Solution of partial differential equations by cosine and sine (CAS) wavelets

by

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A thesis submitted in partial fulfillment of the requirements for the degree of Masters of Science in Mathematics

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MASTER'S THESIS WORK

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Acknowledgement

All praises to Allah, the originator of all knowledge, most gracious, the most compassionate. I offer all gratitude to Almighty Allah (glorified and exalted be He), the creator of all things especially related to all types of knowledge and the Almighty, Who has bestowed upon me the quest for knowledge and granted me determination to accomplish this research.

My individual efforts would not have been sufficient for the accomplishment of this purely technical thesis, various personalities helped me throughout for this thesis work. This help was not only in the form of technical knowledge, but through motivation, time and support, encouragement and pertinent criticism which I was offered by many individuals. My special thanks go to Dr Umer Saeed, my supervisor, for the guidance, thought provoking suggestions, pertinent criticism, and inspiration that he provided me throughout this research. His unending support in my study has been an invaluable asset for me. He not only taught me how to embark on a technical research problem but taught me how to write in an appropriate manner. I would also like to express my deepest gratitude to Mr Irfan shah, my husband. His motivation and support kept me to continue reading relevant material along with my thesis work.

I am particularly indebted to my parents, brothers and sisters for their monumental, unwavering support and encouragement on all fronts. Last but not least, I offer my regards and thanks to NUST for partially sponsoring this study.

Syeda Khadija Gilani

Abstract

In this thesis we develop two numerical methods, CAS wavelet Picard and CAS wavelet quasilinearization technique, for solving nonlinear partial differential equations. The proposed methods are the combination of CAS wavelet and linearization techniques.

CAS wavelet Picard and CAS wavelet quasilinearization methods are proposed by utilizing CAS wavelet in conjunction with Picard and quasilinearization techniques respectively. The operational matrices of integration for CAS wavelet are derived and constructed. According to the methods, we convert the nonlinear partial differential equation into linear partial differential equations and then utilized the operational matrices of CAS wavelet to get the solution on each iteration of Picard or quasilinearization techniques.

We utilized the CAS wavelet method for linear partial differential equations and implemented on wave, diffusion and Klein-Gordon equation. CAS wavelet Picard method is tested on Burger's and Burger-Huxley equation while CAS wavelet quasilinearization technique is applied on Burger-Fisher equation to get the solution.

Error analysis and procedure of implementation for both the methods are given. Comparison analysis for solution of some famous equations is also done. Solutions by present methods are more accurate as compared to the solution obtained by some well-known methods like Adomian decomposition method, variational iteration method and reduced differential transform method.

Contents

Chapter 1

Introduction

An equation involving derivatives of one or more dependent variables with respect to one or more independent variables is called a differential equation. Ordinary differential equations (ODEs) are equations that involve ordinary derivatives of dependent variables.

$$F(x, y, \frac{dy}{dx}, \frac{d^2y}{dx^2}, \ldots, \frac{d^ny}{dx^n}) = 0. \tag{1.0.1}$$

The function $y = f(x)$, defined on the set M (set of real numbers), is said to be solution of equation (1.0.1) if and only if $y = f(x)$ has derivatives up to and include the order n [1]. Whereas an equation containing partial derivatives of one or more dependent variables with respect to more than one independent variable is called partial differential equations (PDEs). A linear PDE is an equation for which the dependent variable and its derivatives occur with degree at most one. Which means that the coefficients are either constants or function of independent variables. The general form of first-order quasilinear PDE with x and y two independent variables and one unknown u, dependent variable is written as

$$f(x, y, u)\frac{\partial u}{\partial x} + g(x, y, u)\frac{\partial u}{\partial y} = h(x, y, u).$$

If the functions f, g and h are independent of unknown function u then we call it linear PDE [2]. While the partial differential equations which do not fall in any of the above categories are known as nonlinear PDEs. They usually contain powers of partial derivatives or product of unknown function with partial derivatives. The partial differential equation is homogeneous, if it contains only partial derivatives of dependent variable and terms involving dependent variables and they are set to 0 as given below:

$$\frac{\partial^2 y}{\partial x^2} + x\frac{\partial y}{\partial x} + y^2 = 0.$$

Nonhomogeneous partial differential equations are the same as homogeneous partial differential equations except they have terms containing x or constants on the right side of equation as

$$\frac{\partial^2 y}{\partial x^2} + x\frac{\partial y}{\partial x} + y^2 = 4x^2 + 9.$$

Consider the second order partial differential equation in two variables as

$$A(x,y)\frac{\partial^2 u}{\partial x^2} + B(x,y)\frac{\partial^2 u}{\partial xy} + C(x,y)\frac{\partial^2 u}{\partial y^2} + D(x,y)\frac{\partial u}{\partial x} + E(x,y)\frac{\partial u}{\partial y} + F(x,y)u = G, \tag{1.0.2}$$

where the coefficients A, B and C do not vanish simultaneously.

There are fundamentally three kinds of PDEs hyperbolic, parabolic and elliptic PDEs. From the physical point of view these PDEs represent the wave propagation, the diffusion processes and the equilibrium processes respectively. The hyperbolic equations model the transport of physical quantity, like fluids or waves. Parabolic problems explain evolutionary phenomena which lead to a steady state presented by an elliptic equation. While elliptic equations are related to a special state of a system, in correspondence to state of energy.

Mathematically, if the discriminant of the equation (1.0.2) is strictly positive, then the roots are real distinct and the equation is called hyperbolic. When the discriminant is equal to zero, we obtain heat equation which is parabolic equation. If the discriminant of the equation (1.0.2) is strictly negative and there are no real characteristics then equation is elliptic i.e. the Laplace equation.

Wavelets

Wavelet means "small wave" [3]. It has different backgrounds in history of mathematics. In 1930's a lot of research was carried out regarding wavelets. A wavelet is a function with compact support and has oscillatory nature for a small period of time. A wavelet is manipulated by two parameters scaling and translation to generate a set of orthonormal basis function which represents a signal. Wavelet is defined by:

$$\psi_{a,b}(x) = \frac{1}{\sqrt{|a|}}\psi(\frac{x-a}{b}) \;\; a,b \;\rightarrow\; R, \;\; a \neq 0, \tag{1.0.3}$$

where a is scaling parameter and b is translation parameter. If $|a| < 1$ then wavelet (1.0.3) is compressed and shows high frequencies. Whereas when $|a| > 1$ then wavelet (1.0.3) is expanded and corresponds to low frequencies.

Admissibility condition is the most important property of wavelets. For the wavelet (1.0.3) it can be shown that integrable functions $\psi(x)$ satisfying the admissibility condition can be used in analyzing and reconstruction of signal.

$$\int \frac{|\hat{\psi}(\omega)|^2}{|\omega|} d\omega \ < +\infty, \tag{1.0.4}$$

where $\hat{\psi}$ stands for the Fourier transform of $\psi(x)$. It satisfies $|\hat{\psi}(\omega)|^2|_{\omega=0} = 0$. There are different kinds of wavelets, some of which are described below:-

Alfred Haar, a Hungarian mathematician was the first entrant in the wavelet theory, who introduced Haar wavelets in 1909. These functions are composed of a short positive pulse followed by a short negative pulse. They are the orthonormal wavelet with compact support. Integrating them analytically is the key feature of Haar wavelet. The n-th uniform Haar wavelet [4] $h_n(x),\ x \ \rightarrow\ [0,1)$ is defined as

$$h_n(x) = \begin{cases} 1, & \text{if } \frac{k}{m} \leq x < \frac{k+0.5}{m}; \\ -1, & \text{if } \frac{k+0.5}{m} \leq x < \frac{k+1}{m}; \\ 0, & \text{otherwise,} \end{cases}$$

where $n = 2^j + k + 1,\ j = 0, 1, 2, \cdots, J,$ is the dilation parameter $m = 2^j$ and $k = 0, 1, 2, 3, \cdots, 2^j - 1$ and J is the level of resolution.

Hariharan Gopalakrishnan [5] developed Haar wavelet method to solve some famous partial differential equations (PDEs) like Nowell-whitehead equation, Cahn-Allen equation, FitzHugh-Nagumo equation, Fishers equation, Burgers equation and the Burgers-Fisher equation. First eight Haar functions and their integrals are presented and the Haar wavelet operational matrices are also given in his paper.

Solution method for fractional nonlinear partial differential equations are presented in [6]. The proposed method utilizes the Haar wavelets operational matrices in conjunction with Picard technique. These matrices are derived and convergence for the proposed technique has also been given. Numerical solution of Burger and Burger-Fisher equations are provided to demonstrate the accuracy of the technique.

Legendre wavelets are obtained when we put Legendre polynomials, $L_q(x)$ of order q, in the discrete wavelet function, with scaling parameter $a = 2^k$ and translation parameter $b = p2^k$. Legendre wavelet on interval $[0, 1)$ is defined as

$$\psi_{p,q}(x) = \begin{cases} (q + \frac{1}{2})^{\frac{1}{2}} 2^{\frac{k}{2}} L_q(2^k x - \hat{p}), & \text{if } \frac{\hat{p}-1}{2^k} \leq x < \frac{\hat{p}+1}{2^k}; \\ 0, & \text{otherwise,} \end{cases}$$

where $k = 2, 3, \ldots\ldots$, is the level of resolution, $p = 1, 2, 3, \cdots, 2^{k-1}$, $\hat{p} = 2p - 1$ is the translation parameter and $q = 0, 1, 2, \cdots, M - 1$ is the order of legendre polynomials, $M > 0$.

To solve fractional partial differential equations with Dirichlet boundary conditions, two-dimensional Legendre wavelets method is used in [7]. The Legendre wavelet operational matrices of integration and derivative of these basis functions are provided. Convergence analysis of present method is also given and few nonlinear PDEs and Laplace equation are solved to show the validity of this scheme.

Nanshan [8] introduced an orthogonal basis on $[-1, 1] \times [-1, 1]$ generated by Legendre polynomials. Some properties of Legendre polynomials are given and the convergence is also proved. Few examples are illustrated to show the efficiency of the scheme.

Ingrid Daubechies discovered the family of wavelets called the Daubechies wavelets. Being non symmetrical, highest number of vanishing moments, compact support and orthonormality are properties of Daubechies wavelets. This wavelet system consists of wavelet function $\psi(x)$ and scaling function $\phi(x)$. Two-scale relation is the most important relation in wavelet theory given as

$$\phi(x) = \sum_{n=0}^{N-1} p_n \psi_n(2x), \tag{1.0.5}$$

and

$$\psi(x) = \sum_{n=2-N}^{1} (-1)^n p_{1-n} \phi_n(2x), \tag{1.0.6}$$

where $\phi_n(x) = \psi(x-n)$, N(an even integer) is the number and p_n is wavelet filter coefficients in the refinement relations (1.0.5) and (1.0.6). The interval $[0, N-1]$ is support of $\phi(x)$ and $[1 - \frac{N}{2}, \frac{N}{2}]$ is support of $\psi(x)$.

The Daubechies wavelet based on spectral finite element method [9] is generated to solve linear transient dynamics and elastic wave propagation problems. Few $2D$ and $3D$ examples are included to show the convergence and implementation of the scheme. Possible approaches for speed up the method are also discussed.

Multi-scale Daubechies (DB) wavelet method [10] is used for solution of 2-D elastic problems. A method for evaluation of integrals is given to treat general boundaries. The numerical examples show the efficiency of the method. Furthermore, the present method can be easily introduced to other mechanics problem.

The Chebyshev wavelets defined on interval $[a, b]$ by

$$\psi_{p,q}(x) = \begin{cases} 2^{\frac{k}{2}}\sqrt{\frac{4}{(b-a)\pi}}T_q(2^k x - \hat{p}), & \text{if } a + (b-a)\frac{\hat{p}-1}{2^k} \leq x < a + (b-a)\frac{\hat{p}+1}{2^k}; \\ 0, & \text{otherwise,} \end{cases}$$

where $k = 1, 2, 3 \cdots$, is the level of resolution, $p = 1, 2, 3, \cdots , 2^{k-1}$, $\hat{p} = 2p-1$ is the translation parameter and $q = 1, 2, 3, \cdots , M-1$ is the order of Chebyshev polynomials, $M > 0$. While the shifted Chebyshev polynomials $T_m(x)$, of order m defined on interval $[a, b]$ and recurrence formulae is given below $T_0(x) = 1$, $T_1(x) = \frac{2x-(b-a)}{b-a}$ and $T_{m+1}(x) = 2(\frac{2x-(b-a)}{b-a})T_m(x) - T_{m-1}(x)$, where $m = 1, 2, 3, \cdots$.

The second order linear hyperbolic telegraph equation is solved by Chebyshev wavelet method [11]. To study wave propagation of electric signals and to model the reaction-diffusion processes this equation is used. The method is developed by using Chebyshev wavelets method with the operational matrices of integration and differentiation. The accuracy and efficiency of the method is explained with the help of examples.

Chapter 2

Solution of linear partial differential equations

Linear and nonlinear PDEs are the main equations in mathematical physics like heat transfer, electro magnetism and quantum mechanics. An effective method is needed to analyze the mathematical model which generates solutions conforming to physical reality. Common analytic scheme linearize the system or suppose that nonlinearities are comparatively insignificant. Sometimes, the solution of the phenomenon is changed due to such assumptions. Thus, finding numerical solution of non-linear and linear PDES is of great importance.

In this chapter, we have defined the CAS wavelets and constructed the operational matrices of integration for CAS wavelet. We have also constructed the operational matrix of integration for boundary value problem. Error analysis for the method is derived. The method is tested on Wave equation, diffusion equation and Klein-Gordon equation. The obtained solutions by present method are in good agreement with the exact solution and better than variational iteration method(VIM) and Adomian decomposition method(ADM).

2.1 CAS wavelet

Wavelets are defined as

$$\psi_{a,b}(x) = \frac{1}{\sqrt{a}}\psi(\frac{x-b}{a}), \qquad a, b \to R,\ a \neq 0,$$

where a is scaling parameter and b is called translation parameter. For discrete values of $a = a_0^{-k}$ and $b = nb_0a_0^{-k}$, n and k are positive integers, we get discrete wavelets. Particularly, if $a_0 = 2$ and $b_0 = 1$ then we have CAS wavelets which are

$$\psi_{n,m}(x) = \begin{cases} 2^{\frac{k}{2}} CAS_m(2^k x - n + 1), & \frac{(n-1)}{2^k} \leq x \leq \frac{n}{2^k}, \\ 0 & \text{elsewhere,} \end{cases}$$

$CAS_m(x) = \cos(2m\pi x) + \sin(2m\pi x)$, the level of resolution is $k = 0, 1, 2, ...,$ $m = -M, -M+1, ... - 1, 0, 1, ..., M-1, M$ and $n = 0, 1, 2, ..., 2^k - 1$ is the translation parameter and k, M are user defined positive numbers. The set of CAS wavelets forms an orthonormal basis for $L^2[0, 1]$. We can expand any function $f(x, t)$ defined over $[0, 1]$ into truncated CAS wavelet series as

$$\begin{aligned} f(x,t) &= \sum_{n=0}^{\infty} \sum_{m \to Z} \sum_{i=0}^{\infty} \sum_{j \to Z} c_{nm,ij} \psi_{n,m}(x) \psi_{i,j}(t), \\ &\simeq \sum_{n=0}^{2^k-1} \sum_{m=-M}^{M} \sum_{i=0}^{2^{k'}-1} \sum_{j=-M'}^{M'} c_{nm,ij} \psi_{n,m}(x) \psi_{i,j}(t), \\ &= \Psi(x) C^T \Psi(t). \end{aligned}$$

Here C and $\Psi(x)$ are $\hat{m} \times 1$ matrices, given as $c_{l,p} =< \psi_l(x), < u(x,t), \psi_p(t) >>$. The index l and p are determined by the equations $l = M(2n+1) + (n+m+1)$ and $p = M'(2i+1)+(i+j+1)$, respectively. Also $\hat{m} = 2^k(2M+1)$ and $\hat{m'} = 2^{k'}(2M'+1)$. The collocation points for the CAS wavelets are taken as $x_i = \frac{2i-1}{2\hat{m}}$ and $t_j = \frac{2i-1}{2\hat{m'}}$ where $i = 1, 2, ..., \hat{m}$, $j = 1, 2, ..., \hat{m'}$. The CAS wavelets matrix $\mathbf{\Psi}_{\hat{m},\hat{m}}$ is given [13] as

$$\mathbf{\Psi}_{\hat{m} \times \hat{m}} = \left[\mathbf{\Psi}\left(\frac{1}{2\hat{m}}\right), \mathbf{\Psi}\left(\frac{3}{2\hat{m}}\right), ..., \mathbf{\Psi}\left(\frac{2\hat{m}-1}{2\hat{m}}\right) \right]. \tag{2.1.1}$$

In particular, we fix $k = k' = 2$, $M = M' = 1$, we have $n = 0, 1, 2, 3$; $m = -1, 0, 1$ and $i = j = 1, 2, ..., 12$, the CAS wavelets matrix $\mathbf{\Psi}_{12 \times 12}$ is given as

$$\begin{pmatrix} -0.7321 & -2.0000 & 2.7321 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -0.7321 & -2.0000 & 2.7321 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -0.7321 & -2.0000 & 2.7321 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -0.7321 & -2.0000 & 2.7321 \\ 2.0000 & 2.0000 & 2.0000 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2.0000 & 2.0000 & 2.0000 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2.0000 & 2.0000 & 2.0000 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2.0000 & 2.0000 & 2.0000 \\ 2.7321 & -2.0000 & -0.7321 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2.7321 & -2.0000 & -0.7321 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2.7321 & -2.0000 & -0.7321 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2.7321 & -2.0000 & -0.7321 \end{pmatrix}$$

Function Approximation by Block Pulse Function (BPF)

An m-set of Block Pulse Functions (BPFs) on interval $[0, t)$ can be expressed as

$$b_i(x) = \begin{cases} 1, & \text{if } \frac{it}{m} \leq x < \frac{(i+1)t}{m}; \quad i = 0, 1, 2, 3, \dots, m-1. \\ 0, & \text{otherwise,} \end{cases}$$

with a positive integer value for m. Now, few properties of BPFs are defined below.

- Disjointness

$$b_i(x)b_j(x) = \begin{cases} b_i(x), & \text{if } i = j \\ 0, & \text{if } i \neq j. \end{cases}$$

- Orthogonality

$$\int_0^1 b_i(x)b_j(x)dx = \begin{cases} \frac{1}{m}, & \text{if } i = j \\ 0, & \text{if } i \neq j. \end{cases}$$

- BPFs have compact support which is $[\frac{i}{m}, \frac{i+1}{m}]$.

By the orthogonal property of BPFs an arbitrary function $u(x,t) \in L_2[0,1) \times [0,1)$, can be expanded into block-pulse functions [12], this means that for every $u(x,t) \to L^2[0,1)$ we can write:

$$u(x,t) \approx \sum_{i=0}^{\hat{m}-1} \sum_{j=0}^{\hat{m}-1} a_{i,j} b_i(x) b_j(t) = \mathbf{B}^T(x)\mathbf{a}\mathbf{B}(t),$$

where $a_{i,j}$ are the coefficients of the block-pulse functions b_i and b_j. The CAS wavelets can be expanded into $\hat{m}-$set of block-pulse functions as:

$$\mathbf{\Psi}(x) = \mathbf{\Psi}_{\hat{m}\times\hat{m}}\mathbf{B}(x). \tag{2.1.2}$$

We can generalize a function $f(t)$ in terms of BPF

$$f(t) = \xi^T \phi_k(t), \tag{2.1.3}$$

where

$$\xi^T = [c_1, c_2, c_3, ..., c_k].$$

We can start the derivation by taking the generalization of repeated integrals, where $n = 1, 2, 3, ..., k$ [12].

$$\int_0^t dt_n \int_0^{t_n} dt_{n-1} \int_0^{t_{n-1}} dt_{n-2} ..... \int_0^{t_2} f(t_1)dt_1 = \frac{1}{(n+1)!} \int_0^t (t-t_1)^{n-1} f(t_1)dt_1.$$

Suppose

$$\int_0^t dt_n \int_0^{t_n} dt_{n-1} \int_0^{t_{n-1}} dt_{n-2} ..... \int_0^{t_2} f(t_1)dt_1 = I^\alpha f(t).$$

By definition of the convolution integral

$$\frac{1}{(n+1)!}\int_0^t (t-t_1)^{n-1} f(t_1)dt_1 = \frac{1}{\Gamma(n)} t^{n-1} f(t),$$

where 0≤t<b, from (2.1.3) we obtain:

$$I^\alpha f(t) = \frac{1}{\Gamma(\alpha)} t^{\alpha-1} \xi^T \phi_k(t) \qquad \alpha > 0. \tag{2.1.4}$$

By applying Laplace transformation on functions we obtain:

$$\mathcal{L}(\frac{1}{\Gamma(\alpha)} t^{\alpha-1} \xi^T \phi_k(t)) = \frac{\xi^T}{s^{\alpha-1}} (e^{-\frac{(n-1)bs}{k}} - e^{-\frac{(n)bs}{k}}).$$

After taking inverse Laplace transformation, the result is:

$$\frac{1}{\Gamma(\alpha)} t^{\alpha-1} \xi^T \phi_k(t) = \frac{\xi^T}{\Gamma(\alpha)} (U(t - \frac{(n-1)b}{k}) \frac{1}{(\alpha+1)!} (t - \frac{(n-1)b}{k})^\alpha - U(t - \frac{(n)b}{k}) \frac{1}{(\alpha+1)!} (t - \frac{(n)b}{k})^\alpha). \tag{2.1.5}$$

We have

$$t^\alpha U(t) \approx \xi^T \phi_k(t).$$

As defined previously, we get:

$$t^\alpha U(t) \approx [c_1, c_2, c_3, ..., c_k] \phi_k(t), \tag{2.1.6}$$

where

$$C_n = (\frac{b}{k})^\alpha \frac{(n)^{\alpha+1} - (n-1)^{\alpha+1}}{\alpha+1}.$$

From orthogonality of BPFs, we have:

$$U(t - \frac{(n-1)b}{k})(t - \frac{(n-1)b}{k})^\alpha \simeq [0, 0, 0, 0, ..., c_1, c_2, c_3, ..., c_{k-n+1}] \phi_k(t).$$

$$U(t - \frac{(n)b}{k})(t - \frac{(n)b}{k})^\alpha \simeq [0, 0, 0, 0, ..., c_1, c_2, c_3, ..., c_{k-n}] \phi_k(t).$$

By putting values of above in (2.1.5), we get:

$$\frac{\xi^T}{\Gamma(\alpha)} (U(t - \frac{(n-1)b}{k}) \frac{1}{(\alpha+1)!} (t - \frac{(n-1)b}{k})^\alpha - U(t - \frac{(n)b}{k}) \frac{1}{(\alpha+1)!} (t - \frac{(n)b}{k})^\alpha) = \frac{1}{\Gamma(\alpha)} [0, 0, 0, 0, ..., c_1, c_2 - c_1, c_3 - c_2, ..., c_{k-n+1} - c_{k-n}] \phi_k(t).$$

Now, we have:

$$c_r - c_{r-1} = \frac{(\frac{b}{k})^{\alpha}(r^{\alpha+1} - 2(r-1)^{\alpha+1} + (r-2)^{\alpha+1})}{\alpha+1}$$

$$\frac{1}{\Gamma(\alpha)} t^{\alpha-1} f(t) \simeq (\frac{b}{k})^{\alpha} \frac{1}{\Gamma(\alpha+1)} [0, 0, 0, ..., \xi_1, \xi_2, ..., \xi_{k-n+1}]\phi_k(t) \tag{2.1.7}$$

$$\xi_1 = 1$$
$$\xi_p = p^{\alpha+1} - 2(p-1)^{\alpha+1} + (p-2)^{\alpha+1}$$
$$p = 2, 3, 4, ..., k - n + 1.$$

This can be written as:-

$$\frac{1}{\Gamma(\alpha)} t^{\alpha-1} f(t) = \mathbf{F}_{\alpha}\phi_k(t),$$

$$\mathbf{F}^{\alpha} = (\frac{b}{k})^{\alpha} \frac{1}{\Gamma(\alpha+1)} \begin{bmatrix} \xi_1 & \xi_2 & \xi_3 & \cdots & \xi_k \\ 0 & \xi_1 & \xi_2 & \cdots & \xi_{k-1} \\ 0 & 0 & \xi_1 & \cdots & \xi_{k-2} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \xi_1 \end{bmatrix} \tag{2.1.8}$$

The r-th integral of block-pulse function can be written as:

$$(\mathcal{I}_x^r \mathbf{B})(x) = \mathbf{F}^r_{\hat{m}\times\hat{m}}\ \mathbf{B}(x), \tag{2.1.9}$$

where r is integer and $\mathbf{F}^r_{\hat{m}\times\hat{m}}$ is derived in 2.1.8, so we have:

$$\mathbf{P}^r_{\hat{m}\times\hat{m}} = \mathbf{\Psi}_{\hat{m}\times\hat{m}} \mathbf{F}^r (\mathbf{\Psi}_{\hat{m}\times\hat{m}})^{-1}. \tag{2.1.10}$$

The CAS wavelets operational matrix of r-th order integration

The CAS wavelets operational matrix of integration $\mathbf{P}^r_{\hat{m}\times\hat{m}}$ of integer order r are utilized for solving differential equations.

In particular, for $k = 2$, $M = 1$, $r = 2$, the CAS wavelet operational matrix of integration $\mathbf{P}^2_{12\times12}$ is given by

$$\begin{pmatrix}
-0.000847 & -0.0074 & -0.0209 & -0.0241 & -0.0241 & -0.0241 & -0.0241 & -0.0241 & -0.0241 & -0.0241 & -0.0241 & -0.0241 \\
0 & 0 & 0 & -0.000847 & -0.0074 & -0.0209 & -0.0241 & -0.0241 & -0.0241 & -0.0241 & -0.0241 & -0.0241 \\
0 & 0 & 0 & 0 & 0 & 0 & -0.000847 & -0.0074 & -0.0209 & -0.0241 & -0.0241 & -0.0241 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -0.000847 & -0.0074 & -0.0209 \\
0.00231 & 0.0162 & 0.044 & 0.0833 & 0.125 & 0.167 & 0.208 & 0.25 & 0.292 & 0.333 & 0.375 & 0.417 \\
0 & 0 & 0 & 0.00231 & 0.0162 & 0.044 & 0.0833 & 0.125 & 0.167 & 0.208 & 0.25 & 0.292 \\
0 & 0 & 0 & 0 & 0 & 0 & 0.00231 & 0.0162 & 0.044 & 0.0833 & 0.125 & 0.167 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.00231 & 0.0162 & 0.044 \\
0.00316 & 0.0167 & 0.0232 & 0.0241 & 0.0241 & 0.0241 & 0.0241 & 0.0241 & 0.0241 & 0.0241 & 0.0241 & 0.0241 \\
0 & 0 & 0 & 0.00316 & 0.0167 & 0.0232 & 0.0241 & 0.0241 & 0.0241 & 0.0241 & 0.0241 & 0.0241 \\
0 & 0 & 0 & 0 & 0 & 0 & 0.00316 & 0.0167 & 0.0232 & 0.0241 & 0.0241 & 0.0241 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.00316 & 0.0167 & 0.0232
\end{pmatrix}$$

This phenomena makes calculations fast because the operational matrices $\mathbf{\Psi}_{\hat{m}\times\hat{m}}$ and $\mathbf{P}^r_{\hat{m}\times\hat{m}}$ contain many zero entries.

The CAS wavelets operational matrix of r-th order integration for boundary value problems

We need another operational matrix of integration[13] while solving boundary value problems. Here, we drive an operational matrix of integration for dealing with the boundary conditions while solving boundary value problem. Let $g(x) \in \ L_2[0,1]$ be a given function, then

$$g(x)I^r_{x=1}\psi_{n,m}(x) \ \ = \tfrac{g(x)}{\Gamma(r)}\int\limits_0^1 (1-s)^{r-1}\psi_{n,m}(s)ds. \tag{2.1.11}$$

Since the CAS wavelets are supported on the intervals $[\frac{n-1}{2^k}, \frac{n}{2^k})$, therefore

$$\begin{aligned} g(x)I^r_{x=1}\psi_{n,m}(x) &= \tfrac{g(x)2^{\frac{k}{2}}}{\Gamma(r)}\int\limits_{\frac{n-1}{2^k}}^{\frac{n}{2^k}} (1-s)^{r-1}CAS_m(2^k s-n+1)ds, \\ &= g(x)Q^r_{n,m}, \end{aligned} \tag{2.1.12}$$

where $Q^r_{n,m} = \frac{2^{\frac{k}{2}}}{\Gamma(r)}\int\limits_{\frac{n-1}{2^k}}^{\frac{n}{2^k}} (1-s)^{r-1}CAS_m(2^k s-n+1)ds.$

Expand the equation (2.1.12) at the collocation points, $x_j = \frac{2j-1}{2\hat{m}}$, where $j = 1,2,...,\hat{m}$, to obtain

$$\mathbf{W^{g,r}_{\hat{m}\times\hat{m}}} = \mathbf{Q^r_{\hat{m}\times 1}}\mathbf{G_{1\times\hat{m}}}, \tag{2.1.13}$$

where $\mathbf{G_{1\times\hat{m}}} = [g(x_1), g(x_2), ..., g(x_{\hat{m}})]$,

$\mathbf{Q^r_{\hat{m}\times 1}} =$
$[Q^r_{0,-M}, Q^r_{0,-M+1}, \cdots, Q^r_{0,M}, Q^r_{1,-M}, Q^r_{1,-M+1}, \cdots, Q^r_{1,M}, \cdots, Q^r_{2^k-1,-M}, Q^r_{2^k-1,-M+1}, \cdots, Q^r_{2^k-1,M}]^T.$

In particular, for $k=2,\ M=1,\ r=2,$ and $g(x) = x^2\sin(x)$, we have $\mathbf{W^{g,2}_{12\times 12}}$ as

$$\begin{pmatrix} -3.515e-6 & -9.459e-7 & -4.350e-8 & -1.181e-3 & -2.478e-3 & -4.448e-3 & -7.193e-3 & -1.078e-4 & -1.527e-4 & -2.065e-4 \\ -3.515e-6 & -9.459e-7 & -4.350e-8 & -1.181e-3 & -2.478e-3 & -4.448e-3 & -7.193e-3 & -1.078e-4 & -1.527e-4 & -2.065e-4 \\ -7.030e-6 & -1.891e-8 & -8.700e-8 & -2.363e-3 & -4.956e-3 & -8.897e-3 & -1.438e-4 & -2.157e-4 & -3.054e-4 & -4.130e-4 \\ -7.030e-6 & -1.891e-8 & -8.700e-8 & -2.363e-3 & -4.956e-3 & -8.897e-3 & -1.438e-4 & -2,157e-4 & -3.054e-4 & -4.130e-4 \\ 6.626e-7 & 1.783e-3 & 8.200e-3 & 2.227e-4 & 4.671e-4 & 8.385e-4 & 1.356e-5 & 2.033e-5 & 2.878e-5 & 3.893e-5 \\ 2.208e-7 & 5.943e-8 & 2.733e-3 & 7.425e-3 & 1.557e-3 & 2.795e-3 & 4.520e-4 & 6.778e-4 & 9.595e-4 & 2.977e-4 \\ 7.030e-6 & 1.891e-8 & 8.700e-8 & 2.363e-3 & 4.956e-3 & 8.897e-3 & 1.438e-4 & 2.157e-4 & 3.054e-4 & 4.130e-4 \\ 7.030e-6 & 1.891e-8 & 8.700e-8 & 2.363e-3 & 4.956e-3 & 8.897e-3 & 1.438e-4 & 2.157e-4 & 3.054e-4 & 4.130e-4 \\ 3.515e-6 & 9.459e-7 & 4.350e-8 & 1.181e-3 & 2.478e-3 & 4.448e-3 & 7.193e-3 & 1.078e-4 & 1.527e-4 & 2.065e-4 \\ 3.515e-6 & 9.459e-7 & 4.350e-8 & 1.181e-3 & 2.478e-3 & 4.448e-3 & 7.193e-3 & 1.078e-4 & 1.527e-4 & 2.065e-4 \end{pmatrix}$$

2.2 Error analysis of CAS wavelet

Theorem: Assume that $u(x,t) \to L^2[0,1]$ is a function with bounded partial derivative on [0,1] that is $\exists\ \gamma > 0; \forall\ x \ \to\ [0,1] : |\frac{\partial^4 u}{\partial^2 p \partial^2 q}| \leq \gamma$. The function $u(x,t)$ is expanded as an infinite sum of the CAS wavelets and the series converges uniformly to $u(x,t)$, that is $u(x,t) = \sum\limits_{n=0}^{\infty} \sum\limits_{m \to Z} \sum\limits_{i=0}^{\infty} \sum\limits_{j \to Z} c_{nm,ij} \psi_{n,m}(x) \psi_{i,j}(t)$. Furthermore, $u^{k,k',M,M'}(x,t) = \sum\limits_{n=0}^{2^k-1} \sum\limits_{m=-M}^{M} \sum\limits_{i=0}^{2^{k'}-1} \sum\limits_{j=-M'}^{M'} c_{nm,ij} \psi_{n,m}(x) \psi_{i,j}(t)$, we have

$$|u^{k,k',M,M'} - u(x,t)| \leq |\sum_{n=2^k}^{\infty} \sum_{m=M+1}^{\infty} \sum_{i=2^k}^{\infty} \sum_{j=M'+1}^{\infty} \frac{\gamma}{(mj\pi^2)^2 (n+1)^{\frac{5}{2}} (i+1)^{\frac{5}{2}}}|$$

Proof: Since $u^{k,k',M,M'}(x,t) = \sum\limits_{n=0}^{2^k-1} \sum\limits_{m=-M}^{M} \sum\limits_{i=0}^{2^{k'}-1} \sum\limits_{j=-M'}^{M'} c_{nm,ij} \psi_{n,m}(x) \psi_{i,j}(t)$ and

$$\begin{aligned} c_{nm,ij} &= \int\limits_0^1 \int\limits_0^1 u(x,t) \psi_{n,m}(x) \psi_{i,j}(t) dx dt \\ &= \int\limits_{\frac{n-1}{2^k}}^{\frac{n}{2^k}} \int\limits_{\frac{n-1}{2^k}}^{\frac{n}{2^k}} 2^{\frac{k}{2}} 2^{\frac{k'}{2}} u_{r+1}(x,t) CAS_m(2^k x - n + 1) CAS_j(2^{k'} t - i + 1) dx dt. \end{aligned}$$

Let $2^k x - n + 1 = p$ and $2^{k'} t - i + 1 = q$, then we have:

$$c_{nm,ij} = \int_0^1 \int_0^1 \frac{1}{2^{\frac{k}{2}} 2^{\frac{k'}{2}}} u(\frac{p+n-1}{2^k}, \frac{q+i-1}{2^{k'}}) CAS_m(p) CAS_j(q) dp dq,$$

$$c_{nm,ij} = \int_0^1 \int_0^1 \frac{1}{2^{\frac{k}{2}} 2^{\frac{k'}{2}}} u(\frac{p+n-1}{2^k}, \frac{q+i-1}{2^{k'}})(\cos(2m\pi p) + \sin(2m\pi p)) \\ (\cos(2j\pi q) + \sin(2j\pi q)) dp dq.$$

Use integration w.r.t p to get

$$c_{nm,ij} = -\frac{1}{2m\pi 2^{\frac{3k}{2}} 2^{\frac{k'}{2}}} \int_0^1 \int_0^1 \frac{\partial u}{\partial p}(\frac{p+n-1}{2^k}, \frac{q+i-1}{2^{k'}})(\sin(2m\pi p) - \cos(2m\pi p)) \\ (\cos(2j\pi q) + \sin(2j\pi q)) dp dq.$$

Now applying integration w.r.t q, we obtain:

$$c_{nm,ij}=\frac{1}{(2m\pi)(2j\pi)2^{\frac{3k}{2}}2^{\frac{3k'}{2}}}\int_0^1\int_0^1\frac{\partial^2 u}{\partial p\partial q}(\frac{p+n-1}{2^k},\frac{q+i-1}{2^{k'}})(\sin(2m\pi p)-\cos(2m\pi p))$$
$$(\sin(2j\pi q)-\cos(2j\pi q))dpdq.$$

Again integrating w.r.t p and q, we obtain:

$$c_{nm,ij}=\frac{1}{(2m\pi)^2(2j\pi)^2 2^{\frac{5k}{2}}2^{\frac{5k'}{2}}}\int_0^1\int_0^1\frac{\partial^4 u}{\partial^2 p\partial^2 q}(\frac{p+n-1}{2^k},\frac{q+i-1}{2^{k'}})$$
$$(-\cos(2m\pi p)-\sin(2m\pi p))(-\cos(2j\pi q)-\sin(2j\pi q))dpdq,$$

or

$$|c_{nm,ij}|^2\leq\left|\frac{1}{(2m\pi)^2(2j\pi)^2 2^{\frac{5k}{2}}2^{\frac{5k'}{2}}}\right|^2\int_0^1\int_0^1\left|\frac{\partial^4 u}{\partial^2 p\partial^2 q}(\frac{p+n-1}{2^k},\frac{q+i-1}{2^{k'}})\right|^2$$
$$|(-\cos(2m\pi p)-\sin(2m\pi p))|^2\,|(-\cos(2j\pi q)-\sin(2j\pi q))|^2\,dp^2dq^2.$$

Since $\left|\frac{\partial^4 u}{\partial^2 p\partial^2 q}\right|\leq\gamma$, we have:

$$|c_{nm,ij}|^2\leq(\frac{\gamma}{(2m\pi)^2(2j\pi)^2 2^{\frac{5k}{2}}2^{\frac{5k'}{2}}})^2\int_0^1\int_0^1|(-\cos(2m\pi p)-\sin(2m\pi p))|^2\,dpdq$$
$$\int_0^1\int_0^1|(-\cos(2j\pi q)-\sin(2j\pi q))|^2\,dpdq.$$

By orthogonality of CAS wavelet as $\int\limits_0^1(CAS_m(x)CAS_m(x))dx=1$ so

$$|c_{nm,ij}|\leq\left|\frac{\gamma}{(2m\pi)^2(2j\pi)^2 2^{\frac{5k}{2}}2^{\frac{5k'}{2}}}\right|.$$

By using above Lemma, the series $2^k\geq n+1$ and $2^{k'}\geq i+1$, we get:

$$|c_{nm,ij}|\leq\left|\frac{\gamma}{(2m\pi)^2(2j\pi)^2(n+1)^{\frac{5}{2}}(i+1)^{\frac{5}{2}}}\right|,$$

hence the series $\sum\limits_{n=0}^{\infty}\sum\limits_{m\to Z}\sum\limits_{i=0}^{\infty}\sum\limits_{j\to Z}c_{nm,ij}$ is absolutely convergent. Also, we can obtain:

$$\left|\sum_{n=0}^{2^k-1}\sum_{m=-M}^{M}\sum_{i=0}^{2^{k'}-1}\sum_{j=-M'}^{M'} c_{nm,ij}\psi_{n,m}(x)\psi_{i,j}(t)\right| \leq \sum_{n=0}^{2^k-1}\sum_{m=-M}^{M}\sum_{i=0}^{2^{k'}-1}\sum_{j=-M'}^{M'} |c_{nm,ij}|\,|\psi_{n,m}(x)|\,|\psi_{i,j}(t)|$$

$$\leq 4\sum_{n=0}^{2^k-1}\sum_{m=-M}^{M}\sum_{i=0}^{2^{k'}-1}\sum_{j=-M'}^{M'} |c_{nm,ij}|,$$

as $\sum_{n=0}^{2^k-1}\sum_{m=-M}^{M}\sum_{i=0}^{2^{k'}-1}\sum_{j=-M'}^{M'} c_{nm,ij}\psi_{n,m}(x)\psi_{i,j}(t)$ converges to $u(x,t)$, so we have:

$$|u^{k,k',M,M'} - u(x,t)| \leq 4\left|\sum_{n=2^k}^{\infty}\sum_{m=M+1}^{\infty}\sum_{i=2^{k'}}^{\infty}\sum_{j=M'+1}^{\infty} c_{n,m,i,j}\psi_{n,m}(x)\psi_{i,j}(t)\right|,$$

or

$$|u^{k,k',M,M'} - u(x,t)| \leq \frac{\gamma}{\pi^4}\left|\sum_{n=2^k}^{\infty}\sum_{m=M+1}^{\infty}\sum_{i=2^{k'}}^{\infty}\sum_{j=M'+1}^{\infty} \frac{1}{(mj)^2(n+1)^{\frac{5}{2}}(i+1)^{\frac{5}{2}}}\right|. \quad (2.2.1)$$

Inequality (2.2.1) exhibits that the absolute error is inversely proportional to k, k', M and M' which implies that $u^{k,k',M,M'}(x,t)$ converges to $u(x,t)$ as $k, k', M, M' \longrightarrow \infty$. This suggest that solution by CAS wavelet technique $u^{k,k',M,M'}(x,t)$ converges to $u(x,t)$ when k,k', M, M' and $\longrightarrow \infty$.

2.3 Applications

In this section, solution of the diffusion equation, the wave equation and linear Klein-Gordon equation is considered. The results by CAS wavelet technique are compared with exact solution and results obtained by Adomian decomposition method and the variational iteration method.

Diffusion equation

Consider the diffusion equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \quad t > 0, \quad 0 < x < 1, \quad (2.3.1)$$

subject to initial and boundary conditions

$$u(x,0) = sinx, \quad u(0,t) = 0, \quad u(1,t) = e^{-t}\sin(1),$$

and the exact solution is $u(x,t) = e^{-t}\sin(x)$.
Apply CAS wavelet technique to equation (2.3.1), we have:

$$\frac{\partial^2 u}{\partial x^2} = \sum_{n=0}^{2^k-1}\sum_{m=-M}^{M}\sum_{i=0}^{2^{k'}-1}\sum_{j=-M'}^{M'} c_{nm,ij}\psi_{n,m}(x)\psi_{i,j}(t) = \Psi^T(x)C\Psi(t). \tag{2.3.2}$$

By integrating and putting initial and boundary conditions, it implies:

$$u_{r+1}(x,t) = I_x^2\Psi^T(x)C^{r+1}\Psi(t) - xI_{x=1}^2\Psi^T(x)C^{r+1}\Psi(t) + xe^{-t}\sin(1). \tag{2.3.3}$$

By substituting the equation (2.3.2) in (2.3.1), we obtain:

$$\frac{\partial u}{\partial t} = \Psi^T(x)C^{r+1}\Psi(t).$$

Applying first order integration, we have:

$$u(x,t) = \Psi^T(x)C^{r+1}I_t^1\Psi(t) + \sin(x). \tag{2.3.4}$$

By comparing equations (2.3.3) and (2.3.4), we get:

$$I_x^2\Psi^T(x)C^{r+1}\Psi(t) - xI_{x=1}^2\Psi^T(x)C^{r+1}\Psi(t) + xe^{-t}\sin(1) = \Psi^T(x)C^{r+1}I_t^1\Psi(t) + \sin(x),$$

for simplification, let $K(x,t) = \sin(x) - xe^{-t}\sin(1)$, we obtain:

$$(I_x^2\Psi^T(x) - xI_{x=1}^2\Psi^T(x))C^{r+1}\Psi(t) - \Psi^T(x)C^{r+1}I_t^1\Psi(t) = K(x,t). \tag{2.3.5}$$

Put collocation points $x_i = \frac{2i-1}{2\hat{m}}$ and $t_j = \frac{2j-1}{2\hat{m'}}$, where $i = 1,2,3,....,\hat{m}$, $j = 1,2,3,....,\hat{m'}$, $\hat{m} = 2^k(2M+1)$ and $\hat{m'} = 2^{k'}(2M'+1)$.

$$(I_x^2\Psi^T(x_i) - x_iI_{x=1}^2\Psi^T(x_i))C^{r+1} - \Psi^T(x_i)C^{r+1}I_t^1\Psi(t_j)\Psi(t_j)^{-1} = K(x_i,t_j)\Psi(t_j)^{-1},$$

which can be written in matrix form as:-

$$(\mathbf{P}^{2,x}_{\hat{m}\times\hat{m'}} - \mathbf{W}^{x,2}_{\hat{m}\times\hat{m'}})\mathbf{C}^{r+1} - \Psi^T\mathbf{C}^{r+1}\mathbf{P}^{1,t}_{\hat{m}\times\hat{m'}}\Psi^{-1} = \mathbf{K}\Psi^{-1}.$$

Further, written as Sylvester equation $\mathbf{vQC}^{r+1} - \mathbf{C}^{r+1}\mathbf{R} = \mathbf{S}$, where $\mathbf{Q} = \mathbf{P}^{2,x}_{\hat{m}\times\hat{m'}} - \mathbf{W}^{x,2}_{\hat{m}\times\hat{m'}}$, $\mathbf{R} = \mathbf{P}^{1,t}_{\hat{m}\times\hat{m'}}\Psi^{-1}$, $\mathbf{S} = \mathbf{K}\Psi^{-1}$ and $\mathbf{v} = (\Psi^T)^{-1}$.

The matrix $\mathbf{K}$ is defined as

$$\mathbf{K} = \begin{pmatrix} k(x_1,y_1) & k(x_1,y_2) & \cdots & k(x_1,y_n) \\ k(x_2,y_1) & k(x_2,y_2) & \cdots & k(x_2,y_n) \\ \vdots & \vdots & \ddots & \vdots \\ k(x_n,y_1) & k(x_n,y_2) & \cdots & k(x_n,y_n) \end{pmatrix},$$

t	x	$E_{Adomian}$ [14]	E_{VIM} [14]	E_{CAS}
0.2	0.25	1.5860e-005	1.5860e-005	2.3414e-006
	0.5	3.0719e-005	3.0720e-005	5.0764e-006
	0.75	4.3679e-005	4.3680e-005	8.3488e-006
0.4	0.25	2.4412e-004	2.4412e-004	3.1933e-006
	0.5	4.7305e-004	4.7306e-004	6.3764e-006
	0.75	6.7257e-004	6.7258e-004	9.5008e-006
0.6	0.25	1.1904e-003	1.1904e-003	2.6847e-006
	0.5	2.3068e-003	2.3068e-003	5.3214e-006
	0.75	3.2798e-003	3.2798e-003	7.8537e-006

Table 2.1: Comparison of the CAS wavelet solution of diffusion equation when $k = k'$=4 and $M = M'$=3 with solution by Adomian decomposition method and variational iteration method.

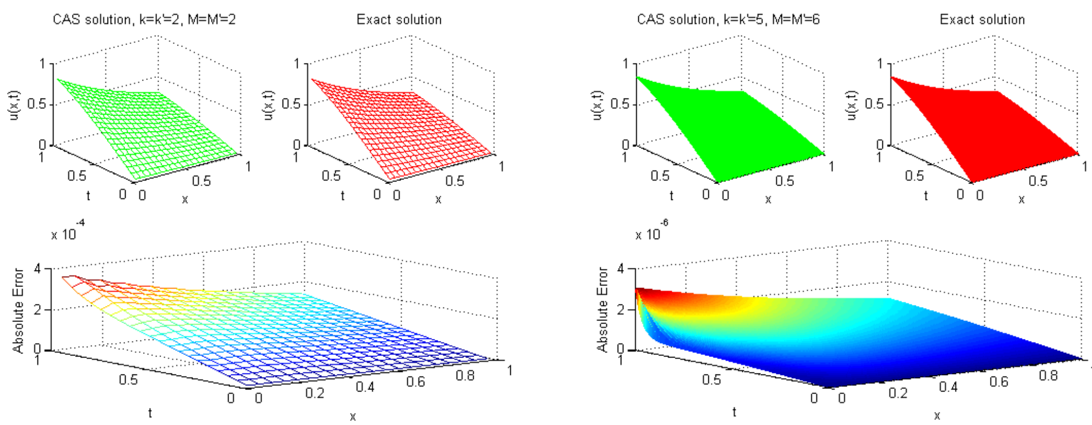

Figure 2.1: Comparison of the numerical results for diffusion equation by present method for different values of k, k', M and M' with the solution by exact solution.

where Ψ^T and Ψ are $\hat{m} \times \hat{m'}$ CAS matrix. From above Sylvester equation, we get the value of $\mathbf{C}^{r+1}$ which is used in equation (2.3.3) to get the solution $u(x,t)$. For different values of x and t, the solution is calculated and shown below:-

In Table 2.1 the solution of the diffusion equation is shown when $k = k' = 4$ and $M = M' = 3$. The results by present method are compared with the solution by ADM and VIM. The Figure 2.1 shows the results of diffusion equation by CAS wavelet technique for different values of k, k', M and M'.

Wave equation

Consider the wave equation

$$\frac{\partial^2 u}{\partial t^2} = \frac{1}{2}x^2\frac{\partial^2 u}{\partial x^2}, \quad t > 0, \quad 0 < x < 1, \tag{2.3.6}$$

subject to initial and boundary conditions

$$u(x,0) = x, \qquad \frac{\partial u}{\partial t}(x,0) = x^2, \qquad u(0,t) = 0, \quad u(1,t) = 1 + \sinh(t),$$

and the exact solution is $u(x,t) = x + x^2\sinh(t)$. Applying CAS wavelet technique to equation (2.3.6), we get:

$$\frac{\partial^2 u}{\partial x^2} = \sum_{n=0}^{2^k-1}\sum_{m=-M}^{M}\sum_{i=0}^{2^{k'}-1}\sum_{j=-M'}^{M'} c_{nm,ij}\psi_{n,m}(x)\psi_{i,j}(t) = \Psi^T(x)C\Psi(t). \tag{2.3.7}$$

By integrating and putting initial and boundary conditions, it implies:

$$u_{r+1}(x,t) = I_x^2\Psi^T(x)C^{r+1}\Psi(t) - xI_{x=1}^2\Psi^T(x)C^{r+1}\Psi(t) + x(1 + \sinh(t)), \tag{2.3.8}$$

after putting the values from equation (2.3.8) in equation (2.3.6), we obtain:

$$\frac{\partial^2 u}{\partial t^2} = \frac{1}{2}x^2\Psi^T(x)C^{r+1}\Psi(t),$$

by integration and putting initial condition we have:

$$u(x,t) = \frac{1}{2}x^2\Psi^T(x)C^{r+1}I_t^2\Psi(t) + x^2t + x. \tag{2.3.9}$$

By comparing equations (2.3.8) and (2.3.9), we obtain:

$$I_x^2\Psi^T(x)C^{r+1}\Psi(t) - xI_{x=1}^2\Psi^T(x)C^{r+1}\Psi(t) + x(1+\sinh(t)) = \frac{1}{2}x^2\Psi^T(x)C^{r+1}I_t^2\Psi(t) + x^2t + x,$$

for simplification, let $K(x,t) = x^2t + x - x(1 + \sinh(t))$ then we get:

$$(I_x^2\Psi^T(x) - xI_{x=1}^2\Psi^T(x))C^{r+1}\Psi(t) - \frac{1}{2}x^2\Psi^T(x)C^{r+1}I_t^2\Psi(t) = K(x,t), \tag{2.3.10}$$

put collocation points $x_i = \frac{2i-1}{2\hat{m}}$ and $t_j = \frac{2j-1}{2\hat{m'}}$, where $i = 1, 2, 3, ..., \hat{m}$, $j = 1, 2, 3, ..., \hat{m'}$, $\hat{m} = 2^k(2M+1)$ and $\hat{m'} = 2^{k'}(2M'+1)$.

$$(I_x^2\Psi^T(x_i) - x_i I_{x=1}^2\Psi^T(x_i))C^{r+1} - \frac{1}{2}x_i^2\Psi^T(x_i)C^{r+1}I_t^2\Psi(t_j)\Psi(t_j)^{-1} = K(x_i, t_j)\Psi(t_j)^{-1},$$

which can be written in matrix form as:-

$$(\mathbf{P}^{2,x}_{\hat{m}\times\hat{m'}} - \mathbf{W}^{x,2}_{\hat{m}\times\hat{m'}})\mathbf{C}^{r+1} - \mathbf{D}\Psi^T\mathbf{C}^{r+1}\mathbf{P}^{2,t}_{\hat{m}\times\hat{m'}}\Psi^{-1} = \mathbf{K}\Psi^{-1}.$$

Further, it can be written as Sylvester equation

$$\mathbf{vQC}^{r+1} - \mathbf{C}^{r+1}\mathbf{R} = \mathbf{S}, \tag{2.3.11}$$

where
$\mathbf{Q} = \mathbf{P}^{2,x}_{\hat{m}\times\hat{m'}} - \mathbf{W}^{x,2}_{\hat{m}\times\hat{m'}}$, $\mathbf{R} = \mathbf{DP}^{2,t}_{\hat{m}\times\hat{m'}}\Psi^{-1}$, $\mathbf{S} = \mathbf{K}\Psi^{-1}$ and $\mathbf{v} = (\Psi^T)^{-1}$.

The diagonal matrix $\mathbf{D}$ is defined as

$$\mathbf{D} = \begin{pmatrix} \frac{1}{2}x_1^2 & 0 & \cdots & 0 \\ 0 & \frac{1}{2}x_2^2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \frac{1}{2}x_n^2 \end{pmatrix}.$$

The equation 2.3.11 is Sylvester equation from which we get the value of $\mathbf{C}^{r+1}$ which is used in equation (2.3.6) to get the solution $u(x,t)$. For different values of x and t the solution is calculated and shown below.

t	x	u_{Exact}	u_{CAS}	$Error$
0.2	0.25	0.26258350	0.26258521	1.71844e-006
	0.5	0.55033400	0.55033599	1.99893e-006
	0.75	0.86325150	0.86325396	2.46640e-006
0.4	0.25	0.27567202	0.27567547	3.45766e-006
	0.5	0.60268808	0.60269196	3.88536e-006
	0.75	0.98104818	0.98105276	4.58369e-006
0.6	0.25	0.28979084	0.28979621	5.36312e-006
	0.5	0.65916339	0.65916943	6.03757e-006
	0.75	1.10811763	1.10812598	8.34081e-006

Table 2.2: Comparison of exact solution of wave equation with the CAS wavelet solution when $k = k'$=4 and $M = M'$=5.

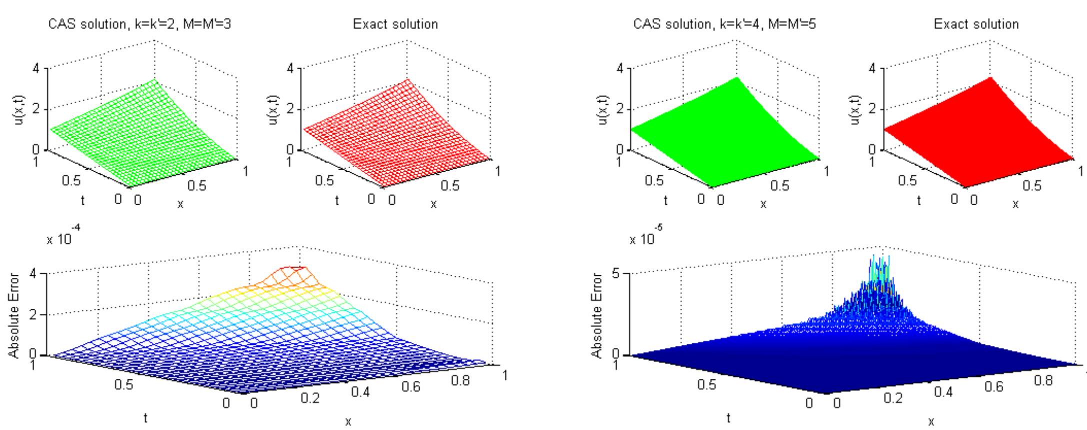

Figure 2.2: The solution by present method for different values of k, k', M and M' is given with the exact solution of the wave equation.

Table 2.2 present the exact and numerical solution calculated by presented scheme. The error is also provided when $k = k' = 4$ and $M = M' = 5$. Figure 2.2 shows compared results of wave equation for different values of k, k', M and M'.

Klein-Gordon equation

Consider the Klein-Gordon equation

$$\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} = u(x,t), \quad t > 0, \quad 0 < x < 1, \tag{2.3.12}$$

subject to initial and boundary conditions

$$u(x,0) = 1+\sin(x), \quad \frac{\partial u}{\partial t}(x,0) = 0, \quad u(0,t) = \cosh(t), \quad u(1,t) = \sin(1)+\cosh(t),$$

and the exact solution is $u(x,t) = \sin(x) + \cosh(t)$. Apply CAS wavelet technique to equation (2.3.12) by supposition as:-

$$\frac{\partial^2 u}{\partial x^2} = \sum_{n=0}^{2^k-1} \sum_{m=-M}^{M} \sum_{i=0}^{2^{k'}-1} \sum_{j=-M'}^{M'} c_{nm,ij}\psi_{n,m}(x)\psi_{i,j}(t) = \Psi^T(x)C\Psi(t). \tag{2.3.13}$$

Integrating above and putting initial and boundary conditions, we get:

$$u_{r+1}(x,t) = I_x^2\Psi^T(x)C^{r+1}\Psi(t) - xI_{x=1}^2\Psi^T(x)C^{r+1}\Psi(t) + x\sin(1) + \cosh(t), \tag{2.3.14}$$

by putting the values from equation (2.3.13) and (2.3.14) in (2.3.12), we obtain:

$$\frac{\partial^2 u}{\partial t^2} - \Psi^T(x)C^{r+1}\Psi(t) = I_x^2\Psi^T(x)C^{r+1}\Psi(t) - xI_{x=1}^2\Psi^T(x)C^{r+1}\Psi(t) + x\sin(1) + \cosh(t).$$

For simplification assume $G = x\sin(1) + \cosh(t)$ where $G \simeq \Psi^T(x)M\Psi(t)$, integrating and putting initial condition, it yields:

$$\begin{aligned} u(x,t) = \Psi^T(x)MI_t^2\Psi(t) + \Psi^T(x)C^{r+1}I_t^2\Psi(t) + I_x^2\Psi^T(x)C^{r+1}I_t^2\Psi(t) \\ -xI_{x=1}^2\Psi^T(x)C^{r+1}I_t^2\Psi(t), \end{aligned} \tag{2.3.15}$$

by comparing equations (2.3.14) and (2.3.15), we have:

$$\begin{aligned} I_x^2\Psi^T(x)C^{r+1}\Psi(t) - xI_{x=1}^2\Psi^T(x)C^{r+1}\Psi(t) + x\sin(1) + \cosh(t) = \Psi^T(x)MI_t^2\Psi(t) \\ +\Psi^T(x)C^{r+1}I_t^2\Psi(t) + I_x^2\Psi^T(x)C^{r+1}I_t^2\Psi(t) - xI_{x=1}^2\Psi^T(x)C^{r+1}I_t^2\Psi(t) + 1 + \sin(x), \end{aligned}$$

for simplification let $H(x,t) = \Psi^T(x)MI_t^2\Psi(t) + 1 + \sin(x) - x\sin(1) - \cosh(t)$, we get:

$$\begin{aligned} (I_x^2\Psi^T(x) - xI_{x=1}^2\Psi^T(x))C^{r+1}\Psi(t) - (\Psi^T(x) - xI_{x=1}^2\Psi^T(x) \\ +I_x^2\Psi^T(x))C^{r+1}I_t^2\Psi(t) = \Psi^T(x)MI_t^1\Psi(t) + H(x,t). \end{aligned} \tag{2.3.16}$$

Putting the collocation points $x_i = \frac{2i-1}{2\hat{m}}$ and $t_j = \frac{2j-1}{2\hat{m'}}$, where $i = 1, 2, 3, ..., \hat{m}$, $j = 1, 2, 3, ..., \hat{m'}$, $\hat{m} = 2^k(2M + 1)$ and $\hat{m'} = 2^{k'}(2M' + 1)$.

$$(I_x^2\Psi^T(x_i) - x_i I_{x=1}^2\Psi^T(x_i))C^{r+1}\Psi(t_j) - (x_i I_x^1\Psi^T(x_i) + \Psi^T(x_i) - x_i I_{x=1}^2\Psi^T(x_i))$$
$$C^{r+1}I_t^2\Psi(t_j) = \Psi^T(x_i)MI_t^2\Psi(t_j) + H(x_i, t_j).$$

in matrix form can be written as:-

$$(\mathbf{P}^{2,x}_{\hat{m}\times\hat{m'}} - \mathbf{W}^{x,2}_{\hat{m}\times\hat{m'}})\mathbf{C}^{r+1} - (\Psi^T + \mathbf{P}^{2,x}_{\hat{m}\times\hat{m'}} - \mathbf{W}^{x,2}_{\hat{m}\times\hat{m'}})\mathbf{C}^{r+1}\mathbf{P}^{2,t}_{\hat{m}\times\hat{m'}}\Psi^{-1} = (\Psi^T\mathbf{M}\mathbf{P}^{2,t}_{\hat{m}\times\hat{m'}} + \mathbf{H})\Psi^{-1}.$$

Further, it can be written as Sylvester equation

$$\mathbf{vQC}^{r+1} - \mathbf{C}^{r+1}\mathbf{R} = \mathbf{S}, \qquad (2.3.17)$$

where
$\mathbf{Q} = \mathbf{P}^{2,x}_{\hat{m}\times\hat{m'}} - \mathbf{W}^{x,2}_{\hat{m}\times\hat{m'}}$, $\mathbf{R} = \mathbf{P}^{2,t}_{\hat{m}\times\hat{m'}}\Psi^{-1}$, $\mathbf{S} = (\Psi^T\mathbf{M}\mathbf{P}^{2,t}_{\hat{m}\times\hat{m'}} + \mathbf{H})\Psi^{-1}$ and
$\mathbf{v} = (\Psi^T + \mathbf{P}^{2,x}_{\hat{m}\times\hat{m'}} - \mathbf{W}^{x,2}_{\hat{m}\times\hat{m'}})^{-1}$.

From the Sylvester equation 2.3.17, we get the value of $\mathbf{C}^{r+1}$ which is used in equation (2.3.12) to get the solution $u(x, t)$. For different values of x and t the solution is calculated and shown below:

x	t	u_{Exact}	u_{CAS}	$Error$
0.3	0.3	1.34085	1.34085	6.0810e-007
	0.6	1.48098	1.48098	1.7003e-007
	0.9	1.72860	1.72860	2.9121e-006
0.6	0.3	1.60998	1.60998	7.4879e-007
	0.6	1.75011	1.75010	2.3627e-006
	0.9	1.99773	1.99772	4.6793e-006
0.9	0.3	1.82866	1.82866	2.6209e-006
	0.6	1.96879	1.96879	3.9361e-006
	0.9	2.21641	2.21641	4.8696e-006

Table 2.3: The CAS wavelet solution of one-dimensional inhomogeneous equation is compared with exact solution when $k = k'$=5 and $M = M'$=6.

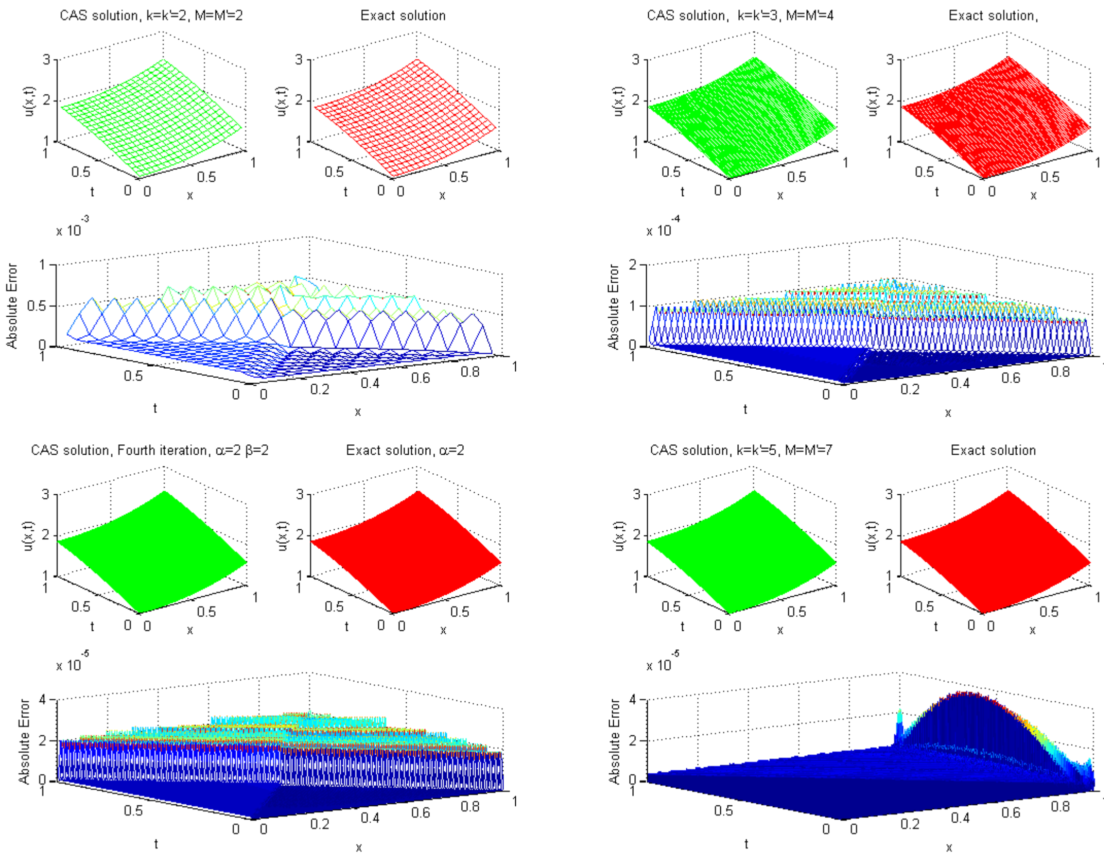

Figure 2.3: The numerical results of linear Klein-Gordian equation by present method for different values of k, k', M and M'.

In Table 2.3 the exact solution and calculated solution by present method are compared and the error is stated. Figure 2.3 shows the compared results graphically for different values of k, k', M and M'.

It clearly shows that CAS wavelet technique gives comparatively good results when applied to different linear partial differential equations. It is observed from Tables 2.1-2.3 and Figures 2.1-2.3 that present scheme give better results than other methods [14]. The numerical solution converges to the exact solution when the values k, k', M and M' are increased.

Chapter 3

Solution of nonlinear partial differential equation by CAS wavelet Picard technique

Nonlinear partial differential equation explains the interaction among convection, diffusion and reaction. Seeking the exact or numerical solution of nonlinear PDE or ODE, various powerful methods have been implemented to get the solution. The Burgers-Huxley equation is utilized to model the interaction between reaction mechanisms and convection effects. Wang [15],[16] used Burger-Huxley equation for nerve pulse propagation in nerve fibers and wall motion liquid crystals. In equation (3.0.1) when $\beta = 0$ and $\delta = 1$, we have Burger equation. Y.C. Hon [17] explained Burger equation by using multiquadric radial basis function. Many famous methods are used to get the solution of the Burgers-Huxley equation like Adomian decomposition method [18], Solitary wave solutions [19], the singular manifold method [20], the homotopy analysis method [21] and spectral collocation method [22]. While variational iteration method [23], explicit and exact-explicit finite difference method [24] were adopted to get the solution of Burgers equation.
The generalized Burgers-Huxley equation is given by [19]

$$\frac{\partial u}{\partial t} + \alpha u^{\delta}\frac{\partial u}{\partial x} - \frac{\partial^2 u}{\partial x^2} = \beta u(1-u^{\delta})(u^{\delta}-\gamma), \qquad \forall\ 0 \le x \le 1,\ t \ge 0, \tag{3.0.1}$$

with initial and boundary conditions

$$u(x,0) := H(x) = (\frac{\gamma}{2} + \frac{\gamma}{2}tanh(A_1 x))^{\frac{1}{\delta}} \tag{3.0.2}$$

$$u(0,t) := I(t) = (\frac{\gamma}{2} + \frac{\gamma}{2}tanh(A_1(-A_2 t)))^{\frac{1}{\delta}} \tag{3.0.3}$$

$$u(1,t) := J(t) = (\frac{\gamma}{2} + \frac{\gamma}{2}tanh(A_1(1-A_2 t)))^{\frac{1}{\delta}}. \tag{3.0.4}$$

The exact solution of the equation is

$$u(x,t) = (\frac{\gamma}{2} + \frac{\gamma}{2}tanh(A_1(x - A_2t)))^{\frac{1}{\delta}},$$

where

$$A_1 = -\frac{\alpha\delta + \delta\sqrt{\alpha^2 + 4\beta(1+\delta)}}{4(1+\delta)}\gamma,$$

$$A_2 = \frac{\gamma\alpha}{(1+\delta)} - \frac{(1+\delta-\gamma)(-\alpha + \sqrt{\alpha^2 + 4\beta(1+\delta)})}{2(1+\delta)},$$

$\alpha, \beta \geq\ 0$ are constants, $\gamma \to [0,1]$ and δ is any positive integer.

In CAS wavelet Picard method, nonlinear partial differential equation is converted into series of linear partial differential equations.

In this chapter, solution of generalized Burger's-Huxley equation is proposed through a numerical method. The nonlinear PDEs are discretized by Picard technique and then CAS wavelet approximation is applied to get the solution. The implementation procedure is provided. Error analysis is done and the results of this method are compared with results from some well-known methods, which support the accuracy and validity of this scheme.

3.1 Picard Technique

Consider non-linear second order partial differential equation [25]

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + g(u, \frac{\partial u}{\partial x}), \qquad (3.1.1)$$

$$t \geq 0, \quad 0 < x < 1, \quad r > 0,$$

with the initial and boundary conditions

$$u(x,0) = g_1(x), \quad u(0,t) = h_1(t), \quad u(1,t) = h_2(t),$$

where $g(u, u_x)$ is nonlinear function. By applying Picard approach for equation (3.1.1), we obtain:

$$\frac{\partial u_{r+1}}{\partial t} = \frac{\partial^2 u_{r+1}}{\partial x^2} + g(u_r, \frac{\partial u_r}{\partial x}), \qquad (3.1.2)$$

with the boundary conditions

$$u_{r+1}(x,0) = g_1(x), \quad u_{r+1}(0,t) = h_1(t), \quad u_{r+1}(1,t) = h_2(t),$$

Above equation (3.1.2) is always a linear partial differential equation which can be solved recursively, by starting with an initial approximation $u_0(x,t)$ in (3.1.2).

3.2 Method of implementation

In this section, the procedure implemented for method is explained. By using Picard technique the partial differential equation is discretized. Then discretized partial differential equation is solved by CAS wavelet operational matrix method. Consider the following discretized nonlinear partial differential equation

$$\frac{\partial u_{r+1}}{\partial t} + a(x)u_r^{\delta}\frac{\partial u_r}{\partial x} + c(x)\frac{\partial^2 u_{r+1}}{\partial x^2} + d(x)u_{r+1} = b(x,t),\ r > 0, \tag{3.2.1}$$

with initial and boundary conditions as:

$$u_{r+1}(x,0) = g_1(x), \quad \frac{\partial u_{r+1}}{\partial t}(x,0) = g_2(x), \quad u_{r+1}(0,t) = h_1(t), \quad u_{r+1}(1,t) = h_2(t).$$

Approximate the highest order term w.r.t x by CAS wavelet Picard method as:-

$$\frac{\partial^2 u_{r+1}}{\partial x^2} = \sum_{n=0}^{2^k-1}\sum_{m=-M}^{M}\sum_{i=0}^{2^{k'}-1}\sum_{j=-M'}^{M'} c_{nm,ij}{}^{r+1}\psi_{n,m}(x)\psi_{i,j}(t) = \Psi^T(x)C^{r+1}\Psi(t),$$

now apply the integral operator on above equation, we have:

$$u_{r+1}(x,t) = (I_x^2\Psi^T(x))C^{r+1}\Psi(t) + p(t)x + q(t),$$

where $p(t)$ and $q(t)$ are

$$p(t) = h_2(t) - h_1(t) - (I_{x=1}^2\Psi^T(x))C^{r+1}\Psi(t)$$
$$q(t) = h_1(t).$$

By putting the values of $p(t)$ and $q(t)$ in $u(x,t)$, we get:

$$u_{r+1}(x,t) = (I_x^2\Psi^T(x))C^{r+1}\Psi(t) + (h_2(t) - h_1(t))x - (I_{x=1}^2\Psi^T(x))C^{r+1}\Psi(t)x + h_1(t), \tag{3.2.2}$$

now put all the values in equation (3.2.1) and have:

$$\frac{\partial u_{r+1}}{\partial t} + a(x)u_r^{\delta}\frac{\partial u_r}{\partial x} + c(x)\Psi^T(x)C^{r+1}\Psi(t) + d(x)I_x^2\Psi^T(x)C^{r+1}\Psi(t)$$
$$+d(x)(h_2(t) - h_1(t))x - d(x)xI_{x=1}^2\Psi^T(x)C^{r+1}\Psi(t) + d(x)h_1(t) = b(x,t).$$

We make some substitution $G = b(x,t) - a(x)u_r^{\delta}\frac{\partial u_r}{\partial x} - d(x)(h_2(t)+h_1(t))x - d(x)h_1(t)$ and $G \simeq \Psi^T(x)M\Psi(t)$, for simplification and get:

$$\frac{\partial u_{r+1}}{\partial t} + c(x)\Psi^T(x)C^{r+1}\Psi(t) + d(x)(I_x^2\Psi^T(x))C^{r+1}\Psi(t)$$
$$-d(x)xI_{x=1}^2\Psi^T(x)C^{r+1}\Psi(t) = \Psi^T(x)M\Psi(t).$$

Apply integration on above equation to get:

$$\begin{aligned} u_{r+1}(x,t) = &-c(x)\Psi^T(x)C^{r+1}I_t\Psi(t) - d(x)I_x^2\Psi^T(x)C^{r+1}I_t\Psi(t) \\ &+d(x)xI_{x=1}^2\Psi^T(x)C^{r+1}I_t\Psi(t) + \Psi^T(x)MI_t\Psi(t) + g_1(x). \end{aligned} \tag{3.2.3}$$

Now by equations (3.2.2) and (3.2.3) and simplification it is,

$$\begin{gathered} (I_x^2\Psi^T(x) - xI_{x=1}^2\Psi^T(x))C^{r+1}\Psi(t) + (c(x)\Psi^T(x) + d(x)I_x^2\Psi^T(x) - \\ d(x)xI_{x=1}^2\Psi^T(x))C^{r+1}I_t\Psi(t) = \Psi^T(x)MI_t\Psi(t) - (h_2(t) - h_1(t))x - h_1(t) + g_1(x). \end{gathered}$$

For simplification, let $K(x,t) = (h_2(t) - h_1(t))x + h_1(t) - g_1(x)$ above equation at collocation points $x_i = \frac{2i-1}{2\hat{m}}$ and $t_j = \frac{2j-1}{2\hat{m'}}$ is, where $i = 1, 2, 3, ..., \hat{m}$, $j = 1, 2, 3, ..., \hat{m'}$, $\hat{m} = 2^k(2M+1)$ and $\hat{m'} = 2^{k'}(2M'+1)$.

$$\begin{gathered} (I_x^2\Psi^T(x_i) - x_iI_{x=1}^2\Psi^T(x_i))C^{r+1}\Psi(t_j) + (c(x_i)\Psi^T(x_i) + d(x_i)I_x^2\Psi^T(x_i) - \\ d(x_i)x_iI_{x=1}^2\Psi^T(x_i))C^{r+1}I_t\Psi(t_j) = \Psi^T(x_i)MI_t\Psi(t_j) - K(x_i,t_j), \end{gathered}$$

which can be written in matrix form as:-

$$(\mathbf{P}^{2,x}_{\hat{m}\times\hat{m'}} - \mathbf{W}^{x,2}_{\hat{m}\times\hat{m'}})\mathbf{C}^{r+1}\Psi + (\mathbf{E}\Psi^T + \mathbf{D}\mathbf{P}^{2,x}_{\hat{m}\times\hat{m'}} - \mathbf{D}\mathbf{W}^{x,2}_{\hat{m}\times\hat{m'}})\mathbf{C}^{r+1}\mathbf{P}^{1,t}_{\hat{m}\times\hat{m'}} = \Psi^T\mathbf{M}\Psi - \mathbf{K},$$

where $\mathbf{E}$ and $\mathbf{D}$ are diagonal matrices given by

$$\mathbf{E} = \begin{pmatrix} c(x_1) & 0 & \cdots & 0 \\ 0 & c(x_2) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & c(x_j) \end{pmatrix} \quad \text{and} \quad \mathbf{D} = \begin{pmatrix} d(x_1) & 0 & \cdots & 0 \\ 0 & d(x_2) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & d(x_j) \end{pmatrix}.$$

The matrix $\mathbf{K}$ is defined as:-

$$\mathbf{K} = \begin{pmatrix} k(x_1,y_1) & k(x_1,y_2) & \cdots & k(x_1,y_n) \\ k(x_2,y_1) & k(x_2,y_2) & \cdots & k(x_2,y_n) \\ \vdots & \vdots & \ddots & \vdots \\ k(x_n,y_1) & k(x_n,y_2) & \cdots & k(x_n,y_n) \end{pmatrix}.$$

let $\mathbf{v} = (\mathbf{E}\Psi^T + \mathbf{D}\mathbf{P}_x^2 - \mathbf{D}\mathbf{W}^{x,2}_{\hat{m}\times\hat{m'}})^{-1}$.
Then, we get:

$$\begin{gathered} \mathbf{v}(\mathbf{P}^{2,x}_{\hat{m}\times\hat{m'}} - \mathbf{W}^{x,2}_{\hat{m}\times\hat{m'}})\mathbf{C}^{r+1}\Psi + (\mathbf{E}\Psi^T + \mathbf{D}\mathbf{P}^{2,x}_{\hat{m}\times\hat{m'}} - \mathbf{D}\mathbf{W}^{x,1}_{\hat{m}\times\hat{m'}})\mathbf{C}^{r+1}\mathbf{P}^{1,t}_{\hat{m}\times\hat{m'}} = \Psi^T\mathbf{M}\Psi - \mathbf{K} \\ \mathbf{v}(\mathbf{P}^{2,x}_{\hat{m}\times\hat{m'}} - \mathbf{W}^{x,2}_{\hat{m}\times\hat{m'}})\mathbf{C}^{r+1} - \mathbf{C}^{r+1}\mathbf{P}^{1,t}_{\hat{m}\times\hat{m'}}\Psi^{-1} = \mathbf{v}(\Psi^T\mathbf{M}\mathbf{P}^{1,t}_{\hat{m}\times\hat{m'}} - \mathbf{K})\Psi^{-1}, \end{gathered}$$

suppose that $\mathbf{Q} = \mathbf{v}(\mathbf{P}^{2,x}_{\hat{m}\times\hat{m'}} - \mathbf{W}^{x,2}_{\hat{m}\times\hat{m'}})$, $\mathbf{R} = \mathbf{P}^{1,t}_{\hat{m}\times\hat{m'}}\Psi^{-1}$ and $\mathbf{S} = \mathbf{v}(\Psi^T\mathbf{M}\mathbf{P}^{1,t}_{\hat{m}\times\hat{m'}} - \mathbf{K})\Psi^{-1}$.

Where Ψ^T and Ψ are $\hat{m}\times\hat{m'}$ CAS matrix. Then we have $\mathbf{Q}(\mathbf{C}^{r+1})-(\mathbf{C}^{r+1})\mathbf{R}=\mathbf{S}$, which is Sylvester equation and from Sylvester equation we get $\mathbf{C}^{r+1}$ which is used in equation (3.2.2) to get the solution $u_{r+1}(x,t)$.

The approximate solution $u_{r+1}(x,t)$ will converge the exact solution when r$\to\infty$.

3.3 Error analysis of CAS wavelet Picard technique

Lemma: If the CAS wavelets expansion of a continuous function $u_{r+1}(x,t)$ converges uniformly, then the CAS wavelets expansion converges to the function $u_{r+1}(x,t)$.
Proof: Let

$$v_{r+1}(x,t)=\sum_{i=0}^{\infty}\sum_{j\to Z}\sum_{m=0}^{\infty}\sum_{n\to Z}c_{nm,ij}^{r+1}\psi_{n,m}(x)\psi_{i,j}(t) \qquad (3.3.1)$$

Multiply both sides of equation (3.3.1) by $\psi_{p,q}(t)$ and $\psi_{r,s}(x)$, then integrating from 0 to 1 with respect to x as well as t, we obtain (3.3.2) by using orthonormality of CAS wavelet

$$\int_0^1\int_0^1 v_{r+1}(x,t)\psi_{p,q}(t)\psi_{r,s}(x)dtdx=c_{pq,rs}^{r+1}. \qquad (3.3.2)$$

Thus $c_{pq,rs}^{r+1}=\langle\langle v_{r+1}(x,t),\psi_{p,q}(x)\rangle,\psi_{r,s}(t)\rangle$ for $p,r\ \to\ N,\ q,s\ \to\ Z$. This implies that $u_{r+1}(x,t)=v_{r+1}(x,t)$.
Theorem: Assume that $u_{r+1}(x,t)\to L^2[0,1]$ is a function with bounded partial derivative on [0,1] that is $\exists\,\gamma>0;\forall\,x\ \to\ [0,1]:|\frac{\partial^4 u_{r+1}}{\partial^2 x\partial^2 t}|\leq\gamma$. The function $u_{r+1}(x,t)$ is expanded as an infinite sum of the CAS wavelets and the series converges uniformly to $u_{r+1}(x,t)$, that is $u_{r+1}(x,t)=\sum_{n=0}^{\infty}\sum_{m\to Z}\sum_{i=0}^{\infty}\sum_{j\to Z}c_{nm,ij}^{r+1}\psi_{n,m}(x)\psi_{i,j}(t)$. Furthermore, $u_{r+1}^{k,k',M,M'}(x,t)=\sum_{n=0}^{2^k-1}\sum_{m=-M}^{M}\sum_{i=0}^{2^{k'}-1}\sum_{j=-M'}^{M'}c_{nm,ij}^{r+1}\psi_{n,m}(x)\psi_{i,j}(t)$, we have:

$$|u_{r+1}^{k,k',M,M'}-u_{r+1}(x,t)|\leq\frac{\gamma}{\pi^4}\left|\sum_{n=2^k}^{\infty}\sum_{m=M+1}^{\infty}\sum_{i=2^{k'}}^{\infty}\sum_{j=M'+1}^{\infty}\frac{1}{(mj)^2(n+1)^{\frac{5}{2}}(i+1)^{\frac{5}{2}}}\right|,$$

$u_{r+1}^{k,k',M,M'}$ converges to $u_{r+1}(x,t)$ as k,k',M and $M'\to\infty$ and $u_{r+1}(x,t)$ converges to $u(x,t)$ as $r\to\infty$.

Proof: Since $u_{r+1}^{k,k',M,M'}(x,t) = \sum\limits_{n=0}^{2^k-1}\sum\limits_{m=-M}^{M}\sum\limits_{i=0}^{2^{k'}-1}\sum\limits_{j=-M'}^{M'} c_{nm,ij}^{r+1}\psi_{n,m}(x)\psi_{i,j}(t)$ and

$$\begin{aligned} c_{nm,ij}^{r+1} &= \int_0^1\int_0^1 u_{r+1}(x,t)\psi_{n,m}(x)\psi_{i,j}(t)dxdt \\ &= \int_{\frac{n-1}{2^k}}^{\frac{n}{2^k}}\int_{\frac{n-1}{2^k}}^{\frac{n}{2^k}} 2^{\frac{k}{2}}2^{\frac{k'}{2}}u_{r+1}(x,t)CAS_m(2^kx-n+1)CAS_j(2^{k'}t-i+1)dxdt. \end{aligned}$$

Let $2^kx-n+1=p$ and $2^{k'}t-i+1=q$, then we have:

$$c_{nm,ij}^{r+1} = \int_0^1\int_0^1 \frac{1}{2^{\frac{k}{2}}2^{\frac{k'}{2}}}u(\frac{p+n-1}{2^k},\frac{q+i-1}{2^{k'}})CAS_m(p)CAS_j(q)dpdq,$$

$$c_{nm,ij}^{r+1} = \int_0^1\int_0^1 \frac{1}{2^{\frac{k}{2}}2^{\frac{k'}{2}}}u(\frac{p+n-1}{2^k},\frac{q+i-1}{2^{k'}})(\cos(2m\pi p)+\sin(2m\pi p))$$
$$(\cos(2j\pi q)+\sin(2j\pi q))dpdq.$$

Use integration w.r.t p, to get:

$$c_{nm,ij}^{r+1} = -\frac{1}{2m\pi 2^{\frac{3k}{2}}2^{\frac{k'}{2}}}\int_0^1\int_0^1 \frac{\partial u}{\partial p}(\frac{p+n-1}{2^k},\frac{q+i-1}{2^{k'}})(\sin(2m\pi p)-\cos(2m\pi p))$$
$$(\cos(2j\pi q)+\sin(2j\pi q))dpdq.$$

Now applying integration w.r.t q, we obtain:

$$c_{nm,ij}^{r+1} = \frac{1}{(2m\pi)(2j\pi)2^{\frac{3k}{2}}2^{\frac{3k'}{2}}}\int_0^1\int_0^1 \frac{\partial^2 u}{\partial p\partial q}(\frac{p+n-1}{2^k},\frac{q+i-1}{2^{k'}})(\sin(2m\pi p)-\cos(2m\pi p))$$
$$(\sin(2j\pi q)-\cos(2j\pi q))dpdq.$$

Again integrating w.r.t p and q, we obtain:

$$c_{nm,ij}^{r+1} = \frac{1}{(2m\pi)^2(2j\pi)^2 2^{\frac{5k}{2}}2^{\frac{5k'}{2}}}\int_0^1\int_0^1 \frac{\partial^4 u}{\partial^2 p\partial^2 q}(\frac{p+n-1}{2^k},\frac{q+i-1}{2^{k'}})$$
$$(-\cos(2m\pi p)-\sin(2m\pi p))(-\cos(2j\pi q)-\sin(2j\pi q))dpdq,$$

or

$$|c_{nm,ij}^{r+1}|^2 \leq \left|\frac{1}{(2m\pi)^2(2j\pi)^2 2^{\frac{5k}{2}}2^{\frac{5k'}{2}}}\right|^2\int_0^1\int_0^1 \left|\frac{\partial^4 u}{\partial^2 p\partial^2 q}(\frac{p+n-1}{2^k},\frac{q+i-1}{2^{k'}})\right|^2$$
$$|(-\cos(2m\pi p)-\sin(2m\pi p))|^2\,|(-\cos(2j\pi q)-\sin(2j\pi q))|^2\,dp^2dq^2.$$

Since $\left|\frac{\partial^4 u}{\partial^2 p \partial^2 q}\right| \leq \gamma$, we have:

$$|c^{r+1}_{nm,ij}|^2 \leq \left(\frac{\gamma}{(2m\pi)^2(2j\pi)^2 2^{\frac{5k}{2}} 2^{\frac{5k'}{2}}}\right)^2 \int_0^1 \int_0^1 |(-\cos(2m\pi p) - \sin(2m\pi p))|^2 \, dpdq$$
$$\int_0^1 \int_0^1 |(-\cos(2j\pi q) - \sin(2j\pi q))|^2 \, dpdq.$$

By orthogonality of CAS wavelet as $\int\limits_0^1 (CAS_m(x) CAS_m(x))dx = 1$ so:

$$|c^{r+1}_{nm,ij}| \leq \left|\frac{\gamma}{(2m\pi)^2(2j\pi)^2 2^{\frac{5k}{2}} 2^{\frac{5k'}{2}}}\right|.$$

By using above Lemma, the series $2^k \geq n+1$ and $2^{k'} \geq i+1$, we get:

$$|c^{r+1}_{nm,ij}| \leq \left|\frac{\gamma}{(2m\pi)^2(2j\pi)^2 (n+1)^{\frac{5}{2}} (i+1)^{\frac{5}{2}}}\right|.$$

Hence, the series $\sum\limits_{n=0}^{\infty} \sum\limits_{m \to Z} \sum\limits_{i=0}^{\infty} \sum\limits_{j \to Z} c^{r+1}_{nm,ij}$ is absolutely convergent. Also, we can obtain:

$$\left|\sum_{n=0}^{2^k-1} \sum_{m=-M}^{M} \sum_{i=0}^{2^{k'}-1} \sum_{j=-M'}^{M'} c_{nm,ij} \psi_{n,m}(x) \psi_{i,j}(t)\right| \leq \sum_{n=0}^{2^k-1} \sum_{m=-M}^{M} \sum_{i=0}^{2^{k'}-1} \sum_{j=-M'}^{M'} |c_{nm,ij}| \, |\psi_{n,m}(x)| \, |\psi_{i,j}(t)|$$
$$\leq 4 \sum_{n=0}^{2^k-1} \sum_{m=-M}^{M} \sum_{i=0}^{2^{k'}-1} \sum_{j=-M'}^{M'} |c_{nm,ij}|,$$

as $\sum\limits_{n=0}^{2^k-1} \sum\limits_{m=-M}^{M} \sum\limits_{i=0}^{2^{k'}-1} \sum\limits_{j=-M'}^{M'} c_{nm,ij} \psi_{n,m}(x) \psi_{i,j}(t)$ converges to $u_{r+1}(x,t)$, so we have:

$$|u^{k,k',M,M'}_{r+1} - u_{r+1}(x,t)| \leq 4 \left|\sum_{n=2^k}^{\infty} \sum_{m=M+1}^{\infty} \sum_{i=2^{k'}}^{\infty} \sum_{j=M'+1}^{\infty} c_{n,m,i,j} \psi_{n,m}(x) \psi_{i,j}(t)\right|,$$

or

$$|u^{k,k',M,M'}_{r+1} - u_{r+1}(x,t)| \leq \frac{\gamma}{\pi^4} \left|\sum_{n=2^k}^{\infty} \sum_{m=M+1}^{\infty} \sum_{i=2^{k'}}^{\infty} \sum_{j=M'+1}^{\infty} \frac{1}{(mj)^2 (n+1)^{\frac{5}{2}} (i+1)^{\frac{5}{2}}}\right|. \tag{3.3.3}$$

Inequality (3.3.3) exhibits that the absolute error at $(r+1)th$ iteration is inversely proportional to k, k', M and M' which implies that $u_{r+1}^{k,k',M,M'}(x,t)$ converges to $u_{r+1}(x,t)$ as $k, k', M, M' \longrightarrow \infty$. Since $u_{r+1}(x,t)$ is obtained at $(r+1)th$ iteration of Picard technique so according to the convergence analysis of Picard technique [26] which states that $u_{r+1}(x,t)$ converges to $u(x,t)$ as r $\longrightarrow \infty$, if there is convergence at all. This suggest that solution by CAS wavelet Picard technique $u_{r+1}^{k,k',M,M'}(x,t)$ converges to $u(x,t)$ when k,k', M, M' and $r \longrightarrow \infty$.

3.4 Applications

In this section, we implement the CAS wavelet Picard method on some nonlinear partial differential equations like Burger Huxley and Burger's equation. The results are compared with exact solution and results obtained by the other methods.

Burger-Huxley equation

Consider the Generalized Burger's-Huxley equation (3.0.1) and its boundary and initial conditions (3.0.2),(3.0.3) and (3.0.4). Applying CAS wavelet Picard method, we get:

$$\frac{\partial u_{r+1}}{\partial t} - \frac{\partial^2 u_{r+1}}{\partial x^2} + \gamma\beta u_{r+1} + \alpha u_r^\delta \frac{\partial u_r}{\partial x} - \beta u_r^{\delta+1} + \beta u_r^{2\delta+1} - \gamma\beta u_r^{\delta+1} = 0. \tag{3.4.1}$$

Applying CAS wavelet method, we take highest derivative term and make supposition as:- $\frac{\partial^2 u_{r+1}}{\partial x^2} = \Psi^T(x)C\Psi(t)$.
By integration and putting boundary conditions, it implies:

$$u_{r+1}(x,t) = I_x^2\Psi^T(x)C^{r+1}\Psi(t) + xJ(t) - xI(t) - xI_{x=1}^2\Psi^T(x)C^{r+1}\Psi(t) + I(t). \tag{3.4.2}$$

By putting the required values in equation (3.4.1), we get:

$$\frac{\partial u_{r+1}}{\partial t} - \Psi^T(x)C^{r+1}\Psi(t) + \gamma\beta(I_x^2\Psi^T(x)C^{r+1}\Psi(t) - xI_{x=1}^2\Psi^T(x)C^{r+1}\Psi(t)) +$$
$$\gamma\beta(xJ(t) - xI(t) + I(t)) + \alpha u_r^\delta \frac{\partial u_r}{\partial x} - \beta u_r^{\delta+1} + \beta u_r^{2\delta+1} - \gamma\beta u_r^{\delta+1} = 0.$$

Here some substitution is done for the simplification of the above equation as $G = \gamma\beta(-xJ(t) + xI(t) - I(t)) - \alpha u_r^\delta \frac{\partial u_r}{\partial x} + \beta u_r^{\delta+1} - \beta u_r^{2\delta+1} + \gamma\beta u_r^{\delta+1}$ and $G \simeq \Psi^T M \Psi$.
By taking integral and putting initial condition, we obtain:

$$u_{r+1}(x,t) = \Psi^T(x)C^{r+1}I_t\Psi(t) - \gamma\beta(I_x^2\Psi^T(x)C^{r+1}I_t\Psi(t) - xI_{x=1}^2\Psi^T(x)C^{r+1}I_t\Psi(t)) +$$
$$\Psi^T(x)MI_t\Psi(t) + H(x). \tag{3.4.3}$$

Equation (3.4.2) and (3.4.3) are equated and simplified as:-

$$(I_x^2\Psi^T(x) - xI_{x=1}^2\Psi^T(x))C^{r+1}\Psi(t) + xJ(t) - xI(t) + I(t) = (\Psi^T(x) - \gamma\beta I_x^2\Psi^T(x) - \gamma\beta xI_{x=1}^2\Psi^T(x))C^{r+1}I_t\Psi(t) + \Psi^T(x)MI_t\Psi(t) + H(x).$$

$$(I_x^2\Psi^T(x) - xI_{x=1}^2\Psi^T(x))C^{r+1}\Psi(t) = (\Psi^T(x) - \gamma\beta I_x^2\Psi^T(x) - \gamma\beta xI_{x=1}^2\Psi^T(x))C^{r+1}I_t\Psi(t) + \Psi^T(x)MI_t\Psi(t) + H(x) - xJ(t) + xI(t) - I(t).$$

Let $K(x,t) = H(x) - xJ(t) + xI(t) - I(t)$, and by put collocation points $x_i = \frac{2i-1}{2\hat{m}}$ and $t_j = \frac{2j-1}{2\hat{m'}}$ above equation in matrix form is written as, where $i = 1, 2, 3, ..., \hat{m}$, $j = 1, 2, 3, ..., \hat{m'}$, $\hat{m} = 2^k(2M+1)$ and $\hat{m'} = 2^{k'}(2M'+1)$.

$$(\mathbf{P}^{2,x}_{\hat{m}\times\hat{m'}} - \mathbf{W}^{x,2}_{\hat{m}\times\hat{m'}})\mathbf{C}^{r+1}\Psi = (\Psi^T - \gamma\beta\mathbf{P}^{2,t}_{\hat{m}\times\hat{m'}} - \gamma\beta\mathbf{W}^{x,2}_{\hat{m}\times\hat{m'}})\mathbf{C}^{r+1}\mathbf{P}^{1,t}_{\hat{m}\times\hat{m'}} + \Psi^T\mathbf{M}\mathbf{P}^{1,t}_{\hat{m}\times\hat{m'}} + \mathbf{K}.$$

after some more simplification and rearrangement we get Sylvester equation, which is,

$$(\mathbf{P}^{2,x}_{\hat{m}\times\hat{m'}} - \mathbf{W}^{x,2}_{\hat{m}\times\hat{m'}})\mathbf{C}^{r+1}\Psi - (\Psi^T - \gamma\beta\mathbf{P}^{2,t}_{\hat{m}\times\hat{m'}} - \gamma\beta\mathbf{W}^{x,2}_{\hat{m}\times\hat{m'}})\mathbf{C}^{r+1}\mathbf{P}^{1,t}_{\hat{m}\times\hat{m'}} = \Psi^T\mathbf{M}\mathbf{P}^{1,t}_{\hat{m}\times\hat{m'}} + \mathbf{K}$$

$$\mathbf{A}\mathbf{C}^{r+1} - \mathbf{C}^{r+1}\mathbf{B} = \mathbf{D}, \qquad (3.4.4)$$

in the equation 3.4.4, we have assumed $\mathbf{u} = (\Psi^T - \gamma\beta\mathbf{P}^{2,x}_{\hat{m}\times\hat{m'}} - \gamma\beta\mathbf{W}^{x,2}_{\hat{m}\times\hat{m'}})^{-1}$, $\mathbf{A} = \mathbf{u}(\mathbf{P}^{2,x}_{\hat{m}\times\hat{m'}} - \mathbf{W}^{x,2}_{\hat{m}\times\hat{m'}})$, $\mathbf{B} = \mathbf{P}^{1,t}_{\hat{m}\times\hat{m'}}\Psi^{-1}$ and $\mathbf{D} = \mathbf{u}(\Psi^T\mathbf{M}\mathbf{P}^{1,t}_{\hat{m}\times\hat{m'}} + \mathbf{K})\Psi^{-1}$. By solving the above equation we obtain $\mathbf{C}^{r+1}$, which is used to get the solution $u_{r+1}(x,t)$. For different values of x and t the solution is calculated and shown below:

x	t	$E_{Adomian}$ [27]	E_{VIM} [28]	E_{CAS}
0.1	0.05	1.87406e-08	1.87405e-08	1.03034e-008
	0.1	3.74812e-08	3.74813e-08	1.50631e-008
0.5	0.05	1.87406e-08	1.87405e-08	2.31345e-008
	0.1	3.74812e-08	1.37481e-08	3.84361e-008
0.9	0.05	1.87406e-08	1.87405e-08	1.03014e-008
	0.1	3.74812e-08	3.74813e-08	1.50601e-008

Table 3.1: Comparison of the CAS wavelet solution of Burgers-Huxley equation when $k = k'$=5 and $M = M'$=7 at $\alpha = 1$, $\beta = 1$, $\gamma = 0.001$ and $\delta = 1$ with solution by Adomian decomposition method and variational iteration method.

x	t	$E_{Adomian}$ [27]	E_{VIM} [28]	E_{CAS}
0.1	0.1	5.51554e-05	5.51580e-05	4.31053e-005
	0.2	1.10342e-04	1.10310e-04	5.60339e-005
	0.3	1.65529e-04	1.65457e-04	6.05142e-005
	0.4	2.20708e-04	2.20598e-04	6.18613e-005
	0.5	2.75950e-04	2.75734e-04	6.20450e-005
0.3	0.1	5.51381e-05	5.51340e-05	9.44550e-005
	0.2	1.10293e-04	1.10262e-04	1.28422e-004
	0.3	1.65458e-04	1.65385e-04	1.40286e-004
	0.4	2.20635e-04	2.20502e-04	1.43952e-004
	0.5	2.75832e-04	2.75614e-04	1.44575e-004
0.5	0.1	5.51134e-05	5.51099e-05	1.09814e-004
	0.2	1.10243e-04	1.10214e-04	1.51828e-004
	0.3	1.65402e-04	1.65313e-04	1.66537e-004
	0.4	2.20543e-04	2.20406e-04	1.71118e-004
	0.5	2.75716e-04	2.75493e-04	1.71940e-004

Table 3.2: Comparison of the approximate solution by CAS wavelet when $M = M'$=8 and $k = k'$=6 for Burgers-Huxley equation at $\alpha = 1,\ \beta = 1,\ \gamma = 0.01$ and $\delta = 2$ with solution by Adomian decomposition and variational iteration method.

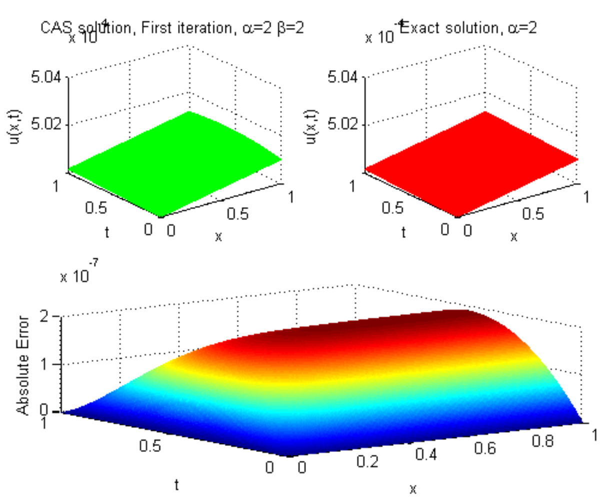

Figure 3.1: Comparison of exact and numerical solution of Burger-Huxley equation by CAS wavelet Picard method when $k = k'$=5, $M = M'$=7 and r= 4.

In the Figure 3.1 the solution of Burger-Huxley equation is presented when $k = k' = 5$, $M = M' = 7$ and $r = 4$.

Burger's equation

In this section, we consider the Burger's equation [29]

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = \frac{\partial^2 u}{\partial x^2} \tag{3.4.5}$$

with the initial and boundary conditions as:-

$$u(x,0) = 2x \qquad u(0,t) = 0 \qquad u(1,t) = \frac{2}{1+2t}$$

where exact solution is

$$u(x,t) = \frac{2x}{1+2t}$$

First we apply Picard technique to (3.4.5), and we get:

$$\frac{\partial u_{r+1}}{\partial t} + u_r\frac{\partial u_r}{\partial x} = \frac{\partial^2 u_{r+1}}{\partial x^2} \tag{3.4.6}$$

with initial and boundary conditions as:-

$$u_{r+1}(x,0) = 2x \qquad u_{r+1}(0,t) = 0 \qquad u_{r+1}(1,t) = \frac{2}{1+2t}.$$

The solution is calculated, where $r = 0, 1, 2, ..., N$. Now, we apply the CAS wavelet Picard technique to discretized equation (3.4.6), as described in section above to get the solution at the collocation points. The calculated results of Burger's equation are shown below:

x	t	u_{EXACT}	u_{CAS}	u_{error}
0.1	0.3	0.1250000	0.1249999	5.3016e-008
	0.6	0.0909090	0.0909090	1.3792e-008
	0.9	0.0714285	0.0714285	3.2702e-009
0.5	0.3	0.6250000	0.6249999	7.2402e-008
	0.6	0.4545454	0.4545454	1.0043e-008
	0.9	0.3571428	0.3571428	2.2690e-008
0.7	0.3	0.8750000	0.8750001	1.8632e-007
	0.6	0.6363636	0.6363637	6.7319e-008
	0.9	0.5000000	0.5000000	4.1963e-008
0.9	0.3	1.1250000	1.1250002	2.6196e-007
	0.6	0.8181818	0.8181819	1.3342e-007
	0.9	0.6428571	0.6428572	6.0216e-008

Table 3.3: Comparison of Approximate solution of Burgers equation by CAS wavelet when $M = M'$=8, $k = k'$=6 and r=8 with exact solution.

Figure 3.2 shows the numerical solution of Burgers equation for four different values of r, when r= 2, 5, 6 and 7 respectively.

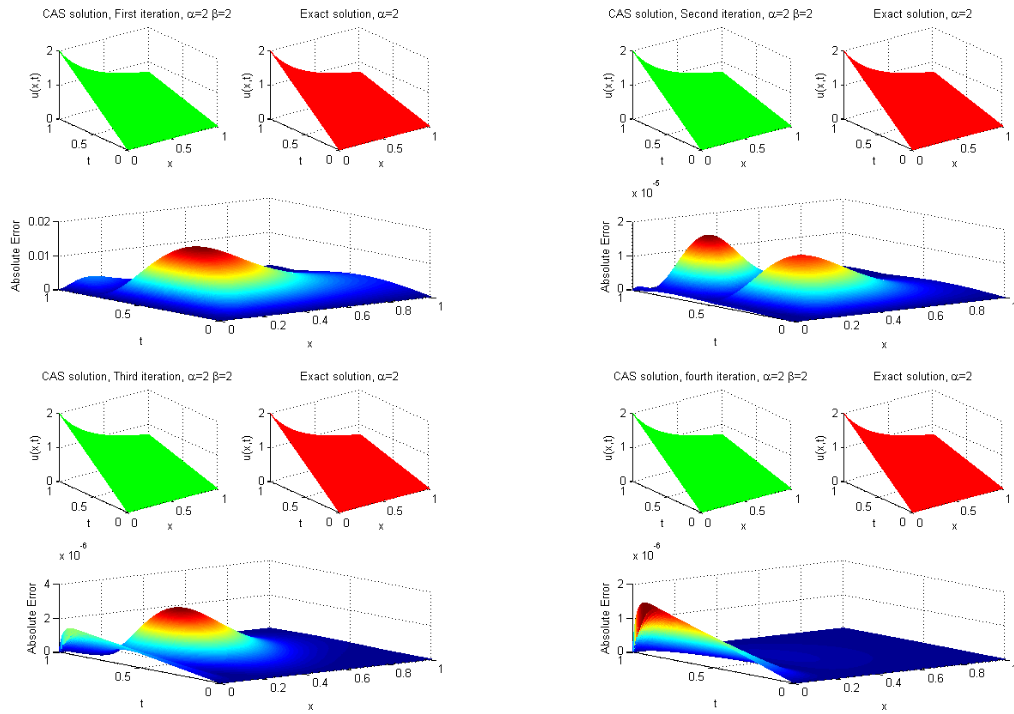

Figure 3.2: Comparison of exact and numerical solution of Burgers equation by CAS wavelet Picard method when k=k'=6 and M=M'=8

It is shown that CAS wavelet Picard technique gives excellent results when applied to different nonlinear initial and boundary value problems. Tables 3.1-3.3 show that present scheme gives better results when compared with results by few other methods. The solution of the nonlinear partial differential equations converge to exact solution when values of k, k', M and M' increases as shown in Figures 3.1 and 3.2.

Chapter 4

Solution of nonlinear partial differential equation by CAS wavelet quasilinearization technique

In wavelet method, Galerkin techniques or the collocation method is used to solve partial differential equations. CAS wavelet algorithm is based on the collocation method. CAS wavelet method is used to solve many famous equations like the fractional nonlinear differential equation [30], Integro-differential equations [18], nonlinear Fredholm integro-differential equations of fractional order [31] and nonlinear Volterra integro-differential equations of arbitrary order [32]. Umer Saeed [30] explained the solution of fractional nonlinear differential equations by CAS wavelet Picard method. The application of CAS wavelets in solving Fredholm-Hammerstein Integral Equations and error analysis is explained by Barzkar, A., P. Assari, and M. A. Mehrpouya [33]. A numerical method for the solution of boundary integral equations with logarithmic singular kernels based on the CAS wavelets approach is introduced by [34].

In this chapter, a numerical method is proposed by combining the CAS wavelet method and the quasilinearization technique for solving nonlinear partial differential equations. Initially, we discretize the nonlinear partial differential equation by quasilinearization technique and then convert the obtained discretized equation into a Sylvester equation by the CAS wavelet quasilinearization technique to get the solution. Then, error analysis and convergence has been derived. The results derived from this method are compared with Adomian decomposition method (ADM), Reduced differential transform method (RDTM), HAAR wavelet method and Variational iteration method (VIM). Generalized Burger-Fisher equation, klein-Gordon equation, time derivative Klein-Gordon equation and Burgers equation are used as test problems.

4.1 Quasilinearization Technique

Quasilinearization technique [26]-[25] is generalization of Newton-Raphson method for functional equations. In this method, we take nonlinear PDEs or ODEs, then we linearize it by considering the first two terms in the Taylor's series expansion. The benefit of this technique is that it converges quadratically to the solution, if there is convergence at all.
Consider non-linear second order differential equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + g(u, \frac{\partial u}{\partial x}), \qquad (4.1.1)$$
$$t \geq 0, \quad 0 < x < 1, \quad r > 0,$$

with the initial and boundary conditions as:

$$u(x,0) = g_1(x) \quad u(0,t) = h_1(t) \quad u(1,t) = h_2(t),$$

where $g(u, u_x)$ is nonlinear function. After applying quasilinearization approach for equation (4.1.1) implies:

$$\begin{aligned} \frac{\partial u_{r+1}}{\partial t} &= \frac{\partial^2 u_{r+1}}{\partial x^2} + g(u_r, \frac{\partial u_r}{\partial x}) + (u_{r+1} - u_r) g(u_r, \frac{\partial u_r}{\partial x}) \\ &+ (\frac{\partial u_{r+1}}{\partial x} - \frac{\partial u_r}{\partial x}) g(u_r, \frac{\partial u_r}{\partial x}), \end{aligned} \qquad (4.1.2)$$

with the boundary conditions as:

$$u_{r+1}(x,0) = g_1(x) \quad u_{r+1}(0,t) = h_1(t) \quad u_{r+1}(1,t) = h_2(t),$$

Above equation (4.1.2) is always a linear differential equation and can be solved recursively, starting with initial approximation $u_0(x,t)$.

4.2 Procedure of Implementation

In this section, we explain the procedure of implementing the method for nonlinear partial differential equation. The procedure begins with the conversion of nonlinear partial differential equation into discretize form by quasilinearization technique. Then, we solve discretized nonlinear partial differential equation by CAS wavelet operational matrix method. Consider the following discretized nonlinear partial differential equation,

$$\frac{\partial u_{r+1}}{\partial t} - a(x)\frac{\partial^2 u_{r+1}}{\partial t^2} + b(x)\frac{\partial u_{r+1}}{\partial x} + d(x)u_{r+1} = f(x,t),\ r > 0, \tag{4.2.1}$$

with initial and boundary conditions as:

$$u_{r+1}(x,0) = g_1(x), \quad \frac{\partial u_{r+1}}{\partial t}(x,0) = g_2(x), \quad u_{r+1}(0,t) = h_1(t), \quad u_{r+1}(1,t) = h_2(x).$$

Approximate the highest order term w.r.t x by CAS wavelet quasilinearization method as:

$$\frac{\partial^2 u_{r+1}}{\partial t^2} = \sum_{n=0}^{2^k-1}\sum_{m=-M}^{M}\sum_{i=0}^{2^{k'}-1}\sum_{j=-M'}^{M'} {c_{nm,ij}}^{r+1}\psi_{n,m}(x)\psi_{i,j}(t) = \Psi^T(x)C^{r+1}\Psi(t), \tag{4.2.2}$$

now apply the integral operator on the equation 4.2.2, we have:

$$\frac{\partial u_{r+1}}{\partial x} = [I_x\Psi^T(x)]C^{r+1}\Psi(t) + p(t)$$
$$u_{r+1}(x,t) = (I_x^2\Psi^T(x))C^{r+1}\Psi(t) + p(t)x + q(t),$$

where $p(t)$ and $q(t)$ are

$$p(t) = h_2(t) - h_1(t) - (I_{x=1}^2\Psi^T(x))C^{r+1}\Psi(t)$$
$$q(t) = h_1(t).$$

By putting the values of $p(t)$ and $q(t)$ in $u(x,t)$, we get:

$$u_{r+1}(x,t) = (I_x^2\Psi^T(x))C^{r+1}\Psi(t) + (h_2(t) - h_1(t))x - (I_{x=1}^2\Psi^T(x))C^{r+1}\Psi(t)x + h_1(t). \tag{4.2.3}$$

Equation (4.2.3) implies that

$$\frac{\partial u_{r+1}}{\partial t} - a(x)\Psi^T(x)C^{r+1}\Psi(t) + b(x)I_x\Psi^T(x)C^{r+1}\Psi(t) + b(x)(h_2(t) - h_1(t)) -$$
$$b(x)(I_{x=1}^2\Psi^T(x))C^{r+1}\Psi(t) + d(x)(I_x^2\Psi^T(x))C^{r+1}\Psi(t) + d(x)(h_2(t) - h_1(t))x -$$
$$d(x)(I_{x=1}^2\Psi^T(x))C^{r+1}\Psi(t)x + d(x)h_1(t) = f(x,t).$$

We make substitution as $G = f(x,t) - b(x)(h_2(t) - h_1(t)) - xd(x)(h_2(t) - h_1(t)) - d(x)h_1(t)$ and $G \simeq \Psi^T(x)M\Psi(t)$ for simplification and get

$$\frac{\partial u_{r+1}}{\partial t} - a(x)\Psi^T(x)C^{r+1}\Psi(t) + b(x)(I_x\Psi^T(x))C^{r+1}\Psi(t) - b(x)(I_{x=1}^2\Psi^T(x))C^{r+1}\Psi(t) +$$
$$d(x)(I_x^2\Psi^T(x))C^{r+1}\Psi(t) - d(x)(I_{x=1}^{\beta}\Psi^T(x))C^{r+1}\Psi(t)x = \Psi^T(x)M\Psi(t). \tag{4.2.4}$$

Apply integration on the equation 4.2.4 to get:

$$u_{r+1}(x,t) = a(x)\Psi^T(x)C^{r+1}(I_t\Psi(t)) - b(x)(I_x\Psi^T(x))C^{r+1}I_t\Psi(t)+$$
$$b(x)(I^2_{x=1}\Psi^T(x))C^{r+1}(I_t\Psi(t)) - d(x)(I^2_x\Psi^T(x))C^{r+1}(I_t\Psi(t))+$$
$$d(x)(I^2_{x=1}\Psi^T(x))C^{r+1}(I_t\Psi(t))x + \Psi^T(x)MI_t\Psi(t) \quad (4.2.5)$$

Now, by equating equations (4.2.3) and (4.2.5) and simplification it is,

$$(I^2_x\Psi^T(x) - xI^2_{x=1}\Psi^T(x))(C^{r+1}) - ((a(x)\Psi^T(x) - b(x)I_x\Psi^T(x) + b(x)I^2_{x=1}\Psi^T(x) -$$
$$d(x)I^2_x\Psi^T(x) + d(x)I^2_{x=1}\Psi^T(x)x))C^{r+1}(I_t\Psi(t))\Psi(t)^{-1} = (\Psi^T(x)MI_t\Psi(t) -$$
$$(h_2(t) - h_1(t))x - h_1(t))\Psi(t)^{-1},$$

let $K(x,t) = (h_2(t) - h_1(t))x - h_1(t)$ above equation at collocation points $x_i = \frac{2i-1}{2\hat{m}}$, $t_j = \frac{2j-1}{2\hat{m'}}$ is, where $i = 1,2,3,....,\hat{m}$, $j = 1,2,3,....,\hat{m'}$, $\hat{m} = 2^k(2M+1)$ and $\hat{m'} = 2^{k'}(2M'+1)$.

$$(I^2_x\Psi^T(x_i) - x_iI^2_{x=1}\Psi^T(x_i))(C^{r+1}) - (a(x_i)\Psi^T(x_i) - b(x_i)I_x\Psi^T(x_i) + b(x_i)I^2_{x=1}\Psi^T(x_i) -$$
$$d(x_i)I^2_x\Psi^T(x_i) + d(x_i)x_iI^2_{x=1}\Psi^T(x_i))C^{r+1}I_t\Psi(t_j)\Psi(t_j)^{-1} = (\Psi^T(x_i)MI_t\Psi(t_j) -$$
$$K(x_i,t_j))\Psi(t)^{-1},$$

which is presented in matrix form as:-

$$(\mathbf{P}^{2,x}_{\hat{m}\times\hat{m'}} - \mathbf{W}^{x,2}_{\hat{m}\times\hat{m'}})\mathbf{C}^{r+1} - (\mathbf{A}\Psi^T - \mathbf{B}\mathbf{P}^{1,x}_{\hat{m}\times\hat{m'}} + \mathbf{B}\mathbf{W}^{x,2}_{\hat{m}\times\hat{m'}} - \mathbf{D}\mathbf{P}^{2,x}_{\hat{m}\times\hat{m'}}$$
$$+\mathbf{D}\mathbf{W}^{x,2}_{\hat{m}\times\hat{m'}})\mathbf{C}^{r+1}\mathbf{P}^{1,t}_{\hat{m}\times\hat{m'}}(\Psi)^{-1} = (\Psi^T\mathbf{M}\Psi - \mathbf{K})\Psi^{-1}$$

where **A, B** and **D** are diagonal matrices given as:-

$$\mathbf{A} = \begin{pmatrix} a(x_1) & 0 & \cdots & 0 \\ 0 & a(x_2) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a(x_j) \end{pmatrix}, \mathbf{B} = \begin{pmatrix} b(x_1) & 0 & \cdots & 0 \\ 0 & b(x_2) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & b(x_j) \end{pmatrix} \text{ and}$$

$$\mathbf{D} = \begin{pmatrix} d(x_1) & 0 & \cdots & 0 \\ 0 & d(x_2) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & d(x_j) \end{pmatrix}.$$

The matrix **K** is defined as:- $\mathbf{K} = \begin{pmatrix} k(x_1,y_1) & k(x_1,y_2) & \cdots & k(x_1,y_n) \\ k(x_2,y_1) & k(x_2,y_2) & \cdots & k(x_2,y_n) \\ \vdots & \vdots & \ddots & \vdots \\ k(x_n,y_1) & k(x_n,y_2) & \cdots & k(x_n,y_n) \end{pmatrix}$. Let $\mathbf{v} = (\mathbf{A}\Psi^T - \mathbf{B}\mathbf{P}^{1,x}_{\hat{m}\times\hat{m'}} + \mathbf{B}\mathbf{W}^{x,2}_{\hat{m}\times\hat{m'}} - \mathbf{D}\mathbf{P}^{2,x}_{\hat{m}\times\hat{m'}} + \mathbf{D}\mathbf{W}^{x,2}_{\hat{m}\times\hat{m'}})^{-1}$.

Then, we get:

$$\mathbf{v}(\mathbf{P}^{2,x}_{\hat{m}\times\hat{m}'} - \mathbf{W}^{x,2}_{\hat{m}\times\hat{m}'})(\mathbf{C}^{r+1}) - (\mathbf{C}^{r+1}\mathbf{P}^{1,t}_{\hat{m}\times\hat{m}'})(\Psi^{-1}) = \mathbf{v}(\Psi^T\mathbf{M}\mathbf{P}^{1,t}_{\hat{m}\times\hat{m}'} - \mathbf{K})\Psi^{-1}.$$

Suppose that $\mathbf{Q} = \mathbf{v}(\mathbf{P}^{2,x}_{\hat{m}\times\hat{m}'} - \mathbf{W}^{x,1}_{\hat{m}\times\hat{m}'})$, $\mathbf{R} = \mathbf{P}^{1,t}_{\hat{m}\times\hat{m}'}\Psi^{-1}$ and $\mathbf{S} = \mathbf{v}(\Psi^T\mathbf{M}\Psi - \mathbf{K})\Psi^{-1}$.

$$\mathbf{Q}(\mathbf{C}^{r+1}) - (\mathbf{C}^{r+1}\mathbf{R}) = \mathbf{S} \tag{4.2.6}$$

(4.2.6) is used for $\mathbf{C}^{r+1}$ and then use it in (4.2.3) to get the solution $u_{r+1}(x,t)$.

4.3 Error analysis of CAS wavelet quasilinearization technique

Theorem: Assume that $u_{r+1}(x,t) \to L^2[0,1]$ is a function with bounded partial derivative on [0,1] that is $\exists\, \gamma > 0; \forall\, x \;\to\; [0,1] : |\frac{\partial^4 u_{r+1}}{\partial^2 p \partial^2 q}| \leq \gamma$. The function $u_{r+1}(x,t)$ is expanded as an infinite sum of the CAS wavelets and the series quadratically converges uniformly to $u_{r+1}(x,t)$, that is
$u_{r+1}(x,t) = \sum\limits_{n=0}^{\infty}\sum\limits_{m\to Z}\sum\limits_{i=0}^{\infty}\sum\limits_{j\to Z} c^{r+1}_{nm,ij}\psi_{n,m}(x)\psi_{i,j}(t)$. Furthermore,
$u^{k,k',M,M'}_{r+1}(x,t) = \sum\limits_{n=0}^{2^k-1}\sum\limits_{m=-M}^{M}\sum\limits_{i=0}^{2^{k'}-1}\sum\limits_{j=-M'}^{M'} c^{r+1}_{nm,ij}\psi_{n,m}(x)\psi_{i,j}(t)$, we have

$$|u^{k,k',M,M'}_{r+1} - u_{r+1}(x,t)| \leq |\sum_{n=2^k}^{\infty}\sum_{m=M+1}^{\infty}\sum_{i=2^k}^{\infty}\sum_{j=M'+1}^{\infty}\frac{\gamma}{(mj\pi^2)^2(n+1)^{\frac{5}{2}}(i+1)^{\frac{5}{2}}}| \tag{4.3.1}$$

Inequality (4.3.1) exhibits that the absolute error at the $(r+1)th$ iteration is inversely proportional to k, k', M and M'. This implies that $u^{k,k',M,M'}_{r+1}(x,t)$ converges to $u_{r+1}(x,t)$ as $k, k', M, M' \longrightarrow \infty$. Since $u_{r+1}(x,t)$ is obtained at $(r+1)th$ iteration of quasilinearization technique so according to the convergence analysis of quasilinearization technique [26] which states that $u_{r+1}(x,t)$ converges to $u(x,t)$ as r $\longrightarrow \infty$, if there is convergence at all. This suggest that solution by CAS wavelet quasilinearization technique $u^{k,k',M,M'}_{r+1}(x,t)$ converges to $u(x,t)$ when k,k', M, M' and $r \;\longrightarrow \infty$. The converges of CAS wavelet quasilinearization method is fast as compared to CAS wavelet Picard method.

4.4 Applications

Numerical solution of some famous problems by CAS wavelet quasilinearization technique are explained in this section. The results obtained by CAS wavelet technique are compared with the results obtained by like Adomian decomposition method (ADM), reduced differential transform method (RDTM) and variational iteration method (VIM).

Generalized Burger-Fisher equation

Consider the Burgers Fisher equation

$$\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} + au^\gamma \frac{\partial u}{\partial x} + bu(u^\gamma - 1) = 0. \tag{4.4.1}$$

where $0 \leq x \leq 1$, $t \geq 1$
with initial condition and boundary conditions as

$$u(x,0) = (\frac{1}{2} - \frac{1}{2}tanh(\frac{a\gamma}{2(1+\gamma)}x))^{\frac{1}{\gamma}},$$
$$u(0,t) = (\frac{1}{2} - \frac{1}{2}tanh(\frac{a\gamma}{2(1+\gamma)}(-(\frac{a^2+b(1+\gamma^2)}{a1+\gamma})t)))^{\frac{1}{\gamma}},$$
$$u(1,t) = (\frac{1}{2} - \frac{1}{2}tanh(\frac{a\gamma}{2(1+\gamma)}(1-(\frac{a^2+b(1+\gamma^2)}{a1+\gamma})t)))^{\frac{1}{\gamma}},$$

whose exact solution is

$$u(x,t) = (\frac{1}{2} - \frac{1}{2}tanh(\frac{a\gamma}{2(1+\gamma)}(x-(\frac{a^2+b(1+\gamma^2)}{a1+\gamma})t)))^{\frac{1}{\gamma}}.$$

Now apply quasilinearization technique to (4.4.1), it becomes:

$$\frac{\partial u_{r+1}}{\partial t} - \frac{\partial^2 u_{r+1}}{\partial x^2} + bu_{r+1} + F(u_{r+1}, \frac{\partial u_r}{\partial x}) = 0, \tag{4.4.2}$$

where

$$F(u_{r+1}, \frac{\partial u_r}{\partial x}) = bu_r^{\gamma+1} + b(\gamma+1)u_{r+1}u_r^\gamma - bu_{r+1} - b(\gamma+1)u_r^{\gamma+1} +$$
$$a\gamma u_r^{\gamma-1}u_{r+1}\frac{\partial u_r}{\partial x} - a\gamma u_r^\gamma \frac{\partial u_r}{\partial x} + au_r^\gamma \frac{\partial u_{r+1}}{\partial x},$$

with the initial and boundary conditions as:-

$$u_{r+1}(x,0) := H(x) = (\frac{1}{2} - \frac{1}{2}tanh(\frac{a\gamma}{2(1+\gamma)}x))^{\frac{1}{\gamma}},$$
$$u_{r+1}(0,t) := J(t) = (\frac{1}{2} - \frac{1}{2}tanh(\frac{a\gamma}{2(1+\gamma)}(-(\frac{a^2+b(1+\gamma^2)}{a1+\gamma})t)))^{\frac{1}{\gamma}},$$
$$u_{r+1}(1,t) := I(t) = (\frac{1}{2} - \frac{1}{2}tanh(\frac{a\gamma}{2(1+\gamma)}(1-(\frac{a^2+b(1+\gamma^2)}{a1+\gamma})t)))^{\frac{1}{\gamma}},$$

where $r = 0,1,2,...,N$.
Now approximate the highest order term w.r.t x in equation (bf2) by CAS wavelet series as:

$$\frac{\partial^2 u_{r+1}}{\partial x^2} = \sum_{n=0}^{2^k-1}\sum_{m=-M}^{M}\sum_{i=0}^{2^{k'}-1}\sum_{j=-M'}^{M'} {c_{nm,ij}}^{r+1}\psi_{n,m}(x)\psi_{i,j}(t) = \Psi^T(x)C^{r+1}\Psi(t).$$

The integration along with the initial conditions yields:

$$\frac{\partial u_{r+1}}{\partial x} = I_x^1\Psi^T(x)C^{r+1}\Psi(t) + I(t) - J(t) - I_{x=1}^2\Psi^T(x)C^{r+1}\Psi(t) \tag{4.4.3}$$
$$u_{r+1}(x,t) = I_x^2\Psi^T(x)C^{r+1}\Psi(t) + xI(t) - xJ(t) - xI_{x=1}^2\Psi^T(x)C\Psi(t) + J(t), \tag{4.4.4}$$

substituting above equations in equation (4.4.2), we obtain:

$$\frac{\partial u_{r+1}}{\partial t} - b(I(t)-J(t)) - a\gamma u_r^\gamma\frac{\partial u_r}{\partial x} - \Psi^T(x)C^{r+1}\Psi(t) + au_r^\gamma\frac{\partial u_{r+1}}{\partial x}$$
$$+ b(\gamma+1)u_r^\gamma(I_x^2\Psi^T(x)C^{r+1}\Psi(t) + xI(t) - xJ(t) + J(t) - xI_{x=1}^2\Psi^T(x)C^{r+1}\Psi(t))$$
$$- b(I_x^2\Psi^T(x)C^{r+1}\Psi(t) - xI_{x=1}^2\Psi^T(x)C^{r+1}\Psi(t) + J(t)) - b(\gamma+1)u_r^{\gamma+1} + bu_r^{\gamma+1}$$
$$+ a\gamma u_r^{\gamma-1}\frac{\partial u_r}{\partial x}(I_x^2\Psi^T(x)C^{r+1}\Psi(t) + xI(t) - xJ(t) - xI_{x=1}^2\Psi^T(x)C^{r+1}\Psi(t) + J(t)) = 0,$$

or

$$\frac{\partial u_{r+1}}{\partial t} + b(\gamma+1)u_r^\gamma(I_x^2\Psi^T(x)C^{r+1}\Psi(t) + xI(t) - xJ(t) + J(t) - xI_{x=1}^2\Psi^T(x)C^{r+1}\Psi(t))$$
$$- b(I_x^2\Psi^T(x)C^{r+1}\Psi(t) - xI_{x=1}^2\Psi^T(x)C^{r+1}\Psi(t) + J(t)) - b(xI(t) - xJ(t))$$
$$+ a\gamma u_r^{\gamma-1}\frac{\partial u_r}{\partial x}(I_x^2\Psi^T(x)C^{r+1}\Psi(t) + xI(t) - xJ(t) - xI_{x=1}^2\Psi^T(x)C^{r+1}\Psi(t) + J(t))$$
$$- b(\gamma+1)u_r^{\gamma+1} + bu_r^{\gamma+1} - a\gamma u_r^\gamma\frac{\partial u_r}{\partial x} - \Psi^T(x)C^{r+1}\Psi(t) + au_r^\gamma\frac{\partial u_{r+1}}{\partial x} = 0. \tag{4.4.5}$$

We make substitution for simplification as:

$$G = -b(\gamma+1)u_r^\gamma(xI(t) - xJ(t) + J(t)) + b(J(t) + xI(t) - xJ(t)) -$$
$$a\gamma u_r^{\gamma-1}\frac{\partial u_r}{\partial x}(xI(t) - xJ(t) + J(t)) + bu_r^{\gamma+1}\gamma + 1 + a\gamma u_r^\gamma\frac{\partial u_r}{\partial x} -$$
$$au_r^\gamma(I(t) - J(t)) - bu_r^{\gamma+1}.$$

Here $G \simeq \Psi^T(x)M\Psi(x)$ on putting this substitution in equation (4.4.5), we have:

$$\frac{\partial u_{r+1}}{\partial t} = \Psi^T(x)C^{r+1}\Psi(t) - b(\gamma+1)u_r^{\gamma}(I_x^2\Psi^T(x)C^{r+1}\Psi(t) - xI_{x=1}^2\Psi^T(x)C^{r+1}\Psi(t))$$
$$+ b(I_x^2\Psi^T(x)C^{r+1}\Psi(t) - xI_{x=1}^2\Psi^T(x)C^{r+1}\Psi(t)) - a\gamma u_r^{\gamma-1}\frac{\partial u_r}{\partial x}(I_x^2\Psi^T(x)C^{r+1}\Psi(t) - xI_{x=1}^2\Psi^T(x)C^{r+1}\Psi(t))$$
$$- au_r^{\gamma}(I_x^2\Psi^T(x)C^{r+1}\Psi(t) - xI_{x=1}^2\Psi^T(x)C^{r+1}\Psi(t)) + \Psi^T(x)M\Psi(t).$$

Again by taking integral and after putting the initial condition, it gives:

$$u_{r+1}(x,t) = \Psi^T(x)C^{r+1}I_t^1\Psi(t) - b(\gamma+1)u_r^{\gamma}(I_x^2\Psi^T(x)C^{r+1}I_t^1\Psi(t) - xI_{x=1}^2\Psi^T(x)C^{r+1}I_t^1\Psi(t))$$
$$- a\gamma u_r^{\gamma-1}\frac{\partial u_r}{\partial x}(I_x^2\Psi^T(x)C^{r+1}I_t^1\Psi(t) - xI_{x=1}^2\Psi^T(x)C^{r+1}I_t^1\Psi(t)) + \Psi^T(x)MI_t^1\Psi(t) + H(x)$$
$$+ b(I_x^2\Psi^T(x)C^{r+1}I_t^1\Psi(t) - xI_{x=1}^2\Psi^T(x)C^{r+1}I_t^1\Psi(t))$$
$$- au_r^{\gamma}(I_x^2\Psi^T(x)C^{r+1}I_t^1\Psi(t) - xI_{x=1}^2\Psi^T(x)C^{r+1}I_t^1\Psi(t)). \tag{4.4.6}$$

Equation (4.4.4) and (4.4.6) are equated to obtain:

$$I_x^2\Psi^T(x)C^{r+1}\Psi(t) + xI(t) - xJ(t) - xI_{x=1}^2\Psi^T(x)C^{r+1}\Psi(t) + u_{r+1}(0,t) =$$
$$- b(\gamma+1)u_r^{\gamma}I_x^2\Psi^T(x)C^{r+1}I_t^1\Psi(t) - b(\gamma+1)u_r^{\gamma}xI_{x=1}^2\Psi^T(x)C^{r+1}I_t^1\Psi(t) + \Psi^T(x)C^{r+1}I_t^1\Psi(t)$$
$$- a\gamma u_r^{\gamma-1}\frac{\partial u_r}{\partial x}(I_x^2\Psi^T(x)C^{r+1}I_t^1\Psi(t) - xI_{x=1}^2\Psi^T(x)C^{r+1}I_t^1\Psi(t)) + \Psi^T(x)MI_t^1\Psi(t)$$
$$+ b(I_x^2\Psi^T(x)C^{r+1}I_t^1\Psi(t) - xI_{x=1}^2\Psi^T(x)C^{r+1}I_t^1\Psi(t)) + J(t)$$
$$- au_r^{\gamma}(I_x^2\Psi^T(x)C^{r+1}I_t^1\Psi(t) - xI_{x=1}^2\Psi^T(x)C^{r+1}I_t^1\Psi(t)).$$

After rearranging and simplification, we get:

$$(I_x^2\Psi^T(x) - xI_{x=1}^2\Psi^T(x))C^{r+1}\Psi(t) - (\Psi^T(x) - b(\gamma+1)u_r^{\gamma}I_x^2\Psi^T(x) - b(\gamma+1)u_r^{\gamma}xI_{x=1}^2\Psi^T(x)$$
$$- a\gamma u_r^{\gamma-1}\frac{\partial u_r}{\partial x}(I_x^2\Psi^T(x) - xI_{x=1}^2\Psi^T(x)) - au_r^{\gamma}(I_x^2\Psi^T(x) - xI_{x=1}^2\Psi^T(x))$$
$$+ b(I_x^2\Psi^T(x) - xI_{x=1}^2\Psi^T(x)))C^{r+1}I_t^1\Psi(t) = \Psi^T(x)MI_t^1\Psi(t) + H(x)$$
$$- xI(t) + xJ(t) - J(t).$$

Let $K(x,t) = H(x) - xI(t) + xJ(t) - J(t)$ and then by putting collocation points $x_i = \frac{2i-1}{2\hat{m}}$ and $t_j = \frac{2j-1}{2\hat{m}}$ in above equation which is further written as:-
where $i = 1,2,3,....,\hat{m}$, $j = 1,2,3,....,\hat{m'}$, $\hat{m} = 2^k(2M+1)$ and $\hat{m'} = 2^{k'}(2M'+1)$.

$$(I_x^2\Psi^T(x_i) - x_iI_{x=1}^2\Psi^T(x_i))C^{r+1}\Psi(t_j) - (\Psi^T(x_i) - b(\gamma+1)u_r^{\gamma}I_x^2\Psi^T(x_i) - b(\gamma+1)u_r^{\gamma}x_iI_{x=1}^2\Psi^T(x_i)$$
$$- a\gamma u_r^{\gamma-1}\frac{\partial u_r}{\partial x_i}(I_x^2\Psi^T(x_i) - x_iI_{x=1}^2\Psi^T(x_i)) - au_r^{\gamma}(I_x^2\Psi^T(x_i) - x_iI_{x=1}^2\Psi^T(x_i))$$
$$+ b(I_x^2\Psi^T(x_i) - x_iI_{x=1}^2\Psi^T(x_i)))C^{r+1}I_t^1\Psi(t) = \Psi^T(x)MI_t^1\Psi(t) + K(x_i,t_j).$$

In matrix form above can be presented as

$$(\mathbf{P}^{2,x}_{\hat{m}\times\hat{m}'} - \mathbf{W}^{x,2}_{\hat{m}\times\hat{m}'})\mathbf{C}^{r+1}\Psi - (\Psi^T - b(\gamma+1)u_r^\gamma \mathbf{P}^{2,x}_{\hat{m}\times\hat{m}'} - b(\gamma+1)u_r^\gamma \mathbf{W}^{x,2}_{\hat{m}\times\hat{m}'} \\ -a\gamma u_r^{\gamma-1}\tfrac{\partial u_r}{\partial x}(\mathbf{P}^{2,x}_{\hat{m}\times\hat{m}'} - \mathbf{W}^{x,2}_{\hat{m}\times\hat{m}'}) - au_r^\gamma(\mathbf{P}^{2,x}_{\hat{m}\times\hat{m}'} - \mathbf{W}^{x,2}_{\hat{m}\times\hat{m}'}) \\ +b(\mathbf{P}^{2,x}_{\hat{m}\times\hat{m}'} - \mathbf{W}^{x,2}_{\hat{m}\times\hat{m}'}))\mathbf{C}^{r+1}\mathbf{P}^{1,t}_{\hat{m}\times\hat{m}'} = \Psi^T\mathbf{M}\mathbf{P}^{1,t}_{\hat{m}\times\hat{m}'} + \mathbf{K}.$$

Now we make supposition to write above equation as Sylvester equation.

$$\mathbf{Q}\mathbf{C}^{r+1} - \mathbf{C}^{r+1}\mathbf{R} = \mathbf{S}, \tag{4.4.7}$$

where

$$\mathbf{v} = (\Psi^T - b(\gamma+1)u_r^\gamma \mathbf{P}^{2,x}_{\hat{m}\times\hat{m}'} - b(\gamma+1)u_r^\gamma \mathbf{W}^{x,2}_{\hat{m}\times\hat{m}'} - a\gamma u_r^{\gamma-1}\tfrac{\partial u_r}{\partial x}(\mathbf{P}^{2,x}_{\hat{m}\times\hat{m}'} - \mathbf{W}^{x,2}_{\hat{m}\times\hat{m}'}) \\ -au_r^\gamma(\mathbf{P}^{2,x}_{\hat{m}\times\hat{m}'} - \mathbf{W}^{x,2}_{\hat{m}\times\hat{m}'}) + b(\mathbf{P}^{2,x}_{\hat{m}\times\hat{m}'} - \mathbf{W}^{x,2}_{\hat{m}\times\hat{m}'}))^{-1}$$

$\mathbf{Q} = \mathbf{v}(\mathbf{P}^{2,x}_{\hat{m}\times\hat{m}'} - \mathbf{W}^{x,2}_{\hat{m}\times\hat{m}'})$, $\mathbf{R} = \mathbf{P}^{1,t}_{\hat{m}\times\hat{m}'}\Psi^{-1}$ and $\mathbf{S} = \mathbf{v}(\Psi^T\mathbf{M}\mathbf{P}^{1,t}_{\hat{m}\times\hat{m}'} + \mathbf{K})\Psi^{-1}$.

By the equation (4.4.7), we get the value of $\mathbf{C}^{r+1}$ which is utilized to get the solution $u_{r+1}(x,t)$. The results are compared with other famous results of generalized Burgers Fisher equation.

$M = M' = 5,\ k = k' = 4$				
x	t	E_{RTDM}[35]	E_{VIM}[35]	E_{CAS}
0.01	0.02	0.4999e-05	2.5031e-03	1.9435e-007
0.01	0.04	0.4999e-05	2.5081e-03	2.7604e-007
0.01	0.06	1.4999e-05	2.5131e-03	3.3814e-007
0.01	0.08	1.9999e-05	2.5181e-03	3.8724e-007
0.04	0.02	0.4997e-05	9.9962e-03	7.1200e-007
0.04	0.04	0.9997e-05	1.0001e-02	1.0346e-006
0.04	0.06	1.4997e-05	1.0006e-02	1.2805e-006
0.04	0.08	1.9997e-05	1.0011e-02	1.4781e-006
0.08	0.02	0.4995e-05	1.9979e-02	1.2555e-006
0.08	0.04	0.9995e-05	1.9984e-02	1.8928e-006
0.08	0.06	1.4995e-05	1.9989e-02	2.3807e-006
0.08	0.08	1.9995e-05	1.9994e-02	2.7727e-006

Table 4.1: Comparison of the approximate solutions of Burger-Fisher equation when a=0.001, b=0.001 and $\gamma = 1$.

$k = k' = 5,\ M = M' = 7$			
x	t	E_{RTDM}[35]	E_{CAS}
0.01	0.02	4.7133e-06	3.17037e-008
0.01	0.04	9.4271e-06	2.72883e-008
0.01	0.06	1.4142e-05	2.64518e-008
0.01	0.08	1.8855e-05	2.63137e-008
0.04	0.02	4.7117e-06	1.20316e-007
0.04	0.04	9.4260e-06	1.05412e-007
0.04	0.06	1.4140e-05	1.02628e-007
0.04	0.08	1.8854e-05	1.02080e-007
0.08	0.02	4.7104e-06	2.27121e-007
0.08	0.04	9.4241e-06	2.01289e-007
0.08	0.06	1.4138e-05	1.96390e-007
0.08	0.08	1.8852e-05	1.95436e-007

Table 4.2: Comparative solution of Burger-Fisher equation for a=0.001, b=0.001 and $\gamma = 2$.

$M = M' = 6,\ k = k' = 4$			
x	t	$E_{Adomian}$[36]	E_{CAS}
0.1	0.005	9.68763e-06	5.45915e-007
0.1	0.01	1.93752e-05	9.23339e-007
0.5	0.005	9.68691e-06	6.87450e-007
0.5	0.01	1.93738e-05	1.31102e-006
0.9	0.005	9.68619e-06	5.46290e-007
0.9	0.01	1.93724e-05	9.23979e-007

Table 4.3: Comparison of the numerical solutions of Burger-Fisher equation for a=0.001, b=0.001 and $\gamma = 1$.

Solution of generalize Burger-Fisher equation for $a = 0.001, b = 0.001$ and $\gamma = 1$ by present method at $M = M' = 5,\ k = k' = 4$ and $r = 4$ is given in Table 4.1 . The obtained results are compared with the results obtained from reduced differential transform method (RDTM) [35] and variational iteration method (VIM) [35].

Table 4.2 is used to list the results of generalized Burger–Fisher equation at $a = 0.001,\ b = 0.001$ and $\gamma = 2$. We implement the proposed method at $M = M' = 7$, $k = k' = 5$ and $r = 3$. We compared our results with the results obtained by reduced differential transform method (RDTM) [35].

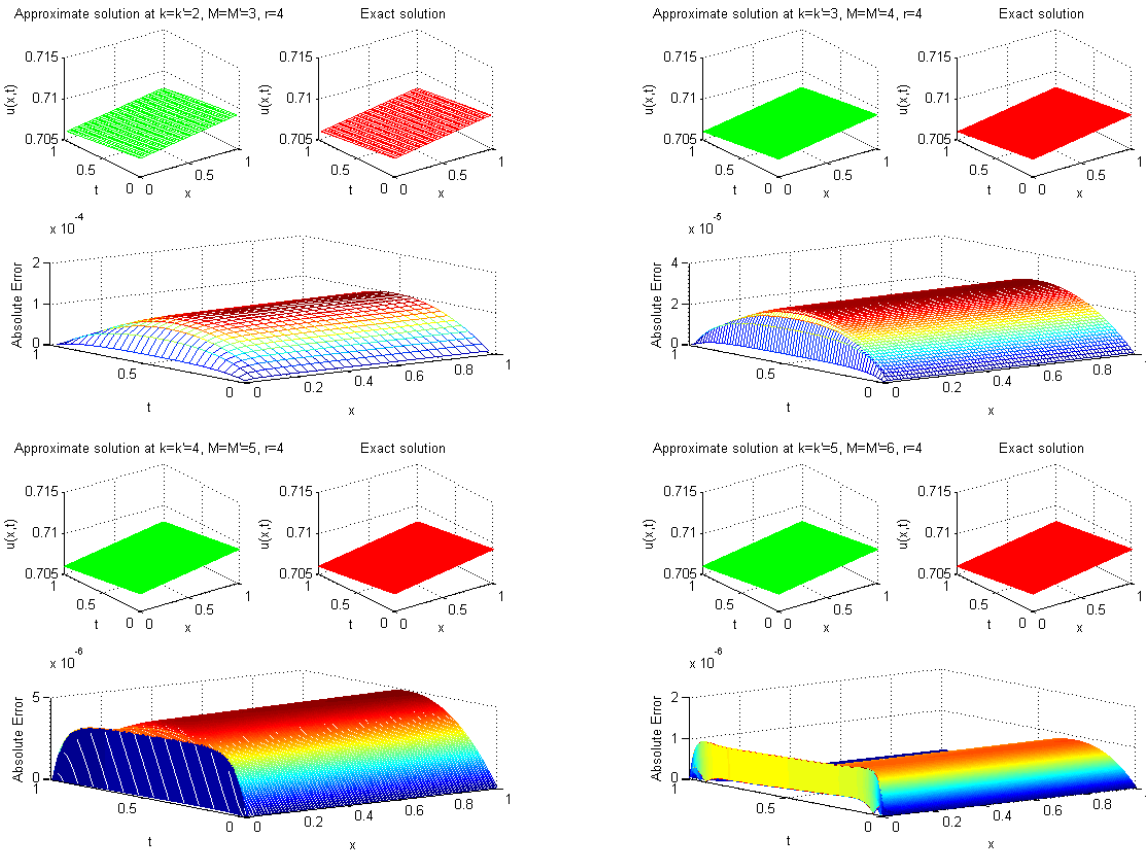

Figure 4.1: Comparison of the numerical results by present method at r = 4 and different values of k, k', M and M' with exact solution of generalized BurgerFisher equation.

Present method at $k = k' = 5,\ M = M' = 7,\ r = 4$ is implemented on generalized Burger–Fisher equation with $a = 0.001,\ b = 0.001$ and $\gamma = 1$. The obtained results are listed in Table 3.

Figure 4.1 is used to plot the exact solution of equation (1.1) with $a = 0.01$ $b = 0.01$ and $\gamma = 2$, solution by CAS wavelet quasilinearization technique at $r = 4$ and different values of k, k', M, and M'.

Klein Gordon equation

In this section, we consider the klein Gordon equation

$$\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} + u^2 = x^2 t^2. \tag{4.4.8}$$

with the initial and boundary conditions as:

$$u(x,0) = 0, \qquad u(0,t) = 0, \qquad u_t(x,0) = x, \qquad u(1,t) = t.$$

where exact solution is

$$u(x,t) = xt.$$

First by applying quasilinearization technique to (4.4.8), we have:

$$\frac{\partial^2 u_{r+1}}{\partial t^2} - \frac{\partial^2 u_{r+1}}{\partial x^2} = x^2 t^2 + u_r^2 - 2u_r u_{r+1}, \tag{4.4.9}$$

with initial and boundary conditions as:-

$$u_{r+1}(x,0) = 0, \qquad u_{r+1}(0,t) = 0, \qquad (u_{r+1})_t(x,0) = x, \qquad u_{r+1}(1,t) = t.$$

Now, approximate the highest order term w.r.t x in equation (4.4.9) by CAS wavelet series as:

$$\frac{\partial^2 u_{r+1}}{\partial x^2} = \sum_{n=0}^{2^k-1} \sum_{m=-M}^{M} \sum_{i=0}^{2^{k'}-1} \sum_{j=-M'}^{M'} c_{nm,ij}^{r+1} \psi_{n,m}(x) \psi_{i,j}(t) = \Psi^T(x) C^{r+1} \Psi(t).$$

By the integration and by putting initial conditions, we get:

$$u_{r+1}(x,t) = I_x^2 \Psi(x)^T C^{r+1} \Psi(t) + xt - x I_{x=1}^2 \Psi^T(x) C^{r+1} \Psi(t). \tag{4.4.10}$$

Now, put equation (4.4.10) in (4.4.9), we have:

$$\frac{\partial^2 u_{r+1}}{\partial t^2} - \Psi^T(x) C^{r+1} \Psi(t) = x^2 t^2 + u_r^2 - 2u_r I_x^2 \Psi^T(x) C^{r+1} \Psi(t) - 2u_r xt + 2u_r x I_{x=1}^2 \Psi^T(x) C^{r+1} \Psi(t).$$

By doing following substitution for simplification $G = 2u_r xt + x^2 t^2 + u_r^2$, Where $G \simeq \Psi^T(x) M \Psi(t)$, after applying integration and by the initial condition, we get:

$$\begin{aligned} u_{r+1}(x,t) = \Psi^T(x) C^{r+1} I_t^2 \Psi(t) + 2u_r I_x^2 \Psi^T(x) C^{r+1} I_t^2 \Psi(t) - 2u_r x I_{x=1}^2 \Psi^T(x) C^{r+1} I_t^2 \Psi(t) \\ + \Psi^T(x) M I_t^2 \Psi(t) + xt. \end{aligned} \tag{4.4.11}$$

By Equating equation (4.4.10) and (4.4), we have:

$$\begin{aligned} I_x^2 \Psi^T(x) C^{r+1} \Psi(t) + xt - x I_{x=1}^2 \Psi^T(x) C^{r+1} \Psi(t) = \Psi^T(x) C^{r+1} I_t^2 \Psi(t) + 2u_r I_x^2 \Psi^T(x) C^{r+1} I_t^2 \Psi(t) - \\ 2u_r x I_{x=1}^2 \Psi^T(x) C^{r+1} I_t^2 \Psi(t) + \Psi^T(x) M I_t^2 \Psi(t) + xt. \end{aligned}$$

Putting collocation points $x_i = \frac{2i-1}{2\hat{m}}$ and $t_j = \frac{2j-1}{2\hat{m'}}$ in above equation which is further written as, where $i = 1, 2, 3, ..., \hat{m}$, $j = 1, 2, 3, ..., \hat{m'}$, $\hat{m} = 2^k(2M + 1)$ and $\hat{m'} = 2^{k'}(2M' + 1)$.

$$I_x^2\Psi^T(x_i)C^{r+1}\Psi(t_j) - x_i I_{x=1}^2\Psi^T(x_i)C^{r+1}\Psi(t_j) = \Psi^T(x_i)C^{r+1}I_t^2\Psi(t_j) + 2u_r I_x^2\Psi^T(x_i)C^{r+1}I_t^2\Psi(t_j) - 2u_r x_i I_{x=1}^2\Psi^T(x_i)C^{r+1}I_t^2\Psi(t_j) + \Psi_x(x_i)MI_t^2\Psi(t_j).$$

$$\mathbf{P}^{2,x}_{\hat{m}\times\hat{m'}}\mathbf{C}^{r+1}\Psi - \mathbf{W}^{x,2}_{\hat{m}\times\hat{m'}}\mathbf{C}^{r+1}\Psi = \Psi^T\mathbf{C}^{r+1}\mathbf{P}^{2,t}_{\hat{m}\times\hat{m'}} + 2u_r\mathbf{P}^{2,x}_{\hat{m}\times\hat{m'}}\mathbf{C}^{r+1}\mathbf{P}^{2,t}_{\hat{m}\times\hat{m'}} - 2u_r\mathbf{P}^{2,x}_{\hat{m}\times\hat{m'}}\mathbf{C}^{r+1}\mathbf{P}^{2,t}_{\hat{m}\times\hat{m'}} + \Psi^T\mathbf{M}\mathbf{P}^{2,t}_{\hat{m}\times\hat{m'}}.$$

Above equation can be written as Sylvester equation, which is given as:-

$$\mathbf{vQC}^{r+1} - \mathbf{C}^{r+1}\mathbf{R} = \mathbf{vS},$$

where $\mathbf{Q} = \mathbf{P}^{2,x}_{\hat{m}\times\hat{m'}} - \mathbf{W}^{x,2}_{\hat{m}\times\hat{m'}}$, $\mathbf{v} = (\Psi^T + 2u_r\mathbf{P}^{2,x}_{\hat{m}\times\hat{m'}} - 2u_r\mathbf{W}^{x,2}_{\hat{m}\times\hat{m'}})^{-1}$, $\mathbf{R} = \mathbf{P}^{2,t}_{\hat{m}\times\hat{m'}}\Psi^{-1}$ and $\mathbf{S} = \Psi^T\mathbf{M}\mathbf{P}^{2,t}_{\hat{m}\times\hat{m'}}\Psi^{-1}$. By solving above equation we get the solution $u_{r+1}(x,t)$. The results obtained by this method are shown below:

x	t	E_{HAAR}	E_{CAS}	E_{abs}
0.1	0.2	0.0000	6.9388e-18	-6.9388e-18
0.1	0.4	0.0000	6.9388e-18	-6.9388e-18
0.1	0.6	6.9389e-18	0.0000	6.9388e-18
0.1	0.8	0.0000	0.0000	0.0000
0.5	0.2	0.0000	0.0000	0.0000
0.5	0.4	0.0000	0.0000	0.0000
0.5	0.6	5.5511e-17	0.0000	5.5511e-17
0.5	0.8	1.3323e-15	0.0000	1.3323e-15
0.9	0.2	2.7756e-17	2.7755e-17	0.0000
0.9	0.4	5.5511e-17	5.5511e-17	0.0000
0.9	0.6	0.0000	1.1102e-16	-1.1102e-16
0.9	0.8	1.5543e-15	1.1102e-16	1.44328e-15

Table 4.4: Comparison of the approximate CAS wavelet solution with HAAR wavelet and Adomian decomposition method solution when $M = M'$=3 and $k = k'$=4.

Table 4.4 is used to compare our results with the results obtained from Haar wavelet method. Figure 4.2 shows the graph of exact solution, solution of CAS wavelet and their absolute errors at $r = 2, 5, 8, 10$ respectively.

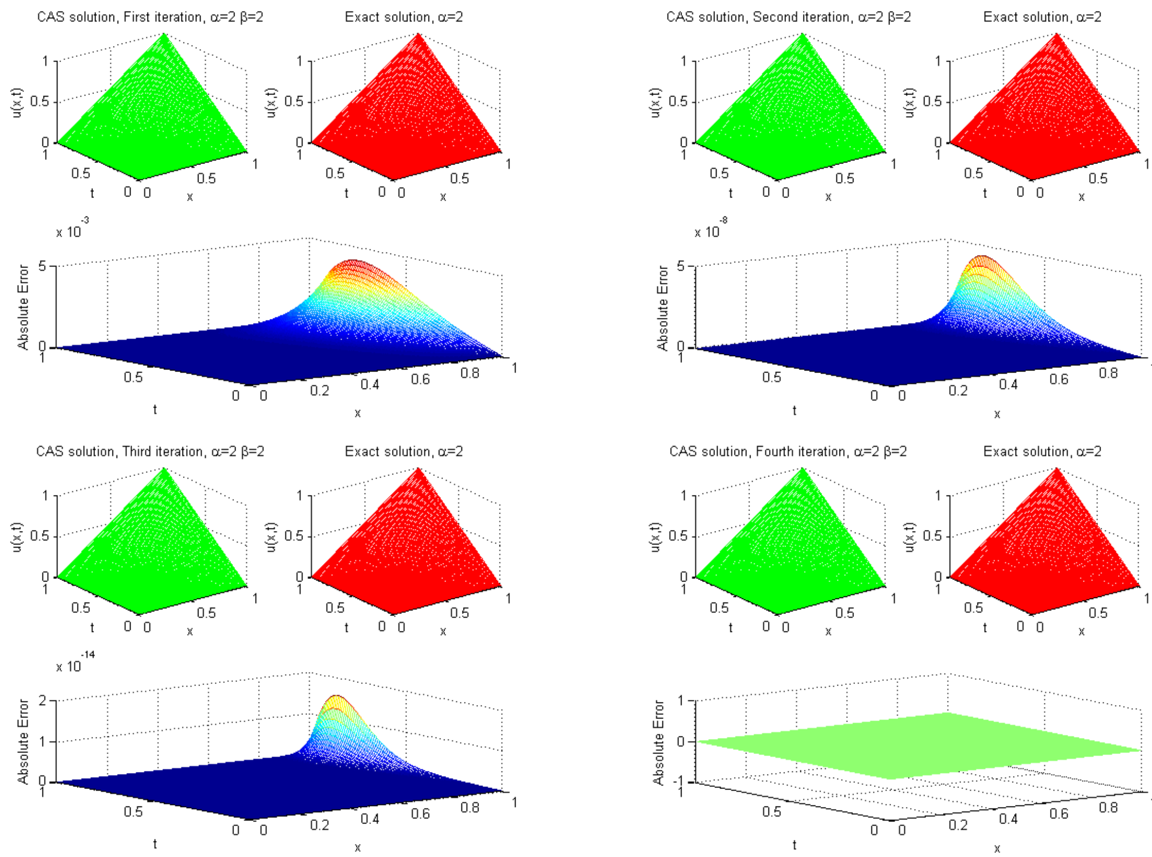

Figure 4.2: Comparison of exact and numerical solution of KleinGordon equation by CAS wavelet quasilinearization method when $M = M'$=3 and $k = k'$=4

Klein Gordon equation

Consider the following form of Klein Gordon equation

$$\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} + u^2 = -x\cos(t) + x^2\cos^2(t), \tag{4.4.12}$$

with the initial and boundary conditions as:-

$$u(x,0) = x, \qquad u_t(x,0) = 0, \qquad u(0,t) = 0, \qquad u(1,t) = \cos(t),$$

where exact solution is

$$u(x,t) = x\cos(t).$$

Apply quasilinearization technique to (4.4.12), we get:

$$\frac{\partial^2 u_{r+1}}{\partial t^2} - \frac{\partial^2 u_{r+1}}{\partial x^2} = x^2t^2 + u_r^2 - 2u_r u_{r+1}, \tag{4.4.13}$$

with initial and boundary conditions as:-

$$u_{r+1}(x,0) = x, \qquad (u_{r+1})_t(x,0) = 0, \qquad u_{r+1}(0,t) = 0, \qquad u_{r+1}(1,t) = \cos(t),$$

where $r = 0, 1, 2, ..., N$. At this stage, we approximate the higher order term in equation (4.4.13) by CAS wavelet series as:

$$\frac{\partial^2 u_{r+1}}{\partial x^2} = \sum_{n=0}^{2^k-1} \sum_{m=-M}^{M} \sum_{i=0}^{2^{k'}-1} \sum_{j=-M'}^{M'} {c_{nm,ij}}^{r+1} \psi_{n,m}(x)\psi_{i,j}(t) = \Psi^T(x)C^{r+1}\Psi(t).$$

The integration along with the initial conditions yields:

$$u_{r+1}(x,t) = I_x^2\Psi^T(x)C^{r+1}\Psi(t) + x\cos(t) - xI_{x=1}^2\Psi^T(x)C^{r+1}\Psi(t). \tag{4.4.14}$$

After putting equation (4.4.14) in (4.4.13), we have:

$$\frac{\partial^2 u_{r+1}}{\partial t^2} - \Psi^T(x)C^{r+1}\Psi(t) = -x\cos(t) + x^2\cos^2(t)$$
$$+u_r^2 - 2u_rI_x^2\Psi^T(x)C^{r+1}\Psi(t) - 2u_rx\cos(t) + 2u_rxI_{x=1}^2\Psi^T(x)C^{r+1}\Psi(t),$$

we do following substitution for simplification $G = 2u_rxt + x^2t^2 + u_r^2$, where $G \simeq \Psi^T(x)M\Psi(t)$, after doing integration and by putting the initial conditions, we get:

$$u_{r+1}(x,t) = \Psi^T(x)C^{r+1}I_t^2\Psi(t) + 2u_rI_x^2\Psi^T(x)C^{r+1}I_t^2\Psi(t) - 2u_rxI_{x=1}^2\Psi^T(x)C^{r+1}I_t^2\Psi(t)$$
$$+\Psi^T(x)MI_t^2\Psi(t) + xt. \tag{4.4.15}$$

By Equating equation (4.4.14) and (4.4.15), we get:

$$I_x^2\Psi^T(x)C^{r+1}\Psi(t) + x\cos(t) - xI_{x=1}^2\Psi^T(x)C^{r+1}\Psi(t) = \Psi^T(x)C^{r+1}I_t^2\Psi(t)$$
$$+2u_rI_x^2\Psi^T(x)C^{r+1}I_t^2\Psi(t) - 2u_rxI_{x=1}^2\Psi^T(x)C^{r+1}I_t^2\Psi(t) + \Psi^T(x)MI_t^2\Psi(t) + xt.$$

By rearranging and simplification, we have:

$$(I_x^2\Psi^T(x) - xI_{x=1}^2\Psi^T(x))(C^{r+1}\Psi(t)) - (\Psi^T(x) + 2u_rI_x^2\Psi^T(x) - 2u_rxI_{x=1}^2\Psi^T(x))(C^{r+1}I_t^2\Psi(t))$$
$$= (\Psi^T(x)MI_t^2\Psi(t) + xt - x\cos(t)).$$

Suppose $K(x,t) = xt - x\cos(t)$ and by putting collocation points $x_i = \frac{2i-1}{2*\hat{m}}$ and $t_j = \frac{2j-1}{2*\hat{m}}$ in above equation which is further written as, where $i = 1, 2, 3, ..., \hat{m}$, $j = 1, 2, 3, ..., \hat{m'}$, $\hat{m} = 2^k(2M+1)$ and $\hat{m'} = 2^{k'}(2M'+1)$.

$$(I_x^2\Psi^T(x_i) - x_iI_{x=1}^2\Psi^T(x_i))(C^{r+1}\Psi(t_j)) - (\Psi^T(x_i) + 2u_rI_x^2\Psi^T(x_i) - 2u_rx_iI_{x=1}^2\Psi^T(x_i))(C^{r+1}I_t^2\Psi(t_j))$$
$$= (\Psi^T(x_i)MI_t^2\Psi(t_j) + K(x_i,t_j)).$$

Further can be written in matrix form as:-

$$(\mathbf{P}^{2,x}_{\hat{m}\times\hat{m}'} - \mathbf{W}^{x,2}_{\hat{m}\times\hat{m}'})\mathbf{C}^{r+1}\Psi - (\Psi^T + 2u_r\mathbf{P}^{2,x}_{\hat{m}\times\hat{m}'} - 2u_r\mathbf{W}^{x,2}_{\hat{m}\times\hat{m}'})\mathbf{C}^{r+1}\mathbf{P}^{2,t}_{\hat{m}\times\hat{m}'} = \Psi^T\mathbf{M}\mathbf{P}^{2,t}_{\hat{m}\times\hat{m}'} + \mathbf{K}. \tag{4.4.16}$$

Now, we make supposition to write the equation (4.4.16) as Sylvester equation. $\mathbf{Q} = \mathbf{P}^{2,x}_{\hat{m}\times\hat{m}'} - \mathbf{W}^{x,2}_{\hat{m}\times\hat{m}'}$, $\mathbf{v} = (\Psi^T + 2u_r\mathbf{P}^{2,x}_{\hat{m}\times\hat{m}'} - 2u_r\mathbf{W}^{x,2}_{\hat{m}\times\hat{m}'})^{-1}$, $\mathbf{R} = \mathbf{P}^{2,t}_{\hat{m}\times\hat{m}'}\Psi^{-1}$ and $\mathbf{S} = (\Psi^T\mathbf{M}\mathbf{P}^{2,t}_{\hat{m}\times\hat{m}'} + \mathbf{K})\Psi^{-1}$. Now, we have:

$$\mathbf{v}\mathbf{Q}\mathbf{C}^{r+1} - \mathbf{C}^{r+1}\mathbf{R} = \mathbf{v}\mathbf{S}.$$

By solving above equation, we obtain values of $\mathbf{C}^{r+1}$, which is used to get solution $u_{r+1}(x,t)$. The obtained solution is shown in Table 4.5 below:

x	t	E_{HAAR}	E_{CAS}	E_{abs}
0.1	0.3	9.9999e-009	2.2579e-008	7.5790e-009
0.1	0.7	1.0000e-008	2.8176e-008	1.7176e-008
0.1	0.9	0.0000	2.2747e-008	7.5253e-008
0.5	0.4	1.9999e-008	8.4851e-008	5.5851e-008
0.5	0.6	2.0000e-008	1.0101e-008	9.8990e-009
0.5	0.8	2.9999e-008	2.2166e-007	1.9166e-007
0.7	0.1	3.0000e-008	1.4447e-007	1.1447e-007
0.7	0.5	7.0000e-008	2.6317e-007	1.9317e-007
0.7	0.9	1.9999e-008	3.1837e-008	1.1838e-008
0.9	0.3	9.0000e-008	1.6256e-007	7.2560e-008
0.9	0.6	3.0000e-008	2.8354e-008	1.6459e-009
0.9	0.9	2.0000e-008	5.3055e-008	3.3055e-008

Table 4.5: Comparison of the approximate solution of time derivative Klein Gordon equation.

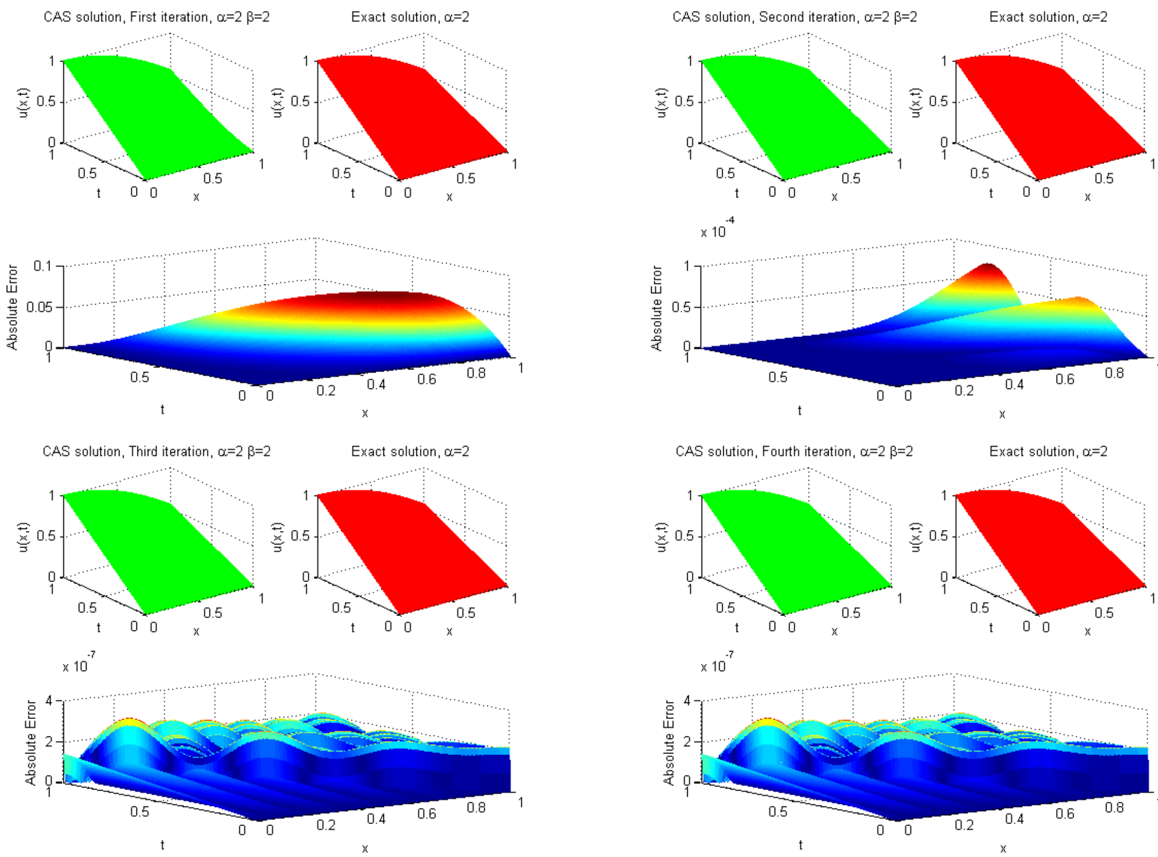

Figure 4.3: Comparison of exact and numerical solution of time derivative Klein Gordon equation by CAS wavelet quasilinearization method when $M = M'$=8 and $k = k'$=5

The results computed as shown in Table 4.5 at $J = 9$ for Haar wavelet, and $M = M' = 8, k = k' = 5$ for CAS wavelet. Figure 4.3 is used to plot the obtained solution by present method at $M = M' = 8$, $k = k' = 5$ and at $r = 1, 2, 3$and 4 respectively.

It is shown that the CAS wavelet quasilinearization technique gives excellent results when applied to different nonlinear partial differential equations. Error by the CAS wavelet quasilinearization technique reduces while iterations increase, as mentioned in Figure 4.1-4.3. The results obtained from the CAS wavelet quasilinearization technique are better from the results obtained by other methods and are in good agreement with exact solutions, as shown in Tables. Different type of nonlinearities can easily be handled by the CAS wavelet quasilinearization technique.

Conclusions

Many numerical methods are implemented to get the solution of nonlinear partial differential equation but till now there is no general technique. The objective of this thesis is to develop new numerical methods and their supporting analysis for solving linear and nonlinear partial differential equations. We have proposed two methods. First method is the combination of CAS wavelet and Picard technique and the second is the combination of CAS wavelet and quasilinearization technique.

In Chapter 1, a brief introduction of differential equations, few numerical methods used in past to solve nonlinear partial differential equations are provided. The equations used in Chapter 2, Chapter 3 and Chapter 4 are explained in chapter 1. Wavelets and wavelet transforms are explained. Few famous wavelets are discussed as well as few wavelet methods are provided which were used to solve partial differential equations.

Linear partial differential equations like wave, diffusion and Klein-Gordon equations are solved using CAS wavelets in chapter 2. We have derived and constructed the CAS wavelets matrix,$\Psi_{\hat{m}\times\hat{m}}$, and the CAS wavelets operational matrix of $N-th$ order integration, $P^{N}_{\hat{m}\times\hat{m}}$, and CAS wavelets operational matrix of integration for boundary value problems, $W^{g,N}_{\hat{m}\times\hat{m}}$. The error analysis for CAS wavelet is also explained. The solution provided in Tables (2.1-2.3) shows that the result by CAS wavelet is more accurate as compared to Adomian decomposition method and variational iteration method. The Figures (fig111-fig131) shows that the solution by CAS wavelets converges to exact solution when k, k', M and M' increases. Which shows that by CAS wavelet we can solve linear partial differential equations easily.

Chapter 3 contains some information about Burger's-Huxley equation. CAS wavelets and picard technique are briefly explained. CAS wavelet matrices for integration and integration for boundary value problem are successfully utilized to solve equations like Burger's-Huxley equation, Burger's equation, generalized Burger Fisher equation and Klein-Gordon equation. According to the Table (3.1-3.2), our results are more accurate as compared to variational iteration method and Adomian decomposition method. In Table (3.3), the numerical solution is compared with exact solution of Burger's equation. Figures (3.1-3.2) shows the solution by present method which converges to the exact solution while increasing r. It is shown that present method gives excellent results when applied to Burger's-Huxley equation and Burger's equation.

In Chapter 4, solution of different differential equations by CAS wavelet quasilinearization method is provided. Quasilinearization technique is explained. Method of implementation and error analysis of the method are explained. From Tables (4.1-4.3), it is clear that results by present methods are more close to the exact solution for generalized Burger Fisher equation. The Table (4.4) explains comparison of solutions of Klein-Gordon equation by Haar wavelet method, Adomian decomposition method and present method, while in Table (4.5), CAS wavelet solution is compared with Haar wavelet solution. It is clearly shown that present method gives better results when applied to generalized Burger-Fisher equation and the klein Gordon equation.

Different types of non-linearities can easily be handled by the present methods. The solution obtained by CAS wavelet picard/ quasilinearization method is more accurate and efficient.

Bibliography

[1] D. Greenspan. Introduction to partial differential equations. *Courier Corporation;* 1961.

[2] K. S. Rao. Introduction to partial differential equations. *PHI Learning Pvt. Ltd.*; 2010.

[3] R. Polikar. The story of wavelets. *Physics and modern topics in mechanical and electrical engineering.* 1999: 192-7.

[4] C. F. Chen, C. H. Hsiao. Haar wavelet method for solving lumped and distributed-parameter systems. *IEE Proceedings-Control Theory and Applications.* 1997 Jan; 144(1): 87-94.

[5] G. Hariharan, K. Kannan. Haar wavelet method for solving some nonlinear parabolic equations. *Journal of mathematical chemistry.* 2010 Nov 1; 48(4): 1044-61.

[6] U. Saeed, M ur Rehman. Haar wavelet Picard method for fractional nonlinear partial differential equations. *Applied Mathematics and Computation.* 2015 Aug 1; 264: 310-22.

[7] M. H. Heydari, M. R. Hooshmandasl, F. Mohammadi. Legendre wavelets method for solving fractional partial differential equations with Dirichlet boundary conditions. *Applied Mathematics and Computation.* 2014 May 15; 234: 267-76.

[8] N. Liu, E. B. Lin. Legendre wavelet method for numerical solutions of partial differential equations. *Numerical Methods for Partial Differential Equations.* 2010 Jan 1; 26(1): 81-94.

[9] L. Pahlavan, C. Kassapoglou, Z. Grdal. Spectral formulation of finite element methods using Daubechies compactly-supported wavelets for elastic wave propagation simulation. *Wave Motion.* 2013 Apr 30; 50(3): 558-78.

[10] Y. Liu, Y. Liu, Z. Cen. Multi-scale Daubechies wavelet-based method for 2-D elastic problems. *Finite Elements in Analysis and Design.* 2011 Apr 30; 47(4): 334-41.

[11] M.H. Heydari, M.R. Hooshmandasl, F.M. Ghaini. A new approach of the Chebyshev wavelets method for partial differential equations with boundary conditions of the telegraph type. *Applied Mathematical Modelling.* 2014 Mar 1; 38(5): 1597-606.

[12] A. Kilicman and Z. A. A. Al Zhour, Kronecker operational matrices for fractional calculus and some applications, *Applied Mathematics and Computation.* 187 (2007) 250–265.

[13] U. Saeed. CAS Picard method for fractional nonlinear differential equation. *Applied Mathematics and Computation.* 2017 Aug 15;307:102-12.

[14] S. Momani, Z. Odibat. Analytical approach to linear fractional partial differential equations arising in fluid mechanics. *Physics Letters A.* 2006 Jul 10; 355(4): 271-9.

[15] X. Y. Wang. Nerve propagation and wall in liquid crystals. *Physics Letters A.* 1985 Nov 11; 112(8): 402-6.

[16] W. Xin-Yi. Brochard-Leger wall in liquid crystals. *Physical Review A.* 1986 Dec 1; 34(6): 5179.

[17] Z. Wu. Dynamical knot and shape parameter setting for simulating shock wave by using multi-quadric quasi-interpolation. *Engineering analysis with boundary elements.* 2005 Apr 30; 29(4): 354-8.

[18] H. Danfu and S. Xufeng. Numerical solution of integro-differential equations by using CAS wavelet operational matrix of integration. *Applied mathematics and computation.* 194.2 (2007): 460-466.

[19] X. Y. Wang, Z. S. Zhu, and Y. K. Lu. Solitary wave solutions of the generalised Burgers-Huxley equation. *Journal of Physics A: Mathematical and General* 23.3 (1990): 271.

[20] P. G. Estevez, Non-classical symmetries and the singular manifold method: the Burgers and the Burgers-Huxley equations. *Journal of Physics A: Mathematical and General* 27.6 (1994); 2113.

[21] A. Molabahrami, and F. Khani. The homotopy analysis method to solve the BurgersHuxley equation. *Nonlinear Analysis: Real World Applications* 10.2 (2009): 589-600.

[22] M. Javidi, A numerical solution of the generalized BurgersHuxley equation by spectral collocation method. *Applied Mathematics and Computation.* 178.2 (2006): 338-344.

[23] M. A. Abdou, and A. A. Soliman. Variational iteration method for solving Burger's and coupled Burger's equations. *Journal of Computational and Applied Mathematics* 181.2 (2005): 245-251.

[24] S. Kutluay, A. R. Bahadir, and A. zde. Numerical solution of one-dimensional Burgers equation: explicit and exact-explicit finite difference methods. *Journal of Computational and Applied Mathematics* 103.2 (1999): 251-261.

[25] E.S. Lee. Quasilinearization And Invarint Imbedding. *Academic Press New York and London* 1968.

[26] R. E. Bellman and R. E. Kalaba, Quasilinearization and Nonlinear Boundary-Value Problems. *Elsevier Publishing Company, New York* 1965.

[27] I. Hashim, M. S. M. Noorani, M. S. Al-Hadidi. Solving the generalized BurgersHuxley equation using the Adomian decomposition method. *Mathematical and Computer Modelling.* 2006 Jun 30; 43(11): 1404-11.

[28] B. Batiha, M. S. M. Noorani, I. Hashim. Application of variational iteration method to the generalized BurgersHuxley equation. *Chaos, Solitons and Fractals.* 2008 May 31; 36(3): 660-3.

[29] Y. C. Hon, X. Z. Mao. An efficient numerical scheme for Burgers' equation. *Applied Mathematics and Computation.* 1998 Sep 1; 95(1): 37-50.

[30] U. Saeed. CAS Picard method for fractional nonlinear differential equation. *Applied Mathematics and Computation.* 2017 Aug 15; 307: 102-12.

[31] H. Saeedi, M. M. Moghadam, N. Mollahasani, G. N. Chuev. A CAS wavelet method for solving nonlinear Fredholm integro-differential equations of fractional order. *Communications in Nonlinear Science and Numerical Simulation.* 2011 Mar 31; 16(3): 1154-63.

[32] H. Saeedi and M. M. Moghadam. Numerical solution of nonlinear Volterra integro-differential equations of arbitrary order by CAS wavelets. *Communications in Nonlinear Science and Numerical Simulation* 16.3 (2011): 1216-1226.

[33] A. Barzkar, P. Assari, and M. A. Mehrpouya. Application of the CAS Wavelet in Solving Fredholm-Hammerstein Integral Equations of the Second Kind with Error Analysis 1. (2012).

[34] M. M. Shamooshaky, P. Assari and H. Adibi, CAS wavelet method for the numerical solution of boundary integral equations with logarithmic singular kernels, *International Journal of Mathematical Modelling and Computations* Vol. 04, No. 04, Fall 2014, 377387.

[35] D. Kocacoban, et al. A better approximation to the solution of Burger-Fisher equation. *Proceedings of the World Congress on Engineering.* Vol. 1. 2011.

[36] H. N. Ismail, K. Raslan, A. A. A. Rabboh. Adomian decomposition method for Burger'sHuxley and Burger'sFisher equations. *Applied Mathematics and Computation* 159(1); 291-301; (2004).