Terminal Value Problems for Fractional Differential Equations:

Existence, Uniqueness and Stability

by

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Abstract

Fractional differential equations have been studied as by three decades and have discovered various applications in different fields of physical science and engineering. This thesis aims at existence, uniqueness and stability theory of terminal value problems for fractional differential equations. We discuss brief history and some basic facts about fractional calculus. Applications of fractional calculus and some results from functional analysis are also discussed. In this work, we focus on two mainly used approaches i.e. The Riemann-Liouville approach and The Caputo's approach are also discussed.
In this thesis we review well posedness concept for the Caputo's type fractional differential equations. We review impact of perturbed data and stability of change of location of starting point of initial value problem for fractional differential equations. Stability of solutions for terminal value problems is also discussed. We establish new results for existence and uniqueness of terminal problem on the finite interval with the help of the Schauder's fixed-point theorem, the Schaefer's fixed-point theorem and the Banach fixed-point theorem. Some examples are also added for the applicability of our results.
Finally we establish sufficient conditions for existence and uniqueness of solutions of terminal value problems for fractional differential equations on infinite interval. Some illustrative examples are comprehended to demonstrate the proficiency and utility of our results.

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Syed Afaq Hussain Shah

Dedicated to
My Beloved Parents
and Sister

Contents

Chapter 1

Introduction

Fractional calculus is a field of mathematics that exited the conventional definitions of the integral calculus and differential operator in just about the comparable route as fractional exponent is the idea originating from integer order exponents. At the point when one contemplates the physical significance of the exponent, agreeing to general idea exponents are short documentation for rehashed augmentation of a numerical value. This idea in itself is straightforward. On the other hand, this physical definition can get to be confounded when we take non integer value of exponents. While practically anybody can check that $u^5 = u \cdot u \cdot u \cdot u \cdot u$, by what method would we be able to depict the physical significance of $u^{7/9}$ or of the 2^{π} . One can't imagine what it may be like to multiply a number or a value without anyone else's $\frac{7}{9}$ times, or π times, while these expressions have a definite value for any value u, can be computed by infinite series expansion, or all the more effortlessly, by calculators.

Now, in the similar way, consider the integral and derivative of non-integer order. Despite the fact that they are surely concepts of a higher complexity by nature, it is still genuinely simple to physically speak to their importance. For any integer value of n, completing n integrations can turn out to be as orderly as multiplication. Question can be asked that what will happen if one does not take n to be a integer value? In the same way, one can easily experience the differential operators $\frac{d}{dx}$, $\frac{d^2}{dx^2}$, $\frac{d^3}{dx^3}$ etc. be that as it may, but by applying ordinary definition of derivative one can't explain $\frac{d^{1/2}}{dx^{1/2}}$, $\frac{d^{3/2}}{dx^{3/2}}$ $\frac{d^{5/3}}{dx^{5/3}}$, etc. Indeed, these were the inquiries that gave another dimension in the field of mathematics and furthermore occupied the considerations of mathematicians to another idea which leads to generalization of classical calculus, known as fractional calculus.

1.1 Brief history

The discovery of differential calculus is attributed to Isaac Newton $(1642-1727)$ and G.W. Leibniz $(1646-1716)$ who developed the foundations of the subject in the seventieth century. For nth order derivative of a function f, Leibniz introduced notation $\frac{d^n f}{dx^n}$ with the assumption that $n \geq 0$. L'Hospital $(1661-1704)$ posed the question to Leibniz "What would be the result if $n = 1/2$?" Leibniz replied "This is an apparent paradox from which one day useful consequences will be drawn". Most authors on this subject refer to a specific date $30th$, September 1695 as the birthday of alleged fractional calculus.

Taking after L'Hopital's and Liebniz's first investigation, fractional calculus was mainly a study for the best talented minds in mathematics. Fourier $(1768-1830)$, Euler $(1707-1783)$ and Laplace $(1749-1827)$ are among the numerous that dallied with fractional calculus and the mathematical outcomes. Numerous mathematicians utilizing their own particular documentation and system discovered the definitions that gives a justification for the idea of a non-integer order integrals and derivatives.

1.2 The Gamma function

The gamma function was first introduced by Swiss mathematician Leonhard Euler $(1707-1783)$ in his ambition to generalize the factorial to non integer values. Later, as a result of its incredible significance, it was contemplated by other prominent mathematicians like Adrien-Marie Legendre $(1752-1833)$, Carl Friedrich Gauss $(1777-1855)$, Christoph Gudermann $(1798-1852)$, Joseph Liouville $(1809-1882)$, Karl Weierstrass $(1815-1897)$, Charles Hermite $(1822-1901)$ and many others. Factorial of a nonnegative integer n is product of all positive integers less then or equal to n. The factorial function is defined as $n! = n \cdot (n-1) \cdot (n-2) \cdots 3 \cdot 2 \cdot 1$ for all $n \in \mathbb{Z}^+$. In fact, the gamma function is generalization of factorial function. For $x > 0$, the gamma function [1] $\Gamma(x) : (0, \infty) \to \mathbb{R}$ is defined by

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} dt. \tag{1.1}$$

The integral in (1.1) is uniformly convergent for all $x \in [a, b]$ where $0 < a \leq b < \infty$ and hence $\Gamma(x)$ is a continuous function for all $x > 0$. Integrating (1.1) by parts we can easily prove that

$$\Gamma(x+1) = x\Gamma(x), \quad x > 0. \tag{1.2}$$

Which is known as functional equation of the gamma function.

1.3 The Mittag-Leffler function

The exponential function plays an important and significant role in theory of classical calculus. Since as we found that the gamma function is generalization of factorial function, in the same way Mittag-Leffler function is generalization of exponential series. The Mittag-Leffler function was introduced by G.M. Mittag-Leffler [2]. When solving fractional differential equation, the Mittag-Leffler function occurs naturally as their solution.

Definition 1.3.1. *Let $\alpha > 0$. The function E_α*

$$E_\alpha(z) := \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(k\alpha + 1)} \qquad \alpha \in \mathbb{R}, \quad z \in \mathbb{C}. \tag{1.3}$$

is called Mittag-Leffler function of the order α.

Definition 1.3.2. *Let $\alpha, \beta > 0$. The function $E_{\alpha,\beta}$*

$$E_{\alpha,\beta}(z) := \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(k\alpha + \beta)} \qquad \alpha, \beta \in \mathbb{R}, \quad z \in \mathbb{C}. \tag{1.4}$$

is called two-parameter Mittag-Leffler function.

The generalization (1.4) was studied by A. Wiaman in 1905, Agarwal and Humbert in 1953 and others (see [3]). Taking different value of the parameter α, β, some special relations may be established as,

$$\begin{aligned}
E_{1,1}(z) &= \exp(z), \\
E_{1,2}(z) &= \frac{\exp(z) - 1}{z}, \\
E_{\alpha,1}(z) &= E_\alpha(z), \\
E_{2,1}(z^2) &= \cosh(z), \\
E_{2,2}(z^2) &= \frac{\sinh(z)}{z}.
\end{aligned}$$

Theorem 1.3.1. *[4] Consider the two parameter Mittage-Leffler function $E_{\alpha,\beta}$ for some $\alpha, \beta > 0$. The power series defining $E_{\alpha,\beta}(z)$ is convergent for all $z \in \mathbb{C}$. In other words $E_{\alpha,\beta}$ is entire function.*

1.4 Fractional integrals and derivatives

The main phenomenon of classical calculus are derivatives and integrals of functions. If we begin with arbitrary function $f(x)$ and put its integrals on the left-hand side and on the right-hand side we continue with its derivative, we get a both-side infinite sequence,

$$\cdots \int_a^x \int_a^{x_1} f(x_0)dx_0dx_1, \int_a^x f(x_0)dx_0, f(x), \frac{df(x)}{dx}, \frac{d^2f(x)}{dx^2} \cdots .$$

Fractional calculus interpolates this sequence so this operation brings together classical integrals and derivatives and generalizes them for arbitrary order α. In this section we give some definitions and important results for fractional integrals and derivatives.

1.4.1 Riemann-Liouville fractional integrals

Riemann-Liouville approach is only based on well known Cauchy formula (1.5) for the n^{th} integral which uses only a simple integration, so it provides necessary basis for generalization. Repeated integration yields

$$\begin{aligned}
I_a^1 f(x) &= \int_a^x f(x_0)dx_0, \\
I_a^2 f(x) &= \int_a^x \int_a^{x_1} f(x_0)dx_0dx_1, \\
I_a^3 f(x) &= \int_a^x \int_a^{x_2} \int_a^{x_1} f(x_0)dx_0dx_1dx_2, \\
&\vdots \\
I_a^n f(x) &= \int_a^x \int_a^{x_{n-1}} \cdots \int_a^{x_2} \int_a^{x_1} f(x_0)dx_0dx_1dx_2 \cdots dx_n.
\end{aligned}$$

By Fubini's theorem, we may interchange order of integration

$$\begin{aligned}
I_a^2 f(x) &= \int_a^x \int_a^{x_1} f(x_0)dx_0dx_1 \\
&= \int_a^x f(x_0) \left(\int_{x_0}^x dx_1 \right) dx_0 = \int_a^x (x - x_0) f(x_0)dx_0, \\
I_a^3 f(x) &= \int_a^x \int_a^{x_2} \int_a^{x_1} f(x_0)dx_0dx_1dx_2 \\
&= \int_a^x \left(\int_a^{x_2} (x_2 - x_0)dx_0 \right) dx_2 \\
&= \int_a^x \frac{(x - x_0)^{3-1}}{(3-1)!} f(x_0)dx_0.
\end{aligned}$$

In general,

$$I_a^n f(x) = \int_a^x \frac{(x - x_0)^{n-1}}{(n-1)!} f(x_0)dx_0. \qquad \text{(Cauchy formula)} \tag{1.5}$$

Definition 1.4.1. *[4, 5] Let $\alpha > 0$. The operator $I_{a^+}^{\alpha}$ defined on $L_1[a, b]$ by*

$$I_{a^+}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-s)^{\alpha-1} f(s)\, ds, \quad x > a \tag{1.6}$$

is called left sided Riemann-Liouville fractional integral operator of order α.

Definition 1.4.2. *[4, 5] Let $\alpha > 0$. The operator $I_{b^-}^{\alpha}$, defined on $L_1[a, b]$ by*

$$I_{b^-}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (s-x)^{\alpha-1} f(s)\, ds, \quad x < b \tag{1.7}$$

is called right sided Riemann-Liouville fractional integral.

Properties of Riemann-Liouville fractional integral
Riemann-Liouville fractional integral operator has the following properties: [4, 5]

(*i*) For $\alpha = 0$, upon setting $I_{a^+}^0 := I$ and $I_{b^-}^0 := I$, the identity operator.

(*ii*) Semigroup Property: Let $\alpha, \beta \geq 0$ and $f \in L_1[a, b]$, then

$$I_{a^+}^{\alpha} I_{a^+}^{\beta} f = I_{a^+}^{\beta} I_{a^+}^{\alpha} f = I_{a^+}^{\alpha+\beta} f$$

holds almost everywhere on $[a, b]$. If additionally, $f \in C[a, b]$ or $\alpha + \beta \geq 1$, then the identity holds everywhere on $[a, b]$. Similar result holds for right sided Riemann-Liouville fractional integral operator.

(*iii*) Linearity: Let f and g be two functions defined on $[a, b]$, such that $I_{a^+}^{\alpha} f$ and $I_{a^+}^{\alpha} g$ exists almost everywhere. Then fractional Riemann-Liouville integrals are linear almost everywhere i.e. for $a, b \in \mathbb{R}$

$$I_{a^+}^{\alpha}(af(x) + bg(x)) = aI_{a^+}^{\alpha} f(x) + bI_{a^+}^{\alpha} g(x).$$

Similar result holds for right sided Riemann-Liouville fractional integral operator.

Example 1.4.1. *Let $f(x) = (x-a)^{\beta}$ for some $\beta > -1$ and $\alpha > 0$. Then,*

$$I_{a^+}^{\alpha}(x-a)^{\beta} = \frac{\Gamma(\beta+1)}{\Gamma(\beta+1+\alpha)}(x-a)^{\beta+\alpha}.$$

In particular

$$I_{0^+}^{1/2} x^{3/2} = \frac{3\sqrt{\pi}}{8} x^2.$$

1.4.2 Riemann-Liouville fractional derivatives

Let f be continuous on $[a,b]$ and define $F:[a,b]\to\mathbb{R}$ as

$$F(x)=\int_a^x f(s)ds$$

then, F is differentiable and

$$\frac{d}{dx}F(x)=f(x),$$

or

$$f(x)=\frac{d}{dx}\left(\int_a^x f(s)ds\right),$$

or

$$f=DI_{a^+}f, \tag{1.8}$$

where $D=\frac{d}{dx}$. Repeated application of (1.8) gives

$$D^2I_{a^+}^2f=D(DI_a^+)I_{a^+}f=D((I_{a^+}f))=DI_{a^+}f=f$$

$$f=D^nI_{a^+}^nf, \quad \text{where} \quad D^n=\frac{d^n}{dt^n}, \qquad n=0,1,2,\cdots. \tag{1.9}$$

Replacing n by $m-n$, with $n<m$ and applying D^n on both sides, we have

$$\begin{aligned}D^nf&=D^nD^{m-n}I_{a^+}^{m-n}f\\&=D^mI_{a^+}^{m-n}f.\end{aligned}$$

This relation is still valid for some class of functions,if n is replaced by $\alpha\in\mathbb{R}^+$, provided $m-\alpha>0$. That is

$$D_{a^+}^{\alpha}f=D^mI_{a^+}^{m-\alpha}f.$$

Thus, the Riemann-Liouville fractional differential operator is equivalent to the composition of the same operator $m-\alpha$-fold integration and m^{th} order differentiation but in reverse order.

Definition 1.4.3. *Let $\alpha>0, x>a$ $m=\lceil\alpha\rceil$ and $\alpha\in\mathbb{R}^+$, then*

$$D_{a^+}^{\alpha}f(x):=\frac{1}{\Gamma(m-\alpha)}\frac{d^m}{dx^m}\int_a^x(x-s)^{m-\alpha-1}f(s)\,ds, \quad x>a \tag{1.10}$$

is called left sided Riemann-Liouville fractional differential operator of order α.

Definition 1.4.4. *[5] Let $\alpha > 0$ and $m = \lceil \alpha \rceil$. The operator $D^{\alpha}_{b^-}$*

$$\begin{aligned} D^{\alpha}_{b^-} f(x) &:= D^m I^{m-\alpha}_{b^-} f(x) \\ &= \frac{1}{\Gamma(m-\alpha)} \left(-\frac{d}{dx}\right)^m \int_x^b (s-x)^{m-\alpha-1} f(s)\, ds, \quad x < b \end{aligned} \tag{1.11}$$

is called right sided Riemann-Liouville fractional derivative of order α.

Properties of left sided Riemann-Liouville fractional derivative
Left sided Riemann-Liouville fractional differential operator has the following properties.

(i) For $\alpha = 0$, upon setting $D^0_{a^+} := I$, the identity operator.

(ii) Let $\alpha, \beta > 0$ and $\beta > \alpha$ then,

$$D^{\alpha}_{a^+} I^{\beta}_{a^+} f = I^{\beta-\alpha}_{a^+} f.$$

In particular,

$$D^{\alpha}_{a^+} I^{\alpha}_{a^+} f = f.$$

(iii) Riemann-Liouville fractional derivative is non-zero for constant function $f(x) = c$ if $\alpha \notin \mathbb{N}$ i.e.

$$D^{\alpha}_{0^+} c = \frac{x^{-\alpha}}{\Gamma(1-\alpha)} c \neq 0,$$

where c is arbitrary constant.

(iv) Linearity: Let f and g be two functions defined on $[a, b]$, such that $D^{\alpha}_{a^+} f$ and $D^{\alpha}_{a^+} g$ exists almost everywhere. Moreover, let $a, b \in \mathbb{R}$. Then $D^{\alpha}_{a^+}(af(x) + bg(x))$ exists everywhere, and

$$D^{\alpha}_{a^+}(af(x) + bg(x)) = aD^{\alpha}_{a^+} f(x) + bD^{\alpha}_{a^+} g(x).$$

Similar properties hold for right sided Riemann-Liouville fractional differential operator.

Example 1.4.2. *Let $f(x) = (x-a)^{\beta}$ for some $\beta > -1$ and $\alpha > 0$. Then,*

$$D^{\alpha}_{a^+}(x-a)^{\beta} = \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)} (x-a)^{\beta-\alpha}.$$

In particular,

$$D^{1/2}_{0^+} x^{5/2} = \frac{15\sqrt{\pi}}{16} x^2.$$

Theorem 1.4.1. *[4] Let $\alpha > 0$ and $m = \lceil \alpha \rceil$. Assume that f is such that $I_a^{m-\alpha} f \in A^m[a,b]$. Then,*

$$I_{a^+}^{\alpha} D_{a^+}^{\alpha} f(x) = f(x) - \sum_{k=0}^{m-1} \frac{(x-a)^{\alpha-k-1}}{\Gamma(\alpha-k)} \lim_{\tau \to a^+} D^{m-k-1} I_{a^+}^{m-\alpha} f(\tau).$$

Specifically, for $0 < \alpha < 1$ we have,

$$I_{a^+}^{\alpha} D_{a^+}^{\alpha} f(x) = f(x) - \frac{(x-a)^{\alpha-1}}{\Gamma(\alpha)} \lim_{\tau \to a^+} I_{a^+}^{1-\alpha} f(\tau).$$

Theorem 1.4.2. *[4] Let $\alpha, \beta \geq 0$, $\phi \in L_1[a,b]$ and $f := I_{a^+}^{\alpha+\beta} \phi$. Then*

$$D_{a^+}^{\alpha} D_{a^+}^{\beta} f = D_{a^+}^{\alpha+\beta} f. \tag{1.12}$$

Proof.

$$\begin{aligned} D_{a^+}^{\alpha+\beta} f &= D_{a^+}^{\alpha+\beta} I_{a^+}^{\alpha+\beta} \phi \\ &= \phi. \end{aligned}$$

By using property $D_{a^+}^{\alpha} I_{a^+}^{\alpha} f = f$ and by definition of Riemann-Liouville definition

$$\begin{aligned} D_{a^+}^{\alpha} D_{a^+}^{\beta} f &= D_{a^+}^{\alpha} D_{a^+}^{\beta} I_{a^+}^{\alpha+\beta} \phi \\ &= D_{a^+}^{\lceil \alpha \rceil} I_{a^+}^{\lceil \alpha \rceil - \alpha} D_{a^+}^{\lceil \beta \rceil} \cdot I_{a^+}^{\lceil \beta \rceil - \beta} I_{a^+}^{\alpha+\beta} \phi. \end{aligned}$$

By semigroup property,

$$D_{a^+}^{\alpha} D_{a^+}^{\beta} f = D_{a^+}^{\lceil \alpha \rceil} I_{a^+}^{\lceil \alpha \rceil - \alpha} D_{a^+}^{\lceil \beta \rceil} I_{a^+}^{\lceil \beta \rceil + \alpha} \phi.$$

Using property $D_{a^+}^{\alpha} I_{a^+}^{\beta} f = I_{a^+}^{\beta-\alpha} f$, we get

$$\begin{aligned} D_{a^+}^{\alpha} D_{a^+}^{\beta} f &= D_{a^+}^{\lceil \alpha \rceil} I_{a^+}^{\lceil \alpha \rceil - \alpha} I_{a^+}^{\alpha} \phi \\ &= D_{a^+}^{\lceil \alpha \rceil} I_{a^+}^{\lceil \alpha \rceil} \phi \\ &= \phi. \end{aligned}$$

Hence (1.12) holds. □

For example, let $f(x) = \frac{x^3}{3}$ and $\alpha = \beta = 1/2$. Then $D_{0^+}^{1/2} f(x) = \frac{\Gamma(4)}{3\Gamma(7/2)} x^{5/2}$ and $D_{0^+}^{1/2} D_{0^+}^{1/2} f(x) = x^2$. Also $D_{0^+}^{1/2+1/2} f(x) = Df(x) = x^2$.
It is possible to have

$$D_{a^+}^{\alpha} D_{a^+}^{\beta} f = D_{a^+}^{\beta} D_{a^+}^{\alpha} f \neq D_{a^+}^{\alpha+\beta} f,$$

and

$$D^{\alpha}_{a^+}D^{\beta}_{a^+}f \neq D^{\beta}_{a^+}D^{\alpha}_{a^+}f = D^{\alpha+\beta}_{a^+}f.$$

For example, let $f(x) = x^{-3/2}$ and $\alpha = \beta = 3/2$. Then $D^{\alpha}_{0^+}f(x) = D^{\beta}_{0^+}f(x) = 0$. So this implies $D^{\alpha}_{0^+}D^{\beta}_{0^+}f(x) = D^{\beta}_{0^+}D^{\alpha}_{0^+}f(x) = 0$, but $D^{\alpha+\beta}_{0^+}f(x) = -\frac{105}{8}x^{-9/2}$.
If we consider $f(x) = x^{3/2}$ and let $\alpha = 3/2, \quad \beta = 5/2$, then $D^{\alpha}_{0^+}f(x) = \frac{3\sqrt{\pi}}{4}$ and $D^{\beta}_{0^+}f(x) = 0$. So this implies $D^{\alpha}_{0^+}D^{\beta}_{0^+}f(x) = 0$, $D^{\alpha+\beta}_{0^+}f(x) = \frac{9}{16}x^{-5/2}$, but also $D^{\beta}_{0^+}D^{\alpha}_{0^+}f(x) = \frac{9}{16}x^{-5/2} = D^{\alpha+\beta}_{0^+}f(x)$.

Theorem 1.4.3. *[4] Let $\alpha > 0$ and $m = \lceil \alpha \rceil$. Assume that f is such that $I^{m-\alpha}_{b^-}f \in A^m[a,b]$. Then,*

$$I^{\alpha}_{b^-}D^{\alpha}_{b^-}f(x) = f(x) - \sum_{k=0}^{m-1} \frac{(-1)^{\alpha-k-1}(b-x)^{\alpha-k-1}}{\Gamma(\alpha-k)} \lim_{\tau \to b^-} D^{m-k-1}I^{m-\alpha}_{b^-}f(\tau).$$

1.4.3 The Caputo fractional derivatives

The definition of Riemann-Liouville fractional differential operator have played significant role in advancement of theory of fractional differentiation and integration and for its applications in mathematics. It turns out that the Riemann-Liouville fractional differential operator have certain complications when one try to model a real-world phenomena with fractional differential equations. We might thusly now talk about modified approach of a fractional derivatives i.e. The Caputo derivative. M. Caputo proposed the new approach in 1967 in his paper [6] and then after two years in his book [7]. Comparing both approaches, the second one is much easier and suitable for such tasks.

Definition 1.4.5. *Let $\alpha > 0, x > a$ and $m = \lceil \alpha \rceil$, then*

$$D^{\alpha}_{*a^+}f(x) := I^{m-\alpha}_{a^+}D^{\alpha}f(x) = \begin{cases} \frac{1}{\Gamma(m-\alpha)}\int_a^x (x-s)^{m-\alpha-1}f^{(m)}(s)\,ds, & \text{if } m-1 < \alpha < m, \\ \frac{d^m}{dx^m}f(x) & \text{if } \alpha = m, \end{cases}$$

is called left sided Caputo fractional differential operator of order α.

Definition 1.4.6. *Let $\alpha > 0$ and $m = \lceil \alpha \rceil$. The fractional derivative $D^{\alpha}_{*b^-}$ of order α*

$$D^{\alpha}_{*b^-}f(x) := \begin{cases} \frac{(-1)^m}{\Gamma(m-\alpha)}\int_x^b (s-x)^{m-\alpha-1}f^{(m)}(s)\,ds, & \text{if } m-1 < \alpha < m, \\ \frac{d^m}{dx^m}f(x) & \text{if } \alpha = m, \end{cases} \tag{1.13}$$

is called right sided Caputo's fractional differential operator of order α.

Properties of left sided Caputo's fractional derivatives
The left sided Caputo fractional differential operator has the following properties.

(i) Linearity: Let f and g be two functions defined on $[a, b]$, such that $D^{\alpha}_{*a^+}f$ and $D^{\alpha}_{*a^+}g$ exists almost everywhere. Moreover, let $a, b \in \mathbb{R}$. Then $D^{\alpha}_{*a^+}(af(x) + bg(x))$ exists almost everywhere, and

$$D^{\alpha}_{*a^+}(af(x) + bg(x)) = aD^{\alpha}_{*a^+}f(x) + bD^{\alpha}_{*a^+}g(x).$$

(ii) Let $\alpha > 0$ and $m = \lceil \alpha \rceil$ if $\alpha \notin \mathbb{N}$ and let $m = \alpha$ if $\alpha \in \mathbb{N}$. If $f(x) \in C^m[a, b]$, then

$$I^{\alpha}_{a^+}D^{\alpha}_{*a^+}f(x) = f(x) - \sum_{j=0}^{m-1} \frac{f^{(j)}(a)}{j!}(x-a)^j.$$

In particular, if $0 < \alpha \leq 1$ and $f(x) \in C[a, b]$, then

$$I^{\alpha}_{a^+}D^{\alpha}_{*a^+}f(x) = f(x) - f(a).$$

Similar properties hold for right sided Caputo differential operator.

Example 1.4.3. *Let $f(x) = (x-a)^{\beta}$. Then the right sided Caputo's derivative for some $\beta \geq 0$,*

$$D^{\alpha}_{*a^+}f(x) = \begin{cases} 0 & \text{if } \beta \in \{0, 1, 2 \cdots, m-1\}, \\ \frac{\Gamma(\beta+1)}{\beta+1-\alpha}(x-a)^{\beta-\alpha} & \text{if } \beta \in \mathbb{N} \text{ and } \beta \geq m, \\ & \text{or } \beta \notin \mathbb{N} \text{ and } \beta > m-1. \end{cases}$$

Definition 1.4.7. *The Taylor's polynomial of the n^{th} order centered at h is defined as*

$$T_n[f(x); h] = f(h) + (x-h)f'(h) + \frac{(x-h)^2 f''(h)}{2!} + \cdots + \frac{(x-h)^n f^{(n)}(h)}{n!}.$$

Theorem 1.4.4. *[4] Let $\alpha \geq 0$ and $m = \lceil \alpha \rceil$. Moreover, suppose that $f \in A^m[a, b]$. Then,*

$$D^{\alpha}_{*a^+}f = D^{\alpha}_{a^+}[f - T_{m-1}[f; a]] \tag{1.14}$$

almost everywhere. Here, $T_{m-1}[f; a]$ denotes the Taylor's polynomial of the $(m-1)^{th}$ order centered at a, in the case if $m = 0$, then $T_{m-1}[f; a] := 0$.

Note that if $D^{\alpha}_{a^+}f$ exists and f possess $m-1$ times derivative at a then the right hand side of (1.14) exists. Theorem 1.4.4 gives relation between the Riemann-Liouville fractional derivative and the Caputo fractional derivatives.
Also note that if $m = \alpha$, then

$$D^{\alpha}_{*a^+}f = D^{\alpha}_{a^+}[f - T_{\alpha-1}[f; a]] = D^{\alpha}_{a^+}f - D^{\alpha}_{a^+}(T_{\alpha-1}[f; a]) = D^{\alpha}_{a^+}f,$$

because $T_{\alpha-1}[f; a]$ is Taylor's polynomial of degree $\alpha - 1$ that is obliterated by classical operator $D^{\alpha}_{a^+}$.

Theorem 1.4.5. *If f is continuous and $\alpha \geq 0$, then*

$$D^{\alpha}_{*a^+} I^{\alpha}_{a^+} f = f.$$

Theorem 1.4.6. *Assume that $\alpha \geq 0$, $m = \lceil \alpha \rceil$ and $f \in A^m[a, b]$. Then*

$$I^{\alpha}_{a^+} D^{\alpha}_{*a^+} f(x) = f(x) - \sum_{k=0}^{m-1} \frac{f^{(k)}(a)}{k!}(x-a)^k.$$

1.4.4 Relation between Riemann-Liouville and Caputo's fractional derivatives

The Riemann-Liouville derivative of the constant function is $D^{\alpha}_{a^+} c = \frac{(x-a)^{-\alpha}}{\Gamma(1-\alpha)} c \neq 0$, while Caputo derivative of the constant function $D^{\alpha}_{*a^+} c = 0$, where c is arbitrary constant. Let $\alpha \geq 0$ and $m = \lceil \alpha \rceil$.
Assume that f is such that both $D^{\alpha}_{*a^+} f$ and $D^{\alpha}_{a^+} f$ exists. Then the Riemann-Liouville and the Caputo's fractional derivatives coincides $D^{\alpha}_{*a^+} f = D^{\alpha}_{a^+} f$, if and only if f has an m-fold zero at a i.e. if and only if $D^k f(a) = 0$ for $k = 0, 1, 2, \ldots, m-1$.
Relation between the left sided Caputo derivatives and left sided Riemann-Liouville derivatives for a differential functions is defined as, let $\alpha \geq 0$ and $m = \lceil \alpha \rceil$. Assume that f is such that both $D^{\alpha}_{*a^+} f$ and $D^{\alpha}_{a^+} f$ exists. Then,

$$D^{\alpha}_{*a^+} f(x) = D^{\alpha}_{a^+} f(x) - \sum_{k=0}^{m-1} \frac{f^{(k)}(a)}{\Gamma(k-\alpha+1)}(x-a)^{k-\alpha}. \tag{1.15}$$

Relation between the right sided Caputo derivatives and right sided Riemann-Liouville derivatives for a differential functions is defined as, let $\alpha \geq 0$ and $m = \lceil \alpha \rceil$. Assume that f is such that both $D^{\alpha}_{*b^-} f$ and $D^{\alpha}_{b^-} f$ exists. Then,

$$D^{\alpha}_{*b^-} f(x) = D^{\alpha}_{b^-} f(x) - \sum_{k=0}^{m-1} \frac{f^{(k)}(b)}{\Gamma(k-\alpha+1)}(b-x)^{k-\alpha}. \tag{1.16}$$

1.5 Applications of fractional calculus

In this section we describe the applications of fractional calculus. The first application of fractional calculus was made by Niels Henrik Abel $(1802 - 1829)$ when he was dealing with the formulation of tautochronous problem to solve the integral equation see [8]. Later on, during last decades of nineteenth century, without any strict mathematical argument, Oliver Heaviside $(1850-1925)$ developed operational calculus in 1892, in which he introduced the fundamental and basic idea of fractional

derivatives in his work in electric transmission lines [8].
Recently many of the scientists and researchers focused their attention to fractional calculus due to its wider and numerous applications in viscoelasticity and damping, diffusion and wave propagation, electromagnetism, chaos and fractals, heat transfer, biology, electronics, signal processing, robotics, system identification, traffic system, genetic algorithm, porous media, modeling and identification, telecommunication, electro-chemistry, physics, control system, rheology, economy and finance, edge detection in image processing and in fluid mechanics (see [9–18]).

1.6 Some results from analysis

In this section, some basic definitions and familiar results from [19] are to be given here which will be used throughout in this thesis to establish the existence and uniqueness of solutions of terminal value problems for fractional differential equations on finite and infinite interval respectively.

Definition 1.6.1. *A function $f : [a, b] \to \mathbb{R}$ is absolutely continuous on $[a, b]$ if for $\epsilon > 0$ there exists a $\delta > 0$ such that,*

$$\sum_{k=1}^{N} |f(b_k) - f(a_k)| < \epsilon$$

for any finite collection $\{[a_k, b_k]; 1 \leq k \leq N\}$ of non-overlapping sub-interval $[a_k, b_k]$ of $[a, b]$ with

$$\sum_{k=1}^{N} |b_k - a_k| < \delta.$$

Definition 1.6.2. *A set $\mathcal{S} \subset C[a, b]$ is said to be equicontinuous if and only if for each $\epsilon > 0$, there exist a $\delta > 0$ such that, $|f(x_1) - f(x_2)| < \epsilon$ for all $x_1, x_2 \in [a, b]$, whenever $|x_1 - x_2| < \delta$ for all $f \in \mathcal{S}$.*

Definition 1.6.3. *A complete normed vector space is called Banach space.*

Definition 1.6.4. *Let $\mathcal{S}$ be a closed subset of the Banach space $\mathbb{X}$. A mapping $\mathcal{A} : \mathcal{S} \to \mathcal{S}$ is called contraction, if there exists a non-negative real number $q < 1$, such that for any $x_1, x_2 \in \mathcal{S}$*

$$\|\mathcal{A}(x_1) - \mathcal{A}(x_2)\| \leq q\|x_1 - x_2\|.$$

Definition 1.6.5. *Let $\mathcal{S}$ be a closed subset of the Banach space $\mathbb{X}$. $\mathcal{A}$ is said to satisfy Lipschitz condition on $\mathcal{S}$ with Lipschitz constant L, iff there is a $L < \infty$, such that for any $x, y \in \mathcal{S}$*

$$\|\mathcal{A}(x) - \mathcal{A}(y)\| \leq L\|x - y\|.$$

Definition 1.6.6. *A point $\bar{f}$ is said to be a fixed point of the operator $\mathcal{A}$ iff $\bar{f} = \mathcal{A}\bar{f}$.*

Theorem 1.6.1. *(Banach Fixed Point Theorem) Let $\mathcal{S}$ be a closed subset of the Banach space $\mathbb{X}$ and $\mathcal{A} : \mathcal{S} \to \mathcal{S}$ is a contraction. Then $\mathcal{A}$ has a unique fixed point in $\mathcal{S}$.*

Definition 1.6.7. *A class of sets in a Banach space $\mathbb{X}$ is said to cover a given set $\mathcal{S}$ if and only if each point of $\mathcal{S}$ lies at least one of sets.*

Definition 1.6.8. *Let $\mathcal{S}$ be a subset of the Banach space $\mathbb{X}$. $\mathcal{S}$ is said to be compact if and only if every class of open sets which covers $\mathcal{S}$ has a finite subclass which also cover $\mathcal{S}$.*

Definition 1.6.9. *Suppose that $\mathcal{S}$ is a subset of the Banach space $\mathbb{X}$. $\mathcal{S}$ is relatively compact if and only if its closure $\overline{\mathcal{S}}$ is compact.*

Definition 1.6.10. *Suppose that $\mathcal{S}$ is a subset of the Banach space $\mathbb{X}$. An operator $\mathcal{A} : \mathcal{S} \to \mathbb{X}$ is compact if and only if it is continuous and it maps every bounded subset of $\mathcal{S}$ into a relatively compact set.*

Theorem 1.6.2. *(Arzel-Ascoli Theorem) Let Ω be a bounded subset of $\mathbb{R}^n$, and let $\mathcal{M}$ be a subset of $C(\Omega)$. Then $\mathcal{M}$ is relatively compact if and only if it is bounded and equicontinuous.*

Theorem 1.6.3. *(Schauder's Fixed Point Theorem) Let $\mathcal{S}$ be a non-empty closed bounded convex subset of the Banach space $\mathbb{X}$, and suppose that $\mathcal{A} : \mathcal{S} \to \mathbb{X}$ is compact and maps $\mathcal{S}$ into itself. Then the operator $\mathcal{A}$ has a fixed point in $\mathcal{S}$.*

Theorem 1.6.4. *(Schaefer's Fixed Point Theorem) Let $\mathbb{X}$ be a Banach space and the operator $\mathcal{F} : \mathbb{X} \to \mathbb{X}$ is continuous and compact. Assume further the set $\{x \in \mathbb{X};\ \ x = \lambda\mathcal{F}(x) \ \text{ for some } \ \lambda \in [0, 1]\}$ is bounded. Then $\mathcal{F}$ has a fixed point.*

Chapter 2

Well-posedness concept for fractional differential equations of Caputo's type

It is surely understood that, under standard suppositions, initial value problem for fractional differential equations involving Caputo derivative is well-posed if a unique solution exists and this solution continuously depends on the given data i.e. the source function, the initial values and order of the derivative [20, 21]. This chapter concerns with well posedness of initial value problem of fractional differential equation. We begin with initial value problem,

$$\begin{aligned} D^{\alpha}_{*a^+}u(x) &= f(x, u(x)), \\ u^k(a) &= u_k, \quad (k = 0, 1, ..., \lceil\alpha\rceil - 1) \end{aligned} \tag{2.1}$$

for $\alpha > 0$ on interval $[a, b]$, where $D^{\alpha}_{*a^+}$ represents the left sided Caputo differential operator of order α [4].
Section 2.1 deals with uniqueness of solution of (2.1). Section 2.2 concerns with perturbation in given data i.e. small changes in any of the given data yield small changes in the solution. In section 2.3, the continuity of the solution to the initial value problem (2.1) with respect to location of initial point is to be discussed. Examples are included to demonstrate the significance of the results.

2.1 Uniqueness of solution

In this section, there holds the following existence and uniqueness result for this initial value problem (2.1) [21, 22].

Lemma 2.1.1. *[4] Let $\alpha \geq 0$ and $m = \lceil \alpha \rceil$. Suppose that $u_0, u_1, u_2 \dots u_{m-1} \in \mathbb{R}, K > 0$ and $h^* > 0$. Define*

$$G := \{(x, u) : x \in [0, h^*], |u - \sum_{k=0}^{\lceil \alpha \rceil - 1} u_k x^k / k!| \leq K\}$$

and let the function $f : G \to \mathbb{R}$ is continuous. Furthermore, define $M := \sup_{(x,u)\in G} |f(x, u)|$ and

$$h := \begin{cases} h^* & \text{if } M = 0, \\ \min\{h^*, (K\Gamma(\alpha + 1)/M)^{1/\alpha}\} & \text{else.} \end{cases}$$

Then the function $u \in C[0, h]$ is a solution of

$$\begin{aligned} D^{\alpha}_{*0^+} u(x) &= f(x, u(x)), \\ D^k u(0) &= u_k, \quad (k = 0, 1, ..., m - 1), \end{aligned} \tag{2.2}$$

if and only if it is solution of the Volterra integral equation

$$u(x) = \sum_{k=0}^{m-1} \frac{u_k x^k}{k!} + \frac{1}{\Gamma(\alpha)} \int_0^x (x - s)^{\alpha - 1} f(s, u(s))\, ds. \tag{2.3}$$

Proof. Let u be the solution of (2.3), then applying $D^{\alpha}_{*0^+}$ on both sides of (2.3)

$$\begin{aligned} D^{\alpha}_{*0^+} u(x) &= \sum_{k=0}^{m-1} \frac{u_k D^{\alpha}_{*0^+} x^k}{k!} + D^{\alpha}_{*0^+} I^{\alpha}_{0^+} f(x, u(x)), \\ &= \sum_{k=0}^{m-1} \frac{u_k x^{k-\alpha}}{\Gamma(k + 1 - \alpha)} + f(x, u(x)). \end{aligned}$$

Since the gamma function in summands becomes negative for all value of k, therefore the summands vanishes and we get $D^{\alpha}_{*0^+} u(x) = f(x, u(x))$. Now for initial condition, applying D^k_0 on both sides of (2.3)

$$\begin{aligned} D^k_0 u(x) &= \sum_{j=0}^{m-1} \frac{u_j D^k_0 x^j}{j!} + D^k_0 I^{\alpha}_{0^+} f(x, u(x), \\ &= \sum_{j=0}^{m-1} \frac{u_j x^{j-k}}{\Gamma(j - k + 1)} + I^{\alpha - k}_{0^+} f(x, u(x). \end{aligned}$$

The summands vanishes identically for $k > j$. Moreover for $k < j$ the summands vanishes if $x = 0$. For $k = j$

$$D_0^k u(x) = \sum_{k=0}^{m-1} u_k + I_{0+}^{\alpha-k} f(x, u(x)),$$
$$D_0^k u(0) = \sum_{k=0}^{m-1} u_k + I_{0+}^{\alpha-k} f(0, u(0)),$$
$$D_0^k u(0) = \sum_{k=0}^{m-1} u_k.$$

Hence u solves given initial value problem (3.1).
Conversely, define $v(x) := f(x, u(x))$ and note that $v \in C[0, h]$ by our assumption on f and u. Using definition of the Caputo differential operator, differential equation can be rewritten as

$$\begin{aligned} v(x) = f(x, u(x)) &= D_{*0+}^{\alpha} u(x) = D_{0+}^{\alpha}(u - T_{m-1}[u; 0])(x), \\ &= D_{0+}^{m} I_{0+}^{m-\alpha}(u - T_{m-1}[u; 0])(x). \end{aligned}$$

Applying I_0^m on both sides of the above equation we get

$$I_0^m v(x) = I_{0+}^{m-\alpha}(u - T_{m-1}[u; 0])(x) + q(x)$$

with some polynomial q of degree not exceeding $m - 1$. Since v is continuous, the function $I_0^m v$ on left hand side of equation has a zero of order m at origin. Moreover the difference $u - T_{m-1}[u; 0]$ has the same property by construction, and therefore $I_{0+}^{m-\alpha}(u - T_{m-1}[u; 0])(x)$ on the right hand side of eqution must have such an m^{th} order zero too and the degree of the polynomial q is not more than $m-1$, we deduce that $q = 0$. Conseqently,

$$I_0^m v(x) = I_{0+}^{m-\alpha}(u - T_{m-1}[u; 0])(x).$$

Applying left sided Riemann-Liouville differential operator $D_{0+}^{m-\alpha}$ to this equation,

$$\begin{aligned} u(x) - T_{m-1}[u; 0](x) &= D_{0+}^{m-\alpha} I_0^m v(x) = D^1 I_0^{1+\alpha-m} I_0^m v(x) = D I_0^{1+\alpha} v(x) \\ &= I_0^{\alpha} v(x). \end{aligned}$$

Therefore

$$\begin{aligned} u(x) &= T_{m-1}[u; 0](x) + I_0^{\alpha} v(x), \\ &= \sum_{k=0}^{m-1} \frac{u_k x^k}{k!} + \frac{1}{\Gamma(\alpha)} \int_0^x (x - s)^{\alpha-1} f(s, u(s))\, ds, \end{aligned}$$

which completes the proof. □

Theorem 2.1.1. *[23] Assume the function $f : [a, b] \times \mathbb{R} \to \mathbb{R}$ be continuous and bounded, and assume that it satisfies the Lipschitz condition and $\frac{L}{\Gamma(\alpha+1)}(b-a)^\alpha < 1$ holds. Then, the initial value problem (2.1) has a unique solution on $[a, b]$.*

Proof. Assume the set

$$\widetilde{V} := \{u \in C[a, b] : \|u(x) - \mathcal{T}(x)\| < C_1\}$$

where $C_1 := \frac{M}{\Gamma(\alpha+1)}(b-a)^\alpha$ and $C_2 := \frac{L}{\Gamma(\alpha+1)}(b-a)^\alpha$ and introduce the polynomial

$$\mathcal{T}(x) = \sum_{k=0}^{\lceil\alpha\rceil-1} \frac{(x-a)^k}{k!} u_k.$$

Let us define the norm $\|u\| := sup_{x\in[a,b]}|u(x)|$ and $\mathbb{X} = \{u(x) : u \in C[a, b]\}$, then $(\mathbb{X}, \|.\|)$ is Banach space. We define an operator $\mathcal{B}$ on $\mathbb{X}$ as

$$(\mathcal{B}u)(x) := \mathcal{T}(x) + \frac{1}{\Gamma(\alpha)} \int_a^x (x-s)^{\alpha-1} f(s, u(s))\, ds. \tag{2.4}$$

First we show that $\mathcal{B} : \widetilde{V} \to \widetilde{V}$.

$$\begin{aligned}
\|(\mathcal{B}u)(x) - \mathcal{T}(x)\| &= \sup_{x\in[a,b]} |(\mathcal{B}u)(x) - \mathcal{T}(x)| \\
&= \frac{1}{\Gamma(\alpha)} \sup_{x\in[a,b]} \Big| \int_a^x (x-s)^{\alpha-1} f(s, u(s))\, ds \Big| \\
&\leq \frac{M}{\Gamma(\alpha+1)} \sup_{x\in[a,b]} (x-a)^\alpha \\
&\leq \frac{M}{\Gamma(\alpha+1)} (b-a)^\alpha = C_1.
\end{aligned}$$

Hence, operator $\mathcal{B}$ maps set $\widetilde{V}$ to itself. Now we show that $\mathcal{B}$ is a contraction mapping on $\mathbb{X}$. For any u , $v \in \mathbb{X}$

$$\begin{aligned}
\|\mathcal{B}u(x) - \mathcal{B}v(x)\| &= \sup_{x\in[a,b]} |\mathcal{B}u(x) - \mathcal{B}v(x)| \\
&= \sup_{x\in[a,b]} \frac{1}{\Gamma(\alpha)} \Big| \int_a^x (x-s)^{\alpha-1} [f(s, u(s)) - f(s, v(s))]\, ds \Big| \\
&\leq \frac{L}{\Gamma(\alpha)} \sup_{x\in[a,b]} \int_a^x (x-s)^{\alpha-1} [|u-v|]\, ds \\
&\leq \frac{L\|u-v\|}{\Gamma(\alpha)} \int_a^x \sup_{x\in[a,b]} (x-s)^{\alpha-1}\, ds \\
&\leq \frac{L\|u-v\|}{\Gamma(\alpha+1)} (b-a)^\alpha \\
\|\mathcal{B}u(x) - \mathcal{B}v(x)\| &\leq C_2 \|u-v\|.
\end{aligned}$$

Hence, $\mathcal{B}$ is contraction mapping. Thus by the Banach fixed point theorem, a unique fixed point exist of operator $\mathcal{B}$. Consequently, (2.1) has a unique solution. $\square$

Example 2.1.1. *Consider an initial value problem,*

$$\begin{aligned} D^{\alpha}_{*1^-} u &= \frac{u \exp(-x)}{(105 + \exp(-x))(1 + |u|)}, \quad 0 < \alpha \leq 1, \quad x \in [0,1], \\ u(0) &= u_0, \quad u_0, \in \mathbb{R}. \end{aligned} \tag{2.5}$$

Here $\alpha = 1/2$ and

$$f(x,u) := \frac{u \exp(-x)}{(105 + \exp(-x))(1 + |u|)}.$$

It is obvious that function $f(x,u)$ is continuous. Now

$$\begin{aligned} \Big| f(x,u_1) - f(x,u_2) \Big| &= \Big| \frac{u_1 \exp(-x)}{(105 + \exp(-x))(1 + |u_1|)} - \frac{u_2 \exp(-x)}{(105 + \exp(-x))(1 + |u_2|)} \Big| \\ &\leq \frac{\exp(-x)}{(105 + \exp(-x))} \Big| \frac{u_1}{1 + |u_1|} - \frac{u_2}{1 + |u_2|} \Big| \\ &\leq \frac{1}{105} \cdot \frac{|u_1 - u_2|}{(|1 + |u_1|)(|1 + |u_2|)}. \end{aligned}$$

Since $\frac{1}{(1+|u_1|)(1+|u_2|)} < 1$, therefore

$$\Big| f(x,u_1) - f(x,u_2) \Big| < \frac{1}{105} |u_1 - u_2|.$$

Hence $f(x,u,u')$ satisfies Lipschitz condition. Also, clearly $\frac{L}{\Gamma(\alpha+1)}(b-a)^{\alpha} = \frac{1/105}{\Gamma(5/2)} = 0.007164312 << 1$. Therefore, by Theorem 2.1.1, there exists a unique solution on [0,1].

2.2 Impact of perturbed data

In this section, we will discuss the stability of solutions of the problem (2.1) with respect to initial values, the source function and the order of derivative respectively but first we state a useful lemma.

Lemma 2.2.1. *[4] Assume that $\zeta : [a,b] \to \mathbb{R}$ is a continuous function which satisfies the inequality,*

$$|\zeta(x)| \leq \eta_1 + \frac{\eta_2}{\Gamma(\alpha)} \int_a^x (x-s)^{\alpha-1} |\zeta(s)| \, ds,$$

for all $x \in [a,b]$. Then

$$|\zeta(x)| \leq \eta_1 E_{\alpha}(\eta_2 (x-a)^{\alpha})$$

for $x \in [a,b]$, where E_{α} denotes Mittag-Leffler function of order α.

Theorem 2.2.1. *[4] Let u be solution of* (2.1)*, and let $\tilde{u}$ be the solution of the initial value problem*

$$\begin{aligned} D^{\alpha}_{*a^+}\tilde{u}(x) &= f(x, \tilde{u}(x)), \\ \tilde{u}^k(a) &= \tilde{u}_k, \quad (k = 0, 1, ..., \lceil\alpha\rceil - 1), \end{aligned} \tag{2.6}$$

where f is assumed to satisfy the hypothesis of Theorem 2.1.1. Moreover let $\varepsilon := \max_{k=0,1,...,\lceil\alpha\rceil-1} |u_k - \tilde{u}_k|$. Then, if ε is sufficiently small, there exists some $b > a$ such that the functions u and $\tilde{u}$ are defined on [a,b], and

$$\sup_{a\le x\le b} |u(x) - \tilde{u}(x)| = O\Big(\max_{k=0,1,...\lceil\alpha\rceil-1} |u_k - \tilde{u}_k|\Big).$$

Proof. Let us define $\zeta(x) := u(x) - \tilde{u}(x)$ and observe that ζ is solution of initial value problem

$$\begin{aligned} D^{\alpha}_{*a^+}\zeta(x) &= f(x, u(x)) - f(x, \tilde{u}(x)), \\ \zeta^k(a) &= u_k - \tilde{u}_k, \quad (k = 0, 1, ..., \lceil\alpha\rceil - 1). \end{aligned} \tag{2.7}$$

From Lemma 2.1.1, initial value problem (2.7) is equivalent to

$$\begin{aligned} \zeta(x) &= \sum_{k=0}^{\lceil\alpha\rceil-1} \frac{(x-a)^k}{k!}\Big(u_k - \tilde{u}_k\Big) + \frac{1}{\Gamma(\alpha)}\int_a^x (x-s)^{\alpha-1}(f(s, u(s)) - f(s, \tilde{u(s)}))\,ds, \\ &\le \Big|\sum_{k=0}^{\lceil\alpha\rceil-1} \frac{(x-a)^k}{k!}\Big(u_k - \tilde{u}_k\Big)\Big| + \frac{1}{\Gamma(\alpha)}\Big|\int_a^x (x-s)^{\alpha-1}(f(s, u(s)) - f(s, \tilde{u(s)}))\,ds\Big|. \end{aligned}$$

Using Holder's inequality for sums and Lipschitz condition on the function f,

$$|\zeta(x)| \le \varepsilon \sum_{k=0}^{\lceil\alpha\rceil-1} \frac{(b-a)^k}{k!} + \frac{1}{\Gamma(\alpha)}\int_a^x (x-s)^{\alpha-1}|\zeta(s)|\,ds$$

where L is Lipschitz constant of the function f. Thus, by Lemma 2.2.1

$$|\zeta(x)| \le O(\varepsilon)E_\alpha(L(b-a)^\alpha) = O(\varepsilon).$$

□

Now we look at the impact of changes in given function on right hand side of the differential equation.

Theorem 2.2.2. *[4] Let u be solution of* (2.1)*, and let $\tilde{u}$ be the solution of the initial value problem*

$$\begin{aligned} D^{\alpha}_{*a^+}\tilde{u}(x) &= \tilde{f}(x,\tilde{u}(x)), \\ \tilde{u}^k(a) &= u_k, \quad (k = 0,1,...,\lceil\alpha\rceil - 1), \end{aligned} \tag{2.8}$$

where f and $\tilde{f}$ is assumed to satisfy the hypothesis of Theorem 2.1.1. Let

$$\varepsilon := \max_{(x_1,x_2)\in\widetilde{V}} |f(x,u(x)) - \tilde{f}(x,\tilde{u}(x))|.$$

Then, if ε is sufficiently small, there exists some $b > a$ such that the functions u and $\tilde{u}$ are defined on [a,b], and

$$\sup_{a\le x\le b} |u(x) - \tilde{u}(x)| = O\Big(\max_{(x_1,x_2)\in\widetilde{V}} |f(x,u(x)) - \tilde{f}(x,\tilde{u}(x))|\Big).$$

Proof. To prove the required inequality, we moreover proceed in a practically identical way. Once again define $\zeta(x) := u(x) - \tilde{u}(x)$ and observe that ζ is solution of initial value problem

$$\begin{aligned} D^{\alpha}_{*a^+}\zeta(x) &= f(x,u(x)) - \tilde{f}(x,\tilde{u}(x)), \\ \zeta^k(a) &= 0, \quad (k = 0,1,...,\lceil\alpha\rceil - 1). \end{aligned} \tag{2.9}$$

From Lemma 2.1.1, the initial value problem (2.9) is equivalent to the following integral equation

$$\zeta(x) = \frac{1}{\Gamma(\alpha)}\int_a^x (x-s)^{\alpha-1}(f(s,u(s)) - \tilde{f}(s,\tilde{u(s)}))\,ds.$$

Since f and $\tilde{f}$ satisfies Lipschitz condition, so taking absolute values on both sides, we find

$$\begin{aligned} |\zeta(x)| &= \frac{1}{\Gamma(\alpha)}\Big|\int_a^x (x-s)^{\alpha-1}(f(s,u(s)) - \tilde{f}(s,\tilde{u(s)}))\,ds\Big|, \\ &\le \frac{1}{\Gamma(\alpha)}\int_a^x (x-s)^{\alpha-1}[|(f(s,u(s)) - f(s,\tilde{u}(s))| + |f(s,\tilde{u}(s)) - \tilde{f}(s,\tilde{u(s)}))|]\,ds, \\ &\le \frac{L}{\Gamma(\alpha)}\int_a^x (x-s)^{\alpha-1}|\zeta(s)|\,ds + \frac{1}{\Gamma(\alpha)}\int_a^x (x-s)^{\alpha-1}|f(x,\tilde{u}(x)) - \tilde{f}(s,\tilde{u(s)})|\,ds, \\ &\le \frac{L}{\Gamma(\alpha)}\int_a^x (x-s)^{\alpha-1}|\zeta(s)|\,ds + \varepsilon\frac{1}{\Gamma(\alpha)}\int_a^x (x-s)^{\alpha-1}\,ds, \\ &\le \frac{L}{\Gamma(\alpha)}\int_a^x (x-s)^{\alpha-1}|\zeta(s)|\,ds + \varepsilon\frac{(b-a)^\alpha}{\Gamma(\alpha+1)}. \end{aligned}$$

Therefore, by Lemma 2.2.1,

$$|\zeta(x)| \leq O(\varepsilon)E_\alpha(L(b-a)^\alpha) = O(\varepsilon).$$

$\square$

Now we shall discuss about the stability of (2.1) with respect to order.

Theorem 2.2.3. *[4] Let u be solution of (2.1), and let $\tilde{u}$ be the solution of the initial value problem*

$$\begin{aligned} D^{\tilde{\alpha}}_{*a^+}\tilde{u}(x) &= f(x,\tilde{u}(x)), \\ \tilde{u}^k(a) &= u_k, \quad (k=0,1,...,\lceil\tilde{\alpha}\rceil-1), \end{aligned} \tag{2.10}$$

where f is assumed to satisfy the hypothesis of Theorem 2.1.1. Moreover let $\varepsilon_1 := \tilde{\alpha}-\alpha$ and

$$\varepsilon_2 = \begin{cases} 0 & \text{if } \lceil\alpha\rceil = \lceil\tilde{\alpha}\rceil, \\ \max\left\{|u_k| : \lceil\alpha\rceil \leq k \leq \lceil\tilde{\alpha}\rceil - 1\right\} & \text{else.} \end{cases}$$

Then, if ε_1 and ε_2 is sufficiently small, there exists some $b > a$ such that the functions u and $\tilde{u}$ are defined on [a,b], and we have that

$$\sup_{a\leq x\leq b}|u(x)-\tilde{u}(x)| = O(\alpha-\tilde{\alpha}) + O\Big(\max\left\{0, \max\left\{|u_k| : \lceil\alpha\rceil \leq k \leq \lceil\tilde{\alpha}\rceil-1\right\}\right\}\Big).$$

Proof. To prove the required inequality, we moreover proceed in a practically similar way. Once again define

$$\begin{aligned} \zeta(x) &:= u(x) - \tilde{u}(x) \\ &= \sum_{k=0}^{\lceil\alpha\rceil-1}\frac{(x-a)^k}{k!}u_k + \frac{1}{\Gamma(\alpha)}\int_a^x (x-s)^{\alpha-1}f(s,u(s))\,ds - \sum_{k=0}^{\lceil\tilde{\alpha}\rceil-1}\frac{(x-a)^k}{k!}u_k \\ &\quad + \frac{1}{\Gamma(\tilde{\alpha})}\int_a^x (x-s)^{\tilde{\alpha}-1}f(s,\tilde{u}(s))\,ds, \\ &= -\sum_{k=\lceil\alpha\rceil}^{\lceil\tilde{\alpha}\rceil-1}\frac{(x-a)^k}{k!}u_k + \frac{1}{\Gamma(\alpha)}\int_a^x (x-s)^{\alpha-1}(f(s,u(s))-f(s,\tilde{u}(s)))\,ds \\ &\quad + \int_a^x \Big(\frac{(x-s)^{\alpha-1}}{\Gamma(\alpha)} - \frac{(x-s)^{\tilde{\alpha}-1}}{\Gamma(\tilde{\alpha})}\Big)f(s,\tilde{u}(s))\,ds. \end{aligned}$$

Since f satisfies Lipschitz condition, this implies

$$\begin{aligned} |\zeta(x)| \leq& \sum_{k=\lceil\alpha\rceil}^{\lceil\tilde{\alpha}\rceil-1}\frac{(b-a)^k}{k!}|u_k| + \max_{(x_1,x_2)\in\tilde{V}}|f(x_1,x_2)|\int_a^x\Big|\frac{(x-s)^{\alpha-1}}{\Gamma(\alpha)} - \frac{(x-s)^{\tilde{\alpha}-1}}{\Gamma(\tilde{\alpha})}\Big|\,ds \\ &+ \frac{L}{\Gamma(\alpha)}\int_a^x (x-s)^{\alpha-1}|\zeta(s)|\,ds. \end{aligned} \tag{2.11}$$

Moreover, we can bound first integral according to

$$\int_a^x \Big|\frac{(x-s)^{\alpha-1}}{\Gamma(\alpha)} - \frac{(x-s)^{\tilde{\alpha}-1}}{\Gamma(\tilde{\alpha})}\Big|\, ds \le \int_a^b \Big|\frac{(x-s)^{\alpha-1}}{\Gamma(\alpha)} - \frac{(x-s)^{\tilde{\alpha}-1}}{\Gamma(\tilde{\alpha})}\Big|\, ds.$$

The zero of integrand can be found easily, as

$$\frac{(x-s)^{\tilde{\alpha}-1}}{(x-s)^{\alpha-1}} = \frac{\Gamma(\tilde{\alpha})}{\Gamma(\alpha)}$$
$$(x-s) = \Big(\frac{\Gamma(\tilde{\alpha})}{\Gamma(\alpha)}\Big)^{\frac{1}{(\tilde{\alpha}-\alpha)}}.$$

Thus, the zero of integrand is at $\mathcal{C} := \Big(\frac{\Gamma(\tilde{\alpha})}{\Gamma(\alpha)}\Big)^{\frac{1}{(\tilde{\alpha}-\alpha)}}$. If b is smaller than $\mathcal{C}$, then there is no change of sign in integrand, and absolute value operation can be moved outside of the integral. Else we proceed as follows,

$$\begin{aligned}
&\int_a^{\mathcal{C}} \Big|\frac{(x-s)^{\alpha-1}}{\Gamma(\alpha)} - \frac{(x-s)^{\tilde{\alpha}-1}}{\Gamma(\tilde{\alpha})}\Big|\, ds + \int_{\mathcal{C}}^b \Big|\frac{(x-s)^{\alpha-1}}{\Gamma(\alpha)} - \frac{(x-s)^{\tilde{\alpha}-1}}{\Gamma(\tilde{\alpha})}\Big|\, ds \\
&= \Big|\frac{(x-\mathcal{C})^{\alpha}}{\Gamma(\alpha+1)} - \frac{(x-a)^{\alpha}}{\Gamma(\alpha+1)} - \frac{(x-\mathcal{C})^{\tilde{\alpha}}}{\Gamma(\tilde{\alpha}+1)} + \frac{(x-a)^{\tilde{\alpha}}}{\Gamma(\tilde{\alpha}+1)}\Big| + \Big|\frac{(x-b)^{\alpha}}{\Gamma(\alpha+1)} - \frac{(x-\mathcal{C})^{\alpha}}{\Gamma(\alpha+1)} \\
&\quad - \frac{(x-b)^{\tilde{\alpha}}}{\Gamma(\tilde{\alpha}+1)} + \frac{(x-\mathcal{C})^{\tilde{\alpha}}}{\Gamma(\tilde{\alpha}+1)}\Big|.
\end{aligned}$$

Using Mean Value Theorem, we get

$$\begin{aligned}
&\int_a^{\mathcal{C}} \Big|\frac{(x-s)^{\alpha-1}}{\Gamma(\alpha)} - \frac{(x-s)^{\tilde{\alpha}-1}}{\Gamma(\tilde{\alpha})}\Big|\, ds + \int_{\mathcal{C}}^b \Big|\frac{(x-s)^{\alpha-1}}{\Gamma(\alpha)} - \frac{(x-s)^{\tilde{\alpha}-1}}{\Gamma(\tilde{\alpha})}\Big|\, ds \\
&\le \Big|\frac{\alpha\xi^{\alpha-1}(a-\mathcal{C})}{\Gamma(\alpha+1)} - \frac{\tilde{\alpha}\xi^{\tilde{\alpha}-1}(a-\mathcal{C})}{\Gamma(\tilde{\alpha}+1)}\Big| + \Big|\frac{\alpha\xi^{\alpha-1}(\mathcal{C}-b)}{\Gamma(\alpha+1)} - \frac{\tilde{\alpha}\xi^{\tilde{\alpha}-1}(\mathcal{C}-b)}{\Gamma(\tilde{\alpha}+1)}\Big| \\
&\le \Big|\frac{(\tilde{\alpha}\xi^{\tilde{\alpha}-1} - \alpha\xi^{\alpha-1})(a-\mathcal{C})}{\Gamma(\alpha+1)}\Big| + \Big|\frac{(\tilde{\alpha}\xi^{\tilde{\alpha}-1} - \alpha\xi^{\alpha-1})(\mathcal{C}-b)}{\Gamma(\alpha+1)}\Big| \\
&\le \Big|\frac{(\tilde{\alpha}-\alpha)\xi^{\tilde{\alpha}-1}(a-\mathcal{C})}{\Gamma(\alpha+1)}\Big| + \Big|\frac{(\tilde{\alpha}-\alpha)\xi^{\tilde{\alpha}-1}(\mathcal{C}-b)}{\Gamma(\alpha+1)}\Big| \\
&\le \frac{(\tilde{\alpha}-\alpha)\xi^{\tilde{\alpha}-1}(a-b)}{\Gamma(\alpha+1)} = O(\varepsilon_1).
\end{aligned}$$

Thus (2.11) becomes,

$$|\zeta(x)| \le O(\varepsilon_1) + O(\varepsilon_2) + \frac{L}{\Gamma(\alpha)} \int_a^x (x-s)^{\alpha-1} |\zeta(s)|\, ds.$$

Therefore by Lemma 2.2.1, find that

$$|\zeta(x)| \le (O(\varepsilon_1) + O(\varepsilon_2)) E_{\alpha}(L(b-a)^{\alpha}).$$

□

2.3 Stability of the change of location of starting point

In this section, we broaden well-posedness idea to the degree that we moreover permit the location of starting point of the differential operator to be changed, furthermore, we demonstrate that the solution depends upon this parameter in a continuous as well on the off chance that the standard assumptions are fulfilled. The fundamental consequence of the chapter is the accompanying statement that permits us to presume that the solution to the initial value problem (2.1) is for sure continuous as for location of starting point.

Theorem 2.3.1. *[23] Let $a \leq \tilde{a} < b$, and consider the initial value problem* (2.1) *and*

$$\begin{aligned} D^{\alpha}_{*\tilde{a}^+}\tilde{u}(x) &= \tilde{f}(x, \tilde{u}(x)), \\ \tilde{u}^k(\tilde{a}) &= \tilde{u}_k, \quad (k = 0, 1, ..., \lceil\alpha\rceil - 1), \end{aligned} \tag{2.12}$$

under the assumption of Theorem 2.1.1. The solutions u and $\tilde{u}$ to these initial value problems satisfy the relation

$$\sup_{x\in[\tilde{a},b]} |u(x) - \tilde{u}(x)| = O\Big(|a - \tilde{a}|^{\min\{\alpha,1\}}\Big). \tag{2.13}$$

Proof. Since from Lemma 2.1.1 the initial value problem (2.1) is equivalent to the Volterra integral equation

$$u(x) := \sum_{k=0}^{\lceil\alpha\rceil-1} \frac{(x-a)^k}{k!}u_k + \frac{1}{\Gamma(\alpha)}\int_a^x (x-s)^{\alpha-1} f(s, u(s))\, ds, \tag{2.14}$$

for $x \in [a, b]$ and (2.12) is equivalent to

$$\tilde{u}(x) := \sum_{k=0}^{\lceil\alpha\rceil-1} \frac{(x-\tilde{a})^k}{k!}u_k + \frac{1}{\Gamma(\alpha)}\int_{\tilde{a}}^x (x-s)^{\alpha-1} f(s, \tilde{u(s)})\, ds, \tag{2.15}$$

for $x \in [\tilde{a}, b] \subseteq [a, b]$. Therefore, subtracting (2.15) from (2.14), we obtain for $x \in [\tilde{a}, b]$

$$\zeta(x) := |u(x) - \tilde{u}(x)| \leq |\mathcal{D}_1| + |\mathcal{D}_2| + |\mathcal{D}_3| \tag{2.16}$$

where

$$\mathcal{D}_1 = \sum_{k=1}^{\lceil\alpha\rceil-1} \frac{u_k}{k!}[(x-a)^k - (x-\tilde{a})^k], \tag{2.17}$$

$$\mathcal{D}_2 = \frac{1}{\Gamma(\alpha)} \int_a^{\tilde{a}} (x-s)^{\alpha-1} f(s,u(s))\, ds, \tag{2.18}$$

$$\mathcal{D}_3 = \frac{1}{\Gamma(\alpha)} \int_{\tilde{a}}^{x} (x-s)^{\alpha-1} [f(s,u(s)) - f(s,\tilde{u(s)})]\, ds. \tag{2.19}$$

Since f satisfies Lipschitz condition, therefore

$$|\mathcal{D}_3| \leq \frac{1}{\Gamma(\alpha)} \int_{\tilde{a}}^{x} (x-s)^{\alpha-1} |f(s,u(s)) - f(s,\tilde{u(s)})|\, ds$$

$$|\mathcal{D}_3| \leq \frac{L}{\Gamma(\alpha)} \int_{\tilde{a}}^{x} (x-s)^{\alpha-1} |u(x) - \tilde{u}(x)|\, ds$$

$$|\mathcal{D}_3| \leq \frac{L}{\Gamma(\alpha)} \int_{\tilde{a}}^{x} (x-s)^{\alpha-1} \zeta(s)\, ds. \tag{2.20}$$

We now need to recognize two cases. First we will look on case $0 < \alpha \leq 1$. Clearly $\mathcal{D}_1 = 0$ and

$$\begin{aligned}
|\mathcal{D}_2| &\leq \frac{1}{\Gamma(\alpha)} \int_a^{\tilde{a}} (x-s)^{\alpha-1} |f(s,u(s))|\, ds \\
&\leq \frac{M}{\Gamma(\alpha)} \int_a^{\tilde{a}} (x-s)^{\alpha-1}\, ds \\
&\leq \frac{M}{\Gamma(\alpha)} \int_a^{\tilde{a}} (\tilde{a}-s)^{\alpha-1}\, ds = \frac{M}{\Gamma(\alpha+1)} (\tilde{a}-a)^{\alpha}.
\end{aligned}$$

If $\alpha > 1$, using Mean Value Theorem in (2.17)

$$|\mathcal{D}_1| \leq |\tilde{a} - a| \sum_{k=1}^{\lceil\alpha\rceil-1} \frac{u_k}{k-1!} (x-\xi_k)^{k-1},$$

where $\xi_k \in [a, \tilde{a}]$. Thus

$$|\mathcal{D}_1| \leq |\tilde{a} - a| \sum_{k=1}^{\lceil\alpha\rceil-1} \frac{u_k}{k-1!} (b-a)^{k-1} = \mathcal{F}|\tilde{a} - a|.$$

Where $\mathcal{F}$ is constant independent of $a - \tilde{a}$ and x. Also in this case

$$|\mathcal{D}_2| \leq \frac{1}{\Gamma(\alpha)} \int_a^{\tilde{a}} (x-s)^{\alpha-1} |f(s,u(s))|\, ds \leq \frac{M}{\Gamma(\alpha)} \int_a^{\tilde{a}} (x-s)^{\alpha-1}\, ds$$
$$\leq \frac{M}{\Gamma(\alpha)} \int_a^{\tilde{a}} (b-a)^{\alpha-1}\, ds = \frac{M}{\Gamma(\alpha)} (b-a)^{\alpha-1}(\tilde{a}-a).$$

Follows from (2.16) and (2.20) that

$$\zeta(x) \leq |\mathcal{D}_1| + |\mathcal{D}_2| + \frac{L}{\Gamma(\alpha)} \int_{\tilde{a}}^{x} (x-s)^{\alpha-1} \zeta(s)\, ds. \tag{2.21}$$

In either case, our appraisals above infer that

$$|\mathcal{D}_1| + |\mathcal{D}_2| \leq \mathcal{F}|\tilde{a} - a| + \frac{M}{\Gamma(\alpha)}(b-a)^{\alpha-1}(\tilde{a}-a)$$
$$|\mathcal{D}_1| + |\mathcal{D}_2| \leq \mathcal{F}^*|\tilde{a} - a|^q \qquad \text{with } q := \min\{1, \alpha\},$$

where $\mathcal{F}^*$ is an absolute constant. By Lemma 2.2.1, we find that

$$\zeta(x) \leq \mathcal{F}^*|\tilde{a} - a|^q E_\alpha(L(x-\tilde{a})^\alpha) \leq \mathcal{F}^*|\tilde{a} - a|^q E_\alpha(L(b-a)^\alpha).$$

□

We point out attention to that estimate of equation (2.13) is best conceivable.

Theorem 2.3.2. *[23]Under the assumption of Theorem 2.1.1, the estimate of equation* (2.13) *can't be improved. i.e. the exponent* $\min\{1, \alpha\}$ *in the* $O-$*term on right hand side of* (2.13) *cannot be replaced by a larger number.*

To prove this statement, it suffices to develop examples where the order showed in this $O-$term is really achieved. We might do this independently for two cases $0 < \alpha \leq 1$ and $\alpha > 1$.

Example 2.3.1. *[23] Consider initial value problem with starting point* $a = 0$ *for the case* $0 < \alpha \leq 1$.

$$D^{\alpha}_{*0^+} u(x) = u(x), \qquad u(0) = 1, \tag{2.22}$$

and

$$D^{\alpha}_{*\tilde{a}^+} \tilde{u}(x) = \tilde{u}(x), \qquad \tilde{u}(\tilde{a}) = 1. \tag{2.23}$$

Whose solutions can easily be found that,

$$u(x) = E_\alpha(x^\alpha) \quad \text{and} \quad \tilde{u}(x) = E_\alpha(x-\tilde{a})^\alpha,$$

respectively. Where $E_\alpha(v)$ denotes the one-parameter Mittag-Leffler function. For arbitrary $b > \tilde{a}$

$$\sup_{x\in[\tilde{a},b]} |u(x) - \tilde{u}(x)| \geq u(\tilde{a}) - \tilde{u}(\tilde{a}) = E_\alpha(\tilde{a}^\alpha) - E_\alpha(\tilde{a}-\tilde{a})^\alpha$$
$$\geq E_\alpha(\tilde{a}^\alpha) - 1 = \sum_{k=1}^{\infty} \frac{\tilde{a}^{\alpha k}}{\Gamma(\alpha k+1)}$$
$$\sup_{x\in[\tilde{a},b]} |u(x) - \tilde{u}(x)| \geq \frac{\tilde{a}^\alpha}{\Gamma(\alpha+1)} = \frac{|a-\tilde{a}|^\alpha}{\Gamma(\alpha+1)}.$$

Since $\alpha = \min\{1, \alpha\}$, therefore gives the required results.

Example 2.3.2. *[23]*

For the case $\alpha > 1$, we use the same initial value problems as in Example 2.3.1 above and choose an arbitrary $b > \tilde{a}$.

$$\sup_{x\in[\tilde{a},b]} |u(x) - \tilde{u}(x)| \geq u(b) - \tilde{u}(b) = E_\alpha(b^\alpha) - E_\alpha(b-\tilde{a})^\alpha$$
$$= \sum_{k=1}^{\infty} \frac{b^{\alpha k} - (b-\tilde{a})^{\alpha k}}{\Gamma(\alpha k+1)} \geq \frac{b^\alpha - (b-\tilde{a})^\alpha}{\Gamma(\alpha+1)}.$$

Using Mean Value Theorem, we get

$$\sup_{x\in[\tilde{a},b]} |u(x) - \tilde{u}(x)| = \frac{\xi^{\alpha-1}}{\Gamma(\alpha)}\tilde{a} \geq \frac{(b-\tilde{a})^{\alpha-1}}{\Gamma(\alpha)}|a-\tilde{a}|$$
$$\geq \frac{(b-\tilde{a})^{\alpha-1}}{\Gamma(\alpha)}|a-\tilde{a}|.$$

Since in this case, $1 = \min\{1, \alpha\}$, therefore it gives the required result.

Chapter 3

Well-posedness concept for terminal value problems of fractional differential equations of Caputo's type

3.1 Terminal value problems

Terminal value problems emerges naturally in the simulation of techniques that are watched (i.e. measured) at a later point, eventually after the methodology has started. Existence theory for classical terminal value problems have been investigated by several researchers [24–26]. In particular K. Diethelm [23, 27] established existence, uniqueness and stability results for fractional terminal value problems on finite interval and it has been observed that initial value problems and terminal value problems have in general different solution structure. To the best of our knowledge, one does know the estimations of the quantities of interest at $x = 0$. We begin with following terminal value problem

$$D_{*a}^{\alpha} u(x) = f(x, u(x)), \quad u^k(b) = u_k, \quad (k = 0, 1, ..., \lceil \alpha \rceil - 1), \tag{3.1}$$

where $b > a$. The inquiry for the well posedness of such issues in the traditional sense, i.e. in the event that initial value problem are differed, has been tended to [20]. In the context of the problem under thought here, it is then extremely characteristic to consider an as an extra obscure and to ask under which conditions it is conceivable to distinguish some suitable extra information given which one can close that it is conceivable to uniquely focus both the solution to the terminal value problem furthermore, the starting point a. Another point of enthusiasm for such an association would be to discover how the solution responds to a small change of the

value b.
Inquiries like these emerges rather actually if one tries to model a sensation watched outside of a research facility by a fractional differential equations and does not know when the procedure has begun i.e. earthquakes [6, 28, 29] have been modeled using fractional differential equations where the exact value of starting point is unknown but have a good approximation. Spreading of diseases like dengue fever can be modeled by fractional differential equations [30]. Distribution of pollutants in ground water can be modeled by fractional differential equations see [31, 32]. Everyone is mostly interested in finding out when pollutants mixed in ground water, usually a very rough idea of starting point have to assume for modeling such problem. The main purpose of the chapter is to exhibit that a small change in this value additionally just leads to a small change in the solution, not just in the neighborhood of the starting point yet all through the complete interval $[a, b]$ where the solution exists. In subsection 3.1.1 the existence and uniqueness result is discussed. Subsection 3.1.2 deals with stability of terminal value problems for fractional differential equations.

3.1.1 Existence and uniqueness of solution for terminal value problems

The theory for the existence and uniqueness of solutions to such problems i.e. (3.1) under the standard assumptions have been addressed in [33, 34].

Theorem 3.1.1. *[33, 34] Let $f : [a, b] \times [c, d] \to \mathbb{R}$ be continuous and satisfy a Lipschitz condition with respect to the second variable, then there exists a unique solution of* (3.1) *on $[a, b]$.*

3.1.2 Stability of terminal value problems

We have already seen that the stability of solution of initial value problem of fractional differential equation of order α in chapter 2. This section deals with the stability of solution of terminal value problem of fractional differential equation of order α. We might first express the result for the case that b varies.

Lemma 3.1.1. *[4] Let $0 < \alpha < 1$ and assume $f : [0, b] \times [c, d] \to \mathbb{R}$ to be continuous and satisfy a Lipschitz condition with respect to second variable and let $0 < x^* \leq b$. Then the fractional differential equation*

$$D^{\alpha}_{*0^+} u(x) = f(x, u(x)), \qquad u(x^*) = u^*, \tag{3.2}$$

is equivalent to the weakly singular integral equation

$$u(x) = u^* + \frac{1}{\Gamma(\alpha)} \int_0^{x^*} G(x, s) f(s, u(s)) \, ds,$$

where

$$G(x,s) = \begin{cases} -(x^* - s)^{\alpha-1} & \text{for } s > x, \\ (x-s)^{\alpha-1} - (x^* - s)^{\alpha-1} & \text{for } s \le x. \end{cases}$$

Proof. Let us assume that fractional differential equation (3.2) has a solution $u \in C[0, x^*]$. Then applying integral operator I_0^α to both sides

$$I_0^\alpha D_{*0+}^\alpha u(x) = I_0^\alpha f(x, u(x)),$$
$$u(x) - u(0) = I_0^\alpha f(x, u(x)),$$

$$u(x) = u(0) + \frac{1}{\Gamma(\alpha)} \int_0^x (x-s)^{\alpha-1} f(s, u(s))\, ds \tag{3.3}$$

for all $x \in [0, x^*]$. Upon setting $x = x^*$ and using the condition $u(x^*) = u^*$, we get

$$u^* = u(x^*) = u(0) + \frac{1}{\Gamma(\alpha)} \int_0^{x^*} (x^* - s)^{\alpha-1} f(s, u(s))\, ds \tag{3.4}$$

which implies

$$u(0) = u^* - \frac{1}{\Gamma(\alpha)} \int_0^{x^*} (x^* - s)^{\alpha-1} f(s, u(s))\, ds.$$

Substituting this relation in (3.3), we obtain

$$\begin{aligned}
u(x) &= u^* + \frac{1}{\Gamma(\alpha)} \int_0^x (x-s)^{\alpha-1} f(s,u(s))\, ds - \frac{1}{\Gamma(\alpha)} \int_0^{x^*} (x^*-s)^{\alpha-1} f(s,u(s))\, ds, \\
&= u^* + \int_0^s \frac{(x-s)^{\alpha-1}}{\Gamma(\alpha)} f(s,u(s))\, ds + \int_s^x \frac{(x-s)^{\alpha-1}}{\Gamma(\alpha)} f(s,u(s))\, ds \\
&\quad - \int_0^s \frac{(x^*-s)^{\alpha-1}}{\Gamma(\alpha)} f(s,u(s))\, ds - \int_s^{x^*} \frac{(x^*-s)^{\alpha-1}}{\Gamma(\alpha)} f(s,u(s))\, ds, \\
&= u^* + \int_0^s \frac{(x-s)^{\alpha-1} - (x^*-s)^{\alpha-1}}{\Gamma(\alpha)} f(s,u(s))\, ds + \int_s^x \frac{(x-s)^{\alpha-1}}{\Gamma(\alpha)} f(s,u(s))\, ds \\
&\quad - \int_s^x \frac{(x^*-s)^{\alpha-1}}{\Gamma(\alpha)} f(s,u(s))\, ds - \int_x^{x^*} \frac{(x^*-s)^{\alpha-1}}{\Gamma(\alpha)} f(s,u(s))\, ds, \\
&= u^* + \int_0^s \frac{(x-s)^{\alpha-1} - (x^*-s)^{\alpha-1}}{\Gamma(\alpha)} f(s,u(s))\, ds \\
&\quad + \int_s^x \frac{(x-s)^{\alpha-1} - (x^*-s)^{\alpha-1}}{\Gamma(\alpha)} f(s,u(s))\, ds - \int_x^{x^*} \frac{(x^*-s)^{\alpha-1}}{\Gamma(\alpha)} f(s,u(s))\, ds, \\
&= u^* + \frac{1}{\Gamma(\alpha)} \int_0^x [(x-s)^{\alpha-1} - (x^*-s)^{\alpha-1}] f(s,u(s))\, ds \\
&\quad - \frac{1}{\Gamma(\alpha)} \int_x^{x^*} (x^*-s)^{\alpha-1} f(s,u(s))\, ds.
\end{aligned}$$

Thus it can be written as

$$u(x) = u^* + \frac{1}{\Gamma(\alpha)} \int_0^{x^*} G(x,s) f(s, u(s))\, ds,$$

where

$$G(x,s) = \begin{cases} -(x^* - s)^{\alpha-1} & \text{for } \; s > x, \\ (x-s)^{\alpha-1} - (x^* - s)^{\alpha-1} & \text{for } \; s \le x, \end{cases}$$

which completes the proof. □

Theorem 3.1.2. *[23] Consider the terminal value problems of fractional order* α

$$D^{\alpha}_{*a^+} u(x) = f(x, u(x)), \qquad u(b) = u^*, \tag{3.5}$$

and

$$D^{\alpha}_{*a^+} \tilde{u}(x) = f(x, \tilde{u}(x)), \qquad \tilde{u}(\tilde{b}) = u^*, \tag{3.6}$$

for some $\alpha \in (0,1)$ *and* $a < b \le \tilde{b}$, *where the function* f *is assumed to satisfy the hypothesis of Theorem 2.1.1. The solutions* u *and* $\tilde{u}$ *satisfy the relation,*

$$\sup_{x \in [a,b]} |u(x) - \tilde{u}(x)| = O(|b - \tilde{b}|^{\alpha}). \tag{3.7}$$

Proof. Define $\zeta(x) := u(x) - \tilde{u}(x)$ and by Lemma 3.1.1 we can write (3.5) and (3.6) as their equivalent integral equations,

$$u(x) = u^* + \frac{1}{\Gamma(\alpha)} \int_a^b G(x,s) f(s, u(s))\, ds \tag{3.8}$$

and

$$\tilde{u}(x) = u^* + \frac{1}{\Gamma(\alpha)} \int_a^{\tilde{b}} \tilde{G}(x,s) f(s, \tilde{u}(s))\, ds, \tag{3.9}$$

respectively, where

$$G(x,s) = \begin{cases} -(b - s)^{\alpha-1} & \text{for } \; s > x, \\ (x-s)^{\alpha-1} - (b - s)^{\alpha-1} & \text{for } \; s \le x, \end{cases}$$

and

$$\tilde{G}(x,s) = \begin{cases} -(\tilde{b} - s)^{\alpha-1} & \text{for } \; s > x, \\ (x-s)^{\alpha-1} - (\tilde{b} - s)^{\alpha-1} & \text{for } \; s \le x. \end{cases}$$

Subtracting (3.8) and (3.9),

$$\begin{aligned} u(x) - \tilde{u}(x) = {} & \frac{1}{\Gamma(\alpha)} \int_a^b [G(x,s) f(s, u(s)) - \tilde{G}(x,s) f(s, \tilde{u}(s))]\, ds \\ & - \frac{1}{\Gamma(\alpha)} \int_b^{\tilde{b}} \tilde{G}(x,s) f(s, \tilde{u}(s))\, ds \end{aligned} \tag{3.10}$$

for $a \le x \le b$. Taking absolute values on (3.10),

$$\begin{aligned} |u(x) - \tilde{u}(x)| \le \frac{1}{\Gamma(\alpha)} \Big| \int_a^b [G(x,s)f(s,u(s)) - \tilde{G}(x,s)f(s,\tilde{u}(s))]\, ds \Big| \\ + \frac{1}{\Gamma(\alpha)} \Big| \int_b^{\tilde{b}} \tilde{G}(x,s)f(s,\tilde{u}(s))\, ds \Big|. \end{aligned} \tag{3.11}$$

Adding and subtracting the factor "$G(x,s)f(s,\tilde{u}(s))$" in the first integral of (3.11),

$$\begin{aligned} \int_a^b [G(x,s)f(s,u(s)) - \tilde{G}(x,s)f(s,\tilde{u}(s))]\, ds = \int_a^b G(x,s)[f(s,u(s)) - f(s,\tilde{u}(s))]\, ds \\ + \int_a^b [G(x,s) - \tilde{G}(x,s)]f(s,\tilde{u}(s))\, ds. \end{aligned}$$

Which can be bounded as,

$$\begin{aligned} \Big| \int_a^b [G(x,s)f(s,u(s)) - \tilde{G}(x,s)f(s,\tilde{u}(s))]\, ds \Big| \le \Big| \int_a^b G(x,s)[f(s,u(s)) - f(s,\tilde{u}(s))]\, ds \Big| \\ + \Big| \int_a^b [G(x,s) - \tilde{G}(x,s)]f(s,\tilde{u}(s))\, ds \Big|. \end{aligned} \tag{3.12}$$

For the first integral of inequality (3.12), using Lipschitz condition on f, we get

$$\begin{aligned} \Big| \int_a^b G(x,s)[f(s,u(s)) - f(s,\tilde{u}(s))]\, ds \Big| &\le \int_a^b |G(x,s)| \cdot |f(s,u(s)) - f(s,\tilde{u}(s))|\, ds \\ &\le L \int_a^b |G(x,s)| \cdot |u(s) - \tilde{u}(s)|\, ds. \end{aligned}$$

For second integral of inequality(3.12)

$$\begin{aligned} \Big| \int_a^b [G(x,s) - \tilde{G}(x,s)]f(s,\tilde{u}(s))\, ds \Big| &\le M \int_a^b |G(x,s) - \tilde{G}(x,s)|\, ds \\ &= M \int_a^b |(b-s)^{\alpha-1} - (\tilde{b}-s)^{\alpha-1}|\, ds \\ &= \frac{M}{\alpha} |(b-a)^\alpha + (\tilde{b}-b)^\alpha - (\tilde{b}-a)^\alpha| \\ &\le \frac{M}{\alpha} |b - \tilde{b}|^\alpha. \end{aligned}$$

Therefore (3.12) becomes,

$$\Big|\int_a^b [G(x,s)f(s,u(s)) - \tilde{G}(x,s)f(s,\tilde{u}(s))]\,ds\Big| \le L\int_a^b |G(x,s)|\cdot|u(s)-\tilde{u}(s)|\,ds + \frac{M|b-\tilde{b}|^\alpha}{\alpha}. \tag{3.13}$$

Now for the second integral of (3.11)

$$\begin{aligned}\Big|\int_b^{\tilde{b}} \tilde{G}(x,s)f(s,\tilde{u}(s))\,ds\Big| &\le M\int_b^{\tilde{b}} |(\tilde{b}-s)^{\alpha-1}|\,ds \\ &\le \frac{M|b-\tilde{b}|^\alpha}{\alpha}.\end{aligned}$$

So,

$$|u(x)-\tilde{u}(x)| = |\zeta(x)| \le \frac{2M|b-\tilde{b}|^\alpha}{\Gamma(\alpha+1)} + \frac{L}{\Gamma(\alpha)}\int_a^b |G(x,s)|\cdot|\zeta(s)|\,ds. \tag{3.14}$$

So as to conclude the sought inequality (3.7) from this connection, we have to summon a Gronwall type result [46]. We found our required result. □

The case that a varies can be dealt in a similar (but not exactly and precisely identical) way, and we can demonstrate the following result that is formally essentially the same as the previous theorem.

Theorem 3.1.3. *[23] Consider the terminal value problems of fractional order α*

$$D^\alpha_{*a^+}u(x) = f(x,u(x)), \qquad u(b) = u^*, \tag{3.15}$$

and

$$D^\alpha_{*\tilde{a}^+}\tilde{u}(x) = f(x,\tilde{u}(x)), \qquad \tilde{u}(b) = u^*, \tag{3.16}$$

for some $\alpha \in (0,1)$ and $a \le \tilde{a} < b$, where the function f is assumed once again to satisfy the hypothesis of Theorem 2.1.1. Then the solutions u and $\tilde{u}$ satisfy the following relation,

$$\sup_{x\in[\tilde{a},b]} |u(x)-\tilde{u}(x)| = O(|a-\tilde{a}|^\alpha). \tag{3.17}$$

Proof. Define $\zeta(x) := u(x) - \tilde{u}(x)$ and by Lemma 3.1.1, we can rewrite (3.15) and (3.16) as their equivalent integral equations,

$$u(x) = u^* + \frac{1}{\Gamma(\alpha)}\int_a^b G(x,s)f(s,u(s))\,ds \tag{3.18}$$

and

$$\tilde{u}(x) = u^* + \frac{1}{\Gamma(\alpha)} \int_{\tilde{a}}^{b} G(x,s) f(s, \tilde{u}(s))\, ds \tag{3.19}$$

respectively, where

$$G(x,s) = \begin{cases} -(b-s)^{\alpha-1} & \text{for } s > x, \\ (x-s)^{\alpha-1} - (b-s)^{\alpha-1} & \text{for } s \leq x. \end{cases}$$

As kernel function $G(x,t)$ depends only on b, so there is no change of b in this case. Thus in both integral, the kernel $G(x,t)$ is same. Subtracting (3.18) and (3.19) and then taking absolute values, we get

$$\begin{aligned} |u(x) - \tilde{u}(x)| \leq & \frac{1}{\Gamma(\alpha)} \int_{a}^{\tilde{a}} |G(x,s) f(s,u(s))|\, ds \\ & + \frac{1}{\Gamma(\alpha)} \int_{\tilde{a}}^{b} |G(x,s)| \cdot |f(s,u(s)) - f(s,\tilde{u}(s))|\, ds. \end{aligned} \tag{3.20}$$

The first integral of (3.20) can be assessed by,

$$\begin{aligned} \frac{1}{\Gamma(\alpha)} \int_{a}^{\tilde{a}} |G(x,s) f(s,u(s))|\, ds \leq & \frac{M}{\Gamma(\alpha)} \int_{a}^{\tilde{a}} |G(x,s)|\, ds \\ = & \frac{M}{\Gamma(\alpha)} \int_{a}^{\tilde{a}} |(x-s)^{\alpha-1} - (b-s)^{\alpha-1}|\, ds \\ = & \frac{M}{\Gamma(\alpha+1)} |(x-a)^{\alpha} - (x-\tilde{a})^{\alpha} - (b-a)^{\alpha} + (b-\tilde{a})^{\alpha}|. \end{aligned}$$

Using Holder's condition of order α and Mean Value Theorem, we get,

$$\begin{aligned} \frac{1}{\Gamma(\alpha)} \int_{a}^{\tilde{a}} |G(x,s) f(s,u(s))|\, ds \leq & \frac{M}{\Gamma(\alpha+1)} [\mathcal{H}|\tilde{a}-a|^{\alpha} + \alpha|b-\xi|^{\alpha-1}|\tilde{a}-a|] \\ \leq & B_1|a-\tilde{a}|^{\alpha} + B_2|a-\tilde{a}| \leq B|a-\tilde{a}|^{\alpha} \end{aligned}$$

where $B_1 = \frac{M\mathcal{H}}{\Gamma(\alpha+1)}$, $B_2 = \frac{M\alpha|b-\xi|^{\alpha-1}}{\Gamma(\alpha+1)}$ and B are certain constants due to Holder's condition of order α and Mean Value Theorem for the second integral of (3.20), using Lipschitz condition on f with respect to second variable, we get

$$\frac{1}{\Gamma(\alpha)} \int_{\tilde{a}}^{b} |G(x,s)| \cdot |f(s,u(s)) - f(s,\tilde{u}(s))|\, ds \leq \frac{L}{\Gamma(\alpha)} \int_{\tilde{a}}^{b} |G(x,s)| \cdot |u(s) - \tilde{u}(s)|\, ds.$$

Thus (3.20) becomes,

$$|u(x) - \tilde{u}(x)| = |\zeta(x)| \leq B|a-\tilde{a}|^{\alpha} + \frac{L}{\Gamma(\alpha)} \int_{\tilde{a}}^{b} |G(x,s)| \cdot |\zeta(s)|\, ds$$

we might again contend with help of Gronwall inequality for fredholm operator [46] and acquire the result (3.18). □

3.2 Existence and uniqueness of solution of terminal value problems for fractional differential equations

Amid a decade ago, a very few of papers have been focused on the study of existence and uniqueness of fractional differential equations [4, 5, 35–41]. The purpose of this section of the chapter is to examine the results of existence and uniqueness of solutions of terminal value problem for fractional differential equations. The existence hypothesis for initial value problem for fractional differential equations have gotten extensive consideration in most recent two decades [41–44]. However the existence hypothesis for terminal value problem for fractional differential equations in its beginning phases of advancement see [24–26, 45]. We begin with following terminal value problem

$$D^{\alpha}_{*b^-}u(x) = f(x, u(x), u'(x)), \quad 1 < \alpha \leq 2, \quad x \in [a, b], \tag{3.21}$$

$$u(b) = u_0, \quad u'(b) = u_1, \qquad u_0, u_1 \in \mathbb{R}, \tag{3.22}$$

where the function $f : [a, b] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is assumed to be continuous and $D^{\alpha}_{*b^-}$ represents right sided Caputo derivative of order α. Let the norm defined as $\|u\| = sup_{x\in[a,b]}|u(x)| + sup_{x\in[a,b]}|u'(x)|$ and $\mathbb{X} = \{u(x) : u \in C[a, b]; u' \in C[a, b]\}$. Then $(\mathbb{X}, \|.\|)$ is Banach space. In section 3.2.1 we will establish results for existence of solutions of terminal value problems for fractional differential equations. Section 3.2.2 deals with uniqueness results for solutions of terminal value problems for fractional differential equations. Examples are added to demonstrate the significance of our result.

3.2.1 Existence of solutions

Lemma 3.2.1. *Assume the function $f : [a, b] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is continuous. Then $u \in C[a, b]$ is solution of terminal value problem* (3.21) *if and only if it is solution of integral equation*

$$u(x) = u_0 + u_1(b - x) + \frac{1}{\Gamma(\alpha)} \int_x^b (s - x)^{\alpha-1} f(s, u(s), u'(s))\, ds. \tag{3.23}$$

Proof. Assume $u \in C[a, b]$ is solution of (3.21), then applying $I^{\alpha}_{b^-}$on both sides of (3.21)

$$I^{\alpha}_{b^-}D^{\alpha}_{*b^-}u(x) = I^{\alpha}_{b^-}f(x, u(x), u'(x)).$$

By Theorem 1.4.6

$$u(x) - u(b) - u'(b)(b - x) = \frac{1}{\Gamma(\alpha)} \int_x^b (s - x)^{\alpha-1} f(s, u(s), u'(s))\, ds.$$

Using terminal condition $u(b) = u_0$ and $u'(b) = u_1$

$$u(x) = u_0 + u_1(b - x) + \frac{1}{\Gamma(\alpha)} \int_x^b (s - x)^{\alpha-1} f(s, u(s), u'(s))\, ds.$$

Hence u is solution of (3.23).
Conversely, assume that $u \in C[a, b]$ is solution of integral equation (3.23) then applying $D^{\alpha}_{*b^-}$ on both sides of (3.23), we get

$$D^{\alpha}_{*b^-} u(x) = D^{\alpha}_{*b^-} (u_0 + u_1(b - x) + \frac{1}{\Gamma(\alpha)} \int_x^b (s - x)^{\alpha-1} f(s, u(s), u'(s))\, ds),$$

$$D^{\alpha}_{*b^-} u(x) = D^{\alpha}_{*b^-} (u_0) + D^{\alpha}_{*b^-} (u_1(b - x)) + D^{\alpha}_{*b^-} I^{\alpha}_{b^-} f(x, u(x), u'(x)).$$

The Caputo differential operator of a constant function is zero, therefore first two terms will vanish, thus by Theorem 1.4.5,

$$D^{\alpha}_{*b^-} u(x) = f(x, u(x), u'(x)).$$

Now to satisfy terminal condition (3.22), when limit $x \to b$ in (3.23) we get

$$u(b) = u_0.$$

Differentiating (3.23) once, we obtained

$$u'(x) = u'(b) + D I^{\alpha}_{b^-} f(x, u(x), u'(x)).$$

Since $D^m I^{\alpha} = I^{\alpha-m}$, so

$$u'(x) = u'(b) + \frac{1}{\Gamma(\alpha - 1)} \int_x^b (s - x)^{\alpha-2} f(s, u(s), u'(s))\, ds.$$

Thus when limit $x \to b$

$$u'(b) = u_1.$$

Therefore, second terminal condition $u'(b) = u_1$ is also satisfied. Hence u is solution of (3.23), which completes the proof. □

Theorem 3.2.1. *Assume $f : [a, b] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is continuous and $|f(x, u, u')| \le M$ where $M \ge 0$, then there exist at least one solution of the terminal value problem* (3.21).

Proof. Define

$$\mathbf{U} := \left\{ u \in C[a, b] : |u - u_0 - (b - x)u_1| \le K \quad, |u' - u_1| \le \widetilde{K} \right\},$$

where $K := \frac{M}{\Gamma(\alpha+1)}(b-a)^{\alpha}$ and $\widetilde{K} := \frac{M(b-a)^{\alpha-1}}{\Gamma(\alpha)}$. The set $\mathbf{U}$ is closed convex subset of Banach space $\mathbb{X}$. We define an operator $\mathcal{A}$ on $\mathbb{X}$ as

$$(\mathcal{A}u)(x) := u_0 + (b-x)u_1 + \frac{1}{\Gamma(\alpha)}\int_x^b (s-x)^{\alpha-1} f(s, u(s), u'(s))\, ds. \tag{3.24}$$

Using (3.24), the Voletrra integral equation (3.23) can be written as

$$u = \mathcal{A}u.$$

Thus we have to show that $\mathcal{A}$ has a fixed point. The fixed point of operator $\mathcal{A}$ are solution of (3.21). To study properties of operator $\mathcal{A}$, first we show that $\mathcal{A} : \mathbf{U} \to \mathbf{U}$. Now

$$\begin{aligned}
|(\mathcal{A}u)(x) - u_0 - (b-x)u_1| &= \frac{1}{\Gamma(\alpha)}\Big|\int_x^b (s-x)^{\alpha-1} f(s, u(s), u'(s))\, ds\Big| \\
&\leq \frac{M}{\Gamma(\alpha)}\Big|\int_x^b (s-x)^{\alpha-1}\, ds\Big| \\
&= \frac{M}{\Gamma(\alpha+1)}(b-x)^{\alpha} \\
&\leq \frac{M}{\Gamma(\alpha+1)}(b-a)^{\alpha} = K.
\end{aligned}$$

Also,

$$\begin{aligned}
|(\mathcal{A}u)'(x) - u_1| &= \frac{1}{\Gamma(\alpha-1)}|\int_x^b (s-x)^{\alpha-2} f(s, u(s), u'(s))\, ds| \\
&\leq \frac{M}{\Gamma(\alpha-1)}\int_x^b |(s-x)^{\alpha-2}|\, ds \\
|(\mathcal{A}u)'(x) - u_1| &\leq \frac{M}{\Gamma(\alpha)}(b-x)^{\alpha-1} \leq \frac{M(b-a)^{\alpha-1}}{\Gamma(\alpha)} = \widetilde{K}.
\end{aligned}$$

Thus, we have shown that $\mathcal{A}u \in \mathbf{U}$ if $u \in \mathbf{U}$, i.e. $\mathcal{A}$ maps the set $\mathbf{U}$ to itself. Now we have to show that $\mathcal{A}\mathbf{U} := \{\mathcal{A} : u \in \mathbf{U}\}$ is relatively compact set. For $\omega \in \mathcal{A}U$

$$\begin{aligned}
|\omega(x)| = |(\mathcal{A}u)(x)| &\leq |u_0 - (b-x)u_1| \\
&\quad + \frac{1}{\Gamma(\alpha)}\int_x^b (s-x)^{\alpha-1}|f(s, u(s), u'(s))|\, ds \\
&\leq |u_0 - (b-x)u_1| + \frac{M}{\Gamma(\alpha+1)} \cdot (b-x)^{\alpha} \\
&\leq |u_0| + |(b-a)u_1| + \frac{M}{\Gamma(\alpha+1)} \cdot (b-a)^{\alpha} \\
\sup_{x\in[a,b]} |\omega(x)| = \sup_{x\in[a,b]} |(\mathcal{A}u)(x)| &\leq |u_0| + |(b-a)u_1| + K
\end{aligned}$$

and

$$\begin{aligned}|\omega'(x)| = |(\mathcal{A}u)'(x)| &\leq |u_1| + \frac{1}{\Gamma(\alpha-1)}\int_x^b (s-x)^{\alpha-2}|f(s,u(s),u'(s))|\,ds \\ &\leq |u_1| + \frac{M}{\Gamma(\alpha)}\cdot(b-x)^{\alpha-1} \\ &\leq |u_1| + \frac{M}{\Gamma(\alpha)}\cdot(b-a)^{\alpha-1}\end{aligned}$$

$$\sup_{x\in[a,b]}|\omega'(x)| = \sup_{x\in[a,b]}|(\mathcal{A}u)'(x)| \leq |u_1| + \widetilde{K}.$$

$$\begin{aligned}\|\omega(x)\| &= \sup_{x\in[a,b]}|\omega(x)| + \sup_{x\in[a,b]}|\omega'(x)| \\ &\leq |u_0| + |(b-a)u_1| + |u_1| + K + \widetilde{K}, \text{which shows boundedness.}\end{aligned}$$

Now we have to prove that $\mathcal{A}u$ is continuous. For this, we begin with $a \leq x_1 \leq x_2 \leq b$,

$$\begin{aligned}\Big|(\mathcal{A}u)(x_1) - (\mathcal{A}u)(x_2)\Big| = \Big|&(b-x_1)u_1 + \frac{1}{\Gamma(\alpha)}\int_{x_1}^b (s-x_1)^{\alpha-1}f(s,u(s),u'(s))\,ds \\ &- (b-x_2)u_1 - \frac{1}{\Gamma(\alpha)}\int_{x_2}^b (s-x_2)^{\alpha-1}f(s,u(s),u'(s))\,ds\Big|\end{aligned}$$

$$\begin{aligned}\Big|(\mathcal{A}u)(x_1) - (\mathcal{A}u)(x_2)\Big| &= \Big|(x_1-x_2)u_1 + \frac{1}{\Gamma(\alpha)}\Big(\int_{x_1}^{x_2} (s-x_1)^{\alpha-1}f(s,u(s),u'(s))\,ds \\ &\quad + \int_{x_2}^b (s-x_1)^{\alpha-1}f(s,u(s),u'(s))\,ds \\ &\quad + \int_{x_2}^b (s-x_2)^{\alpha-1}f(s,u(s),u'(s))\,ds\Big)\Big| \\ &\leq |(x_1-x_2)u_1| + \frac{1}{\Gamma(\alpha)}\Big|\int_{x_1}^{x_2} (s-x_1)^{\alpha-1}f(s,u(s),u'(s))\,ds \\ &\quad + \int_{x_2}^b ((s-x_1)^{\alpha-1} - (s-x_2)^{\alpha-1})f(s,u(s),u'(s))\,ds\Big|.\end{aligned}$$

Since f is bounded, therefore

$$\begin{aligned}\Big|(\mathcal{A}u)(x_1) - (\mathcal{A}u)(x_2)\Big| \leq |(x_2-x_1)u_1| + \frac{M}{\Gamma(\alpha)}\Big\{&\int_{x_1}^{x_2} (s-x_1)^{\alpha-1}\,ds \\ &+ \int_{x_2}^b |(s-x_1)^{\alpha-1} - (s-x_2)^{\ \alpha-1}|\,ds\Big\}.\end{aligned} \tag{3.25}$$

The first integral of (3.25) has value $\frac{M}{\Gamma(\alpha+1)}(x_2-x_1)^\alpha$. For second integral

$$\frac{M}{\Gamma(\alpha)}\int_{x_2}^{b}|(s-x_1)^{\alpha-1}-(s-x_2)^{\alpha-1}|\,ds=\frac{M}{\Gamma(\alpha)}\int_{x_2}^{b}[(s-x_2)^{\alpha-1}-(s-x_1)^{\alpha-1}]\,ds$$

$$=\frac{M}{\Gamma(\alpha+1)}[(b-x_2)^\alpha-(b-x_1)^\alpha+(x_2-x_1)^\alpha]. \tag{3.26}$$

Using mean value theorem in (3.26), we get

$$\frac{M}{\Gamma(\alpha)}\int_{x_2}^{b}|(s-x_1)^{\alpha-1}-(s-x_2)^{\alpha-1}|\,ds=\frac{M}{\Gamma(\alpha+1)}[(x_2-x_1)^\alpha+\alpha(b-\xi)^{\alpha-1}(x_2-x_1)].$$

Consequently, (3.25) can be written as

$$\Big|(\mathcal{A}u)(x_1)-(\mathcal{A}u)(x_2)\Big|\le|x_2-x_1|u_1+\frac{M}{\Gamma(\alpha+1)}[2(x_2-x_1)^\alpha+\alpha(b-\xi)^{\alpha-1}(x_2-x_1)]. \tag{3.27}$$

Also,

$$\begin{aligned}\Big|(\mathcal{A}u)'(x_1)-(\mathcal{A}u)'(x_2)\Big|&=\Big|\frac{1}{\Gamma(\alpha-1)}\int_{x_1}^{b}(s-x_1)^{\alpha-2}f(s,u(s),u'(s))\,ds\\&\quad-\frac{1}{\Gamma(\alpha-1)}\int_{x_2}^{b}(s-x_2)^{\alpha-2}f(s,u(s),u'(s))\,ds\Big|\\&=\Big|\frac{1}{\Gamma(\alpha-1)}\Big(\int_{x_1}^{x_2}(s-x_1)^{\alpha-2}f(s,u(s),u'(s))\,ds\\&\quad+\int_{x_2}^{b}(s-x_1)^{\alpha-2}f(s,u(s),u'(s))\,ds\\&\quad+\int_{x_2}^{b}(s-x_2)^{\alpha-2}f(s,u(s),u'(s))\,ds\Big)\Big|\\&\le\frac{1}{\Gamma(\alpha-1)}\Big|\int_{x_1}^{x_2}(s-x_1)^{\alpha-2}f(s,u(s),u'(s))\,ds\\&\quad+\int_{x_2}^{b}((s-x_1)^{\alpha-2}-(s-x_2)^{\alpha-2})f(s,u(s),u'(s))\,ds\Big|.\end{aligned}$$

Since f is bounded, therefore

$$\begin{aligned}\Big|(\mathcal{A}u)'(x_1)-(\mathcal{A}u)'(x_2)\Big|&\le\frac{M}{\Gamma(\alpha-1)}\Big\{\int_{x_1}^{x_2}(s-x_1)^{\alpha-2}\,ds\\&\quad+\int_{x_2}^{b}|(s-x_1)^{\alpha-2}-(s-x_2)^{\alpha-2}|\,ds\Big\}.\end{aligned} \tag{3.28}$$

The first integral of (3.28) has value $\frac{M}{\Gamma(\alpha)}(x_2 - x_1)^{\alpha-1}$. For second integral

$$\frac{M}{\Gamma(\alpha-1)} \int_{x_2}^{b} |(s-x_1)^{\alpha-2} - (s-x_2)^{\alpha-2}|\, ds = \frac{M}{\Gamma(\alpha-1)} \int_{x_2}^{b} [(s-x_2)^{\alpha-2} - (s-x_1)^{\alpha-2}]\, ds$$

$$= \frac{M}{\Gamma(\alpha)}[(b-x_2)^{\alpha-1} - (b-x_1)^{\alpha-1} + (x_2-x_1)^{\alpha-1}]. \quad (3.29)$$

Using mean value theorem in (3.29), we get

$$\frac{M}{\Gamma(\alpha-1)} \int_{x_2}^{b} |(s-x_1)^{\alpha-2} - (s-x_2)^{\alpha-2}|\, ds = \frac{M}{\Gamma(\alpha)}[(x_2-x_1)^{\alpha-1} + (\alpha-1)(b-\xi)^{\alpha-2}(x_2-x_1)].$$

Therefore (3.28) can be written as

$$\Big|(\mathcal{A}u)'(x_1) - (\mathcal{A}u)'(x_2)\Big| \leq \frac{M}{\Gamma(\alpha)}[2(x_2-x_1)^{\alpha-1} + (\alpha-1)(b-\xi)^{\alpha-2}(x_2-x_1)]. \quad (3.30)$$

Thus when limit $x_1 \to x_2$, then (3.27) and (3.30) converges to zero, which proves that $\mathcal{A}u$ is continuous. For equicontinuity, if $|x_2 - x_1| < \delta$, then (3.27) becomes

$$\Big|(\mathcal{A}u)(x_1) - (\mathcal{A}u)(x_2)\Big| \leq \delta u_1 + \frac{M}{\Gamma(\alpha+1)}[2\delta^{\alpha} + \alpha(b-\xi)^{\alpha-1}\delta]. \quad (3.31)$$

Similarly, (3.30) becomes

$$\Big|(\mathcal{A}u)'(x_1) - (\mathcal{A}u)'(x_2)\Big| \leq \frac{M}{\Gamma(\alpha)}[2\delta^{\alpha-1} + (\alpha-1)(b-\xi)^{\alpha-2}\delta]. \quad (3.32)$$

Since right hand side of (3.31) and (3.32) is independent of u, x_1 and x_2, therefore set $\mathcal{A}$(U) is equicontinuous. Thus, from Arzel-Ascoli Theorem $\mathcal{A}$ is relatively compact. Therefore by Schauder's fixed point theorem $\mathcal{A}$ has a fixed point. Hence, terminal value problem (3.21) has at least one solution. □

Theorem 3.2.2. *Assume $f : [a,b] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is continuous and*

$$|f(x,u,u')| \leq \phi(x) + \psi(x)|u| + \chi(x)|\nu| \quad (3.33)$$

where $\phi(x), \psi(x)$ and $\chi(x)$ are non-negative continuous and defined on $[a,b]$, then there exist at least one solution of the terminal value problem (3.21).

Proof. Define

$$\mathbf{V} := \Big\{ u \in C[a,b] : |u - u_0 - (b-x)u_1| \le K_1, \quad |u' - u_1| \le K_2 \Big\},$$

where $K_1 := \frac{(M_1+M_2\|u\|+M_3\|u'\|)(b-a)^\alpha}{\Gamma(\alpha+1)}$ and $K_2 := \frac{(M_1+M_2\|u\|+M_3\|u'\|)(b-a)^{\alpha-1}}{\Gamma(\alpha)}$. Where $M_1 = \max_{a\le x\le b}\phi(x)$, $M_2 = \max_{a\le x\le b}\psi(x)$ and $M_3 = \max_{a\le x\le b}\chi(x)$. The set $\mathbf{V}$ is closed convex subset of Banach space $\mathbb{X}$. We have to show that (3.24) has a fixed point.

First we show that $\mathcal{A} : \mathbf{V} \to \mathbf{V}$. Now

$$\begin{aligned}
|(\mathcal{A}u)(x) - u_0 - (b-x)u_1| &= \frac{1}{\Gamma(\alpha)} \Big| \int_x^b (s-x)^{\alpha-1} f(s, u(s), u'(s))\, ds \Big| \\
&\le \frac{1}{\Gamma(\alpha)} \int_x^b (s-x)^{\alpha-1} (\phi(s) + \psi(s)|u| + \chi(s)|u'|)\, ds \\
&\le \frac{1}{\Gamma(\alpha)} \Big\{ \int_x^b (s-x)^{\alpha-1}\phi(s)\, ds \\
&\quad + \|u\| \int_x^b (s-x)^{\alpha-1}\psi(s)\, ds \\
&\quad + \|u'\| \int_x^b (s-x)^{\alpha-1}\chi(s)\, ds \Big\}.
\end{aligned}$$

Since ϕ, ψ andχ are defined and continuous on $[a,b]$, therefore there exists $M_i, \quad i = 1,2,3$ such that,

$$\begin{aligned}
|(\mathcal{A}u)(x) - u_0 - (b-x)u_1| &\le \frac{M_1 + M_2\|u\| + M_3\|u'\|}{\Gamma(\alpha+1)} (b-x)^\alpha \\
&\le \frac{M_1 + M_2\|u\| + M_3\|u'|}{\Gamma(\alpha+1)} (b-a)^\alpha = K_1.
\end{aligned}$$

Also,

$$\begin{aligned}
|(\mathcal{A}u)'(x) - u_1| &= \frac{1}{\Gamma(\alpha-1)} \Big| \int_x^b (s-x)^{\alpha-2} f(s, u(s), u'(s))\, ds \Big| \\
&\le \frac{1}{\Gamma(\alpha-1)} \int_x^b |(s-x)^{\alpha-2}| \cdot (\phi(s) + \psi(s)|u| + \chi(s)|u'|)\, ds \\
&\le \frac{1}{\Gamma(\alpha)} \Big\{ \int_x^b (s-x)^{\alpha-2}\phi(s)\, ds + \|u\| \int_x^b (s-x)^{\alpha-2}\psi(s)\, ds \\
&\quad + \|u'\| \int_x^b (s-x)^{\alpha-2}\chi(s)\, ds \Big\}
\end{aligned}$$

$$
\begin{aligned}
|(\mathcal{A}u)'(x) - u_1| &\le \frac{M_1 + M_2\|u\| + M_3\|u'\|}{\Gamma(\alpha)}(b-x)^{\alpha-1} \\
&\le \frac{(M_1 + M_2\|u\| + M_3\|u'\|)(b-a)^{\alpha-1}}{\Gamma(\alpha)} = K_2.
\end{aligned}
$$

Thus, we have shown that $\mathcal{A}u \in \mathbf{V}$ if $u \in \mathbf{V}$, i.e. $\mathcal{A}$ maps the set $\mathbf{V}$ to itself.
Now we have to show that $\mathcal{A}\mathbf{V} := \{\mathcal{A} : u \in \mathbf{V}\}$ is relatively compact set. For $\omega \in \mathcal{A}V$

$$
\begin{aligned}
|\omega(x)| = |(\mathcal{A}u)(x)| &\le |u_0 - (b-x)u_1| \\
&\quad + \frac{1}{\Gamma(\alpha)} \int_x^b (s-x)^{\alpha-1} |f(s, u(s), u'(s))| \, ds \\
&\le |u_0 - (b-x)u_1| \\
&\quad + \frac{1}{\Gamma(\alpha)} \int_x^b (s-x)^{\alpha-1} (\phi(s) + \psi(s)|u| + \chi(s)|u'|) \, ds \\
&\le |u_0 - (b-x)u_1| + \frac{1}{\Gamma(\alpha)} \Big\{ \int_x^b (s-x)^{\alpha-1}\phi(s) \, ds \\
&\quad + \|u\| \int_x^b (s-x)^{\alpha-1}\psi(s) \, ds \\
&\quad + \|u'\| \int_x^b (s-x)^{\alpha-1}\chi(s) \, ds \Big\} \\
&\le |u_0 - (b-x)u_1| + \frac{(M_1 + M_2\|u\| + M_3\|u'\|}{\Gamma(\alpha+1)} \cdot (b-x)^{\alpha} \\
&\le |u_0 - (b-x)u_1| + \frac{(M_1 + M_2\|u\| + M_3\|u'\|}{\Gamma(\alpha+1)} \cdot (b-a)^{\alpha}
\end{aligned}
$$

$$
\sup_{x \in [a,b]} |\omega(x)| = \sup_{x \in [a,b]} |(\mathcal{A}u)(x)| \le |u_0 - (b-a)u_1| + K_1,
$$

$$
\begin{aligned}
|\omega'(x)| = |(\mathcal{A}u)'(x)| &\le |u_1| + \frac{1}{\Gamma(\alpha-1)} \int_x^b (s-x)^{\alpha-2} |f(s, u(s), u'(s))| \, ds \\
&\le |u_1| + \frac{1}{\Gamma(\alpha-1)} \int_x^b (s-x)^{\alpha-2} (\phi(s) + \psi(s)|u| + \chi(s)|u'|) \, ds
\end{aligned}
$$

$$
\begin{aligned}
|\omega'(x)| = |(\mathcal{A}u)'(x)| &\le |u_1| + \frac{1}{\Gamma(\alpha-1)}\Big\{ \int_x^b (s-x)^{\alpha-2}\phi(s)\,ds \\
&\quad + \|u\| \int_x^b (s-x)^{\alpha-2}\psi(s)\,ds \\
&\quad + \|u'\| \int_x^b (s-x)^{\alpha-2}\chi(s)\,ds \Big\} \\
&\le |u_1| + \frac{(M_1 + M_2\|u\| + M_3\|u'\|}{\Gamma(\alpha)} \cdot (b-x)^{\alpha-1} \\
&\le |u_1| + \frac{(M_1 + M_2\|u\| + M_3\|u'\|}{\Gamma(\alpha)} \cdot (b-a)^{\alpha-1}
\end{aligned}
$$

$$
\sup_{x\in[a,b]} |\omega'(x)| = \sup_{x\in[a,b]} |(\mathcal{A}u)'(x)| \le |u_1| + K_2.
$$

Thus,

$$
\begin{aligned}
\|\omega(x)\| &= \sup_{x\in[a,b]} |\omega(x)| + \sup_{x\in[a,b]} |\omega'(x)| \\
&\le |u_0 - (b-a)u_1| + K_1 + |u_1| + K_2 \\
&\le |u_0 - (b-a)u_1| + |u_1| + K_1 + K_2, \text{which shows boundedness.}
\end{aligned}
$$

Now we have to prove that $\mathcal{A}u$ is continuous. For this, we begin with $a \le x_1 \le x_2 \le b$,

$$
\begin{aligned}
\Big|(\mathcal{A}u)(x_1) - (\mathcal{A}u)(x_2)\Big| &= \Big|(b-x_1)u_1 + \frac{1}{\Gamma(\alpha)} \int_{x_1}^b (s-x_1)^{\alpha-1} f(s,u(s),u'(s))\,ds \\
&\quad - (b-x_2)u_1 - \frac{1}{\Gamma(\alpha)} \int_{x_2}^b (s-x_2)^{\alpha-1} f(s,u(s),u'(s))\,ds\Big| \\
&= \Big|(x_1-x_2)u_1 + \frac{1}{\Gamma(\alpha)}\Big(\int_{x_1}^{x_2} (s-x_1)^{\alpha-1} f(s,u(s),u'(s))\,ds \\
&\quad + \int_{x_2}^b (s-x_1)^{\alpha-1} f(s,u(s),u'(s))\,ds \\
&\quad + \int_{x_2}^b (s-x_2)^{\alpha-1} f(s,u(s),u'(s))\,ds\Big)\Big| \\
&\le |(x_1-x_2)u_1| + \frac{1}{\Gamma(\alpha)}\Big| \int_{x_1}^{x_2} (s-x_1)^{\alpha-1} f(s,u(s),u'(s))\,ds \\
&\quad + \int_{x_2}^b ((s-x_1)^{\alpha-1} - (s-x_2)^{\alpha-1}) f(s,u(s),u'(s))\,ds\Big|.
\end{aligned}
$$

Using (3.33), we have

$$\begin{aligned}
\Big|(\mathcal{A}u)(x_1) - (\mathcal{A}u)(x_2)\Big| &\leq |(x_1 - x_2)u_1| \\
&\quad + \frac{1}{\Gamma(\alpha)}\Big\{ \int_{x_1}^{x_2} (s - x_1)^{\alpha-1}(\phi(s) + \psi(s)|u| + \chi(s)|u'|)\, ds \\
&\quad + \int_{x_2}^{b} |(s - x_1)^{\alpha-1} - (s - x_2)^{\alpha-1}| \cdot |\phi(s) + \psi(s)|u| + \chi(s)|u'||\, ds \Big\} \\
&\leq |(x_1 - x_2)u_1| + \frac{1}{\Gamma(\alpha)}\Big\{ \int_{x_1}^{x_2} (s - x_1)^{\alpha-1}\phi(s)\, ds \\
&\quad + \|u\| \int_{x_1}^{x_2} (s - x_1)^{\alpha-1}\psi(s)\, ds \\
&\quad + \|u'\| \int_{x_1}^{x_2} (s - x_1)^{\alpha-1}\chi(s)\, ds \\
&\quad + \int_{x_2}^{b} |(s - x_1)^{\alpha-1} - (s - x_2)^{\alpha-1}| \cdot \phi(s)\, ds \\
&\quad + \|u\| \int_{x_2}^{b} |(s - x_1)^{\alpha-1} - (s - x_2)^{\alpha-1}|\psi(s)\, ds \\
&\quad + \|u'\| \int_{x_2}^{b} |(s - x_1)^{\alpha-1} - (s - x_2)^{\alpha-1}|\chi(s)\, ds \Big\}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\Big|(\mathcal{A}u)(x_1) - (\mathcal{A}u)(x_2)\Big| &\leq (x_2 - x_1)u_1 + \frac{M_1 + M_2\|u\| + M_3\|u'\|}{\Gamma(\alpha)}\Big\{ \int_{x_1}^{x_2} (s - x_1)^{\alpha-1}\, ds \\
&\quad + \int_{x_2}^{b} |(s - x_1)^{\alpha-1} - (s - x_2)^{\alpha-1}|\, ds \Big\}.
\end{aligned} \tag{3.34}$$

The first integral of (3.34) has value $\frac{M_1+M_2\|u\|+M_3\|u'\|}{\Gamma(\alpha+1)}(x_2 - x_1)^{\alpha}$. For second integral

$$\begin{aligned}
\int_{x_2}^{b} |(s - x_1)^{\alpha-1} - (s - x_2)^{\alpha-1}|\, ds &= \int_{x_2}^{b} [(s - x_2)^{\alpha-1} - (s - x_1)^{\alpha-1}]\, ds \\
&= [(b - x_2)^{\alpha} - (b - x_1)^{\alpha} + (x_2 - x_1)^{\alpha}].
\end{aligned} \tag{3.35}$$

Using mean value theorem in (3.35), we get

$$\int_{x_2}^{b} |(s - x_1)^{\alpha-1} - (s - x_2)^{\alpha-1}|\, ds = [(x_2 - x_1)^{\alpha} + \alpha(b - \xi)^{\alpha-1}(x_2 - x_1)].$$

Consequently, (3.34) can be written as

$$\Big|(\mathcal{A}u)(x_1)-(\mathcal{A}u)(x_2)\Big| \leq (x_2-x_1)u_1+\frac{M_1+M_2\|u\|+M_3\|u'\|}{\Gamma(\alpha+1)}[(2(x_2-x_1)^{\alpha} \\ +\alpha(b-\xi)^{\alpha-1}(x_2-x_1)]. \tag{3.36}$$

And

$$\begin{aligned}\Big|(\mathcal{A}u)'(x_1)-(\mathcal{A}u)'(x_2)\Big| &= \Big|\frac{1}{\Gamma(\alpha-1)}\int_{x_1}^{b}(s-x_1)^{\alpha-2}f(s,u(s),u'(s))\,ds \\ &\quad -\frac{1}{\Gamma(\alpha-1)}\int_{x_2}^{b}(s-x_2)^{\alpha-2}f(s,u(s),u'(s))\,ds\Big| \\ &= \Big|\frac{1}{\Gamma(\alpha-1)}\Big(\int_{x_1}^{x_2}(s-x_1)^{\alpha-2}f(s,u(s),u'(s))\,ds \\ &\quad +\int_{x_2}^{b}(s-x_1)^{\alpha-2}f(s,u(s),u'(s))\,ds \\ &\quad +\int_{x_2}^{b}(s-x_2)^{\alpha-2}f(s,u(s),u'(s))\,ds\Big)\Big| \\ &\leq \frac{1}{\Gamma(\alpha-1)}\Big|\int_{x_1}^{x_2}(s-x_1)^{\alpha-2}f(s,u(s),u'(s))\,ds \\ &\quad +\int_{x_2}^{b}((s-x_1)^{\alpha-2}-(s-x_2)^{\alpha-2})f(s,u(s),u'(s))\,ds\Big|.\end{aligned}$$

Using (3.33), we have

$$\begin{aligned}\Big|(\mathcal{A}u)'(x_1)-(\mathcal{A}u)'(x_2)\Big| &\leq \frac{1}{\Gamma(\alpha-1)}\Big\{\int_{x_1}^{x_2}(s-x_1)^{\alpha-2}(\phi(s)+\psi(s)|u|+\chi(s)|u'|)\,ds \\ &\quad +\int_{x_2}^{b}|(s-x_1)^{\alpha-2}-(s-x_2)^{\alpha-2}|\cdot(\phi(s)+\psi(s)|u|+\chi(s)|u'|)\,ds\Big\}. \\ &\leq \frac{1}{\Gamma(\alpha-1)}\Big\{\int_{x_1}^{x_2}(s-x_1)^{\alpha-2}\phi(s)\,ds+\|u\|\int_{x_1}^{x_2}(s-x_1)^{\alpha-2}\psi(s)\,ds \\ &\quad +\|u'\|\int_{x_1}^{x_2}(s-x_1)^{\alpha-2}\chi(s)\,ds \\ &\quad +\int_{x_2}^{b}|(s-x_1)^{\alpha-2}-(s-x_2)^{\alpha-2}|\cdot\phi(s)\,ds \\ &\quad +\|u\|\int_{x_2}^{b}|(s-x_1)^{\alpha-2}-(s-x_2)^{\alpha-2}|\psi(s)\,ds \\ &\quad +\|u'\|\int_{x_2}^{b}|(s-x_1)^{\alpha-2}-(s-x_2)^{\alpha-2}|\chi(s)\,ds\Big\}.\end{aligned}$$

$$\Big|(\mathcal{A}u)'(x_1)-(\mathcal{A}u)'(x_2)\Big| \leq \frac{M_1+M_2\|u\|+M_3\|u'\|}{\Gamma(\alpha-1)}\Big\{\int_{x_1}^{x_2}(s-x_1)^{\alpha-2}\,ds + \int_{x_2}^{b}|(s-x_1)^{\alpha-2}-(s-x_2)^{\alpha-2}|\,ds\Big\}. \tag{3.37}$$

The first integral of (3.37) has value $\frac{M_1+M_2\|u\|+M_3\|u'\|}{\Gamma(\alpha)}(x_2-x_1)^{\alpha-1}$. For second integral

$$\int_{x_2}^{b}|(s-x_1)^{\alpha-2}-(s-x_2)^{\alpha-2}|\,ds = \int_{x_2}^{b}[(s-x_2)^{\alpha-2}-(s-x_1)^{\alpha-2}]\,ds$$

$$= [(b-x_2)^{\alpha-1}-(b-x_1)^{\alpha-1}+(x_2-x_1)^{\alpha-1}]. \tag{3.38}$$

Using mean value theorem in (3.38), we get

$$\int_{x_2}^{b}|(s-x_1)^{\alpha-2}-(s-x_2)^{\alpha-2}|\,ds = [(x_2-x_1)^{\alpha-1} + (\alpha-1)(b-\xi)^{\alpha-2}(x_2-x_1)].$$

Consequently, (3.37) can be written as

$$\Big|(\mathcal{A}u)'(x_1)-(\mathcal{A}u)'(x_2)\Big| \leq \frac{M_1+M_2\|u\|+M_3\|u'\|}{\Gamma(\alpha)}\cdot[2(x_2-x_1)^{\alpha-1} +(\alpha-1)(b-\xi)^{\alpha-2}(x_2-x_1)]. \tag{3.39}$$

Thus when limit $x_1 \to x_2$, then (3.36) and (3.39) converges to zero, which proves that $\mathcal{A}u$ is continuous. For equicontinuity, if $|x_2-x_1|<\delta$, then (3.36) becomes

$$\Big|(\mathcal{A}u)(x_1)-(\mathcal{A}u)(x_2)\Big| \leq \delta u_1 + \frac{M_1+M_2\|u\|+M_3\|u'\|}{\Gamma(\alpha+1)}[2\delta^{\alpha}+\alpha(b-\xi)^{\alpha-1}\delta]. \tag{3.40}$$

Similarly, (3.39) becomes

$$\Big|(\mathcal{A}u)'(x_1)-(\mathcal{A}u)'(x_2)\Big| \leq \frac{M_1+M_2\|u\|+M_3\|u'\|}{\Gamma(\alpha)}[2\delta^{\alpha-1}+(\alpha-1)(b-\xi)^{\alpha-2}\delta]. \tag{3.41}$$

Since right hand side of (3.40) and (3.41) is independent of u, x_1 and x_2, therefore set $\mathcal{A}$(U) is equicontinuous. Thus, from Arzel-Ascoli Theorem $\mathcal{A}$ is relatively compact. Now assume the set

$$\mathcal{D} := \{u \in C[a,b]: \quad u = \lambda\mathcal{A}(u) \quad for \quad some \quad 0 \leq \lambda \leq 1\}.$$

We have to show that this set $\mathcal{D}$ is bounded.
Let $u \in \mathcal{D}$, then $u = \lambda\mathcal{A}(u)$ for some $\lambda \in [0,1]$. Thus, for each $x \in [a,b]$, we have

$$(\mathcal{A}u)(x) = \lambda u_0 + \lambda(b-x)u_1 + \frac{\lambda}{\Gamma(\alpha)}\int_x^b (s-x)^{\alpha-1}f(s,u(s),u'(s))\,ds,$$

$$\left|(\mathcal{A}u)(x)\right| \le |\lambda u_0| + |\lambda(b-x)u_1| + \frac{\lambda}{\Gamma(\alpha)}\int_x^b (s-x)^{\alpha-1}|f(s,u(s),u'(s))|\,ds.$$

Using (3.33), we have

$$\begin{aligned}\left|(\mathcal{A}u)(x)\right| &\le |\lambda u_0| + |\lambda(b-x)u_1| \\ &\quad + \frac{\lambda}{\Gamma(\alpha)}\int_x^b (s-x)^{\alpha-1}(\phi(s)+\psi(s)|u|+\chi(s)|u'|)\,ds \\ &\le |\lambda u_0| + |\lambda(b-x)u_1| + \frac{\lambda(M_1+M_2\|u\|+M_3\|u'\|)}{\Gamma(\alpha)}\int_x^b (s-x)^{\alpha-1}\,ds \\ &= |\lambda u_0| + |\lambda(b-x)u_1| + \frac{\lambda(M_1+M_2\|u\|+M_3\|u'\|)}{(\alpha)\Gamma(\alpha)}(b-x)^{\alpha}\,ds\end{aligned}$$

$$\sup_{x\in[a,b]}|(\mathcal{A}u)(x)| \le |\lambda u_0| + |\lambda(b-a)u_1| + \frac{\lambda(M_1+M_2\|u\|+M_3\|u'\|)}{\Gamma(\alpha+1)}(b-a)^{\alpha}$$

and

$$(\mathcal{A}u)'(x) = -\lambda(b-x)u_1 - \frac{\lambda}{\Gamma(\alpha-1)}\int_x^b (s-x)^{\alpha-2}f(s,u(s),u'(s))\,ds,$$

$$\left|(\mathcal{A}u)'(x)\right| \le |\lambda u_1| + \frac{\lambda}{\Gamma(\alpha-1)}\int_x^b (s-x)^{\alpha-2}|f(s,u(s),u'(s))|\,ds.$$

Using (3.33), we have

$$\begin{aligned}\left|(\mathcal{A}u)'(x)\right| &\le |\lambda u_1| + \frac{\lambda}{\Gamma(\alpha-1)}\int_x^b (s-x)^{\alpha-2}(\phi(s)+\psi(s)|u|+\chi(s)|u'|)\,ds \\ &\le |\lambda u_1| + \frac{\lambda(M_1+M_2\|u\|+M_3\|u'\|)}{\Gamma(\alpha-1)}\int_x^b (s-x)^{\alpha-2}\,ds \\ &= |\lambda u_1| + \frac{\lambda(M_1+M_2\|u\|+M_3\|u'\|)}{(\alpha-1)\Gamma(\alpha-1)}(b-x)^{\alpha-1}\end{aligned}$$

$$\sup_{x\in[a,b]}|(\mathcal{A}u)'(x)| \le |\lambda u_1| + \frac{\lambda(M_1+M_2\|u\|+M_3\|u'\|)}{\Gamma(\alpha)}(b-a)^{\alpha-1}.$$

$$
\begin{aligned}
\Big\|(\mathcal{A}u)(x)\Big\| &= \sup_{x\in[a,b]}\Big|(\mathcal{A}u)(x)\Big| + \sup_{x\in[a,b]}\Big|(\mathcal{A}u)'(x)\Big| \\
&\le |\lambda u_0| + |\lambda(b-a)u_1| + \frac{\lambda(M_1+M_2\|u\|+M_3\|u'\|)}{\Gamma(\alpha+1)}(b-a)^{\alpha} + |\lambda u_1| \\
&\quad + \frac{\lambda(M_1+M_2\|u\|+M_3\|u'\|)}{\Gamma(\alpha)}(b-a)^{\alpha-1} \\
&\le |\lambda u_0| + |\lambda(b-a)u_1| + |\lambda u_1| + \frac{(b-a+\alpha)\cdot(b-a)^{\alpha-1}\lambda}{\Gamma(\alpha+1)}[M_1 \\
&\quad + M_2\|u\| + M_3\|u'\|].
\end{aligned}
$$

This shows that the set $\mathcal{D}$ is bounded. Therefore by Schaefer's fixed point theorem $\mathcal{A}$ has a fixed point. Hence, this fixed point of $\mathcal{A}$ is solution of (3.21). □

Example 3.2.1. *Consider a terminal value problem,*

$$
\begin{aligned}
D^{\alpha}_{*1^-}u &= x\sin(u+u'), \quad 1<\alpha\le 2, \quad x\in[0,1], \\
u(1) &= u_0, \quad u'(1)=u_1, \quad u_0,u_1\in\mathbb{R}.
\end{aligned} \tag{3.42}
$$

Here $\alpha = 3/2$ and $f(x,u,u') = x\sin(u+u')$. All the conditions of Theorem 3.2.1 are satisfied, hence there exists at least one solution of (3.42) on [0,1].

Example 3.2.2. *Consider a terminal value problem,*

$$
\begin{aligned}
D^{\alpha}_{*1^-}u &= x\cos(x) + u\sin^2(x) + u'x^2, \quad 1<\alpha\le 2, \quad x\in[0,1] \\
u(1) &= u_0, \quad u'(1)=u_1, \quad u_0,u_1\in\mathbb{R}.
\end{aligned} \tag{3.43}
$$

Here $\alpha = 3/2$ and $f(x,u,u') = x\cos(x) + u\sin^2(x) + u'x^2$.

$$
\begin{aligned}
|f(x,u,u')| &= \Big|x\cos(x) + u\sin^2(x) + u'x^2\Big| \\
&\le x\cos(x) + |u|\sin^2(x) + |u'|x^2 \\
|f(x,u,u')| &\le \phi(x) + \psi(x)|u| + \chi(x)|u'|
\end{aligned}
$$

where $\phi(x) = x\cos(x)$, $\psi(x) = \sin^2(x)$ and $\chi(x) = x^2$. Therefore, by Theorem 3.2.2, there exists at least one solution on [0,1].

3.2.2 Uniqueness of solution

In this section we will establish results for uniqueness of solution to terminal value problem (3.21). For this, let us define $\widetilde{K_1} := \frac{1}{\Gamma(\alpha)}\int_a^b (s-a)^{\alpha-1}L(s)\,ds$ and $\widetilde{K_2} := \frac{1}{\Gamma(\alpha-1)}\int_a^b (s-a)^{\alpha-2}L(s)\,ds$, which we will use in following theorem.

Theorem 3.2.3. *Assume $f : [a,b] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is continuous and satisfies*

$$\Big| f(x,u,u') - f(x,v,v') \Big| \leq L(x)[|u-v| + |u'-v'|], \tag{3.44}$$

with respect to second and third variable, where $L(x)$ is measurable function and $\widetilde{K_1} + \widetilde{K_2} < 1$ holds. Then there exists a unique solution of terminal value problem (3.21) *on [a,b].*

Proof. Since as we defined the norm $\|u\| = \sup_{x\in[a,b]} |u(x)| + \sup_{x\in[a,b]} |u'(x)|$ and $\mathbb{X} = \{u(x) : u \in C[a,b]; u' \in C[a,b]\}$, then $(\mathbb{X}, \|.\|)$ is banach space. Also we defined the operator $\mathcal{A}$ on $\mathbb{X}$ as

$$(\mathcal{A}u)(x) := u_0 + (b-x)u_1 + \frac{1}{\Gamma(\alpha)} \int_x^b (s-x)^{\alpha-1} f(s,u(s),u'(s))\, ds.$$

First we show that $\mathcal{A}$ is a contraction mapping on $\mathbb{X}$. For any u , $v \in \mathbb{X}$

$$\|\mathcal{A}u(x) - \mathcal{A}v(x)\| = \sup_{x\in[a,b]} |\mathcal{A}u(x) - \mathcal{A}v(x)| + \sup_{x\in[a,b]} |\mathcal{A}u'(x) - \mathcal{A}v'(x)| \tag{3.45}$$

We will first find $\sup_{x\in[a,b]} |\mathcal{A}u(x) - \mathcal{A}v(x)|$,

$$\begin{aligned}
\sup_{x\in[a,b]} |\mathcal{A}u(x) - \mathcal{A}v(x)| &= \sup_{x\in[a,b]} \frac{1}{\Gamma(\alpha)} \Big| \int_x^b (s-x)^{\alpha-1} [f(s,u(s),u'(s)) - f(s,v(s),v'(s))]\, ds \Big| \\
&\leq \frac{1}{\Gamma(\alpha)} \int_a^b (s-a)^{\alpha-1} [L(s)(|u-v| + |u'-v'|)]\, ds \\
&\leq \frac{\|u-v\|}{\Gamma(\alpha)} \int_a^b (s-a)^{\alpha-1} L(s)\, ds \\
&\leq \widetilde{K_1} \|u-v\|.
\end{aligned}$$

Now for $\sup_{x\in[a,b]} |\mathcal{A}u'(x) - \mathcal{A}v'(x)|$,

$$\begin{aligned}
\sup_{x\in[a,b]} |\mathcal{A}u'(x) - \mathcal{A}v'(x)| &= \sup_{x\in[a,b]} \frac{1}{\Gamma(\alpha-1)} \Big| \int_x^b (s-x)^{\alpha-2} [f(s,u(s),u'(s)) - f(s,v(s),v'(s))]\, ds \Big| \\
&\leq \frac{1}{\Gamma(\alpha-1)} \int_a^b (s-a)^{\alpha-2} [L(s)(|u-v| + |u'-v'|)]\, ds \\
&\leq \frac{\|u-v\|}{\Gamma(\alpha-1)} \int_a^b (s-a)^{\alpha-2} L(s)\, ds \\
&\leq \widetilde{K_2} \|u-v\|.
\end{aligned}$$

Therefore (3.45) become,

$$\begin{aligned}\|\mathcal{A}u(x)-\mathcal{A}v(x)\| &= \sup_{x\in[a,b]}|\mathcal{A}u(x)-\mathcal{A}v(x)| + \sup_{x\in[a,b]}|\mathcal{A}u'(x)-\mathcal{A}v'(x)|\\ &\le \widetilde{K_1}\|u-v\| + \widetilde{K_2}\|u-v\|\\ &\le [\widetilde{K_1}+\widetilde{K_2}]\|u-v\|.\end{aligned}$$

Hence, $\mathcal{A}$ is contraction mapping. Thus by the Banach fixed point theorem, a unique fixed point exist of operator $\mathcal{A}$. Consequently, (3.21) has a unique solution. □

Example 3.2.3. *Consider a terminal value problem,*

$$D^{\alpha}_{*1^-}u = \frac{\cos\sqrt{x}}{12}\left(\frac{u}{1+|u|}+\frac{u'}{1+|u'|}\right), \quad 1<\alpha\le 2, \quad x\in[0,1], \tag{3.46}$$

$$u(1)=u_0, \quad u'(1)=u_1 \quad u_0,u_1\in\mathbb{R}. \tag{3.47}$$

Here $\alpha = 3/2$ and

$$f(x,u,u') := \frac{\cos\sqrt{x}}{12}\left(\frac{u}{1+|u|}+\frac{u'}{1+|u'|}\right).$$

It is obvious that function $f(x,u,u')$ is continuous. Now

$$\begin{aligned}\Big|f(x,u_1,u_1')-f(x,u_2,u_2')\Big| &= \Big|\frac{u_1\cos\sqrt{x}}{12(1+|u_1|)}+\frac{u_1'\cos\sqrt{x}}{12(1+|u_1'|)}-\frac{u_2\cos\sqrt{x}}{12(1+|u_2|)}\\ &\quad -\frac{u_2'\cos\sqrt{x}}{12(1+|u_2'|)}\Big|\\ &\le \frac{\cos\sqrt{x}}{12}\Big[\Big|\frac{u_1}{1+|u_1|}-\frac{u_2}{1+|u_2|}\Big|+\Big|\frac{u_1'}{1+|u_1'|}-\frac{u_2'}{1+|u_2'|}\Big|\Big]\\ &\le \frac{\cos\sqrt{x}}{12}\Big[\frac{|u_1-u_2|}{(|1+|u_1|)(|1+|u_2|)}+\frac{|u_1'-u_2'|}{(|1+|u_1'|)(1+|u_2'|)}\Big].\end{aligned}$$

Since $\frac{1}{(1+|u_1|)(1+|u_2|)} < 1$, therefore

$$\Big|f(x,u_1,u_1')-f(x,u_2,u_2')\Big| \le L(x)[|u_1-u_2|+|u_1'-u_2'|]$$

where $L(x)=\frac{\cos\sqrt{x}}{12}$. Hence $f(x,u,u')$ satisfies (3.44). Also,

$$\begin{aligned}\widetilde{K_1}+\widetilde{K_2} &= \frac{1}{\Gamma(\alpha)}\int_a^b[(s-a)^{\alpha-1}+(s-a)^{\alpha-2}]L(s)\,ds\\ &= \frac{1}{12\Gamma(3/2)}\int_0^1[\sqrt{s}\cos\sqrt{s}+\frac{\cos\sqrt{s}}{\sqrt{s}}]\,ds = 0.2217545 < 1.\end{aligned}$$

Therefore, by Theorem 3.2.3, there exists a unique solution on [0,1].

Chapter 4

Existence and uniqueness of solution of terminal value problems for fractional differential equations on infinite interval

The existence hypothesis for boundary value problems for fractional differential equations on infinite interval have been considered extensively, see [47–50]. However the existence hypothesis for terminal value problems for fractional differential equation on infinite interval is never been studied so far. The related theory of terminal value problems for fractional differential equations is more complicated and have received attention quite recently. In many cases the domain of governing equation for certain physical phenomena is an infinite or semi-infinite interval. Infinite or semi-infinite interval problems require special treatment. The existence theory for infinite interval problems involving fractional derivative have been studied by number of authors [51–54]. However, to the best of our knowledge the existence hypothesis for terminal value problems for fractional differential equation on infinite interval is never been studied so far. Motivated by [25], the objective of this chapter is to analyze and develop the results of existence and uniqueness of solutions of terminal value problem for fractional differential equations of order α, where $1 < \alpha \leq 2$, on infinite interval. This work has been accepted for publication see [55]. We consider the following terminal value problem

$$\begin{aligned} D^{\alpha}_{*\infty^-}u(x) &= f(x, u(x), u'(x)), \quad 1 < \alpha \leq 2, \quad x \in [a, \infty), \\ u(\infty) &= \mu, \quad u'(\infty) = 0, \qquad \mu \in \mathbb{R}, \end{aligned} \tag{4.1}$$

where non-linear function $f : [a, \infty) \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is assumed to be continues. Let $C^1[a, \infty)$ be the space of continuously differentiable functions on $[a, \infty)$ and norm $\|\cdot\|$ defined as $\|u\| = \sup_{x\in[a,\infty)} |u(x)| + \sup_{x\in[a,\infty)} |u'(x)|$ and $\mathbb{X} = \{u(x) : u \in C^1[a, \infty)\}$.

Then $(\mathbb{X}, \|\cdot\|)$ is a Banach space. Following assumptions will be used to establish existence results for solutions of terminal value problem of fractional differential equation. For convenience, we introduce following notations:

$$P := \{(x, u, u') : x \in [a, \infty), \lim_{x\to\infty} u(x) \text{ and } \lim_{x\to\infty} u'(x) \text{ exists}\}.$$

Following assumptions will be used in the sequel to establish existence results for solutions of terminal value problem of fractional differential equation.

(H_1) $f \in C(P); f(x, u_1, u_1') \geq f(x, u_2, u_2')$ for $u_1 \geq u_2$, $u_1' \leq u_2'$,

(H_2) $f \in C(P); f(x,0,0) = 0$ and $\lim_{x\to\infty} \int_x^\infty (s-x)^{\alpha-2} f(s, u(s), u'(s))\, ds = 0$, for $x \in [a, \infty)$,

(H_3) $|f(x, u, u') - f(x, v, v')| \leq L(x)[|u-v| + |u'-v'|]$, where $L(x)$ is measurable function.

In section 4.1 we will establish results for existence of unbounded solutions of terminal value problem of fractional differential equation. Section 4.2 concerns with uniqueness results for unbounded solutions of terminal value problem of fractional differential equation.

4.1 Existence of solutions

Lemma 4.1.1. *Assume that the function f satisfies hypothesis (H_1), (H_2) and suppose that a function $u \in C^1[a, \infty)$ satisfies the condition*

$$u(x)u'(x) \leq 0, \tag{4.2}$$

then u is solution of (4.1) if and only if u is solution of

$$u(x) = \mu + \frac{1}{\Gamma(\alpha)} \int_x^\infty (s-x)^{\alpha-1} f(s, u(s), u'(s))\, ds. \tag{4.3}$$

Proof. Assume that u is solution of integral equation (4.1) then applying $D^\alpha_{*\infty^-}$ on both sides of (4.1), we get

$$D^\alpha_{*\infty^-} u(x) = D^\alpha_{*\infty^-}(\mu + \frac{1}{\Gamma(\alpha)} \int_x^\infty (s-x)^{\alpha-1} f(s, u(s), u'(s))\, ds),$$

$$D^\alpha_{*\infty^-} u(x) = D^\alpha_{*\infty^-}(\mu) + D^\alpha_{*\infty^-} I^\alpha_{\infty^-} f(x, u(x), u'(x)).$$

The Caputo differential operator of a constant function is zero, therefore first term will vanish and thus by Theorem 1.4.5, we have

$$D^\alpha_{*\infty^-} u(x) = f(x, u(x), u'(x)).$$

Now to satisfy terminal condition, when limit $x \to \infty$ in (4.1) we get

$$u(\infty) = \mu.$$

Further differentiating (4.1) once, we obtained

$$u'(x) = DI^{\alpha}_{\infty^-}(f(x, u(x), u'(x))).$$

Since $D^m_{\infty^-} I^{\alpha}_{\infty^-} = I^{\alpha-m}_{\infty^-}$, so

$$u'(x) = -\frac{1}{\Gamma(\alpha-1)} \int_x^{\infty} (s-x)^{\alpha-2} f(s, u(s), u'(s))\, ds.$$

Thus when limit $x \to \infty$, by (H_2) we have $u'(\infty) \to 0$. Therefore, second terminal condition is also satisfied. Hence u is solution of (4.7).
Conversely, assume $u \in C^1[a, \infty)$ is solution of (4.1), then applying $I^{\alpha}_{\infty^-}$on both sides of (4.1)

$$I^{\alpha}_{\infty^-} D^{\alpha}_{*\infty^-} y(x) = I^{\alpha}_{\infty^-} f(x, y(x), y'(x)).$$

By Theorem 1.4.6

$$\lim_{b\to\infty} \{u(x) - u(b) - (u'(b)(b-x)\} = \lim_{b\to\infty} \left\{\frac{1}{\Gamma(\alpha)} \int_x^b (s-x)^{\alpha-1} f(s, u(s), u'(s))\, ds\right\}. \tag{4.4}$$

Condition (4.2) restricts graph of u from crossing the $x-$axis. Therefore without loss of generality suppose that $u(x) \geq 0$, $u'(x) \leq 0$. Also $\mu \geq 0$. Thus, the monotone properties of f (as f satisfies H_1 and H_2) we can write $g(x) = u(x) - \mu$ therefore $g''(x) \geq 0$ and also when $x \to \infty$, $g \to 0$ also $g'(x) = u'(x)$, $xg'(x) = xu'(x)$ therefore as $x \to \infty$, $xu'(x) \to 0$. Thus $\lim_{x\to\infty} xu'(x) = 0$. So as $b \to \infty$, (4.4) becomes

$$\begin{aligned} u(x) &= \lim_{x\to\infty} [u(b) + u'(b)(x-b)] + \lim_{x\to\infty} \frac{1}{\Gamma(\alpha)} \int_x^b (s-x)^{\alpha-1} f(s, u(s), u'(s))\, ds, \\ &= \mu + \frac{1}{\Gamma(\alpha)} \int_x^{\infty} (s-x)^{\alpha-1} f(s, u(s), u'(s))\, ds. \end{aligned}$$

which completes the proof. □

Corollary 4.1.1. *Let f satisfies (H_1) and (H_2) and $lim_{x\to\infty} xu'(x) = 0$. Then u is solution of* (4.1) *if and only if u satisfies* (4.3).

Theorem 4.1.1. *Let $v, w \in C[\mathbb{R}, \mathbb{R}]$ such that $v(\infty), w(\infty)$ exists and f is continuous, then*

$$v(x) \leq v(\infty) + \frac{1}{\Gamma(\alpha)} \int_x^{\infty} (s-x)^{\alpha-1} f(s, v(s), v'(s))\, ds, \tag{4.5}$$

$$w(x) \geq w(\infty) + \frac{1}{\Gamma(\alpha)} \int_x^\infty (s-x)^{\alpha-1} f(s, w(s), w'(s))\, ds \tag{4.6}$$

and one of the inequality is strict. Suppose $v(\infty) < w(\infty)$, then $v(x) < w(x)$.

Proof. Suppose conclusion is not true. Then there exists $x_1 \in [0, \infty)$ such that $v(x_1) = w(x_1)$ and $v(x) < w(x)$ for $x \in [x_1, \infty)$. Suppose inequality (4.6) is strict. Thus

$$w(x_1) > w(\infty) + \frac{1}{\Gamma(\alpha)} \int_{x_1}^\infty (s-x_1)^{\alpha-1} f(s, w(s), w'(s))\, ds,$$

$$w(x_1) > v(\infty) + \frac{1}{\Gamma(\alpha)} \int_{x_1}^\infty (s-x_1)^{\alpha-1} f(s, v(s), v'(s))\, ds,$$

$$w(x_1) > v(x_1)$$

which is contradiction. Hence, $v(x) < w(x)\ \forall x$. Therefore $w(\infty) > v(\infty) \Rightarrow w(x) > v(x)$. □

Theorem 4.1.2. *Assume that function f satisfies (H_1) and (H_2). Let $w(x)$ be a solution of*

$$\begin{aligned} D^\alpha_{*\infty^-} w(x) &= f(x, w(x), w'(x)), \quad 1 < \alpha \leq 2, \quad x \in [a, \infty), \\ w(\infty) &= \tau, \quad w'(\infty) = 0, \qquad \tau \in \mathbb{R}, \end{aligned} \tag{4.7}$$

such that for all $x \in [a, \infty)$,

$$\tau w'(x) \leq 0. \tag{4.8}$$

Then there exist at least one solution of (4.1) for each μ between 0 and τ.

Proof. Clearly inequality (4.8) restricts graph of w from crossing the $x-$axis. Therefore we without loss of generality suppose that $w(x) \geq 0 \Rightarrow w'(x) \leq 0$ also $\tau \geq 0$. Now we define a recursive sequence of functions. For $\mu \in [0, \tau]$, let

$$u_n(x) = \begin{cases} \mu & \text{for } n = 1, x \in [a, \infty), \\ \mu + \frac{1}{\Gamma(\alpha)} \int_x^\infty (s-x)^{\alpha-1} f(s, u_{n-1}(s), u'_{n-1}(s))\, ds & \text{for } n \geq 2, x \in [a, \infty). \end{cases} \tag{4.9}$$

Using induction hypothesis, it is easy to show that, for $n \geq 2$ and for all $x \in [a, \infty)$

$$\mu \leq u_{n-1}(x) \leq u_n(x) \leq w(x),$$

and

$$0 \geq u'_{n-1}(x) \geq u'_n(x) \geq w'(x).$$

By using the hypothesis of (H_1) and (H_2). Thus $u_n(x)$ and $u'_n(x)$ approaches to its limit when $n \to \infty$. Differentiating (4.9) we get

$$u'_n(x) = -\frac{1}{\Gamma(\alpha-1)} \int_x^\infty (s-x)^{\alpha-2} f(s, u_{n-1}(s), u'_{n-1}(s))\, ds. \tag{4.10}$$

Now using the Monotone-Convergence theorem in (4.9), we get.

$$\begin{aligned}\lim_{n\to\infty} u_n(x) &= \mu + \frac{1}{\Gamma(\alpha)} \lim_{n\to\infty} \int_x^\infty (s-x)^{\alpha-1} f(s, u_{n-1}(s), u'_{n-1}(s))\, ds,\\ u(x) &= \mu + \frac{1}{\Gamma(\alpha)} \int_x^\infty (s-x)^{\alpha-1} \lim_{n\to\infty} f(s, u_{n-1}(s), u'_{n-1}(s))\, ds,\end{aligned}$$

$$u(x) = \mu + \frac{1}{\Gamma(\alpha)} \int_x^\infty (s-x)^{\alpha-1} f(s, u(s), u'(s))\, ds. \tag{4.11}$$

Similarly (4.10) becomes,

$$u'(x) = -\frac{1}{\Gamma(\alpha-1)} \int_x^\infty (s-x)^{\alpha-2} f(s, u(s), u'(s))\, ds. \tag{4.12}$$

Consequently, if we differentiate (4.11) once, it is clear that it will become (4.12). Since (4.11) and (4.7) are equivalent, therefore there exists at least one solution u of (4.1). □

Example 4.1.1.

Consider a terminal value problem,

$$\begin{aligned}&D^\alpha_{*\infty^-} u = \frac{\Gamma(1-\alpha)u^2}{(x^\alpha+1)^2} - \frac{u'x}{\alpha\Gamma(1-\alpha)}, \quad 1<\alpha\le 2, \quad x\in[0,\infty),\\ &u(\infty) = 1, \quad u'(\infty) = 0.\end{aligned} \tag{4.13}$$

Then (4.13) has a solution $u(x) = (x^\alpha + 1)x^{-\alpha}$ and

$$\begin{aligned}&D^\alpha_{*\infty^-} v = \frac{v'x}{\alpha\Gamma(1-\alpha)} - \frac{\Gamma(1-\alpha)v^2}{(x^\alpha+1)^2}, \quad 1<\alpha\le 2, \quad x\in[0,\infty),\\ &v(\infty) = -1, \quad v'(\infty) = 0.\end{aligned} \tag{4.14}$$

Then (4.14) has a solution $v(x) = -u(x)$.

$$\begin{aligned}&D^\alpha_{*\infty^-} w = \frac{\Gamma(1-\alpha)w^2}{(x^\alpha+1)^2} - \frac{w'x}{\alpha\Gamma(1-\alpha)}, \quad 1<\alpha\le 2, \quad x\in[0,\infty),\\ &w(\infty) = \mu, \quad w'(\infty) = 0.\end{aligned} \tag{4.15}$$

Hence, by Theorem 4.1.2, (4.15) has a solution for all $\mu \in [-1, 1]$ and the iteration methodology utilized as a part of proof of Theorem 4.1.2 offers a method for acquiring it.

4.2 Uniqueness of solution

Lemma 4.2.1. *Suppose f satisfies $(H_1)-(H_3)$ and $\frac{1}{\Gamma(\alpha-1)}\int_a^\infty sL(s)(s-a)^{\alpha-1}\,ds<\infty$. Let u and v be solutions of differential equation in* (4.1) *satisfying* $\lim_{x\to\infty}u(x)=\alpha$ *and* $\lim_{x\to\infty}v(x)=\beta$, *then* $\lim_{x\to\infty}[x(u'(x)-v'(x))]=0$.

Proof. Using the given condition, $\lim_{x\to\infty}f(x)=b,\quad |f(x)-b|<\epsilon$ for $x\geq b$, we have $\frac{1}{\Gamma(\alpha-1)}\int_x^\infty sL(s)(s-a)^{\alpha-1}\,ds<\epsilon, |u(x)-\beta|<\epsilon, |v(x)-\gamma|<\epsilon$, $|u'(x)|<\epsilon, |v'(x)|<\epsilon$. Thus as $u'(x)=-\frac{1}{\Gamma(\alpha-1)}\int_x^\infty (s-x)^{\alpha-2}f(s,u(s),u'(s))\,ds$ and $v'(x)=-\frac{1}{\Gamma(\alpha-1)}\int_x^\infty (s-x)^{\alpha-2}f(s,v(s),v'(s))\,ds$ then for $x\geq b$,

$$\begin{aligned}|x(u'(x)-v'(x))| &\leq \left|\frac{x}{\Gamma(\alpha-1)}\int_x^\infty (s-x)^{\alpha-2}[f(s,v(s),v'(s))-f(s,u(s),u'(s))]\,ds\right|\\ &\leq \frac{x}{\Gamma(\alpha-1)}\int_x^\infty (s-x)^{\alpha-2}|f(s,v(s),v'(s))-f(s,u(s),u'(s))|\,ds\Big|\\ &\leq \frac{x}{\Gamma(\alpha-1)}\int_x^\infty (s-x)^{\alpha-2}L(s)[|u(s)-v(s)|+|u'(s)-v'(s)|]\,ds.\end{aligned}$$

Therefore,

$$\begin{aligned}|x(u'(x)-v'(x))| &\leq \frac{(|\beta-\gamma|+4\epsilon)}{\Gamma(\alpha-1)}\int_x^b x(s-x)^{\alpha-2}L(s)\,ds\\ &\leq \frac{(|\beta-\gamma|+4\epsilon)}{\Gamma(\alpha-1)}\int_x^b s(s-a)^{\alpha-2}L(s)\,ds\\ &< (|\beta-\gamma|+4\epsilon)\epsilon.\end{aligned}$$

Which gives $\lim_{x\to\infty}[x(u'(x)-v'(x))]=0$. □

Define $\widetilde{K_3}:=\frac{1}{\Gamma(\alpha)}\int_a^\infty (s-a)^{\alpha-1}L(s)\,ds$ and $\widetilde{K_4}:=\frac{1}{\Gamma(\alpha-1)}\int_a^\infty (s-a)^{\alpha-2}L(s)\,ds$.

Theorem 4.2.1. *Suppose $(H_1)-(H_3)$ are satisfied and $\widetilde{K_3}+\widetilde{K_4}<1$. Then a function u satisfying* (4.2) *is a unique solution of the terminal value problem* (4.1) *on $[a,\infty)$.*

Proof. Let us define a set

$$\widetilde{U}:=\{u\in[a,\infty):u(\infty)\ \text{ exist }\ \text{ and }\ u'(\infty)=0\}$$

and norm $\|u\|=sup_{x\in[a,\infty)}|u(x)|+sup_{x\in[a,\infty)}|u'(x)|$. Then this norm makes our set $\widetilde{U}$ a subset of the Banach Space $\mathbb{X}$. We define an operator $\widetilde{\mathcal{A}}$ on $\mathbb{X}$ as

$$(\widetilde{\mathcal{A}}u)(x):=\mu+\frac{1}{\Gamma(\alpha)}\int_x^\infty (s-x)^{\alpha-1}f(s,u(s),u'(s))\,ds. \tag{4.16}$$

Using (4.16), the Voletrra integral equation (4.7) can be written as

$$u = \widetilde{\mathcal{A}}u.$$

First we show that $\widetilde{\mathcal{A}} : \widetilde{U} \to \widetilde{U}$.

$$\begin{aligned}
\Big|u(x) - \mu - w(x) + \tau\Big| &= \frac{1}{\Gamma(\alpha)}\Big|\int_x^\infty (s-x)^{\alpha-1}[f(s,u(s),u'(s)) - f(s,w(s),w'(s))]\,ds\Big| \\
&\le \frac{1}{\Gamma(\alpha)}\int_a^\infty (s-a)^{\alpha-1}|f(s,u(s),u'(s)) - f(s,w(s),w'(s))|\,ds \\
&\le \frac{1}{\Gamma(\alpha)}\int_a^\infty (s-a)^{\alpha-1}L(s)[|u(s)-w(s)| + |u'(s)-w'(s)|]\,ds \\
\Big|u(x) - \mu + \tau\Big| - |w(x)| &\le \frac{\|u-w\|}{\Gamma(\alpha)}\int_a^\infty (s-a)^{\alpha-1}L(s)\,ds \\
\Big|u(x) - \mu + \tau\Big| &\le |w(x)| + \frac{\|u-w\|}{\Gamma(\alpha)}\int_a^\infty (s-a)^{\alpha-1}L(s)\,ds.
\end{aligned}$$

Which implies that $u(\infty) = \mu$. Since $u'(x) = \frac{1}{\Gamma(\alpha-1)}\int_x^\infty (s-x)^{\alpha-2}f(s,u(s),u'(s))\,ds$ implies $u'(\infty) = 0$. Hence, operator $\widetilde{\mathcal{A}}$ maps set $\widetilde{U}$ to itself.
Now we have to show that $\widetilde{\mathcal{A}}$ is contraction mapping on $\widetilde{U}$. For any u , $v \in \widetilde{U}$,

$$\|\widetilde{\mathcal{A}}u(x) - \widetilde{\mathcal{A}}v(x)\| = \sup_{x\in[a,\infty)} |\widetilde{\mathcal{A}}u(x) - \widetilde{\mathcal{A}}v(x)| + \sup_{x\in[a,\infty)} |\mathcal{A}u'(x) - \mathcal{A}v'(x)| \tag{4.17}$$

$$\begin{aligned}
\sup_{x\in[a,\infty)} |\widetilde{\mathcal{A}}u(x) - \widetilde{\mathcal{A}}v(x)| &= \sup_{x\in[a,\infty)} \frac{1}{\Gamma(\alpha)}\Big|\int_x^\infty (s-x)^{\alpha-1}[f(s,u(s),u'(s)) - f(s,v(s),v'(s))]\,ds\Big| \\
&\le \frac{1}{\Gamma(\alpha)}\int_a^\infty (s-a)^{\alpha-1}[L(s)(|u-v| + |u'-v'|)]\,ds \\
&\le \frac{\|u-v\|}{\Gamma(\alpha)}\int_a^\infty (s-a)^{\alpha-1}L(s)\,ds \\
&\le \widetilde{K_3}\|u-v\|.
\end{aligned}$$

$$\begin{aligned}
\sup_{x\in[a,\infty)} |\widetilde{\mathcal{A}}u'(x) - \widetilde{\mathcal{A}}v'(x)| &= \sup_{x\in[a,\infty)} \frac{1}{\Gamma(\alpha-1)}\Big|\int_x^\infty (s-x)^{\alpha-2}[f(s,u(s),u'(s)) - f(s,v(s),v'(s))]\,ds\Big| \\
&\le \frac{1}{\Gamma(\alpha-1)}\int_a^\infty (s-a)^{\alpha-2}[L(s)(|u-v| + |u'-v'|)]\,ds \\
&\le \frac{\|u-v\|}{\Gamma(\alpha-1)}\int_a^\infty (s-a)^{\alpha-2}L(s)\,ds \\
&\le \widetilde{K_4}\|u-v\|.
\end{aligned}$$

Therefore (4.17) become,

$$\begin{aligned}\|\widetilde{\mathcal{A}}u(x) - \widetilde{\mathcal{A}}v(x)\| &= \sup_{x\in[a,\infty)} |\mathcal{A}u(x) - \mathcal{A}v(x)| + \sup_{x\in[a,\infty)} |\mathcal{A}u'(x) - \mathcal{A}v'(x)| \\ &\leq \widetilde{K_3}\|u-v\| + \widetilde{K_4}\|u-v\| \\ &\leq [\widetilde{K_3} + \widetilde{K_4}]\|u-v\|.\end{aligned}$$

Hence, $\widetilde{\mathcal{A}}$ is a contraction mapping. Thus by the Banach fixed point theorem, a unique fixed point of operator $\widetilde{\mathcal{A}}$ exists. Consequently, by Lemma 4.1.1 unique solution of integral equation (4.3) is also solution of (4.1). Now we prove that this is only solution of (4.1). Suppose there exists an other solution $v(x)$ of (4.1), then by Lemma 4.2.1, $\lim_{x\to\infty}(x(u'(x) - v'(x))) = 0$, which implies $lim_{x\to\infty} xu'(x) = lim_{x\to\infty} xv'(x)$. Using (4.2)we have $0 = lim_{x\to\infty} xv'(x)$. Thus Corollary 4.1.1, $v(x)$ satisfies (4.2). Hence $v(x) = u(x)$. □

Example 4.2.1. *Consider a terminal value problem,*

$$\begin{aligned}&D^{\alpha}_{*\infty^-}u = \frac{\sqrt{x}\exp(-x^2)}{20}\left(\frac{u}{1+|u|} + \frac{u'}{1+|u'|}\right), \quad 1<\alpha\leq 2, \quad x\in[0,\infty), \\ &u(\infty) = u_0, \quad u'(\infty) = 0.\end{aligned} \tag{4.18}$$

Here $\alpha = 3/2$ and

$$f(x,u,u') := \frac{\sqrt{x}\exp(-x^2)}{20}\left(\frac{u}{1+|u|} + \frac{u'}{1+|u'|}\right).$$

It is obvious that function $f(x,u,u')$ is continuous. Now

$$\begin{aligned}\Big|f(x,u_1,u_1') - f(x,u_2,u_2')\Big| &= \Big|\frac{u_1\sqrt{x}\exp(-x^2)}{20(1+|u_1|)} + \frac{u_1'\sqrt{x}\exp(-x^2)}{20(1+|u_1'|)} - \frac{u_2\sqrt{x}\exp^{-x^2}}{20(1+|u_2|)} \\ &\quad - \frac{u_2'\sqrt{x}\exp^{-x^2}}{20(1+|u_2'|)}\Big| \\ &\leq \frac{\sqrt{x}\exp(-x^2)}{20}\Big[\Big|\frac{u_1}{1+|u_1|} - \frac{u_2}{1+|u_2|}\Big| + \Big|\frac{u_1'}{1+|u_1'|} - \frac{u_2'}{1+|u_2'|}\Big|\Big] \\ &\leq \frac{\sqrt{x}\exp(-x^2)}{20}\Big[\frac{|u_1-u_2|}{(|1+|u_1|)(|1+|u_2|)} + \frac{|u_1'-u_2'|}{(|1+|u_1'|)(1+|u_2'|)|}\Big].\end{aligned}$$

Since $\frac{1}{(1+|u_1|)(1+|u_2|)} < 1$, therefore

$$\Big|f(x,u_1,u_1') - f(x,u_2,u_2')\Big| \leq L(x)[|u_1-u_2| + |u_1'-u_2'|]$$

where $L(x) = \frac{\sqrt{x}\exp(-x^2)}{20}$. Hence $f(x, u, u')$ satisfies (H_3). Also,

$$\begin{aligned}
\widetilde{K_3} + \widetilde{K_4} &= \frac{1}{\Gamma(\alpha)} \int_a^{\infty} [(s-a)^{\alpha-1} + (s-a)^{\alpha-2}] L(s)\, ds \\
&= \frac{1}{\Gamma(3/2)} \int_0^{\infty} [(s)^{1/2} + (s)^{-1/2}] L(s)\, ds \\
&= \frac{1}{20\Gamma(3/2)} \int_0^{\infty} [s\exp(-s^2) + \exp(-s^2)]\, ds \\
&= \frac{1}{20\Gamma(3/2)} \Big[\frac{1}{2} + \frac{3\sqrt{\pi}}{4}\Big] = 0.103209 < 1.
\end{aligned}$$

Therefore, by Theorem 4.2.1, there exists a unique solution on $[0,\infty)$.

Chapter 5

Conclusions

In the first chapter, we introduced some spacial functions that were required for the development of our results. Some fundamental concepts and definitions from fractional calculus including some basic results about the Riemann–Liouville approach and the Caputo's approach are provided. In the remaining sections of the Chapter 1 applications of fractional calculus and some fixed point theorems are stated.
In Chapter 2, some known results of existence, uniqueness and stability of some classes of initial value problems for fractional differential equations have been reviewed. The impact of perturbed data and stability of change of location of starting point of initial value problem for fractional differential equations is also reviewed.
In Chapter 3, Well posedness of terminal value problem for fractional differential equations have also been discussed. Terminal value problems had been widely used in modelling of spreading of epidemics, distributions of pollutants in ground water etc. In the third section of Chapter 3, we established results for existence and uniqueness of terminal problem on the finite interval with the help of the Schauder's fixed-point theorem, the Schaefer's fixed-point theorem and the Banach fixed-point theorem. Some examples are also added for the applicability of our results.
We established sufficient conditions for existence and uniqueness of solutions of terminal value problems for fractional differential equations on infinite interval. By this, we can say something about under what conditions can we be sure that a solutions of terminal value problems for fractional differential equations on infinite interval to exist and one can check under what conditions, this solution is unique.
In future, the stability results of (3.21) and (4.1) have to establish. Furthermore, sufficient conditions for existence and uniqueness of solutions of some classes of boundary value problems for fractional differential equations on infinite interval will also establish.

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