# Partition Dimension and Strong Metric Dimension of Chain Cycle

by

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**Supervised by**

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***To my most loving parents***

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# Abstract

Let $G$ be a connected graph with edge set $E(G)$ and vertex set $V(G)$. The distance between a vertex $v \in V(G)$ and a set $Q \subseteq V(G)$ is defined by $d(v, Q) = \min\{d(v, u) \mid u \in Q\}$. For $t$-partition $\Pi = \{Q_1, \ldots, Q_t\}$ of $V(G)$, which is an ordered partition, the representation of a vertex $v \in V(G)$ with respect to $\Pi$ is the $t$-vector $r(v|\Pi) = (d(v, Q_1), \ldots, d(v, Q_t))$. The partition $\Pi$ is a resolving partition if $r(u|\Pi) \neq r(v|\Pi)$, for each pair of distinct vertices $u, v \in V(G)$. The smallest $t$ for which there is a resolving $t$-partition of $V(G)$ is the partition dimension of $G$. A vertex $w \in V(G)$ strongly resolves two distinct vertices $u, v \in V(G)$ if $u$ belongs to a shortest $v - w$ path or $v$ belongs to a shortest $u - w$ path. The set $W = \{w_1, \ldots, w_k\} \subseteq V(G)$ is a strong resolving ordered set for $G$ if for every two unique vertices $u$ and $v$ of $G$ $\exists\ w \in W$ that strongly resolves $u$ and $v$. A strong metric basis of $G$ is a strong resolving set of smallest cardinality. The cardinality of a strong metric basis is called strong metric dimension of $G$. In this thesis, we survey some results from partition dimension and strong metric dimension. We also determine the partition dimension of a chain cycle constructed by even cycles and a chain cycle constructed by odd cycles. We also calculate the strong metric dimension of a chain cycle constructed by even cycles.

# Contents

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# Chapter 1

# Introduction to graph theory

In this chapter, we discuss some preliminary concepts of graph theory with examples. Our goal is to make the reader familiar with basic concepts and applications of graph theory.

## 1.1 Brief history

Euclid, the father of geometry, primarily introduced some basic concepts of geometry regarding point, line and plane. This geometry was very simple unlike that of today. He mostly discussed flat spaces whereas today we understand curved spaces and advanced mathematical concepts. This was the beginning of discrete mathematics but we realized it after many centuries. Euclid postulated some results for construction of geometry and one of them was to draw a straight line from one point to an other. A Swiss mathematician Leonhard Euler developed graph theory after almost 2000 years of Euclid by redrawing Euclid's geometry in an abstract way.

In 17th century, the work of Pascal and De Moivre [21, 26] gave rise to the idea of combinatorics. In 18th century, with continuation of their work, Euler solved the Königsberg's bridge problem [9] which lead to new branch of mathematics called graph theory. The city of Königsberg, situated in Russia, comprised of four sectors connected by seven bridges. The bridge problem entailed the traversal of all seven bridges exactly once in a single round trip. Euler studied the model and proved that no such route exist. It was the first problem of graph theory that opened a new world. He also worked on partitioning and its enumeration. His interest in latin squares further lead to combinatorics [8, 13].

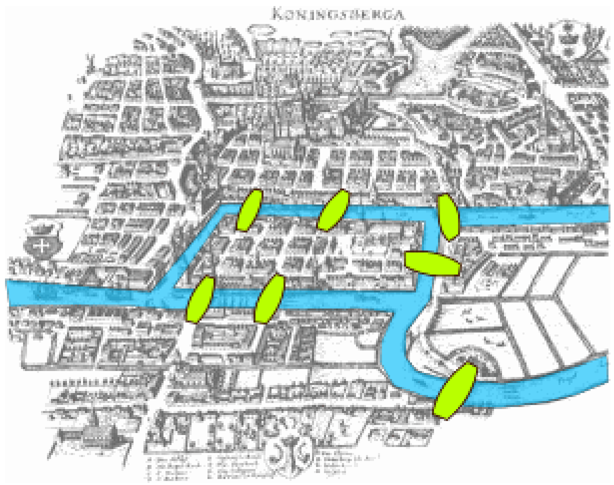



Figure 1.1: The seven bridges

There are some problems of practical interest which can be answered by constructing the corresponding graph models. Here we list few such problems. How can we assign tasks to construction companies to link all the roads from countryside to cities with minimum cost? What is the shortest route for trains from national capital to each state's capital? How can government provide jobs to its citizens with optimal total utility? How can a salesman save his fuel by choosing shortest path from warehouse to different shops in city? How can we differentiate countries on world map by choosing different and minimum colors for neighboring countries?

In the real world, we can easily explain these problems by diagrams made of points and lines joining these points. These points could represent number of robots in an industry, number of people in a party or communication towers and lines may represent set of commands, pairs of friends or links between towers, respectively. Such mathematical abstraction of situations give rise to the idea of a graph.

## 1.2 Basics of graph

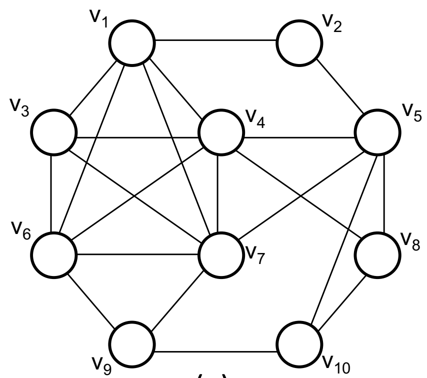
A graph $G = (V(G), E(G))$ containing a finite set of vertices $V(G)$ and a set of edges $E(G)$. The total number of edges and vertices in a graph is called its size and order, respectively. For $a, b \in V(G)$, an edge with end-points $a$ and $b$ is denoted by $ab$ or $ba$. In a graph $G$, if the vertex $v \in V(G)$ is the end-vertex of some edge $e \in E(G)$ then $e$ is said to be incident on $v$. A vertex which has exactly one edge incident on it is called a leaf. For $v \in V(G)$, the set of all vertices adjacent to $v$ is called open neighbourhood or simply neighbourhood of the vertex $v$, denoted by $N(v)$. The degree of a vertex $v$ in $G$ is defined as the cardinality of its neighbourhood $|N(v)|$, denoted by $deg(v)$. The closed neighborhood of $v$ is defined as $N[v] = N(v) \cup \{v\}$. The maximum and minimum degree of some vertex in $G$ is denoted by $\Delta(G)$ and $\delta(G)$, respectively. A loop is defined as the end-vertices of an edge are same. A loop adds 2 to the degree of its end-vertex. An isolated vertex has degree zero. If there are two or more edges between any two vertices then such edges are called multiple edges. If $G$ does not have any loop or multiple edges then it is called a simple graph. If $\delta(G) = \Delta(G) = k$ then every vertex in $G$ has same degree and the graph is called $k$-regular. For example, a cubic graph is 3-regular.

Let $G$ is a graph with $V(G) = \{v_1, \ldots, v_n\}$. The matrix of adjacency of $G$ is the $n \times n$ defined as $a_{ij} = 1$ iff $v_i$ and $v_j$ are adjacent, otherwise $a_{ij} = 0$, denoted by $A_{n\times n}$. In Figure 1.2, we give an example of adjacency matrix of a graph with ten vertices.

**Example 1.1.** The graph $G = (V(G), E(G))$, where $E(G) = \{g, h, i, j, k,\ l, m, n\}$ and $V(G) = \{p, q, r, s, t\}$ as shown in Figure 1.3. The incidence function $\psi_G$ is defined by

$$\psi_G(g) = pq, \quad \psi_G(h) = pp, \quad \psi_G(i) = qr, \quad \psi_G(j) = rt,$$
$$\psi_G(k) = qs, \quad \psi_G(l) = rs, \quad \psi_G(m) = ps, \quad \psi_G(n) = st.$$

In the graph shown in Figure 1.3, the vertex $p$ has degree 4 and the edge $h$ is a loop on vertex $p$. The edges $l$ and $j$ are multiple edges and $t$ is a leaf. In $G$, we have $\delta(G) = 1$ and $\Delta(G) = 5$. The neighborhood of $q$ is $N(q) = \{p, r, s\}$.




(a)

$$\begin{pmatrix} 0 & 1 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 \end{pmatrix}$$

(b)

Figure 1.2: (a) A simple graph, (b) The adjacency matrix of graph in (a)

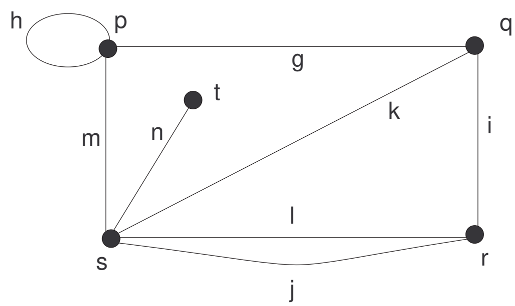



Figure 1.3: A graph $G$

A walk is a finite alternating list $v_0e_1v_1e_2v_2\ldots e_nv_n$ of vertices and edges such that, for $1 \leq i \leq n$, the end-points of $e_i$ are $v_{i-1}$ and $v_i$. A $(v, w)$-walk is a walk with $v$ and $w$ as its first and last vertices. The edges and vertices in a walk are not necessarily distinct. If the edges $e_1, \ldots, e_n$ of a walk are distinct, then the walk is said to be a trail and a trail without repetition of its vertices is called a path. A path containing $n$ vertices is denoted by $P_n$. A $(v, w)$-path can be defined analogously. A trail whose first and last vertices are same is a closed trail and is called a circuit. Moreover, a cycle containing $n$ vertices is a closed path, we denote it $C_n$. If $G$ has a $(v, w)$-path $\forall v, w \in V(G)$ then it is connected graph.

**Example 1.2.** Paths $P_4$, $P_6$ and cycle $C_8$ are shown in Figure 1.4.

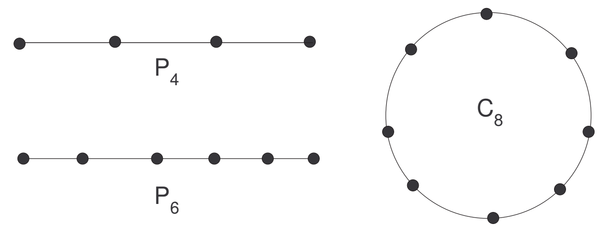



Figure 1.4: Paths $P_4, P_6$ and cycle $C_8$

If $V(G) = \{y_1, \ldots, y_m\}$ then the sequence $\{d(y_1), d(y_2), \ldots, d(y_m)\}$ is called the degree sequence of $G$. We usually arrange vertex degrees in monotone decreasing or increasing order. Since every edge has two vertices as its ends, thus the entirety of degrees of all vertices is double the quantity of edges in it. That is

$$\sum_{i=1}^{m} d(y_i) = 2e(G),$$

which shows that the sum of degrees of $G$ is even.

The above equation is called handshaking lemma which states: At any gathering, the quantity of individuals who shake hands an odd number of times is even.

**Corollary 1.3.** *For a graph $G$, the vertices of that has odd degree are in even numbers.*

**Theorem 1.4** (Chartrand and Zhang [7])**.** *If the vertices of graph $G$ has at least degree 2. Then $G$ will have a cycle.*

**Example 1.5.** In Figure 1.6, we represent graph $G$ by deleting a vertex and an edge from $G$, where $e$ is a bridge and $v$ is a cut-vertex.

### 1.2.1 Distances

We define the notion of distance between vertices in a graph. Moreover, we give results related to radius, diameter and eccentricity.

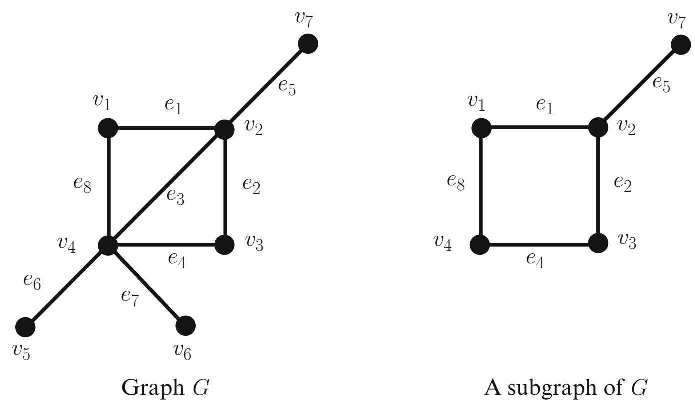



Figure 1.5: Example of subgraph

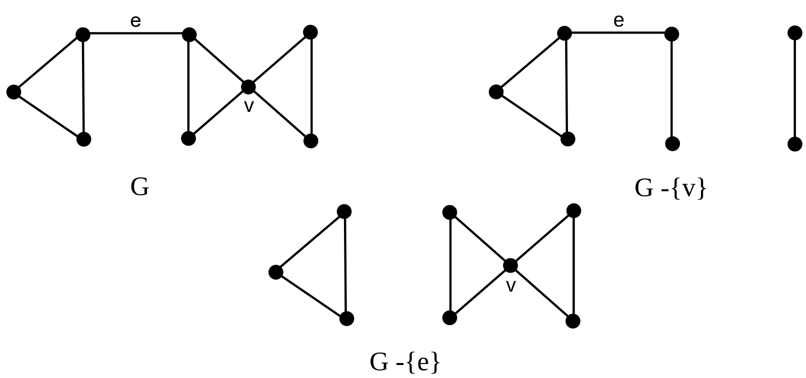



Figure 1.6: Edge and vertex-deleted subgraphs of a graph $G$

**Definition 1.6.** The eccentricity of vertex $x$ of graph(connected) $G$ can be defined as the distance from $x$ to a vertex furthest from $x$. It is denoted by $e(x)$. That is $e(x) = \max\{d(x, y) \mid y \in V(G)\}$.

**Definition 1.7.** The largest eccentricity between every vertices of graph $G$ is known as diameter of graph. That is, $\text{diam}(G) = \max\{e(z) : z \in V(G)\}$.

**Definition 1.8.** The smallest eccentricity between every vertices of graph $G$ is known as radius of graph and denoted by $\text{rad}(G)$. That is, $\text{rad}(G) = \min\{e(z) \mid z \in V\}$.

**Example 1.9.** Let $G$ be graph as shown in Figure 1.7. The eccentricities of vertices are given as $e(b) = e(d) = e(f) = e(h) = 3$, $e(a) = e(c) = e(g) = e(i) = 4$ and $e(e) = 2$. We can see that $2\text{rad}(H) = \text{diam}(H)$. In graph $G$, the eccentricity of every vertex is 3. We see that $\text{rad}(G) = \text{diam}(G)$.

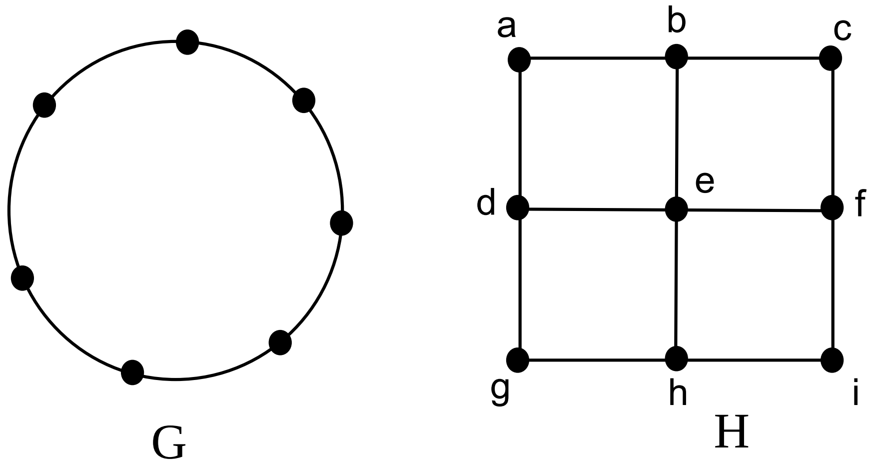



Figure 1.7: Two simple graphs

### 1.2.2 Coloring

Vertex coloring is a process of assigning colors to each vertex in a graph. A coloring is called proper if any adjacent vertex do not share same color in graph. We find smallest integer $r$, then $G$ has a $r$-coloring.

**Example 1.10.** Figure 1.8 shows that graph of ten vertices and twenty one edges. Here, we can see that this coloring is proper as all pairs of vertices have been alloted unlike colors.

### 1.2.3 Operations on graphs

Consider two graphs $G$ and $H$, where $|V(G)| = m_1$ and $|V(H)| = m_2$, with $V(G) = \{x_1, x_2, \ldots, x_n\}$. The graph acquired from $G$ and $H$ by choosing single replica of $G$ and

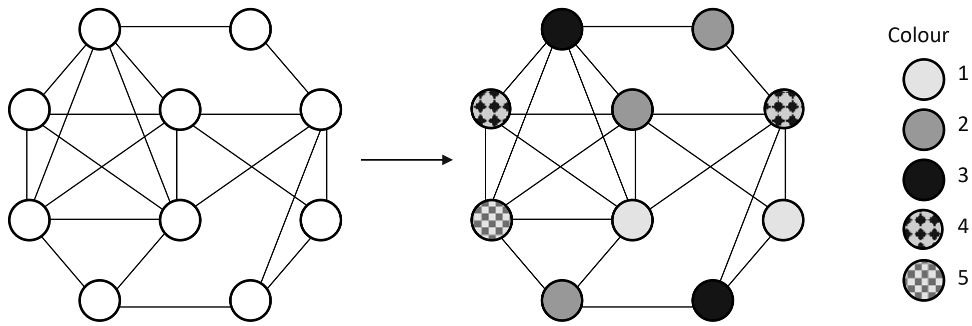



Figure 1.8: (a) Simple graph, (b) 5-coloring graph

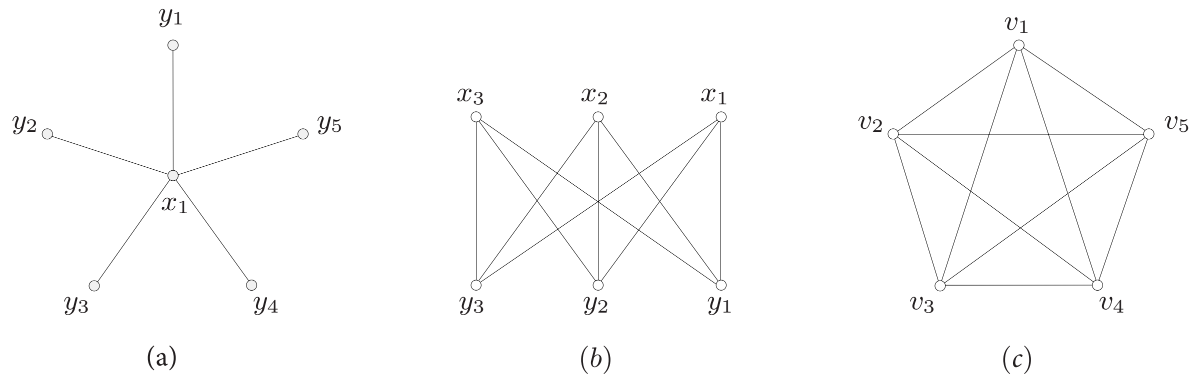



Figure 1.9: (a)Star graph , (b) $K_{3,3}$ (c) $K_5$

$m_1$ replicas of $H$ and connecting by an edge every vertex from the jth-replica of $H$ with the jth-vertex of $G$ is called corona product $G \odot H$. The join $G + H$ of graphs $G$ and $H$ can be defined as the graph acquired by taking disjoint union of $G$ and $H$ and adding all edges $pq$ such that $p \in V(G)$ and $q \in V(H)$.

## 1.3 Some applications

We discuss some important applications of bipartite graphs and coloring of graphs. We will see how graph theory help us in real world problems.

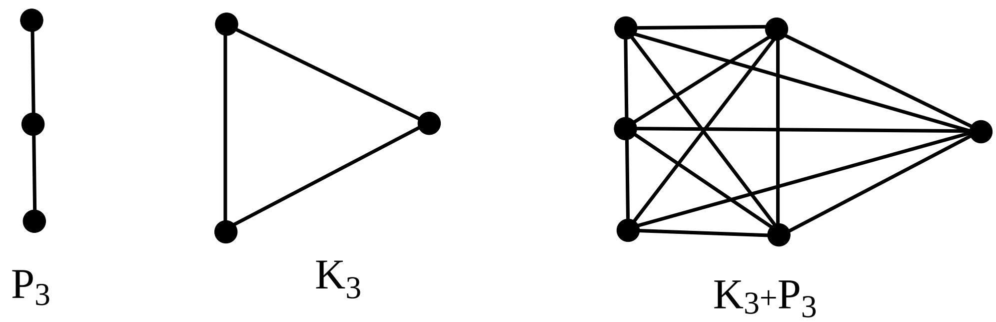



Figure 1.10: Join of $K_3$ and $P_3$

### 1.3.1 Bipartite graphs in real world

*Query-document pair and bipartite graphs*: [1] In the modern world, E-commerce business is current trend and bipartite graphs have major role in online world particularly in advertising and e-commerce. Search engines like Google, Bing, etc can track record when we click any document for our queries.

These search engines use query-document pair for their business models. In a system there is data of millions query-document pair per day. It is not necessary that user always find relevant result but at-least user decides to click the document. A click graph consists of a document (advertise) as one partition, queries as the other partition and set of edges connect a document node to querry node that represents clicks, is also a bipartite graph. We use this graph like clustering to find alike advertisers and related queries.

Two queries are similar if users click on the same documents. We consider an example. A click graph on very small scale is shown in Figure 1.11. We have not considered the weight of edges for convenience and simplicity. An edge demonstrates that no less than a single click from a query to document. For the graph, the queries 'laptop' and 'notebook' are attached by a common document hence they must be considered similar. Now, we can see that queries 'notebook' and 'printer' are attached through two same documents, since they can be taken similar. In disparity, 'laptop' and 'fax' are not attached by any document. However both 'laptop' and 'fax' are attached by document with queries 'notebook' and 'printer' as we have already seen that they are similar.

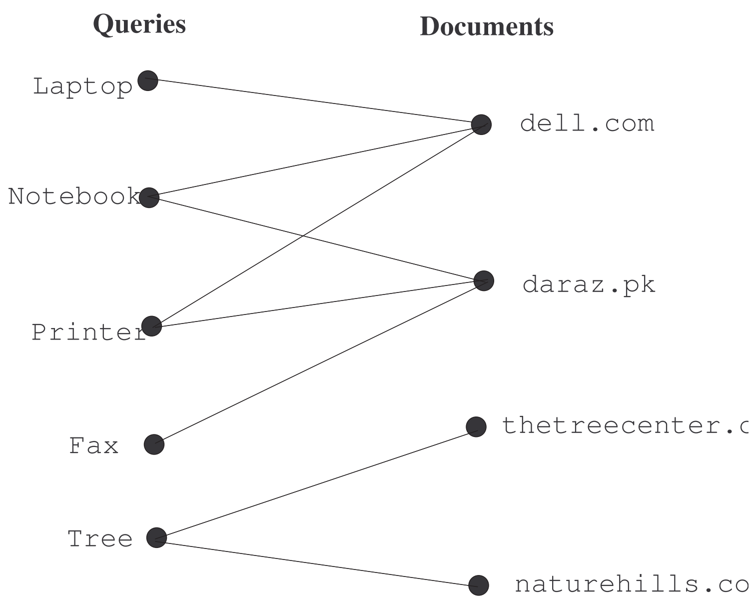



Figure 1.11: A click graph

Now we have a small amount of authentication that somehow 'laptop' and 'fax' are similar, because they bring the user on same documents by clicks. In last, let us consider the queries 'Fax' and 'tree'. There does not exist any path that connects these two queries in the click graph and hence we found that these queries are not similar.

### 1.3.2 Coloring problem

*Scheduling Cabs*: [20] In real-world problem, we use graph coloring vastly. These type of problems arise when we deal with scheduling of task such that every task has start and finish time. Let us suppose a scenario that cab company has received $m$ number of reservations in a day. Each cab has starting time indicating that when the cab will depart and each have finish time signifying that when the cab is anticipated that would return. How can we allot $m$ reservations to cabs such that smallest number of cabs is needed. We explain it by a diagram. In the Figure 1.12, we have ten reservations for cabs. For explanatory purpose to reader, they are in order according to their start time from top to bottom. It can easily be seen from the Figure 1.12, that booking 3 covers with reservation 1, 2 and 4; thus any cab accomplishing reservation 3 would not allow to work for reservation 1, 2 and 4.

From the given data, we can build a graph such that choosing single vertex for all reservation and now we add edges between any vertex pair for overlapping reservations. A graph of 3-colouring is given in Figure 1.12(b), and the corresponding task of the reservation to 3 cabs (the smallest color have been used), see Figure 1.12 (c).

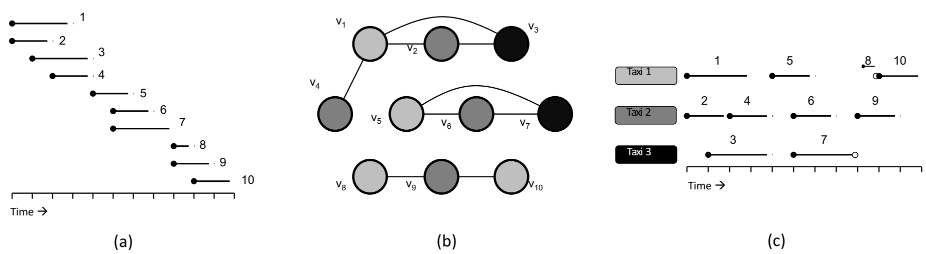



Figure 1.12: (a) Set of cab travel over time, (b) 3-coloring graph with corresponding interval, and (c) the analogous task of traveling with cabs

## 1.4 Overview

The arrangement of this exposition is as follows:

In Chapter 2, we survey some fundamental results associated with resolvability of graphs. These results have been obtained from [2, 5, 18, 27].

In Chapter 3, We find the partition dimension of a chain cycle constructed by even cycles and a chain cycle constructed by odd cycles. We also calculate the strong metric dimension of a chain cycle constructed by even cycles.

# Chapter 2

# Resolvability and metric dimension

This chapter includes discussion on the concepts and applications of resolvability. It also discusses basic results from metric dimension. Moreover, the concept of partition dimension and strong metric dimension is also discussed.

## 2.1 Introduction to resolvability

The length of a shortest $(x, y)$-path in graph(connected) $G$ is the distance between 2 vertices $x, y \in G$. Let $W = \{x_1, x_2, \ldots, x_k\} \subseteq V(G)$ be ordered set and $v \in G$, we denote the $k$-vector $r_W(z)$ defined by

$$r_W(z) = (d(z, x_1), d(z, x_2), \ldots, d(z, x_k))$$

as the code of $v$ with respect to $W$. If distinct vertices have distinct codes then set $W$ is called a resolving set. A resolving set of smallest cardinality for a graph $G$ is called a basis for $G$. Metric dimension is the number of vertices in basis is denoted by $dim(G)$.

Melter and Harary [12] solo introduced the metric dimension of a graph in 1970s, and also by Slater [32]. Resolvability of digraphs was first researched by Chartrand et al. [3] in 2000 and further in [4]. For more detail see [2, 12, 14, 17, 35]. Fehr et al. [10] studied the resolvability of Cayley digraphs. The idea of locating set was introduced by Slater and he used the term locating set, which is now called resolving set. Slater mentioned to cardinality of smallest resolving set as its locating number denoted by $loc(G)$. He [31] explained the practicality of these concepts while working with coast guard LORAN stations and U.S. sonar. Metric dimension is a framework that has seemed in many applications, with complicated as network problems and verification, combinatorial optimization, methods for mastermind program, molecular chemistry and so on. It is shown in literature that calculating the metric dimension is an NP-complete problem. For understanding of NP-complete, NP-hard and other decision problems, the reader is referred to [11]. The motivation for these concepts is derived from chemistry. When we study chemical structures a basic problem arises of mathematical classification of chemical compounds. In this scenario, the compounds are modeled through mathematical tools and compound are studied by means of these mathematical object. We can study chemical structures easily by doing research on classification of compounds. We can represent chemical compound by graphs naturally. Vertex represents atoms of the molecule and edges of graph represent the valence bond between pair of atoms.

Let us consider an example of ethane of formula $C_2H_6$, where $C_2$ indicates two carbon atoms and $H_6$ indicate six hydrogen atoms. An ethane molecule can be shown by graph in Figure 2.1.

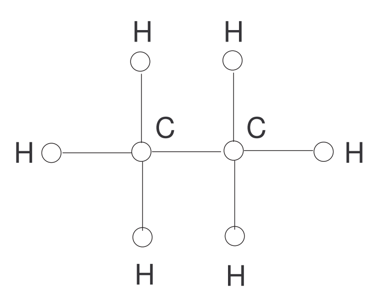



Figure 2.1: Ethane

If a molecule has different arrangements and same number of atoms is called isomer. For example, both ethanol and methoxymethane have the same atomic formula $C_2H_6O$

($O$ represents oxygen atom) and they have different chemical properties. The graphs of these compounds are shown in Figure 2.2.

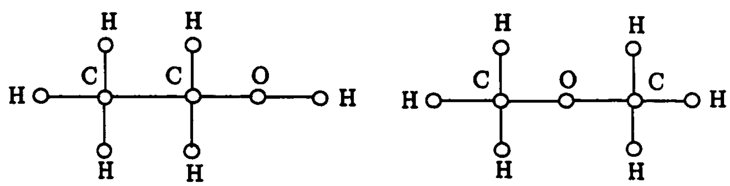



Figure 2.2: Ethanol and Methoxymethane

For further explanation, we consider an example from a doctoral thesis [33]. Assume a hotel that has five sections $S_1, S_2, S_3, S_4, S_5$ on ground floor as shown in Figure 2.3. The space from $S_2$ and $S_4$ is $2m$ and from $S_1$ and $S_3$ is also $2m$. The distance betwixt all other set of two distinct section is $1m$. The distance betwixt a section and itself is $0m$. Assume that a green fire alarm is placed in any arbitrary section. If a fire should take place in that section, then the alarm can check the distance from section with the alarm to section that has fire. For better understanding, assume that the alarm is placed in $S_1$. If a fire occurs in $S_3$, then the alarm warns us that a fire has taken place in a section at distance $2m$ from $S_1$. Then the fire is in $S_3$, now $S_3$ is the only section at distance $2m$ from $S_1$. If fire occurs in $S_1$, when alarm shows that fire has been occurred in hotel section at distance $0m$ from $S_1$. Then fire takes place in $S_1$. However, if fire occurs in the other 3 sections. Then alarm shows us that, fire takes place in a room at distance $1m$ from $S_1$. Resultantly, we cannot tell precisely in which section the fire has been take place. Actually, no section exist where alarm can be placed to distinguish the precise position of a fire. Furthermore, if we set the green fire alarm in $S_1$ and a red fire alarm in $S_2$, and a fire has been occurred in $S_4$, then the green alarm in $S_1$ indicate us that there is a fire in a section at distance $1m$ from $S_1$, at the same time red alarm indicate us that the fire is in a section at distance $2m$ from $S_2$, that is, $S_4$ has the code $(1, 2)$. Hence the representations are distinct for each of section. The smallest number of alarms required to check the precise position of any fire is two. Although $2m$ is the answer. We have to

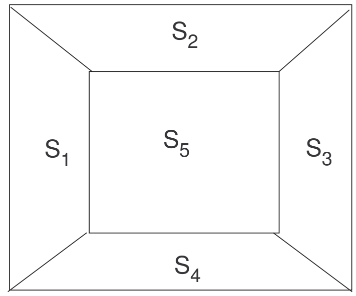



Figure 2.3: Hotel with five sections

care, where the two alarms are placed. For example, we cannot place alarms in $S_1$ and $S_3$ since, in this case, the representation of $S_2$, $S_4$, and $S_5$ are all same, and we cannot identify the exact location of the fire.

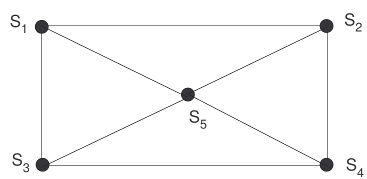



Figure 2.4: Graph representing hotel with five sections

For checking a subset(ordered) $W \subset V(G)$ is a resolving set, only we need to check with $V(G) \setminus W$ set. Hence, for graph $G$ and $\forall\ v \in G$, the $V(G)$ and $V(G) - \{v\}$ are resolving set. Let $G$ is connected graph with $|V(G)| = n \geq 2$. Then we have this inequality $1 \leq dim(G) \leq n - 1$. Contrastingly, if diameter of $G$ is known then one can easily get an better lower bound as well as upper bound in general for metric dimension. Choose arbitrary positive integers $b$ and $n$, we can interpret $g(n, b)$ is the smallest positive integer $k$, this inequality holds $k + b^k \geq n$.

**Theorem 2.1** (Chartrand et al. [2]). *Let $H$ is graph(connected), where $|V(H)| = n \geq 2$ and diameter $d_1$. Then we have $g(n, d_1) \leq dim(H) \leq n - d_1$.*

Metric dimension of path can be explained from this theorem.

**Theorem 2.2** (Chartrand et al. [2])**.** *If* $G$ *is a graph(connected), where* $|V(G)| = n$*. Then we have* $G = P_n$ *iff* $dim(G) = 1$*.*

We have observed in Theorem 2.2 that if $G \cong P_n$ then metric dimension will always be 1 and an end-vertex of path makes a basis. Paths are only graphs of metric dimension 1 [2]. In contrast, let $G$ be complete graph denoted by $K_n$, where $n \geq 2$. If $W$ is a basis for $G$ and $u \notin W$, now each coordinate of $r(u|W)$ is one. Hence, each resolving set for $G$ must have each vertex of $G$ except one, we can conclude that $dim(K_n) = n - 1$. The following theorem gives a classification of graph with $dim(G) = n - 2$.

**Theorem 2.3** (Chartrand et al. [2])**.** *If* $G$ *is graph(connected), where* $|V(G)| = n \geq 4$*. Then* $G = K_p + (K_q \cup K_p)(p, q \geq 1)$ *or* $G = K_{p,q}$ $(p, q \geq 1)$*,* $G = K_p + \overline{K_q}(p \geq 1, p \geq 2)$ *iff* $dim(G) = n - 2$*.*

Furthermore, the lower bound in Theorem 2.1 is only attainable for graphs of diameter 2 or 3. We now see a result for strong lower bound for metric dimension in terms of its maximum degree, thus bounds can be improved further.

**Theorem 2.4** (Chartrand et al. [6])**.** *For any graph(connected)* $H$*, where* $|V(H)| \geq 2$*. Then we have* $\lceil \log_3(\Delta(H) + 1) \rceil \leq dim(H) \leq n - diam(H)$*.*

## 2.2 Partition dimension of graphs

In this section, we discuss few fundamental concepts of partition dimension. We will also survey some results, given by Chartrand, Salehi and Zhang [5].

Assume that $G$ is graph, which is also connected. Consider a subset $X \subseteq V(G)$ and a vertex $y \in G$. Then we can define distance $d(y, X)$ between $y$ and $X$ is given by equation

$$d(y, X) = \min\{d(y, z) \mid z \in S\}$$

Consider an $k$-partition (ordered) $\Pi = \{X_1, X_2, \ldots, X_k\} \subseteq V(G)$ and a vertex $y \in G$. Then we define the characterization of $y$ with respect to $\Pi$ from the $k$-vector

$$r(y|\Pi) = (d(y, X_1), d(y, X_2), \ldots, d(y, X_k)).$$

We call $\Pi$ a resolving partition, if the $k$-vectors $r(y|\Pi)$ are distinct, $\forall y \in V(G)$. The smallest $k$ for that we have resolving $k$-partition of $V(G)$ is called partition dimension of graph $G$. We denote it by $pd(G)$. See [5, 15, 24, 28, 34] for more results.

It is obvious that the metric and partition dimension are associated with each other. For graph $G$, we have Theorem 2.5, which explains the relationship betwixt metric and partition dimension.

**Theorem 2.5** (Chartrand et al. [5]). *Let $G$ is graph(connected). Then we have*

$$pd(G) \leq dim(G) + 1.$$

**Theorem 2.6** (Chartrand et al. [5]). *For each pair $c, d \in \mathbb{Z}_+$. For this $\lceil \frac{d}{2} \rceil + 1 \leq c \leq d+1$, there exists a graph $G$, which is connected, such that $dim(G) = d$ and $pd(G) = c$.*

Theorem 2.6 raises a question. Could this inequality be true $pd(G) \geq \frac{dim(G)}{2} + 1$ for every connected graph $G$? We also know that finding the metric dimension is an NP-complete problem in graph theory.

**Proposition 2.7** (Chartrand et al. [5]). *For a graph(connected) $G$, where $|V(G)| = n \geq 2$. Then we have $G = P_n$ if and only if $pd(G) = 2$.*

Contrastingly, there is single graph which has $n$ order, $\forall\ n \geq 2$, with partition dimension $n$. Following lemma is established before the next Proposition 2.8.

**Proposition 2.8** (Chartrand et al. [5]). *Let $G$ is graph(connected), where $|V(G)| = n$. Then $G = K_n$ iff $pd(G) = n$.*

If $G$ is graph and its order is greater than 3, then from Proposition **??** and 2.8 which is neither complete graph nor path, then this inequality is true $3 \leq pd(G) \leq n - 1$. It is seen from Proposition 2.7 that if $G$ is not taken as path where $pd(G) = 3$. Hence, it is enough to show the occurrence of a resolving 3-partition for $V(G)$. We could check the partition dimension of the Petersen graph and $n$-cycle is 4 and 3, respectively.

**Theorem 2.9** (Chartrand et al. [5]). *For graph $G$ where $G$ is connected bipartite graph. That has 2 sets $Y_1$ and $Y_2$ where $p$ and $q$ are cardinalities, respectively. Then $G$ is complete bipartite iff*

1. $pd(G) \leq \max\{p, q\}$, *if* $p \neq q$, *and*

2. $pd(G) \leq p + 1$, *if* $p = q$.

*Furthermore, equality holds in 2 or 1.*

For $n, d \in \mathbb{Z}$ such that $n > d \geq 2$, we say that $g(n, d)$ is the smallest positive integer $k$ where $(d + 1)^k \geq n$.

**Theorem 2.10** (Chartrand et al. [5]). *Let $G$ is a graph, where $|V(G)| = n \geq 3$ and diameter $d$. Then we have $g(n, d) \leq pd(G) \leq n - d + 1$.*

## 2.3 Strong metric dimension of graphs

A set $S \subseteq V(G)$ such that every edge of graph $G$ has not less than one of member of $S$ as an endpoint is known as vertex cover. The vertex set of a graph is also called a vertex cover. The minimum vertex cover for a graph $G$ is called smallest vertex covering and cardinality of set is called the vertex covering number which is denoted by $\alpha(G)$. For more detail, see [16, 19, 27, 29].

Suppose $G$ is a graph, which is connected. Consider $x, y$ are vertices of $G$. The interval $I_G[x, y]$ betwixt $x$ and $y$ can be defined as the collection of each of vertex that belongs to smallest $(x, y)$-path. A vertex $z$ is resolved strongly if $y \in I_G[x, z]$ or $x \in I_G[y, z]$ . If each of 2 vertices of $G$ are strongly resolved by arbitrary vertex of $S$ then set $X$ of vertices in a graph $G$ is a strong resolving set. The strong resolving set of smallest cardinality is called a strong metric basis of $G$ and the cardinality of that set is called its strong metric dimension, denoted by $dim_s(G)$.

The number of vertices in a largest clique in $G$ denoted by $\Omega(G)$. If $N_G[u_1] = N_G[u_2]$ then 2 distinct vertices $u_1, u_2$ are called true twins. If the induced subgraph from $Y \subset V$

is clique and we have for each $u_1, u_2$ in $Y$ such that $N_G[u_1] \neq N_G[u_2]$ then $Y$ is twin-free clique. The largest size between all twin-free cliques is denoted by $\overline{\Omega}(G)$. So, $\overline{\Omega}(G) \leq \Omega(G)$. [18]

**Theorem 2.11** (Kuziak et al. [18]). *If $H$ is graph(connected), where $|V(H)| = k \geq 2$. Then*

$$dim_s(H) \leq k - \overline{\Omega}(H).$$

*Furthermore, if $diam(H)$ is 2, then*

$$dim_s(H) = k - \overline{\Omega}(H).$$

**Lemma 2.12** (Kuziak et al. [18]). *Consider two graphs(connected) $G$ and $H$, where $|V(G)| = n_1 \geq 2$ and $|V(H)| = n_2 \geq 2$, and maximum degree are given by $\Delta_1(G)$ and $\Delta_2(G)$, respectively.*

1. *If $\Delta_1(G) \neq n_1 - 1$ or $\Delta_2(G) \neq n_2 - 1$, then*

   $\overline{\Omega}(G + H) = \overline{\Omega}(G) + \overline{\Omega}(H).$

2. *If $\Delta_1(G) = n_1 - 1$ and $\Delta_2(G) = n_2 - 1$, then*

   $\overline{\Omega}(G + H) = \overline{\Omega}(G) + \overline{\Omega}(H) - 1.$

Consider two complete graphs $G$ and $H$ where order of $G$ is $p$ and order of $H$ is $q$, respectively, then $G + H = K_{p+q}$ and $dim_s(G + H) = dim_s(K_{p+q}) = p + q - 1$. From Theorem 2.11 and Lemma 2.12, following results are obtained.

**Theorem 2.13** (Kuziak et al. [18]). *Consider two graphs(connected) $H$ and $G$, where $|V(H)| = p \geq 2$ and $|V(G)| = q \geq 2$, and maximum degree are given by $\Delta_1(H)$ and $\Delta_2(H)$, respectively.*

1. *If $\Delta_1(H) \neq p - 1$ or $\Delta_2(H) \neq q - 1$, then we have*

   $dim_s(H + G) = p + q - \overline{\Omega}(H) - \overline{\Omega}(G) \geq dim_s(H) + dim_s(G).$

2. *If* $H$ *and* $G$ *are graphs, where* $diam(H) = diam(G) = 2$ *&* $\overline{\Omega}_1 \neq P-1$ *or* $\overline{\Omega}_2 \neq q-1$, *then we have*

   $dim_s(H+G) = dim_s(H) + dim_s(G)$.

3. *If* $\overline{\Omega}_1 = p-1$ *and* $\overline{\Omega}_2 = q-1$, *then we have*

   $dim_s(H+G) = dim_s(H) + dim_s(G) + 1$.

**Corollary 2.14** (Kuziak et al. [18])**.** *Let* $G$ *is graph(connected), where* $|V(G)| = n$, $dim_s(G \odot K_1) = n-1$.

**Theorem 2.15** (Kuziak et al. [18])**.** *Let* $G$ *is graph(connected),* $|V(G)| = q$. *Consider another graph* $H$, *where* $|V(H)| = p$ *and* $\Delta(G)$ *is largest degree.*

1. *If* $\Delta(G) = p-1$, *then* $dim_s(K_1 + H) = p + 1 - \overline{\Omega}(H)$.

2. *If* $\Delta(G) \leq p-2$ *or* $q \geq 2$, *then* $dim_s(G \odot H) = qp - \overline{\Omega}(H)$.

**Corollary 2.16** (Kuziak et al. [18])**.** *If* $G$ *is graph(connected),* $|V(G)| = n_1$. *If* $H$ *is a triangle free graph* $|V(H)| = n_2 \geq 3$ *&* $\Delta(G)$ *is the largest degree. Let* $n_1 \geq 2$ *or* $\Delta(G) \leq n_2 - 2$. *Then*

$$dim_s(G \odot H) = n_1 n_2 - 2.$$

# Chapter 3

# Resolvability of chain cycle

In this chapter, we compute the partition dimension of a chain cycle constructed by even cycles and a chain cycle constructed by odd cycles. We also calculate the strong metric dimension of a chain cycle constructed by even cycles.

## 3.1 Introduction

Let $G$ be a finite connected and simple graph. In this section, the diameter of a graph is denoted by $d(G)$. A cycle of lenght $n$ is denoted by $C_n$.

A vertex $u \in V(G)$ is maximally distant from $v \in V(G)$ which is denoted by $u$MD$v$, if $\forall\ w \in N(u)$ we have this inequality $d_G(v,w) \leq d_G(u,v)$. If $u$ is maximally distant from $v$ and vice versa, then we say that $u$ and $v$ are mutually maximally distant and is denoted by $u$MMD$v$. The strong resolving graph of $G$ is a graph $G_{SR}$ whose vertex set is $V(G)$ and two vertices $u, v \in V(G)$ are adjacent in $G_{SR}$ iff $u$MMD$v$. Oellermann and Peters-Fransen [27] proved that determining the strong metric dimension of a graph $G$ is analogous to determining the vertex cover number of $G_{SR}$.

**Theorem 3.1** (Oellermann and Peters-Fransen [27]). *Let $G$ is graph(connected). Then $dim_s(G) = \alpha(G_{SR})$.*

We will use Theorem 3.1 for solving our problem. We identify mutually maximally distant vertices for particular class of graph, then we generate a strong resolving graph $G_{SR}$ from mutually maximally distant vertices $V(G_{SR})$. Lastly, we find the vertex cover $\alpha(G_{SR})$ of graph.

Let $\{G_i\}_{i=1}^m$ be finite set of disjoint simple connected graphs occuring in pairs. The chain graph

$$\mathcal{C}(G_1, G_2, \ldots, G_m) = \mathcal{C}(G_1, G_2, ..., G_m; x_1, w_1, x_2, w_2, \ldots, x_m, w_m)$$

of $\{G_i\}_{i=1}^m$ with respect to the vertices $\{x_i, w_i \in V(G_i) \mid i = 1, 2, \ldots, m\}$ is the graph acquired from the graphs $G_1, \ldots, G_m$ by identifying the vertex $w_i$ and the vertex $x_{i+1}$, as shown in Figure 3.1, for all $i \in \{1, 2, \ldots, m-1\}$. For more results and detail about chain graph, see [22, 25].

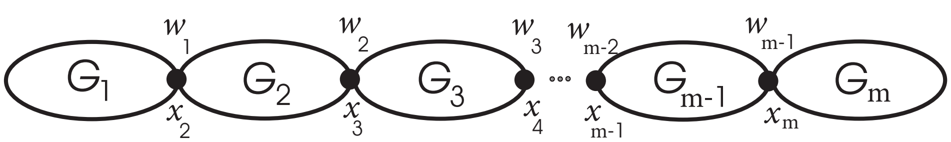



Figure 3.1: A chain graph

Let $\{C_{n_i}\}_{i=1}^m$ be finite set of disjoint simple cycles occuring in pairs. Let $V(C_{n_i}) = \{v_j^i \mid j \equiv 1, 2, \ldots, n_i\}$, where $i \in \{1, 2, \ldots, m\}$. Assume that $n_i$ is even for each $i = 1, 2, \ldots, m$. We consider a chain cycle of $\{C_{n_i}\}_{i=1}^m$ given by

$$\mathcal{C}(C_{n_1}, C_{n_2}, \ldots, C_{n_m}) = \mathcal{C}\left(C_{n_1}, C_{n_2}, \ldots, C_{n_m}; v_1^1, v_1^2, v^1_{\frac{n_1}{2}+1}, v_1^3, v^2_{\frac{n_2}{2}+1}, \ldots, v_1^m, v^{m-1}_{\frac{n_{m-1}}{2}+1}, v^m_{\frac{n_m}{2}+1}\right)$$

with respect to the vertices $\{v^i_{\frac{n_i}{2}+1}, v_1^{i+1} \mid i = 1, 2, \ldots, m-1\}$. A chain cycle of $\{C_8, C_{10}, C_8\}$ with respect to vertices $\{v_5^1, v_1^2, v_6^2 v_1^3\}$ is shown in Figure 3.2.

Now, assume that $n_i$ is odd for each $i = 1, 2, \ldots, m$. We consider a chain cycle of $\{C_{n_i}\}_{i=1}^m$ given by

$$\mathcal{C}(C_{n_1}, C_{n_2}, \ldots, C_{n_m}) = \mathcal{C}\left(C_{n_1}, C_{n_2}, \ldots, C_{n_m}; v_1^1, v_1^2, v^1_{\frac{n_1+1}{2}+1}, v_1^3, v^2_{\frac{n_2+1}{2}+1}, \ldots, v_1^m, v^{m-1}_{\frac{n_{m-1}+1}{2}+1}, v^m_{\frac{n_m+1}{2}+1}\right)$$

with respect to the vertices $\{v^i_{\frac{n_i+1}{2}+1}, v^{i+1}_1 \mid i = 1, 2, \ldots, m-1\}$. A chain cycle of $\{C_5, C_7, C_5\}$ with respect to vertices $\{v^1_4, v^2_1, v^2_5, v^3_1\}$ is shown in Figure 3.3.

Through out the chapter, we denote the edge set and the vertex set of chain cycle by $E(\mathcal{C})$ & $V(\mathcal{C})$ instead of $E(C_{n_1}, C_{n_2}, \ldots, C_{n_m})$ & $V(C_{n_1}, C_{n_2}, \ldots, C_{n_m})$, respectively.

## 3.2 Partition dimension of chain cycles

In this section, we find the partition dimension of chain cycle constructed by even cycles and chain cycle constructed by odd cycles. Theorem 2.5 and Proposition 2.7 are important tools for proving our results.

In the Theorem 3.2, we compute the partition dimension of chain cycle constructed by even cycles.

**Theorem 3.2.** *The partition dimension of chain cycle constructed by even cycles* $\mathcal{C}(C_{n_1}, C_{n_2}, \ldots, C_{n_m}) = \mathcal{C}\left(C_{n_1}, C_{n_2}, \ldots, C_{n_m}; v^1_1, v^2_1, v^1_{\frac{n_1}{2}+1}, v^3_1, v^2_{\frac{n_2}{2}+1}, \ldots, v^m_1, v^{m-1}_{\frac{n_{m-1}}{2}+1}, v^m_{\frac{n_m}{2}+1}\right)$ *is* 3.

*Proof.* Let $\Pi = \{Q_1, Q_2, Q_3\}$, where $Q_1 = \{v^1_1, \ldots, v^1_{\frac{n_1}{2}-1}, v^1_{\frac{n_1}{2}+2}, \ldots, v^1_{n_1}\}$, $Q_2 = \{v^2_{\frac{n_2}{2}+3}, v^2_{\frac{n_2}{2}+4}, \ldots, v^2_{n_2}, v^3_{\frac{n_3}{2}+3}, v^3_{\frac{n_3}{2}+4}, \ldots, v^3_{n_3}, \ldots, v^{m-1}_{\frac{n_{m-1}}{2}+3}, v^{m-1}_{\frac{n_{m-1}}{2}+4}, \ldots, v^{m-1}_{n_{m-1}}\} \cup \{v^m_{n_m}\}$ and $Q_3 = V(\mathcal{C}) \setminus \{Q_1 \cup Q_2\}$ be a partition of $V(\mathcal{C})$. We show that $\Pi$ is a resolving partition of $V(\mathcal{C})$ with smallest cardinality. The representation of each vertex of $V(\mathcal{C})$ with respect to $\Pi$ is given as:

$$r(v^1_{\frac{n_1}{2}}|\Pi) = (1, 2, 0), \quad r(v^1_{\frac{n_1}{2}+1}|\Pi) = (1, 1, 0), \quad r(v^m_{n_m}|\Pi) = \left(\sum_{k=2}^{m} \frac{n_k}{2} + 2, 0, 1\right).$$

$$r(v^1_j|\Pi) = \begin{cases} (0, n_1 - j - 1, n_1 - j - 3) & \text{if } 1 \leq j \leq \frac{n_1}{2} - 1 \\ (0, j - \frac{n_1}{2}, j - \frac{n_1}{2} - 1) & \text{if } \frac{n_1}{2} + 2 \leq j \leq n_1, \end{cases}$$

$$r(v^i_j|\Pi) = \begin{cases} (j, j, 0) & \text{if } 1 \leq j \leq \lceil \frac{n_2}{4} \rceil \\ (j, \frac{n_2}{2} - j + 2, 0) & \text{if } \lceil \frac{n_2}{4} \rceil + 1 \leq j \leq \frac{n_2}{2}, \end{cases}$$

$$r(v^i_j|\Pi) = \begin{cases} \left(\sum_{k=3}^{m} \frac{n_k}{2} + j, j, 0\right) & \text{if } 1 \leq j \leq \lceil \frac{n_i}{4} \rceil, 3 \leq i \leq m \\ \left(\sum_{k=3}^{m} \frac{n_k}{2} + j, \frac{n_i}{2} - j + 2, 0\right) & \text{if } \lceil \frac{n_i}{4} \rceil + 1 \leq j \leq \frac{n_i}{2}, 3 \leq i \leq m, \end{cases}$$

$$r(v^i_{\frac{n_i}{2}+j}|\Pi) = \begin{cases} \left(\sum\limits_{k=2}^{m-1} \frac{n_k}{2}, 1, 0\right) & \text{if } 2 \le i \le m-1, j=2 \\ \left(\sum\limits_{k=2}^{m-1} \frac{n_k}{2} + n_m - j + 2, n_i - j, 0\right) & \text{if } i = m, \frac{n_i}{2}+1 \le j \le n_i - 1, \end{cases}$$

$$r(v^i_j|\Pi) = \begin{cases} (n_i + 2 - j,\ 0\ , j - \frac{n_i}{2} - 2) & \text{if } \frac{n_2}{2} + 3 \le j \le \lceil \frac{3n_2}{4} \rceil + 1 \\ (n_i + 2 - j,\ 0\ , n_i + 1 - j) & \text{if } \lceil \frac{3n_2}{4} \rceil + 2 \le j \le n_2, \end{cases}$$

$$r(v^i_j|\Pi) = \begin{cases} \left(\sum\limits_{k=3}^{m} \frac{n_k}{2} + n_i - j,\ 0\ , j - \frac{n_i}{2} - 2\right) & \text{if } \frac{n_i}{2} + 3 \le j \le \lceil \frac{3n_i}{4} \rceil + 1, 3 \le i \le m \\ \left(\sum\limits_{k=3}^{m} \frac{n_k}{2} + n_i - j,\ 0\ , n_i + 1 - j\right) & \text{if } \lceil \frac{3n_i}{4} \rceil + 2 \le j \le n_i, 3 \le i \le m. \end{cases}$$

It is easily seen that the presentation of each vertex with respect to $\Pi$ is distinct. This shows that $\Pi$ is a resolving partition of $\mathcal{C}(C_{n_1}, C_{n_2}, \ldots, C_{n_m})$. Thus $pd(\mathcal{C}(C_{n_1}, C_{n_2}, \ldots, C_{n_m})) \le 3$.

On the other hand, by Proposition 2.7 it follows that $pd(\mathcal{C}(C_{n_1}, C_{n_2}, \ldots, C_{n_m})) \ge 3$. Hence $pd(\mathcal{C}(C_{n_1}, C_{n_2}, \ldots, C_{n_m})) = 3$. □

In the Example 3.3, we find the partition dimension of a chain cycle constructed by $C_8$, $C_{10}$ and $C_8$.

**Example 3.3.** Let $m = 3$ and $n_1 = 8$, $n_2 = 10$ and $n_3 = 8$. The chain cycle constructed by $C_8$, $C_{10}$ and $C_8$ with respect to the vertices $\{v^1_5, v^2_1, v^2_6 v^3_1\}$ is denoted by $\mathcal{C}(C_8, C_{10}, C_8) = \mathcal{C}(C_8, C_{10}, C_8; v^1_1, v^2_1, v^1_5, v^3_1,\ v^2_6, v^3_5)$ and is given in Figure 3.2.

Using Theorem 3.2, we construct a resolving partition of $\mathcal{C}(C_8, C_{10}, C_8)$ as $\Upsilon = \{Q_1, Q_2, Q_3\}$, where $Q_1 = \{v^1_1, v^1_2, v^1_3, v^1_6, v^1_7, v^1_8\}$, $Q_2 = \{v^2_8, v^2_9, v^2_{10}, v^3_8\}$ and $Q_3 = \{v^1_4, v^2_5, v^1_5, v^2_3, v^2_2, v^2_4, v^2_6, v^2_7, v^3_2, v^3_3, v^3_5, v^3_4, v^3_7, v^3_6\}$. Again by Theorem 3.2, we note that each vertex of $\mathcal{C}(C_8, C_{10}, C_8)$ has distinct representation with respect to $\Upsilon$, as shown in Table 1. Hence $pd(\mathcal{C}(C_8, C_{10}, C_8)) = 3$.

In Theorem 3.4, we compute the partition dimension of chain cycle constructed by odd cycles.

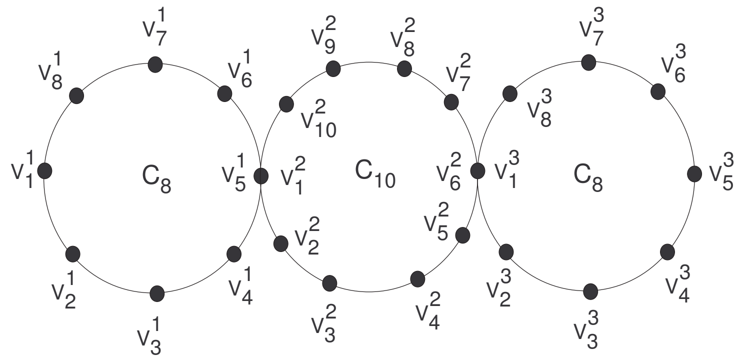



Figure 3.2: Chain cycles of $C_8$, $C_{10}$ and $C_8$

| | | |
|---|---|---|
| $s(v_1^1\|\Upsilon) = (0,5,3)$ | $s(v_2^2\|\Upsilon) = (2,2,0)$ | $s(v_{10}^2\|\Upsilon) = (2,0,1)$ |
| $s(v_2^1\|\Upsilon) = (0,4,2)$ | $s(v_3^2\|\Upsilon) = (3,3,0)$ | $s(v_2^3\|\Upsilon) = (7,2,0)$ |
| $s(v_3^1\|\Upsilon) = (0,3,1)$ | $s(v_4^2\|\Upsilon) = (4,3,0)$ | $s(v_3^3\|\Upsilon) = (8,3,0)$ |
| $s(v_4^1\|\Upsilon) = (1,2,0)$ | $s(v_5^2\|\Upsilon) = (5,2,0)$ | $s(v_4^3\|\Upsilon) = (9,4,0)$ |
| $s(v_5^1\|\Upsilon) = (1,1,0)$ | $s(v_6^2\|\Upsilon) = (6,1,0)$ | $s(v_5^3\|\Upsilon) = (10,3,0)$ |
| $s(v_6^1\|\Upsilon) = (0,2,1)$ | $s(v_7^2\|\Upsilon) = (5,1,0)$ | $s(v_6^3\|\Upsilon) = (9,2,0)$ |
| $s(v_7^1\|\Upsilon) = (0,3,2)$ | $s(v_8^2\|\Upsilon) = (4,0,1)$ | $s(v_7^3\|\Upsilon) = (8,1,0)$ |
| $s(v_8^1\|\Upsilon) = (0,4,3)$ | $s(v_9^2\|\Upsilon) = (3,0,2)$ | $s(v_8^3\|\Upsilon) = (7,0,1)$ |

Table 3.1: Representation of $v_j^i$ with respect to $\Upsilon$

**Theorem 3.4.** *The partition dimension of chain cycle constructed by odd cycles* $\mathcal{C}(C_{n_1}, C_{n_2}, \ldots, C_{n_m}) = \mathcal{C}\left(C_{n_1}, C_{n_2}, \ldots, C_{n_m}; v_1^1, v_1^2, v^1_{\frac{n_1+1}{2}+1}, v_1^3, v^2_{\frac{n_2+1}{2}+1}, \ldots, v_1^m, v^{m-1}_{\frac{n_{m-1}+1}{2}+1}, v^m_{\frac{n_m+1}{2}+1}\right)$ *is* 3.

*Proof.* Let $\Pi = \{Q_1, Q_2, Q_3\}$, where $Q_1 = \{v_1^1, \ldots, v^1_{\lceil\frac{n_1}{2}\rceil-1}, v^1_{\lceil\frac{n_1}{2}\rceil+2}, \ldots, v^1_{n_1}\}$, $Q_2 = \{v^2_{\lceil\frac{n_2}{2}\rceil+3}, v^2_{\lceil\frac{n_2}{2}\rceil+4}, \ldots, v^2_{n_2}, v^3_{\lceil\frac{n_3}{2}\rceil+3}, v^3_{\lceil\frac{n_3}{2}\rceil+4}, \ldots, v^3_{n_3}, \ldots, v^{m-1}_{\lceil\frac{n_{m-1}}{2}\rceil+3}, v^{m-1}_{\lceil\frac{n_{m-1}}{2}\rceil+4}, \ldots, v^{m-1}_{n_{m-1}}\} \cup \{v^m_{n_m}\}$ and $Q_3 = V(\mathcal{C}) \setminus \{Q_1 \cup Q_2\}$ be a partition of $V(\mathcal{C})$. We show that $\Pi$ is a resolving partition of $V(\mathcal{C})$ with smallest cardinality. The representation of each vertex of $V(\mathcal{C})$ with respect to $\Pi$ is given as:

$$r(v^1_{\lceil \frac{n_1}{2}\rceil}|\Pi) = (1,2,0), \quad r(v^1_{\lceil \frac{n_1}{2}\rceil+1}|\Pi) = (1,1,0), \quad r(v^m_{n_m}|\Pi) = \left(\sum_{k=2}^{m}\lfloor \frac{n_k}{2}\rfloor + 2, 0, 1\right).$$

$$r(v^1_j|\Pi) = \begin{cases} (0, n_1 - \lfloor \frac{n_1}{2}\rfloor, n_1 - \lfloor \frac{n_1}{2}\rfloor - 1) & \text{if j=1} \\ (0, n_1 - j - 1, n_1 - j - 3) & \text{if } 2 \le j \le \lceil \frac{n_1}{2}\rceil - 1 \\ (0, j - \lceil \frac{n_1}{2}\rceil, j - \lceil \frac{n_1}{2}\rceil - 1) & \text{if } \lceil \frac{n_1}{2}\rceil + 2 \le j \le n_1, \end{cases}$$

$$r(v^i_j|\Pi) = \begin{cases} (j, j, 0) & \text{if } 1 \le j \le \lceil \frac{n_2}{4}\rceil + 1 \\ (j, \lceil \frac{n_2}{2}\rceil - j + 3, 0) & \text{if } \lceil \frac{n_2}{4}\rceil + 2 \le j \le \lceil \frac{n_2}{2}\rceil + 1, \end{cases}$$

$$r(v^i_j|\Pi) = \begin{cases} \left(\sum\limits_{k=3}^{m}\lfloor \frac{n_k}{2}\rfloor + j, j, 0\right) & \text{if } 1 \le j \le \lceil \frac{n_i}{4}\rceil + 1, 3 \le i \le m \\ \left(\sum\limits_{k=3}^{m}\lfloor \frac{n_k}{2}\rfloor + j, \lceil \frac{n_i}{2}\rceil - j + 3, 0\right) & \text{if } \lceil \frac{n_i}{4}\rceil + 2 \le j \le \lceil \frac{n_i}{2}\rceil + 1, 3 \le i \le m, \end{cases}$$

$$r(v^i_{\frac{n_i}{2}+j}|\Pi) = \begin{cases} \left(\sum\limits_{k=2}^{m-1}\lfloor \frac{n_k}{2}\rfloor, 1, 0\right) & \text{if } 2 \le i \le m-1, j = 2 \\ \left(\sum\limits_{k=2}^{m-1}\lfloor \frac{n_k}{2}\rfloor + n_m - j + 2, n_i - j, 0\right) & \text{if } i = m, \lceil \frac{n_i}{2}\rceil + 2 \le j \le n_i - 1, \end{cases}$$

$$r(v^i_j|\Pi) = \begin{cases} (n_i + 2 - j, 0, j - \lceil \frac{n_i}{2}\rceil - 2) & \text{if } \lceil \frac{n_2}{2}\rceil + 3 \le j \le \lceil \frac{3n_2}{4}\rceil + 1 \\ (n_i + 2 - j, 0, n_i + 1 - j) & \text{if } \lceil \frac{3n_2}{4}\rceil + 2 \le j \le n_2, \end{cases}$$

$$r(v^i_j|\Pi) = \begin{cases} \left(\sum\limits_{k=3}^{m}\lceil \frac{n_k}{2}\rceil + n_i - j,\ 0\ , j - \lceil \frac{n_i}{2}\rceil - 2\right) & \text{if } \lceil \frac{n_i}{2}\rceil + 3 \le j \le \lceil \frac{3n_i}{4}\rceil + 1, 3 \le i \le m \\ \left(\sum\limits_{k=3}^{m}\lceil \frac{n_k}{2}\rceil + n_i - j,\ 0\ , n_i + 1 - j\right) & \text{if } \lceil \frac{3n_i}{4}\rceil + 2 \le j \le n_i, 3 \le i \le m. \end{cases}$$

It is easily seen that the presentation of each vertex with respect to $\Pi$ is distinct. This shows that $\Pi$ is a resolving partition of $\mathcal{C}(C_{n_1}, C_{n_2}, \ldots, C_{n_m})$. Thus *pd*$(\mathcal{C}(C_{n_1}, C_{n_2}, \ldots, C_{n_m}))$ $\le 3$.

On the other hand, by Proposition 2.7 it follows that *pd*$(\mathcal{C}(C_{n_1}, C_{n_2}, \ldots, C_{n_m})) \ge 3$. Hence *pd*$(\mathcal{C}(C_{n_1}, C_{n_2}, \ldots, C_{n_m})) = 3$. □

In the Example 3.5, we compute the partition dimension of the chain cycle constructed by $C_5$, $C_7$ and $C_5$.

| | |
|---|---|
| $s(v_1^1|\Upsilon) = (0,3,2)$ | $s(v_5^2|\Upsilon) = (4,1,0)$ |
| $s(v_2^1|\Upsilon) = (0,3,1)$ | $s(v_6^2|\Upsilon) = (3,1,0)$ |
| $s(v_3^1|\Upsilon) = (1,2,0)$ | $s(v_7^2|\Upsilon) = (2,0,1)$ |
| $s(v_4^1|\Upsilon) = (1,1,0)$ | $s(v_2^3|\Upsilon) = (5,2,0)$ |
| $s(v_5^1|\Upsilon) = (0,2,1)$ | $s(v_3^3|\Upsilon) = (6,2,0)$ |
| $s(v_2^2|\Upsilon) = (2,2,0)$ | $s(v_4^3|\Upsilon) = (6,1,0)$ |
| $s(v_3^2|\Upsilon) = (3,3,0)$ | $s(v_5^3|\Upsilon) = (5,0,1)$ |
| $s(v_4^2|\Upsilon) = (4,2,0)$ | |

Table 3.2: Representation of $v_j^i$ with respect to $\Upsilon$

**Example 3.5.** Let $m = 3$ and $n_1 = 5$, $n_2 = 7$ and $n_3 = 5$. The chain cycle constructed by $C_5$, $C_7$ and $C_5$ with respect to vertices $\{v_4^1, v_1^2, v_5^2, v_1^3\}$ is denoted by $\mathcal{C}(C_5, C_7, C_5) = \mathcal{C}(C_5, C_7, C_5; v_1^1, v_1^2, v_4^1, v_1^3, v_5^2, v_4^3)$ and is given in Figure 3.3.

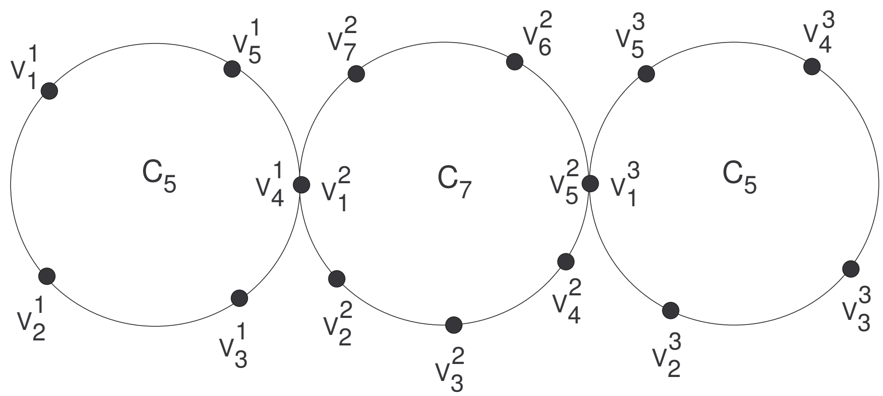



Figure 3.3: Chain cycles of $C_5$, $C_7$ and $C_5$

Using Theorem 3.4, we construct a resolving partition of $\mathcal{C}(C_5, C_7, C_5)$ as $\Upsilon = \{Q_1, Q_2, Q_3\}$, where $Q_1 = \{v_1^1, v_2^1, v_5^1\}$, $Q_2 = \{v_7^2, v_5^3\}$ and $Q_3 = \{v_3^1, v_4^1, v_2^2, v_3^2, v_4^2, v_5^2, v_6^2, v_2^3, v_3^3, v_4^3\}$. Again by Theorem 3.4, we note that each vertex of $\mathcal{C}(C_5, C_7, C_5)$ has distinct representation with respect to $\Upsilon$, as shown in Table 2. Hence $pd(\mathcal{C}(C_5, C_7, C_5)) = 3$.

## 3.3 Strong metric dimension of chain cycle

In this section, we find the strong metric dimension of chain cycle $\mathcal{C}(C_{n_1}, C_{n_2}, \ldots, C_{n_m})$, with each $n_i$ is even, with respect to the vertices $\{v^i_{\frac{n_i}{2}+1}, v^{i+1}_1 \mid i = 1, 2, \ldots, m-1\}$. Let $V_1 = \{v^2_1, v^3_1, \ldots, v^m_1\}$ and $V_2 = V(\mathcal{C}) \setminus V_1$. Furthermore, we denote $U_1(C_{n_i}) = \{v^i_1, v^i_2, \ldots, v^i_{\frac{n_i}{2}}\}$ and $U_2(C_{n_i}) = \{v^i_{\frac{n_i}{2}+1}, v^i_{\frac{n_i}{2}+2}, \ldots, v^i_{n_i}\}, i \in \{1, 2, \ldots, m\}$. Through out the section, we denote the strong resolving graph of a chain cycle $\mathcal{C}(C_{n_1}, C_{n_2}, \ldots, C_{n_m})$ by $\mathcal{C}_{SR}(C_{n_1}, C_{n_2}, \ldots, C_{n_m})$. Furthermore, we identify the edge set and the vertex set of the strong resolving graph of a chain cycle $\mathcal{C}(C_{n_1}, C_{n_2}, \ldots, C_{n_m})$ by $E(\mathcal{C}_{SR})$ and $V(\mathcal{C}_{SR})$, respectively.

Lemma 3.6 and Lemma 3.7 are easy observations from the structure of a cycle $C_n$ and a chain cycle constructed by even cycles, respectively.

**Lemma 3.6.** *Let $C_n$ be a cycle. Then for two distinct vertices $v_i, v_j \in V(C_n)$ we have $v_i MMD v_j$ if and only if $d(v_i, v_j) = d(C_n)$.*

*Proof.* Let $v_i$, $v_j \in V(C_n)$ such that $d(v_i, v_j) = d(C_n)$. Then it is clear that $v_i$MMD$v_j$.

Conversely, let $v_i$MMD$v_j$. Note that, the set of neighbors of $v_i$ and $v_j$ are $N(v_i) = \{v_{i-1}, v_{i+1}\}$ and $N(v_j) = \{v_{j-1}, v_{j+1}\}$, respectively. Then by definition

$$d(v_{i-1}, v_j) \leq d(v_i, v_j) \text{ and } d(v_{i+1}, v_j) \leq d(v_i, v_j),$$

and

$$d(v_i, v_{j-1}) \leq d(v_i, v_j) \text{ and } d(v_i, v_{j+1}) \leq d(v_i, v_j).$$

We show that $d(v_i, v_j) = d(C_n)$. On contrary, suppose that $d(u_i, u_j) = l < d(C_n)$. Consider a shortest path $P = v_i v_{i-1} \ldots v_{j+1} v_j$ from $v_i$ to $v_j$. Then from $N(v_i)$, $N(v_j)$ and $P$, we have $d(v_i, v_j) < d(v_{i+1}, v_j)$. Thus $v_i$ is not MD from $v_j$ and hence are not MMD, which is a contradiction to our supposition. □

**Lemma 3.7.** *Let $x \in V_1$ and $y \in V(\mathcal{C})$. Then $x$ and $y$ are not mutually maximally distant.*

*Proof.* Assume that $x \in V_1$ and $y \in V(\mathcal{C})$. Then $x$ and $y$ are not MMD. We discuss two cases.

Case 1: Let $x, y \in V_1$, say $x = v_1^{k'}$ and $y = v_1^k$ where $k \neq k' \in \{1, 2, \ldots, m-1\}$ and $k' < k$. Then, $N(v_1^k)$=$\{v_2^k, v_{n_k}^k, v^{k+1}_{\frac{n_{k+1}}{2}}, v^{k+1}_{\frac{n_{k+1}}{2}+2}\}$ and $N(v_1^{k'})$=$\{v_2^{k'}, v_{n_{k'}}^{k'}, v^{k'+1}_{\frac{n_{k'+1}}{2}}, v^{k'+1}_{\frac{n_{k'+1}}{2}+2}\}$ be the set of neighbors of $v_1^k$ and $v_1^{k'}$ respectively. Consider a shortest path $P = v_1^k v_2^k \ldots v^{k'}_{\frac{n_{k'+1}}{2}}$ $v_{\frac{n_{k'+2}}{2}+1}$ from $v_1^k$ to $v_1^{k'}$. Then from $N(v_1^k)$, $N(v_1^{k'})$ and $P$, we have $d(v^{k+1}_{\frac{n_{k+1}}{2}}, v_1^{k'}) > d(v_1^{k'}, v_1^k)$ and $d(v^{k+1}_{\frac{n_{k+1}}{2}+2}, v_1^{k'}) > d(v_1^{k'}, v_1^k)$. Thus $v_1^k$ and $v_1^{k'}$ are not maximally distant. Hence $v_1^k$ and $v_1^{k'}$ are not mutually maximally distant.

Case 2: Let $x \in V_1$ and $y \in V(\mathcal{C}) \setminus V_1$, say $x = v_i^{k'}$ and $y = v_1^k$ where $i \in \{2, 3, \ldots n_{k'}\}$ and $k \neq k' \in \{1, 2, \ldots, m-1\}$ and $k' < k$. Since $N(v_1^k)$=$\{v_2^1, v_{n_k}^k, v^{k+1}_{\frac{n_{k+1}}{2}}, v^{k+1}_{\frac{n_{k+1}}{2}+2}\}$ and $N(v_i^{k'}) = \{v_{i-1}^{k'}, v_{i+1}^{k'}\}$.

Subcase 2.1: Let $v_i^{k'} \in U_2(C_{n_i})$. Then $P = v_1^k v_{n_k}^k \ldots v_{i-1}^{k'} v_i^{k'}$ be the shortest path from $v_1^k$ to $v_i^{k'}$. Then from $P$, we have $d(v^k_{\frac{n_{k+1}}{2}}, v_i^{k'}) > d(v_1^k, v_i^{k'})$ and $d(v^{k+1}_{\frac{n_{k+1}}{2}+2}, v_i^{k'}) > d(v_1^k, v_i^{k'})$. Thus $v_1^k$ and $v_i^{k'}$ are not maximally distant. Hence $v_1^k$ and $v_i^{k'}$ are not MMD.

Subcase 2.2: Let $v_i^{k'} \in U_1(C_{n_i})$. Then $P = v_1^k v_2^k \ldots v_{i+1}^{k'} v_i^{k'}$ be the shortest path from $v_1^k$ to $v_i^{k'}$. Then from $P$, we have $d(v^k_{\frac{n_{k+1}}{2}}, v_i^{k'}) > d(v_1^k, v_i^{k'})$ and $d(v^{k+1}_{\frac{n_{k+1}}{2}+2}, v_i^{k'}) > d(v_1^k, v_i^{k'})$. Thus $v_1^k$ and $v_i^{k'}$ are not maximally distant. Hence $v_1^k$ and $v_i^{k'}$ are not MMD. □

In the next theorem, we find the MMD vertices in chain cycle $\mathcal{C}(C_{n_1}, C_{n_2}, \ldots, C_{n_m})$, with each $n_i$ is even, with respect to the vertices $\{v^i_{\frac{n_i}{2}+1}, v_1^{i+1} \mid i = 1, 2, \ldots, m-1\}$.

**Theorem 3.8.** *Let $v_j^i, v_l^k \in V_2$, where $i, k \in \{1, 2, \ldots, m\}$, in a chain cycle $\mathcal{C}(C_{n_1}, C_{n_2}, \ldots, C_{n_m})$ constructed by even cycles with respect to the vertices $\{v^i_{\frac{n_i}{2}+1}, v_1^{i+1} \mid i = 1, 2, \ldots, m-1\}$.*

(a). *Let $i = k$. Then $v_j^i MMD v_l^i$ if and only if $d(v_j^i, v_l^i) = d(C_{n_i})$.*

(b). *Let $i \neq k$. Then $v_j^i MMD v_l^k$ if and only if $d(v_j^i, v_l^k) = d(\mathcal{C}(C_{n_1}, C_{n_2}, \ldots, C_{n_m}))$.*

*Proof.* (*a*). Let $d(v_j^i, v_l^i) = d(C_{n_i})$. Then from Lemma 3.6, we have $v_j^i$MMD$v_l^i$.

Conversely, let $v^i_j$MMD$v^i_l$ and $j < l$. On contrary, assume that $d(v^i_j, v^i_l) < d(C_{n_i})$. Since $N(v^i_j) = \{v^i_{j-1}, v^i_{j+1}\}$ and $N(v^i_l) = \{v^i_{l-1}, v^i_{l+1}\}$. Note that either $v^i_j v^i_{j+1} \dots v^i_l$ or $v^i_j v^i_{j-1} \dots v^i_1 v^i_{n_i} \dots v^i_l$ is a shortest path from $v^i_j$ to $v^i_l$. This shows that $v^i_j$ and $v^i_l$ are not mutually maximally distant which contradicts to our supposition that $v^i_j$MMD$v^i_l$.

$(b)$. Let $d(v^i_j, v^k_l) = d(\mathcal{C}(C_{n_1}, C_{n_2}, \dots, C_{n_m}))$. Then clearly $v^i_j$MMD$v^k_l$.

Conversely, let $u^i_j$MMD$u^k_l$ and let $d(v^i_j, v^k_l) < d(\mathcal{C}(C_{n_1}, C_{n_2}, \dots, C_{n_m}))$. Since $N(v^i_j) = \{v^i_{j-1},\ v^i_{j+1}\}$ and $N(v^k_l) = \{v^k_{l-1}, u^k_{l+1}\}$. If $v^i_j \in U_1(C_{n_i})$ and $v^k_l \in U_1(C_{n_k})$, then $P_1 = v^i_j v^i_{j+1} \dots v^{i+1}_1 v^{i+1}_2 \dots\ v^k_1 v^k_2 \dots v^k_l$ is a shortest path from $v^i_j$ to $v^k_l$. This clearly shows that $v^i_j$ is not mutually maximally distant to $v^k_l$. Similarly, if $v^i_j \in U_2(C_{n_i})$ and $v^k_l \in U_2(C_{n_k})$, then $P_2 = v^i_j v^i_{j-1} \dots v^{i+1}_1 v^{i+1}_{n_{i+1}} \dots v^k_1\ v^k_{n_k} \dots v^k_l$ is a shortest path from $v^i_j$ to $v^k_l$. This shows that $v^i_j$ is not mutually maximally distant to $v^k_l$. Moreover, If $v^i_j \in U_1(C_{n_i})$ and $v^k_l \in U_2(C_{n_k})$, then $R_1 = v^i_j v^i_{j+1} \dots v^{i+1}_1 v^{i+1}_2 \dots v^k_1,\ v^k_{n_k} \dots v^k_l$ is a shortest path from $v^i_j$ to $v^k_l$. This clearly shows that $v^i_j$ is not mutually maximally distant to $v^k_l$. Similarly, if $v^i_j \in U_2(C_{n_i})$ and $v^k_l \in U_1(C_{n_k})$, then $R_2 = v^i_j v^i_{j-1} \dots v^{i+1}_1 v^{i+1}_2 \dots v^k_1,\ v^k_2 \dots v^k_l$ is a shortest path from $v^i_j$ to $v^k_l$, which shows that $v^i_j$ is not mutually maximally distant to $v^k_l$. □

For each $i \in \{1, 2, \dots, m\}$, Theorem 3.8 $(a)$ implies

$$A = \{v^i_j v^i_{j+\frac{n_i}{2}} \mid j = 2, 3, \dots, \frac{n_i}{2}, \frac{n_i}{2} + 2, \frac{n_i}{2} + 3, \dots, n_i\} \subseteq E(\mathcal{C}_{SR}), \tag{3.1}$$

where $j + \frac{n_i}{2}$ are integers modulo $n_i$. Similarly, Theorem 3.8 $(b)$ implies $v^1_1 v^m_{\frac{n_m}{2}+1} \in E(\mathcal{C}_{SR})$. Thus $E(\mathcal{C}_{SR}) = A \cup \{v^1_1 v^m_{\frac{n_m}{2}+1}\}$.

**Lemma 3.9.** *Let $\mathcal{C}(C_{n_1}, C_{n_2}, \dots, C_{n_m})$ ba a chain cycle constructed by even cycles with respect to the vertices $\{v^i_{\frac{n_i}{2}+1}, v^{i+1}_1 \mid i = 1, 2, \dots, m-1\}$ and each $n_i \geq 4$. Then $\alpha(\mathcal{C}_{SR}(C_{n_1}, C_{n_2}\ , \dots, C_{n_m}) = 1 + \sum_{i=1}^{m} \frac{n_i - 2}{2}$.*

*Proof.* We construct a vertex cover of strong resolving graph of $\mathcal{C}(C_{n_1}, C_{n_2}, \dots, C_{n_m})$ with minimum cardinality. From (3.1), we note that the vertices $\{v^i_j, v^i_{j+\frac{n_i}{2}} \mid j = 2, 3, \dots, \frac{n_i}{2}, \frac{n_i}{2} + 2, \frac{n_i}{2} + 3, \dots, n_i\}$, for each $i \in \{1, 2, \dots, m\}$, form $\sum_{i=1}^{m} \frac{n_i - 2}{2}$ copies of $K_2$. Thus, the $\sum_{i=1}^{m} \frac{n_i - 2}{2}$ vertices $\{v^i_j \mid j = 2, 3, \dots, \frac{n_i}{2}, \frac{n_i}{2} + 2, \frac{n_i}{2} + 3, \dots, n_i\}$, for each $i \in \{1, 2, \dots, m\}$, are minimum number of vertices to conver the edges of $A$. Let $S = \{v_j^i \mid j = 2, 3, \ldots, \frac{n_i}{2}, \frac{n_i}{2} + 2, \frac{n_i}{2} + 3, \ldots, n_i\}$. Furthermore, since $v_1^1 v^m_{\frac{n_m}{2}+1} \in E(\mathcal{C}_{SR})$. Thus, the vertex cover of the strong resolving graph of chain cycle $\mathcal{C}(C_{n_1}, C_{n_2}, \ldots, C_{n_m})$ with minimum cardinality is $S := S \cup \{v_1^1\}$. Hence $\alpha(\mathcal{C}_{SR}(C_{n_1}, C_{n_2}, \ldots, C_{n_m})) = 1 + \sum_{i=1}^{m} \frac{n_i - 2}{2}$. □

**Theorem 3.10.** *Let $\{C_{n_i}\}_{i=1}^m$ be $m$ disjoint cycles with each $n_i$ is even and $n_i \geq 4$, then $dim_s(\mathcal{C}\ (C_{n_1}, \ldots, C_{n_m})) = 1 + \sum_{i=1}^{m} \frac{n_i-2}{2}$.*

*Proof.* The proof follows from Lemma 3.9 and Theorem 3.1. □

In the Example 3.11, we find the strong metric dimension of a chain cycle constructed by $C_8$, $C_{10}$ and $C_8$.

**Example 3.11.** Let $m = 3$ and $n_1 = 8$, $n_2 = 10$ and $n_3 = 8$. The chain cycle constructed by $C_8$, $C_{10}$ and $C_8$ with respect to the vertices $\{v_5^1, v_1^2, v_6^2 v_1^3\}$ is denoted by $\mathcal{C}(C_8, C_{10}, C_8) = \mathcal{C}(C_8, C_{10}, C_8; v_1^1, v_1^2, v_5^1, v_1^3, v_6^2, v_5^3)$ and is given in Figure 3.4.

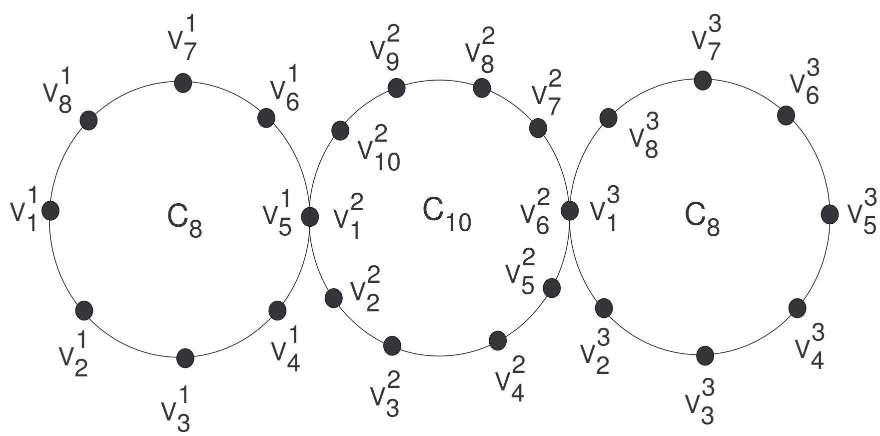



Figure 3.4: Chain cycles of $C_8$, $C_{10}$ and $C_8$

From Theorem 3.10 we have,

$dim_s(\mathcal{C}\ (C_{n_1}, \ldots, C_{n_m})) = 1 + \sum_{i=1}^{m} \frac{n_i-2}{2}$

where $dim_s(\mathcal{C}\ (C_{n_1}, \ldots, C_{n_m})) = 1 + 3 + 4 + 3 = 11$.

# Conclusion

One can build chain graph from a collection of connected graphs out of them. In the chemistry world, there are many applications of graph theory as we have already discussed in the introduction of Chapter 2 of this thesis. We study atoms and group of atoms (molecule) in chemistry. The concept of chain graph also comes from the mixing of these molecules or atoms. Mathematicians develop theories and chemist use those for the welfare of human being, that can also improve efficiency of medicine. For our reader, we explain the definition of chain in perspective of chemists. The group of atoms combine a part of essential structure of a chain. Spiro-chain is well known example for chemists. Spiro-chain comprising of an continuous sequence of rings, where adjacent rings have only single atom in common. Mansour and Schork, in [23], have given examples of Spiro-chain. They have calculated Wiener, hyper-detour, detour and hyper-Wiener indices for chain and bridge graphs. In 2017, D. Nilanjan calculated Hyper Zagreb Index of chain and bridge Grpahs in [25]. These results help chemists for making improved medicines and this is also a positive contribution in the world.

We calculated partition dimension and strong metric dimension of chain cycle in this thesis. Further, we can find the metric dimension and strong partition dimension of chain cycle. We can also study the partition and strong metric dimension for more general case, where even and odd cycles are mixed together. Furthermore, we can also calculate the metric dimension, partition and strong partition dimension of chain of different classes of graphs which has direct relation in chemical industry. The outcome of this research adds to a growing understanding of the resolvability problems. It is also emphasized here that the purpose for solving this problem is to understand that how resolvability of any graph can be calculated and proved for a general case. It also opens an new area of research as resolvability of many chain graphs are still to be determined.

# References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