Some Methods of Solution for Nonlinear Differential Equations

By

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Dedicated to

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Abstract

The field of nonlinear differential equations has great importance in science and technology. It is increasing day by day in the diverse fields such as economics, space flight and astronomy. The purpose of this thesis is to study a few methods to solve nonlinear ordinary differential equations and partial differential equations.

The perturbation, Adomian decomposition, homotopy analysis methods for the solution of nonlinear ODEs are described and are applied to a few examples. Furthermore, the method of successive approximations for non linear initial value problem of first order and the boundary value problems for second order nonlinear differential equations is described in detail. We introduced a general case of the Riccati equation, and solved it numerically. The solutions are compared and the results are plotted graphically.

Some theorems are described which are helpful to establish existence of the unique solution of differential equations in an interval. Moreover, we shall find bounds on the solution.

To find the exact solution of nonlinear partial differential equations we shall explain the tanh and the sine-cosine methods and use them to solve Kortweg-de Vries equation and nonlinear dispersive equation, respectively.

Finally, existence of the solutions for boundary value problems is discussed via contraction mapping theorem, lower and upper solutions.

Contents

Chapter 1

Introduction

The study of differential equations was initiated in 1675, when Gottfried Wilhelm von Leibniz wrote the equation [8]:

$$\int x dx = \frac{1}{2}x^2.$$

The search for general methods of integrating differential equations began with Isaac Newton classification of first order differential equation into three classes two of which contain ordinary derivatives with respect to a single independent variable, of one or more dependent variables and are known as ordinary differential equations. Third class contain partial derivative, therefore called partial differential equations.

Real progress in this direction started with the work of two great men in science, Henri Poincare and Alexander Mikhailovich Liapunov. Most of the modern theory of differential equations is based on work of these two mathematicians. Poincare dealt primarily with the geometric properties of solutions, along with with some techniques for the computation of special solutions. On the other hand, Liapunov generally sought as much information as possible about the stability properties of solutions without knowing explicit solutions.

There are two types of approximate solutions, numerical and analytic solution. As far as numerical solutions are concerned, they require a lot of work to get a general view of the solution of the problem. Each solution corresponds to a set of values of the parameter present in the problem. Presence of singular points or multiple solutions, causes additional difficulties. Numerical methods may provide a solution valid over whole domain. Analytic solution provides a series which may converge in a smaller domain. Analytic solutions give overview of the solution in short time and there is no need to carry large calculations repeatedly. That is why both techniques are helpful to analyze a nonlinear problem.

1.1 Perturbation Method

The most widely used method to find analytic approximate solutions of nonlinear problems is the perturbation method. It has been used to study various problems employed in different branches of mathematics, physics, and technology, see for example references [15, 16]. Using perturbation method a nonlinear problem is replaced by an infinite number of linear equations. This significant feature is common to most analytic techniques.

1.2 Adomian Decomposition Method

Adomian decomposition method is an improved form of the artificial small parameter method [1-2]. First of all a solution of non linear operator equation is decomposed into a series of functions i.e.

$$y = \sum_{n=0}^{\infty} y_n. \tag{1.1}$$

Then non linear term $f(y)$ is re-arranged as a series of B_n Adomian polynomials,

$$f(y) = \sum_{n=0}^{\infty} B_n, \tag{1.2}$$

where

$$\begin{aligned}
B_0 &= f(y_0),\\
B_1 &= y_1 f'(y_0),\\
B_2 &= y_2 f'(y_0) + \frac{y_1^2}{2!} f''(y_0),\\
B_3 &= y_3 f'(y_0) + y_1 y_2 f''(y_0) + \frac{y_1^3}{3!} f'''(y_0),\\
B_4 &= y_4 f'(y_0) + \frac{1}{2!}(2y_1y_3 + y_2^2) f''(y_0\\
&+ \frac{1}{2!} y_1^2 y_2 f^{(3)}(y_0) + \frac{y_1^4}{4!} f^{(4)}(y_0),\\
&\vdots
\end{aligned}$$

Components y_n for $n = 0, 1, 2, \ldots,$ of the solution of the under consideration equation can be successively calculated.

1.3 Homotopy Analysis Method

Liao has developed homotopy analysis method (HAM) which is more advantageous than previous techniques as it has some free parameters which can be selected to match the solution with the exact solution. Thus analytic expression is obtained which on comparison with earlier methods describes the physical problem over wider domain [13].

Assume that the series upto m terms is obtained with help of Adomian decomposition method, for a problem comes out to be

$$y_m(x) = \sum_{n=0}^{m} b_n x^n. \tag{1.3}$$

If the linear operator in homotopy analysis method (HAM) is taken the same as in the Adomian method, the solution produced by HAM is

$$y_m(x,h) = \sum_{n=0}^{m} \mu_{m,n}(h) b_n x^n. \tag{1.4}$$

The approaching function $\mu_{m,n}(h)$ is defined as

$$\mu_{m,n}(h) = (-h)^n \sum_{k=0}^{m-n} \binom{n-1+k}{k} (1+h)^k, \tag{1.5}$$

where $h \in (-2, 0)$. The value of h can be chosen so that Eq. (1.4) gives a closer approximation to the exact solution as compared with Eq. (1.3). This idea was proposed by F. Ahmad to get arbitrary sets of approaching function which produce polynomial approximation for arbitrary analytic functions [3].

1.4 Successive Approximation Method

R. Bellman introduced the idea of qausilinearization that solution to certain nonlinear differential equations can be represented by maximum operation applied to the solution of associated linear differential equations [4]. This method provides an explicit approach for calculating approximate solutions to non linear differential equations. When the function involved is convex, it gives point-wise lower approximations of the solution of given problem. Using two properties, i.e. , monotonicity and fast convergence it can be used as basis of a successive approximation method.

1.5 The Tanh and Sine-Cosine Methods

Suppose we have to solve a nonlinear PDE with the help of tanh method. Then introducing wave variable ξ changes this partial differential equation into an ordinary differential equation. The ODE obtained is integrated until all terms contain derivative. A new independent variable $Z = \tanh(\mu\xi)$ is introduced which changes derivatives and using these derivatives and series $S(Z)$ in integrated ODE, an equation is obtained in powers of Z. To determine the highest power of Z, m, linear term of highest order is balanced with highest order nonlinear term of resulting equation. With the help of this value of m, coefficients of powers of Z are collected which have to vanish. In this way, a system of equations involving μ, c, a_k is obtained. Solving this system gives analytic solution $v(x,t)$.

In order to solve a nonlinear problem using the sine-cosine method, wave variable ξ changes nonlinear PDE into nonlinear ODE. The resulting ODE is integrated until all terms contain derivative. The solutions can be written in the form

$$v(x,t) = \begin{cases} \{\lambda \sin^{\alpha}[\nu(x-ct)]\}, & \text{if } |x-ct| \leq \frac{\pi}{\mu}, \\ 0 & \text{otherwise.} \end{cases} \tag{1.6}$$

or

$$v(x,t) = \begin{cases} \{\lambda \cos^{\alpha}[\nu(x-ct)]\}, & \text{if } |x-ct| \leq \frac{\pi}{2\mu}, \\ 0 & \text{otherwise.} \end{cases} \tag{1.7}$$

here λ, α, ν are unknowns to be determined and can be obtained using the values of $v(x,t)$ and its derivatives in simplified ODE.

1.6 Existence of Solution of Boundary Value Problems

Second order, ordinary differential equations and associated boundary value problems, have been widely studied in the literature since Newton's time, not only due to its intrinsic mathematical importance, but also due to the huge variety of phenomena in their nature they may be used to model.

A problem may occur about BVP's that they may or may not have a solution. Therefore there is a need to discuss the existence of solution of BVP's.

1.7 Plan of the Thesis

In this dissertation, we discuss a few methods to solve non linear ODE's, PDE's and a method to find upper and lower solutions in an interval. The dissertation consists of seven chapters.

In Chapter 1 introduction to the basic concepts of perturbation, Adomian decomposition, homotopy analysis, successive approximation, tanh methods.

Chapter 2 presents three examples of the application of the perturbation method. This is done to provide a base for the introduction of non perturbative methods.

Chapter 3 deals with the Adomian's decomposition method and include some examples of this method.

In chapter 4 we discuss the homotopy analysis method. We apply this method to an initial value problem and then to the Lane Emden equation. Approximate solutions are found for different values of the free parameter h.

In chapter 5 we will use quasilinearization technique [4] for some equations like the Riccati equation. The successive approximation method will be applied to solve a special form of the Riccati equation and second order nonlinear differential equation. We developed successive approximation method for a general case of the Riccati equation. Also upper or lower bounds will be sorted out using these methods.

Chapter 6 will explain the tanh and the sine-cosine method for solving nonlinear PDE's. "Kortweg-de Vries equation" model for water waves is investigated by the tanh method for different values of n. Nonlinear dispersive equation will be solved through sine-cosine method for different values of m and n.

Finally, in chapter 7 we discuss the contraction mapping theorem and its applications to boundary value problems. We use the theory to solve some examples. Further with the help of lower and upper solutions of a BVP in a given interval, unique solution can be found.

Chapter 2

Perturbation methods

2.1 Perturbation Series

Consider a nonlinear problem of the form

$$R[y] = 0, \tag{2.1}$$

where R is a nonlinear operator. First of all try to identify a small parameter, usually denoted by ϵ and then arrange Eq. (2.1) as follows

$$Qy + \epsilon R_0 y = 0. \tag{2.2}$$

Here $\epsilon R_0 y$ is called a perturbation term. Then

$$Qy = 0, \tag{2.3}$$

is a simpler equation than Eq. (2.1) and whose solution can be found easily. Here Q is linear operator in case of differential equation and nonlinear operator for polynomial equation. Moreover, Eq. (2.3) is free of ϵ. Let y_0 be a solution of Eq. (2.3). Assume that the solution of Eq. (2.2) can be expressed as under

$$y = y_0 + \epsilon y_1 + \epsilon^2 y_2 + \epsilon^3 y_3 + ... \tag{2.4}$$

Equation (2.4) is known as perturbation series for y. By substituting Eq. (2.4) in Eq. (2.2) and arranging the terms in such a way that a series in ascending powers of ϵ is obtained.

$$P_0(y_0) + \epsilon P_1(y_0, y_1) + \epsilon^2 P_2(y_0, y_1, y_2) + ... = 0. \tag{2.5}$$

Now we can obtain an infinite set of equations, by comparing coefficients of each power of ϵ to zero.

$$\begin{aligned}
P_0(y_0) &= 0, \\
P_1(y_0, y_1) &= 0, \\
P_2(y_0, y_1, y_2) &= 0, \\
&\vdots
\end{aligned}$$

The above system of equations can be solved successively.

2.2 Applications

In this section we solve some equations using perturbation method. A simple polynomial equation in Example (1), then a nonlinear differential equation (DE) in Example (2) and finally duffing equation in Example (3) is solved with the help of perturbation method.

Example 1

Consider the equation

$$y^3 - (2+\epsilon)y^2 + 6 + 3y = 0. \tag{2.6}$$

Let

$$Qy = y^3 - 2y^2 + 3y + 6 = 0. \tag{2.7}$$

Then Eq. (2.6) can be written as

$$Qy + \epsilon R_0 y = 0, \tag{2.8}$$

where

$$R_0 y = -y^2. \tag{2.9}$$

The equation (2.7) has a solution

$$y = -1.$$

Substituting Eq. (2.4) in Eq. (2.6), We get

$$(-1 + \epsilon y_1 + \epsilon^2 y_2 + ...)^3 - (2+\epsilon)(-1 + \epsilon y_1 + \epsilon^2 y_2 + ...)^2 + 6 + 3(-1 + \epsilon y_1 + \epsilon^2 y_2 + ...) = 0,$$

which can be simplified as

$$(10y_1 - 1)\epsilon + (3y_1^2 + 4y_2 - 2y_1^2 + 2y_1 + 3y_2)\epsilon^2 + \ldots = 0. \tag{2.10}$$

Equating coefficients of ϵ and ϵ^2 to zero in Eq. (2.10), we obtain

$$10y_1 - 1 = 0, \tag{2.11}$$

$$3y_1^2 + 4y_2 - 2y_1^2 + 2y_1 + 3y_2 = 0. \tag{2.12}$$

This gives

$$y_1 = 1/10,$$

$$y_2 = -3/70.$$

Hence the approximate solution is as follows

$$y = -1 + \frac{1}{10}\epsilon - \frac{3}{70}\epsilon^2 + \ldots. \tag{2.13}$$

Let $\epsilon = 0.5$, then the exact solution obtained is -0.953472, ignoring imaginary solution. Substitution of value of ϵ in Eq. (2.13) gives $y = -0.960714$.

Example 2

Consider

$$\frac{dy}{dx} + y + \epsilon y^2 = 0,\ y(0) = 1. \tag{2.14}$$

Let

$$Qy = \frac{dy}{dx} + y = 0. \tag{2.15}$$

Then Eq. (2.14) can be written as

$$Qy + \epsilon y^2 = 0. \tag{2.16}$$

The solution of Eq. (2.15) is .

$$y_0 = ce^{-x}. \tag{2.17}$$

After substituting Eq. (2.4) in Eq. (2.14), we obtain

$$(\dot{y}_0 + \epsilon\dot{y}_1 + \epsilon^2\dot{y}_2 + \ldots) + (y_0 + \epsilon y_1 + \epsilon^2 y_2 + \ldots) + \epsilon(y_0 + \epsilon y_1 + \epsilon^2 y_2 + \ldots)^2 = 0, \tag{2.18}$$

which can be simplified as

$$\dot{y}_0 + y_0 + \epsilon(\dot{y}_1 + y_1 + y_0^2) + \epsilon^2(\dot{y}_2 + y_2 + 2y_0y_1) + \ldots = 0. \tag{2.19}$$

Equating coefficients of ϵ and ϵ^0 in Eq. (2.19), gives

$$\dot{y_0} + y_0 = 0, \tag{2.20}$$

$$\dot{y_1} + y_1 + y_0^2 = 0. \tag{2.21}$$

General solution of (2.20) is Eq. (2.17). Substitution of Eq. (2.17) in (2.21), gives

$$\dot{y_1} + y_1 = -c^2 e^{-2x}, \tag{2.22}$$

This is a linear equation and easy to solve. Particular solution of this equation is

$$y_1 = c^2 e^{-2x}.$$

Substitute value of y_0 and y_1 in (2.4)

$$y = ce^{-x} + \epsilon c^2 e^{-2x} + \ldots. \tag{2.23}$$

Let $\epsilon = \frac{1}{4}$, then exact solution is $y = \frac{4}{-1+5e^x}$. Take $c = 0.83$ which satisfies the initial condition in Eq. (2.14), then Eq. (2.23) becomes

$$y = 0.83e^{-x} + 0.17e^{-2x} + \ldots. \tag{2.24}$$

The exact and approximate solutions are compared in Figure 2.1.

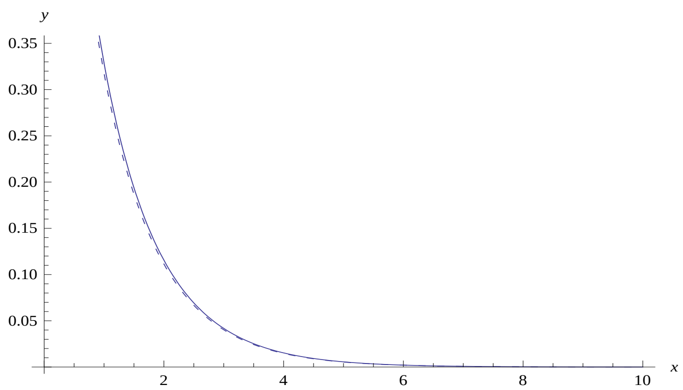

Figure 2.1: Comparison of exact solution (dotted line) with the solution by perturbation method (solid line)

Example 3

Consider the Duffing equation with initial conditions

$$\frac{d^2y}{dt^2} + y + \epsilon y^3 = 0, \; y(0) = a, \; \frac{dy(0)}{dt} = 0. \tag{2.25}$$

Let

$$Qy = \frac{d^2y}{dt^2} + y = 0. \tag{2.26}$$

Re-writing the equation (2.25), we get

$$Qy + \epsilon y^3 = 0.$$

The solution of Eq. (2.26) is

$$y_0 = a\cos(t+\beta). \tag{2.27}$$

Substituting Eq. (2.4) in (2.25), we obtain

$$(\ddot{y_0} + \epsilon\ddot{y_1} + \epsilon^2\ddot{y_2} + \cdots) + (y_0 + \epsilon y_1 + \epsilon^2 y_2 + \cdots) + \epsilon(y_0 + \epsilon y_1 + \epsilon^2 y_2 + \cdots)^3 = 0,$$

which can be written as

$$\ddot{y_0} + y_0 + \epsilon(\ddot{y_1} + y_1 + y_0^3) + \epsilon^2(\ddot{y_2} + y_2 + 3y_0^2 y_1) + \cdots = 0. \tag{2.28}$$

Comparing coefficients of ϵ^0 and ϵ in Eq. (2.28)

$$\ddot{y_0} + y_0 = 0, \tag{2.29}$$

$$\ddot{y_1} + y_1 + y_0^3 = 0. \tag{2.30}$$

The general solution of Eq. (2.29) is Eq. (2.27). Substitution of this value in Eq. (2.30), gives

$$\ddot{y_1} + y_1 = -a^3\cos^3(t+\beta). \tag{2.31}$$

which is a linear equation and can be easily solved. The particular solution of Eq. (2.31) is

$$y_1 = -\frac{3}{8}a^3 t\cos(t+\beta) + \frac{a^3}{32}\cos 3(t+\beta). \tag{2.32}$$

In Eq. (2.32), y_1 have secular term (first term) due to which the solution is unbounded. Lindsteht Poincare method is used to overcome this problem. Let $\tau = wt$ where constant w depends on ϵ. Therefore Eq. (2.25) becomes

$$w^2\frac{d^2y}{d\tau^2} + y + \epsilon y^3 = 0.$$

or equivalently

$$w^2 y'' + y + \epsilon y^3 = 0. \tag{2.33}$$

Here both y and w are unknown. To obtain approximate solution for them in the form of power series of ϵ, let

$$y = y_0(\tau) + \epsilon y_1(\tau) + \epsilon^2 y_2(\tau) + \cdots . \tag{2.34}$$

$$w = 1 + \epsilon w_1 + \epsilon^2 w_2 + \cdots . \tag{2.35}$$

Substitution of Eqs. (2.34) and (2.35) in Eq. (2.33), gives

$$(1+\epsilon w_1+\epsilon^2 w_2+\cdots)^2(y_0''+\epsilon y_1''+\epsilon^2 y_2''+\cdots)+(y_0+\epsilon y_1+\epsilon^2 y_2+\cdots)+\epsilon(y_0+\epsilon y_1+\epsilon^2 y_2+\cdots)^3 = 0,$$

which can be simplified as

$$y_0''+y_0+\epsilon(2w_1 y_0''+y_1''+y_1+y_0^3)+\epsilon^2(2w_1 y_1''+w_1^2 y_0''+y_2+3y_0^2 y_1+y_2''+2w_2 y_0''+\cdots = 0. \tag{2.36}$$

In Eq. (2.36), equating coefficients of ϵ^0 and ϵ to zero, we get

$$y_0'' + y_0 = 0, \tag{2.37}$$

$$y_1'' + y_1 = -y_0^3 - 2w_1 y_0''. \tag{2.38}$$

The general solution of Eq. (2.37) is

$$y_0 = a\cos(\tau+\beta). \tag{2.39}$$

With the help of Eq. (2.39), Eq. (2.38) becomes

$$y_1'' + y_1 = -a^3\cos^3(\tau+\beta) + 2w_1 a\cos(\tau+\beta). \tag{2.40}$$

The particular solution of Eq. (2.40) is

$$y_1 = \frac{1}{2}\left(2w_1 a - \frac{3}{4}a^3\right)\tau\sin(\tau+\beta) + \frac{1}{32}a^3\cos 3(\tau+\beta). \tag{2.41}$$

Note that the particular solution y_1 contains mixed-secular term (first term of Eq. (2.41)), which makes the expansion non-uniform. For a uniform expansion, we take

$$2w_1 a - \frac{3}{4}a^3 = 0,$$

which gives

$$w_1 = \frac{3}{8}a^2. \tag{2.42}$$

Eq. (2.42) causes the secular term to disappear and solution becomes uniform. Similarly w_2 is chosen by comparing the coefficient of ϵ^2 in Eq. (2.36). Then Eq. (2.41) becomes

$$y_1 = \frac{1}{32}a^3\cos 3(\tau+\beta). \tag{2.43}$$

Substitution of Eqs. (2.39) and (2.43) into Eq. (2.34), gives

$$y = a\cos(\tau+\beta) + \frac{1}{32}a^3\cos 3(t+\beta) + \dots . \tag{2.44}$$

Now using Eq. (2.42) into Eq. (2.35), we obtain

$$w = 1 + \frac{3}{8}\epsilon a^2 + \dots . \tag{2.45}$$

Since

$$\tau = wt = (1 + \frac{3}{8}\epsilon a^2 + \cdots)t$$

Rewriting Eq. (2.44) as

$$y = a\cos\left[(1+\frac{3}{8}\epsilon a^2)t + \beta\right] + \frac{1}{32}a^3\epsilon\cos\left[3(1+\frac{3}{8}\epsilon a^2)t + 3\beta\right] + \dots, \tag{2.46}$$

which is a uniform solution. Hu and Tang considered the problem of a cubic quintic Duffing equation [7]

$$\frac{d^2u}{dt^2} + u + \epsilon u^3 + \lambda u^5 = 0,\ u(0) = a,\ u'(0) = 0.$$

They found the period of solution as

$$T = \frac{2\pi}{\Omega},$$

where

$$\Omega = \frac{2048 + 3072\epsilon a^2 + 1104\epsilon^2 a^4 + 2640\lambda^4 + 1860\epsilon\lambda a^6 + 775\lambda^2 a^8}{2048 + 1536\epsilon a^2 + 1360\lambda a^4}. \tag{2.47}$$

The expression (2.46) is valid for small values of a, but Eq. (2.47) holds for small as well as large values of the amplitude. In 2012 Khan found an asymptotic solution of the Duffing equation and used it to find a simple uniform expression for the period of the Duffing oscillator [9].

Chapter 3

Adomian Decomposition Method

3.1 Adomian Polynomials

The perturbation method is only applicable when a small parameter is present in the problem. However most nonlinear problems do not contain such a parameter so we require alternative methods to solve such problems. George Adomian introduced a new technique in 1980's, which is known as Adomian decomposition method (ADM) [1, 2].

Consider a nonlinear problem governed by operator equation

$$N[y(x)] = f(x), \tag{3.1}$$

here N is a nonlinear operator and y is unknown function to be determined. Suppose it is possible to divide N into two parts, i.e. ,

$$N = L_0 + N_0, \tag{3.2}$$

where L_0 and N_0 are linear and nonlinear operator, respectively, and L_0^{-1} exists. Using Eq.(3.2), Eq. (3.1) becomes

$$L_0[y(x)] + N_0(y(x)) = f(x). \tag{3.3}$$

With the help of Adomian decomposition method, solution $y(x)$ of the problem (3.1) can be written as

$$y(x) = y_0(x) + \sum_{n=1}^{\infty} y_n(x) \tag{3.4}$$

In Eq. (3.4), y_0 is the solution of linear problem $L_0 y = f$, i.e. ,

$$y(x) = L_0^{-1}[f(x)]. \tag{3.5}$$

The solution of nonlinear problem is

$$y_n(x) = -L_0^{-1}[B_{n-1}(x)], \tag{3.6}$$

for $n = 1,\ 2, \ldots.$ B_n are Adomian polynomials defined by

$$B_n(x) = \frac{1}{n!}[\frac{\partial^n}{\partial p^n} N_0[y_0(x) + \sum_{n=1}^{\infty} y_n(x)p^n]]|_{p=0}. \tag{3.7}$$

3.2 Applications

In example (1) Adomian decomposition method is applied to a function then to a nonlinear DE in example (2) and finally in example (3) Lane-Emden type equation is solved using ADM.

Example 1 Consider a function to elaborate the method,

$$f(y) = y^3. \tag{3.8}$$

Let

$$y = y_0 + py_1 + p^2y_2 + \ldots. \tag{3.9}$$

Substitution of Eq. (3.9) in Eq. (3.8), gives

$$\begin{aligned} f(y,p) &= (y_0 + py_1 + p^2y_2 + \ldots)^3 \\ &= \underbrace{y_0^3}_{\mathrm{B}_0} + \underbrace{3y_0^2y_1}_{\mathrm{B}_1} p + \underbrace{3y_0y_1^2 + 3y_0^2y_2}_{\mathrm{B}_2} p^2 \\ &\quad + \underbrace{(y_1^3 + 6y_0y_1y_2 + 3y_0^2y_3)}_{\mathrm{B}_3} p^3 + \underbrace{(3y_1^2y_2 + 3y_0y_2^2 + 6y_0y_1y_3 + 3y_0^2y_4)}_{\mathrm{B}_4} p^4 + \ldots. \end{aligned}$$

Now put $p = 1$, we get

$$f(y,1) = f(y) = \sum_{i=1}^{\infty} B_i, \tag{3.10}$$

where B_i are called Adomian polynomials and

$$
\begin{aligned}
B_0 &= y_0^3, \\
B_1 &= 3y_0^2 y_1, \\
B_2 &= 3y_0 y_1^2 + 3y_0^2 y_2, \\
B_3 &= y_1^3 + 6y_0 y_1 y_2 + 3y_0^2 y_3, \\
B_4 &= 3y_1^2 y_2 + 3y_0 y_2^2 + 6y_0 y_1 y_3 + 3y_0^2 y_4, \\
&\vdots
\end{aligned}
$$

Using values of B_i's in Eq. (3.10), the solution of Eq. (3.8) can be written as

$$y(x) = y_0^3 + 3y_0^2 y_1 + 3y_0 y_1^2 + 3y_0^2 y_2 + y_1^3 + 6y_0 y_1 y_2 + 3y_0^2 y_3 + 3y_1^2 y_2 + 3y_0 y_2^2 + 6y_0 y_1 y_3 + 3y_0^2 y_4 + \dots. \tag{3.11}$$

Example 2 Consider the initial value problem

$$y' = -2 - y + y^2, \; y(0) = 1. \tag{3.12}$$

Eq. (3.12) can be written in operator form as

$$L_0 y = -2 - y - y^2, \tag{3.13}$$

where L_o is a differential operator $\frac{d}{dx}$ and L_0^{-1} is integration defined by

$$L_0^{-1}(.) = \int_0^x (.)dx.$$

Applying L_0^{-1} to both sides of Eq. (3.13) and using initial condition given in Eq. (3.12), we get

$$y(x) = 1 + L_0^{-1}(-2 - y + y^2). \tag{3.14}$$

Comparing Eqs.(3.4) and (3.14)

$$y_0(x) = 1,$$

and the recursive relations for y_{k+1} are

$$y_{k+1}(x) = L_0^{-1}(B_k), \; k \geq 0. \tag{3.15}$$

Now we obtain Adomian polynomials for $f(y) = -2 - y - y^2$, using the same technique as in the previous example.

$$\begin{aligned}
B_0 &= -2 - y_0 + y_0^2, \\
B_1 &= -y_1 + 2y_0 y_1, \\
B_2 &= -y_2 + y_1^2 + 2y_0 y_2, \\
B_3 &= -y_3 + 2y_0 y_3 + 2y_1 y_2, \\
B_4 &= -y_4 + y_2^2 + 2y_1 y_3 + 2y_0 y_4, \\
&\vdots
\end{aligned}$$

Putting Adomian polynomials from above in Eq. (3.15) gives

$$\begin{aligned}
y_0(x) &= 1, \\
y_1(x) &= -2x, \\
y_2(x) &= -x^2, \\
y_3(x) &= x^3, \\
y_4(x) &= \frac{5}{4}x^4, \\
y_5(x) &= -\frac{7}{20}x^5, \\
&\vdots
\end{aligned}$$

Therefore solution of Eq. (3.12) is

$$y(x) = 1 - 2x - x^2 + x^3 + \frac{5}{4}x^4 - \frac{7}{20}x^5 + \dots. \tag{3.16}$$

The exact and approximate solutions are compared in Figure 3.1.

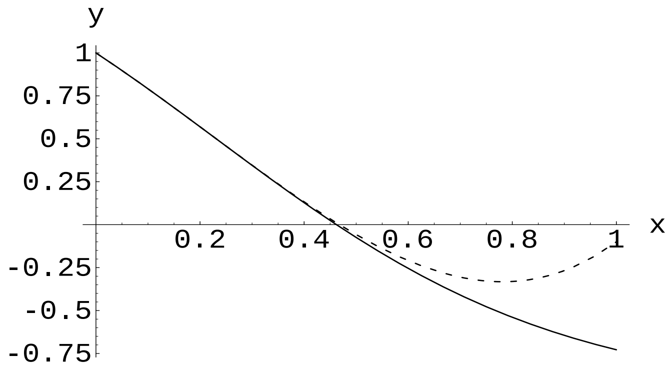

Figure 3.1: Comparison of exact solution, found with the help of mathematica(dotted line) with the solution by Adomian decomposition method (solid line).

Example 3 Consider Lane-Emden type equation

$$y'' + \frac{2}{x}y' + \cos y = 0, \tag{3.17}$$

along with initial conditions

$$y(0.0001) = 1, \ y'(0.0001) = 0. \tag{3.18}$$

Writing Eq. (3.17) in operator form becomes

$$L_0 y = -\cos y, \tag{3.19}$$

where $L_0 = x^{-2}\frac{d}{dx}(x^2\frac{d}{dx})$, and $L_0^{-1}(.) = \int_{0.0001}^{x} x^{-2} \int_{0.0001}^{x} x^2(.)dxdx$. Apply L_0^{-1} on both sides of Eq. (3.19) and substituting initial conditions, we get

$$y(x) = 1 - L_0^{-1}(\cos y). \tag{3.20}$$

Comparison of Eqs. (3.4) and (3.20), gives

$$y_0(x) = 1,$$

and

$$y_{k+1}(x) = -L_0^{-1}(B_k), \ k \geq 0. \tag{3.21}$$

Now we obtain Adomian polynomial for $f(y) = \cos y$, using the same technique as in previous example

$$\begin{aligned}
B_0 &= \cos y_0, \\
B_1 &= -y_1 \sin y_0, \\
B_2 &= -y_2 \sin y_0 - \frac{1}{2} y_1^2 \cos y_0, \\
B_3 &= -y_3 \sin y_0 - y_1 y_2 \cos y_0 + \frac{1}{6} y_1^3 \sin y_0, \\
B_4 &= -y_4 \sin y_0 - y_1 y_3 \cos y_0 - \frac{1}{2} y_2^2 \cos y_0 \\
&+ \frac{1}{2} y_1^2 y_2 \sin y_0 + \frac{1}{4!} y_1^4 \cos y_0, \\
&\vdots
\end{aligned}$$

Substituting values of B_n in Eq. (3.21), we get

$$\begin{aligned}
y_0(x) &= 1, \\
y_1(x) &= 2.70151 \times 10^{-9} - 1.80101 \times 10^{-13}/x - 0.0900504x^2, \\
y_2(x) &= 5.68311 \times 10^{-18} - 1.5155 \times 10^{-22}/x - 7.57748 \times 10^{-14}x + 3.78874 \times 10^{-10}x^2 \\
&- 0.00378874x^4 \\
y_3(x) &= 9.10316 \times 10^{-26} + 1.61005 \times 10^{-31}/x - 1.95203 \times 10^{-22}x + 1.12563 \times 10^{-18}x^2 \\
&- 4.5833 \times 10^{-15}x^3 + 9.36855 \times 10^{-12}x^4 - 0.0000237485x^6 + 8.7627 \times 10^{-27} Log[x]
\end{aligned}$$

The solution of Eq. (3.17) can be written as follows.

$$\begin{aligned}
y(x) &= 1 - 1.80101 \times 10^{-13}/x - 7.57748 \times 10^{-14}x - 0.0900504x^2 - 4.5833 \times 10^{-15}x^3 \\
&- 0.00378874x^4 - 0.0000237485x^6 + 8.7627 \times 10^{-27} Log[x] \ldots .
\end{aligned}$$

The exact and approximate solutions are compared in Figure 3.2.

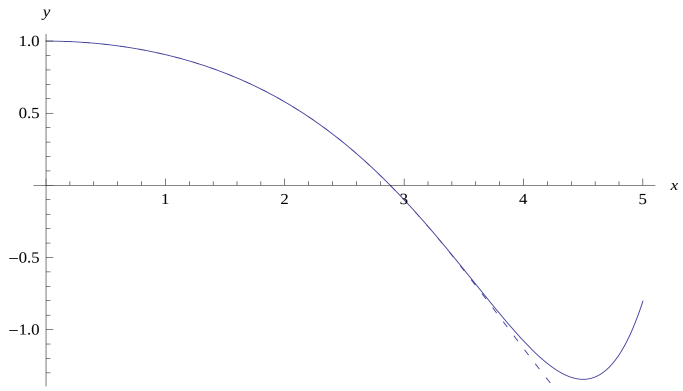

Figure 3.2: Comparison of exact solution, found with the help of mathematica(dotted line) with the solution by Adomian decomposition method (solid line).

Chapter 4

Homotopy Analysis Method

4.1 Liao's Method

HAM generalizes the basic ideas of artificial small parameter of Lyapunov and Adomian decomposition method [13]. This method has been widely used to solve analytically several nonlinear equations. For a given non-linear differential equation

$$N[y(x)] = f(x), \tag{4.1}$$

where N is a non linear operator and $y(x)$ is unknown function. After introducing a new variable p consider the equation

$$(1-p)L[\Phi(x,p) - y_0(x)] = hp\{N[\Phi(x,p)] - f(x)\}, \tag{4.2}$$

here L is a linear operator chosen suitably, $y_0(x)$ is an initial approximation to the solution and h is a parameter to be selected later. In Eq. (4.2) put $p = 0$, we get

$$\Phi(x,0) = y_0(x). \tag{4.3}$$

If we let $p = 1$ in Eq. (4.2) then we obtain

$$N\Phi(x,1) = f(x), \tag{4.4}$$

whose solution is the required function $y(x)$. As p varies continuously from 0 to 1, the initial approximation $y_0(x)$ evolves to the desired solution $y(x)$ of the problem under consideration. Let

$$y_n(x) = \frac{1}{n!}\frac{\partial^n}{\partial p^n}\Phi(x,p)|_{p=0}, \tag{4.5}$$

then

$$\Phi(x,p) = y_0(x) + \sum_{n=1}^{\infty} y_n(x)p^n \ . \tag{4.6}$$

At $p = 1$, if the series converges, then

$$y(x) = \Phi(x,1) = y_0(x) + \sum_{n=1}^{\infty} y_n(x). \tag{4.7}$$

We can successively find the functions $y_n(x)$ from the equations

$$L[y_1(x)] = h\{N[y_0(x)] - f(x)\}. \tag{4.8}$$

If $n = 1$ and

$$L[y_n(x) - y_{n-1}(x)] = hR_n(x). \tag{4.9}$$

If $n \geq 2$ with

$$R_n(x) = \frac{1}{(n-1)!}\frac{\partial^{n-1}}{\partial p^{n-1}} N[\Phi(x,p)]|_{p=0}. \tag{4.10}$$

4.2 Applications

In example (1) HAM is applied to a simple differential equation and then Lane-Emden equation in example (2) is solved with the help of homotopy analysis method.

Example 1 Given

$$y'' + 4y = 0, \tag{4.11}$$

$$y(0) = 0, \ y'(0) = 2, \tag{4.12}$$

whose exact solution is

$$y(x) = \sin 2x. \tag{4.13}$$

After introducing parameter h in this problem we get

$$(1-p)L[\Phi(x,p) - y_0(x)] = hpN(\Phi(x,p)), \tag{4.14}$$

or

$$(1-p)L[y_0 + py_1 + p^2 y_2 + \ldots - y_0] = hpN(y_0 + py_1 + p^2 y_2 + \ldots). \tag{4.15}$$

Let $y_0 = x + \sin x$, which satisfies initial conditions. The above relation give values of y_i's

$$y_1 = h[\frac{2}{3}x^3 - 3\sin x + 3x],$$

$$y_2 = 3hx - 9h^2x + h[\frac{2}{3}x^3 - 3\sin x] + h[\frac{8}{3}hx^3 + \frac{2}{15}hx^5 + 9h\sin x],$$

$$y_3 = 3hx - 9h^2x + 9h^2(-1+3h)x + h[\frac{2}{3}x^3 - 3\sin x] + h[\frac{8}{3}hx^3 + \frac{2}{15}hx^5 + 9h\sin x]$$
$$+ \frac{1}{315}h^2[2x^3\{21(20+x^2) + h(-525 + 105x^2 + 2x^4\} - 2835(-1+3h)\sin x],$$

$$\vdots$$

Using these values of y_i's, we get

$$y(x) = (1 + 9h - 27h^2 + 27h^3)x + \frac{2}{3}h(3 + 12h - 5h^2)x^3$$
$$+ \frac{2}{15}h^2(3+5h)x^5 + \frac{4}{315}h^3x^7 - (-1+3h)^3 Sin[x] + \ldots.$$

For different values of h a few graphs are given below

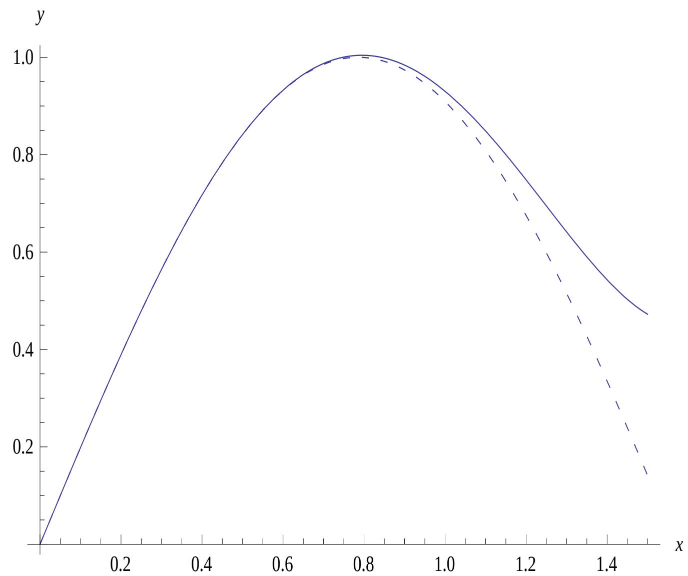

Figure 4.1: Comparison of exact solution (dotted line) with the solution by the homotopy analysis method (solid line) for $h = -1$.

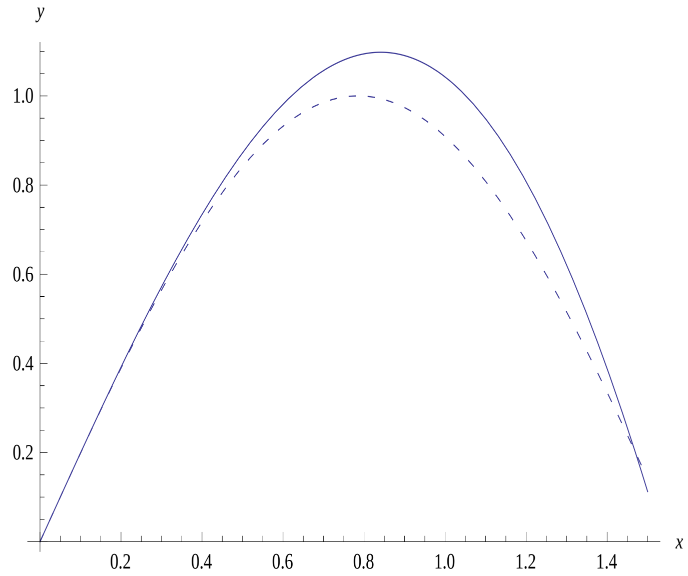

Figure 4.2: Comparison of exact solution (dotted line) with the solution by the homotopy analysis method (solid line) for $h = -0.5$.

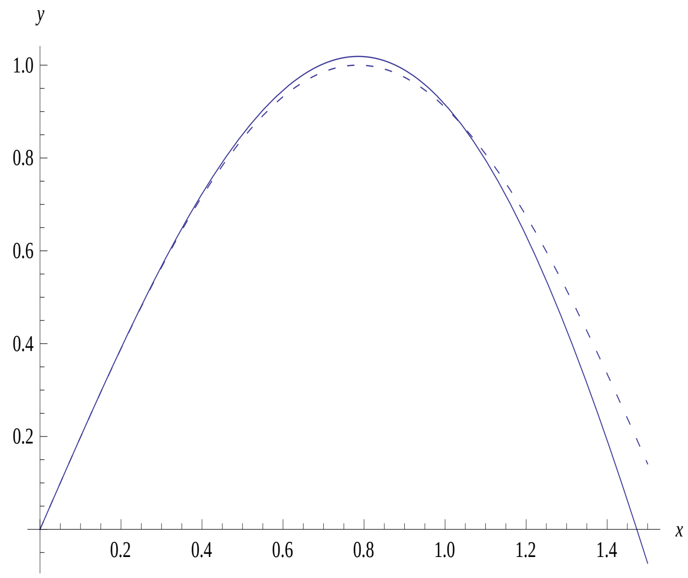

Figure 4.3: Comparison of exact solution (dotted line) with the solution by the homotopy analysis method (solid line) for $h = -0.7$.

Above graphs show that $h = -1$ gives better approximation.

Example 2

$$y'' + \frac{2}{x}y' + y = 0, \tag{4.16}$$

with $x > 0$, and initial conditions

$$y(0) = 1,\ y'(0) = 0, \tag{4.17}$$

whose exact solution is

$$y(x) = \frac{\sin x}{x}. \tag{4.18}$$

Let initial approximation be $y_0 = 1$, which satisfy the initial conditions in Eq. (4.17). Eq. (4.15) gives values of y_i's

$$y_1 = \frac{hx^2}{2},$$

$$y_2 = \frac{hx^2}{2} + h[\frac{3hx^2}{2} + \frac{hx^4}{24}],$$

$$y_3 = \frac{hx^2}{2} + h[\frac{3hx^2}{2} + \frac{hx^4}{24}] + \frac{1}{720}h^2x^2\{30(36 + x^2) + h(3240 + 140x^2 + x^4)\},$$

$$\begin{aligned} y_4 = & \frac{hx^2}{2} + h[\frac{3hx^2}{2} + \frac{hx^4}{24}] + \frac{1}{720}h^2x^2\{30(36 + x^2) \\ & + h(3240 + 140x^2 + x^4)\} + \frac{1}{604800}[h^2x^2\{25200(36 + x^2) + 1680h(3240 + 140x^2 + x^4) \\ & + h^2(8164800 + 422800x^2 + 5096x^4 + 15x^6)\}], \end{aligned}$$

$$\begin{aligned} y_5 \;=\; & \frac{hx^2}{2} + h[\frac{3hx^2}{2} + \frac{hx^4}{24}] \\ + \; & \frac{1}{720}h^2x^2\{30(36 + x^2) \\ + \; & h(3240 + 140x^2 + x^4)\} + \frac{1}{604800}[h^2x^2\{25200(36 + x^2) + 1680h(3240 + 140x^2 + x^4) \\ + \; & h^2(8164800 + 422800x^2 + 5096x^4 + 15x^6)\}] \\ + \; & h(\frac{3hx^2}{2} + \frac{27h^2x^2}{2} + \frac{81h^3x^2}{2} + \frac{81h^4x^2}{2} \\ + \; & \frac{hx^4}{24} + \frac{7h^2x^4}{12} + \frac{151h^3x^4}{72} + \frac{371h^4x^4}{162} \\ + \; & \frac{h^2x^6}{240} + \frac{91h^3x^6}{3600} + \frac{2843h^4x^6}{81000} \\ + \; & \frac{h^3x^8}{13440} + \frac{193h^4x^8}{1058400} + \frac{h^4x^{10}}{3628800}), \end{aligned}$$

$$
\begin{aligned}
y(x) &= 1+\frac{5}{2}hx^2+\frac{1}{6}h^2x^2(36+x^2)+1/240h^2x^2(30(36+x^2)+h(3240+140x^2+x^4)) \\
&+ (h^2x^2(25200(36+x^2)+1680h(3240+140x^2+x^4) \\
&+ h^2(8164800+422800x^2+5096x^4+15x^6)))/302400 \\
&+ h(1/24hx^2(36+x^2)+1/240h^2x^2(3240+140x^2+x^4) \\
&+ h^3((81x^2)/2+(151x^4)/72+(91x^6)/3600+x^8/13440) \\
&+ h^4((81x^2)/2+(371x^4)/162+(2843x^6)/81000+(193x^8)/1058400+x^{10}/3628800))+\ldots.
\end{aligned}
$$

Here are some graphs for different values of h

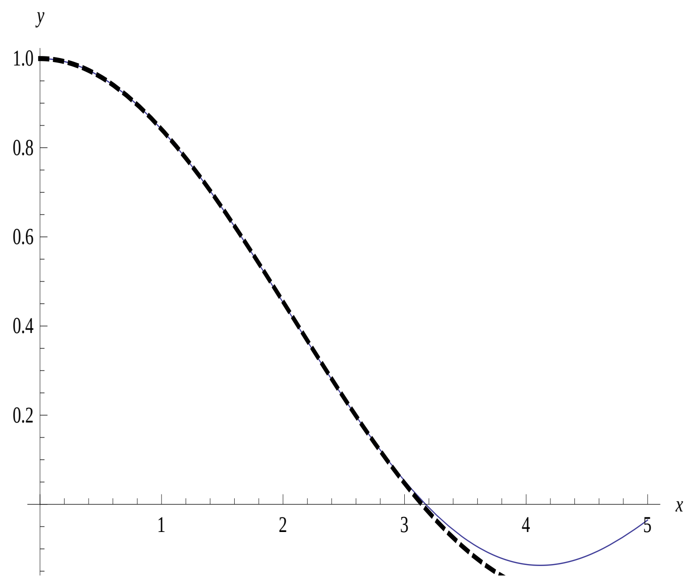

Figure 4.4: Comparison of exact solution (dotted line) with the solution by the homotopy analysis method (solid line) for $h=-\frac{1}{3}$.

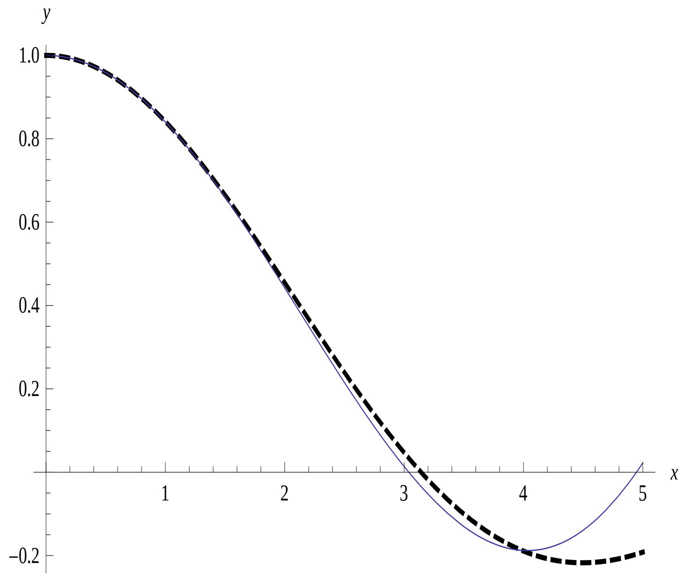

Figure 4.5: Comparison of exact solution (dotted line) with the solution by the homotopy analysis method (solid line) for $h = -1/4$.

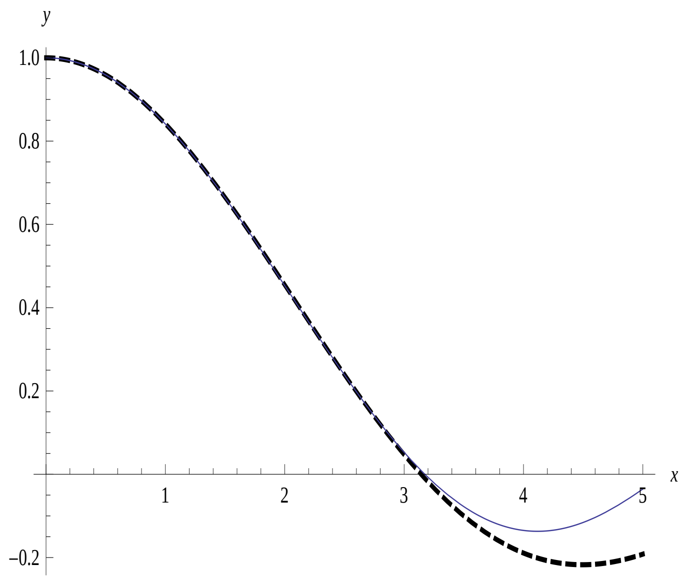

Figure 4.6: Comparison of exact solution (dotted line) with the solution by the homotopy analysis method (solid line) for $h = -1/2$.

The graphs above show that $h = -\frac{1}{3},\ \frac{1}{2}$ give almost same result.

Chapter 5

Successive Approximation Method

5.1 A Few Basic Results

In this chapter, we describe few basic results which are very useful to simplify nonlinear IVP's and BVP's.

5.1.1 A Basic Identity

This identity is very simple but it will be widely helpful to simplify quadrature functions.

$$y^2 = \max_x (2yx - x^2). \tag{5.1}$$

Proof: The proof is simple and obvious.

Consider

$$\begin{aligned} y^2 &= (x + y - x)^2, \\ &= x^2 + 2x(y - x) + (y - x)^2, \\ &\geq x^2 + 2x(y - x) = 2xy - x^2, \end{aligned}$$

equality holds only for $x = y$.

5.1.2 The Inequality $y' \leq a(x)y + f(x)$

Consider the first-order linear differential inequality.

$$\frac{dy}{dx} \leq a(x)y + f(x), \;\; y(0) = c, \tag{5.2}$$

which is easy to handle. Inequality (5.2) is equivalent to the following equation

$$\frac{dy}{dx} = a(x)y + f(x) - p(x), \;\; y(0) = c, \tag{5.3}$$

where $p(x)$ is some function satisfying the condition $p(x) \geq 0$, for $x \geq 0$. Associated equation of inequality (5.2) is

$$\frac{dz}{dx} = a(x)z + f(x), \;\; z(0) = c. \tag{5.4}$$

Our aim is to examine whether or not solution of associated equation and the set of functions satisfying inequality (5.2) have a relationship in the form of an inequality.

Theorem 5.1. *Let y be a function satisfying inequality (5.2) in the interval $[0, x_0]$. Then in this interval*

$$y(x) \leq z(x)$$

Proof: *The proof readily follows from the explicit representation of the solution of Eq. (5.3)*

$$\begin{aligned} y &= c\exp\left[\int_0^x a(x_1)dx_1\right] + \exp\left[\int_0^x a(x_1)dx_1\right] \times \{\int_0^x \exp\left[-\int_0^{x_1} a(s)ds\right]\left[f(x_1) - p(x_1)\right]dx_1\}, \\ &= z - \exp\left[\int_0^x a(x_1)dx_1\right]\int_0^x \exp\left[-\int_0^{x_1} a(s)ds\right]p(x_1)dx_1. \end{aligned}$$

Since $p \geq 0$, by assumption we get $y \leq z$.

Theorem 5.2. *If we have following relations*

a$\cdot$ $y'' + p(x)y' - q(x)y > 0, \;\; x \geq 0,$

b$\cdot$ $z'' + p(x)z' - q(x)z = 0, \;\; x \geq 0,$

c$\cdot$ $q(x) \geq 0, \;\; x \geq 0,$

d$\cdot$ $y(0) = z(0), \;\; y'(0) = z'(0),$

then $y > z$ for $x > 0$.

Proof: *Subtract equation (b) from inequality (a) gives*

$$w'' + p(x)w' - q(x)w > 0, \tag{5.5}$$

where $w = y - z$, with $w(0) = w'(0) = 0$. It follows that $w > 0$ in some initial interval $[0, x_0]$. Suppose that w becomes zero at some subsequent point x_1, and let x_1 be the first

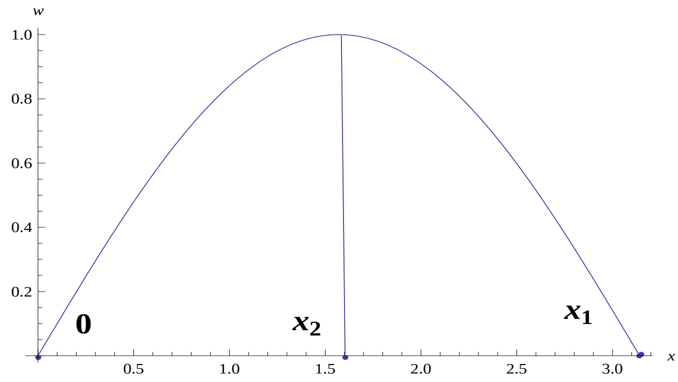

Figure 5.1: Local maximum is x_2 and x_1 is zero.

such point, as indicated in Figure 5.1. Then w has a local maximum at some intermediate point x_2. Using this point in inequality (5.5), we have

$$w'' + 0 - q(x_2)w > 0,$$

here $w'' \geq 0$. This contradicts our assumption that x_2 provides a local maximum.

5.1.3 The Riccati Equation

Consider the Riccati equation with an initial condition

$$\frac{dy}{dx} = y^2 + a(x), \ y(0) = c. \tag{5.6}$$

With the help of identity (5.1), Eq. (5.6) can be written in quasilinear form

$$\frac{dy}{dx} = \max_z [2yz - z^2 + a(x)]. \tag{5.7}$$

Due to maximum property Eq. (5.7) can be written as

$$\frac{dy}{dx} \geq [2yz - z^2 + a(x)],$$

$z(x)$ is any function. By Theorem 5.1, we can conclude that for any function z, we get the inequality

$$y \geq w(x, z), \tag{5.8}$$

where $w(x, z)$ is the solution of

$$\frac{dw}{dx} = [2wz - z^2 + a(x)], \ w(0) = c. \tag{5.9}$$

Explicit representation of Eq. (5.9) is as follows

$$w(x, z) = [c \exp(2 \int_0^x z ds) + \int_0^x \exp(2 \int_{x_1}^x z ds)[a(x_1) - z^2(x_1)dx_1].$$

5.1.4 Explicit Representation

Considering Eq. (5.8), one can conclude that the relation $y \geq w(x, z)$ holds for all z. Moreover equality holds for one function, $y = z$. Therefore, we can write

$$y = \max_z w(x, z).$$

This provides an explicit analytic representation in terms of the elementary operations of analysis and maximum operation for the Riccati equation [6].

5.2 Successive Approximation Method and its Properties

In Eq. (5.7) the function y itself maximizes on the right. This emphasize that a method of successive approximations should be used in which z is chosen at each step to get an estimate of the desired solution y. Let y_0 be an initial approximation to y and calculate y_1 from the equation

$$y_1' = 2y_1y_0 - y_0^2 + a(x), \; y_1(0) = c. \tag{5.10}$$

After getting y_1 in this way, let y_2 be determined from the equation

$$y_2' = 2y_2y_1 - y_1^2 + a(x), \; y_2(0) = c. \tag{5.11}$$

Continuing in the same fashion, we can construct the sequence of functions $\{y_n\}$ where

$$y_{n+1}' = 2y_{n+1}y_n - y_n^2 + a(x), \; y_{n+1}(0) = c. \tag{5.12}$$

The resulting sequence will be monotonically increasing

$$y_1 \leq y_2 \leq y_3 \leq \ldots \leq y_n \leq y_{n+1} \leq \ldots, \tag{5.13}$$

and bounded by the common bound for sequence $y(x)$,

$$y_1 \leq y_2 \leq y_3 \leq \ldots \leq y_n \leq y_{n+1} \leq \ldots \leq y(x). \tag{5.14}$$

We have to show that

$$y_n(x) \leq y(x), \tag{5.15}$$

for $n = 1, 2, \ldots$. This relation shows that $\{y_n\}$ is well defined in any interval $[0, x_0]$ within which the solution $y(x)$ exists. To demonstrate this relation, for any n,

$$(y - y_n)^2 \geq 0, \tag{5.16}$$

or

$$y^2 \geq 2yy_n - y_n^2. \tag{5.17}$$

Hence

$$y' = y^2 + a(x) \geq 2yy_n - y_n^2 + a(x). \tag{5.18}$$

Comparing inequality (5.18) and Eq. (5.12), and using Theorem (5.1) we get $y \geq y_{n+1}$ in interval of existence of solution $y(x)$.

Now to check whether Eq. (5.13) relation holds or not, we can see

$$y_n' = 2y_n y_{n-1} - y_{n-1}^2 + a(x) \leq 2y_n y_n - y_n^2 + a(x). \tag{5.19}$$

Comparison of Eq. (5.19) and Eq. (5.12) gives $y_n \leq y_{n+1}$, which is our required inequality. The sequence $\{y_n(x)\}$ is monotonically increasing and uniformly bounded.

Therefore we can conclude that it is a convergent sequence for $x \in [0, x_0]$. It is important to note that whatever the the interval of existence of solution $y(x)$ of Eq. (5.9), this will be interval of convergence of $\{y_n(x)\}$. Also $\{y_n(x)\}$ has maximum interval of convergence of any sequence obtained using successive approximation method.

Example 1

Consider the IVP

$$y' = y^2 + 1, \; y(0) = 1. \tag{5.20}$$

With help of identity (5.1), we can write

$$y' = \max_z [2yz - z^2 + 1], \; y(0) = 1, \tag{5.21}$$

the function y is maximized. We shall use method of successive approximation in which z is chosen at each stage to be an initial guess of the desired solution.

Let y_0 be an initial approximation to y and y_1 can be found by the equation

$$y_1' = 2y_1 y_0 - y_0^2 + 1, y_1(0) = 1. \tag{5.22}$$

Take $y_0 = 1$. Then Eq. (5.22) becomes,

$$y_1' = 2y_1 - 1 + 1, y_1(0) = 1, \tag{5.23}$$

which gives value of $y_1 = e^{2x}$. For calculating y_2, we have to solve following equation

$$y_2' = 2y_2e^{2x} - e^{4x} + 1, y_2(0) = 1, \tag{5.24}$$

Solving equations in similar fashion, we can clearly see the increasing behavior of the sequence $\{y_i\}$ in Fig. 5.2. Curves for y_3 and y_{ex} are not distinguishable. It is also clear that the sequence is bounded.

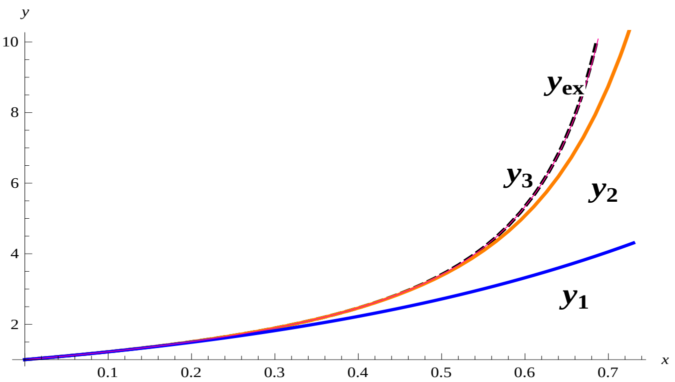

Figure 5.2: Comparison of curves for y_1, y_2 and y_3 with exact solution.

Example 2

Consider the differential equation

$$y' = y^2 + \sin x, \tag{5.25}$$

with an initial condition

$$y(0) = 1.$$

Let y_0 be an initial approximation to y and y_1 can be found by the equation

$$y_1' = 2y_1y_0 - y_0^2 + \sin x,\ y_1(0) = 1. \tag{5.26}$$

Let $y_0 = 1$. Eq. (5.26) becomes,

$$y_1' = 2y_1 - 1 + \sin x,\ y_1(0) = 1, \tag{5.27}$$

which gives value of y_1 on solving. Putting this value of y_1, we get equation in which y_2 is to be determined. Such equations can be solved in similar fashion. We can clearly see the increasing behavior of the sequence $\{y_i\}$ in Fig. 5.3. It is also clear that the sequence is bounded.

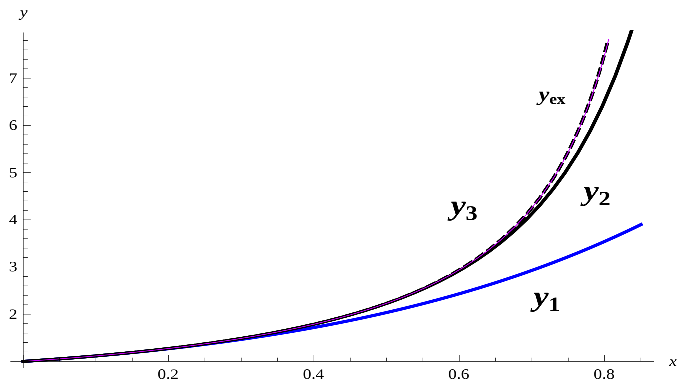

Figure 5.3: Comparison of curves for y_1, y_2 and y_3 with exact solution.

Here at third iteration the solution coincide with the exact one.

5.3 A Generalization to the Successive Approximation Method in 5.2

In this section the successive approximation method is applied to a general form of above discussed Riccati equation as a new achievement. Consider equation

$$\frac{dy}{dx} = y^2 f(x) + a(x), \; y(0) = c, \tag{5.28}$$

where $f(x)$ is continuous over the whole domain. Moreover, $f(x) > 0$ in $[0, x_0]$ (which is interval of existence of solution of Eq. (5.28)).

Proof: With help of identity (5.1), we can write Eq. (5.28) as under

$$y' = \max_z [2yz - z^2] f(x) + a(x). \tag{5.29}$$

Due to maximum property we can write Eq. (5.29) as follows

$$\frac{dy}{dx} \geq [(2yz - z^2) f(x) + a(x)], \; y(0) = c. \tag{5.30}$$

Note that, when we write

$$y' = \max_z (2yz - y^2) f(x) + a(x), \; y(0) = c. \tag{5.31}$$

then the maximum function on right is y itself. It strongly recommends that we use a method of successive approximation in which, we choose z at each stage to be an initial guess of the desired solution y. Let y_0 be an initial guess of y and find y_1 by the equation

$$y_1' = (2y_1y_0 - y_0^2)f(x) + a(x),\ y_1(0) = c. \tag{5.32}$$

In this way y_1 is obtained, let y_2 can be found by the equation

$$y_2' = (2y_2y_1 - y_1^2)f(x) + a(x),\ y_2(0) = c. \tag{5.33}$$

Working in a similar way, we can obtain the sequence of functions $\{y_n\}$ where

$$y_{n+1}' = (2y_{n+1}y_n - y_n^2)f(x) + a(x),\ y_{n+1}(0) = c. \tag{5.34}$$

We have to establish the basic result that the sequence obtained is monotonically increasing and has a bound i.e. $y(x)$. To show that (5.15) holds for $n = 1, 2, \ldots$ i.e $\{y_n\}$ is bounded and well defined in $[0, x_0]$. Consider Eq. (5.17) for any n, then

$$y^2 f(x) \geq (2y_ny_n - y_n^2)f(x), \tag{5.35}$$

$$y' = y^2 f(x) + a(x) \geq (2y_ny_n - y_n^2)f(x) + a(x),\ y_n(0) = c. \tag{5.36}$$

Comparing Eqs. (5.34) and (5.36), we get that relation (5.15) holds. Now to show monotonically increasing behavior of the sequence, consider

$$y_n' = (2y_ny_{n-1} - y_{n-1}^2)f(x) + a(x) \leq (2y_ny_n - y_n^2)f(x) + a(x). \tag{5.37}$$

Since

$$(y_n - y_{n-1}^2) \geq 0 \Rightarrow 2y_ny_{n-1} - y_{n-1}^2 \leq 2y_ny_n - y_n^2, \tag{5.38}$$

and $f(x) > 0$. Comparison of Eqs. (5.34) and (5.37) give the required inequality, which shows that sequence is monotonically increasing.

Example 3

Consider IVP

$$y' = y^2x^2 + x,\ y(0) = 0. \tag{5.39}$$

Let y_0 be an initial approximation to y and y_1 can be found from the equation

$$y_1' = (2y_1y_0 - y_0^2)x^2 + x,\ y_1(0) = 0. \tag{5.40}$$

Let $y_0 = 1$. Eq. (5.40) becomes

$$y_1' = (2y_1 - 1)x^2 + x,\ y_1(0) = 0, \tag{5.41}$$

which gives value of y_1. Solving equations in similar fashion, we can clearly see the increasing behavior of the sequence $\{y_n\}$ in Fig. 5.4. It is also clear that the sequence is bounded.

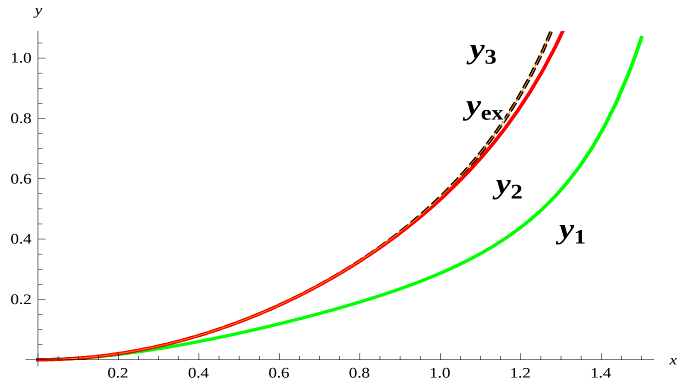

Figure 5.4: Curve for y_3 and exact solution y_{ex} coincide.

Example 4

Consider

$$y' = y^2 x^6 + x^2, \tag{5.42}$$

with an initial condition

$$y(0) = 1.$$

Let y_0 be an initial approximation to y and y_1 can be found from the equation

$$y_1' = (2y_1 y_0 - y_0^2)x^6 + x, \; y_1(0) = 1. \tag{5.43}$$

Let $y_0 = 1$. Eq. (5.43) becomes

$$y_1' = (2y_1 - 1)x^6 + x^2, \; y_1(0) = 1, \tag{5.44}$$

which gives value of y_1. Solving equations in similar fashion, we can clearly see the increasing behavior of the sequence $\{y_n\}$ in Fig. 5.5. It can be seen that the sequence is bounded.

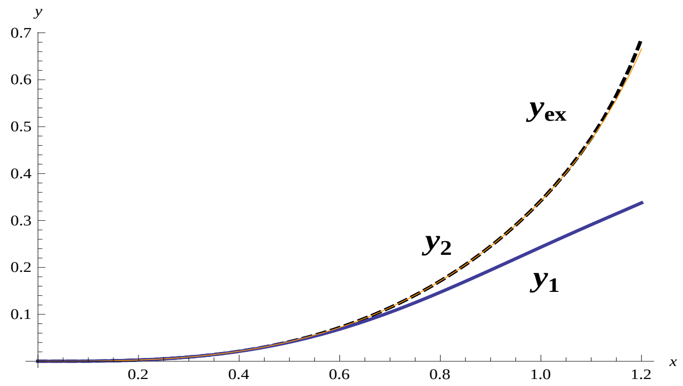

Figure 5.5: Curve for y_2 and curve for exact solution coincide.

5.4 Second Order Boundary Value Problem

Consider the equation of the form

$$y'' = f(x, y), \tag{5.45}$$

subject to boundary condition

$$y(0) = c_1, \ y(T) = c_2. \tag{5.46}$$

Let us start with the case when we have established existence of a unique solution by some other means. Moreover assume that $f(x, y)$ is strictly convex function as a function of y for $x \in [0, T]$. If a function is strictly convex and twice differentiable, then we can write[10]

$$y'' = \max_z [f(x, z) + (y - z)f_y(x, z)], \tag{5.47}$$

where

$$y'' \geq [f(x, z) + (y - z)f_y(x, z)], \tag{5.48}$$

for any z. Particularly, we have

$$y'' \geq f(x, y_0) + (y - y_0)f_y(x, y_0). \tag{5.49}$$

We want to compare this inequality with the equation for y_1, which is as under

$$y_1'' = f(x, y_0) + (y_1 - y_0)f_y(x, y_0), \tag{5.50}$$

subject to the same boundary conditions as in Eq. (5.46)

$$y_1(0) = c_1, \ y_1(T) = c_2. \tag{5.51}$$

The comparison above depend upon the properties of the equation

$$w'' - f_y(x, y_0)w = g(x), \tag{5.52}$$

subject to the boundary conditions $w(0) = w(T) = 0$, since this equation is satisfied by $w = y - y_1$ with a positive forcing function. If this equation has non negative associated Green's function, then we get

$$y - y_1 > 0. \tag{5.53}$$

Moreover, to demonstrate the fact that y_1 is uniquely determined by Eq. (5.50), we shall use this property of Green's function. For this nonnegativity, there is a sufficient condition, $f_y > 0$ for all y. We need a condition such as

$$f_y > -\lambda_1(T), \tag{5.54}$$

here the smallest characteristic value is $\lambda_1(T)$ associated with the Sturm-Liouville equation

$$w'' + \lambda w = 0,\ w(0) = w(T) = 0. \tag{5.55}$$

where $\lambda_1(T) = \pi^2/T^2$. Continuing in a similar way, we can show monotonically decreasing behavior which can also be seen in Figure 5.6,

$$y \leq y_n \leq y_{n-1} \leq \cdots \leq y_1.$$

Example 5 Consider

$$y'' = y + y^3, \tag{5.56}$$

with boundary condition

$$y(0) = 0,\ y(4) = 1.7. \tag{5.57}$$

Then

$$y_1'' = f(x, y_0) + (y_1 - y_0)f_y(x, y_0),\ y_1(0) = 0,\ y_1(4) = 1.7. \tag{5.58}$$

Here $f(x, y) = y + y^3,\ f_y(y_0, x) = 1 + 3y_0^2$. Using these values in Eq. (5.58), we get

$$y_1'' = y_1 + 3y_1y_0^2 - 2y_0^3,\ y_1(0) = 0,\ y_1(4) = 1.7. \tag{5.59}$$

Let $y_0 = 0.1$, be approximate solution of y, then Eq. (5.59) becomes

$$y_1'' = 1.03y_1 - .002,\ y_1(0) = 0,\ y_1(4) = 1.7. \tag{5.60}$$

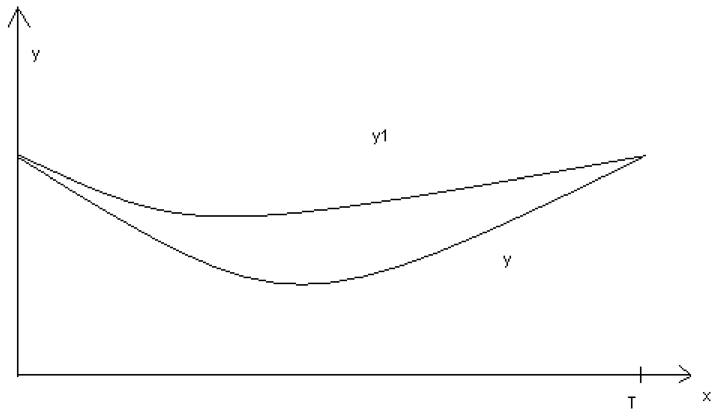

Figure 5.6: Graph shows monotonically non increasing behavior of sequence y_n.

Solving this linear equation we obtain y_1, in a similar way we can find y_2 and y_3. For obtaining exact solution, we shall find $y_3'(0)$. Now we have to deal with a nonlinear second order differential equation with two initial values. Here is a graph which shows exact solution coincide with solution at second step i.e. y_2. Moreover, curves show the monotonically non increasing behavior of sequence $\{y_n\}$ in Figure 5.7.

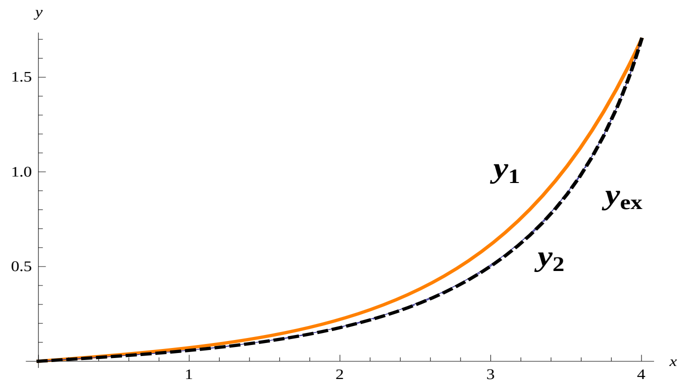

Figure 5.7: Curves for y_2 is not distinguishable from the exact solution curve y_{ex}.

Example 6 Consider

$$y'' = y + y^5, \tag{5.61}$$

with boundary condition

$$y(0) = 0, \ y(2) = 2.5. \tag{5.62}$$

Then

$$y_1'' = f(y_0, x) + (y_1 - y_0) f_y(y_0, x), \ y_1(0) = 0, \ y_1(2) = 2.5. \tag{5.63}$$

Here $f(y, x) = y + y^5, \ f_y(y_0, x) = 1 + 5y_0^4$. Using these values in Eq. (5.63), we get

$$y_1'' = y_1 + 5y_1 y_0^4 - 4y_0^5, \ y_1(0) = 0, \ y_1(2) = 2.5. \tag{5.64}$$

Let $y_0 = 1$, be initial guess of y, then Eq. (5.64) becomes

$$y_1'' = y_1 - 4, \ y_1(0) = 0, \ y_1(2) = 2.5. \tag{5.65}$$

Solving this linear equation we obtain y_1, in a similar way we can find y_2 and y_3. For obtaining exact solution, we shall find $y_3'(0)$. Now we have to deal with a nonlinear second order differential equation with two initial values. Here is a graph which shows exact solution coincide with solution at second step i.e. y_2. Moreover, curves show the monotonically non increasing behavior of sequence $\{y_n\}$ (see Figure 5.8).

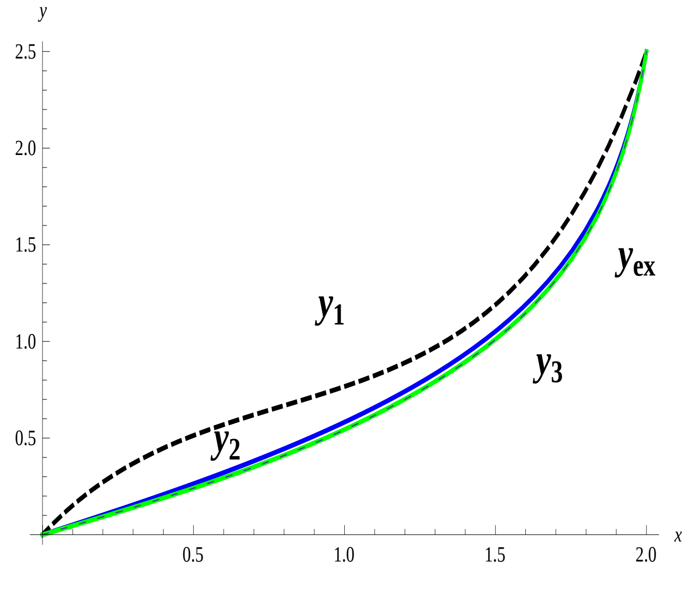

Figure 5.8: The curves for y_2, y_3 are not distinguishable from the exact solution curve y_{ex}.

Chapter 6

The Tanh and Sine-Cosine Methods

6.1 The Tanh Method

For computation of exact traveling wave solutions the tanh method is a powerful method [14, 17]. This method has various extensions for example, nonlinear heat conduction and Burger Fisher equations [17, 18]. First of all a power series as an ansatz was used in tanh method to get analytic solutions of certain nonlinear evolution equations that were traveling wave type.

Substitution of wave variable $\xi = x - ct$ or $\xi = x + y - ct$ converts following nonlinear PDE

$$P_d(v, v_t, v_x, v_{xx}, v_{xxx}, v_{xxxx}, \cdots) = 0, \tag{6.1}$$

into a nonlinear ODE

$$O_d(v, v', v'', v''', \cdots) = 0. \tag{6.2}$$

Eq. (6.2) then integrated until all terms contain derivatives, constants of integration are neglected. Introducing new independent variable

$$Z = \tanh(\mu\xi), \tag{6.3}$$

which changes derivatives as follows

$$\frac{d}{d\xi} = \mu(1 - Z^2)\frac{d}{dZ}, \tag{6.4}$$

$$\frac{d^2}{d\xi^2} = \mu^2(1-Z^2)(-2Z\frac{d}{dZ} + (1-Z^2)\frac{d^2}{dZ^2}), \tag{6.5}$$

$$\begin{aligned}\frac{d^3}{d\xi^3} &= 4\mu^3(1-Z^2)Z^2\frac{d}{dZ} - 2\mu^3(1-Z^2)^2\frac{d}{dZ} \\ &- 2\mu^3(1-Z^2)Z\Big((1-Z^2)+1+(1-Z^2)^3\Big)\frac{d^2}{dZ^2} - \mu^3(1-Z^2)^2\frac{d^3}{dZ^3}.\end{aligned}$$

Similarly other derivatives can be found easily. Then introduce a series expansion as under

$$v(\mu\xi) = S(Z) = \sum_{k=0}^{m} a_k Z^k, \tag{6.6}$$

where m is a positive integer. Putting Eqs. (6.4), (6.5) and (6.6) into simplified ODE, gives an equation involving powers of Z. To find the value of m, usually the highest order linear term is balanced with highest order nonlinear term of the resulting ODE. Using this value of m, a series $v(x,t)$ is obtained. Substituting this series in the last ODE and arranging such that coefficients of all powers of Z vanish. In such a way a system of algebraic equations with variables μ, c, a_k is obtained. Using these values an analytic solution $v(x,t)$ is obtained. If m is not an integer then an appropriate transformation will be used to get integer value of m.

The tanh method is applied to KdV equation for $n = 2, 3$ in Example (1) and (2) respectively.

Example 1

Consider the KdV equation for $n = 2$

$$v_t + 3av^2v_x + bv_{xxx} = 0. \tag{6.7}$$

Introduce $\xi = x - ct$ in this equation, we get

$$-cv' + a(v^3)' + bv''' = 0, \tag{6.8}$$

where $v' = \frac{dv}{d\xi}$. Integration of Eq. (6.8) yields

$$-cv + a(v^3) + bv'' = 0. \tag{6.9}$$

Now substituting value of v and v'' in Eq. (6.9) and then balancing the powers of Z in resulting terms of v^3 and v'', we get

$$m = 1. \tag{6.10}$$

Therefore

$$v(x,t) = S(Z) = a_0 + a_1 Z. \tag{6.11}$$

Substituting this series in Eq. (6.9) and then rearranging coefficients of Z and then comparing these with other side of equation gives a system of equations with variables $a_0,\ a_1,\ \mu$

$$\begin{aligned}
Y^0 \text{ coefficient} &: \quad -ca_0 + aa_0^3 = 0,\\
Y^1 \text{ coefficient} &: \quad -ca_1 + 3aa_0^2 a_1 - 2b\mu^2 a_1 = 0,\\
Y^2 \text{ coefficient} &: \quad 3aa_0^2 a_1 = 0,\\
Y^3 \text{ coefficient} &: \quad aa_1^3 + 2b\mu^2 a_1 = 0.
\end{aligned}$$

Solving this system simultaneously we get

$$\begin{aligned}
a_0 &= 0,\\
a_1 &= \sqrt{\frac{c}{a}},\\
\mu &= \sqrt{\frac{-c}{2b}},\ \frac{c}{2b} < 0.
\end{aligned}$$

With the help of above values, the solution for $\frac{c}{2b} < 0$, can be written as

$$v(x,t) = \sqrt{\frac{c}{a}} \tanh\left(\sqrt{\frac{-c}{2b}}(x - ct)\right),$$

and

$$v(x,t) = \sqrt{\frac{c}{a}} \coth\left(\sqrt{\frac{-c}{2b}}(x - ct)\right).$$

For $\frac{c}{2b} > 0$, complex solutions are obtained as follows

$$v(x,t) = i\sqrt{\frac{c}{a}} \tan\left(\sqrt{\frac{c}{2b}}(x - ct)\right),$$

and

$$v(x,t) = i\sqrt{\frac{c}{a}} \cot\left(\sqrt{\frac{c}{2b}}(x - ct)\right).$$

Here $i = \sqrt{-1}$.

Example 2

Consider the KdV equation for $n = 3$

$$v_t + 4av^3 v_x + bv_{xxx} = 0. \tag{6.12}$$

Introduce $\xi = x - ct$ in this equation, we get

$$-cv' + a(v^4)' + bv''' = 0, \tag{6.13}$$

where $v' = \frac{dv}{d\xi}$. Integration of Eq. (6.13) yields

$$-cv + av^4 + bv'' = 0. \tag{6.14}$$

Substituting the value of v and v'' in Eq. (6.14) and then balancing the highest powers of Z in resulting terms of v^4 and v'', we get

$$m = \frac{2}{3}. \tag{6.15}$$

As m must be a positive integer. To obtain such a solution, now we will use a transformation formula. After substituting

$$v(x,t) = u^{\frac{1}{3}}(x,t),$$

into Eq. (6.14), we get

$$-9cu^2 + 9au^3 - 2b(u')^2 + 3uu'' = 0. \tag{6.16}$$

Balancing u^3 and uu'', we obtain

$$m = 2. \tag{6.17}$$

Using Eq. (6.17), we can write

$$u(x,t) = S(Z) = a_0 + a_1 Z + a_2 Z^2.$$

Substituting this value into reduced ODE (6.16) and arranging the coefficients of Z, then comparing to zero, gives the system of algebraic equations in terms of $a_0,\ a_1,\ a_2,\ \mu$.

Y^0 coefficient : $-9ca_0^2 + 9aa_0^3 + 6ba_2a_0\mu^2 - 2b\mu^2a_1^2 = 0,$

Y^1 coefficient : $-18ca_1a_0 + 27aa_0^2a_1 - 6ba_0a_1\mu^2 - 2ba_1a_2\mu^2 = 0,$

Y^2 coefficient : $-9ca_1^2 - 18ca_0a_2 + 27aa_0a_1^2 + 27aa_0^2a_2 - 24ba_0a_2\mu^2 - 2ba_1^2\mu^2 - 2b\mu^2a_2^2 = 0,$

Y^3 coefficient : $-18ca_1a_2 + 9aa_1^3 + 6ba_1a_0\mu^2 - 8ba_1a_2\mu^2 - 6ba_2a_1\mu^2 + 54aa_0a_1a_2 = 0,$

Y^4 coefficient : $-9ca_2^2 + 27aa_1^2a_2 + 27aa_0a_2^2 + 12ba_0a_2\mu^2 + 4ba_1^2\mu^2 - 8ba_2^2\mu^2 + 6ba_0a_2\mu^2 = 0,$

Y^5 coefficient : $27aa_1a_2^2 + 16ba_1a_2\mu^2 = 0,$

Y^6 coefficient : $9aa_2^3 + 10ba_2^2\mu^2 = 0.$

Solving this system simultaneously we get

$$\begin{aligned}
a_0 &= \frac{5c}{2a}, \\
a_1 &= 0, \\
a_2 &= -\frac{5c}{2a}, \\
\mu &= -\frac{3}{2}\sqrt{\frac{c}{b}},\ \frac{c}{b} > 0.
\end{aligned}$$

As we have used $v = u^{\frac{1}{3}}$ and using above values of $a_0,\ a_1,\ a_2,\ \mu.$, the solution for $\frac{c}{b} > 0$, can be expressed as follows

$$v(x,t) = \{\frac{5c}{2a}\sec h^2(\frac{3}{2}\sqrt{\frac{c}{b}}(x-ct))\}^{\frac{1}{3}},$$

and

$$v(x,t) = \{-\frac{5c}{2a}\csc h^2(\frac{3}{2}\sqrt{\frac{c}{b}}(x-ct))\}^{\frac{1}{3}}.$$

For $\frac{c}{b} < 0$, following solutions are obtained

$$v(x,t) = \{\frac{5c}{2a}\sec^2(\frac{3}{2}\sqrt{-\frac{c}{b}}(x-ct))\}^{\frac{1}{3}},$$

and

$$v(x,t) = \{\frac{5c}{2a}\csc^2(\frac{3}{2}\sqrt{-\frac{c}{b}}(x-ct))\}^{\frac{1}{3}}.$$

6.2 The Sine-Cosine Method

Consider a nonlinear partial differential equation (PDE) which define a physical model

$$N(v, v_t, v_x, v_{xx}, v_{xt}, v_{tt}, \cdots) = 0. \tag{6.18}$$

Then substitution of wave variable $\xi = x - ct$, to obtain the traveling-type wave solution $v(x,t)$ of Eq. (6.18) bring following changes

$$v_t = -cv',\ v_{tt} = c^2v'',\ v_x = v',\ v_{xx} = v'', \tag{6.19}$$

where $v' = \frac{dv}{d\xi}$. Let $v(x,t) = v(\xi)$. With help of these changes the PDE become an ODE

$$Q(v, v', v'', \cdots) = 0. \tag{6.20}$$

Eq. (6.20) is then integrated unless all terms contain derivatives, constants of integration are neglected. Using the derivations made in [19, 20], the solutions can be expressed in the form of Eqs. (1.6) and (1.7). Derivatives of Eqs. (1.6) and (1.7) are as follows

$$\begin{aligned} v(\xi) &= \lambda \sin^{\alpha}(\nu\xi), \\ v^n(\xi) &= \lambda^n \sin^{n\alpha}(\nu\xi), \\ (v^n)_{\xi} &= n\nu\alpha\lambda^n \cos(\nu\xi)\sin^{n\alpha-1}(\nu\xi), \\ (v^n)_{\xi\xi} &= -n^2\nu^2\alpha^2\lambda^n \sin^{n\alpha}(\nu\xi) + n\nu^2\lambda^n\alpha(n\alpha-1)\sin^{n\alpha-2}(\nu\xi), \end{aligned}$$

and

$$\begin{aligned} v(\xi) &= \lambda \cos^{\alpha}(\nu\xi), \\ v^n(\xi) &= \lambda^n \cos^{n\alpha}(\nu\xi), \\ (v^n)_{\xi} &= -n\mu\beta\lambda^n \sin(\mu\xi)\cos^{n\beta-1}(\mu\xi), \\ (v^n)_{\xi\xi} &= -n^2\nu^2\alpha^2\lambda^n \cos^{n\alpha}(\nu\xi) + n\nu^2\lambda^n\alpha(n\alpha-1)\cos^{n\alpha-2}(\nu\xi). \end{aligned}$$

Example 3

Consider the nonlinear dispersive equation $K(m,n)$

$$v_t + a(v^m)_x + b(v^n)_{xxx} = 0, \;\; b > 0. \tag{6.21}$$

Take $m = n = 2$

$$v_t + a(v^2)_x + b(v^2)_{xxx} = 0, \;\; b > 0. \tag{6.22}$$

Introduce $\xi = x - ct$ in this equation, we get

$$-cv' + a(v^2)' + b(v^2)''' = 0, \tag{6.23}$$

which on integration becomes

$$-cv + av^2 + b(v^2)'' = 0. \tag{6.24}$$

Now substituting values of $v,\ v^2$ and $(v^2)''$ in Eq. (6.24), gives

$$-c\lambda\sin^{\alpha}(\nu\xi) + a\lambda^2\sin^{2\alpha}(\nu\xi) + b[-4\alpha^2\lambda^2\nu^2\sin^{2\alpha}(\nu\xi) + 2\alpha\lambda^2\nu^2(2\alpha-1)\sin^{2\alpha-2}(\nu\xi) = 0. \tag{6.25}$$

This equation holds only if

$$
\begin{aligned}
2\alpha - 2 &= \alpha, \\
-c\lambda &= 2b\alpha\lambda^2\nu^2(2\alpha - 1), \\
a\lambda^2 &= -4b\alpha^2\lambda^2\nu^2.
\end{aligned}
$$

Solving above system, we obtain

$$
\begin{aligned}
\alpha &= 2, \\
\lambda &= \frac{4c}{3a}, \\
\nu &= \frac{1}{4}\sqrt{\frac{a}{b}}.
\end{aligned}
$$

This result will hold even if we use cosine instead of sine method. Substitution of these values of $\lambda,\ \alpha$ and ν in Eqs. (1.6) and (1.7), gives

$$v(x,t) = \frac{4c}{3a}\sin^2[\frac{1}{4}\sqrt{\frac{a}{b}}(x - ct)],\ \text{if } a > 0, \tag{6.26}$$

and

$$v(x,t) = \frac{4c}{3a}\cos^2[\frac{1}{4}\sqrt{\frac{a}{b}}(x - ct)],\ \text{if } a > 0. \tag{6.27}$$

and for $a < 0$, we get

$$v(x,t) = -\frac{4c}{3a}\sinh^2[\frac{1}{4}\sqrt{\frac{-a}{b}}(x - ct)],\ \text{if } a < 0, \tag{6.28}$$

$$v(x,t) = \frac{4c}{3a}\cosh^2[\frac{1}{4}\sqrt{\frac{-a}{b}}(x - ct)],\ \text{if } a < 0. \tag{6.29}$$

Example 4

Let $m = 3,\ n = 3$

$$v_t + a(v^3)_x + b(v^3)_{xxx} = 0,\ b > 0. \tag{6.30}$$

Introduce $\xi = x - ct$ in this equation, we get

$$-cv' + a(v^3)' + b(v^3)''' = 0. \tag{6.31}$$

Integration of Eq. (6.31), gives

$$-cv + av^3 + b(v^3)'' = 0. \tag{6.32}$$

Now substituting values of $v,\ v^3$ and $(v^3)''$

$$-c\lambda\sin^{\alpha}(\nu\xi)+a\lambda^3\sin^{3\alpha}(\nu\xi)+b[-9\alpha^2\lambda^3\nu^2\sin^{3\alpha}(\nu\xi)+3\alpha\lambda^3\nu^2(3\alpha-1)\sin^{3\alpha-2}(\nu\xi)]=0. \tag{6.33}$$

This equation holds only if

$$\begin{aligned} 3\alpha-2&=\alpha,\\ -c\lambda&=3b\alpha\lambda^3\nu^2(3\alpha-1),\\ a\lambda^3&=-9b\alpha^2\lambda^3\nu^2. \end{aligned}$$

Solving this system, we obtain

$$\begin{aligned} \alpha &= 1,\\ \lambda &= \sqrt{\frac{3c}{2a}},\\ \nu &= \frac{1}{3}\sqrt{\frac{-a}{b}}. \end{aligned}$$

This result will hold even if we use cosine instead of sine method. For this set of values in Eqs. (1.6) and (1.7)

$$v(x,t)=\sqrt{\frac{3c}{2a}}\sin[\frac{1}{3}\sqrt{\frac{-a}{b}}(x-ct)],\ \text{if } a<0, \tag{6.34}$$

and

$$v(x,t)=\sqrt{\frac{3c}{2a}}\cos[\frac{1}{3}\sqrt{\frac{-a}{b}}(x-ct)],\ \text{if } a<0. \tag{6.35}$$

and for $a>0$, we get

$$v(x,t)=-\sqrt{\frac{3c}{2a}}\sinh^2[\frac{1}{3}\sqrt{\frac{a}{b}}(x-ct)],\ \text{if } a>0, \tag{6.36}$$

$$v(x,t)=\sqrt{\frac{3c}{2a}}\cosh^2[\frac{1}{3}\sqrt{\frac{a}{b}}(x-ct)],\ \text{if } a>0. \tag{6.37}$$

Example 5

Consider $m=2,\ n=1$

$$v_t+a(v^2)_x+b(v)_{xxx}=0.\ b>0, \tag{6.38}$$

Let $\xi=x-ct$, then we get

$$-cv'+a(v^2)'+b(v)'''=0. \tag{6.39}$$

Integrating equation (6.39), we get

$$-cv + av^2 + b(v)'' = 0. \tag{6.40}$$

Now using values of $v,\ v^2$ and $(v)''$

$$-c\lambda \sin^{\alpha}(\nu\xi) + a\lambda^2 \sin^{2\alpha}(\nu\xi) + b[-\alpha^2\lambda\nu^2 \sin^{\alpha}(\nu\xi) + \alpha\lambda\nu^2(\alpha-1)\sin^{\alpha-2}(\nu\xi)] = 0. \tag{6.41}$$

This equation is satisfied only if

$$2\alpha = \alpha - 2,$$
$$c\lambda = b\alpha^2\lambda\nu^2,$$
$$a\lambda^2 = b\lambda\nu^2\alpha(\alpha-1).$$

Simultaneously solution of this system, gives

$$\begin{aligned} \alpha &= -2, \\ \lambda &= \frac{3c}{2a}, \\ \nu &= \frac{1}{2}\sqrt{\frac{c}{b}}. \end{aligned}$$

The same result will be obtained if we use cosine instead of sine method. Using these values in Eqs. (1.6) and (1.7), we obtain

$$v(x,t) = \frac{3c}{2a}\csc^2[\frac{1}{2}\sqrt{\frac{c}{b}}(x-ct)],\ \text{if } c > 0, \tag{6.42}$$

and

$$v(x,t) = \frac{3c}{2a}\sec^2[\frac{1}{2}\sqrt{\frac{c}{b}}(x-ct)],\ \text{if } c > 0. \tag{6.43}$$

and for $c < 0$, we get

$$v(x,t) = -\frac{3c}{2a}\csc h^2[\frac{1}{2}\sqrt{\frac{c}{b}}(x-ct)],\ \text{if } c < 0, \tag{6.44}$$

$$v(x,t) = \frac{3c}{2a}\sec h^2[\frac{1}{2}\sqrt{\frac{c}{b}}(x-ct)],\ \text{if } c < 0. \tag{6.45}$$

Chapter 7

Existence of a Solution for a BVP

7.1 Definitions:

Contraction Mapping: A mapping $f\colon \mathbb{Y} \to \mathbb{Y}$ is called a contraction mapping on a normed linear space $\mathbb{Y}$, if a constant $\beta \in (0,\ 1)$ exists such that

$$\|fy - fz\| \leq \beta \|y - z\|,\ \forall\ y,\ z \in \mathbb{Y}.$$

Moreover $\overline{y}$ is a fixed point of f if

$$f\overline{y} = \overline{y}.$$

Cauchy Sequence: A sequence $\{y_n\} \subset \mathbb{Y}$ is called a Cauchy sequence, if for every $\epsilon > 0$ there is a positive integer K such that

$$\|y_k - y_n\| < \epsilon,\ \forall\ k,\ n \geq K.$$

Banach Space: A normed linear space $\mathbb{Y}$ is called a Banach space if every Cauchy sequence in $\mathbb{Y}$ converges to an element of $\mathbb{Y}$.

Contraction mapping theorem has a significant importance in the field of differential equations[12].

7.2 Contraction Mapping Theorem

If $f\colon \mathbb{Y} \to \mathbb{Y}$ is a contraction mapping on a Banach space $\mathbb{Y}$ with contraction constant β, where $\beta \in (0, 1)$ then f has a fixed point $\overline{y} \in \mathbb{Y}$ which is unique. Also if $y_0 \in \mathbb{Y}$ and taking

$y_{n+1} = fy_n$, for $n \geq 0$, then

$$\lim_{n\to\infty} y_n = \overline{y}.$$

Moreover, for $n \geq 1$

$$\|y_n - \overline{y}\| \leq \frac{\beta^n}{1-\beta}\|y_1 - y_0\|. \tag{7.1}$$

7.3 Application of Contraction Mapping Theorem to BVPs

Consider a nonlinear BVP

$$y'' = g(x, y, y'),\ y(a) = c_1,\ y(b) = c_2, \tag{7.2}$$

where $a < b$ and $c_1,\ c_2 \in \mathbb{R}$. We will use CMT to show existence of a unique solution for a class of functions g.

Theorem 7.1. *Let $g\colon\ [a,b]\times\mathbb{R} \to \mathbb{R}$ be continuous. With Lipschitz constant L, g satisfies a Lipschitz condition w.r.t y on $[a,b]\times\mathbb{R}$, i.e.*

$$|g(x,y) - g(x,z)| \leq L|y - z|, \tag{7.3}$$

for all $(x,y),\ (x,z) \in [a,b]\times\mathbb{R}$. If

$$b - a < \frac{2\sqrt{2}}{L}, \tag{7.4}$$

then there is a unique solution of boundary value problem (7.2).

Proof:- *Let $\mathbb{Y}$ be a Banach space (of continuous function on $[a,b]$) with norm, which is max norm, and can be defined as follows*

$$\|y\| = \max\{|y(x)| : a \leq x \leq b\}. \tag{7.5}$$

Here y is a solution of BVP (7.2) iff y is a solution of the nonhomogeneous, linear BVP

$$y'' = h(x) := g(x, y(x)),\ y(a) = c_1,\ y(b) = c_2. \tag{7.6}$$

This BVP has a solution iff following integral equation has a solution

$$y(x) = w(x) + \int_a^b G(x,s)g(s, y(s))ds, \tag{7.7}$$

where w is the solution of BVP

$$w'' = 0,\ w(a) = c_1,\ w(b) = c_2, \tag{7.8}$$

and G is the Green function for the following BVP

$$y'' = 0,\ y(a) = 0,\ y(b) = 0. \tag{7.9}$$

Define a mapping $F : \mathbb{Y} \to \mathbb{Y}$ by

$$Fy(x) = w(x) + \int_a^b G(x,s)g(s,y(s))ds, \tag{7.10}$$

for $x \in [a,b]$. Therefore the BVP (7.2) has a unique solution if and only if the operator F has a unique fixed point. We want to show that F has a unique fixed point in $\mathbb{Y}$ using contraction mapping theorem (CMT).

Consider $y, z \in \mathbb{Y}$ and

$$\begin{aligned}
|Fy(x) - Fz(x)| &\leq \int_a^b |G(x,s)||g(s,y(s)) - g(s,z(s))|ds, \\
&\leq \int_a^b |G(x,s)|L|y(s) - z(s)|ds, \\
&\leq L\int_a^b |G(x,s)|ds\|y - z\|, \\
&\leq L\frac{(b-a)^2}{8}\|y - z\|,
\end{aligned}$$

for $x \in [a,b]$. It implies that

$$\|Fy - Fz\| \leq \beta\|y - z\|,$$

where from Eq. (7.4), we can see

$$\beta = L\frac{(b-a)^2}{8} < 1.$$

Hence F is a contraction mapping on $\mathbb{Y}$ and by contraction mapping theorem we conclude that the BVP has a unique solution.

Example 1

Consider the BVP

$$y'' = \cos y,$$

$$y(0) = 0,\ y(1) = 0. \tag{7.11}$$

Here $g(x,y) = \cos y$, and

$$|g_y(x,y)| = |-\sin y| \leq L = 1.$$

Since

$$b - a = 1 < \frac{2\sqrt{2}}{\sqrt{L}} = 2\sqrt{2},$$

by Theorem 7.1 the BVP (7.11) has a unique solution. Let $y_0(x) = 0$, be initial approximation, then the first Picard iterate y_1 solves the BVP as under:

$$y_1'' = g(x, y(x)) = g(x, 0) = \cos 0 = 1,$$

$$y_1(0) = 0, \; y_1(1) = 0.$$

Now solving this BVP, the first Picard iterate y_1. i.e.

$$y_1(x) = \frac{x}{2}(x - 1), \; x \in [0, 1] .$$

Next, we will check how good approximation y_1 is to actual solution of the BVP.

Contraction constant β is

$$\beta = L\frac{(b-a)^2}{8} = \frac{1}{8}$$

Hence using Eq. (7.1)

$$\|y - y_1\| \leq \frac{\beta}{\beta - 1}\|y_1 - y_0\| = \frac{1}{7}\|y_1\| = \frac{1}{56},$$

since $\|.\|$ is max norm. It follows that

$$|y(x) - y_1(x)| \leq \frac{1}{56}, \; x \in [0, 1] ,$$

and we can see that $y_1(x) = \frac{1}{2}x(x-1)$ is a very good approximation of the actual solution y of the BVP (7.11).

7.4 Lower and Upper solutions

This section comprises of study of lower and upper solutions of BVP's, that implies existence of a solution of a BVP as well as gives bounds on the solution.

7.4.1 Lower Solution:

β is called a lower solution of $y'' = g(x, y, y')$ on an interval I, if for $x \in I$,

$$\beta''(x) \geq g(x, \beta(x), \beta'(x)). \tag{7.12}$$

7.4.2 Upper Solution:

γ is called an upper solution of $y'' = g(x, y, y')$ on an interval I, if for $x \in I$,

$$\gamma''(x) \leq g(x, \gamma(x), \gamma'(x)). \tag{7.13}$$

We assume that $\beta(x) \leq \gamma(x), x \in [b, c]$. Due to this, we can call β a lower solution and γ an upper solution.

Theorem 7.2. *Suppose $g\colon [b, c] \times \mathbb{R}^2 \to \mathbb{R}$ is continuous and bounded. Then there exist a solution to boundary value problem*

$$y'' = g(x, y, y'), \tag{7.14}$$

$y(b) = B$, $y(c) = C$, where B, C are constants [11].

Theorem 7.3. *Suppose g is continuous on $[b, c] \times \mathbb{R}$ and lower and upper solutions of $y'' = g(x, y, y')$, are β, γ, respectively, where $\beta(x) \leq \gamma(x)$ on $[b, c]$. Let B and C be constants such that $\beta(b) \leq B \leq \gamma(b)$ and $\beta(c) \leq C \leq \gamma(c)$. Then the BVP*

$$y'' = g(x, y), \; y(b) = B, \; y(c) = C, \tag{7.15}$$

has a solution y, which satisfy the following relation on $[b, c]$

$$\beta(x) \leq y(x) \leq \gamma(x).$$

Example 2

Consider the BVP

$$y'' = -\sin y,$$

$$y(0) = 0 = y(1).$$

This system has a solution by Theorem 7.2. To find bounds on a solution of this BVP we will now use Theorem 7.3. Let $\beta(x) := 0$ and $\gamma(x) := \frac{x(1-x)}{2}$, for $x \in [0, 1]$ be lower and upper solutions, respectively, of $y'' = g(x, y, y')$ on $[0, 1]$ satisfying conditions of Theorem 7.3, $\beta(x) \leq \gamma(x)$ on $[0, 1]$ and $\beta(0) = 0 \leq B \leq \gamma(0) = 0$, and $\beta(1) = 0 \leq C \leq \gamma(1) = 0$. Therefore there is a solution y of this BVP satisfying

$$0 \leq y(x) \leq \frac{x(1-x)}{2},$$

for $x \in [0, 1]$.

Theorem 7.4. *Suppose that $g\colon [b,c] \times \mathbb{R}^2 \to \mathbb{R}$ is continuous and for each fixed $(x, y') \in [b,c] \times \mathbb{R}$, $g(x, y, y')$ is strictly increasing with respect to y. Then the BVP (7.15) has at most one solution.*

Conclusion

In this thesis we have discussed a few methods for the solution of nonlinear differential equations.

We have found approximate solutions using different methods such as perturbation, Adomian decomposition and homotopy analysis methods to get analytic solution in the form of an infinite series. The tanh and sine-cosine methods were used to find closed form solution of KdV and a nonlinear dispersive equation respectively. A method of successive approximations is described which replaces a nonlinear equation by a linear differential equation. This equation is solved recursively to get a sequence of functions which converges monotonically to the exact solution.

Finally, the contraction mapping theorem is applied to discuss existence of a solution for a class of nonlinear BVP's. The lower and upper solutions are used to find existence of a solution in an interval and bounds on it.

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