here $k = \frac{8\pi G}{c^4}$ is constant, which is determined under the condition where the field equation reduces to Newton's law in weak field limit. $\pounds_m$ is the Lagrangian that is a function of x^μ and $\dot{x}^\mu$ and describes dynamics of a system [14]. Now we vary the action I inside the infinitesimally small region V under the condition that the variation at the boundary vanishes i.e; $\delta g_{\mu\nu}|_{\delta v} = 0$ and also $\delta g_{\mu\nu,\lambda}|_{\delta v} = 0$. Now calculating I_G and I_M one by one as

$$\delta I_G = \frac{1}{2k}\int_V \delta(R_{\mu\nu}g^{\mu\nu}\sqrt{-g})d^4x, \tag{2.2}$$

$$= \frac{1}{2k}\int_V [R_{\mu\nu}\delta(\sqrt{-g}g^{\mu\nu}) + \sqrt{-g}g^{\mu\nu}\delta R_{\mu\nu}]d^4x. \tag{2.3}$$

Consider a local inertial frame with vanishing of christoffel symbol in infinitesimal region, ($\Gamma^\lambda_{\mu\nu} = 0$ and $\Gamma^\lambda_{\mu\nu,\eta} \neq 0$) the components of Ricci tensor reduce to

$$R_{\mu\nu} = \Gamma^\lambda_{\mu\nu,\lambda} - \Gamma^\lambda_{\mu\lambda,\nu}. \tag{2.4}$$

As $\Gamma^\lambda_{\mu\nu}$ is not a tensor, so we can replace partial derivative with covariant derivative

$$g^{\mu\nu}\delta R_{\mu\nu} = (g^{\mu\nu}\delta\Gamma^\lambda_{\mu\nu})_{;\lambda} - (g^{\mu\nu}\delta\Gamma^\lambda_{\mu\lambda})_{;\nu}, \tag{2.5}$$

$$g^{\mu\nu}\delta R_{\mu\nu} = (g^{\mu\nu}\delta\Gamma^\lambda_{\mu\nu} - g^{\mu\nu}\delta\Gamma^\nu_{\mu\nu})_{;\lambda}. \tag{2.6}$$

Consider a vector A^λ defined as

$$A^\lambda = g^{\mu\nu}\delta\Gamma^\lambda_{\mu\nu} - g^{\mu\nu}\delta\Gamma^\nu_{\mu\nu}. \tag{2.7}$$

Using Eq.(2.7) in Eq.(2.6), one gets

$$g^{\mu\nu}\delta R_{\mu\nu} = A^{\lambda}_{;\lambda}, \tag{2.8}$$

$$\int_V \sqrt{-g}g^{\mu\nu}\delta R_{\mu\nu}d^4x = \int_V \sqrt{-g}A^{\lambda}_{;\lambda}d^4x. \tag{2.9}$$

Now applying *Guass divergence theorem* on right side of Eq.(2.9) defined as

$$\int_v \sqrt{-g}A^{\lambda}_{;\lambda}d^4x = \int_{\partial V} U_n A^{\lambda}\sqrt{-g}d^3y, \tag{2.10}$$

here $A^{\lambda}_{;\lambda}$ is the divergence of A^{λ} and U_n is the unit normal to the surface [15], so the Eq.(2.9) become

$$\int_V \sqrt{-g}g^{\mu\nu}\delta R_{\mu\nu}d^4x = 0. \tag{2.11}$$

Consider

$$\delta g = -g g_{\mu\nu}\delta g^{\mu\nu} = g g^{\mu\nu}\delta g_{\mu\nu}, \tag{2.12}$$

$$\delta\sqrt{-g} = -\frac{1}{2\sqrt{-g}}\delta g, \tag{2.13}$$

$$= -\frac{1}{2\sqrt{-g}}(-g g_{\mu\nu}\delta g^{\mu\nu}), \tag{2.14}$$

$$= -\frac{1}{2}\sqrt{-g}g_{\mu\nu}\delta g^{\mu\nu}, \tag{2.15}$$

or

$$\delta(g^{\mu\nu}\sqrt{-g}) = \sqrt{-g}\delta g^{\mu\nu} + g^{\mu\nu}(-\frac{1}{2}\sqrt{-g}g_{\alpha\beta}\delta g^{\alpha\beta}). \tag{2.16}$$

Using Eqs.(2.11) and (2.16) in Eq.(2.3), one gets

$$\delta I_G = \int_V \frac{1}{2k}\sqrt{-g}R_{\mu\nu}(\delta g^{\mu\nu} - \frac{1}{2}g^{\mu\nu}g_{\alpha\beta}\delta g^{\alpha\beta})d^4x. \tag{2.17}$$

Now consider

$$\delta I_M = \int_V \delta\sqrt{-g}\pounds_m d^4x, \tag{2.18}$$

$$= \int_V (\delta \pounds_m)\sqrt{-g} + \pounds_m(\delta\sqrt{-g}) d^4x. \qquad (2.19)$$

Using Eqs.(2.13) and (2.15) in Eq.(2.19), we get

$$\delta I_M = -\frac{1}{2}\int [g_{\mu\nu}\pounds_m - 2\frac{\delta \pounds_m}{\delta g^{\mu\nu}}]\sqrt{-g}\delta g^{\mu\nu} d^4x. \qquad (2.20)$$

Let us define energy momentum tensor $T_{\mu\nu}$, which describes the flux and density of momentum and energy in spacetime as

$$T_{\mu\nu} = g_{\mu\nu}\pounds_m - 2\frac{\delta \pounds_m}{\delta g^{\mu\nu}}, \qquad (2.21)$$

Using Eqs.(2.17), (2.20) and (2.21) in Eq.(2.1)), we obtain

$$\delta I = \int_V [\frac{1}{2k}(\sqrt{-g}R_{\mu\nu}(\delta g^{\mu\nu} - \frac{1}{2}g^{\mu\nu}g_{\alpha\beta}\delta g^{\alpha\beta}) + \sqrt{-g}A^{\lambda}_{,\lambda}]d^4x - \frac{1}{2}\int_V \sqrt{-g}T_{\mu\nu}\delta g^{\mu\nu}d^4x, \qquad (2.22)$$

$$= \frac{1}{2k}\int_V \sqrt{-g}(R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu})\delta g^{\mu\nu}d^4x - \frac{1}{2}\int_V \sqrt{-g}T_{\mu\nu}\delta g^{\mu\nu}d^4x, \qquad (2.23)$$

$$= \frac{1}{2k}(\int_V \sqrt{-g}(R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} - kT_{\mu\nu})\delta g^{\mu\nu}d^4x. \qquad (2.24)$$

For an arbitrary variation $\delta I = 0$, we have $\partial g_{\mu\nu}= 0$ and obtain the EFEs as [16]

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = kT_{\mu\nu}. \qquad (2.25)$$

$\varepsilon_{\mu\nu}= R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu}$ is the Einstein tensor which is second rank symmetric tensor. The EFEs with cosmological constant Λ are given as

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} + \Lambda g_{\mu\nu} = kT_{\mu\nu}. \qquad (2.26)$$

2.2 The Schwarzschild Solution

The first exact solution of the EFEs was found by Karl Schwarzschild in (1916). Schwarzschild sought the metric g_{ab} representing the static, spherically symmetric gravitational field in the empty space surrounding some massive spherical objects.

Consider the most general spherically symmetric static metric as

$$ds^2 = -A(r)c^2dt^2 + B(r)dr^2 + r^2d\Omega^2, \tag{2.27}$$

where

$$d\Omega^2 = d\theta^2 + r^2\sin^2\theta d\phi^2. \tag{2.28}$$

As $T_{\mu\nu}$=0 so we have $R_{\mu\nu} = 0$. The metric tensor $g_{\mu\nu}$ can be written as

$$\mathsf{g}_{\mu\nu} = \begin{pmatrix} -A(r) & 0 & 0 & 0 \\ 0 & B(r) & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2\sin^2\theta \end{pmatrix}, \tag{2.29}$$

the inverse metric tensor $g^{\mu\nu}$ is

$$\mathsf{g}^{\mu\nu} = \begin{pmatrix} -\frac{1}{A(r)} & 0 & 0 & 0 \\ 0 & \frac{1}{B(r)} & 0 & 0 \\ 0 & 0 & \frac{1}{r^2} & 0 \\ 0 & 0 & 0 & \frac{1}{r^2\sin^2\theta} \end{pmatrix}. \tag{2.30}$$

The non-zero *Christoffel* symbols for Eq.(2.27) are

$$\Gamma^0_{01} = \frac{A'(r)}{2A(r)}, \quad \Gamma^2_{33} = -\sin\theta\cos\theta, \quad \Gamma^3_{23} = \cot\theta, \tag{2.31}$$

$$\Gamma^2_{12} = \frac{1}{r} = \Gamma^3_{13}, \quad \Gamma^1_{11} = \frac{B'(r)}{2B(r)}, \quad \Gamma^1_{22} = -\frac{r}{B(r)}, \tag{2.32}$$

$$\Gamma^1_{00} = \frac{A'(r)}{2B(r)}, \quad \Gamma^1_{33} = -\frac{r\sin^2\theta}{B(r)}. \tag{2.33}$$

By using above mentioned *Christoffel* symbol the surviving EFEs are

$$R_{00} = \frac{A''(r)}{2B(r)} + \frac{A'(r)}{4B(r)}\left(\frac{A'(r)}{A} + \frac{B'(r)}{B(r)}\right) - \frac{A'(r)}{rB(r)} = 0, \tag{2.34}$$

$$R_{11} = \frac{A''(r)}{2A(r)} - \frac{A'(r)}{4A(r)}\left(\frac{(A'(r)}{A(r)} + \frac{B'(r)}{B(r)}\right) - \frac{B'(r)}{rB(r)} = 0, \tag{2.35}$$

$$R_{22} = \frac{1}{B(r)} - 1 + \frac{r}{2B(r)}\left(\frac{A'(r)}{A(r)} - \frac{B'(r)}{B(r)}\right) = 0, \tag{2.36}$$

$$R_{33} = R_{22}\sin^2\theta = 0. \tag{2.37}$$

Adding Eqs.(2.35) and Eq.(2.36), one yeilds

$$-\frac{A'(r)}{rA(r)} - \frac{B'(r)}{rB(r)} = 0, \tag{2.38}$$

by simplifying and integrating Eq.(2.38), we have

$$B(r) = \frac{\alpha}{A(r)}, \tag{2.39}$$

where α is the constant of integration.

Taking derivative of Eq.(2.39), we get

$$B'(r) = -\frac{\alpha A'(r)}{A^2(r)}. \tag{2.40}$$

Using Eqs.(2.39) and Eq.(2.40) in Eq.(2.37), one gets

$$\frac{A(r)}{\alpha} - 1 + \frac{rA(r)}{2\alpha}\left(\frac{A'(r)}{A(r)} + \frac{A'(r)}{A(r)}\right) = 0,$$

$$\frac{A(r)}{\alpha} + \frac{rA'(r)}{\alpha} = 1,$$

$$A(r) = \alpha + \frac{\beta}{r}, \tag{2.41}$$

where β is constant of integration. Using Eq.(2.41) in Eq.(2.39), one gets

$$B(r) = \frac{\alpha}{\alpha + \frac{\beta}{r}}. \tag{2.42}$$

With Eq.(2.41) and Eq.(2.42), our metric become

$$ds^2 = -(\alpha + \frac{\beta}{r})c^2dt^2 + \alpha(\alpha + \frac{\beta}{r})^{-1}dr^2 + r^2d\Omega^2. \tag{2.43}$$

In order to find the values of α and β, one uses the weak field limit given as

$$\begin{aligned} g_{00} &= -(1 + \frac{2\Phi}{c^2}), \\ g_{11} &= (1 + \frac{2\Phi}{c^2}), \end{aligned} \tag{2.44}$$

where $\Phi = -\frac{GM}{r}$ is the potential. Comparing Eq.(2.41) with Eq.(2.44), we get

$$\begin{aligned} \alpha &= 1, \\ \beta &= \frac{-2GM}{c^2}. \end{aligned}$$

Our final result is the Schwarzschild metric given as

$$ds^2 = -(1 - \frac{2GM}{rc^2})c^2dt^2 + (1 - \frac{2GM}{rc^2})^{-1}dr^2 + r^2d\Omega^2. \tag{2.45}$$

In gravitational units ($c = G = 1$) the Schwarzschild metric [17] can be written as

$$ds^2 = -(1 - \frac{2M}{r})c^2dt^2 + (1 - \frac{2M}{r})^{-1}dr^2 + r^2d\Omega^2. \tag{2.46}$$

It describes static spherically symmetric geometry. The event horizon of this metric is located where $g_{11} \longrightarrow \infty$, and at

$$r = 2M, \tag{2.47}$$

At $r = 2M$ there is coordinate singularity and at $r = 0$ there is an essential singularity.

2.3 The Reissner-Nordström Solution

The EFEs for electromagnetic fields are

$$R_{\mu\nu} = \alpha T^{em}_{\mu\nu}, \tag{2.48}$$

where $T^{em}_{\mu\nu}$ is the electromagnetic stress energy tensor. Electromagnetic field tensor is given as

$$\mathsf{F}_{\mu\nu} = \begin{pmatrix} 0 & \frac{E_1(r)}{c} & 0 & 0 \\ -\frac{E_1(r)}{c} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \tag{2.49}$$

and the inverse is

$$\mathsf{F}^{\mu\nu} = \begin{pmatrix} 0 & -\frac{E_1(r)}{A(r)B(r)c} & 0 & 0 \\ \frac{E_1(r)}{A(r)B(r)c} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{2.50}$$

From the stress energy momentum tensor given by Eq.(1.31), one yeilds

$$T^{em}_{00} = -\frac{E_1(r)^2}{2\mu_0 c^2 B(r)}, \tag{2.51}$$

$$T^{em}_{11} = \frac{E_1(r)^2}{2\mu_0 c^2 A(r)}, \tag{2.52}$$

$$T^{em}_{22} = -\frac{r^2 E_1(r)^2}{2\mu_0 c^2 A(r)B(r)}, \tag{2.53}$$

$$T^{em}_{33} = -\frac{r^2 E_1(r)^2}{2\mu_0 c^2 A(r)B(r)} \sin^2\theta. \tag{2.54}$$

Using Eqs.(2.51), (2.52), (2.53) and (2.54) in Eq.(2.48), to have

$$-\frac{A(r)''}{2B(r)}+\frac{A(r)'}{4B(r)}(\frac{A(r)'}{A(r)}+\frac{B(r)'}{B(r)})-\frac{A(r)'}{rB(r)}=-\frac{\alpha E_1^2}{2\mu_0 B(r)c^2}, \quad (2.55)$$

$$\frac{A''}{2A}-\frac{A'}{4A}(\frac{A'}{A}+\frac{B'}{B})-\frac{B'}{rB}=\frac{\alpha E_1^2}{2\mu_0 A(r)c^2}, \quad (2.56)$$

$$\frac{1}{B(r)}-1+\frac{r}{2B(r)}(\frac{A(r)'}{A(r)}-\frac{B(r)'}{B(r)}-\frac{B(r)'}{rB(r)}=\frac{-\alpha r^2 E_1^2}{2\mu_0 A(r)B(r)c^2}, \quad (2.57)$$

$$\sin^2\theta(\frac{1}{B(r)}-1+\frac{r}{2B(r)}(\frac{A(r)'}{A(r)}-\frac{B(r)'}{B(r)})-\frac{B(r)'}{rB(r)})=\frac{-\alpha r^2 E_1^2}{2\mu_0 A(r)B(r)c^2}. \quad (2.58)$$

Multiplying Eq.(2.55) by $\frac{B(r)}{A(r)}$ and adding in Eq.(2.56), one have

$$A(r)B'(r)+A'(r)B(r)=0, \quad (2.59)$$

$$(A(r)B(r))'=0, \quad (2.60)$$

$$B(r)=\frac{-k_1}{A(r)}, \quad (2.61)$$

where k_1 is the constant of integration. Now using Eq.(2.61) in Eq.$eqref$51, one gets

$$A(r)+rA'(r)=k_1-\frac{-\alpha r^2 E_1^2}{2\mu_0 c^2}, \quad (2.62)$$

where α=$\frac{8\pi G}{c^4}$. Integrating Eq.(2.62), we obtain

$$A(r)=k_1+\frac{GQ^2}{4\pi\epsilon_0 c^4 r^2}+\frac{k_2}{r}, \quad (2.63)$$

where k_2 is constant of integration. For the values of k_1 and k_2 we use weak field limit

$$g_{00}=-(1-2GMc^2). \quad (2.64)$$

Now by comparing $Eqs.$(2.63) and (2.64) one yields k_1 and k_2=$2GMc^2$. Using the values of k_1 and k_2 in Eq.(2.63), one gets

$$A(r) = 1 - \frac{2GM}{c^2 r} + \frac{GQ^2}{4\pi\epsilon_0 c^4 r^2}. \tag{2.65}$$

In gravitational units ($c = G = 1$) also by takins $\frac{1}{4\pi\epsilon_0}$=1, Ar and $B(r)$ become

$$A(r) = 1 - \frac{2M}{r} + \frac{Q^2}{r^2}, \tag{2.66}$$

$$B(r) = (1 - \frac{2M}{r} + \frac{Q^2}{r^2})^{-1}. \tag{2.67}$$

Now using Eqs.(2.66) and (2.67) in Eq.(2.27), we get

$$ds^2 = -(1 - \frac{2M}{r} + \frac{Q^2}{r^2})dt^2 + (1 - \frac{2M}{r} + \frac{Q^2}{r^2})^{-1} dr^2 + r^2 d\Omega^2. \tag{2.68}$$

The Eq.(2.68) is the Reissner [18] and Nordström [19] solution of the EFEs. It describes the static spherically symmetric geometry due to a charged massive point. The event horizons of this metric are located where $g_{11} \longrightarrow \infty$, and are at

$$r_\pm = M \pm \sqrt{M^2 - Q^2}. \tag{2.69}$$

$r_\pm$ are coordinate singularities, r_- defines the inner horizon of the black hole and r_+ defines the outer horizon of the black hole. At $r =\ 0$ there is an essential singularity.

2.4 The Kerr Solution

This type of black hole possess only mass and angular momentum. The Kerr black hole is an uncharged black hole which rotates about a central axis. It is stationary and axially symmetric [20]. It is named after the Roy Kerr, who was the first person to solve the EFE's for a rotating source in 1963. The Kerr solution has the following distinct regions:

1. Ring singularity,
2. Inner and outer horizons,
3. Ergospheres,
4. Static limit.

The Kerr solution of the EFEs in Boyer-Lindquist coordinates is given as

$$ds^2 = -(1 - \frac{2Mr}{\rho^2})dt^2 - \frac{4Mar}{\rho^2}\sin^2\theta d\phi dt + \frac{\rho^2}{\Delta}dr^2 \tag{2.70}$$

$$+\rho^2 d\theta^2 + (r^2 + a^2 + \frac{2Ma^2 r\sin^2\theta}{\rho^2})\sin^2\theta d\phi^2, \tag{2.71}$$

where

$$\rho^2 = r^2 + a^2\cos^2\theta, \tag{2.72}$$

and

$$\Delta = r^2 - 2Mr + a^2, \tag{2.73}$$

where a is the angular momentum per unit mass and m is the point mass. Horizon exist when $g_{11} \longrightarrow \infty$, and are at

$$r_{\pm} = M \pm \sqrt{M^2 - a^2}. \tag{2.74}$$

Three cases arises here:

$$M = a \quad (Extreme\ Kerr\ solution),$$

$$M > a \quad (Regular\ Kerr\ BH\ solution),$$

$$M < a \quad (No\ horizon,\ Naked\ singularity).$$

In the limit $a \longrightarrow 0$ the metric (2.70) reduces to the Schwarzschild metric. There Exist a coordinate singularity for $\Delta = 0$ and an essential sigularity for $\rho = 0$.

2.5 The Kerr Newmann Solution

The Kerr-Newman solution of EFEs in GR describes the spacetime geometry in the region surrounding a charged, rotating mass. It is generalization of the Kerr metric. It is also stationary and axially symmetric [21]. Mathematically it can be described as

$$\begin{aligned} ds^2 &= -(\frac{\Delta - a^2 \sin^2\theta}{\Sigma})dt^2 - \frac{2asin^2\theta(r^2 + a^2 - \Delta)}{\Sigma}dtd\phi + \frac{\Sigma}{\Delta}dr^2 + \Sigma d\theta^2 \\ &\quad +(\frac{(r^2+a^2)^2 - \Delta a^2 \sin^2\theta}{\Sigma})\sin^2\theta d\phi, \end{aligned} \tag{2.75}$$

with

$$\Delta = r^2 - 2Mr + a^2 + Q^2, \quad \Sigma^2 = r^2 + a^2\cos^2\theta, \quad \text{and} \quad a = \frac{J}{M}. \tag{2.76}$$

Chapter 3

Killing Vectors

3.1 Lie Differentiation

Consider a curve γ, its tangent vector $v^a = \frac{dx^\alpha}{d\lambda}$, where λ is the affine parameter and B^α a vector field defined in a neighbourhood of γ. The point P shall have the coordinates x^a, while the point Q shall be at $(x^\alpha + dx^\alpha)$. [22]

$$x^{'\alpha} \equiv x^\alpha + dx^\alpha = x^\alpha + v^\alpha d\lambda. \tag{3.1}$$

Eq.(3.1) can be interpreted as an infinitesimal coordinate transformation from the system x to $x^{'}$. Under transformation (3.1), the vector field B^α takes the form

$$\begin{aligned} B^{'\alpha}(x^{'}) =& \frac{\partial x^{'\alpha}}{\partial x^\beta} B^\beta(x), \\ =& (\delta^\alpha_\beta + v^\alpha_{,\beta} d\lambda) B^\beta(x), \\ =& B^\alpha(x) + v^\alpha_{,\beta} B^\beta(x) d\lambda, \end{aligned} \tag{3.2}$$

or

$$B^{'\alpha}(Q) = B^\alpha(P) + v^\alpha_{,\beta} B^\beta(P) d\lambda. \tag{3.3}$$

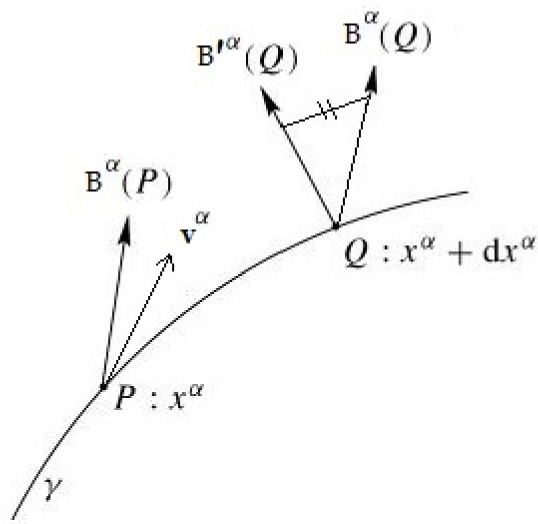


Figure 3.1: Lie Differentiation

Also, $B^{\alpha}(Q)$, the value of the original vector field at point Q, can be calculated as

$$\begin{aligned} B^{\alpha}(Q) =& B^{\alpha}(x + dx), \\ =& B^{\alpha}(x) + B^{\alpha}_{,\beta}(x) dx^{\beta}, \\ =& B^{\alpha}(p) + v^{\beta} B^{\alpha}_{,\beta}(P) d\lambda. \end{aligned} \tag{3.4}$$

In general, $B'^{\alpha}(Q)$ and $B^{\alpha}(Q)$ will not be equal. Their difference defines the Lie derivative [23] of the vector B^{α} along the curve γ as

$$\pounds_{v} B^{\alpha}(P) \equiv \frac{B^{\alpha}(Q) - B'^{\alpha}(Q)}{d\lambda}. \tag{3.5}$$

Now by combining Eqs.(3.3), (3.4) and (3.5), one yields

$$\pounds_{v} B^{\alpha} = B^{\alpha}_{,\beta} v^{\beta} - v^{\alpha}_{,\beta} B^{\beta}. \tag{3.6}$$

The Lie derivative of a covariant vector is given as

$$\pounds_v B_\alpha = B_{\alpha,\beta} v^\beta + v^\alpha_{,\beta} B_\beta, \tag{3.7}$$

and the Lie derivative of the mixed tensor is given as

$$\pounds_v T^\alpha_\beta = T^\alpha_{\beta,\gamma} v^\gamma - v^\alpha_{,\gamma} T^\gamma_\beta + v^\gamma_{,\beta} T^\alpha_\gamma. \tag{3.8}$$

3.2 Killing Vectors

If, in a given coordinate system, the metric tensor remains invariant when Lie transported along a curve γ then the tangent vector χ^α to the curve is called a Killing vector. Thus, the Lie derivative of the metric tensor along the tangent vector must be zero [14]

$$\pounds_\chi g_{\alpha\beta} = 0. \tag{3.9}$$

The condition for χ^α to be a Killing vector is that

$$0 = \pounds_\chi g_{\alpha\beta} = \chi_{\alpha;\beta} + \chi_{\beta;\alpha}. \tag{3.10}$$

Thus, the tensor $\chi_{\alpha;\beta}$ is antisymmetric if χ^α is a Killing vector. The alternative form of Eq.(3.10) is

$$\pounds_\chi g_{\alpha\beta} = \chi^a g_{\alpha\beta,a} + g_{\alpha a} \chi^a_{,\beta} + g_{\beta a} \chi^a_{,\alpha} = 0. \tag{3.11}$$

Killing vectors are usually used to evaluate constants of motion along a geodesic $x^\alpha(\lambda)$. If $v^a = \frac{dx^a}{d\lambda}$ is a tangent vector to the geodesic under consideration and χ_α is a Killing vector, then

$$\frac{d}{d\lambda}(v^\alpha \chi_\alpha) = (v^\alpha \chi_\alpha)_{;\beta} v^\beta, \tag{3.12}$$

$$=(v^{\alpha}_{;\beta}u^{\beta})\chi_{\alpha}+\chi_{\alpha;\beta}v^{\alpha}v^{\beta}, \tag{3.13}$$

$$=0. \tag{3.14}$$

The first term of Eq.(3.13) becomes zero by virtue of the geodesic equation and second term vanishes because it is a product of symmetric and anti-symmetric tensors.

3.3 Killing Vectors of the Schwarzschild Metric

The Killing Eq.(3.11) for Eq.(2.46) reduce to

$$k^{r}+\frac{r(r-2m)}{m}k^{t}_{,t}=0, \tag{3.15}$$

$$(1-\frac{2m}{r})^{2}k^{t}_{,r}-k^{r}_{,t}=0, \tag{3.16}$$

$$(1-\frac{2m}{r})k^{t}_{,\theta}-r^{2}k^{\theta}_{,t}=0, \tag{3.17}$$

$$(1-\frac{2m}{r})k^{t}_{,\phi}-r^{2}\sin^{2}\theta k^{\phi}_{,t}=0, \tag{3.18}$$

$$k^{r}-\frac{r(r-2m)}{m}k^{r}_{,r}=0, \tag{3.19}$$

$$r(r-2m)k^{\theta},r+k^{r}_{,\theta}=0, \tag{3.20}$$

$$r(r-2m)\sin^{2}\theta k^{\phi}_{,r}+k^{r}_{,\phi}=0, \tag{3.21}$$

$$k^{r}+rk^{\theta}_{,\theta}=0, \tag{3.22}$$

$$\sin^{2}\theta k^{\phi}_{,\theta}+k^{\theta}_{,\phi}=0, \tag{3.23}$$

$$\sin\theta k^{r}+r\cos\theta k^{\theta}+r\sin\theta k^{\phi}_{,\phi}=0. \tag{3.24}$$

Eqs.(3.15) to (3.24) represent a system of ten non-linear partial differential equations.

Differentiating Eqs.(3.16) and (3.17) w.r.t θ and r respectively and using Eq.(3.20) in the resulting equation obtained from Eq.(3.16), one gets

$$\frac{(r-2m)^2}{r^2}k^t_{,r\theta}+r(r-2m)k^\theta_{,rt}=0, \tag{3.25}$$

$$-\frac{2m(r-2m)}{r}k^t_{,\theta}-\frac{(r-2m)^2}{r^2}k^t_{,r\theta}+2(r-2m)k^\theta_{,t}+r(r-2m)k^\theta_{,tr}=0. \tag{3.26}$$

Adding Eq.(3.25) and Eq.(3.26), one gets

$$k^\theta_{,t}+rk^\theta_{,tr}=0. \tag{3.27}$$

Integrating Eq.(3.27) w.r.t t and then w.r.t r, one gets

$$k^\theta=\frac{1}{r}\int A_1(r,\theta,\phi)dr+\frac{1}{r}A_2(t,\theta,\phi), \tag{3.28}$$

where A_1 and A_2 are the functions of integration.

Differentiating Eqs.(3.17) and (3.18) w.r.t ϕ and θ respectively and using Eq.(3.23) in the resulting equation obtained from Eq.(3.17), one gets

$$\frac{r-2m}{r}k^t_{,\theta\phi}+r^2\sin^2\theta k^\phi_{,t\theta}=0, \tag{3.29}$$

$$\frac{r-2m}{r}k^t_{,\theta\phi}-r^2\sin^2\theta k^\phi_{,t\theta}-r^2\sin 2\theta k^\phi_{,t}=0. \tag{3.30}$$

Subtracting Eqs.(3.29) and (3.30), one gets

$$\sin\theta k^\phi_{,t\theta}+\cos\theta k^\phi_{,t}=0. \tag{3.31}$$

Integrating Eq.(3.31) w.r.t t and then w.r.t θ, one gets

$$k^\phi=\csc\theta\int A_3(r,\theta,\phi)d\theta+\csc\theta A_4(t,r,\phi), \tag{3.32}$$

where A_3 and A_4 are functions of integration. Now $Eq.$(3.19) is solved to obtain k^r as explicit function of t, θ and ϕ as

$$k^r = \frac{\sqrt{r-2m}}{\sqrt{r}} A_5(t,\theta,\phi), \tag{3.33}$$

where A_5 is the function of integration. Using Eq.(3.33) in Eq.(3.15), to have

$$k^t = \frac{-m}{r^{\frac{3}{2}}\sqrt{r-2m}} \int A_5(t,\theta,\phi) + A_6(r,\theta,\phi), \tag{3.34}$$

where A_6 is the function of integration. Now differentiating Eq.(3.16) w.r.t t and using Eqs.(3.33) and (3.34), one gets

$$r^4 A_{5,tt}(t,\theta,\phi) - m(2r-3m)A_5(t,\theta,\phi) = 0. \tag{3.35}$$

Since $A_5(t,\theta,\phi)$ and $A_{5,tt}(t,\theta,\phi)$ are independent of r, therefore, from Eq.(3.35) $A_5(t,\theta,\phi) = 0$, that yeilds

$$k^r = 0, \tag{3.36}$$

and

$$k^t = A_6(r,\theta,\phi) - \frac{mc_5}{r^{\frac{3}{2}}\sqrt{r-2m}}, \tag{3.37}$$

where c_5 is the constant of integration. Considering Eq.(3.16) and differentiating w.r.t θ, one gets

$$A_{6,\theta}(r,\theta,\phi) = B_6(\theta,\phi), \tag{3.38}$$

where $B_6(\theta,\phi)$ is the integration function. Now using Eqs.(3.28) and (3.34) in Eq.(3.17), to have

$$(r-2m)A_{6,\theta}(r,\theta,\phi) - rA_{2,t}(t,\theta,\phi) = 0. \tag{3.39}$$

Using Eq.(3.38) in Eq.(3.39), one gets

$$(r-2m)B_6(\theta,\phi) - r^2 A_{2,t}(t,\theta,\phi) = 0. \tag{3.40}$$

Since $B_6(\theta,\phi)$ and $A_{2,t}(t,\theta,\phi)$ are independent of r, therefore, from Eq.(3.40) $B_6(\theta,\phi)=0$ and $A_{2,t}(t,\theta,\phi)=0$, that leads to

$$A_{6,\theta}(r,\theta,\phi) = A_6(r,\phi),$$
$$A_2(t,\theta,\phi) = A_2(\theta,\phi). \tag{3.41}$$

Using Eqs.(3.41) in Eqs.(3.28) and (3.34) become

$$k^\theta = \frac{1}{r}\int A_1(r,\theta,\phi)dr + \frac{1}{r}A_2(\theta,\phi), \tag{3.42}$$
$$k^t = A_6(r,\phi) - \frac{mc_5}{r^{\frac{3}{2}}\sqrt{r-2m}}. \tag{3.43}$$

Considering Eq.(3.18) and using in Eqs.(3.32) and (3.34), one gets

$$-r^2\sin\theta A_{4,t}(t,r,\phi) + (1-\frac{2m}{r})A_{6,\phi}(r,\phi) = 0. \tag{3.44}$$

Comparing the coefficients of $\sin\theta$, one gets

$$A_{4,t}(t,r,\phi) = 0, \tag{3.45}$$
$$A_4(t,r,\phi) = A_4(r,\phi). \tag{3.46}$$

Also by comparing the constant coefficients, one gets

$$(1-\frac{2m}{r})A_{6,\phi}(r,\phi) = 0. \tag{3.47}$$

Integrating Eq.(3.47), to have

$$A_6(r,\phi) = A_6(r). \tag{3.48}$$

Now Eqs.(3.43) and (3.32) become

$$k^t = A_6(r) - \frac{mc_5}{r^{\frac{3}{2}}\sqrt{r-2m}}, \tag{3.49}$$

$$k^{\phi} = \csc\theta \int A_3(r,\theta,\phi)d\theta + \csc\theta A_4(r,\phi). \tag{3.50}$$

Using Eqs.(3.36) and (3.43) in Eq.(3.16), one gets

$$A_{6,r}(r) = c_5\left(\frac{m}{r^{\frac{3}{2}}\sqrt{r-2m}}\right)_{,r}. \tag{3.51}$$

Integrating above equation w.r.t r and using the resulting equation in Eq.(3.49), one obtains

$$k^t = c_1. \tag{3.52}$$

using Eq.(3.28) and (3.36) in Eq.(3.20) and differentiating the resulting equation twice w.r.t r, one gets

$$rA_{1,r}(r,\theta,\phi) = 0. \tag{3.53}$$

integrating Eq.(3.53) w.r.t r to get

$$A_1(r,\theta,\phi) \equiv A_1(\theta,\phi). \tag{3.54}$$

Subsituting Eq.(3.54) in Eq.(3.42), to have

$$k^{\theta} = A_1(\theta,\phi) + \frac{1}{r}A_2(\theta,\phi) + \frac{c*}{r}. \tag{3.55}$$

Differentiating Eq.(3.55) w.r.t r and using in Eq.(3.20) also using Eq.(3.36), to get

$$A_2(\theta,\phi) = 0. \tag{3.56}$$

Now using Eq.(3.36) in Eq.(3.22), one gets

$$K^{\theta}_{,\theta} = 0, \tag{3.57}$$

by subsituting Eq.(3.56) in Eq.(3.55) and then using in Eq.(3.57), one has

$$A_1(\theta,\phi) \equiv A_1(\phi). \tag{3.58}$$

Using Eqs.(3.56) and (3.58) in Eq.(3.55), to obtain

$$k^{\theta} = A_1(\phi) + \frac{c^*}{r}. \tag{3.59}$$

Using Eqs.(3.32) and (3.59) in Eq.(3.23), one gets

$$-\cos\theta \int A_3(r,\theta,\phi)d\theta + \sin\theta A_3(r,\theta,\phi) - \cos\theta A_4(r,\phi) + A_{1,\phi}(\phi) = 0. \tag{3.60}$$

Differentiating Eq.(3.60) twice w.r.t θ, to obtain

$$A_3(r,\theta,\phi) = B_3(r,\theta)cos\theta + C_3(r,\phi)\sin\theta. \tag{3.61}$$

Using Eq.(3.61) in Eq.(3.50), the resulting equation is

$$k^{\phi} = B_3(r,\phi) - \cot\theta C_3(r,\phi) + \csc\theta A_4(r,\phi). \tag{3.62}$$

Consider Eq.(3.21) after using Eq.(3.36), the resulting equation is of the form

$$\sin\theta B_{3,r} - \cos\theta C_{3,r}(r,\phi) + A_{4,r}(r,\phi) = 0. \tag{3.63}$$

Comparing the coefficients of $\sin\theta$, $\cos\theta$ and constant in Eq.(3.63), one gets

$$B_3(r,\phi) \equiv B_3(\phi) \tag{3.64}$$

$$C_3(r,\phi) \equiv C_3(\phi) \tag{3.65}$$

$$A_4(r,\phi) \equiv A_4(\phi). \tag{3.66}$$

Using Eq.(3.64) in Eq.(3.62) the equation reduces to

$$k^{\phi} = B_3(\phi) - \cot\theta C_3(\phi) + \csc\theta A_4(\phi). \tag{3.67}$$

Consider Eq.(3.23) and use Eqs.(3.59) and (3.64), and differentiating w.r.t θ the resulting equation is

$$A_4(\phi) = 0. \tag{3.68}$$

Using Eq.(3.66) in Eq.(3.65), one obtains

$$k^{\phi} = B_3(\phi) - \cot\theta C_3(\phi). \tag{3.69}$$

Substituting Eq.(3.67) in Eq.(3.23), one gets

$$A_{1,\phi}(\phi) = -C_3(\phi). \tag{3.70}$$

Using Eqs.(3.36), (3.59) and (3.67) in Eq.(3.24), one has

$$\cos\theta((A_1(\phi) - C_{3,\phi}(\phi)) + \sin\theta B_{3,\phi}(\phi) = 0. \tag{3.71}$$

Comparing the coefficients of $\cos\theta$ and $\sin\theta$ in Eq.(3.69) and differentiating w.r.t ϕ, one obtains

$$C_3(\phi) = C_3^* \sin\phi + C_4^* \cos\phi, \tag{3.72}$$

$$B_3(\phi) = c_2. \tag{3.73}$$

Using Eqs.(3.70) and (3.71) in Eq.(3.65), the resulting equation is

$$K^{\phi} = c_2 - \cot\theta(c_3^* \sin\phi + c_4^* \cos\phi). \tag{3.74}$$

Similarly

$$k^{\theta} = c_3^* \cos\phi - c_4^* \sin\phi. \tag{3.75}$$

Finally, we have

$$k^t = c_1, \tag{3.76}$$

$$k^r = 0, \tag{3.77}$$

$$k^{\theta} = c_3^* \cos\phi - c_4^* \sin\phi, \tag{3.78}$$

$$k^{\phi} = c_2 - \cot\theta(c_3^* \sin\phi + c_4^* \cos\phi). \tag{3.79}$$

Therefore,

$$K = c_1\frac{\partial}{\partial t}+c_2\frac{\partial}{\partial\phi}+c_3(-\cos\phi\frac{\partial}{\partial\theta}+\cot\theta\sin\phi\frac{\partial}{\partial\phi})+c_4(\sin\phi\frac{\partial}{\partial\theta}+\cot\theta\cos\phi\frac{\partial}{\partial\phi}), \tag{3.80}$$

where $c_3 = -c_3^*$ and $c_4 = -c_4^*$. So there are four linearly independent Killing vectors which are $\frac{\partial}{\partial t}$, $\frac{\partial}{\partial\phi}$, $-\cos\phi\frac{\partial}{\partial\theta}+\cot\theta\sin\phi\frac{\partial}{\partial\phi}$ and $\sin\phi\frac{\partial}{\partial\theta}+\cot\theta\cos\phi\frac{\partial}{\partial\phi}$.

3.3.1 Killing Vectors of the Kerr Metric

The Killing equations for (2.70) Kerr metric are

$$\begin{aligned} 0 =&(\frac{-2Mr^2+2Ma^2\cos^2\theta}{\rho^2})k^r+(\frac{2Mra^2\sin 2\theta}{\rho^2})k^\theta \\ &+(\frac{4Mr}{\rho^2}-2)k^t_{,t}-(\frac{4Mra\sin^2\theta}{\rho^2})k^\phi_{,t}, \end{aligned} \tag{3.81}$$

$$0 = (\frac{2Mr}{\rho^2}-1)k^t_{,r}-(\frac{2Mra^2\sin^2\theta}{\rho^2})k^\phi_{,r}+\frac{\rho^2}{\Delta}k^r_{,t}, \tag{3.82}$$

$$0 = (\frac{2Mr}{\rho^2}-1)k^t_{,\theta}-(\frac{2Mra\sin^2\theta}{\rho^2})k^\phi_{,\theta}+\rho^2k^\theta_{,t}0, \tag{3.83}$$

$$\begin{aligned} 0 =&(\frac{2Ma\sin^2\theta(r^2-a^2\cos^2\theta)}{\rho^2})k^r-((\frac{2Ma\sin 2\theta(r^2+a^2)}{\rho^2})k^\theta \\ &+(\frac{2Mr}{\rho^2}-1)k^t_{,\phi}+(r^2+a^2+\frac{2mra^2\sin^2\theta}{\rho^2})\sin^2\theta \\ &k^\phi_{,t}-\frac{2Mra\sin^2\theta}{\rho^2}(k^t_{,t}+k^\phi_{,\phi}), \end{aligned} \tag{3.84}$$

$$0 =(\frac{2ra^2\sin^2\theta-2M(r^2-a^2\cos^2\theta}{\Delta^2})k^r-\frac{a^2\sin 2\theta}{\Delta}+\frac{2\rho^2}{\Delta}k^r_{,r}, \tag{3.85}$$

$$0 =\rho^2k^\theta_{,r}+\frac{\rho^2}{\Delta}k^r_{,\theta}, \tag{3.86}$$

$$0 =(r^2 + a^2 + \frac{2mra^2 \sin^2\theta}{\rho^2}) \sin^2\theta k^{\phi}_{,r} + \frac{\rho^2}{\Delta} k^r_{,\phi} - \frac{2mra \sin^2\theta}{\rho^2} k^t_{,r}, \qquad (3.87)$$

$$0 =rk^r - a^2 \cos\theta \sin\theta k^{\theta} + \rho^2 k^{\theta}_{,\theta}, \qquad (3.88)$$

$$0 =(r^2 + a^2 + \frac{2mra^2 \sin^2\theta}{\rho^2}) \sin^2\theta k^{\phi}_{,\theta} + \rho^2 k^{\theta}_{,\phi} - \frac{2mra \sin^2\theta}{\rho^2} k^t_{,\theta}, \qquad (3.89)$$

$$\begin{aligned} 0 =&(2r - \frac{2ma^2 \sin^2\theta (r^2 - a^2 \cos^2\theta)}{\rho^2}) \sin^2\theta k^r + [(\frac{2mra^2 \sin^2\theta (r^2 + a^2)}{\rho^2}) \sin^2\theta \\ &+ (r^2 + a^2 + \frac{2ma^2 \sin^2\theta}{\rho^2}) \sin 2\theta] k^{\theta} + 2(r^2 + a^2 + \frac{2mra^2 \sin^2\theta}{\rho^2}) \sin^2\theta k^{\phi}_{,\phi} \qquad (3.90) \\ &- \frac{4mra \sin^2\theta}{\rho^2}) k^t_{,\phi}. \end{aligned}$$

We can find k^t, k^r, k^{θ}, and k^{ϕ}, by solving system of $Eqs.$(3.81)-(3.90). Since $g_{\mu\nu}$ is independent of t and ϕ so the Kerr metric must have the Killing vector χ_t and χ_{θ} which satisfy the given system of equations.

Chapter 4

Surface Gravity

The surface gravity κ of an object is the gravitational acceleration experienced at its surface. It is the acceleration due to gravity experienced by a test particle which is very close to the object's surface and which in order not to disturb the system, has negligible mass. The units of surface gravity are same as that of acceleration which, in SI system, are meter per second square. Consider a test particle with mass m on the surface of the earth with mass M, then according to the law of gravitational force

$$\begin{aligned} \mathbf{F} &= \frac{GMm}{R^2}, \\ mg &= \frac{GMm}{R^2}, \\ \mathbf{g} &= \frac{GM}{R^2}, \end{aligned} \tag{4.1}$$

where $\mathbf{g}$ represents the surface gravity.

4.1 Surface Gravity of a Black Hole

The Newtonian concept of acceleration turns out not to be clear in relativity. For a black hole, which can only be treated relativistically, we cannot define a surface gravity as the acceleration experienced by a test body at the object's surface. The reason is because in relativity the acceleration of a test body at the event horizon of a black hole becomes infinite. When someone talks about the surface gravity of a black hole, one is defining a notion that behaves analogously to the Newtonian surface gravity, but is not the same. In fact, the surface gravity of a general black hole is not well defined. However, one can define the surface gravity for a black hole whose event horizon is a Killing horizon [24] as
"Surface gravity κ is the force, as exerted at infinity, needed to keep an onject at the horizon".
The surface gravity is an expression of the acceleration of gravity at the Killing horizon of a black hole. Mathematically, if χ^a is a suitably normalized Killing vector, then the surface gravity is defined by [24]

$$\nabla^a(\chi^b\chi_b) = -2\kappa\chi^a, \tag{4.2}$$

where ∇^a is the contravariant derivative and the equation is evaluated at the horizon. Eq.(4.2) can be expressed as

$$(\nabla^a\chi^b)\chi_b + \chi^b(\nabla^a\chi_b) = -\,2\kappa\chi^a, \tag{4.3}$$

$$(g^{ac}\nabla_c\chi^b)\chi_b + \chi^b(g^{ad}\nabla_d\chi_b) = -\,2\kappa\chi^a, \tag{4.4}$$

$$(g^{ac}\nabla_c g^{be}\chi_e)\chi_b + \chi^b(g^{ad}\nabla_d\chi_b) = -\,2\kappa\chi^a, \tag{4.5}$$

$$g^{ac}g^{be}(\nabla_c\chi_e)\chi_b + \chi^b(g^{ad}\nabla_d\chi_b) = -\,2\kappa\chi^a, \tag{4.6}$$

$$g^{ad}[(\nabla_d\chi_e)\chi^e + \chi^b(\nabla_d\chi_b)] = -\,2\kappa\chi^a, \tag{4.7}$$

$$g_{ac}g^{ad}[(\nabla_d\chi_e)\chi^e + \chi^b(\nabla_d\chi_b)] = -\,2\kappa g_{ac}\chi^a, \tag{4.8}$$

$$\delta^d_c[2(\nabla_d\chi_b)\chi^b] = -\,2\kappa\chi_c, \tag{4.9}$$

$$(\nabla_c\chi_b)\chi^b = -\,\kappa\chi_c. \tag{4.10}$$

Now by using Killing vector equation,

$$\nabla_a\chi_b + \nabla_b\chi_a = 0, \tag{4.11}$$

Eq.(4.10) becomes

$$(\nabla_b\chi_d)\chi^b = \kappa\chi_d, \tag{4.12}$$

which can also be written as

$$(\nabla^b\chi^d) = \kappa\chi^d. \tag{4.13}$$

Since X_a is a hypersurface orthogonal to the horizon, by Frobenius's theorem, on the horizon given as

$$\chi_{[a\nabla_b\chi_c]} = 0. \tag{4.14}$$

Using Killing's equation this implies

$$\chi_c\nabla_a\chi_b = -2\chi_{[a\nabla_b]}\chi_c, \tag{4.15}$$

on the horizon. Contracting with $\nabla^a\chi^b$, we have

$$\chi_c(\nabla^a\chi^b)(\nabla_a\chi_b) = -\,2\chi_{[a\nabla_b]}\chi_c(\nabla^a\chi^b), \tag{4.16}$$

$$\chi_c(\nabla^a\chi^b)(\nabla_a\chi_b) = -\,2(\frac{1}{2}(\chi_a\nabla_b - \chi_b\nabla_a))\chi_c(\nabla^a\chi^b), \tag{4.17}$$

$$\chi_c(\nabla^a\chi^b)(\nabla_a\chi_b) = -\,2(\chi_a\nabla_b\chi_c)\nabla^a\chi^b, \tag{4.18}$$

$$\chi_c(\nabla^a\chi^b)(\nabla_a\chi_b) = -\,2(\chi_a\nabla^a\chi^b)\nabla_b\chi_c, \tag{4.19}$$

$$\chi_c(\nabla^a\chi^b)(\nabla_a\chi_b) = -2(\kappa\chi^b)\nabla_b\chi_c, \tag{4.20}$$

$$\chi_c(\nabla^a\chi^b)(\nabla_a\chi_b) = -2\kappa^2\chi_c, \tag{4.21}$$

or

$$\kappa^2 = -\frac{1}{2}(\nabla^a\chi^b)(\nabla_a\chi_b), \tag{4.22}$$

where evaluation on the horizon is understood.

4.2 Surface Gravity of Static Spherically Symmetric Spacetime

Consider a general static and spherically symmetric metric of the form

$$ds^2 = g_{00}(r)dt^2 + g_{11}(r)dr^2 + r^2d\theta^2 + r^2\sin^2\theta d\phi^2. \tag{4.23}$$

As we are considering static metric so we know none of the metric coefficients is a function of time, therefore there must exist a time-like Killing vector $\chi^b = \frac{\partial}{\partial t}$. In (t, r, θ, ϕ) coordinates

$$\chi^b =(\chi^0, \chi^1, \chi^2, \chi^3), \tag{4.24}$$

$$\chi^b =(1, 0, 0, 0), \tag{4.25}$$

$$\chi_b =(g_{00}, 0, 0, 0), \tag{4.26}$$

$$\chi^b\chi_b =g_{00}, \tag{4.27}$$

Hence, κ can be calculated by Eq.(4.22). The only surviving terms in the case of metric (4.23) in $\nabla_a\chi_b$ are $\nabla_r\chi_t$ and $\nabla_t\chi_r$. By using Killing equation, we have

$$\nabla_1\chi_0 = -\nabla_0\chi_1, \tag{4.28}$$

then Eq.(4.22) yeilds

$$\kappa^2 = -(\nabla^0 \chi^1)(\nabla_0 \chi_1). \tag{4.29}$$

Now,

$$\nabla_1 \chi_0 = \chi_{0,1} - \Gamma^{\sigma}_{10} \chi_{\sigma}, \tag{4.30}$$

$$\nabla^1 \chi^0 = g^{1c} \nabla_c \chi^1, \tag{4.31}$$

$$\Gamma^0_{10} = \frac{1}{2} g^{00} g_{00,1}, \tag{4.32}$$

$$\Gamma^1_{00} = -\frac{1}{2} g^{11} g_{00,1}. \tag{4.33}$$

Using the Eqs.(4.30), (4.31), (4.32) and (4.33) in Eq.(4.29), we have

$$\kappa^2 = -\frac{1}{4} g^{00} g^{11} (g_{00,1})^2, \tag{4.34}$$

$$\kappa = \frac{1}{2} \sqrt{-\frac{g^{11}}{g^{00}}} g^{00,1} g^{00}. \tag{4.35}$$

or

$$\kappa = -\frac{1}{2} \sqrt{-\frac{g^{11}}{g^{00}}} \frac{g^{00,1}}{g^{00}}. \tag{4.36}$$

In a particular case when

$$g_{00} = -f(r), \tag{4.37}$$

$$g_{11} = \frac{1}{f(r)}. \tag{4.38}$$

Using the value of g_{00} and g_{11} in Eq.(4.36), we have

$$\kappa = -\frac{1}{2} \sqrt{f^2(r)} \frac{-f'(r)}{f^2(r)} f(r), \tag{4.39}$$

$$\kappa = \frac{f'(r)}{2}. \tag{4.40}$$

4.2.1 Examples:

Surface Gravity of the Schwarzschild Black Hole

For the Schwarzschild metric Eq.(2.46), using Eq.(4.40) with $f(r) = 1 - \frac{2m}{r}$, we have

$$\kappa = \frac{1}{2}\frac{\partial}{\partial r}(1 - \frac{2M}{r})\Bigg|_{r=2m},$$

$$= \frac{M}{r^2}\Bigg|_{r=2m}, \tag{4.41}$$

$$\kappa = \frac{1}{4M}. \tag{4.42}$$

Surface Gravity of the Reissner-Nordström Black Hole

For the Reissner-Nordström metric given by Eq.(2.68) and using the formula given by Eq.(4.40) with $f(r) = 1 - \frac{2M}{r} + \frac{Q^2}{r^2}$, we have

$$\kappa = \frac{1}{2}\frac{\partial}{\partial r}(1 - \frac{2M}{r} + \frac{Q^2}{r^2})\Bigg|_{r_+=M+\sqrt{M^2-Q^2}},$$

$$= \frac{rM - Q^2}{r^3}\Bigg|_{r_+=M+\sqrt{M^2-Q^2}}, \tag{4.43}$$

$$\kappa = \frac{(M + \sqrt{M^2 - Q^2})M - Q^2}{[M + \sqrt{(M^2 - Q^2)}]^3}. \tag{4.44}$$

Notice that at $Q = 0$ the expression reduces to value of the Schwarzschild surface gravity.

4.3 Surface Gravity of General Axially Symmetric Stationary Spacetime

Let us consider an axially symmetric stationary spacetime given as

$$ds^2 = g_{00}(r,\theta)dt^2 + g_{11}(r,\theta)dr^2 + g_{22}(r,\theta)d\theta^2 + g_{33}(r,\theta)d\phi^2 + 2g_{03}(r,\theta)dtd\phi. \tag{4.45}$$

Being stationary and axially symmetric, the metric admits two Killing vector fields

$$\vec{k} \equiv \frac{\partial}{\partial t} = \partial_t, \tag{4.46}$$

$$\vec{m} \equiv \frac{\partial}{\partial \phi} = \partial_\phi. \tag{4.47}$$

If $\vec{k}$ and $\vec{m}$ are the only Killing vector fields then, any Killing vector field, χ^a is a linear combination of these. Therefore,

$$\chi^a = \partial_t + \Omega_H \partial_\phi, \tag{4.48}$$

where $\Omega_H = -\frac{g_{03}}{g_{33}}$. In (t, r, θ, ϕ) coordinates

$$\chi^a = (1, 0, 0, \Omega_H), \tag{4.49}$$

or

$$\chi^a = (1, 0, 0, -\frac{g_{03}}{g_{33}}). \tag{4.50}$$

Also

$$\chi_a = (g_{00} + g_{03}\Omega_H, 0, 0, g_{03} + g_{33}\Omega_H), \tag{4.51}$$

or

$$\chi_a = (g_{00} - \frac{(g_{00})^2}{g_{33}}, 0, 0, 0). \tag{4.52}$$

For the Killing vector given by Eqs.(4.49) and (4.51) leads to

$$\kappa^2 = -[(\nabla_0\chi_1)(\nabla^0\chi^1) + (\nabla_0\chi_2)(\nabla^0\chi^2) + (\nabla_1\chi_3)(\nabla^1\chi^3) + (\nabla_2\chi_3)(\nabla^2\chi^3)]. \tag{4.53}$$

4.3.1 Surface Gravity of the Kerr Black Hole

For the Kerr metric given in Eq.(2.70) the non zero components of the metric tensor are

$$g_{00} = -(1 - \frac{2Mr}{\rho^2}), \tag{4.54}$$

$$g_{03} = g_{30} = -\frac{2Mar\sin^2\theta}{\rho^2}, \tag{4.55}$$

$$g_{33} = (r^2 + a^2 + \frac{2Mra^2\sin^2\theta}{\rho^2})\sin^2\theta, \tag{4.56}$$

$$g_{11} = \frac{r^2 + a^2\cos^2\theta}{r^2 - 2Mr + a^2}, \tag{4.57}$$

$$g_{22} = r^2 + a^2\cos^2\theta, \tag{4.58}$$

and the non-zero components of the inverse metric are

$$g^{00} = -\frac{1}{\Delta}(r^2 + a^2 + \frac{2Mra^2\sin^2\theta}{\rho^2}), \tag{4.59}$$

$$g^{03} = g^{30} = -\frac{2Mra}{\Delta\rho^2}, \tag{4.60}$$

$$g^{33} = \frac{\Delta - a^2\sin^2\theta}{\rho^2\Delta\sin^2\theta}, \tag{4.61}$$

$$g^{11} = \frac{r^2 - 2Mr + a^2}{r^2 + a^2\cos^2\theta}, \tag{4.62}$$

$$g^{22} = r^2 + a^2\cos^2\theta. \tag{4.63}$$

Using Eq.(4.53) with Eqs.(4.49) and (4.51), we have

$$\begin{aligned}\kappa^2 = &-(\frac{-1}{2}g^{00}g_{00,1}\chi_0 - \frac{1}{2}g^{03}g_{03,1}\chi_0)(\frac{-1}{2}g^{00}g^{11}g_{00,1}\chi^0 \\ &- \frac{1}{2}g^{00}g^{11}g_{03,1}\chi^3 - \frac{1}{2}g^{03}g^{11}g_{03,1}\chi^0 - \frac{1}{2}g^{03}g^{11}g_{33,1}\chi^3) \\ &- (\frac{-1}{2}g^{00}g_{00,2}\chi_0 - \frac{1}{2}g_{03,2}g^{03}\chi_0)(\frac{-1}{2}g^{00}g^{22}g_{00,2}\chi^0 \\ &- \frac{1}{2}g^{00}g^{22}g_{03,2}\chi^3 - \frac{1}{2}g^{03}g^{22}g_{03,2}\chi^0 - \frac{1}{2}g^{03}g^{22}g_{33,2}\chi^3) \\ &- (-\frac{1}{2}g^{00}g_{03,1}\chi_0 - \frac{1}{2}g^{03}g_{33,1}\chi_0)(g^{11}\chi^3_{,1} \\ &- \frac{1}{2}g^{11}g^{03}g_{00,1}\chi^0 - \frac{1}{2}g^{11}g^{33}g_{30,1}\chi^0 - \frac{1}{2}g^{11}g^{03}g_{03,1}\chi^3 \\ &- \frac{1}{2}g^{11}g^{33}g_{33,1}\chi^3) - (-\frac{1}{2}g^{00}g_{03,2}\chi_0 - \frac{1}{2}g^{03}g_{33,2}\chi_0)(g^{22}\chi^3_{,2} \\ &+ \frac{1}{2}g^{22}g^{03}g_{00,2}\chi^0 + \frac{1}{2}g^{22}g^{33}g_{03,2}\chi^0 + \frac{1}{2}g^{22}g^{03}g_{03,2}\chi^3 + \frac{1}{2}g^{22}g^{33}g_{33,2}\chi^3)\Bigg|_{r=r_+}\end{aligned} \tag{4.64}$$

and simplifying, we get the surface gravity of the Kerr black hole as

$$\kappa = \frac{(M^2 - a^2)^{\frac{1}{2}}}{2M[M + (M^2 - a^2)^{\frac{1}{2}}]}. \tag{4.65}$$

Notice that for $a = 0$ the expression reduces to the value of the Schwarzschild.

Chapter 5

Conclusion

Black holes were at first only an idea as a result of the calculations by Laplace in 1795, when he talked about the classical body with escape velocity greater than that of light, but his idea did not attract much attention. According to GR, a black hole is a region of spacetime from which nothing, including light, can escape. Around a black hole there is a boundary in spacetime beyond which events cannot affect an outside observer and is called the event horizon. Light emitted beyond the horizon can never reach the observer. The most general solution in four dimensions of the EFEs is the Kerr-Newman solution which contains all parameters i.e. mass, charge and rotation. If we set rotation equal to zero in the Kerr-Newman solution we get the Reissner-Nordström black hole, if we set charge equal to zero, we get the Kerr black hole, and if we set rotation and charge both equal to zero we get the Schwarzschild black hole.

Surface gravity plays an important role in black hole thermodynamics. We have studied the concept of the surface gravity. In literature there are different formulae available to calculate the surface gravity of different black hole spacetimes. However, it is not made explicit under what conditions these formulae can be used. We have also seen generally the Eddington-Finkelstein coordinates are used to find the surface gravity of black hole spacetimes, the reason is not made clear. Therefore, by using killing vectors we have, explicitly derived the formula to calculate the surface gravity for general

(1) spherically symmetric static spacetime,

(2) axially symmetric spacetime.

Surface gravity of the Schwarzschild, the Reissner-Nordström and the Kerr spacetimes are calculated as examples.

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