Discrete Symmetry Analysis of Some Partial Differential Equations and Related Exact Solutions

A thesis presented for the degree of
MS Mathematics

Submitted by: Syed Tahirul Husnain
Supervisor: Dr. Tooba Feroze

Department of Mathematics
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Islamabad, Pakistan
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Contents

Preamble

There are many ingenious techniques for solving differential equations. Surprisingly, many of these techniques emanate from a unified theory of continuous symmetries of differential equations. These numerous techniques are actually specific cases of symmetry methods for solving differential equations. Symmetry methods for differential equations were first studied by Marius Sophus Lie (17 December 1842 - 18 February 1899). He was a Norwegian mathematician. He largely created the theory of continuous symmetry, and applied it to the study of geometry and differential equations. His primary inspiration was Galois's theory. In nineteenth century, Evariste Galois (25 October 1811 - 31 May 1832) had used group theory to solve algebraic (polynomial) equations that were quadratic, cubic, and quartic. Lie initiated his program on the basis of analogy. If finite groups were required to decide on the solvability of finite-degree polynomial equations, then infinite groups, groups depending continuously on one or more real or complex parameters, would probably be involved in the treatment of ordinary and partial differential equations.

It turned out that the continuous groups which Lie sought were the groups of symmetries of the differential equations, which depended continuously on one or more real or complex parameters. Such groups of symmetries were later called Lie groups. Lie developed conditions for finding such symmetries. Symmetries of differential equations which are outside such Lie groups are called discrete symmetries. There are many applications of discrete symmetries of differential equations, some major applications are discussed in [1-4].

Many techniques have been developed for finding discrete symmetries, but often, either the symmetry condition is too difficult to solve, that is, the resulting system of determining equations is very difficult to solve, or the technique does not give all the discrete symmetries of the differential equation. Peter H. Hydon has developed an indirect method, for constructing discrete symmetries of differential equations, having a finite dimensional Lie algebra of infinitesimal generators of its (one-parameter) Lie groups of point symmetries [1-3, 5-8] His method is based on the observation that every point symmetry yields an automorphism of the Lie algebra of the Lie point symmetry generators. The method not only results in an easier system of determining equations, uncoupled, but it is also exhaustive, that is, all discrete symmetries are obtained.

My work concerns reviewing Peter H. Hydon's method, and calculating all discrete point symmetries of two partial differential equations using his method. As an immediate application of discrete symmetries, group invariant solutions of the equations, corresponding to the basis vectors of the Lie algebra, will be investigated for further solutions under transformations due to the discrete symmetries.

Chapter 1

Lie Point Symmetries of Differential Equations

The purpose of this chapter is to concisely review some fundamental concepts related to Lie point symmetries of differential equations. Basic definitions and notations are presented. All the theorems are stated without proofs. Appropriate references are given for detail. Our goal is to be able to find Lie groups of point symmetries of differential equations. Some prerequisite definitions and concepts are discussed in the appendix.

1.1 One-Parameter Lie Groups of Point Transformations and their Infinitesimal Generators

We start this section with the definition of an *r-parameter Lie group* [9].

Definition 1.1.1

*An **r-parameter Lie group** is a group G which also carries the structure of an r-dimensional smooth manifold in such a way, that both, the group operation*

$$m : G \times G \longmapsto G\ , \quad m(g,h) = g.h\ , \quad g,h \in G\ ,$$

and the inversion

$$i : G \longmapsto G\ , \quad i(g) = g^{-1}\ , \quad g \in G\ ,$$

are smooth maps between manifolds.

For applications of Lie groups to differential equations we consider the transformation of coordinates. Such transformations form a *Lie transformation group* [10].

Definition 1.1.2

Let M be a sooth manifold and G be a Lie group. The group G is called a ***Lie transformation group*** *of M if there is a smooth map*

$$\phi : G \times M \longmapsto M\ , \quad \phi(g, z) = gz\ ,$$

such that the following conditions are satisfied

(1) $(g_1.g_2)\, z = g_1(g_2 z)$ *for all $z \in M$ and $g_1, g_2 \in G$,*
(2) $ez = z$ *for all $z \in M$ where e is the identity element of G.*

Now we discuss a specific Lie transformation group called **one-parameter Lie transformation group** [11]. Consider

$$\hat{\mathbf{z}} = \boldsymbol{\varphi}(\mathbf{z}, \epsilon)\ , \tag{1.1}$$

where $\boldsymbol{\varphi} = (\varphi_1, \varphi_2, \cdots, \varphi_n)$. $\hat{\mathbf{z}} = (\hat{z}^1, \hat{z}^2, \cdots, \hat{z}^n)$ and $\mathbf{z} = (z^1, z^2, \cdots, z^n)$ belong to an open domain $D \subset \mathbb{R}^n$. The group operation is $\phi(\epsilon, \delta)$ where $\epsilon, \delta \in I \subset \mathbb{R}$ are the group parameters. If $\hat{\mathbf{z}} = \boldsymbol{\varphi}(\mathbf{z}, \epsilon)$ and $\hat{\hat{\mathbf{z}}} = \boldsymbol{\varphi}(\hat{\mathbf{z}}, \delta)$ then

$$\hat{\hat{\mathbf{z}}} = \boldsymbol{\varphi}(\mathbf{z}, \phi(\epsilon, \delta))\ , \tag{1.2}$$

where $\hat{\mathbf{z}}$ and $\hat{\hat{\mathbf{z}}}$ lie in D.

In its **standard form**, such a group can be re-parameterized such that the group operation is given by

$$\phi(\epsilon, \delta) = \epsilon + \delta\ , \tag{1.3}$$

the identity of the transformation group is $\epsilon = 0$ and the inverse is $-\epsilon$.

Writing eq.(1.1) in its Taylor series at $\epsilon = 0$, we obtain

$$\hat{\mathbf{z}} = \mathbf{z} + \epsilon \, \frac{\partial \boldsymbol{\varphi}}{\partial \epsilon}(\mathbf{z}, \epsilon)\Big|_{\epsilon=0} + \mathcal{O}(\epsilon^2) \; . \tag{1.4}$$

From this we define

$$\boldsymbol{\xi}(\mathbf{z}) = \frac{\partial \boldsymbol{\varphi}}{\partial \epsilon}(\mathbf{z}, \epsilon)\Big|_{\epsilon=0} \; . \tag{1.5}$$

This definition is used in the following theorem, known as **Lie's first fundamental theorem** [11]. This theorem provides us with an algorithmic method to re-parameterize a one-parameter transformation group such that it is of the standard form.

Theorem 1.1.3

There exists a parameterization $\tau(\epsilon)$ such that the Lie group of transformations eq.(1.1) *is equivalent to the solution of the initial value problem for the autonomous system of first order ordinary differential equations*

$$\frac{d\hat{\mathbf{z}}}{d\tau} = \boldsymbol{\xi}(\hat{\mathbf{z}}) \; , \tag{1.6}$$

with $\hat{\mathbf{z}} = \mathbf{z}$ when $\tau = 0$. In particular

$$\tau(\epsilon) = \int_0^{\epsilon} \beta(\epsilon') \, d\epsilon' \; , \tag{1.7}$$

where

$$\beta(\epsilon) = \frac{\partial \phi}{\partial b}(a, b)\Big|_{(a,b)=(\epsilon,\epsilon^{-1})} \; , \quad \beta(0) = 1 \; . \tag{1.8}$$

In terms of ϵ the one-parameter group is given by the solution of the initial value problem

$$\frac{d\hat{\mathbf{z}}}{d\epsilon} = \beta(\epsilon) \, \boldsymbol{\xi}(\hat{\mathbf{z}}) \; . \tag{1.9}$$

We now introduce a representation of a one-parameter Lie group of transformations by way of a group *generator* [10, 11].

Definition 1.1.4

*The **infinitesimal generator** of the one-parameter Lie group of transformations is defined by the linear differential operator*

$$\mathbf{X} = \boldsymbol{\xi}(\mathbf{z}).\nabla = \xi_i(\mathbf{z})\frac{\partial}{\partial z^i} \,, \tag{1.10}$$

where

$$\boldsymbol{\xi}(\mathbf{z}) = (\xi_1(\mathbf{z}), \xi_2(\mathbf{z}), \cdots, \xi_n(\mathbf{z})) \,,$$

and ∇ is the gradient operator.

The following theorem [10, 11] shows how can we write the one-parameter Lie group of transformations in terms of this generator.

Theorem 1.1.5

The one-parameter Lie group of transformations eq.(1.1) *can be written as*

$$\hat{\mathbf{z}} = \boldsymbol{\varphi}(\mathbf{z}, \epsilon) = e^{\epsilon\mathbf{X}}\mathbf{z} = \mathbf{z} + \epsilon\,\mathbf{X}\mathbf{z} + \frac{\epsilon^2}{2!}\,\mathbf{X}^2\mathbf{z} + \mathcal{O}(\epsilon^3) \tag{1.11}$$

$$= \sum_{k=0}^{\infty}\frac{\epsilon^k}{k!}\,\mathbf{X}^k\mathbf{z} \,, \tag{1.12}$$

where the linear operator $\mathbf{X}$ is defined by eq.(1.10) *and $\mathbf{X}^k = \mathbf{X}\mathbf{X}^{k-1}$.*

This theorem can be generalized [10, 11] to any analytic function as following.

Theorem 1.1.6

Let f be an analytic function. For a one-parameter Lie group of transformations eq.(1.1) *with infinitesimal generator* eq.(1.10) *it follows that*

$$f(\hat{\mathbf{z}}) = f(e^{\epsilon\mathbf{X}}\mathbf{z}) = e^{\epsilon\mathbf{X}}f(\mathbf{z}) \,. \tag{1.13}$$

1.2 Multi-Parameter Lie Groups of Point Transformations and their Infinitesimal Generators

We now generalize one-parameter Lie group of transformations to ***r*-parameter Lie group of transformations**. Consider

$$\hat{\mathbf{z}} = \boldsymbol{\varphi}(\mathbf{z}, \boldsymbol{\epsilon}) \; . \tag{1.14}$$

As before, $\boldsymbol{\varphi} = (\varphi_1, \varphi_2, \cdots, \varphi_n)$. $\hat{\mathbf{z}} = (\hat{z}^1, \hat{z}^2, \cdots, \hat{z}^n)$ and $\mathbf{z} = (z^1, z^2, \cdots, z^n)$ belong to an open domain $D \subset \mathbb{R}^n$, but $\boldsymbol{\epsilon} = (\epsilon_1, \epsilon_2, \cdots, \epsilon_n) \in I \subset \mathbb{R}^n$. The group operation is given by $\boldsymbol{\phi}(\boldsymbol{\epsilon}, \boldsymbol{\delta})$.

Such groups also have an analogous version of *Lie's first fundamental theorem* [10, 11].

The vector $\boldsymbol{\xi}(\mathbf{z})$ becomes a matrix $\xi_{\alpha i}(\mathbf{x})$ in case of r-parameter group, where $\alpha = 1, 2, \cdots, r$ and $i = 1, 2, \cdots, n$.

Now we define the infinitesimal generators of an r-parameter Lie group of transformations [10, 11].

Definition 1.2.1

The infinitesimal generator $\mathbf{X}_\alpha$, *corresponding to the parameter* ϵ_α *of the* r*-parameter Lie group of transformations* eq.(1.14) *is*

$$\mathbf{X}_\alpha = \xi_{\alpha i}(\mathbf{z}) \frac{\partial}{\partial z^i} \; , \quad \alpha = 1, 2, \cdots, r \; . \tag{1.15}$$

The r-parameter Lie group of transformations can be written as

$$\hat{\mathbf{z}} = \boldsymbol{\varphi}(\mathbf{z}, \boldsymbol{\epsilon}) = \prod_{\alpha=1}^{r} \exp(\epsilon_\alpha \mathbf{X}_\alpha) \, \mathbf{z} \; . \tag{1.16}$$

1.3 How to Find Lie Point Symmetries of an Ordinary Differential Equation

In this section we discuss how to find Lie point symmetries of an ordinary differential equation (ODE). Similar theory exists for partial differential equations (PDE) also, its detail can be read from [12], for instance.

1.3.1 Lie Point Transformations and their Prolongations

Since there is one dependent and one independent variable in an ODE, eq.(1.1) becomes

$$\hat{x} = \hat{x}(x, y, \epsilon) \ , \quad \hat{y} = \hat{y}(x, y, \epsilon) \ . \tag{1.17}$$

Likewise eq.(1.5) becomes

$$\xi(x, y) = \frac{\partial \hat{x}}{\partial \epsilon}(x, y, \epsilon)\Big|_{\epsilon=0} \ , \quad \eta(x, y) = \frac{\partial \hat{y}}{\partial \epsilon}(x, y, \epsilon)\Big|_{\epsilon=0} \ . \tag{1.18}$$

Henceforth, the entire group eq.(1.17) will be denoted by $\Gamma(\epsilon)$ and a particular member of this group will be denoted by Γ_ϵ. That is

$$\Gamma_\epsilon : (x, y) \longmapsto (\hat{x}(x, y, \epsilon), \hat{y}(x, y, \epsilon)). \tag{1.19}$$

If we want to apply eq.(1.17) to an ODE

$$H(x, y, y', \ \ldots, y^{(n)}) = 0 \ , \tag{1.20}$$

we must prolong the point transformation eq.(1.17) to the derivatives $y^{(k)}$, $k = 1, 2, ..., n$. We calculate $\hat{y}^{(k)}$ recursively as

$$\hat{y}^{(k)} \equiv \frac{d^k \hat{y}}{d\hat{x}^k} = \frac{d\hat{y}^{(k-1)}}{d\hat{x}} = \frac{D_x \hat{y}^{(k-1)}}{D_x \hat{x}} \ , \quad \hat{y}^{(0)} \equiv \hat{y} \ , \tag{1.21}$$

where D_x is the total derivative with respect to x

$$D_x = \frac{\partial}{\partial x} + y' \frac{\partial}{\partial y} + y'' \frac{\partial}{\partial y'} + \ \cdots \ . \tag{1.22}$$

Corresponding to eq.(1.4) we also have

$$
\begin{aligned}
\hat{x} &= x + \epsilon\, \xi(x,y) + O(\epsilon^2), \\
\hat{y} &= y + \epsilon\, \eta(x,y) + O(\epsilon^2), \\
\hat{y'} &= y' + \epsilon\, \eta'(x,y,y') + O(\epsilon^2), \\
&\vdots \\
\hat{y}^{(n)} &= y^{(n)} + \epsilon\, \eta^{(n)}(x,y,y',\, ...\, , y^{(n)}) + O(\epsilon^2),
\end{aligned}
\tag{1.23}
$$

where $\eta', \eta'', \cdots, \eta^{(n)}$ are defined by

$$
\eta' = \frac{\partial \hat{y}'}{\partial \epsilon}\Big|_{\epsilon=0}, \eta'' = \frac{\partial \hat{y}''}{\partial \epsilon}\Big|_{\epsilon=0}, \cdots, \eta^{(n)} = \frac{\partial \hat{y}^{n}}{\partial \epsilon}\Big|_{\epsilon=0}, \tag{1.24}
$$

and can be computed [12], using

$$
\eta^{(k)} = D_x \eta^{(k-1)} - y^{(k)} D_x \xi\,, \quad \eta^{(0)} \equiv \eta. \tag{1.25}
$$

For example

$$
\eta' = \eta_x + (\eta_y - \xi_x) y' - \xi_y {y'}^2\,, \tag{1.26}
$$

$$
\begin{aligned}
\eta'' &= \eta_{xx} + (2\eta_{xy} - \xi_{xx}) y' - (\eta_{yy} - 2\xi_{xy}) {y'}^2 \\
&\quad - \xi_{yy} {y'}^3 + (\eta_y - 2\xi_x - 3\xi_y y') y''\,.
\end{aligned}
\tag{1.27}
$$

Now we seek prolongation of $\mathbf{X}$, the infinitesimal generator of eq.(1.17), such that eq.(1.23) can be written as

$$
\begin{aligned}
\hat{x} &= x + \epsilon\, \mathbf{X}x + O(\epsilon^2), \\
\hat{y} &= y + \epsilon\, \mathbf{X}y + O(\epsilon^2), \\
\hat{y'} &= y' + \epsilon\, \mathbf{X}y' + O(\epsilon^2), \\
&\vdots \\
\hat{y}^{(n)} &= y^{(n)} + \epsilon\, \mathbf{X}y^{(n)} + O(\epsilon^2).
\end{aligned}
\tag{1.28}
$$

This prolonged generator [12], is given by

$$
\mathbf{X}^{(n)} = \xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y} + \eta' \frac{\partial}{\partial y'} + \cdots + \eta^{(n)} \frac{\partial}{\partial y^{(n)}}\,. \tag{1.29}
$$

When there is no ambiguity $\mathbf{X}^{(n)}$ is written as $\mathbf{X}$.

1.3.2 Symmetry Condition

We are now equipped with all the mathematical machinery to state the *symmetry condition* [12] for an ODE and use it to find its Lie point symmetries.

Theorem 1.3.1

An ODE

$$H(x, y, y', \dots, y^{(n)}) = 0 \, ,$$

admits a group of symmetries with generator $\mathbf{X}$ *if and only if*

$$\mathbf{X}H = 0,$$

holds.

Example 1.3.2

As an example [12] we now calculate the symmetries of the second order ODE $y'' = 0$. Here $H(x, y, y', y'') = y''$. We first prolong $\mathbf{X}$ to $\mathbf{X}^{(2)}$. That is

$$\mathbf{X} = \xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y} + \eta' \frac{\partial}{\partial y'} + \eta'' \frac{\partial}{\partial y''} \, . \tag{1.30}$$

Applying $\mathbf{X}$ to H we obtain $\mathbf{X}H = \eta''$. Therefore, the symmetry condition is

$$\begin{aligned} &\eta'' = 0, \\ &\eta'' = \eta_{xx} + (2\eta_{xy} - \xi_{xx})y' - (\eta_{yy} - 2\xi_{xy})y'^2 \\ &\quad - \xi_{yy}y'^3 + (\eta_y - 2\xi_x - 3\xi_y y')y'' = 0 \, . \end{aligned} \tag{1.31}$$

This leads to the system of PDEs

$$\begin{aligned} \eta_{xx} &= 0, \\ 2\eta_{xy} - \xi_{xx} &= 0, \\ \eta_{yy} - 2\xi_{xy} &= 0, \\ \xi_{yy} &= 0. \end{aligned} \tag{1.32}$$

Solving the system we obtain

$$\begin{aligned} \xi(x, y) &= a_1 + a_2 x + a_3 y + a_4 xy + a_5 x^2, \\ \eta(x, y) &= a_6 + a_7 x + a_8 y + a_5 xy + a_4 y^2. \end{aligned} \tag{1.33}$$

Here $a_i,\ i = 1, \cdots, 8$ are arbitrary constants.

So the most general one-parameter Lie group of point symmetries of the ODE, $y'' = 0$ is obtained by the infinitesimal generator

$$\mathbf{X} = (a_1 + a_2 x + a_3 y + a_4 xy + a_5 x^2)\frac{\partial}{\partial x} + (a_6 + a_7 x + a_8 y + a_5 xy + a_4 y^2)\frac{\partial}{\partial y}. \quad (1.34)$$

1.3.3 Lie Algebra of Infinitesimal Generators

We define the algebraic structure *Lie algebra* [10] before we say more about eq.(1.34).

Definition 1.3.3

A **Lie algebra** $\mathcal{L}$ *is a vector space over a field* F *on which a product* [,] *called the* Lie *bracket or commutator is defined with the properties*

(1) $[\mathbf{X}, \mathbf{Y}] \in \mathcal{L}$ *for all* $\mathbf{X}, \mathbf{Y} \in \mathcal{L}$.

(2) $[\mathbf{X}, \alpha\mathbf{Y} + \beta\mathbf{Z}] = [\mathbf{X}, \alpha\mathbf{Y}] + [\mathbf{X}, \beta\mathbf{Z}]$ *for all* $\mathbf{X}, \mathbf{Y}, \mathbf{Z} \in \mathcal{L}$ *and for all* $\alpha, \beta \in F$.

(3) $[\mathbf{X}, \mathbf{Y}] = -[\mathbf{Y}, \mathbf{X}]$ *for all* $\mathbf{X}, \mathbf{Y} \in \mathcal{L}$.

(4) $[\mathbf{X}, [\mathbf{Y}, \mathbf{Z}]] + [\mathbf{Z}, [\mathbf{X}, \mathbf{Y}]] + [\mathbf{Y}, [\mathbf{Z}, \mathbf{X}]] = 0$ *for all* $\mathbf{X}, \mathbf{Y}, \mathbf{Z} \in \mathcal{L}$.

From (3) it follows that $[\mathbf{X}, \mathbf{X}] = 0$. A Lie algebra is a called **abelian** if $[\mathbf{X}, \mathbf{Y}] = 0$ for all $\mathbf{X}, \mathbf{Y} \in \mathcal{L}$.

Infinitesimal generators form a Lie algebra, with the commutator defined as

$$[\mathbf{X}, \mathbf{Y}] = \mathbf{XY} - \mathbf{YX}. \quad (1.35)$$

For example the generators eq.(1.34) form a Lie algebra. The basis for this

Lie algebra are, $\mathbf{X}_i$, infinitesimal generators corresponding to $a_i = 1$ and $a_{j\neq i} = 0$. Namely

$$\begin{aligned}
\mathbf{X}_1 &= \frac{\partial}{\partial x}, & \mathbf{X}_5 &= x^2\frac{\partial}{\partial x} + xy\frac{\partial}{\partial y}, \\
\mathbf{X}_2 &= x\frac{\partial}{\partial x}, & \mathbf{X}_6 &= \frac{\partial}{\partial y}, \\
\mathbf{X}_3 &= y\frac{\partial}{\partial x}, & \mathbf{X}_7 &= x\frac{\partial}{\partial y}, \\
\mathbf{X}_4 &= xy\frac{\partial}{\partial x} + y^2\frac{\partial}{\partial y}, & \mathbf{X}_8 &= y\frac{\partial}{\partial y}.
\end{aligned} \tag{1.36}$$

Since $[\mathbf{X}_i, \mathbf{X}_j] \in \mathcal{L}$ we express these commutators as a linear combination of the basis elements. That is $\kappa^i \mathbf{X}_i$. We give all κ^i a special name, **structure constants** and also give them a special notation

$$[\mathbf{X}_i, \mathbf{X}_j] = c_{ij}^k \mathbf{X}_k, \tag{1.37}$$

For example in eq.(1.36) $[\mathbf{X}_1, \mathbf{X}_5] = 2\mathbf{X}_2 + \mathbf{X}_8$, so $c_{15}^2 = 2$ and $c_{15}^8 = 1$.

Axiom (3) of Definition (1.3.3) gives $c_{ij}^k = -c_{ji}^k$.

We conclude this section with listing all the nonzero structure constants for eq.(1.36). They completely determine the Lie algebra of eq.(1.36).

$$\begin{array}{llllll}
c_{12}^1 = 1, & c_{21}^1 = -1, & c_{27}^7 = 1, & c_{72}^7 = -1, & c_{46}^8 = -2, & c_{64}^8 = 2, \\
c_{14}^3 = 1, & c_{41}^3 = -1, & c_{35}^4 = 1, & c_{53}^4 = -1, & c_{47}^5 = -1, & c_{74}^5 = 1, \\
c_{15}^2 = 2, & c_{51}^2 = -2, & c_{36}^1 = -1, & c_{63}^1 = 1, & c_{48}^4 = -1, & c_{84}^4 = 1, \\
c_{15}^8 = 1, & c_{51}^8 = -1, & c_{37}^8 = 1, & c_{73}^8 = -1, & c_{56}^7 = -1, & c_{65}^7 = 1, \\
c_{17}^6 = 1, & c_{71}^6 = -1, & c_{37}^2 = -1, & c_{73}^2 = 1, & c_{68}^6 = 1, & c_{86}^6 = -1, \\
c_{23}^3 = -1, & c_{32}^3 = 1, & c_{38}^3 = -1, & c_{83}^3 = 1, & c_{78}^7 = 1, & c_{87}^7 = -1, \\
c_{25}^5 = 1, & c_{52}^5 = -1, & c_{46}^2 = -1, & c_{64}^2 = 1. & &
\end{array}$$

Chapter 2

Discrete Symmetries Analysis of Differential Equations

Many techniques have been developed for finding discrete symmetries of differential equations, but often, either the symmetry condition is too difficult to solve, that is, the resulting system of determining equations is very difficult to solve or the technique does not give all the discrete symmetries of the differential equation. Peter H. Hydon has developed an indirect method for constructing discrete symmetries of differential equations, [1-3, 5-8] having a finite dimensional Lie algebra of infinitesimal generators of its (one-parameter) Lie groups of point symmetries. This chapter describes Peter H. Hydon's method. The method not only results in an easier system of determining equations; uncoupled, but it is also exhaustive, that is, all discrete symmetries are obtained.

Only the method for ODEs will be explained. The method for PDEs is almost the same. Explaining the two methods simultaneously requires the use of generalized variables which would make understanding difficult. Appropriate additional detail will be given for PDEs where required.

2.1 Core Theory

This section presents the core theory, that is, definitions, main theorems, their proofs and some exmaples for the elaboration of the theory.

We start the section with the definition of a *discrete symmetry* [1-3].

Definition 2.1.1

*A point symmetry of a differential equation which does not belong to any (one-parameter) Lie group of point symmetries of the differential equation is called a **discrete symmetry**.*

In other words, discrete point symmetries are outside the r-parameter Lie group of point symmetries. They may still be related by continuous parameters, but they can never form a Lie group. One simple reason is that, these point symmetries do not have the identity symmetry among them, which is obviously inside the Lie group. Corresponding to discrete symmetry, the point symmetries inside the r-parameter Lie group are also called continuous point symmetries.

Example 2.1.2

As an example [1] consider the Chazy equation

$$y''' = 2yy'' - 3(y')^2 + \lambda(6y' - y^2)^2. \tag{2.1}$$

Three of its discrete symmetries are

$$(\hat{x}, \hat{y}) \in \Big\{(-x, -y), (-\frac{1}{x}, x^2y + 6x), (\frac{1}{x}, -x^2y - 6x)\Big\}. \tag{2.2}$$

Now we recall some notation and definitions from chapter 1. Consider an ODE

$$y^{(n)} = \omega(x, y, y', ..., y^{(n-1)}), \quad n \geq 2. \tag{2.3}$$

A point symmetry of eq.(2.3), continuous or discrete, will be denoted as

$$\Gamma : (x, y) \longmapsto (\hat{x}(x, y), \hat{y}(x, y)), \tag{2.4}$$

$$\hat{x} = \hat{x}(x, y), \quad \hat{y} = \hat{y}(x, y). \tag{2.5}$$

If such a point symmetry belongs to a (one-parameter) Lie group of point symmetries $\Gamma(\epsilon)$, then the infinitesimal generator of the group is

$$\mathbf{X} = \xi(x, y)\frac{\partial}{\partial x} + \eta(x, y)\frac{\partial}{\partial y}. \tag{2.6}$$

As stated in the beginning of this chapter, eq.(2.3) is required to have a finite dimensional Lie algebra, $\mathcal{L}$, of infinitesimal generators of its (one-parameter) Lie groups of point symmetries. If the dimension of $\mathcal{L}$ is R then the basis elements of $\mathcal{L}$ are denoted by $\mathbf{X}_i, i = 1, \cdots, R$.

For emphasis Theorem (1.1.6) is being re-stated here specialized to two variables, (x, y).

Theorem 2.1.3

If $F(x, y)$ is an infinitely differentiable function, then for a (one-parameter) Lie group of point symmetries $\Gamma(\epsilon)$ with infinitesimal generator eq.(2.6), *we have*

$$F(\hat{x}, \hat{y}) = F(e^{\epsilon\mathbf{X}}x, e^{\epsilon\mathbf{X}}y) = e^{\epsilon\mathbf{X}}F(x, y) = \Gamma F(x, y).$$

The following definition [1] is motivated by the above theorem.

Definition 2.1.4

Action of a point symmetry, Γ, *continuous or discrete, on an infinitely differentiable function F, is defined as*

$$\Gamma F(x, y) = F(\hat{x}, \hat{y}).$$

Before proceeding further we list some properties of the action of a point symmetry on an infinitely differentiable function.

$$\begin{aligned} \Gamma\big(cF(x, y)\big) &= c\Gamma F(x, y), \\ \Gamma\big(F(x, y) + G(x, y)\big) &= \Gamma F(x, y) + \Gamma G(x, y), \\ \Gamma^{-1}F(\hat{x}, \hat{y}) &= F(x, y), \\ \Gamma\Gamma^{-1}F(\hat{x}, \hat{y}) &= F(\hat{x}, \hat{y}), \\ \Gamma^{-1}\Gamma F(x, y) &= F(x, y), \\ \Gamma x &= \hat{x}, \\ \Gamma y &= \hat{y}, \\ \Gamma^{-1}\hat{x} &= x, \\ \Gamma^{-1}\hat{y} &= y. \end{aligned} \tag{2.7}$$

Now we state the two main theorems [1-3] on which the entire theory of the method is based.

Theorem 2.1.5

Let $\mathcal{L}$ be the Lie algebra of infinitesimal generators of (one-parameter) Lie groups of point symmetries of a differential equation. Let $\mathbf{X} \in \mathcal{L}$ and $\Gamma : (x, y) \longmapsto (\hat{x}(x, y), \hat{y}(x, y))$ be any symmetry, continuous or discrete, of the differential equation. Also, let $\Gamma(\delta)$ be the Lie group of point symmetries generated by $\mathbf{X}$. Then, $\Gamma\Gamma(\delta)\Gamma^{-1}$ is also a Lie group of point symmetries of the differential equation and its generator is $\Gamma\mathbf{X}\Gamma^{-1} \in \mathcal{L}$.

Theorem 2.1.6

For an arbitrary Γ, continuous or discrete, if $\{\mathbf{X}_i\}_{i=1}^{R}$ is a basis of $\mathcal{L}$ then $\{\Gamma\mathbf{X}_i\Gamma^{-1}\}_{i=1}^{R}$ is a basis of $\mathcal{L}$ as well.

The following example [1] will help us understand the preceding two theorems.

Example 2.1.7

Consider the Chazy equation again,

$$y''' = 2yy'' - 3(y')^2 + \lambda(6y' - y^2)^2. \tag{2.8}$$

It has a three dimensional Lie algebra of infinitesimal generators of its (one-parameter) Lie groups of point symmetries. Following are the basis elements

$$\mathbf{X}_1 = \frac{\partial}{\partial x}, \tag{2.9}$$

$$\mathbf{X}_2 = x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y}, \tag{2.10}$$

$$\mathbf{X}_3 = x^2\frac{\partial}{\partial x} - (2xy + 6)\frac{\partial}{\partial y}. \tag{2.11}$$

Following are the corresponding Lie groups

$$\Gamma_1(\epsilon) \quad (\hat{x},\hat{y}) = (x+\epsilon,\ y), \tag{2.12}$$
$$\Gamma_2(\epsilon) \quad (\hat{x},\hat{y}) = (e^{\epsilon}x,\ e^{-\epsilon}y), \tag{2.13}$$
$$\Gamma_3(\epsilon) \quad (\hat{x},\hat{y}) = \left(\frac{x}{1-\epsilon x}\ ,\ [\,y(1-\epsilon x)-6\epsilon\,](1-\epsilon x)\right). \tag{2.14}$$

We use the discrete symmetry

$$\Gamma\ (\hat{x},\hat{y}) = (-x,-y), \tag{2.15}$$

of eq.(2.8) to construct another basis for Lie algebra of infinitesimal generators of (one-parameter) Lie groups of point symmetries of eq.(2.8). In this case

$$\Gamma = \Gamma^{-1}. \tag{2.16}$$

$$\Gamma\Gamma_1(\epsilon)\Gamma^{-1}\ (\hat{x},\hat{y}) = (\ -((-x)+\epsilon),\ -((-y))\ ), \tag{2.17}$$
$$= (\ x-\epsilon,\ y\ ). \tag{2.18}$$

By Theorem (2.1.5) this is a (one-parameter) Lie group of point symmetries of eq.(2.8) and its generator is $\Gamma\mathbf{X}_1\Gamma^{-1}$.

Now we calculate the generator

$$\left.\frac{d\hat{x}}{d\epsilon}\right|_{\epsilon=0} = (-1)|_{\epsilon=0} = -1. \tag{2.19}$$
$$\left.\frac{d\hat{y}}{d\epsilon}\right|_{\epsilon=0} = (0)|_{\epsilon=0} = 0. \tag{2.20}$$

Therefore

$$\Gamma\mathbf{X}_1\Gamma^{-1} = -\frac{\partial}{\partial x}. \tag{2.21}$$

Similarly

$$\Gamma\Gamma_2(\epsilon)\Gamma^{-1}\ (\hat{x},\hat{y}) = (\ -(e^{\epsilon}(-x)),\ -(e^{-\epsilon}(-y))\ ), \tag{2.22}$$
$$= (e^{\epsilon}x,\ e^{-\epsilon}y). \tag{2.23}$$

In this case $\Gamma\Gamma_2(\epsilon)\Gamma^{-1} = \Gamma_2(\epsilon)$. Therefore

$$\Gamma \mathbf{X}_2 \Gamma^{-1} = x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y}. \tag{2.24}$$

Also

$$\Gamma\Gamma_3(\epsilon)\Gamma^{-1} \quad \hat{x} = -\bigg(\frac{(-x)}{1-\epsilon\,(-x)}\bigg), \tag{2.25}$$

$$\hat{y} = -\bigg([\,(-y)(1-\epsilon\,(-x)) - 6\epsilon\,](1-\epsilon\,(-x))\bigg), \tag{2.26}$$

$$\hat{x} = \frac{x}{1+\epsilon x}, \tag{2.27}$$

$$\hat{y} = [\,y(1+\epsilon x) + 6\epsilon\,](1+\epsilon x). \tag{2.28}$$

By Theorem (2.1.5) this is a (one-parameter) Lie group of point symmetries of eq.(2.8) and its generator is $\Gamma\mathbf{X}_3\Gamma^{-1}$.

Now we calculate the generator

$$\frac{d\hat{x}}{d\epsilon}\bigg|_{\epsilon=0} = \frac{-x^2}{(1+\epsilon x)^2}\bigg|_{\epsilon=0} = -x^2. \tag{2.29}$$

$$\frac{d\hat{y}}{d\epsilon}\bigg|_{\epsilon=0} = 2xy + 6. \tag{2.30}$$

Therefore

$$\Gamma\mathbf{X}_3\Gamma^{-1} = -x^2\frac{\partial}{\partial x} + (2xy+6)\frac{\partial}{\partial y}. \tag{2.31}$$

Next we will show that $\Gamma\mathbf{X}_1\Gamma^{-1}$, $\Gamma\mathbf{X}_2\Gamma^{-1}$ and $\Gamma\mathbf{X}_3\Gamma^{-1}$ are linearly independent and therefore forms a basis.

Consider

$$c_1\Gamma\mathbf{X}_1\Gamma^{-1} + c_2\Gamma\mathbf{X}_2\Gamma^{-1} + c_3\Gamma\mathbf{X}_3\Gamma^{-1} = 0, \tag{2.32}$$

$$c_1\bigg(-\frac{\partial}{\partial x}\bigg) + c_2\bigg(x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y}\bigg) + c_3\bigg(-x^2\frac{\partial}{\partial x} + (2xy+6)\frac{\partial}{\partial y}\bigg) = 0, \tag{2.33}$$

$$\Big(- c_1 + c_2 x - c_3 x^2 \Big) \frac{\partial}{\partial x} + \Big(- c_2 y + c_3 (2xy + 6) \Big) \frac{\partial}{\partial y} = 0. \tag{2.34}$$

This implies

$$-c_1 + c_2 x - c_3 x^2 = 0, \tag{2.35}$$
$$-c_2 y + c_3 (2xy + 6) = 0. \tag{2.36}$$

So

$$c_1 = c_2 = c_3 = 0. \tag{2.37}$$

Therefore, $\Gamma \mathbf{X}_1 \Gamma^{-1}$, $\Gamma \mathbf{X}_2 \Gamma^{-1}$ and $\Gamma \mathbf{X}_3 \Gamma^{-1}$ are linearly independent.

In terms of the original basis

$$(\Gamma \mathbf{X}_1 \Gamma^{-1},\ \Gamma \mathbf{X}_2 \Gamma^{-1},\ \Gamma \mathbf{X}_3 \Gamma^{-1}) = (-\mathbf{X}_1,\ \mathbf{X}_2,\ -\mathbf{X}_3). \tag{2.38}$$

In matrix form

$$\begin{bmatrix} \mathbf{X}_1 \\ \mathbf{X}_2 \\ \mathbf{X}_3 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} \Gamma \mathbf{X}_1 \Gamma^{-1} \\ \Gamma \mathbf{X}_2 \Gamma^{-1} \\ \Gamma \mathbf{X}_3 \Gamma^{-1} \end{bmatrix}. \tag{2.39}$$

It should be noted that the symmetry $\Gamma\ (\hat{x}, \hat{y}) = (-x, -y)$ has induced an **automorphism** of the Lie algebra of infinitesimal generators represented by the matrix

$$B = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}. \tag{2.40}$$

How these automorphisms are related to the point symmetries and what role do they play in finding point symmetries is the subject of the next section.

2.2 Basic Method

Our aim is not to construct an alternative basis (Example 2.1.7) for the Lie algebra using a continuous or discrete point symmetry but it is almost the opposite. We will use the basis correponding to a point symmetry to find the point symmetry itself, which is what we discuss in this section.

Theorems (2.1.5) and (2.1.6) say that for a fixed basis $\{\mathbf{X}_i\}_{i=1}^{R}$ of $\mathcal{L}$ and an arbitrary but fixed symmetry Γ, continuous or discrete, we obtain the corresponding basis $\{\Gamma\mathbf{X}_i\Gamma^{-1}\}_{i=1}^{R}$ of $\mathcal{L}$. Since both are bases of the same Lie algebra, we can write each $\mathbf{X}_i$ as a linear combination of $\Gamma\mathbf{X}_i\Gamma^{-1}$'s. The following Lemma [1-3] summarizes this argument.

Lemma 2.2.1

Every point symmetry Γ*, continuous or discrete, of* eq.(2.3) *induces an automorphism of the Lie algebra of infinitesimal generators of (one-parameter) Lie groups of point symmetries of* eq.(2.3). *For each such* Γ*, there exists a constant nonsingular* $N \times N$ *matrix* $B = (b_i^l)$ *such that*

$$\mathbf{X}_i = b_i^l\, \Gamma\mathbf{X}_l\Gamma^{-1}. \tag{2.41}$$

This lemma gives us a method for calculating the discrete point symmetries of a given differential equation, with a known finite dimensional Lie algebra. The most basic and direct approach is discussed in this section. Improvements in the method will be introduced in the next section.

The method has two stages. First, we apply the lemma to obtain the following first-order PDEs which every point symmetry, continuous or discrete, of the eq.(2.3) must satisfy.

$$\mathbf{X}_i\hat{x} = b_i^l\, \Gamma\mathbf{X}_l\Gamma^{-1}\hat{x}, \tag{2.42}$$
$$\mathbf{X}_i\hat{x} = b_i^l\, \Gamma\mathbf{X}_l x, \tag{2.43}$$
$$\mathbf{X}_i\hat{x} = b_i^l\, \Gamma\, \xi_l(x, y), \tag{2.44}$$
$$\mathbf{X}_i\hat{x} = b_i^l\, \xi_l(\hat{x}, \hat{y}), \tag{2.45}$$
$$\mathbf{X}_i\hat{x} = b_i^l\, \hat{\xi}_l. \tag{2.46}$$

Likewise

$$\begin{aligned}
\mathbf{X}_i\hat{y} &= b_i^l\, \Gamma\mathbf{X}_l\Gamma^{-1}\hat{y}, &(2.47)\\
\mathbf{X}_i\hat{y} &= b_i^l\, \Gamma\mathbf{X}_l y, &(2.48)\\
\mathbf{X}_i\hat{y} &= b_i^l\, \Gamma\, \eta_l(x,y), &(2.49)\\
\mathbf{X}_i\hat{y} &= b_i^l\, \eta_l(\hat{x},\hat{y}), &(2.50)\\
\mathbf{X}_i\hat{y} &= b_i^l\, \hat{\eta}_l. &(2.51)
\end{aligned}$$

Eq.(2.46) and eq.(2.51) together constitute a system of first order PDEs. This system can be solved by the method of characteristics to obtain $(\hat{x},\hat{y})$ in terms of x, y, b_i^l and some unknown constants or functions. The solutions of this system always include the trivial symmetry $(\hat{x},\hat{y}) = (x,y)$, corresponding to $b_i^l = \delta_i^l$.

In the second stage, we apply the symmetry condition on the general solution of eq.(2.46) and eq.(2.51). By construction every point symmetry Γ, continuous or discrete, of eq.(2.3) satisfies this system of PDEs but it may have solutions which are not point symmetries of eq.(2.3). This second stage separates such non-symmetry solutions from the general solution of this system of PDEs.

This two-stage process gives a complete list of the point symmetries of eq.(2.3). Since we already know about the Lie point symmetries, any symmetries other than them are the discrete symmetries of eq.(2.3).

Before we solve an example of the method we learn to write eq.(2.46) and eq.(2.51) in matrix form and discuss some of its salient features.

Eq.(2.46) can be written in matrix form as

$$\begin{bmatrix}\mathbf{X}_1\hat{x}\\ \mathbf{X}_2\hat{x}\\ \vdots\\ \mathbf{X}_n\hat{x}\end{bmatrix} = \begin{bmatrix} b_1^1 & b_1^2 & \cdots & b_1^n\\ b_2^1 & b_2^2 & \cdots & b_2^n\\ \vdots & \vdots & \ddots & \vdots\\ b_n^1 & b_n^2 & \cdots & b_n^n\end{bmatrix} \begin{bmatrix}\hat{\xi}_1\\ \hat{\xi}_2\\ \vdots\\ \hat{\xi}_n\end{bmatrix}. \qquad (2.52)$$

Likewise, eq.(2.51) can be written as

$$\begin{bmatrix} \mathbf{X}_1\hat{y} \\ \mathbf{X}_2\hat{y} \\ \vdots \\ \mathbf{X}_n\hat{y} \end{bmatrix} = \begin{bmatrix} b_1^1 & b_1^2 & \cdots & b_1^n \\ b_2^1 & b_2^2 & \cdots & b_2^n \\ \vdots & \vdots & \ddots & \vdots \\ b_n^1 & b_n^2 & \cdots & b_n^n \end{bmatrix} \begin{bmatrix} \hat{\eta}_1 \\ \hat{\eta}_2 \\ \vdots \\ \hat{\eta}_n \end{bmatrix}. \tag{2.53}$$

Combining the two, we obtain

$$\begin{bmatrix} \mathbf{X}_1\hat{x} & \mathbf{X}_1\hat{y} \\ \mathbf{X}_2\hat{x} & \mathbf{X}_2\hat{y} \\ \vdots & \vdots \\ \mathbf{X}_n\hat{x} & \mathbf{X}_n\hat{y} \end{bmatrix} = \begin{bmatrix} b_1^1 & b_1^2 & \cdots & b_1^n \\ b_2^1 & b_2^2 & \cdots & b_2^n \\ \vdots & \vdots & \ddots & \vdots \\ b_n^1 & b_n^2 & \cdots & b_n^n \end{bmatrix} \begin{bmatrix} \hat{\xi}_1 & \hat{\eta}_1 \\ \hat{\xi}_2 & \hat{\eta}_2 \\ \vdots & \vdots \\ \hat{\xi}_n & \hat{\eta}_1 \end{bmatrix}. \tag{2.54}$$

This system of equations will henceforth be called system of determining equations.

Following are some salient features of the system (2.54) which makes it solving easier.

(1) All the partial derivatives in the system are of first order.

(2) All the PDEs in the system are linear.

(3) All the PDEs in the system are un-coupled.

(4) If the parameters in the symmetry condition are allowed to take complex values then this method also gives the complex discrete symmetries.

If we are to find the discrete symmetries of a PDE instead of the ODE (2.3) there will be additional columns for other independent variables. For example, a PDE with one dependent variable u, and two independent variables (x, t), system (2.54) will take the form

$$\begin{bmatrix} \mathbf{X}_1\hat{x} & \mathbf{X}_1\hat{t} & \mathbf{X}_1\hat{u} \\ \mathbf{X}_2\hat{x} & \mathbf{X}_2\hat{t} & \mathbf{X}_2\hat{u} \\ \vdots & \vdots & \vdots \\ \mathbf{X}_n\hat{x} & \mathbf{X}_n\hat{t} & \mathbf{X}_n\hat{u} \end{bmatrix} = \begin{bmatrix} b_1^1 & b_1^2 & \cdots & b_1^n \\ b_2^1 & b_2^2 & \cdots & b_2^n \\ \vdots & \vdots & \ddots & \vdots \\ b_n^1 & b_n^2 & \cdots & b_n^n \end{bmatrix} \begin{bmatrix} \hat{\xi}_1 & \hat{\tau}_1 & \hat{\eta}_1 \\ \hat{\xi}_2 & \hat{\tau}_2 & \hat{\eta}_2 \\ \vdots & \vdots & \vdots \\ \hat{\xi}_n & \hat{\tau}_n & \hat{\eta}_n \end{bmatrix}, \tag{2.55}$$

where

$$\mathbf{X}_i = \xi_i(x,t,u)\frac{\partial}{\partial x} + \tau_i(x,t,u)\frac{\partial}{\partial t} + \eta_i(x,t,u)\frac{\partial}{\partial u}. \tag{2.56}$$

The rest of the method will remain exactly the same.

Now we give a detailed but simple example [1, 2] of the method.

Example 2.2.2

Consider the ODE

$$y'' = \tan y'. \tag{2.57}$$

It has a two dimensional Lie algebra of infinitesimal generators of (one-parameter) Lie groups of its point symmetries

$$\mathbf{X}_1 = \frac{\partial}{\partial x}, \tag{2.58}$$

$$\mathbf{X}_2 = \frac{\partial}{\partial y}. \tag{2.59}$$

For this ODE the system of determining equations (2.54) becomes

$$\begin{bmatrix} \mathbf{X}_1\hat{x} & \mathbf{X}_1\hat{y} \\ \mathbf{X}_2\hat{x} & \mathbf{X}_2\hat{y} \end{bmatrix} = \begin{bmatrix} b_1^1 & b_1^2 \\ b_2^1 & b_2^2 \end{bmatrix} \begin{bmatrix} \hat{\xi}_1 & \hat{\eta}_1 \\ \hat{\xi}_2 & \hat{\eta}_2 \end{bmatrix}, \tag{2.60}$$

$$\begin{bmatrix} \mathbf{X}_1\hat{x} & \mathbf{X}_1\hat{y} \\ \mathbf{X}_2\hat{x} & \mathbf{X}_2\hat{y} \end{bmatrix} = \begin{bmatrix} b_1^1 & b_1^2 \\ b_2^1 & b_2^2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \tag{2.61}$$

$$\begin{bmatrix} \mathbf{X}_1\hat{x} & \mathbf{X}_1\hat{y} \\ \mathbf{X}_2\hat{x} & \mathbf{X}_2\hat{y} \end{bmatrix} = \begin{bmatrix} b_1^1 & b_1^2 \\ b_2^1 & b_2^2 \end{bmatrix}. \tag{2.62}$$

In addition to some other solutions, which are not point symmetries, the general solution of this system eq.(2.62) has all the point symmetries of eq.(2.57).

Solving the system, we have

$$\mathbf{X}_1\hat{x} = b_1^1, \tag{2.63}$$

$$\frac{\partial \hat{x}}{\partial x} = b_1^1, \tag{2.64}$$

$$\hat{x} = b_1^1 x + f(y). \tag{2.65}$$

We also have

$$\mathbf{X}_2\hat{x} = b_2^1, \tag{2.66}$$

$$\frac{\partial \hat{x}}{\partial y} = b_2^1 = f'(y), \tag{2.67}$$

$$f(y) = b_2^1 y + c_1. \tag{2.68}$$

This implies

$$\hat{x} = b_1^1 x + b_2^1 y + c_1. \tag{2.69}$$

Likewise

$$\hat{y} = b_1^2 x + b_2^2 y + c_2. \tag{2.70}$$

Therefore, the general solution of eq.(2.62) is

$$(\hat{x}, \hat{y}) = (b_1^1 x + b_2^1 y + c_1, b_1^2 x + b_2^2 y + c_2). \tag{2.71}$$

Now we separate the non-symmetry solutions from the general solution eq.(2.71). By definition eq.(2.71) is a symmetry of eq.(2.57), if and only if

$$y'' = \tan y' \quad \Longrightarrow \quad \hat{y}'' = \tan \hat{y}'. \tag{2.72}$$

We calculate $\hat{y}'$ and $\hat{y}''$.

$$\hat{y}' = \frac{d\hat{y}}{d\hat{x}} = \frac{d(b_1^2 x + b_2^2 y + c_2)}{d(b_1^1 x + b_2^1 y + c_1)} = \frac{b_1^2 + b_2^2 \frac{dy}{dx}}{b_1^1 + b_2^1 \frac{dy}{dx}} = \frac{b_1^2 + b_2^2 y'}{b_1^1 + b_2^1 y'}. \tag{2.73}$$

$$\begin{aligned}\hat{y}'' = \frac{d\hat{y}'}{d\hat{x}} = \frac{d(\frac{b_1^2 + b_2^2 y'}{b_1^1 + b_2^1 y'})}{d(b_1^1 x + b_2^1 y + c_1)} \\ = \frac{(b_1^1 b_2^2 - b_1^2 b_2^1) y''}{(b_1^1 + b_2^1 y')^3} = \frac{(b_1^1 b_2^2 - b_1^2 b_2^1) \tan y'}{(b_1^1 + b_2^1 y')^3}.\end{aligned} \tag{2.74}$$

Therefore, the symmetry condition is

$$\frac{(b_1^1 b_2^2 - b_1^2 b_2^1)\tan y'}{(b_1^1 + b_2^1 y')^3} = \tan\left(\frac{b_1^2 + b_2^2 y'}{b_1^1 + b_2^1 y'}\right). \tag{2.75}$$

Now we find out what values of b_i^l satisfy eq.(2.75).

Differentiating both sides with respect to $y'$and comapiring cefficients of $\tan y'$ and $\tan^2 y'$ we find that $b_2^1 = 0, b_1^1 = 1$ and $b_2^2 = \alpha = \pm 1$.

Therefore, eq.(2.75) becomes

$$\alpha \tan y' = \tan(b_1^2 + \alpha y'). \tag{2.76}$$

This further implies $b_1^2 = k\pi,\ k \in \mathbb{Z}$. So symmetry condition is satisfied if and only if

$$\begin{bmatrix} b_1^1 & b_1^2 \\ b_2^1 & b_2^2 \end{bmatrix} = \begin{bmatrix} 1 & k\pi \\ 0 & \pm 1 \end{bmatrix}. \tag{2.77}$$

Therefore, all the point symmetries of eq.(2.57) are

$$(\hat{x}, \hat{y}) = (x + c_1, k\pi x + \alpha y + c_2), \quad \alpha = \pm 1 \quad k \in \mathbb{Z}. \tag{2.78}$$

Since we already know the Lie point symmetries of eq.(2.57), we can separate them from eq.(2.78) to obtain the discrete symmetries of eq.(2.57).

Following are the discrete symmetries

$$(\hat{x}, \hat{y}) = (x + c_1, k\pi x - y + c_2), \quad k \in \mathbb{Z}, \tag{2.79}$$
$$(\hat{x}, \hat{y}) = (x + c_1, k\pi x + y + c_2), \quad k \in \mathbb{Z} - \{0\}. \tag{2.80}$$

The only case left, that is, $\alpha = 1$ and $k = 0$, are precisely the continuous point symmetries

$$(\hat{x}, \hat{y}) = (x + c_1, y + c_2). \tag{2.81}$$

2.3 Improvements in the Method

This section discusses improvements in the basic method which was introduced in the last section. Little improvement can be made if the Lie algebra of infinitesimal generators of the (one-parameter) Lie groups of point symmetries is abelian. On the contrary, when the Lie algebra is non-abelian the system of determining equations can be considerably simplified in two stages [1-3].

2.3.1 Canonical Coordinates

The only improvement which can be made in the basic method, when the Lie algebra is abelian, is the use of canonical coordinates. This way atleast one generator in the basis is simplified. This is especially useful when the dimension of the Lie algebra is one [2].

Canonical coordinates $s(x, y)$, $r(x, y)$, satisfy

$$\mathbf{X}_1 s = 1, \quad \mathbf{X}_1 r = 0, \tag{2.82}$$

so that

$$\mathbf{X}_1 = \frac{\partial}{\partial s}\,. \tag{2.83}$$

In this case, the system of determining equations eq.(2.54) becomes

$$\begin{bmatrix}\mathbf{X}_1\hat{s} & \mathbf{X}_1\hat{r}\end{bmatrix} = [b_1^1]\begin{bmatrix}1 & 0\end{bmatrix}, \tag{2.84}$$

$$\frac{\partial \hat{s}}{\partial s} = b_1^1 \neq 0, \quad \frac{\partial \hat{r}}{\partial s} = 0\,. \tag{2.85}$$

General solution of eq.(2.85) is

$$\hat{s} = b_1^1 s + g(r), \quad \hat{r} = f(r), \tag{2.86}$$

for some functions f and g. The application of the symmetry condition on this transformation determines which functions f, g and constants b_1^1 are allowable.

Example 2.3.1

For example [2], consider the ODE

$$y'' = \frac{y'}{x} + \frac{4y^2}{x^3}\ . \tag{2.87}$$

This ODE has a one dimensional Lie algebra of point symmetry generators, with a basis

$$\mathbf{X}_1 = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\ . \tag{2.88}$$

In canonical coordinates

$$s = \ln|x|, \quad r = \frac{y}{x}, \tag{2.89}$$

eq.(2.87) becomes

$$\frac{d^2 r}{ds^2} = 4r^2 + r. \tag{2.90}$$

The symmetry condition says

$$\frac{d^2 r}{ds^2} = 4r^2 + r \quad \Longrightarrow \quad \frac{d^2 \hat{r}}{d\hat{s}^2} = 4\hat{r}^2 + \hat{r}. \tag{2.91}$$

Calculating $d^2\hat{r}/d\hat{s}^2$, we already know $\hat{r} = f(r)$, and after applying the symmetry condition (its calculation is too long, refer to [2], page 5) then converting back to (x, y) coordinates, we obtain the following *inequivalent* discrete symmetries of eq.(2.87)

$$(\hat{x}, \hat{y}) \in \left\{ (x, y),\ (-x, -y),\ (\frac{1}{x},\ \frac{y}{x^2}),\ (-\frac{1}{x},\ -\frac{y}{x^2}) \right\}. \tag{2.92}$$

Note that the identity symmetry has been included here for the sake of completion, because the above set of discrete symmetries is incidently isomorphic to the group $\mathbb{Z}_2 \times \mathbb{Z}_2$.

2.3.2 Nonlinear Constraints

The improvements discussed henceforth apply only when the Lie algebra, $\mathcal{L}$, of infinitesimal generators of the (one-parameter) Lie groups of point symmetries is non-abelian.

In this section we discuss certain constraints, nonlinear algebraic equations involving c_{ij}^k and b_i^l, on the system of determining equations (2.54) which will simplify the system considerably.

Theorem 2.3.2

$$\textit{If}\ \ [\mathbf{X}_i, \mathbf{X}_j] = c_{ij}^k\, \mathbf{X}_k \quad \textit{then} \quad [\, \Gamma\mathbf{X}_i\Gamma^{-1}, \Gamma\mathbf{X}_j\Gamma^{-1}\,] = c_{ij}^k\, \Gamma\mathbf{X}_k\Gamma^{-1}.$$

If $\mathcal{L}$ is non-abelian, then at least some of the equations $[\mathbf{X}_i, \mathbf{X}_j] = c_{ij}^k\, \mathbf{X}_k$ are nontrivial. Also, the above theorem [1-3] says $\Gamma\mathbf{X}_i\Gamma^{-1}$ satisfy the same commutator relations as $\mathbf{X}_i$. This leads us to the following theorem [1-3].

Theorem 2.3.3

The structure constants c_{ij}^k and the elements of $B = (b_i^l)$ satisfy the following equations

$$c_{lm}^n b_i^l b_j^m = c_{ij}^k b_k^n, \quad \textit{all indices are from 1 to } dim(\mathcal{L}).$$

If $dim(\mathcal{L}) = R$ then these are R^3 equations, but

$$c_{ij}^k = -c_{ji}^k \quad \text{and} \quad c_{ii}^k = 0 \tag{2.93}$$

so we have

$$\frac{R(R^2 - R)}{2} \tag{2.94}$$

distinct equations and it is sufficient to restrict attention to $i < j$.

These constraints on the elements of the matrix $B = (b_i^l)$ simplify the system of determining equations (2.54) considerably before we solve it. Thereby making the system easier to solve.

The following table lists the number of equations against $dim(\mathcal{L}) = R$, for $2 \leq R \leq 10$.

R	Number of equations
2	2
3	9
4	24
5	50
6	90
7	147
8	224
9	324
10	450

The number of equations increase very rapidly as R increases so a very systematic approach needs to be followed to avoid any complexities. Following are some guidelines in this regard.

(1) Note that all these equations are nonlinear algebraic equations.

(2) If there is one or more nonzero structure constant with every value of superscript then the system of equations may become difficult to solve.

(3) If possible start solving the system with a superscript value such that the structure constant with this superscript value is zero. The left hand side of the corresponding equations for such a superscript value becomes zero and all such equations reduce to linear equations. Thereby becoming easier to solve.

(4) Some equations remain unsolved almost always but such equations may be used in later calculations.

(5) When the number of equations is too large use of some computer algebra system is recomended.

Now we give a detailed example [2].

Example 2.3.4

Consider the ODE,

$$y''' = y''(1 - y''). \tag{2.95}$$

It has a three dimensional Lie algebra, with the basis

$$\mathbf{X}_1 = \frac{\partial}{\partial x}, \tag{2.96}$$

$$\mathbf{X}_2 = x\frac{\partial}{\partial y}, \tag{2.97}$$

$$\mathbf{X}_3 = \frac{\partial}{\partial y}. \tag{2.98}$$

The only nonzero structure constants are

$$c_{12}^3 = 1, \quad c_{21}^3 = -1. \tag{2.99}$$

Now we solve the equations

$$c_{lm}^n b_i^l b_j^m = c_{ij}^k b_k^n, \quad \text{all indices are from 1 to 3.} \tag{2.100}$$

As discussed above we have distinct equations only when we keep $i < j$. In this case, $(i, j) = (1, 2), (1, 3), (2, 3)$. Following guideline (3) above, we should start from $n = 1$ or $n = 2$.

For $n = 1$

$$c_{lm}^1 = 0, \qquad l, m = 1, 2, 3. \tag{2.101}$$

The constraints reduce to linear equations

$$0 = c_{ij}^k b_k^1, \tag{2.102}$$

$$0 = c_{ij}^1 b_1^1 + c_{ij}^2 b_2^1 + c_{ij}^3 b_3^1. \tag{2.103}$$

When $(i, j) = (1, 2)$

$$0 = c_{12}^1 b_1^1 + c_{12}^2 b_2^1 + c_{12}^3 b_3^1, \tag{2.104}$$

$$0 = (0)b_1^1 + (0)b_2^1 + (1)b_3^1, \tag{2.105}$$

$$0 = b_3^1. \tag{2.106}$$

When $(i,j) = (1,3)$

$$0 = c_{13}^1 b_1^1 + c_{13}^2 b_2^1 + c_{13}^3 b_3^1, \tag{2.107}$$
$$0 = (0)b_1^1 + (0)b_2^1 + (0)(0), \tag{2.108}$$
$$0 = 0. \tag{2.109}$$

When $(i,j) = (2,3)$

$$0 = c_{23}^1 b_1^1 + c_{23}^2 b_2^1 + c_{23}^3 b_3^1, \tag{2.110}$$
$$0 = (0)b_1^1 + (0)b_2^1 + (0)(0), \tag{2.111}$$
$$0 = 0. \tag{2.112}$$

For $n = 2$

$$c_{lm}^2 = 0, \qquad l, m = 1, 2, 3. \tag{2.113}$$

The constraints reduce to linear equations

$$0 = c_{ij}^k b_k^2, \tag{2.114}$$
$$0 = c_{ij}^1 b_1^2 + c_{ij}^2 b_2^2 + c_{ij}^3 b_3^2. \tag{2.115}$$

When $(i,j) = (1,2)$

$$0 = c_{12}^1 b_1^2 + c_{12}^2 b_2^2 + c_{12}^3 b_3^2, \tag{2.116}$$
$$0 = (0)b_1^2 + (0)b_2^2 + (1)b_3^2, \tag{2.117}$$
$$0 = b_3^2. \tag{2.118}$$

When $(i,j) = (1,3)$

$$0 = c_{13}^1 b_1^2 + c_{13}^2 b_2^2 + c_{13}^3 b_3^2, \tag{2.119}$$
$$0 = (0)b_1^2 + (0)b_2^2 + (0)(0), \tag{2.120}$$
$$0 = 0. \tag{2.121}$$

When $(i,j) = (2,3)$

$$0 = c_{23}^1 b_1^2 + c_{23}^2 b_2^2 + c_{23}^3 b_3^2, \tag{2.122}$$
$$0 = (0)b_1^2 + (0)b_2^2 + (0)(0), \tag{2.123}$$
$$0 = 0. \tag{2.124}$$

For $n = 3$

$$c^3_{lm} = 0, \qquad (l,m) \neq (1,2),(2,1). \tag{2.125}$$

The constraints reduce to nonlinear equations

$$c^3_{12} b^1_i b^2_j + c^3_{21} b^2_i b^1_j = c^k_{ij} b^3_k, \tag{2.126}$$
$$(1) b^1_i b^2_j + (-1) b^2_i b^1_j = c^1_{ij} b^3_1 + c^2_{ij} b^3_2 + c^3_{ij} b^3_3, \tag{2.127}$$
$$b^1_i b^2_j - b^2_i b^1_j = c^1_{ij} b^3_1 + c^2_{ij} b^3_2 + c^3_{ij} b^3_3. \tag{2.128}$$

When $(i,j) = (1,2)$

$$b^1_1 b^2_2 - b^2_1 b^1_2 = c^1_{12} b^3_1 + c^2_{12} b^3_2 + c^3_{12} b^3_3, \tag{2.129}$$
$$b^1_1 b^2_2 - b^2_1 b^1_2 = (0) b^3_1 + (0) b^3_2 + (1) b^3_3, \tag{2.130}$$
$$b^1_1 b^2_2 - b^2_1 b^1_2 = b^3_3. \tag{2.131}$$

When $(i,j) = (1,3)$

$$b^1_1 b^2_3 - b^2_1 b^1_3 = c^1_{13} b^3_1 + c^2_{13} b^3_2 + c^3_{13} b^3_3, \tag{2.132}$$
$$b^1_1(0) - b^2_1(0) = (0) b^3_1 + (0) b^3_2 + (0) b^3_3, \tag{2.133}$$
$$0 = 0. \tag{2.134}$$

When $(i,j) = (2,3)$

$$b^1_2 b^2_3 - b^2_2 b^1_3 = c^1_{23} b^3_1 + c^2_{23} b^3_2 + c^3_{23} b^3_3, \tag{2.135}$$
$$b^1_2(0) - b^2_2(0) = (0) b^3_1 + (0) b^3_2 + (0) b^3_3, \tag{2.136}$$
$$0 = 0. \tag{2.137}$$

We have been able to simplyfy $B = (b^l_i)$ as

$$B = \begin{bmatrix} b^1_1 & b^2_1 & b^3_1 \\ b^1_2 & b^2_2 & b^3_2 \\ 0 & 0 & b^3_3 \end{bmatrix},$$

with further condition

$$b^1_1 b^2_2 - b^2_1 b^1_2 = b^3_3. \tag{2.138}$$

and because $B = (b^l_i)$ is nonsingular, $b^3_3 \neq 0$.

2.3.3 Inequivalent Discrete Symmetries

In this section we improve the method further and learn how to find *inequivalent* discrete symmetries [1, 7].

Definition 2.3.5

Two point symmetries $\tilde{\Gamma}$ *and* Γ *of the* eq.(2.3) *are called* ***equivalent*** *if there exists an* $\mathbf{X} \in \mathcal{L}$ *such that* $\tilde{\Gamma} = e^{\epsilon \mathbf{X}}\Gamma$*. Likewise the corresponding induced automorphisms of the Lie algebra represented with the matrices, say,* $\tilde{B}$ *and* B *respectively are called* ***equivalent*** *as well.*

Since we are reducing the number of discrete symmetries to be found, and hence the number of corresponding matrices which represent the automorphisms, this naturally simplifies the system of determining equations (2.54).

Theoretically this improvement is not restricted to non-abelian Lie algebra, of infinitesimal generators of the (one-parameter) Lie groups of point symmetries, but no substantial simplification of the system (2.54) is obtained when the Lie algebra is abelian. For the non-abelian case improvements discussed in this section and the nonlinear constraints discussed in the preceding section are used simultaneously. We almost always use the nonlinear constraints first to simplify $B = (b_i^l)$.

Now we define some notation [1, 7]. The matrices $C(j)$ and $A(j, \epsilon)$ are defined as

$$(C(j))_i^k = c_{ij}^k \,. \tag{2.139}$$

$$A(j, \epsilon) = \sum_{n=0}^{\infty} \frac{\epsilon^n}{n!}(C(j))^n = e^{\epsilon C(j)}. \tag{2.140}$$

The following theorem [1, 7] forms the basis for finding inequivalent discrete symmetries.

Theorem 2.3.6

(1) The automorphism of the Lie algebra $\mathcal{L}$ *induced by the point symmetry* $\Gamma = e^{\epsilon \mathbf{X}_j}$ *is represented with the matrix* $B = A(j, \epsilon)$*, here* $\mathbf{X}_j$ *is the basis element of* $\mathcal{L}$*.*

(2) If the automorphisms induced by the point symmetries Γ_1 *and* Γ_2 *are represented with the matrices* B_1 *and* B_2 *respectively then the automorphism induced by the point symmetry* $\Gamma_2 \circ \Gamma_1$ *is represented with the matrix* $B_2 B_1$*.*

(3) If Γ_1 *and* $\Gamma_2 = e^{\epsilon \mathbf{X}} \Gamma_1$ *induce automorphisms represented with matrices* B_1 *and* B_2 *respectively then*

$$B_2 = A(1, \epsilon_1) A(2, \epsilon_2) \cdots A(R, \epsilon_R) B_1.$$

here R *is the dimension of* $\mathcal{L}$ *and* ϵ_j *are some parameters.*

Inequivalent discrete symmetries are found by solving the system (2.54) for only the inequivalent matrices. $B_2 = A(1, \epsilon_1) A(2, \epsilon_2) \cdots A(R, \epsilon_R) B_1$ is a generic member of the equivalence class of an arbitrary B_1. Inequivalent symmetries are calculated by solving system (2.54) for only one such B_2, that is, by fixing the values of ϵ_i s. Following a similar argument B_1 can also be replaced by $B_2 = B_1 A(1, \epsilon_1) A(2, \epsilon_2) \cdots A(R, \epsilon_R)$.

We illustrate the procedure with an example [1].

Example 2.3.7

Consider the ODE

$$y''' = \frac{1}{y^3}. \tag{2.141}$$

It has a two dimensional Lie algebra of point symmetry generators, spanned by

$$\mathbf{X}_1 = \frac{\partial}{\partial x}, \tag{2.142}$$

$$\mathbf{X}_2 = x\frac{\partial}{\partial x} + \frac{3y}{4}\frac{\partial}{\partial y}. \tag{2.143}$$

Following are the only nonzero structure constants, they will be used in solving the system of nonlinear constraints.

$$c_{12}^1 = 1, \quad c_{21}^1 = -1. \tag{2.144}$$

Now we substitute the respective values of c_{ij}^k and b_i^l in the system of nonlinear constraints and solve it.

$$c_{lm}^n b_i^l b_j^m = c_{ij}^k b_k^n, \quad \text{all indices are from 1 to 2.} \tag{2.145}$$

For $n = 2$
$$c_{lm}^2 = 0, \qquad l, m = 1, 2. \tag{2.146}$$
The constraints reduce to
$$0 = c_{ij}^k b_k^2, \tag{2.147}$$
$$0 = c_{ij}^1 b_1^2 + c_{ij}^2 b_2^2. \tag{2.148}$$
When $(i, j) = (1, 2)$
$$0 = c_{12}^1 b_1^2 + c_{12}^2 b_2^2, \tag{2.149}$$
$$0 = (1)b_1^2 + (0)b_2^2, \tag{2.150}$$
$$0 = b_1^2. \tag{2.151}$$
For $n = 1$
$$c_{lm}^1 = 0, \qquad (l, m) \neq (1, 2), (2, 1). \tag{2.152}$$
The constraints reduce to
$$c_{12}^1 b_i^1 b_j^2 + c_{21}^1 b_i^2 b_j^1 = c_{ij}^k b_k^1, \tag{2.153}$$
$$(1)b_i^1 b_j^2 + (-1)b_i^2 b_j^1 = c_{ij}^1 b_1^1 + c_{ij}^2 b_2^1. \tag{2.154}$$
When $(i, j) = (1, 2)$
$$(1)b_1^1 b_2^2 + (-1)b_1^2 b_2^1 = c_{12}^1 b_1^1 + c_{12}^2 b_2^1, \tag{2.155}$$
$$(1)b_1^1 b_2^2 + (-1)b_1^2 b_2^1 = (1)b_1^1 + (0)b_2^1, \tag{2.156}$$
$$b_1^1 b_2^2 - b_1^2 b_2^1 = b_1^1. \tag{2.157}$$
Since $b_1^2 = 0$, so $b_1^1 b_2^2 = b_1^1$. Also B is nonsingualr, therefore $b_1^1 \neq 0$ and $b_2^2 = 1$.

Using the nonlinear constraints we have been able to simplyfy $B = (b_i^l)$ as,
$$B = \begin{bmatrix} b_1^1 & 0 \\ b_2^1 & 1 \end{bmatrix}, \tag{2.158}$$
$$b_1^1 \neq 0.$$

Now we find the inequivalent matrices using Theorem (2.3.6). We first calculate the matrices $A(j, \epsilon)$.

$$C(1) = \begin{bmatrix} 0 & 0 \\ -1 & 0 \end{bmatrix}, \tag{2.159}$$

so

$$A(1, \epsilon) = \begin{bmatrix} 1 & 0 \\ -\epsilon & 1 \end{bmatrix}. \tag{2.160}$$

Likewise

$$C(2) = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \tag{2.161}$$

so

$$A(2, \epsilon) = \begin{bmatrix} e^{\epsilon} & 0 \\ 0 & 1 \end{bmatrix}. \tag{2.162}$$

We multiply B with $A(1, \epsilon)$

$$A(1, \epsilon)B = \begin{bmatrix} 1 & 0 \\ -\epsilon & 1 \end{bmatrix} \begin{bmatrix} b_1^1 & 0 \\ b_2^1 & 1 \end{bmatrix} \tag{2.163}$$

$$= \begin{bmatrix} b_1^1 & 0 \\ -\epsilon b_1^1 + b_2^1 & 1 \end{bmatrix}. \tag{2.164}$$

Choosing $\epsilon = \epsilon_1 = \frac{b_2^1}{b_1^1}$,

$$-\epsilon b_1^1 + b_2^1 = -\epsilon_1 b_1^1 + b_2^1 = -\left(\frac{b_2^1}{b_1^1}\right) b_1^1 + b_2^1 = 0,$$

so

$$A(1, \epsilon_1)B = \begin{bmatrix} b_1^1 & 0 \\ 0 & 1 \end{bmatrix}. \tag{2.165}$$

Now

$$A(2, \epsilon)A(1, \epsilon_1)B = \begin{bmatrix} e^{\epsilon} & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} b_1^1 & 0 \\ 0 & 1 \end{bmatrix} \tag{2.166}$$

$$= \begin{bmatrix} e^{\epsilon} b_1^1 & 0 \\ 0 & 1 \end{bmatrix}. \tag{2.167}$$

Choosing $\epsilon = \epsilon_2 = \ln \frac{1}{|b_1^1|}$,

$$e^{\epsilon} b_1^1 = e^{\epsilon_2} b_1^1 = e^{\ln \frac{1}{|b_1^1|}} b_1^1 = \frac{b_1^1}{|b_1^1|},$$

so

$$A(2, \epsilon_2) A(1, \epsilon_1) B = \begin{bmatrix} \frac{b_1^1}{|b_1^1|} & 0 \\ 0 & 1 \end{bmatrix}. \tag{2.168}$$

Let $\tilde{B} = A(2, \epsilon_2) A(1, \epsilon_1) B$. These are the required inequivalent matrices. So the inequivalent symmetries satisfy system of determining equations (2.54) which becomes in this case

$$\begin{bmatrix} \mathbf{X}_1 \hat{x} & \mathbf{X}_1 \hat{y} \\ \mathbf{X}_2 \hat{x} & \mathbf{X}_2 \hat{y} \end{bmatrix} = \begin{bmatrix} \frac{b_1^1}{|b_1^1|} & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ \hat{x} & \frac{3}{4}\hat{y} \end{bmatrix}, \tag{2.169}$$

$$\begin{bmatrix} \mathbf{X}_1 \hat{x} & \mathbf{X}_1 \hat{y} \\ \mathbf{X}_2 \hat{x} & \mathbf{X}_2 \hat{y} \end{bmatrix} = \begin{bmatrix} \frac{b_1^1}{|b_1^1|} & 0 \\ \hat{x} & \frac{3}{4}\hat{y} \end{bmatrix}. \tag{2.170}$$

Its general solution is

$$(\hat{x}, \hat{y}) = (\frac{b_1^1}{|b_1^1|}\, x + c_2 y^{\frac{4}{3}},\ c_1 y), \tag{2.171}$$

where c_i are constants.

Applying the symmetry condition on the general solution we conclude that $\frac{b_1^1}{|b_1^1|}\,(c_1)^4 = 1$ and $c_2 = 0$. Therefore, up to equivalence there is one discrete symmetry of eq.(2.141)

$$(\hat{x}, \hat{y}) = (x,\ -y). \tag{2.172}$$

2.4 Some Well Known Differential Equations and their Inequivalent Discrete Symmetries

This section lists inequivalent discrete symmetries of some differential equations obtained by the method described in the previous three sections.

Harry-Dym equation.

$$\frac{\partial u}{\partial t} = u^3 \frac{\partial^3 u}{\partial t^3}. \tag{2.173}$$

Inequivalent discrete symmetries [1].

$$(\hat{x}, \hat{t}, \hat{u}) \in \left\{ (\alpha x,\ \beta t,\ \alpha\beta u),\ (-\frac{\alpha}{x},\ \beta t,\ \frac{\alpha\beta u}{x^2}) \right\} \quad \alpha, \beta \in \{1, -1\}. \tag{2.174}$$

Burger's equation.

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = \frac{\partial^2 u}{\partial x^2}. \tag{2.175}$$

Inequivalent discrete symmetries [1].

$$(\hat{x}, \hat{t}, \hat{u}) \in \left\{ (\alpha x,\ t,\ \alpha u),\ (\frac{\alpha x}{2t},\ -\frac{1}{4t},\ 2\alpha(ut - x)) \right\} \quad \alpha \in \{1, -1\}. \tag{2.176}$$

Spherical Burger's equation.

$$\frac{\partial u}{\partial t} + \frac{u}{t} + u\frac{\partial u}{\partial x} = \frac{\partial^2 u}{\partial x^2}. \tag{2.177}$$

Inequivalent discrete symmetries [3].

$$(\hat{x}, \hat{t}, \hat{u}) = (-x,\ t,\ -u). \tag{2.178}$$

Euler Poisson Darboux equation.

$$\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} = \frac{p(p+1)}{t^2}\, u. \tag{2.179}$$

Inequivalent discrete symmetries [8].

$$(\hat{x}, \hat{t}, \hat{u}) \in \left\{ (-x, t, u), (x, -t, u), (x, t, -u), \left(\frac{x}{t^2 - x^2}, \frac{t}{t^2 - x^2}, u \right) \right\}. \tag{2.180}$$

Heat equation.

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}. \tag{2.181}$$

The group of inequivalent discrete symmetries [8], of heat equation is isomorphic to $\mathbb{Z}_4$, and is generated by

$$(\hat{x}, \hat{t}, \hat{u}) = \left(\frac{x}{2t},\ -\frac{1}{4t},\ u\,\sqrt{2i\,t}\, e^{\frac{x^2}{4t}} \right). \tag{2.182}$$

2.5 Review of Group Invariant Solutions

The most immediate application of discrete symmetries of a differential equation is mapping known exact solutions to possibly new exact solutions. Since the method for obtaining discrete symmetries discussed in this chapter gives all the discrete symmetries of a differential equation, it is specially useful when a differential equation has some hidden discrete symmetries, which are otherwise difficult to find. In this case the chances of obtaining new exact solutions from known exact solutions are even higher. The safest bet are the group invaraint solutions. Since the symmetry groups, under which these solutions are invaraint, are known, therefore the chances of mapping these solutions to themselves, under a discrete symmetry, or to other group invaraint solutions are less. In later chapters when we find discrete symmetries of some PDEs, their group invaraint solutions will be analysed under the discrete symmetries for possible new solutions.

The purpose of this section is a brief review of the group invaraint solutions of ODEs and PDEs [1]. The underlying theory is not discussed only the direct method of obtaining group invaraint solutions is reviewed.

Definition 2.5.1

*A solution or a family of solutions of a differential equation, ODE or PDE, which remain functionally unchanged when transformed under a (one-parameter) Lie group of point symmetries of the differential equation, are called **group invaraint solutions** of the differential equation.*

Example 2.5.2

As an example [1] consider the ODE

$$y''' = -yy''. \tag{2.183}$$

It has the group invariant solution

$$y = \frac{3}{x}, \tag{2.184}$$

corresponding to the Lie group

$$(\hat{x}, \hat{y}) = (xe^{\epsilon}, ye^{-\epsilon}), \tag{2.185}$$

generated by

$$\mathbf{X} = x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y}. \tag{2.186}$$

Now we verify that, transfroming eq.(2.184) using eq.(2.185)

$$\hat{y}e^{\epsilon} = \frac{3}{\hat{x}e^{-\epsilon}}, \tag{2.187}$$

$$\hat{y} = \frac{3}{\hat{x}}. \tag{2.188}$$

Since the solution eq.(2.184) remains funtionally invariant after the transformation, it is a group invariant solution.

The following two sections illustrate the method for obtaining group invaraint solutions of ODEs and PDEs respectively.

2.5.1 Method for Obtaining Group Invariant Solutions of ODEs

For ODEs, solutions invariant under the Lie group $\Gamma(\epsilon)$, generated by

$$\mathbf{X} = \xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y}, \tag{2.189}$$

satisfy the ODE

$$\eta - \xi\frac{dy}{dx} = 0. \tag{2.190}$$

Usually eq.(2.190) is easier to solve than the original ODE. Also note that, eq.(2.190) is only a necessary condition.

Example 2.5.3

As an example [1] consider the ODE

$$y''' = -yy''. \tag{2.191}$$

We will obtain group invariant solutions of eq.(2.191) with respect to the Lie group

$$(\hat{x}, \hat{y}) = ((x-1)e^{\epsilon} + 1,\ ye^{-\epsilon}), \tag{2.192}$$

generated by

$$\mathbf{X} = (x-1)\frac{\partial}{\partial x} - y\frac{\partial}{\partial y}. \tag{2.193}$$

eq.(2.190) becomes in this case

$$(-y) - (x-1)\frac{dy}{dx} = 0, \tag{2.194}$$

$$\frac{dy}{y} = \frac{dx}{-x+1}. \tag{2.195}$$

Solving eq.(2.195), we obtain

$$y = \frac{c}{x-1}. \tag{2.196}$$

Substituting eq.(2.196) back in eq.(2.191)

$$\frac{-6c}{(x-1)^4} = -\Big(\frac{c}{(x-1)}\Big)\Big(\frac{2c}{(x-1)^3}\Big). \tag{2.197}$$

This implies $c = 0$ or $c = 3$.

Therefore, the group invariant solutions due to eq.(2.192) are

$$y = 0 \quad \text{or} \quad y = \frac{3}{x-1}. \tag{2.198}$$

2.5.2 Method for Obtaining Group Invariant Solutions of PDEs

For PDEs, with one dependent variable u, and two independent variables (x, t), solutions invariant under the Lie group $\Gamma(\epsilon)$, generated by

$$\mathbf{X} = \xi\frac{\partial}{\partial x} + \tau\frac{\partial}{\partial t} + \eta\frac{\partial}{\partial u}, \tag{2.199}$$

satisfy the PDE

$$\eta - \xi\frac{\partial u}{\partial x} - \tau\frac{\partial u}{\partial t} = 0. \tag{2.200}$$

Usually eq.(2.200) is easier to solve than the original PDE. Also note that, eq.(2.200) is only a necessary condition.

Example 2.5.4

As an example [1] consider the PDE

$$\frac{\partial^2 u}{\partial x \partial t} = u. \tag{2.201}$$

We will obtain group invariant solutions of eq.(2.201) with respect to the Lie group

$$(\hat{x}, \hat{t}, \hat{u}) = (xe^{\epsilon},\ te^{-\epsilon},\ ue^{\epsilon}), \tag{2.202}$$

generated by

$$\mathbf{X} = x\frac{\partial}{\partial x} - t\frac{\partial}{\partial t} + u\frac{\partial}{\partial u}. \tag{2.203}$$

eq.(2.200) becomes in this case

$$u - x\frac{\partial u}{\partial x} + t\frac{\partial u}{\partial t} = 0. \tag{2.204}$$

Solving eq.(2.204)

$$\frac{dx}{x} = \frac{dt}{-t} = \frac{du}{u}. \tag{2.205}$$

We obtain the first integrals

$$r = xt, \quad v = ut. \tag{2.206}$$

Therefore, the general solution of eq.(2.204) is

$$v = F(r), \tag{2.207}$$

or

$$u = \frac{1}{t}F(xt), \tag{2.208}$$

This solution will now be substituted in eq.(2.201) to determine F.

$$\frac{\partial u}{\partial x} = F'(r), \quad \frac{\partial^2 u}{\partial t \partial x} = xF''(r). \tag{2.209}$$

$$xF''(r) = \frac{1}{t}F(r), \tag{2.210}$$

$$F''(r) - \frac{1}{r}F(r) = 0. \tag{2.211}$$

Solution of eq.(2.211) substituted in eq.(2.208) gives us the required group invariant solutions.

Chapter 3

Discrete Symmetries Analysis of Korteweg de Vries Equation and Related Exact Solutions

Korteweg de Vries (KdV) equation

$$\frac{\partial^3 u}{\partial x^3} + 6u\frac{\partial u}{\partial x} + \frac{\partial u}{\partial t} = 0,$$

is a mathematical model of waves on shallow water surfaces. Many different variations of the KdV equation have been studied. The mathematical theory behind the KdV equation is a topic of active research. KdV equation was first introduced by Boussinesq (1877) and rediscovered by Diederik Korteweg and Gustav de Vries (1895).

The purpose of this chapter is to find all discrete symmetries of the KdV equation. As an immediate application of discrete symmetries, group invariant solutions of KdV equation corresponding to the basis vectors of the Lie algebra will be investigated for further solutions under transformations due to the discrete symmetries.

3.1 Discrete Symmetries Analysis

In this section we find all the discrete symmetries of the KdV equation.

3.1.1 Infinitesimal Generators of (One-Parameter) Lie Groups of Point Symmetries of the KdV Equation

Following are the infinitesimal generators of (one-parameter) Lie groups of point symmetries of the KdV equation [12]

$$\begin{aligned}
\mathbf{X}_1 &= \frac{\partial}{\partial x}, \\
\mathbf{X}_2 &= x\frac{\partial}{\partial x} + 3t\frac{\partial}{\partial t} - 2u\frac{\partial}{\partial u}, \\
\mathbf{X}_3 &= 6t\frac{\partial}{\partial x} + \frac{\partial}{\partial u}, \\
\mathbf{X}_4 &= \frac{\partial}{\partial t}.
\end{aligned}$$

3.1.2 Corresponding Lie Groups of Point Symmetries

Following are the (one-parameter) Lie groups of point symmetries of the KdV equation

$$\begin{aligned}
&G_1 \quad (\hat{x}, \hat{t}, \hat{u}) = (x + \epsilon,\ t,\ u), \\
&G_2 \quad (\hat{x}, \hat{t}, \hat{u}) = (e^{\epsilon}x,\ e^{3\epsilon}t,\ e^{-2\epsilon}u), \\
&G_3 \quad (\hat{x}, \hat{t}, \hat{u}) = (x + 6t\epsilon,\ t,\ u + \epsilon), \\
&G_4 \quad (\hat{x}, \hat{t}, \hat{u}) = (x,\ t + \epsilon,\ u).
\end{aligned}$$

3.1.3 Nonzero Structure Constants

Following are the nonzero structure constants obtained after calculating the commutators of the basis vectors of the Lie algebra

$$\begin{array}{llll}
c^1_{12} = 1, & c^1_{21} = -1, & c^3_{23} = 2, & c^3_{32} = -2, \\
c^1_{34} = 6, & c^1_{43} = -6, & c^4_{24} = -3, & c^4_{42} = 3.
\end{array}$$

3.1.4 Nonlinear Constraints

We simplify now the matrix $B = (b^l_i)$. We substitute the above nonzero structure constants in the respective nonlinear constraints

$$c^n_{lm} b^l_i b^m_j = c^k_{ij} b^n_k, \quad i, j, k, l, m, n = 1, \cdots, 4\,.$$

For $n = 2$

$$c_{lm}^2 = 0, \qquad l, m = 1, 2, 3, 4.$$

The constraints reduce to

$$\begin{aligned} 0 &= c_{ij}^k b_k^2, \\ 0 &= c_{ij}^1 b_1^2 + c_{ij}^2 b_2^2 + c_{ij}^3 b_3^2 + c_{ij}^4 b_4^2. \end{aligned}$$

When $(i, j) = (1, 2)$

$$\begin{aligned} 0 &= c_{12}^1 b_1^2 + c_{12}^2 b_2^2 + c_{12}^3 b_3^2 + c_{12}^4 b_4^2, \\ 0 &= (1) b_1^2 + (0) b_2^2 + (0) b_3^2 + (0) b_4^2, \\ 0 &= b_1^2. \end{aligned}$$

When $(i, j) = (1, 3)$

$$\begin{aligned} 0 &= c_{13}^1 b_1^2 + c_{13}^2 b_2^2 + c_{13}^3 b_3^2 + c_{13}^4 b_4^2, \\ 0 &= (0)(0) + (0) b_2^2 + (0) b_3^2 + (0) b_4^2, \\ 0 &= 0. \end{aligned}$$

When $(i, j) = (1, 4)$

$$\begin{aligned} 0 &= c_{14}^1 b_1^2 + c_{14}^2 b_2^2 + c_{14}^3 b_3^2 + c_{14}^4 b_4^2, \\ 0 &= (0)(0) + (0) b_2^2 + (0) b_3^2 + (0) b_4^2, \\ 0 &= 0. \end{aligned}$$

When $(i, j) = (2, 3)$

$$\begin{aligned} 0 &= c_{23}^1 b_1^2 + c_{23}^2 b_2^2 + c_{23}^3 b_3^2 + c_{23}^4 b_4^2, \\ 0 &= (0)(0) + (0) b_2^2 + (2) b_3^2 + (0) b_4^2, \\ 0 &= b_3^2. \end{aligned}$$

When $(i, j) = (2, 4)$

$$\begin{aligned} 0 &= c_{24}^1 b_1^2 + c_{24}^2 b_2^2 + c_{24}^3 b_3^2 + c_{24}^4 b_4^2, \\ 0 &= (0)(0) + (0) b_2^2 + (0)(0) + (-3) b_4^2, \\ 0 &= b_4^2. \end{aligned}$$

When $(i,j)=(3,4)$

$$\begin{aligned}0 &= c_{34}^1 b_1^2 + c_{34}^2 b_2^2 + c_{34}^3 b_3^2 + c_{34}^4 b_4^2,\\ 0 &= (6)(0) + (0)b_2^2 + (0)(0) + (0)(0),\\ 0 &= 0.\end{aligned}$$

For $n=3$

$$c_{lm}^3 = 0, \qquad (l,m) \neq (2,3),(3,2).$$

The constraints reduce to

$$\begin{aligned}c_{23}^3 b_i^2 b_j^3 + c_{32}^3 b_i^3 b_j^2 &= c_{ij}^k b_k^3,\\ (2)b_i^2 b_j^3 + (-2)b_i^3 b_j^2 &= c_{ij}^1 b_1^3 + c_{ij}^2 b_2^3 + c_{ij}^3 b_3^3 + c_{ij}^4 b_4^3.\end{aligned}$$

When $(i,j)=(1,2)$

$$\begin{aligned}(2)b_1^2 b_2^3 + (-2)b_1^3 b_2^2 &= c_{12}^1 b_1^3 + c_{12}^2 b_2^3 + c_{12}^3 b_3^3 + c_{12}^4 b_4^3,\\ (2)(0)b_2^3 + (-2)b_1^3 b_2^2 &= (1)b_1^3 + (0)b_2^3 + (0)b_3^3 + (0)b_4^3,\\ 0 &= b_1^3 + 2b_1^3 b_2^2.\end{aligned}$$

When $(i,j)=(1,3)$

$$\begin{aligned}(2)b_1^2 b_3^3 + (-2)b_1^3 b_3^2 &= c_{13}^1 b_1^3 + c_{13}^2 b_2^3 + c_{13}^3 b_3^3 + c_{13}^4 b_4^3,\\ (2)(0)b_3^3 + (-2)b_1^3(0) &= (0)b_1^3 + (0)b_2^3 + (0)b_3^3 + (0)b_4^3,\\ 0 &= 0.\end{aligned}$$

When $(i,j)=(1,4)$

$$\begin{aligned}(2)b_1^2 b_4^3 + (-2)b_1^3 b_4^2 &= c_{14}^1 b_1^3 + c_{14}^2 b_2^3 + c_{14}^3 b_3^3 + c_{14}^4 b_4^3,\\ (2)(0)b_4^3 + (-2)b_1^3(0) &= (0)b_1^3 + (0)b_2^3 + (0)b_3^3 + (0)b_4^3,\\ 0 &= 0.\end{aligned}$$

When $(i,j)=(2,3)$

$$\begin{aligned}(2)b_2^2 b_3^3 + (-2)b_2^3 b_3^2 &= c_{23}^1 b_1^3 + c_{23}^2 b_2^3 + c_{23}^3 b_3^3 + c_{23}^4 b_4^3,\\ (2)b_2^2 b_3^3 + (-2)b_2^3(0) &= (0)b_1^3 + (0)b_2^3 + (2)b_3^3 + (0)b_4^3,\\ 0 &= b_3^3 - b_2^2 b_3^3\end{aligned}$$

When $(i,j)=(2,4)$

$$\begin{aligned}(2)b_2^2b_4^3+(-2)b_2^3b_4^2 &= c_{24}^1b_1^3+c_{24}^2b_2^3+c_{24}^3b_3^3+c_{24}^4b_4^3,\\ (2)b_2^2b_4^3+(-2)b_2^3(0) &= (0)b_1^3+(0)b_2^3+(0)b_3^3+(-3)b_4^3,\\ 2b_2^2b_4^3+3b_4^3 &= 0.\end{aligned}$$

When $(i,j)=(3,4)$

$$\begin{aligned}(2)b_3^2b_4^3+(-2)b_3^3b_4^2 &= c_{34}^1b_1^3+c_{34}^2b_2^3+c_{34}^3b_3^3+c_{34}^4b_4^3,\\ (2)(0)b_4^3+(-2)b_3^3(0) &= (6)b_1^3+(0)b_2^3+(0)b_3^3+(0)b_4^3,\\ 0 &= b_1^3.\end{aligned}$$

For $n=4$

$$c_{lm}^4=0, \qquad (l,m)\neq(4,2),(2,4).$$

The constraints reduce to

$$\begin{aligned}c_{42}^4b_i^4b_j^2+c_{24}^4b_i^2b_j^4 &= c_{ij}^kb_k^4,\\ (3)b_i^4b_j^2+(-3)b_i^2b_j^4 &= c_{ij}^1b_1^4+c_{ij}^2b_2^4+c_{ij}^3b_3^4+c_{ij}^4b_4^4.\end{aligned}$$

When $(i,j)=(1,2)$

$$\begin{aligned}(3)b_1^4b_2^2+(-3)b_1^2b_2^4 &= c_{12}^1b_1^4+c_{12}^2b_2^4+c_{12}^3b_3^4+c_{12}^4b_4^4,\\ (3)b_1^4b_2^2+(-3)(0)b_2^4 &= (1)b_1^4+(0)b_2^4+(0)b_3^4+(0)b_4^4,\\ 3b_1^4b_2^2-b_1^4 &= 0.\end{aligned}$$

When $(i,j)=(1,3)$

$$\begin{aligned}(3)b_1^4b_3^2+(-3)b_1^2b_3^4 &= c_{13}^1b_1^4+c_{13}^2b_2^4+c_{13}^3b_3^4+c_{13}^4b_4^4,\\ (3)b_1^4(0)+(-3)(0)b_3^4 &= (0)b_1^4+(0)b_2^4+(0)b_3^4+(0)b_4^4,\\ 0 &= 0.\end{aligned}$$

When $(i,j)=(1,4)$

$$\begin{aligned}(3)b_1^4b_4^2+(-3)b_1^2b_4^4 &= c_{14}^1b_1^4+c_{14}^2b_2^4+c_{14}^3b_3^4+c_{14}^4b_4^4,\\ (3)b_1^4(0)+(-3)(0)b_4^4 &= (0)b_1^4+(0)b_2^4+(0)b_3^4+(0)b_4^4,\\ 0 &= 0.\end{aligned}$$

When $(i,j) = (2,3)$

$$\begin{aligned}(3)b_2^4 b_3^2 + (-3)b_2^2 b_3^4 &= c_{23}^1 b_1^4 + c_{23}^2 b_2^4 + c_{23}^3 b_3^4 + c_{23}^4 b_4^4,\\ (3)b_2^4(0) + (-3)b_2^2 b_3^4 &= (0)b_1^4 + (0)b_2^4 + (2)b_3^4 + (0)b_4^4,\\ 0 &= 2b_3^4 + 3b_2^2 b_3^4.\end{aligned}$$

When $(i,j) = (2,4)$

$$\begin{aligned}(3)b_2^4 b_4^2 + (-3)b_2^2 b_4^4 &= c_{24}^1 b_1^4 + c_{24}^2 b_2^4 + c_{24}^3 b_3^4 + c_{24}^4 b_4^4,\\ (3)b_2^4(0) + (-3)b_2^2 b_4^4 &= (0)b_1^4 + (0)b_2^4 + (0)b_3^4 + (-3)b_4^4,\\ b_2^2 b_4^4 - b_4^4 &= 0.\end{aligned}$$

When $(i,j) = (3,4)$

$$\begin{aligned}(3)b_3^4 b_4^2 + (-3)b_3^2 b_4^4 &= c_{34}^1 b_1^4 + c_{34}^2 b_2^4 + c_{34}^3 b_3^4 + c_{34}^4 b_4^4,\\ (3)b_3^4(0) + (-3)(0)b_4^4 &= (6)b_1^4 + (0)b_2^4 + (0)b_3^4 + (0)b_4^4,\\ 0 &= b_1^4.\end{aligned}$$

For $n = 1$

$$c_{lm}^1 = 0, \qquad (l,m) \neq (1,2),(2,1),(3,4),(4,3).$$

The constraints reduce to

$$\begin{aligned}c_{12}^1 b_i^1 b_j^2 + c_{21}^1 b_i^2 b_j^1 + c_{34}^1 b_i^3 b_j^4 + c_{43}^1 b_i^4 b_j^3 &= c_{ij}^k b_k^1,\\ (1)b_i^1 b_j^2 + (-1)b_i^2 b_j^1 + (6)b_i^3 b_j^4 + (-6)b_i^4 b_j^3 &= c_{ij}^1 b_1^1 + c_{ij}^2 b_2^1 + c_{ij}^3 b_3^1 + c_{ij}^4 b_4^1.\end{aligned}$$

When $(i,j) = (1,2)$

$$\begin{aligned}(1)b_1^1 b_2^2 + (-1)b_1^2 b_2^1 + (6)b_1^3 b_2^4 + (-6)b_1^4 b_2^3 &= c_{12}^1 b_1^1 + c_{12}^2 b_2^1 + c_{12}^3 b_3^1 + c_{12}^4 b_4^1,\\ (1)b_1^1 b_2^2 + (-1)(0)b_2^1 + (6)(0)b_2^4 + (-6)(0)b_2^3 &= (1)b_1^1 + (0)b_2^1 + (0)b_3^1 + (0)b_4^1,\\ b_1^1 b_2^2 - b_1^1 &= 0.\end{aligned}$$

When $(i,j) = (1,3)$

$$\begin{aligned}(1)b_1^1 b_3^2 + (-1)b_1^2 b_3^1 + (6)b_1^3 b_3^4 + (-6)b_1^4 b_3^3 &= c_{13}^1 b_1^1 + c_{13}^2 b_2^1 + c_{13}^3 b_3^1 + c_{13}^4 b_4^1,\\ (1)b_1^1(0) + (-1)(0)b_3^1 + (6)(0)b_3^4 + (-6)(0)b_3^3 &= (0)b_1^1 + (0)b_2^1 + (0)b_3^1 + (0)b_4^1,\\ 0 &= 0.\end{aligned}$$

When $(i,j) = (1,4)$

$$\begin{aligned}(1)b_1^1 b_4^2 + (-1)b_1^2 b_4^1 + (6)b_1^3 b_4^4 + (-6)b_1^4 b_4^3 &= c_{14}^1 b_1^1 + c_{14}^2 b_2^1 + c_{14}^3 b_3^1 + c_{14}^4 b_4^1,\\ (1)b_1^1(0) + (-1)(0)b_4^1 + (6)(0)b_4^4 + (-6)(0)b_4^3 &= (0)b_1^1 + (0)b_2^1 + (0)b_3^1 + (0)b_4^1,\\ 0 &= 0.\end{aligned}$$

When $(i,j) = (2,3)$

$$\begin{aligned}(1)b_2^1 b_3^2 + (-1)b_2^2 b_3^1 + (6)b_2^3 b_3^4 + (-6)b_2^4 b_3^3 &= c_{23}^1 b_1^1 + c_{23}^2 b_2^1 + c_{23}^3 b_3^1 + c_{23}^4 b_4^1,\\ (1)b_2^1(0) + (-1)b_2^2 b_3^1 + (6)b_2^3 b_3^4 + (-6)b_2^4 b_3^3 &= (0)b_1^1 + (0)b_2^1 + (2)b_3^1 + (0)b_4^1,\\ -b_2^2 b_3^1 + 6b_2^3 b_3^4 - 6b_2^4 b_3^3 &= 2b_3^1.\end{aligned}$$

When $(i,j) = (2,4)$

$$\begin{aligned}(1)b_2^1 b_4^2 + (-1)b_2^2 b_4^1 + (6)b_2^3 b_4^4 + (-6)b_2^4 b_4^3 &= c_{24}^1 b_1^1 + c_{24}^2 b_2^1 + c_{24}^3 b_3^1 + c_{24}^4 b_4^1,\\ (1)b_2^1(0) + (-1)b_2^2 b_4^1 + (6)b_2^3 b_4^4 + (-6)b_2^4 b_4^3 &= (0)b_1^1 + (0)b_2^1 + (0)b_3^1 + (-3)b_4^1,\\ -b_2^2 b_4^1 + 6b_2^3 b_4^4 - 6b_2^4 b_4^3 &= -3b_4^1.\end{aligned}$$

When $(i,j) = (3,4)$

$$\begin{aligned}(1)b_3^1 b_4^2 + (-1)b_3^2 b_4^1 + (6)b_3^3 b_4^4 + (-6)b_3^4 b_4^3 &= c_{34}^1 b_1^1 + c_{34}^2 b_2^1 + c_{34}^3 b_3^1 + c_{34}^4 b_4^1,\\ (1)b_3^1(0) + (-1)(0)b_4^1 + (6)b_3^3 b_4^4 + (-6)b_3^4 b_4^3 &= (6)b_1^1 + (0)b_2^1 + (0)b_3^1 + (0)b_4^1,\\ b_3^3 b_4^4 - b_3^4 b_4^3 &= b_1^1.\end{aligned}$$

So far we have been able to simplyfy the system of nonlinear constraints as

$$\begin{aligned}0 &= b_1^2,\\ 0 &= b_3^2,\\ 0 &= b_4^2,\\ 0 &= b_1^3 + 2b_1^3 b_2^2,\\ 0 &= b_3^3 - b_2^2 b_3^3,\\ 0 &= 2b_2^2 b_4^3 + 3b_4^3,\\ 0 &= b_1^3,\end{aligned}$$

$$
\begin{aligned}
0 &= 3b_1^4 b_2^2 - b_1^4, \\
0 &= 2b_3^4 + 3b_2^2 b_3^4, \\
0 &= b_2^2 b_4^4 - b_4^4, \\
0 &= b_1^4, \\
0 &= b_1^1 b_2^2 - b_1^1, \\
2b_3^1 &= -b_2^2 b_3^1 + 6b_2^3 b_3^4 - 6b_2^4 b_3^3, \\
-3b_4^1 &= -b_2^2 b_4^1 + 6b_2^3 b_4^4 - 6b_2^4 b_4^3, \\
b_1^1 &= b_3^3 b_4^4 - b_3^4 b_4^3.
\end{aligned}
$$

Since $B = (b_i^l)$ is nonsingular and all other elements of the first row of B are zero, therefore $b_1^1 \neq 0$. Now from $0 = b_1^1 b_2^2 - b_1^1$ it follows that $b_2^2 = 1$. Substituting the value of b_2^2 in respective equations we conclude that

$$
B = \begin{bmatrix} b_1^1 & 0 & 0 & 0 \\ b_2^1 & 1 & b_2^3 & b_2^4 \\ b_3^1 & 0 & b_3^3 & 0 \\ b_4^1 & 0 & 0 & b_4^4 \end{bmatrix},
$$

with further conditions

$$
\begin{aligned}
0 &= b_3^1 + 2b_2^4 b_3^3, \\
0 &= b_4^1 + 3b_2^3 b_4^4, \\
0 &= b_1^1 - b_3^3 b_4^4, \\
& b_1^1, b_3^3, b_4^4 \neq 0.
\end{aligned}
$$

3.1.5 Inequivalent Symmetries

We recall from Theorem (2.3.6) that

$$(C(j))_i^k = c_{ij}^k,$$

and

$$\begin{aligned}
A(j,\epsilon) &= e^{\epsilon C(j)},\\
A(j,\epsilon) &= \sum_{n=0}^{\infty} \frac{\epsilon^n}{n!}(C(j))^n,\\
A(j,\epsilon) &= I + \frac{\epsilon}{1!}(C(j)) + \frac{\epsilon^2}{2!}(C(j))^2 + \frac{\epsilon^3}{3!}(C(j))^3 + ...
\end{aligned}$$

Calculating matrices $C(j)$

$$C(1) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

$$C(2) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix},$$

$$C(3) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 \\ -6 & 0 & 0 & 0 \end{bmatrix},$$

$$C(4) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -3 \\ 6 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Calculating matrices $A(j, \epsilon)$

$(C(1))^n = 0 \quad \text{for} \quad 2 \leq n,$

$$A(1, \epsilon) = I + \epsilon\, C(1) + 0,$$

$$A(1, \epsilon) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -\epsilon & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

$$A(2, \epsilon) = \begin{bmatrix} e^{\epsilon} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & e^{-2\epsilon} & 0 \\ 0 & 0 & 0 & e^{3\epsilon} \end{bmatrix},$$

because $C(2)$ is a digonal matrix.

$(C(3))^n = 0 \quad \text{for} \quad 2 \leq n,$

$$A(3, \epsilon) = I + \epsilon\, C(3) + 0,$$

$$A(3, \epsilon) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 2\epsilon & 0 \\ 0 & 0 & 1 & 0 \\ -6\epsilon & 0 & 0 & 1 \end{bmatrix}.$$

$(C(4))^n = 0 \quad \text{for} \quad 2 \leq n,$

$$A(4, \epsilon) = I + \epsilon\, C(4) + 0,$$

$$A(4, \epsilon) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -3\epsilon \\ 6\epsilon & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Now we calculate the inequivalent matrices

$$BA(3,\epsilon) = \begin{bmatrix} b_1^1 & 0 & 0 & 0 \\ b_2^1 & 1 & b_2^3 & b_2^4 \\ b_3^1 & 0 & b_3^3 & 0 \\ b_4^1 & 0 & 0 & b_4^4 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 2\epsilon & 0 \\ 0 & 0 & 1 & 0 \\ -6\epsilon & 0 & 0 & 1 \end{bmatrix}$$
$$= \begin{bmatrix} b_1^1 & 0 & 0 & 0 \\ b_2^1 - b_2^4 6\epsilon & 1 & 2\epsilon + b_2^3 & b_2^4 \\ b_3^1 & 0 & b_3^3 & 0 \\ b_4^1 - 6\epsilon b_4^4 & 0 & 0 & b_4^4 \end{bmatrix}.$$

Choosing $\epsilon = \epsilon_3 = -\frac{1}{2}b_2^3$,

$$b_2^1 - b_2^4 6\epsilon = b_2^1 - b_2^4 6\epsilon_3 = b_2^1 - b_2^4 6(-\frac{1}{2}b_2^3) = b_2^1 + 3b_2^4 b_2^3,$$

$$2\epsilon + b_2^3 = 2\epsilon_3 + b_2^3 = 2(-\frac{1}{2}b_2^3) + b_2^3 = 0,$$

$$b_4^1 - 6\epsilon b_4^4 = b_4^1 - 6\epsilon_3 b_4^4 = b_4^1 - 6(-\frac{1}{2}b_2^3)b_4^4 = b_4^1 + 3b_2^3 b_4^4 = 0,$$

so

$$BA(3,\epsilon_3) = \begin{bmatrix} b_1^1 & 0 & 0 & 0 \\ b_2^1 + 3b_2^4 b_2^3 & 1 & 0 & b_2^4 \\ b_3^1 & 0 & b_3^3 & 0 \\ 0 & 0 & 0 & b_4^4 \end{bmatrix}.$$

Now

$$BA(3,\epsilon_3))A(4,\epsilon) = \begin{bmatrix} b_1^1 & 0 & 0 & 0 \\ b_2^1 + 3b_2^4 b_2^3 & 1 & 0 & b_2^4 \\ b_3^1 & 0 & b_3^3 & 0 \\ 0 & 0 & 0 & b_4^4 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -3\epsilon \\ 6\epsilon & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$
$$= \begin{bmatrix} b_1^1 & 0 & 0 & 0 \\ b_2^1 + 3b_2^4 b_2^3 & 1 & 0 & b_2^4 - 3\epsilon \\ b_3^1 + 6\epsilon b_3^3 & 0 & b_3^3 & 0 \\ 0 & 0 & 0 & b_4^4 \end{bmatrix}.$$

Choosing $\epsilon = \epsilon_4 = \frac{1}{3}b_2^4$,

$$b_2^4 - 3\epsilon = b_2^4 - 3\epsilon_4 = b_2^4 - 3(\frac{1}{3}b_2^4) = 0,$$

$$b_3^1 + 6\epsilon b_3^3 = b_3^1 + 6\epsilon_4 b_3^3 = b_3^1 + 6(\frac{1}{3}b_2^4)b_3^3 = b_3^1 + 2b_2^4 b_3^3 = 0,$$

so

$$BA(3,\epsilon_3)A(4,\epsilon_4) = \begin{bmatrix} b_1^1 & 0 & 0 & 0 \\ b_2^1 + 3b_2^4 b_2^3 & 1 & 0 & 0 \\ 0 & 0 & b_3^3 & 0 \\ 0 & 0 & 0 & b_4^4 \end{bmatrix}.$$

Now

$$\begin{aligned} BA(3,\epsilon_3)A(4,\epsilon_4)A(1,\epsilon) &= \begin{bmatrix} b_1^1 & 0 & 0 & 0 \\ b_2^1 + 3b_2^4 b_2^3 & 1 & 0 & 0 \\ 0 & 0 & b_3^3 & 0 \\ 0 & 0 & 0 & b_4^4 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ -\epsilon & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} b_1^1 & 0 & 0 & 0 \\ b_2^1 + 3b_2^4 b_2^3 - \epsilon & 1 & 0 & 0 \\ 0 & 0 & b_3^3 & 0 \\ 0 & 0 & 0 & b_4^4 \end{bmatrix}. \end{aligned}$$

Choosing $\epsilon = \epsilon_1 = b_2^1 + 3b_2^4 b_2^3$,

$$b_2^1 + 3b_2^4 b_2^3 - \epsilon = b_2^1 + 3b_2^4 b_2^3 - \epsilon_1 = b_2^1 + 3b_2^4 b_2^3 - (b_2^1 + 3b_2^4 b_2^3) = 0,$$

so

$$BA(3,\epsilon_3)A(4,\epsilon_4)A(1,\epsilon_1) = \begin{bmatrix} b_1^1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & b_3^3 & 0 \\ 0 & 0 & 0 & b_4^4 \end{bmatrix}.$$

Now

$$BA(3,\epsilon_3)A(4,\epsilon_4)A(1,\epsilon_1)A(2,\epsilon) = \begin{bmatrix} b_1^1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & b_3^3 & 0 \\ 0 & 0 & 0 & b_4^4 \end{bmatrix} \begin{bmatrix} e^{\epsilon} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & e^{-2\epsilon} & 0 \\ 0 & 0 & 0 & e^{3\epsilon} \end{bmatrix}$$
$$= \begin{bmatrix} b_1^1 e^{\epsilon} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & b_3^3 e^{-2\epsilon} & 0 \\ 0 & 0 & 0 & b_4^4 e^{3\epsilon} \end{bmatrix}.$$

Choosing $\epsilon = \epsilon_2 = -\ln|b_1^1|$,

$$b_1^1 e^{\epsilon} = b_1^1 e^{\epsilon_2} = b_1^1 e^{-\ln|b_1^1|} = \frac{b_1^1}{|b_1^1|},$$

$$b_3^3 e^{-2\epsilon} = b_3^3 e^{-2\epsilon_2} = b_3^3 e^{-2(-\ln|b_1^1|)} = b_3^3 (b_1^1)^2,$$
$$b_4^4 e^{3\epsilon} = b_4^4 e^{3\epsilon_2} = b_4^4 e^{3(-\ln|b_1^1|)} = \frac{b_4^4}{(|b_1^1|)^3},$$

so

$$BA(3,\epsilon_3)A(4,\epsilon_4)A(1,\epsilon_1)A(2,\epsilon_2) = \begin{bmatrix} \frac{b_1^1}{|b_1^1|} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & b_3^3(b_1^1)^2 & 0 \\ 0 & 0 & 0 & \frac{b_4^4}{(|b_1^1|)^3} \end{bmatrix}.$$

Let $\tilde{B} = BA(3,\epsilon_3)A(4,\epsilon_4)A(1,\epsilon_1)A(2,\epsilon_2)$.

These are the required inequivalent matrices.

3.1.6 Solution of the System of Determining Equations

Following is the system of determining equations with the inequivalent matrices calculated in the previous section.

$$\begin{bmatrix} \mathbf{X}_1\hat{x} & \mathbf{X}_1\hat{t} & \mathbf{X}_1\hat{u} \\ \mathbf{X}_2\hat{x} & \mathbf{X}_2\hat{t} & \mathbf{X}_2\hat{u} \\ \mathbf{X}_3\hat{x} & \mathbf{X}_3\hat{t} & \mathbf{X}_3\hat{u} \\ \mathbf{X}_4\hat{x} & \mathbf{X}_4\hat{t} & \mathbf{X}_4\hat{u} \end{bmatrix} = \begin{bmatrix} \frac{b_1^1}{|b_1^1|} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & b_3^3(b_1^1)^2 & 0 \\ 0 & 0 & 0 & \frac{b_4^4}{(|b_1^1|)^3} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ \hat{x} & 3\hat{t} & -2\hat{u} \\ 6\hat{t} & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}.$$

$$\begin{bmatrix} \mathbf{X}_1\hat{x} & \mathbf{X}_1\hat{t} & \mathbf{X}_1\hat{u} \\ \mathbf{X}_2\hat{x} & \mathbf{X}_2\hat{t} & \mathbf{X}_2\hat{u} \\ \mathbf{X}_3\hat{x} & \mathbf{X}_3\hat{t} & \mathbf{X}_3\hat{u} \\ \mathbf{X}_4\hat{x} & \mathbf{X}_4\hat{t} & \mathbf{X}_4\hat{u} \end{bmatrix} = \begin{bmatrix} \frac{b_1^1}{|b_1^1|} & 0 & 0 \\ \hat{x} & 3\hat{t} & -2\hat{u} \\ b_3^3(b_1^1)^2 6\hat{t} & 0 & b_3^3(b_1^1)^2 \\ 0 & \frac{b_4^4}{(|b_1^1|)^3} & 0 \end{bmatrix}.$$

Now we solve the above system of determining equations and seek a solution of the form

$$(\hat{x}, \hat{t}, \hat{u}) = (\ \hat{x}(x,t,u),\ \hat{t}(x,t,u),\ \hat{u}(x,t,u))\,.$$

We have

$$\mathbf{X}_1\hat{u} = 0,$$
$$\frac{\partial \hat{u}}{\partial x} = 0,$$
$$\hat{u} = A(t,u).$$

We also have

$$\mathbf{X}_4\hat{u} = 0,$$
$$\frac{\partial \hat{u}}{\partial t} = 0,$$
$$\frac{\partial \hat{u}}{\partial t} = \frac{\partial A(t,u)}{\partial t} = 0.$$

So

$$\begin{aligned} A(t,u) &= A(u), \\ \hat{u} &= A(u). \end{aligned}$$

Now

$$\begin{aligned} \mathbf{X}_3\hat{u} &= b_3^3(b_1^1)^2, \\ 6t\frac{\partial \hat{u}}{\partial x} + \frac{\partial \hat{u}}{\partial u} &= b_3^3(b_1^1)^2, \\ A'(u) &= b_3^3(b_1^1)^2, \\ A(u) &= b_3^3(b_1^1)^2 u + c_1, \\ \hat{u} &= b_3^3(b_1^1)^2 u + c_1. \end{aligned}$$

But we also have

$$\begin{aligned} \mathbf{X}_2\hat{u} &= -2\hat{u}, \\ x\frac{\partial \hat{u}}{\partial x} + 3t\frac{\partial \hat{u}}{\partial t} - 2u\frac{\partial \hat{u}}{\partial u} &= -2\hat{u}, \\ uA'(u) &= A(u), \\ ub_3^3(b_1^1)^2 &= b_3^3(b_1^1)^2 u + c_1, \\ 0 &= c_1. \end{aligned}$$

So

$$\hat{u} = b_3^3(b_1^1)^2 u.$$

Next we have

$$\begin{aligned} \mathbf{X}_1\hat{t} &= 0, \\ \frac{\partial \hat{t}}{\partial x} &= 0, \\ \hat{t} &= B(t,u). \end{aligned}$$

We also have

$$
\begin{aligned}
\mathbf{X}_3\hat{t} &= 0,\\
6t\frac{\partial \hat{t}}{\partial x} + \frac{\partial \hat{t}}{\partial u} &= 0,\\
\frac{\partial B}{\partial u} &= 0.
\end{aligned}
$$

So

$$
\begin{aligned}
B(t,u) &= B(t),\\
\hat{t} &= B(t).
\end{aligned}
$$

Now

$$
\begin{aligned}
\mathbf{X}_4\hat{t} &= \frac{b_4^4}{(|b_1^1|)^3},\\
\frac{\partial \hat{t}}{\partial t} &= \frac{b_4^4}{(|b_1^1|)^3},\\
B'(t) &= \frac{b_4^4}{(|b_1^1|)^3},\\
B(t) &= \frac{b_4^4}{(|b_1^1|)^3}t + c_2,\\
\hat{t} &= \frac{b_4^4}{(|b_1^1|)^3}t + c_2.
\end{aligned}
$$

But we also have

$$
\begin{aligned}
\mathbf{X}_2\hat{t} &= 3\hat{t},\\
x\frac{\partial \hat{t}}{\partial x} + 3t\frac{\partial \hat{t}}{\partial t} - 2u\frac{\partial \hat{t}}{\partial u} &= 3\hat{t},\\
tB'(t) &= B(t),\\
t\frac{b_4^4}{(|b_1^1|)^3} &= \frac{b_4^4}{(|b_1^1|)^3}t + c_2,\\
0 &= c_2.
\end{aligned}
$$

So

$$\hat{t} = \frac{b_4^4}{(|b_1^1|)^3} t.$$

Next we have

$$\begin{aligned}
\mathbf{X}_4 \hat{x} &= 0, \\
\frac{\partial \hat{x}}{\partial t} &= 0, \\
\hat{x} &= C(x, u).
\end{aligned}$$

We also have

$$\begin{aligned}
\mathbf{X}_1 \hat{x} &= \frac{b_1^1}{|b_1^1|}, \\
\frac{\partial \hat{x}}{\partial x} &= \frac{b_1^1}{|b_1^1|}, \\
\frac{\partial C(x, u)}{\partial x} &= \frac{b_1^1}{|b_1^1|}, \\
C(x, u) &= \frac{b_1^1}{|b_1^1|} x + D(u).
\end{aligned}$$

So

$$\hat{x} = \frac{b_1^1}{|b_1^1|} x + D(u).$$

Now

$$\begin{aligned}
\mathbf{X}_3 \hat{x} &= b_3^3 (b_1^1)^2 6\hat{t}, \\
6t \frac{\partial \hat{x}}{\partial x} + \frac{\partial \hat{x}}{\partial u} &= b_3^3 (b_1^1)^2 6\hat{t}, \\
6t \frac{b_1^1}{|b_1^1|} + D'(u) &= b_3^3 (b_1^1)^2 6 \Big(\frac{b_4^4}{(|b_1^1|)^3} t \Big), \\
D'(u) &= 6t \Big[\frac{b_3^3 b_4^4}{|b_1^1|} - \frac{b_1^1}{|b_1^1|} \Big], \\
D'(u) &= 6t \Big[\frac{b_1^1}{|b_1^1|} - \frac{b_1^1}{|b_1^1|} \Big], \ \text{using } (b_1^1 = b_3^3 b_4^4), \\
D'(u) &= 0, \\
D(u) &= c_3.
\end{aligned}$$

So

$$\hat{x} = \frac{b_1^1}{|b_1^1|}x + c_3.$$

But we also have

$$\begin{aligned} \mathbf{X}_2\hat{x} &= \hat{x}, \\ x\frac{\partial \hat{x}}{\partial x} + 3t\frac{\partial \hat{x}}{\partial t} - 2u\frac{\partial \hat{x}}{\partial u} &= \hat{x}, \\ x\frac{b_1^1}{|b_1^1|} &= \frac{b_1^1}{|b_1^1|}x + c_3, \\ 0 &= c_3. \end{aligned}$$

So

$$\hat{x} = \frac{b_1^1}{|b_1^1|}x.$$

Therefore, the general solution of the system of determining equations is

$$(\hat{x}, \hat{t}, \hat{u}) = (\ \frac{b_1^1}{|b_1^1|}x,\ \frac{b_4^4}{(|b_1^1|)^3}t,\ b_3^3(b_1^1)^2u\ ).$$

3.1.7 Symmetry Condition

To apply the symmetry condition we find

$$\frac{\partial \hat{u}}{\partial \hat{t}} = \frac{\partial(b_3^3(b_1^1)^2u)}{\partial(\frac{b_4^4}{(|b_1^1|)^3}t)} = (b_3^3)^2(b_1^1)^3|b_1^1|\frac{\partial u}{\partial t}.$$

$$\frac{\partial \hat{u}}{\partial \hat{x}} = \frac{\partial(b_3^3(b_1^1)^2u)}{\partial(\frac{b_1^1}{|b_1^1|}x)} = b_3^3(b_1^1)|b_1^1|\frac{\partial u}{\partial x}.$$

$$\frac{\partial^2 \hat{u}}{\partial \hat{x}^2} = \frac{\partial}{\partial \hat{x}}\Big(\frac{\partial \hat{u}}{\partial \hat{x}}\Big) = \frac{\partial}{\partial(\frac{b_1^1}{|b_1^1|}x)}\Big(b_3^3(b_1^1)|b_1^1|\frac{\partial u}{\partial x}\Big) = b_3^3(b_1^1)^2\frac{\partial^2 u}{\partial x^2}.$$

$$\frac{\partial^3 \hat{u}}{\partial \hat{x}^3} = \frac{\partial}{\partial \hat{x}}\Big(\frac{\partial^2 \hat{u}}{\partial \hat{x}^2}\Big) = \frac{\partial}{\partial(\frac{b_1^1}{|b_1^1|}x)}\Big(b_3^3(b_1^1)^2\frac{\partial^2 u}{\partial x^2}\Big) = b_3^3(b_1^1)|b_1^1|\Big(\frac{\partial^3 u}{\partial x^3}\Big).$$

So

$$\frac{\partial^3 \hat{u}}{\partial \hat{x}^3} + 6\hat{u}\frac{\partial \hat{u}}{\partial \hat{x}} + \frac{\partial \hat{u}}{\partial \hat{t}} = 0,$$

is

$$b_3^3(b_1^1)|b_1^1|\Big(\frac{\partial^3 u}{\partial x^3}\Big) + 6(b_3^3(b_1^1)^2 u)b_3^3(b_1^1)|b_1^1|\frac{\partial u}{\partial x} + (b_3^3)^2(b_1^1)^3|b_1^1|\frac{\partial u}{\partial t} = 0.$$

It simplifies to

$$\Big(\frac{\partial^3 u}{\partial x^3}\Big) + (b_3^3(b_1^1)^2)6u\frac{\partial u}{\partial x} + b_3^3(b_1^1)^2\frac{\partial u}{\partial t} = 0.$$

Which implies $b_3^3(b_1^1)^2 = 1$, if the symmetry condition is to be satisfied.

Using

$$0 = b_1^1 - b_3^3 b_4^4,$$

we can also deduce that

$$\frac{b_4^4}{(|b_1^1|)^3} = \frac{b_1^1}{|b_1^1|}.$$

So

$$(\hat{x}, \hat{t}, \hat{u}) = (\ \frac{b_1^1}{|b_1^1|}x,\ \frac{b_1^1}{|b_1^1|}t,\ u\ ).$$

Therefore, the **only** discrete symmetry of the KdV equation **up to equivalence** is

$$\Gamma_D \quad (\hat{x}, \hat{t}, \hat{u}) = (\ -x,\ -t,\ u\ ).$$

This result is exhaustive. If Γ_D is the discrete symmetry as above, then any other discrete symmetry, $\tilde{\Gamma}_D$, of the KdV equation can be obtained as following

$$\tilde{\Gamma}_D = e^{\epsilon \mathbf{X}} \Gamma_D.$$

For example if we choose $\mathbf{X} = \mathbf{X}_3$, then, $e^{\epsilon \mathbf{X}_3}\Gamma_D$ gives us

$$\begin{aligned} \tilde{\Gamma}_D \quad (\hat{x}, \hat{t}, \hat{u}) &= (\,(-x) + 6(-t)\epsilon,\ (-t),\ (u+\epsilon)\,) \\ &= (\,-x - 6t\epsilon,\ -t,\ u+\epsilon\,). \end{aligned}$$

$\mathbf{X}$ could be any infinitesimal generator from the Lie algebra, $\mathcal{L}$, of infinitesimal generators of (one-parameter) Lie groups of point symmetries of the equation.

Some other examples

$$\begin{aligned} e^{\epsilon \mathbf{X}_1}\Gamma_D \quad (\hat{x}, \hat{t}, \hat{u}) &= ((-x) + \epsilon,\ (-t),\ (u)) \\ &= (-x + \epsilon,\ -t,\ u). \end{aligned}$$

$$\begin{aligned} e^{\epsilon \mathbf{X}_4}\Gamma_D \quad (\hat{x}, \hat{t}, \hat{u}) &= ((-x),\ (-t) + \epsilon,\ (u)) \\ &= (-x,\ -t + \epsilon,\ u). \end{aligned}$$

3.2 Related Exact Solutions

Any exact solution of a differential equation can be examined for new exact solutions using discrete symmetries of the differential equation. In this section it has been attempted to find the group invariant solutions of the KdV equation due to the groups generated by $\mathbf{X}_i$, the basis generators. The group invariant solutions will then be transformed using the discrete symmetries obtained in the last section.

Solutions invariant under the group G, generated by

$$\mathbf{X} = \xi\frac{\partial}{\partial x} + \tau\frac{\partial}{\partial t} + \eta\frac{\partial}{\partial u},$$

satisfy the PDE

$$\eta - \xi\frac{\partial u}{\partial x} - \tau\frac{\partial u}{\partial t} = 0.$$

For $\mathbf{X}_1$ we have

$$\xi = 1, \quad \tau = 0, \quad \eta = 0.$$

$$\begin{aligned} (0) - (1)\frac{\partial u}{\partial x} - (0)\frac{\partial u}{\partial t} &= 0, \\ \frac{\partial u}{\partial x} &= 0, \\ u(x,t) &= F(t). \end{aligned}$$

This solution will now be substituted in KdV equation to determine F. Since

$$\frac{\partial^3 u}{\partial x^3} = 0, \quad 6u\frac{\partial u}{\partial x} = 6F(t)(0) = 0, \quad \frac{\partial u}{\partial t} = F'(t),$$

we have

$$\begin{aligned} F'(t) &= 0, \\ F(t) &= c, \\ u(x,t) &= c. \end{aligned}$$

It is trivial to transform $u = c$.

For $\mathbf{X}_2$ we have

$$\xi = x, \quad \tau = 3t, \quad \eta = -2u.$$

$$(-2u) - (x)\frac{\partial u}{\partial x} - (3t)\frac{\partial u}{\partial t} = 0,$$

$$-2u = x\frac{\partial u}{\partial x} + 3t\frac{\partial u}{\partial t},$$

$$\frac{dx}{x} = \frac{dt}{3t} = \frac{du}{-2u},$$

$$\frac{dx}{x} = \frac{dt}{3t}, \qquad \frac{dx}{x} = \frac{du}{-2u},$$

$$\frac{x^3}{t} = r. \qquad v = ux^2.$$

General solution

$$v = F(r),$$

$$u = \frac{1}{x^2}F(\frac{x^3}{t}).$$

We now substitute this solution in KdV equation to determine F.

$$\frac{\partial u}{\partial t} = -\frac{x}{t^2}F'(r), \quad \frac{\partial^3 u}{\partial x^3} = -\frac{24}{x^5}F(r) + \frac{24}{x^2 t}F'(r) + \frac{27x^4}{t^3}F'''(r),$$

$$6u\frac{\partial u}{\partial x} = -\frac{12}{x^5}\Big(F(r)\Big)^2 + \frac{18}{x^2 t}F'(r)F(r).$$

Substituting in KdV equation, we obtain

$$-\frac{24}{x^5}F(r) + \Big(\frac{24t - x^3}{x^2 t^2}\Big)F'(r) + \frac{27x^4}{t^3}F'''(r) - \frac{12}{x^5}\Big(F(r)\Big)^2 + \frac{18}{x^2 t}F'(r)F(r) = 0.$$

This is a third order nonlinear ODE for F, which appears to be very difficult to solve, if at all possible.

For $\mathbf{X}_4$ we have

$$\xi = 0, \quad \tau = 1, \quad \eta = 0.$$

$$\begin{aligned}(0) - (0)\frac{\partial u}{\partial x} - (1)\frac{\partial u}{\partial t} &= 0,\\ \frac{\partial u}{\partial t} &= 0,\\ u(x,t) &= F(x).\end{aligned}$$

This solution will now be substituted in KdV equation to determine F. Since

$$\frac{\partial^3 u}{\partial x^3} = F'''(x), \quad 6u\frac{\partial u}{\partial x} = 6F(x)F'(x), \quad \frac{\partial u}{\partial t} = 0,$$

we have

$$F''(x) + 3\big(F(x)\big)^2 + c = 0.$$

This is a second order nonlinear ODE for F, which is again very difficult to solve. The form of this equation tells us that its solutions are elliptic functions.

For $\mathbf{X}_3$ we have

$$\xi = 6t, \quad \tau = 0, \quad \eta = 1.$$

$$\begin{aligned}(1) - (6t)\frac{\partial u}{\partial x} - (0)\frac{\partial u}{\partial t} &= 0\\ \frac{\partial u}{\partial x} &= \frac{1}{6t},\\ u(x,t) &= \frac{x}{6t} + F(t).\end{aligned}$$

We now substitute this solution in KdV equation to determine F.

$$\frac{\partial u}{\partial t} = -\frac{x}{6t^2} + F'(t), \quad \frac{\partial u}{\partial x} = \frac{1}{6t}, \quad \frac{\partial^3 u}{\partial x^3} = 0.$$

Substituting in KdV equation, we obtain

$$\begin{aligned} 6\Big(\frac{x}{6t}+F(t)\Big)\Big(\frac{1}{6t}\Big)+\Big(-\frac{x}{6t^2}+F'(t)\Big)&=0,\\ tF'(t)+F(t)&=0,\\ F(t)&=\frac{c}{t}. \end{aligned}$$

So

$$\begin{aligned} u(x,t)&=\frac{x}{6t}+\frac{c}{t}\,, \quad \text{or}\\ u(x,t)&=\frac{x+k}{6t}\,, \quad \text{where} \quad k=6c. \end{aligned}$$

This solution can now be transformed using discrete symmetries to obtain more solutions of the KdV equation.

For instance using

$$e^{\epsilon\mathbf{X}_4}\Gamma_D \quad (\hat{x},\hat{t},\hat{u})=(-x,\ -t+\epsilon,\ u),$$

we obtain a two parameter family of solutions

$$\hat{u}=\frac{(-\hat{x})+k}{6(-\hat{t}+\epsilon)}.$$

Removing carets and rearranging, we have

$$u=\frac{x-k}{6(t-\epsilon)}.$$

Chapter 4

Discrete Symmetries Analysis of a Particular Nonlinear Filtration Equation

The class of equations

$$\frac{\partial u}{\partial t} = D\Big(\frac{\partial u}{\partial x}\Big)\frac{\partial^2 u}{\partial x^2},$$

is known as nonlinear filtration (NLF) equations, D is a function of $\frac{\partial u}{\partial x}$. These equations describe the motion of a non-Newtonian, weakly compressible fluid in a porous medium with a nonlinear filtration law

$$V = -\int D\Big(\frac{\partial u}{\partial x}\Big)\ d\Big(\frac{\partial u}{\partial x}\Big)\ ,$$

where V is the speed of filtration and u is the pressure [14, 15]. The function $D\big(\partial u/\partial x\big)$ is known as filtration coefficient. In general the filtration coefficient is not fixed. NLF equations have been solved for various filtration coefficients [14, 15].

We choose

$$D\Big(\frac{\partial u}{\partial x}\Big) = \left(\frac{1}{1+\left(\frac{\partial u}{\partial x}\right)^2}\right),$$

to find the discrete symmetries for the corresponding NLF equation. As an immediate application of discrete symmetries, group invariant solutions of this NLF equation corresponding to the basis vectors of the Lie algebra will be investigated for further solutions under transformations due to the discrete symmetries.

4.1 Discrete Symmetries Analysis

In this section we find all the discrete symmetries of the following NLF equation.

$$\frac{\partial u}{\partial t} = \left(\frac{1}{1 + \left(\frac{\partial u}{\partial x}\right)^2} \right) \frac{\partial^2 u}{\partial x^2}.$$

4.1.1 Infinitesimal Generators of (One-Parameter) Lie Groups of Point Symmetries of the NLF Equation

Following are the infinitesimal generators of (one-parameter) Lie groups of point symmetries of the NLF equation [1]

$$\begin{aligned} \mathbf{X}_1 &= \frac{\partial}{\partial x}, & \mathbf{X}_4 &= x\frac{\partial}{\partial x} + 2t\frac{\partial}{\partial t} + u\frac{\partial}{\partial u}, \\ \mathbf{X}_2 &= \frac{\partial}{\partial t}, & \mathbf{X}_5 &= u\frac{\partial}{\partial x} - x\frac{\partial}{\partial u}. \\ \mathbf{X}_3 &= \frac{\partial}{\partial u}, & & \end{aligned}$$

4.1.2 Corresponding Lie Groups of Point Symmetries

Following are the (one-parameter) Lie groups of point symmetries of the NLF equation

$$\begin{aligned} &G_1 \quad (\hat{x}, \hat{t}, \hat{u}) = (x + \epsilon,\ t,\ u), \\ &G_2 \quad (\hat{x}, \hat{t}, \hat{u}) = (x,\ t + \epsilon,\ u), \\ &G_3 \quad (\hat{x}, \hat{t}, \hat{u}) = (x,\ t,\ u + \epsilon), \\ &G_4 \quad (\hat{x}, \hat{t}, \hat{u}) = (e^{\epsilon} x,\ e^{2\epsilon} t,\ e^{\epsilon} u), \\ &G_5 \quad (\hat{x}, \hat{t}, \hat{u}) = (x\cos\epsilon + u\sin\epsilon,\ t,\ -x\sin\epsilon + u\cos\epsilon). \end{aligned}$$

4.1.3 Nonzero Structure Constants

Following are the nonzero structure constants obtained after calculating the commutators of the basis vectors of the Lie algebra

$$\begin{array}{llll} c_{14}^1 = 1, & c_{41}^1 = -1, & c_{34}^3 = 1, & c_{43}^3 = -1, \\ c_{35}^1 = 1, & c_{53}^1 = -1, & c_{15}^3 = -1, & c_{51}^3 = 1, \\ c_{24}^2 = 2, & c_{42}^2 = -2. & & \end{array}$$

4.1.4 Nonlinear Constraints

We substitute the above nonzero structure constants in the respective nonlinear constraints to simplify the matrix $B = (b_i^l)$

$$c_{lm}^n b_i^l b_j^m = c_{ij}^k b_k^n, \quad i, j, k, l, m, n = 1, \cdots, 5\,.$$

For $n = 4$

$$c_{lm}^4 = 0, \qquad l, m = 1, 2, 3, 4, 5.$$

The constraints reduce to

$$\begin{aligned} 0 &= c_{ij}^k b_k^4, \\ 0 &= c_{ij}^1 b_1^4 + c_{ij}^2 b_2^4 + c_{ij}^3 b_3^4 + c_{ij}^4 b_4^4 + c_{ij}^5 b_5^4. \end{aligned}$$

When $(i, j) = (1, 2)$

$$\begin{aligned} 0 &= c_{12}^1 b_1^4 + c_{12}^2 b_2^4 + c_{12}^3 b_3^4 + c_{12}^4 b_4^4 + c_{12}^5 b_5^4, \\ 0 &= (0) b_1^4 + (0) b_2^4 + (0) b_3^4 + (0) b_4^4 + (0) b_5^4, \\ 0 &= 0. \end{aligned}$$

When $(i, j) = (1, 3)$

$$\begin{aligned} 0 &= c_{13}^1 b_1^4 + c_{13}^2 b_2^4 + c_{13}^3 b_3^4 + c_{13}^4 b_4^4 + c_{13}^5 b_5^4, \\ 0 &= (0) b_1^4 + (0) b_2^4 + (0) b_3^4 + (0) b_4^4 + (0) b_5^4, \\ 0 &= 0. \end{aligned}$$

When $(i, j) = (1, 4)$

$$\begin{aligned} 0 &= c_{14}^1 b_1^4 + c_{14}^2 b_2^4 + c_{14}^3 b_3^4 + c_{14}^4 b_4^4 + c_{14}^5 b_5^4, \\ 0 &= (1) b_1^4 + (0) b_2^4 + (0) b_3^4 + (0) b_4^4 + (0) b_5^4, \\ 0 &= b_1^4. \end{aligned}$$

When $(i,j) = (1,5)$

$$\begin{aligned}
0 &= c_{15}^1 b_1^4 + c_{15}^2 b_2^4 + c_{15}^3 b_3^4 + c_{15}^4 b_4^4 + c_{15}^5 b_5^4,\\
0 &= (0)(0) + (0)b_2^4 + (-1)b_3^4 + (0)b_4^4 + (0)b_5^4,\\
0 &= b_3^4.
\end{aligned}$$

When $(i,j) = (2,3)$

$$\begin{aligned}
0 &= c_{23}^1 b_1^4 + c_{23}^2 b_2^4 + c_{23}^3 b_3^4 + c_{23}^4 b_4^4 + c_{23}^5 b_5^4,\\
0 &= (0)(0) + (0)b_2^4 + (0)(0) + (0)b_4^4 + (0)b_5^4,\\
0 &= 0.
\end{aligned}$$

When $(i,j) = (2,4)$

$$\begin{aligned}
0 &= c_{24}^1 b_1^4 + c_{24}^2 b_2^4 + c_{24}^3 b_3^4 + c_{24}^4 b_4^4 + c_{24}^5 b_5^4,\\
0 &= (0)(0) + (2)b_2^4 + (0)(0) + (0)b_4^4 + (0)b_5^4,\\
0 &= b_2^4.
\end{aligned}$$

When $(i,j) = (2,5)$

$$\begin{aligned}
0 &= c_{25}^1 b_1^4 + c_{25}^2 b_2^4 + c_{25}^3 b_3^4 + c_{25}^4 b_4^4 + c_{25}^5 b_5^4,\\
0 &= (0)(0) + (0)(0) + (0)(0) + (0)b_4^4 + (0)b_5^4,\\
0 &= 0.
\end{aligned}$$

When $(i,j) = (3,4)$

$$\begin{aligned}
0 &= c_{34}^1 b_1^4 + c_{34}^2 b_2^4 + c_{34}^3 b_3^4 + c_{34}^4 b_4^4 + c_{34}^5 b_5^4,\\
0 &= (0)(0) + (0)(0) + (1)(0) + (0)b_4^4 + (0)b_5^4,\\
0 &= 0.
\end{aligned}$$

When $(i,j) = (3,5)$

$$\begin{aligned}
0 &= c_{35}^1 b_1^4 + c_{35}^2 b_2^4 + c_{35}^3 b_3^4 + c_{35}^4 b_4^4 + c_{35}^5 b_5^4,\\
0 &= (1)(0) + (0)(0) + (0)(0) + (0)b_4^4 + (0)b_5^4,\\
0 &= 0.
\end{aligned}$$

When $(i,j) = (4,5)$

$$\begin{aligned}
0 &= c_{45}^1 b_1^4 + c_{45}^2 b_2^4 + c_{45}^3 b_3^4 + c_{45}^4 b_4^4 + c_{45}^5 b_5^4,\\
0 &= (0)(0) + (0)(0) + (0)(0) + (0)b_4^4 + (0)b_5^4,\\
0 &= 0.
\end{aligned}$$

For $n = 5$
$$c_{lm}^5 = 0, \qquad l, m = 1, 2, 3, 4, 5.$$
The constraints reduce to
$$\begin{aligned}
0 &= c_{ij}^k b_k^5, \\
0 &= c_{ij}^1 b_1^5 + c_{ij}^2 b_2^5 + c_{ij}^3 b_3^5 + c_{ij}^4 b_4^5 + c_{ij}^5 b_5^5.
\end{aligned}$$
When $(i, j) = (1, 2)$
$$\begin{aligned}
0 &= c_{12}^1 b_1^5 + c_{12}^2 b_2^5 + c_{12}^3 b_3^5 + c_{12}^4 b_4^5 + c_{12}^5 b_5^5, \\
0 &= (0) b_1^5 + (0) b_2^5 + (0) b_3^5 + (0) b_4^5 + (0) b_5^5, \\
0 &= 0.
\end{aligned}$$
When $(i, j) = (1, 3)$
$$\begin{aligned}
0 &= c_{13}^1 b_1^5 + c_{13}^2 b_2^5 + c_{13}^3 b_3^5 + c_{13}^4 b_4^5 + c_{13}^5 b_5^5, \\
0 &= (0) b_1^5 + (0) b_2^5 + (0) b_3^5 + (0) b_4^5 + (0) b_5^5, \\
0 &= 0.
\end{aligned}$$
When $(i, j) = (1, 4)$
$$\begin{aligned}
0 &= c_{14}^1 b_1^5 + c_{14}^2 b_2^5 + c_{14}^3 b_3^5 + c_{14}^4 b_4^5 + c_{14}^5 b_5^5, \\
0 &= (1) b_1^5 + (0) b_2^5 + (0) b_3^5 + (0) b_4^5 + (0) b_5^5, \\
0 &= b_1^5.
\end{aligned}$$
When $(i, j) = (1, 5)$
$$\begin{aligned}
0 &= c_{15}^1 b_1^5 + c_{15}^2 b_2^5 + c_{15}^3 b_3^5 + c_{15}^4 b_4^5 + c_{15}^5 b_5^5, \\
0 &= (0)(0) + (0) b_2^5 + (-1) b_3^5 + (0) b_4^5 + (0) b_5^5, \\
0 &= b_3^5.
\end{aligned}$$
When $(i, j) = (2, 3)$
$$\begin{aligned}
0 &= c_{23}^1 b_1^5 + c_{23}^2 b_2^5 + c_{23}^3 b_3^5 + c_{23}^4 b_4^5 + c_{23}^5 b_5^5, \\
0 &= (0)(0) + (0) b_2^5 + (0)(0) + (0) b_4^5 + (0) b_5^5, \\
0 &= 0.
\end{aligned}$$

When $(i,j) = (2,4)$

$$\begin{aligned}
0 &= c_{24}^1 b_1^5 + c_{24}^2 b_2^5 + c_{24}^3 b_3^5 + c_{24}^4 b_4^5 + c_{24}^5 b_5^5,\\
0 &= (0)(0) + (2)b_2^5 + (0)(0) + (0)b_4^5 + (0)b_5^5,\\
0 &= b_2^5.
\end{aligned}$$

When $(i,j) = (2,5)$

$$\begin{aligned}
0 &= c_{25}^1 b_1^5 + c_{25}^2 b_2^5 + c_{25}^3 b_3^5 + c_{25}^4 b_4^5 + c_{25}^5 b_5^5,\\
0 &= (0)(0) + (0)(0) + (0)(0) + (0)b_4^5 + (0)b_5^5,\\
0 &= 0.
\end{aligned}$$

When $(i,j) = (3,4)$

$$\begin{aligned}
0 &= c_{34}^1 b_1^5 + c_{34}^2 b_2^5 + c_{34}^3 b_3^5 + c_{34}^4 b_4^5 + c_{34}^5 b_5^5,\\
0 &= (0)(0) + (0)(0) + (1)(0) + (0)b_4^5 + (0)b_5^5,\\
0 &= 0.
\end{aligned}$$

When $(i,j) = (3,5)$

$$\begin{aligned}
0 &= c_{35}^1 b_1^5 + c_{35}^2 b_2^5 + c_{35}^3 b_3^5 + c_{35}^4 b_4^5 + c_{35}^5 b_5^5,\\
0 &= (1)(0) + (0)(0) + (0)(0) + (0)b_4^5 + (0)b_5^5,\\
0 &= 0.
\end{aligned}$$

When $(i,j) = (4,5)$

$$\begin{aligned}
0 &= c_{45}^1 b_1^5 + c_{45}^2 b_2^5 + c_{45}^3 b_3^5 + c_{45}^4 b_4^5 + c_{45}^5 b_5^5,\\
0 &= (0)(0) + (0)(0) + (0)(0) + (0)b_4^5 + (0)b_5^5,\\
0 &= 0.
\end{aligned}$$

For $n = 2$

$$c_{lm}^2 = 0, \qquad (l,m) \neq (2,4),(4,2).$$

The constraints reduce to

$$\begin{aligned}
c_{24}^2 b_i^2 b_j^4 + c_{42}^2 b_i^4 b_j^2 &= c_{ij}^k b_k^2,\\
(2)b_i^2 b_j^4 + (-2)b_i^4 b_j^2 &= c_{ij}^1 b_1^2 + c_{ij}^2 b_2^2 + c_{ij}^3 b_3^2 + c_{ij}^4 b_4^2 + c_{ij}^5 b_5^2.
\end{aligned}$$

When $(i,j)=(1,2)$

$$\begin{aligned}(2)b_1^2b_2^4+(-2)b_1^4b_2^2&=c_{12}^1b_1^2+c_{12}^2b_2^2+c_{12}^3b_3^2+c_{12}^4b_4^2+c_{12}^5b_5^2,\\(2)b_1^2(0)+(-2)(0)b_2^2&=(0)b_1^2+(0)b_2^2+(0)b_3^2+(0)b_4^2+(0)b_5^2,\\0&=0.\end{aligned}$$

When $(i,j)=(1,3)$

$$\begin{aligned}(2)b_1^2b_3^4+(-2)b_1^4b_3^2&=c_{13}^1b_1^2+c_{13}^2b_2^2+c_{13}^3b_3^2+c_{13}^4b_4^2+c_{13}^5b_5^2,\\(2)b_1^2(0)+(-2)(0)b_3^2&=(0)b_1^2+(0)b_2^2+(0)b_3^2+(0)b_4^2+(0)b_5^2,\\0&=0.\end{aligned}$$

When $(i,j)=(1,4)$

$$\begin{aligned}(2)b_1^2b_4^4+(-2)b_1^4b_4^2&=c_{14}^1b_1^2+c_{14}^2b_2^2+c_{14}^3b_3^2+c_{14}^4b_4^2+c_{14}^5b_5^2,\\(2)b_1^2b_4^4+(-2)(0)b_4^2&=(1)b_1^2+(0)b_2^2+(0)b_3^2+(0)b_4^2+(0)b_5^2,\\2b_1^2b_4^4&=b_1^2.\end{aligned}$$

When $(i,j)=(1,5)$

$$\begin{aligned}(2)b_1^2b_5^4+(-2)b_1^4b_5^2&=c_{15}^1b_1^2+c_{15}^2b_2^2+c_{15}^3b_3^2+c_{15}^4b_4^2+c_{15}^5b_5^2,\\(2)b_1^2b_5^4+(-2)(0)b_5^2&=(0)b_1^2+(0)b_2^2+(-1)b_3^2+(0)b_4^2+(0)b_5^2,\\2b_1^2b_5^4&=-b_3^2.\end{aligned}$$

When $(i,j)=(2,3)$

$$\begin{aligned}(2)b_2^2b_3^4+(-2)b_2^4b_3^2&=c_{23}^1b_1^2+c_{23}^2b_2^2+c_{23}^3b_3^2+c_{23}^4b_4^2+c_{23}^5b_5^2,\\(2)b_2^2(0)+(-2)(0)b_3^2&=(0)b_1^2+(0)b_2^2+(0)b_3^2+(0)b_4^2+(0)b_5^2,\\0&=0.\end{aligned}$$

When $(i,j)=(2,4)$

$$\begin{aligned}(2)b_2^2b_4^4+(-2)b_2^4b_4^2&=c_{24}^1b_1^2+c_{24}^2b_2^2+c_{24}^3b_3^2+c_{24}^4b_4^2+c_{24}^5b_5^2,\\(2)b_2^2b_4^4+(-2)(0)b_4^2&=(0)b_1^2+(2)b_2^2+(0)b_3^2+(0)b_4^2+(0)b_5^2,\\b_2^2b_4^4&=b_2^2.\end{aligned}$$

When $(i,j) = (2,5)$

$$\begin{aligned}(2)b_2^2b_5^4 + (-2)b_2^4b_5^2 &= c_{25}^1b_1^2 + c_{25}^2b_2^2 + c_{25}^3b_3^2 + c_{25}^4b_4^2 + c_{25}^5b_5^2,\\ (2)b_2^2b_5^4 + (-2)(0)b_5^2 &= (0)b_1^2 + (0)b_2^2 + (0)b_3^2 + (0)b_4^2 + (0)b_5^2,\\ b_2^2b_5^4 &= 0.\end{aligned}$$

When $(i,j) = (3,4)$

$$\begin{aligned}(2)b_3^2b_4^4 + (-2)b_3^4b_4^2 &= c_{34}^1b_1^2 + c_{34}^2b_2^2 + c_{34}^3b_3^2 + c_{34}^4b_4^2 + c_{34}^5b_5^2,\\ (2)b_3^2b_4^4 + (-2)(0)b_4^2 &= (0)b_1^2 + (0)b_2^2 + (1)b_3^2 + (0)b_4^2 + (0)b_5^2,\\ 2b_3^2b_4^4 &= b_3^2.\end{aligned}$$

When $(i,j) = (3,5)$

$$\begin{aligned}(2)b_3^2b_5^4 + (-2)b_3^4b_5^2 &= c_{35}^1b_1^2 + c_{35}^2b_2^2 + c_{35}^3b_3^2 + c_{35}^4b_4^2 + c_{35}^5b_5^2,\\ (2)b_3^2b_5^4 + (-2)(0)b_5^2 &= (1)b_1^2 + (0)b_2^2 + (0)b_3^2 + (0)b_4^2 + (0)b_5^2,\\ 2b_3^2b_5^4 &= b_1^2.\end{aligned}$$

When $(i,j) = (4,5)$

$$\begin{aligned}(2)b_4^2b_5^4 + (-2)b_4^4b_5^2 &= c_{45}^1b_1^2 + c_{45}^2b_2^2 + c_{45}^3b_3^2 + c_{45}^4b_4^2 + c_{45}^5b_5^2,\\ (2)b_4^2b_5^4 + (-2)b_4^4b_5^2 &= (0)b_1^2 + (0)b_2^2 + (0)b_3^2 + (0)b_4^2 + (0)b_5^2,\\ b_4^2b_5^4 &= b_4^4b_5^2.\end{aligned}$$

For $n = 1$

$$c_{lm}^1 = 0, \qquad (l,m) \neq (1,4),(4,1),(3,5),(5,3).$$

The constraints reduce to

$$\begin{aligned}c_{14}^1b_i^1b_j^4 + c_{41}^1b_i^4b_j^1 + c_{35}^1b_i^3b_j^5 + c_{53}^1b_i^5b_j^3 &= c_{ij}^kb_k^1,\\ (1)b_i^1b_j^4 + (-1)b_i^4b_j^1 + (1)b_i^3b_j^5 + (-1)b_i^5b_j^3 &= c_{ij}^1b_1^1 + c_{ij}^2b_2^1 + c_{ij}^3b_3^1 + c_{ij}^4b_4^1 + c_{ij}^5b_5^1.\end{aligned}$$

When $(i,j) = (1,2)$

$$\begin{aligned}(1)b_1^1b_2^4 + (-1)b_1^4b_2^1 + (1)b_1^3b_2^5 + (-1)b_1^5b_2^3 &= c_{12}^1b_1^1 + c_{12}^2b_2^1 + c_{12}^3b_3^1 + c_{12}^4b_4^1 + c_{12}^5b_5^1,\\ (1)b_1^1(0) + (-1)(0)b_2^1 + (1)b_1^3(0) + (-1)(0)b_2^3 &= (0)b_1^1 + (0)b_2^1 + (0)b_3^1 + (0)b_4^1 + (0)b_5^1,\\ 0 &= 0.\end{aligned}$$

When $(i, j) = (1, 3)$

$$\begin{aligned}(1)b_1^1 b_3^4 + (-1)b_1^4 b_3^1 + (1)b_1^3 b_3^5 + (-1)b_1^5 b_3^3 &= c_{13}^1 b_1^1 + c_{13}^2 b_2^1 + c_{13}^3 b_3^1 + c_{13}^4 b_4^1 + c_{13}^5 b_5^1,\\ (1)b_1^1(0) + (-1)(0)b_3^1 + (1)b_1^3(0) + (-1)(0)b_3^3 &= (0)b_1^1 + (0)b_2^1 + (0)b_3^1 + (0)b_4^1 + (0)b_5^1,\\ 0 &= 0.\end{aligned}$$

When $(i, j) = (1, 4)$

$$\begin{aligned}(1)b_1^1 b_4^4 + (-1)b_1^4 b_4^1 + (1)b_1^3 b_4^5 + (-1)b_1^5 b_4^3 &= c_{14}^1 b_1^1 + c_{14}^2 b_2^1 + c_{14}^3 b_3^1 + c_{14}^4 b_4^1 + c_{14}^5 b_5^1,\\ (1)b_1^1 b_4^4 + (-1)(0)b_4^1 + (1)b_1^3 b_4^5 + (-1)(0)b_4^3 &= (1)b_1^1 + (0)b_2^1 + (0)b_3^1 + (0)b_4^1 + (0)b_5^1,\\ b_1^1 b_4^4 + b_1^3 b_4^5 &= b_1^1.\end{aligned}$$

When $(i, j) = (1, 5)$

$$\begin{aligned}(1)b_1^1 b_5^4 + (-1)b_1^4 b_5^1 + (1)b_1^3 b_5^5 + (-1)b_1^5 b_5^3 &= c_{15}^1 b_1^1 + c_{15}^2 b_2^1 + c_{15}^3 b_3^1 + c_{15}^4 b_4^1 + c_{15}^5 b_5^1,\\ (1)b_1^1 b_5^4 + (-1)(0)b_5^1 + (1)b_1^3 b_5^5 + (-1)(0)b_5^3 &= (0)b_1^1 + (0)b_2^1 + (-1)b_3^1 + (0)b_4^1 + (0)b_5^1,\\ b_1^1 b_5^4 + b_1^3 b_5^5 &= -b_3^1.\end{aligned}$$

When $(i, j) = (2, 3)$

$$\begin{aligned}(1)b_2^1 b_3^4 + (-1)b_2^4 b_3^1 + (1)b_2^3 b_3^5 + (-1)b_2^5 b_3^3 &= c_{23}^1 b_1^1 + c_{23}^2 b_2^1 + c_{23}^3 b_3^1 + c_{23}^4 b_4^1 + c_{23}^5 b_5^1,\\ (1)b_2^1(0) + (-1)(0)b_3^1 + (1)b_2^3(0) + (-1)(0)b_3^3 &= (0)b_1^1 + (0)b_2^1 + (0)b_3^1 + (0)b_4^1 + (0)b_5^1,\\ 0 &= 0.\end{aligned}$$

When $(i, j) = (2, 4)$

$$\begin{aligned}(1)b_2^1 b_4^4 + (-1)b_2^4 b_4^1 + (1)b_2^3 b_4^5 + (-1)b_2^5 b_4^3 &= c_{24}^1 b_1^1 + c_{24}^2 b_2^1 + c_{24}^3 b_3^1 + c_{24}^4 b_4^1 + c_{24}^5 b_5^1,\\ (1)b_2^1 b_4^4 + (-1)(0)b_4^1 + (1)b_2^3 b_4^5 + (-1)(0)b_4^3 &= (0)b_1^1 + (2)b_2^1 + (0)b_3^1 + (0)b_4^1 + (0)b_5^1,\\ b_2^1 b_4^4 + b_2^3 b_4^5 &= 2b_2^1.\end{aligned}$$

When $(i, j) = (2, 5)$

$$\begin{aligned}(1)b_2^1 b_5^4 + (-1)b_2^4 b_5^1 + (1)b_2^3 b_5^5 + (-1)b_2^5 b_5^3 &= c_{25}^1 b_1^1 + c_{25}^2 b_2^1 + c_{25}^3 b_3^1 + c_{25}^4 b_4^1 + c_{25}^5 b_5^1,\\ (1)b_2^1 b_5^4 + (-1)(0)b_5^1 + (1)b_2^3 b_5^5 + (-1)(0)b_5^3 &= (0)b_1^1 + (0)b_2^1 + (0)b_3^1 + (0)b_4^1 + (0)b_5^1,\\ b_2^1 b_5^4 + b_2^3 b_5^5 &= 0.\end{aligned}$$

When $(i, j) = (3, 4)$

$$\begin{aligned}(1)b_3^1 b_4^4 + (-1)b_3^4 b_4^1 + (1)b_3^3 b_4^5 + (-1)b_3^5 b_4^3 &= c_{34}^1 b_1^1 + c_{34}^2 b_2^1 + c_{34}^3 b_3^1 + c_{34}^4 b_4^1 + c_{34}^5 b_5^1,\\ (1)b_3^1 b_4^4 + (-1)(0)b_4^1 + (1)b_3^3 b_4^5 + (-1)(0)b_4^3 &= (0)b_1^1 + (0)b_2^1 + (1)b_3^1 + (0)b_4^1 + (0)b_5^1,\\ b_3^1 b_4^4 + b_3^3 b_4^5 &= b_3^1.\end{aligned}$$

When $(i,j)=(3,5)$

$$\begin{gathered}(1)b_3^1b_5^4+(-1)b_3^4b_5^1+(1)b_3^3b_5^5+(-1)b_3^5b_5^3=c_{35}^1b_1^1+c_{35}^2b_2^1+c_{35}^3b_3^1+c_{35}^4b_4^1+c_{35}^5b_5^1,\\(1)b_3^1b_5^4+(-1)(0)b_5^1+(1)b_3^3b_5^5+(-1)(0)b_5^3=(1)b_1^1+(0)b_2^1+(0)b_3^1+(0)b_4^1+(0)b_5^1,\\b_3^1b_5^4+b_3^3b_5^5=b_1^1.\end{gathered}$$

When $(i,j)=(4,5)$

$$\begin{gathered}(1)b_4^1b_5^4+(-1)b_4^4b_5^1+(1)b_4^3b_5^5+(-1)b_4^5b_5^3=c_{45}^1b_1^1+c_{45}^2b_2^1+c_{45}^3b_3^1+c_{45}^4b_4^1+c_{45}^5b_5^1,\\(1)b_4^1b_5^4+(-1)b_4^4b_5^1+(1)b_4^3b_5^5+(-1)b_4^5b_5^3=(0)b_1^1+(0)b_2^1+(0)b_3^1+(0)b_4^1+(0)b_5^1,\\b_4^1b_5^4-b_4^4b_5^1+b_4^3b_5^5-b_4^5b_5^3=0.\end{gathered}$$

For $n=3$

$$c_{lm}^3=0,\qquad (l,m)\neq(3,4),(4,3),(1,5),(5,1).$$

The constraints reduce to

$$\begin{gathered}c_{34}^3b_i^3b_j^4+c_{43}^3b_i^4b_j^3+c_{15}^3b_i^1b_j^5+c_{51}^3b_i^5b_j^1=c_{ij}^kb_k^3,\\(1)b_i^3b_j^4+(-1)b_i^4b_j^3+(-1)b_i^1b_j^5+(1)b_i^5b_j^1=c_{ij}^1b_1^3+c_{ij}^2b_2^3+c_{ij}^3b_3^3+c_{ij}^4b_4^3+c_{ij}^5b_5^3.\end{gathered}$$

When $(i,j)=(1,2)$

$$\begin{gathered}(1)b_1^3b_2^4+(-1)b_1^4b_2^3+(-1)b_1^1b_2^5+(1)b_1^5b_2^1=c_{12}^1b_1^3+c_{12}^2b_2^3+c_{12}^3b_3^3+c_{12}^4b_4^3+c_{12}^5b_5^3,\\(1)b_1^3(0)+(-1)(0)b_2^3+(-1)b_1^1(0)+(1)(0)b_2^1=(0)b_1^3+(0)b_2^3+(0)b_3^3+(0)b_4^3+(0)b_5^3,\\0=0.\end{gathered}$$

When $(i,j)=(1,3)$

$$\begin{gathered}(1)b_1^3b_3^4+(-1)b_1^4b_3^3+(-1)b_1^1b_3^5+(1)b_1^5b_3^1=c_{13}^1b_1^3+c_{13}^2b_2^3+c_{13}^3b_3^3+c_{13}^4b_4^3+c_{13}^5b_5^3,\\(1)b_1^3(0)+(-1)(0)b_3^3+(-1)b_1^1(0)+(1)(0)b_3^1=(0)b_1^3+(0)b_2^3+(0)b_3^3+(0)b_4^3+(0)b_5^3,\\0=0.\end{gathered}$$

When $(i,j)=(1,4)$

$$\begin{gathered}(1)b_1^3b_4^4+(-1)b_1^4b_4^3+(-1)b_1^1b_4^5+(1)b_1^5b_4^1=c_{14}^1b_1^3+c_{14}^2b_2^3+c_{14}^3b_3^3+c_{14}^4b_4^3+c_{14}^5b_5^3,\\(1)b_1^3b_4^4+(-1)(0)b_4^3+(-1)b_1^1b_4^5+(1)(0)b_4^1=(1)b_1^3+(0)b_2^3+(0)b_3^3+(0)b_4^3+(0)b_5^3,\\b_1^3b_4^4-b_1^1b_4^5=b_1^3.\end{gathered}$$

When $(i,j)=(1,5)$

$$(1)b_1^3b_5^4+(-1)b_1^4b_5^3+(-1)b_1^1b_5^5+(1)b_1^5b_5^1=c_{15}^1b_1^3+c_{15}^2b_2^3+c_{15}^3b_3^3+c_{15}^4b_4^3+c_{15}^5b_5^3,$$
$$(1)b_1^3b_5^4+(-1)(0)b_5^3+(-1)b_1^1b_5^5+(1)(0)b_5^1=(0)b_1^3+(0)b_2^3+(-1)b_3^3+(0)b_4^3+(0)b_5^3,$$
$$b_1^3b_5^4-b_1^1b_5^5=-b_3^3.$$

When $(i,j)=(2,3)$

$$(1)b_2^3b_3^4+(-1)b_2^4b_3^3+(-1)b_2^1b_3^5+(1)b_2^5b_3^1=c_{23}^1b_1^3+c_{23}^2b_2^3+c_{23}^3b_3^3+c_{23}^4b_4^3+c_{23}^5b_5^3,$$
$$(1)b_2^3(0)+(-1)(0)b_3^3+(-1)b_2^1(0)+(1)(0)b_3^1=(0)b_1^3+(0)b_2^3+(0)b_3^3+(0)b_4^3+(0)b_5^3,$$
$$0=0.$$

When $(i,j)=(2,4)$

$$(1)b_2^3b_4^4+(-1)b_2^4b_4^3+(-1)b_2^1b_4^5+(1)b_2^5b_4^1=c_{24}^1b_1^3+c_{24}^2b_2^3+c_{24}^3b_3^3+c_{24}^4b_4^3+c_{24}^5b_5^3,$$
$$(1)b_2^3b_4^4+(-1)(0)b_4^3+(-1)b_2^1b_4^5+(1)(0)b_4^1=(0)b_1^3+(2)b_2^3+(0)b_3^3+(0)b_4^3+(0)b_5^3,$$
$$b_2^3b_4^4-b_2^1b_4^5=2b_2^3.$$

When $(i,j)=(2,5)$

$$(1)b_2^3b_5^4+(-1)b_2^4b_5^3+(-1)b_2^1b_5^5+(1)b_2^5b_5^1=c_{25}^1b_1^3+c_{25}^2b_2^3+c_{25}^3b_3^3+c_{25}^4b_4^3+c_{25}^5b_5^3,$$
$$(1)b_2^3b_5^4+(-1)(0)b_5^3+(-1)b_2^1b_5^5+(1)(0)b_5^1=(0)b_1^3+(0)b_2^3+(0)b_3^3+(0)b_4^3+(0)b_5^3,$$
$$b_2^3b_5^4-b_2^1b_5^5=0.$$

When $(i,j)=(3,4)$

$$(1)b_3^3b_4^4+(-1)b_3^4b_4^3+(-1)b_3^1b_4^5+(1)b_3^5b_4^1=c_{34}^1b_1^3+c_{34}^2b_2^3+c_{34}^3b_3^3+c_{34}^4b_4^3+c_{34}^5b_5^3,$$
$$(1)b_3^3b_4^4+(-1)(0)b_4^3+(-1)b_3^1b_4^5+(1)(0)b_4^1=(0)b_1^3+(0)b_2^3+(1)b_3^3+(0)b_4^3+(0)b_5^3,$$
$$b_3^3b_4^4-b_3^1b_4^5=b_3^3.$$

When $(i,j)=(3,5)$

$$(1)b_3^3b_5^4+(-1)b_3^4b_5^3+(-1)b_3^1b_5^5+(1)b_3^5b_5^1=c_{35}^1b_1^3+c_{35}^2b_2^3+c_{35}^3b_3^3+c_{35}^4b_4^3+c_{35}^5b_5^3,$$
$$(1)b_3^3b_5^4+(-1)(0)b_5^3+(-1)b_3^1b_5^5+(1)(0)b_5^1=(1)b_1^3+(0)b_2^3+(0)b_3^3+(0)b_4^3+(0)b_5^3,$$
$$b_3^3b_5^4-b_3^1b_5^5=b_1^3.$$

When $(i,j)=(4,5)$

$$(1)b_4^3b_5^4+(-1)b_4^4b_5^3+(-1)b_4^1b_5^5+(1)b_4^5b_5^1=c_{45}^1b_1^3+c_{45}^2b_2^3+c_{45}^3b_3^3+c_{45}^4b_4^3+c_{45}^5b_5^3,$$
$$(1)b_4^3b_5^4+(-1)b_4^4b_5^3+(-1)b_4^1b_5^5+(1)b_4^5b_5^1=(0)b_1^3+(0)b_2^3+(0)b_3^3+(0)b_4^3+(0)b_5^3,$$
$$b_4^3b_5^4-b_4^4b_5^3-b_4^1b_5^5+b_4^5b_5^1=0.$$

The system of nonlinear constraints simplifies as

$$
\begin{aligned}
0 &= b_1^4, && (4a) & b_1^1 b_4^4 + b_1^3 b_4^5 &= b_1^1, && (1a)\\
0 &= b_3^4, && (4b) & b_1^1 b_5^4 + b_1^3 b_5^5 &= -b_3^1, && (1b)\\
0 &= b_2^4, && (4c) & b_2^1 b_4^4 + b_2^3 b_4^5 &= 2b_2^1, && (1c)\\
 & && & b_2^1 b_5^4 + b_2^3 b_5^5 &= 0, && (1d)\\
0 &= b_1^5, && (5a) & b_3^1 b_4^4 + b_3^3 b_4^5 &= b_3^1, && (1e)\\
0 &= b_3^5, && (5b) & b_3^1 b_5^4 + b_3^3 b_5^5 &= b_1^1, && (1f)\\
0 &= b_2^5, && (5c) & b_4^1 b_5^4 + b_4^3 b_5^5 &= b_5^1 b_4^4 + b_5^3 b_4^5, && (1g)
\end{aligned}
$$

$$
\begin{aligned}
2b_1^2 b_4^4 &= b_1^2, && (2a) & b_1^3 b_4^4 - b_1^1 b_4^5 &= b_1^3, && (3a)\\
2b_1^2 b_5^4 &= -b_3^2, && (2b) & b_1^3 b_5^4 - b_1^1 b_5^5 &= -b_3^3, && (3b)\\
b_2^2 b_4^4 &= b_2^2, && (2c) & b_2^3 b_4^4 - b_2^1 b_4^5 &= 2b_2^3, && (3c)\\
b_2^2 b_5^4 &= 0, && (2d) & b_2^3 b_5^4 - b_2^1 b_5^5 &= 0, && (3d)\\
2b_3^2 b_4^4 &= b_3^2, && (2e) & b_3^3 b_4^4 - b_3^1 b_4^5 &= b_3^3, && (3e)\\
2b_3^2 b_5^4 &= b_1^2, && (2f) & b_3^3 b_5^4 - b_3^1 b_5^5 &= b_1^3, && (3f)\\
b_4^2 b_5^4 &= b_4^4 b_5^2, && (2g) & b_4^3 b_5^4 - b_4^1 b_5^5 &= b_5^3 b_4^4 - b_5^1 b_4^5. && (3g)
\end{aligned}
$$

So far we have been able to simplyfy $B = (b_i^l)$ as

$$
B = \begin{bmatrix} b_1^1 & b_1^2 & b_1^3 & 0 & 0 \\ b_2^1 & b_2^2 & b_2^3 & 0 & 0 \\ b_3^1 & b_3^2 & b_3^3 & 0 & 0 \\ b_4^1 & b_4^2 & b_4^3 & b_4^4 & b_4^5 \\ b_5^1 & b_5^2 & b_5^3 & b_5^4 & b_5^5 \end{bmatrix}.
$$

Now we separate the matrices B according to $b_2^2 = 0$ or $b_2^2 \neq 0$ and simplify the above system further for both cases.

When $\mathbf{b_2^2 = 0}$

When $b_2^2 = 0$ both b_2^1 and b_2^3 cannot be simultaneously zero because B is a nonsingular matrix. That is $(b_2^1)^2 + (b_2^3)^2 \neq 0$.

Multiplying $(1c)$ with b_2^1 and $(3c)$ with b_2^3 and adding the two equations we conclude that $b_4^4 = 2$. Likewise from $(1d)$ and $(3d)$ we conclude that $b_5^4 = 0$. By substituting the values of b_4^4 and b_5^4 in the respective equations we further conclude that

$$
\begin{array}{lll}
b_1^2 = 0, & b_1^3 b_4^5 = -b_1^1, & b_1^1 b_4^5 = b_1^3, \\
b_3^2 = 0, & b_1^3 b_5^5 = -b_3^1, & b_1^1 b_5^5 = b_3^3, \\
b_5^2 = 0, & b_2^3 b_4^5 = 0, & b_2^1 b_4^5 = 0, \\
 & b_2^3 b_5^5 = 0, & b_2^1 b_5^5 = 0, \\
 & b_3^3 b_4^5 = -b_3^1, & b_3^1 b_4^5 = b_3^3, \\
 & b_3^3 b_5^5 = b_1^1, & -b_3^1 b_5^5 = b_1^3, \\
 & b_4^3 b_5^5 = 2b_5^1 + b_5^3 b_4^5, & -b_4^1 b_5^5 = 2b_5^3 - b_5^1 b_4^5.
\end{array}
$$

Simplifying further we obtain

$$
\begin{array}{lll}
b_1^3 = 0, & b_2^3 b_4^5 = 0, & b_2^1 b_4^5 = 0, \\
b_3^1 = 0, & b_2^3 b_5^5 = 0, & b_2^1 b_5^5 = 0, \\
b_1^1 = 0, & b_4^3 b_5^5 = 2b_5^1 + b_5^3 b_4^5, & -b_4^1 b_5^5 = 2b_5^3 - b_5^1 b_4^5. \\
b_3^3 = 0, & &
\end{array}
$$

So $B = (b_i^l)$ simplyfies as

$$
B = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ b_2^1 & 0 & b_2^3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ b_4^1 & b_4^2 & b_4^3 & 2 & b_4^5 \\ b_5^1 & 0 & b_5^3 & 0 & b_5^5 \end{bmatrix}.
$$

These are clearly singular matrices. We discard all such matrices because they do not correspond to any automorphism of the Lie algebra.

When $\mathbf{b_2^2 \neq 0}$

When $b_2^2 \neq 0$ $(2c)$ implies $b_4^4 = 1$ and $(2d)$ implies $b_5^4 = 0$. By substituting the values of b_4^4 and b_5^4 in the respective equations we further conclude that

$$\begin{aligned}
b_1^2 &= 0, & b_1^1 b_4^5 &= 0,\\
b_3^2 &= 0, & b_1^1 b_5^5 &= b_3^3,\\
b_5^2 &= 0, & -b_2^1 b_4^5 &= b_2^3,\\
& & b_2^1 b_5^5 &= 0,\\
b_1^3 b_4^5 &= 0, & b_3^1 b_4^5 &= 0,\\
b_1^3 b_5^5 &= -b_3^1, & -b_3^1 b_5^5 &= b_1^3,\\
b_2^3 b_4^5 &= b_2^1, & -b_5^3 - b_4^1 b_5^5 + b_5^1 b_4^5 &= 0.\\
b_2^3 b_5^5 &= 0, & &\\
b_3^3 b_4^5 &= 0, & &\\
b_3^3 b_5^5 &= b_1^1, & &\\
-b_5^1 + b_4^3 b_5^5 - b_5^3 b_4^5 &= 0, & &
\end{aligned}$$

Simplifying further we obtain

$$\begin{aligned}
b_2^1 &= 0, & b_1^1 b_4^5 &= 0,\\
b_2^3 &= 0, & b_1^1 b_5^5 &= b_3^3,\\
& & b_3^1 b_4^5 &= 0,\\
b_1^3 b_4^5 &= 0, & -b_3^1 b_5^5 &= b_1^3,\\
b_1^3 b_5^5 &= -b_3^1, & -b_5^3 - b_4^1 b_5^5 + b_5^1 b_4^5 &= 0.\\
b_3^3 b_4^5 &= 0, & &\\
b_3^3 b_5^5 &= b_1^1, & &\\
-b_5^1 + b_4^3 b_5^5 - b_5^3 b_4^5 &= 0, & &
\end{aligned}$$

It should be noted that any nonzero value of b_4^5 will make $b_1^1 = b_1^3 = 0$ and $b_3^1 = b_3^3 = 0$ thereby causing B to be singular. Therefore $b_4^5 = 0$. Now b_5^5 must be nonzero. Following a similar line of argument we further conclude that $b_5^5 \pm 1$.

So $B = (b_i^l)$ simplyfies as

$$B = \begin{bmatrix} b_1^1 & 0 & b_1^3 & 0 & 0 \\ 0 & b_2^2 & 0 & 0 & 0 \\ b_3^1 & 0 & b_3^3 & 0 & 0 \\ b_4^1 & b_4^2 & b_4^3 & 1 & 0 \\ b_5^1 & 0 & b_5^3 & 0 & b_5^5 \end{bmatrix},$$

with further conditions

$$\begin{aligned} b_1^3 b_5^5 &= -b_3^1, \\ b_3^3 b_5^5 &= b_1^1, \\ -b_5^1 + b_4^3 b_5^5 &= 0, \\ \\ b_1^1 b_5^5 &= b_3^3, \\ -b_3^1 b_5^5 &= b_1^3, \\ b_5^3 + b_4^1 b_5^5 &= 0. \end{aligned}$$

$$b_2^2 \neq 0, \quad b_5^5 = \pm 1.$$

4.1.5 Inequivalent Symmetries

We recall from Theorem (2.3.6) that

$$(C(j))_i^k = c_{ij}^k,$$

and

$$\begin{aligned} A(j,\epsilon) &= e^{\epsilon C(j)}, \\ A(j,\epsilon) &= \sum_{n=0}^{\infty} \frac{\epsilon^n}{n!}(C(j))^n, \\ A(j,\epsilon) &= I + \frac{\epsilon}{1!}(C(j)) + \frac{\epsilon^2}{2!}(C(j))^2 + \frac{\epsilon^3}{3!}(C(j))^3 + ... \end{aligned}$$

Calculating matrices $C(j)$

$$C(1) = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix},$$

$$C(2) = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$C(3) = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$C(4) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$C(5) = \begin{bmatrix} 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Calculating matrices $A(j,\epsilon)$

$(C(1))^n = 0 \quad$ for $\quad 2 \leq n$,

$$A(1,\epsilon) = I + \epsilon\, C(1) + 0,$$

$$A(1,\epsilon) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ -\epsilon & 0 & 0 & 1 & 0 \\ 0 & 0 & \epsilon & 0 & 1 \end{bmatrix}.$$

$(C(2))^n = 0 \quad$ for $\quad 2 \leq n$,

$$A(2,\epsilon) = I + \epsilon\, C(2) + 0,$$

$$A(2,\epsilon) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & -2\epsilon & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

$(C(3))^n = 0 \quad$ for $\quad 2 \leq n$,

$$A(3,\epsilon) = I + \epsilon\, C(3) + 0,$$

$$A(3,\epsilon) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -\epsilon & 1 & 0 \\ -\epsilon & 0 & 0 & 0 & 1 \end{bmatrix}.$$

$$A(4,\epsilon) = \begin{bmatrix} e^{\epsilon} & 0 & 0 & 0 & 0 \\ 0 & e^{2\epsilon} & 0 & 0 & 0 \\ 0 & 0 & e^{\epsilon} & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix},$$

because $C(4)$ is a digonal matrix.

$$A(5,\epsilon) = \begin{bmatrix} \cos\epsilon & 0 & -\sin\epsilon & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ \sin\epsilon & 0 & \cos\epsilon & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix},$$

because $C(5)$ is a skew symmetric matrix. The details are following.

We show that

$$A(5,\epsilon) = e^{\epsilon C(5)} = I + (\sin\epsilon)C(5) + (1-\cos\epsilon)(C(5))^2.$$

It can be proved by induction that

$$(C(5))^{2m+1} = (-1)^m C(5), \quad (C(5))^{2m} = (-1)^{m-1}(C(5))^2.$$

Now

$$\begin{aligned} A(5,\epsilon) &= e^{\epsilon C(5)} \\ &= \sum_{n=0}^{\infty} \frac{\epsilon^n}{n!}(C(5))^n \\ &= I + \sum_{m=0}^{\infty} \frac{\epsilon^{2m+1}}{(2m+1)!}(C(5))^{2m+1} + \sum_{m=1}^{\infty} \frac{\epsilon^{2m}}{(2m)!}(C(5))^{2m} \\ &= I + \sum_{m=0}^{\infty} \frac{\epsilon^{2m+1}}{(2m+1)!}(-1)^m C(5) + \sum_{m=1}^{\infty} \frac{\epsilon^{2m}}{(2m)!}(-1)^{m-1}(C(5))^2 \\ &= I + (\sin\epsilon)\, C(5) + (1-\cos\epsilon)\, (C(5))^2. \end{aligned}$$

We substitute $C(5)$ and $(C(5))^2$ in the final equation and get $A(5,\epsilon)$.

Now we calculate the inequivalent matrices

$$BA(2,\epsilon) = \begin{bmatrix} b_1^1 & 0 & b_1^3 & 0 & 0 \\ 0 & b_2^2 & 0 & 0 & 0 \\ b_3^1 & 0 & b_3^3 & 0 & 0 \\ b_4^1 & b_4^2 & b_4^3 & 1 & 0 \\ b_5^1 & 0 & b_5^3 & 0 & b_5^5 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & -2\epsilon & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$
$$= \begin{bmatrix} b_1^1 & 0 & b_1^3 & 0 & 0 \\ 0 & b_2^2 & 0 & 0 & 0 \\ b_3^1 & 0 & b_3^3 & 0 & 0 \\ b_4^1 & b_4^2 - 2\epsilon & b_4^3 & 1 & 0 \\ b_5^1 & 0 & b_5^3 & 0 & b_5^5 \end{bmatrix}.$$

Choosing $\epsilon = \epsilon_2 = \frac{1}{2}b_4^2$,

$$b_4^2 - 2\epsilon = b_4^2 - 2\epsilon_2 = b_4^2 - 2\left(\frac{1}{2}b_4^2\right) = 0,$$

so

$$BA(2,\epsilon_2) = \begin{bmatrix} b_1^1 & 0 & b_1^3 & 0 & 0 \\ 0 & b_2^2 & 0 & 0 & 0 \\ b_3^1 & 0 & b_3^3 & 0 & 0 \\ b_4^1 & 0 & b_4^3 & 1 & 0 \\ b_5^1 & 0 & b_5^3 & 0 & b_5^5 \end{bmatrix}.$$

Now

$$BA(2,\epsilon_2)A(1,\epsilon) = \begin{bmatrix} b_1^1 & 0 & b_1^3 & 0 & 0 \\ 0 & b_2^2 & 0 & 0 & 0 \\ b_3^1 & 0 & b_3^3 & 0 & 0 \\ b_4^1 & 0 & b_4^3 & 1 & 0 \\ b_5^1 & 0 & b_5^3 & 0 & b_5^5 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ -\epsilon & 0 & 0 & 1 & 0 \\ 0 & 0 & \epsilon & 0 & 1 \end{bmatrix}$$
$$= \begin{bmatrix} b_1^1 & 0 & b_1^3 & 0 & 0 \\ 0 & b_2^2 & 0 & 0 & 0 \\ b_3^1 & 0 & b_3^3 & 0 & 0 \\ b_4^1 - \epsilon & 0 & b_4^3 & 1 & 0 \\ b_5^1 & 0 & b_5^3 + b_5^5\epsilon & 0 & b_5^5 \end{bmatrix}.$$

Choosing $\epsilon = \epsilon_1 = b_4^1$,

$$b_4^1 - \epsilon = b_4^1 - \epsilon_1 = b_4^1 - b_4^1 = 0,$$

$$b_5^3 + b_5^5\epsilon = b_5^3 + b_5^5\epsilon_1 = b_5^3 + b_5^5 b_4^1 = 0,$$

so

$$BA(2,\epsilon_2)A(1,\epsilon_1) = \begin{bmatrix} b_1^1 & 0 & b_1^3 & 0 & 0 \\ 0 & b_2^2 & 0 & 0 & 0 \\ b_3^1 & 0 & b_3^3 & 0 & 0 \\ 0 & 0 & b_4^3 & 1 & 0 \\ b_5^1 & 0 & 0 & 0 & b_5^5 \end{bmatrix}.$$

Now

$$\begin{aligned} BA(2,\epsilon_2)A(1,\epsilon_1)A(3,\epsilon) &= \begin{bmatrix} b_1^1 & 0 & b_1^3 & 0 & 0 \\ 0 & b_2^2 & 0 & 0 & 0 \\ b_3^1 & 0 & b_3^3 & 0 & 0 \\ 0 & 0 & b_4^3 & 1 & 0 \\ b_5^1 & 0 & 0 & 0 & b_5^5 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -\epsilon & 1 & 0 \\ -\epsilon & 0 & 0 & 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} b_1^1 & 0 & b_1^3 & 0 & 0 \\ 0 & b_2^2 & 0 & 0 & 0 \\ b_3^1 & 0 & b_3^3 & 0 & 0 \\ 0 & 0 & b_4^3 - \epsilon & 1 & 0 \\ b_5^1 - b_5^5\epsilon & 0 & 0 & 0 & b_5^5 \end{bmatrix}. \end{aligned}$$

Choosing $\epsilon = \epsilon_3 = b_4^3$,

$$b_4^3 - \epsilon = b_4^3 - \epsilon_3 = b_4^3 - b_4^3 = 0,$$

$$b_5^1 - b_5^5\epsilon = b_5^1 - b_5^5\epsilon_3 = b_5^1 - b_5^5 b_4^3 = 0,$$

so

$$BA(2,\epsilon_2)A(1,\epsilon_1)A(3,\epsilon_3) = \begin{bmatrix} b_1^1 & 0 & b_1^3 & 0 & 0 \\ 0 & b_2^2 & 0 & 0 & 0 \\ b_3^1 & 0 & b_3^3 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & b_5^5 \end{bmatrix}.$$

Now

$$BA(2,\epsilon_2)A(1,\epsilon_1)A(3,\epsilon_3)A(5,\epsilon) = \begin{bmatrix} b_1^1 & 0 & b_1^3 & 0 & 0 \\ 0 & b_2^2 & 0 & 0 & 0 \\ b_3^1 & 0 & b_3^3 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & b_5^5 \end{bmatrix} \begin{bmatrix} \cos\epsilon & 0 & -\sin\epsilon & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ \sin\epsilon & 0 & \cos\epsilon & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} b_1^1\cos\epsilon + b_1^3\sin\epsilon & 0 & -b_1^1\sin\epsilon + b_1^3\cos\epsilon & 0 & 0 \\ 0 & b_2^2 & 0 & 0 & 0 \\ b_3^1\cos\epsilon + b_3^3\sin\epsilon & 0 & -b_3^1\sin\epsilon + b_3^3\cos\epsilon & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & b_5^5 \end{bmatrix}.$$

Choosing $\epsilon = \epsilon_5 = \frac{\pi}{4}$,

so

$$BA(2,\epsilon_2)A(1,\epsilon_1)A(3,\epsilon_3)A(5,\epsilon_5) = \begin{bmatrix} \frac{b_1^1+b_1^3}{\sqrt{2}} & 0 & \frac{-b_1^1+b_1^3}{\sqrt{2}} & 0 & 0 \\ 0 & b_2^2 & 0 & 0 & 0 \\ \frac{b_3^1+b_3^3}{\sqrt{2}} & 0 & \frac{-b_3^1+b_3^3}{\sqrt{2}} & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & b_5^5 \end{bmatrix}.$$

We know

$$\frac{b_1^1+b_1^3}{\sqrt{2}} = \left(\frac{-b_3^1+b_3^3}{\sqrt{2}}\right)b_5^5,$$

$$\frac{-b_1^1+b_1^3}{\sqrt{2}} = -\left(\frac{b_3^1+b_3^3}{\sqrt{2}}\right)b_5^5.$$

So we have

$$BA(2,\epsilon_2)A(1,\epsilon_1)A(3,\epsilon_3)A(5,\epsilon_5) = \begin{bmatrix} \left(\frac{-b_3^1+b_3^3}{\sqrt{2}}\right)b_5^5 & 0 & -\left(\frac{b_3^1+b_3^3}{\sqrt{2}}\right)b_5^5 & 0 & 0 \\ 0 & b_2^2 & 0 & 0 & 0 \\ \frac{b_3^1+b_3^3}{\sqrt{2}} & 0 & \frac{-b_3^1+b_3^3}{\sqrt{2}} & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & b_5^5 \end{bmatrix}.$$

Now

$$BA(2,\epsilon_2)A(1,\epsilon_1)A(3,\epsilon_3)A(5,\epsilon_5)A(4,\epsilon)$$

$$= \begin{bmatrix} \left(\frac{-b_3^1+b_3^3}{\sqrt{2}}\right)b_5^5 & 0 & -\left(\frac{b_3^1+b_3^3}{\sqrt{2}}\right)b_5^5 & 0 & 0 \\ 0 & b_2^2 & 0 & 0 & 0 \\ \frac{b_3^1+b_3^3}{\sqrt{2}} & 0 & \frac{-b_3^1+b_3^3}{\sqrt{2}} & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & b_5^5 \end{bmatrix} \begin{bmatrix} e^{\epsilon} & 0 & 0 & 0 & 0 \\ 0 & e^{2\epsilon} & 0 & 0 & 0 \\ 0 & 0 & e^{\epsilon} & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \left(\frac{-b_3^1+b_3^3}{\sqrt{2}}\right)b_5^5 e^{\epsilon} & 0 & -\left(\frac{b_3^1+b_3^3}{\sqrt{2}}\right)b_5^5 e^{\epsilon} & 0 & 0 \\ 0 & b_2^2 e^{2\epsilon} & 0 & 0 & 0 \\ \frac{b_3^1+b_3^3}{\sqrt{2}}e^{\epsilon} & 0 & \frac{-b_3^1+b_3^3}{\sqrt{2}}e^{\epsilon} & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & b_5^5 \end{bmatrix}.$$

Choosing $\epsilon = \epsilon_4 = \ln \frac{1}{\sqrt{|b_2^2|}}$,

so

$$BA(2,\epsilon_2)A(1,\epsilon_1)A(3,\epsilon_3)A(5,\epsilon_5)A(4,\epsilon_4) = \begin{bmatrix} \left(\frac{-b_3^1+b_3^3}{\sqrt{2\,|b_2^2|}}\right)b_5^5 & 0 & -\left(\frac{b_3^1+b_3^3}{\sqrt{2\,|b_2^2|}}\right)b_5^5 & 0 & 0 \\ 0 & \frac{b_2^2}{|b_2^2|} & 0 & 0 & 0 \\ \frac{b_3^1+b_3^3}{\sqrt{2\,|b_2^2|}} & 0 & \frac{-b_3^1+b_3^3}{\sqrt{2\,|b_2^2|}} & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & b_5^5 \end{bmatrix}.$$

Let $\quad \tilde{B} = BA(2,\epsilon_2)A(1,\epsilon_1)A(3,\epsilon_3)A(5,\epsilon_5)A(4,\epsilon_4).$

These are the required inequivalent matrices.

4.1.6 Solution of the System of Determining Equations

Following is the system of determining equations with the inequivalent matrices calculated in the previous section.

$$\begin{bmatrix} \mathbf{X}_1\hat{x} & \mathbf{X}_1\hat{t} & \mathbf{X}_1\hat{u} \\ \mathbf{X}_2\hat{x} & \mathbf{X}_2\hat{t} & \mathbf{X}_2\hat{u} \\ \mathbf{X}_3\hat{x} & \mathbf{X}_3\hat{t} & \mathbf{X}_3\hat{u} \\ \mathbf{X}_4\hat{x} & \mathbf{X}_4\hat{t} & \mathbf{X}_4\hat{u} \\ \mathbf{X}_5\hat{x} & \mathbf{X}_5\hat{t} & \mathbf{X}_5\hat{u} \end{bmatrix} = \begin{bmatrix} \left(\frac{-b_3^1+b_3^3}{\sqrt{2\,|b_2^2|}}\right)b_5^5 & 0 & -\left(\frac{b_3^1+b_3^3}{\sqrt{2\,|b_2^2|}}\right)b_5^5 & 0 & 0 \\ 0 & \frac{b_2^2}{|b_2^2|} & 0 & 0 & 0 \\ \frac{b_3^1+b_3^3}{\sqrt{2\,|b_2^2|}} & 0 & \frac{-b_3^1+b_3^3}{\sqrt{2\,|b_2^2|}} & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & b_5^5 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ \hat{x} & 2\hat{t} & \hat{u} \\ \hat{u} & 0 & -\hat{x} \end{bmatrix},$$

$$\begin{bmatrix} \mathbf{X}_1\hat{x} & \mathbf{X}_1\hat{t} & \mathbf{X}_1\hat{u} \\ \mathbf{X}_2\hat{x} & \mathbf{X}_2\hat{t} & \mathbf{X}_2\hat{u} \\ \mathbf{X}_3\hat{x} & \mathbf{X}_3\hat{t} & \mathbf{X}_3\hat{u} \\ \mathbf{X}_4\hat{x} & \mathbf{X}_4\hat{t} & \mathbf{X}_4\hat{u} \\ \mathbf{X}_5\hat{x} & \mathbf{X}_5\hat{t} & \mathbf{X}_5\hat{u} \end{bmatrix} = \begin{bmatrix} \left(\frac{-b_3^1+b_3^3}{\sqrt{2\,|b_2^2|}}\right)b_5^5 & 0 & -\left(\frac{b_3^1+b_3^3}{\sqrt{2\,|b_2^2|}}\right)b_5^5 \\ 0 & \frac{b_2^2}{|b_2^2|} & 0 \\ \frac{b_3^1+b_3^3}{\sqrt{2\,|b_2^2|}} & 0 & \frac{-b_3^1+b_3^3}{\sqrt{2\,|b_2^2|}} \\ \hat{x} & 2\hat{t} & \hat{u} \\ b_5^5\hat{u} & 0 & -b_5^5\hat{x} \end{bmatrix}.$$

Now we solve the above system of determining equations and seek a solution of the form

$$(\hat{x},\hat{t},\hat{u}) = (\ \hat{x}(x,t,u),\ \hat{t}(x,t,u),\ \hat{u}(x,t,u))\,.$$

We have

$$\begin{aligned} \mathbf{X}_1\hat{t} &= 0, \\ \frac{\partial \hat{t}}{\partial x} &= 0, \\ \hat{t} &= A(t,u). \end{aligned}$$

We also have

$$\begin{aligned}
\mathbf{X}_3\hat{t} &= 0,\\
\frac{\partial \hat{t}}{\partial u} &= 0,\\
\frac{\partial \hat{t}}{\partial u} &= \frac{\partial A(t,u)}{\partial u} = 0.
\end{aligned}$$

So

$$\begin{aligned}
A(t,u) &= A(t),\\
\hat{t} &= A(t).
\end{aligned}$$

Now

$$\begin{aligned}
\mathbf{X}_2\hat{t} &= \frac{b_2^2}{|b_2^2|},\\
\frac{\partial \hat{t}}{\partial t} &= \frac{b_2^2}{|b_2^2|},\\
A'(t) &= \frac{b_2^2}{|b_2^2|},\\
A(t) &= \frac{b_2^2}{|b_2^2|}t + c_1,\\
\hat{t} &= \frac{b_2^2}{|b_2^2|}t + c_1.
\end{aligned}$$

But we also have

$$\begin{aligned}
\mathbf{X}_4\hat{t} &= 2\hat{t},\\
x\frac{\partial \hat{t}}{\partial x} + 2t\frac{\partial \hat{t}}{\partial t} + u\frac{\partial \hat{t}}{\partial u} &= 2\hat{t},\\
2t\Big(\frac{b_2^2}{|b_2^2|}\Big) &= 2t\Big(\frac{b_2^2}{|b_2^2|}\Big) + 2c_1,\\
0 &= c_1.
\end{aligned}$$

So

$$\hat{t} = \frac{b_2^2}{|b_2^2|}t.$$

Also, $\hat{t}$ just obtained satisfies

$$
\begin{aligned}
\mathbf{X}_5\hat{t} &= 0,\\
u\frac{\partial \hat{t}}{\partial x} - x\frac{\partial \hat{t}}{\partial u} &= 0,\\
u\frac{\partial}{\partial x}\Big(\frac{b_2^2}{|b_2^2|}t\Big) - x\frac{\partial}{\partial u}\Big(\frac{b_2^2}{|b_2^2|}t\Big) &= 0,\\
0 &= 0.
\end{aligned}
$$

Next we have

$$
\begin{aligned}
\mathbf{X}_2\hat{u} &= 0,\\
\frac{\partial \hat{u}}{\partial t} &= 0,\\
\hat{u} &= B(x,u).
\end{aligned}
$$

We also have

$$
\begin{aligned}
\mathbf{X}_3\hat{u} &= \frac{-b_3^1 + b_3^3}{\sqrt{2\,|b_2^2|}},\\
\frac{\partial \hat{u}}{\partial u} &= \frac{-b_3^1 + b_3^3}{\sqrt{2\,|b_2^2|}},\\
\frac{\partial B(x,u)}{\partial u} &= \frac{-b_3^1 + b_3^3}{\sqrt{2\,|b_2^2|}},\\
B(x,u) &= \left(\frac{-b_3^1 + b_3^3}{\sqrt{2\,|b_2^2|}}\right)u + C(x),\\
\hat{u} &= \left(\frac{-b_3^1 + b_3^3}{\sqrt{2\,|b_2^2|}}\right)u + C(x).
\end{aligned}
$$

But we also have

$$
\begin{aligned}
\mathbf{X}_1\hat{u} &= -\left(\frac{b_3^1 + b_3^3}{\sqrt{2\,|b_2^2|}}\right)b_5^5,\\
\frac{\partial \hat{u}}{\partial x} &= -\left(\frac{b_3^1 + b_3^3}{\sqrt{2\,|b_2^2|}}\right)b_5^5,
\end{aligned}
$$

$$C'(x) = -\left(\frac{b_3^1 + b_3^3}{\sqrt{2\,|b_2^2|}}\right) b_5^5,$$
$$C(x) = -\left(\frac{b_3^1 + b_3^3}{\sqrt{2\,|b_2^2|}}\right) b_5^5\, x + c_2.$$
$$\hat{u} = \left(\frac{-b_3^1 + b_3^3}{\sqrt{2\,|b_2^2|}}\right) u - \left(\frac{b_3^1 + b_3^3}{\sqrt{2\,|b_2^2|}}\right) b_5^5\, x + c_2.$$

Now

$$\mathbf{X}_4 \hat{u} = \hat{u},$$
$$x\frac{\partial \hat{u}}{\partial x} + 2t\frac{\partial \hat{u}}{\partial t} + u\frac{\partial \hat{u}}{\partial u} = \hat{u},$$
$$x\left(-\left(\frac{b_3^1 + b_3^3}{\sqrt{2\,|b_2^2|}}\right) b_5^5\right) + u\left(\frac{-b_3^1 + b_3^3}{\sqrt{2\,|b_2^2|}}\right) = \left(\frac{-b_3^1 + b_3^3}{\sqrt{2\,|b_2^2|}}\right) u - \left(\frac{b_3^1 + b_3^3}{\sqrt{2\,|b_2^2|}}\right) b_5^5\, x + c_2,$$
$$0 = c_2.$$

So

$$\hat{u} = \left(\frac{-b_3^1 + b_3^3}{\sqrt{2\,|b_2^2|}}\right) u - \left(\frac{b_3^1 + b_3^3}{\sqrt{2\,|b_2^2|}}\right) b_5^5\, x.$$

Next we have

$$\mathbf{X}_2 \hat{x} = 0,$$
$$\frac{\partial \hat{x}}{\partial t} = 0,$$
$$\hat{x} = D(x, u).$$

We also have

$$\mathbf{X}_3 \hat{x} = \frac{b_3^1 + b_3^3}{\sqrt{2\,|b_2^2|}},$$
$$\frac{\partial \hat{x}}{\partial u} = \frac{b_3^1 + b_3^3}{\sqrt{2\,|b_2^2|}},$$
$$\frac{\partial D(x,u)}{\partial u} = \frac{b_3^1 + b_3^3}{\sqrt{2\,|b_2^2|}},$$
$$D(x,u) = \left(\frac{b_3^1 + b_3^3}{\sqrt{2\,|b_2^2|}}\right) u + E(x),$$
$$\hat{x} = \left(\frac{b_3^1 + b_3^3}{\sqrt{2\,|b_2^2|}}\right) u + E(x).$$

But we also have

$$\begin{aligned}
\mathbf{X}_1\hat{x} &= \left(\frac{-b_3^1+b_3^3}{\sqrt{2\,|b_2^2|}}\right)b_5^5,\\
\frac{\partial \hat{x}}{\partial x} &= \left(\frac{-b_3^1+b_3^3}{\sqrt{2\,|b_2^2|}}\right)b_5^5,\\
E'(x) &= \left(\frac{-b_3^1+b_3^3}{\sqrt{2\,|b_2^2|}}\right)b_5^5,\\
E(x) &= \left(\frac{-b_3^1+b_3^3}{\sqrt{2\,|b_2^2|}}\right)b_5^5\,x + c_3.
\end{aligned}$$

$$\hat{x} = \left(\frac{b_3^1+b_3^3}{\sqrt{2\,|b_2^2|}}\right)u + \left(\frac{-b_3^1+b_3^3}{\sqrt{2\,|b_2^2|}}\right)b_5^5\,x + c_3.$$

Now

$$\begin{aligned}
\mathbf{X}_4\hat{x} &= \hat{x},\\
x\frac{\partial \hat{x}}{\partial x} + 2t\frac{\partial \hat{x}}{\partial t} + u\frac{\partial \hat{x}}{\partial u} &= \hat{x},\\
x\left(\frac{-b_3^1+b_3^3}{\sqrt{2\,|b_2^2|}}\right)b_5^5 + u\left(\frac{b_3^1+b_3^3}{\sqrt{2\,|b_2^2|}}\right) &= \left(\frac{b_3^1+b_3^3}{\sqrt{2\,|b_2^2|}}\right)u + \left(\frac{-b_3^1+b_3^3}{\sqrt{2\,|b_2^2|}}\right)b_5^5\,x + c_3,\\
0 &= c_3.
\end{aligned}$$

So

$$\hat{x} = \left(\frac{b_3^1+b_3^3}{\sqrt{2\,|b_2^2|}}\right)u + \left(\frac{-b_3^1+b_3^3}{\sqrt{2\,|b_2^2|}}\right)b_5^5\,x.$$

Now two equations still remain to be used

$$\mathbf{X}_5\hat{x} = b_5^5\hat{u}, \quad \mathbf{X}_5\hat{u} = -b_5^5\hat{x}.$$

$$\begin{aligned}
\mathbf{X}_5\hat{x} &= b_5^5\hat{u},\\
u\frac{\partial \hat{x}}{\partial x} - x\frac{\partial \hat{x}}{\partial u} &= b_5^5\hat{u},
\end{aligned}$$

$$u\left(\frac{-b_3^1+b_3^3}{\sqrt{2\,|b_2^2|}}\right)b_5^5 - x\left(\frac{b_3^1+b_3^3}{\sqrt{2\,|b_2^2|}}\right) = b_5^5\Big(\left(\frac{-b_3^1+b_3^3}{\sqrt{2\,|b_2^2|}}\right)u - \left(\frac{b_3^1+b_3^3}{\sqrt{2\,|b_2^2|}}\right)b_5^5\,x\Big),$$

$$u\left(\frac{-b_3^1+b_3^3}{\sqrt{2\,|b_2^2|}}\right)b_5^5 - x\left(\frac{b_3^1+b_3^3}{\sqrt{2\,|b_2^2|}}\right) = \left(\frac{-b_3^1+b_3^3}{\sqrt{2\,|b_2^2|}}\right)b_5^5 u - \left(\frac{b_3^1+b_3^3}{\sqrt{2\,|b_2^2|}}\right)(b_5^5)^2\,x,$$

$$0 = 0.$$

Since $(b_5^5)^2 = (\pm 1)^2 = 1$.

$$\mathbf{X}_5\hat{u} = -b_5^5\hat{x},$$

$$u\frac{\partial \hat{u}}{\partial x} - x\frac{\partial \hat{u}}{\partial u} = -b_5^5\hat{x},$$

$$u\Big(-\left(\frac{b_3^1+b_3^3}{\sqrt{2\,|b_2^2|}}\right)b_5^5\Big) - x\left(\frac{-b_3^1+b_3^3}{\sqrt{2\,|b_2^2|}}\right) = -b_5^5\Big(\left(\frac{b_3^1+b_3^3}{\sqrt{2\,|b_2^2|}}\right)u + \left(\frac{-b_3^1+b_3^3}{\sqrt{2\,|b_2^2|}}\right)b_5^5\,x\Big),$$

$$-u\left(\frac{b_3^1+b_3^3}{\sqrt{2\,|b_2^2|}}\right)b_5^5 - x\left(\frac{-b_3^1+b_3^3}{\sqrt{2\,|b_2^2|}}\right) = -b_5^5\left(\frac{b_3^1+b_3^3}{\sqrt{2\,|b_2^2|}}\right)u - \left(\frac{-b_3^1+b_3^3}{\sqrt{2\,|b_2^2|}}\right)(b_5^5)^2\,x,$$

$$0 = 0.$$

Since $(b_5^5)^2 = (\pm 1)^2 = 1$.

Both equations are satisfied.

Therefore, the general solution of the system of determining equations is

$$\hat{x} = \left(\frac{b_3^1+b_3^3}{\sqrt{2\,|b_2^2|}}\right)u + \left(\frac{-b_3^1+b_3^3}{\sqrt{2\,|b_2^2|}}\right)b_5^5\,x.$$

$$\hat{t} = \frac{b_2^2}{|b_2^2|}\,t.$$

$$\hat{u} = \left(\frac{-b_3^1+b_3^3}{\sqrt{2\,|b_2^2|}}\right)u - \left(\frac{b_3^1+b_3^3}{\sqrt{2\,|b_2^2|}}\right)b_5^5\,x.$$

4.1.7 Symmetry Condition

To apply the symmetry condition we find

$$\begin{aligned}\frac{\partial \hat{u}}{\partial \hat{t}} &= \frac{\partial\Big(\Big(\frac{-b_3^1+b_3^3}{\sqrt{2\,|b_2^2|}}\Big)u - \Big(\frac{b_3^1+b_3^3}{\sqrt{2\,|b_2^2|}}\Big)b_5^5\,x\Big)}{\partial\Big(\frac{b_2^2}{|b_2^2|}\,t\Big)} \\ &= \frac{|b_2^2|}{\sqrt{2\,|b_2^2|}}\frac{(-b_3^1+b_3^3)}{b_2^2}\frac{\partial u}{\partial t}.\end{aligned}$$

$$\begin{aligned}\frac{\partial \hat{u}}{\partial \hat{x}} &= \frac{\partial\Big(\Big(\frac{-b_3^1+b_3^3}{\sqrt{2\,|b_2^2|}}\Big)u - \Big(\frac{b_3^1+b_3^3}{\sqrt{2\,|b_2^2|}}\Big)b_5^5\,x\Big)}{\partial\Big(\Big(\frac{b_3^1+b_3^3}{\sqrt{2\,|b_2^2|}}\Big)u + \Big(\frac{-b_3^1+b_3^3}{\sqrt{2\,|b_2^2|}}\Big)b_5^5\,x\Big)} \\ &= \frac{(-b_3^1+b_3^3)\frac{\partial u}{\partial x} - (b_3^1+b_3^3)b_5^5}{(b_3^1+b_3^3)\frac{\partial u}{\partial x} + (-b_3^1+b_3^3)b_5^5}.\end{aligned}$$

$$\begin{aligned}\frac{\partial^2 \hat{u}}{\partial \hat{x}^2} &= \frac{\partial}{\partial \hat{x}}\Big(\frac{\partial \hat{u}}{\partial \hat{x}}\Big) = \frac{\partial\Big(\frac{(-b_3^1+b_3^3)\frac{\partial u}{\partial x} - (b_3^1+b_3^3)b_5^5}{(b_3^1+b_3^3)\frac{\partial u}{\partial x} + (-b_3^1+b_3^3)b_5^5}\Big)}{\partial\Big(\Big(\frac{b_3^1+b_3^3}{\sqrt{2\,|b_2^2|}}\Big)u + \Big(\frac{-b_3^1+b_3^3}{\sqrt{2\,|b_2^2|}}\Big)b_5^5\,x\Big)} \\ &= \frac{(\sqrt{2\,|b_2^2|})\Big((-b_3^1+b_3^3)^2 - (b_3^1+b_3^3)^2\Big)b_5^5\frac{\partial^2 u}{\partial x^2}}{\Big((b_3^1+b_3^3)\frac{\partial u}{\partial x} + (-b_3^1+b_3^3)b_5^5\Big)^3}.\end{aligned}$$

So

$$\frac{\partial \hat{u}}{\partial \hat{t}} = \left(\frac{1}{1+\Big(\frac{\partial \hat{u}}{\partial \hat{x}}\Big)^2}\right)\frac{\partial^2 \hat{u}}{\partial \hat{x}^2},$$

is

$$\frac{|b_2^2|}{\sqrt{2\,|b_2^2|}}\frac{(-b_3^1+b_3^3)}{b_2^2}\frac{\partial u}{\partial t} =$$

$$\frac{1}{1+\left(\frac{(-b_3^1+b_3^3)\frac{\partial u}{\partial x}-(b_3^1+b_3^3)b_5^5}{(b_3^1+b_3^3)\frac{\partial u}{\partial x}+(-b_3^1+b_3^3)b_5^5}\right)^2}\left(\frac{(\sqrt{2\,|b_2^2|})\Big((-b_3^1+b_3^3)^2-(b_3^1+b_3^3)^2\Big)b_5^5\frac{\partial^2 u}{\partial x^2}}{\Big((b_3^1+b_3^3)\frac{\partial u}{\partial x}+(-b_3^1+b_3^3)b_5^5\Big)^3}\right).$$

If the symmetry condition is to be satisfied, $(b_3^1+b_3^3)=0$ and $b_3^3 \neq 0$ otherwise the transformed equation will have $\frac{\partial u}{\partial x}$ and $\frac{\partial^3 u}{\partial x^3}$ terms in it.

After simplification we have

$$\frac{\partial u}{\partial t} = \frac{1}{1+\left(\frac{\partial u}{\partial x}\right)^2}\left(\frac{b_2^2}{2(b_3^3)^2}\right)\frac{\partial^2 u}{\partial x^2}.$$

Therefore, $\frac{b_2^2}{2(b_3^3)^2}$ must be equal to 1. So

$$(\hat{x},\hat{t},\hat{u}) = (\ \frac{b_3^3}{|b_3^3|}b_5^5\,x,\ t,\ \frac{b_3^3}{|b_3^3|}u\ ).$$

Since $b_5^5 \pm 1$, these are essentially four symmetries one of which is the identity.

Therefore, the **only** discrete symmetries of the Nonlinear Filtration Equation **up to equivalence** are

$$\begin{aligned}
\Gamma_{D1} \quad & (\hat{x},\hat{t},\hat{u}) = (\ -x,\ t,\ -u\ ),\\
\Gamma_{D2} \quad & (\hat{x},\hat{t},\hat{u}) = (\ x,\ t,\ -u\ ),\\
\Gamma_{D3} \quad & (\hat{x},\hat{t},\hat{u}) = (\ -x,\ t,\ u\ ).
\end{aligned}$$

This result is exhaustive. If Γ_{Di} is any discrete symmetry from the above three, then any other discrete symmetry, $\tilde{\Gamma}_D$, of the Nonlinear Filtration Equation can be obtained as

$$\tilde{\Gamma}_D = e^{\epsilon \mathbf{X}}\Gamma_{Di}.$$

For example, if we choose $\mathbf{X} = \mathbf{X}_1$, and $i = 1$, then, $e^{\epsilon \mathbf{X}_1}\Gamma_{D1}$ gives us

$$\begin{aligned} \tilde{\Gamma}_D \quad (\hat{x}, \hat{t}, \hat{u}) &= (\,(-x) + \epsilon,\ (t),\ (-u)\,) \\ &= (\,-x + \epsilon,\ t,\ -u\,). \end{aligned}$$

$\mathbf{X}$ could be any infinitesimal generator from the Lie Algebra, $\mathcal{L}$, of infinitesimal generators of (one-parameter) Lie groups of point symmetries of the equation.

Some other examples

$$\begin{aligned} e^{\epsilon \mathbf{X}_5}\Gamma_{D1} \quad (\hat{x}, \hat{t}, \hat{u}) &= ((-x)\cos\epsilon + (-u)\sin\epsilon,\ (t),\ -(-x)\sin\epsilon + (-u)\cos\epsilon) \\ &= (-x\cos\epsilon - u\sin\epsilon,\ t,\ x\sin\epsilon - u\cos\epsilon). \end{aligned}$$

$$\begin{aligned} e^{\epsilon \mathbf{X}_4}\Gamma_{D2} \quad (\hat{x}, \hat{t}, \hat{u}) &= (e^{\epsilon}(x),\ e^{2\epsilon}(t),\ e^{\epsilon}(-u)) \\ &= (e^{\epsilon}x,\ e^{2\epsilon}t,\ -e^{\epsilon}u). \end{aligned}$$

$$\begin{aligned} e^{\epsilon \mathbf{X}_2}\Gamma_{D1} \quad (\hat{x}, \hat{t}, \hat{u}) &= ((-x),\ (t + \epsilon),\ (-u)) \\ &= (-x,\ t + \epsilon,\ -u). \end{aligned}$$

4.2 Related Exact Solutions

Any exact solution of a differential equation can be examined for new exact solutions using discrete symmetries of the differential equation. In this section it has been attempted to find the group invariant solutions of the NLF equation due to the groups generated by $\mathbf{X}_i$, the basis generators. The group invariant solutions will then be transformed using the discrete symmetries obtained in the last section.

Solutions invariant under the group G, generated by

$$\mathbf{X} = \xi\frac{\partial}{\partial x} + \tau\frac{\partial}{\partial t} + \eta\frac{\partial}{\partial u},$$

satisfy the PDE

$$\eta - \xi\frac{\partial u}{\partial x} - \tau\frac{\partial u}{\partial t} = 0.$$

For $\mathbf{X}_1$

$$\xi = 1, \quad \tau = 0, \quad \eta = 0.$$

$$\begin{aligned}(0) - (1)\frac{\partial u}{\partial x} - (0)\frac{\partial u}{\partial t} &= 0,\\ \frac{\partial u}{\partial x} &= 0,\\ u(x,t) &= F(t).\end{aligned}$$

This solution will now be substituted in NLF equation to determine F. Since

$$\frac{\partial u}{\partial t} = F'(t), \quad \frac{\partial u}{\partial x} = 0, \quad \frac{\partial^2 u}{\partial x^2} = 0,$$

we have

$$\begin{aligned}F'(t) &= 0,\\ F(t) &= c,\\ u(x,t) &= c.\end{aligned}$$

It is trivial to transform $u = c$.

For $\mathbf{X}_2$

$$\xi = 0, \quad \tau = 1, \quad \eta = 0.$$

$$\begin{aligned}(0) - (0)\frac{\partial u}{\partial x} - (1)\frac{\partial u}{\partial t} &= 0,\\ \frac{\partial u}{\partial t} &= 0,\\ u(x,t) &= F(x).\end{aligned}$$

We now substitute this solution in NLF equation to determine F.

$$\frac{\partial u}{\partial t} = 0, \quad \frac{\partial u}{\partial x} = F'(x), \quad \frac{\partial^2 u}{\partial x^2} = F''(x).$$

Substituting in NLF equation, we obtain

$$\begin{aligned}0 &= F''(x),\\ F''(x) &= c_1 x + c_2,\\ u(x,t) &= c_1 x + c_2.\end{aligned}$$

Trying all discrete symmetries of the form $e^{\epsilon \mathbf{X}_j}\Gamma_{Di}$ shows, that atleast such symmetries do not transform the above two parameter family of solutions to any new family of solutions.

For instance, using

$$e^{\epsilon \mathbf{X}_5}\Gamma_{D1} \quad (\hat{x}, \hat{t}, \hat{u}) = (-x\cos\epsilon - u\sin\epsilon,\ t,\ x\sin\epsilon - u\cos\epsilon),$$

we obtain

$$(-\hat{x}\sin\epsilon - \hat{u}\cos\epsilon) = c_1(-\hat{x}\cos\epsilon + \hat{u}\sin\epsilon) + c_2.$$

Removing carets and rearranging

$$u = \frac{c_1\cos\epsilon - \sin\epsilon}{\cos\epsilon + c_1\sin\epsilon}\ x - \frac{c_2}{\cos\epsilon + c_1\sin\epsilon}\ .$$

For $\mathbf{X}_3$

$$\xi = 0, \quad \tau = 0, \quad \eta = 1.$$

$$\begin{aligned}(1) - (0)\frac{\partial u}{\partial x} - (0)\frac{\partial u}{\partial t} &= 0,\\ 1 &= 0.\end{aligned}$$

Therefore, no group invariant solutions due to $\mathbf{X}_3$.

For $\mathbf{X}_4$

$$\xi = x, \quad \tau = 2t, \quad \eta = u.$$

$$\begin{aligned}(u) - (x)\frac{\partial u}{\partial x} - (2t)\frac{\partial u}{\partial t} &= 0,\\ x\frac{\partial u}{\partial x} + 2t\frac{\partial u}{\partial t} &= u,\end{aligned}$$

$$\frac{dx}{x} = \frac{dt}{2t} = \frac{du}{u},$$

$$\begin{aligned}\frac{dx}{x} &= \frac{dt}{2t},\\ \frac{x}{\sqrt{t}} &= r.\end{aligned}$$

Likewise

$$v = \frac{u}{\sqrt{t}}.$$

General solution

$$\begin{aligned}v &= F(r),\\ u &= \sqrt{t}F(\frac{x}{\sqrt{t}}).\end{aligned}$$

This solution will now be substituted in NLF equation to determine F.

Since

$$\frac{\partial u}{\partial x} = F'(r), \quad \frac{\partial^2 u}{\partial x^2} = \frac{1}{\sqrt{t}} F''(r), \quad \frac{\partial u}{\partial t} = \frac{1}{2\sqrt{t}} F(r) - \frac{x}{2(\sqrt{t})^2} F'(r),$$

we have

$$2F''(r) = F(r) - rF'(r) + \Big(F'(r)\Big)^2 F(r) - r\Big(F'(r)\Big)^3.$$

This is a second order nonlinear ODE for F, which appears to be very difficult to solve, if at all possible.

For $\mathbf{X}_5$

$$\xi = u, \quad \tau = 0, \quad \eta = -x.$$

$$\begin{aligned} (-x) - (u)\frac{\partial u}{\partial x} - (0)\frac{\partial u}{\partial t} &= 0, \\ u\frac{\partial u}{\partial x} &= -x, \end{aligned}$$

$$\frac{dx}{u} = \frac{dt}{0} = \frac{du}{-x},$$

$$\begin{aligned} \frac{dx}{u} &= \frac{dt}{x}, \\ xdx &= -udu, \\ v &= u^2 + x^2, \end{aligned}$$

and

$$t = r.$$

General solution

$$\begin{aligned} v &= F(r), \\ u^2 + x^2 &= F(t), \\ u &= \pm\sqrt{F(t) - x^2}. \end{aligned}$$

We now substitute this solution in NLF equation to determine F.

$$
\begin{aligned}
\frac{\partial u}{\partial t} &= \frac{1}{2\sqrt{F(t)-x^2}}F'(t)\,,\\
\frac{\partial u}{\partial x} &= \frac{-x}{\sqrt{F(t)-x^2}}\,,\\
\frac{\partial^2 u}{\partial x^2} &= \frac{-F(t)}{(F(t)-x^2)\sqrt{F(t)-x^2}}\,.
\end{aligned}
$$

Substituting in NLF equation, we obtain

$$F(t) = -2t + c.$$

So

$$u(x,t) = \sqrt{c_1 - 2t - x^2},$$

and

$$u(x,t) = -\sqrt{c_2 - 2t - x^2}.$$

These solutions can now be transformed using discrete symmetries to obtain more solutions of the NLF equation.

For instance, using

$$e^{\epsilon \mathbf{X}_1}\Gamma_{D1} \quad (\hat{x}, \hat{t}, \hat{u}) = (-x+\epsilon,\ t,\ -u),$$

to transform

$$u(x,t) = \sqrt{c_1 - 2t - x^2},$$

we obtain a two parameter family of solutions

$$(-\hat{u}) = \sqrt{c_1 - 2\hat{t} - (-\hat{x}+\epsilon)^2}\,.$$

Removing carets and rearranging, we have

$$u = -\sqrt{c_1 - 2t - (x-\epsilon)^2}\,.$$

Summary

Peter H. Hydon's method calculates discrete symmetries of differential equations, having a finite dimensional Lie algebra of infinitesimal generators of its (one-parameter) Lie groups of point symmetries. His method is based on the observation that every point symmetry yields an automorphism of the Lie algebra of the Lie point symmetry generators. The method not only results in an easier system of determining equations, uncoupled, but it is also exhaustive, that is, all discrete symmetries are obtained.

The method in its basic form has two stages, solving the system of determining equations and application of symmetry condition on the solution of this system.

Little improvements in the method can be made when the Lie algebra of Lie point symmetry generators is abelian. In this case we can use canonical coordinates. This way at least one generator in the basis is simplified.

Two further improvements can be made when the Lie algebra of Lie point symmetry generators is non-abelian, the use of the nonlinear constraints and solving the system of determining equations for only the inequivalent matrices B. Although these two improvements are theoretically independent of each other but they are almost always employed simultaneously.

All the discrete symmetries of KdV and NLF equations are calculated. As an immediate application of discrete symmetries, group invariant solutions of the equations, corresponding to the basis vectors of the Lie algebra, are investigated for further solutions under transformations due to the discrete symmetries.

Appendix

This appendix discusses some prerequisite definitions and concepts.

Group

A **group** G is a set of elements with a binary operation ϕ on G such that the following conditions are satisfied

(1) $\phi(a, b) \in G$, for all $a, b \in G$.

(2) $\phi(a, \phi(b, c)) = \phi(\phi(a, b), c)$, for all $a, b, c \in G$.

(3) There exists a unique element $e \in G$ such that,
$\phi(e, a) = \phi(a, e) = a$, for all $a \in G$.

(4) For every $a \in G$ there exist a unique $a^{-1} \in G$ such that,
$\phi(a^{-1}, a) = \phi(a, a^{-1}) = e$.

A group is **abelian** if $\phi(a, b) = \phi(b, a)$, for all $a, b \in G$.

Topological Space

A **topological space** is a pair (S, T) where S is a set and T is a class of subsets of S such that

(1) S and $\emptyset$ belong to T.
(2) The union of any number of sets in T belongs to T.
(3) The intersection of any two sets in T belongs to T.

T is called a **topology** on S and elements of T are called **open sets**. A collection B of open sets of S is a **basis** for the topology of S if every open set is the union of sets of B. If S has a countable basis it is said to be **second countable**.

A topological space is a **Hausdorff** space if any two distinct points have disjoint open neighbourhoods.

Let A and B be topological spaces. A map $f : A \longrightarrow B$ is **continuous** if the inverse image of any open set in B is an open set in A. The map f is a **homeomorphism** if it is bijective and both f and f^{-1} are continuous, in which case $U \subset A$ is open if and only if $f(U) \subset B$ is open.

Manifold

An n-dimensional **topological manifold** is a Hausdorff space M with a countible basis such that every point $p \in M$ has a open neighbourhood U which is homeomorphic with an open subset of $\mathbb{R}^n$.

A **chart** for a topological manifold is the pair (U, h), where U is an open subset of M and $h : U \longrightarrow V$ is a homeomorphism, with V open in $\mathbb{R}^n$.

An **atlas** on an n-dimensional topological manifold is a collection of charts $\{(U_\alpha, h_\alpha) : \alpha \in I\}$, where I is a countable index set, such that $M = \bigcup_{\alpha \in I} U_\alpha$.

An n-dimensional **smooth manifold** M is an n-dimensional topological manifold which satisfy the following conditions

(1) There exists an atlas on M.

(2) On the overlap of any pair of charts $U_\alpha \cap U_\beta$ the following composite map is smooth,

$$h_\beta \circ h_\alpha^{-1} : h_\alpha(U_\alpha \cap U_\beta) \longrightarrow h_\beta(U_\alpha \cap U_\beta).$$

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