MULTIPLICITY RESULTS OF THREE POINT BOUNDARY VALUE PROBLEM AT NONRESONANCE

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Chapter 1

Preliminaries

1.1 Introduction

In this chapter, we recall some basic definitions and known results which are necessary to establish the existence of multiple solutions of the nonlinear three-point boundary value problem (BVP)

$$\begin{aligned} u''(t) &= f(t, u(t)), t \in (0,1), \\ u(0) &= \epsilon u'(0),\ u(1) = \theta u(\eta), \end{aligned} \tag{1.1}$$

where $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ is a continuous function and $\epsilon \in [0, \infty)$, $\theta \in (0, \infty)$, $\eta \in (0,1)$ are given constants with non-resonance condition $\theta(1 + \epsilon) = \eta + \epsilon$.

The winding number, degree theory and the method of lower and upper solutions are discussed in detail. The existence of a solution of the given problem is proved by the method of upper and lower solutions. The basic idea of the method is to modify the given problem suitably and then employ Leray-Schauder theory or some known results together with the theory of differential and integral inequalities to establish existence of a solution. The function f is modified in such a way that solutions of the modified problem lie in a region where f is unmodified and hence are solutions of the original BVP.

1.2 Some basic definitions and notions

We need the following definitions for our work. For details see [21].

Definition: A point set γ in the complex plane is said to be a *curve* if it is the range of some complex-valued function $z = z(t) \in \mathbb{C}$, $a \leq t \leq b$, where $a, b \in \mathbb{R}$.

Definition: A curve γ is called *simple,* if it does not cross itself, except possibly at its initial and terminal points in which case it is called *closed*.

Definition: A simple closed curve is called *Jordan curve.*

Definition: A point set γ in the complex plan is said to be *smooth curve,* if it is the range of some continuous complex-valued function $z(t)$ define on the interval $[a, b]$ such that $z(t)$ satisfies the following conditions

(**i**) $z(t) = x(t) + iy(t)$ has a continuous derivative on $[a, b]$;

(**ii**) $z'(t)$ never vanishes on $[a, b]$;

(**iii**) $z(t)$ is one to one on $[a, b]$.

Moreover, it is a smooth close curve if $z(a) = z(b)$. Consequently a smooth curve has no corners or cusps.

Definition: A *contour* (path) Γ is either a single point z_0 or a finite sequence of directed smooth curves, $\gamma_1, \gamma_2, ..., \gamma_n$, such that the terminal point of γ_k coincides with the initial point of γ_{k+1} for each $k = 1, 2, 3, ..., n-1$.

Definition: A set is *connected* if any two points in it can be connected by an unbroken curve. An open connected set is called a *domain*.

Definition: Any domain D with a property that every loop (closed contour) in D can be continuously deformed to a single point, is called a *simply connected domain* (roughly speaking, a simply connected domain can't have a hole).

Definition: A complex-valued function $f(z)$ is said to be *analytic* on an open set G, if it has a derivative at every point of G. If $f(z)$ is analytic on the whole complex plane, it is said to be *entire.* For example, all polynomial functions of z are entire. Analytic functions are also called *holomorphic* or *regular.*

Definition: A *homotopy* H, between two functions f and g from a space X to a space Y is a continuous mapping $H : [0,1] \times X \to Y$ such that

$$\begin{aligned} H(0,x) &= f(x), \\ H(1,x) &= g(x). \end{aligned}$$

Definition: For given r, R with $0 \leq r < R$, the *annulus* centered at z_0 with radii r and R is defined by

$$A(z_0, r, R) = \{z : r < |z - z_0| < R\}.$$

Definition: An *open disc* D with center at z_0 and radius R is defined by

$$D_R(z_0) = \{z : |z - z_0| < R\}.$$

Definition: The disc $D_R^*(z_0)$ with center at z_0 and radius R defined by

$$D_R^*(z_0) = \{z : 0 < |z - z_0| < R\},$$

is known as the punctured disc.

Definition: If $f(z)$ is analytic at z_0, then the *Taylor's series* is defined as

$$f(z) = \sum_{k=0}^{\infty} \frac{f^{(k)}(z_0)}{k!}(z - z_0)^k, \text{ for all } z \in D_R(z_0).$$

Definition: Let f be analytic within the punctured disc $D_R^*(z_0)$, then the series

$$f(z) = \sum_{-\infty}^{\infty} c_k (z - z_0)^k = \sum_{k=1}^{\infty} c_{-k}(z - z_0)^{-k} + \sum_{k=0}^{\infty} c_k (z - z_0)^k,$$

is known as the *Laurent series.*

Definition: A point p is called a *singular point* of the complex-valued function f, if f is not analytic at the point p and every neighborhood $D_R(p)$ of p contains at least one point at which f is analytic. For example, $z = 1$ is singular point of $f(z) = \frac{1}{1-z}$. Moreover, the origin and each points of the negative real axis are singularities of the following function

$$f(z) = Logz.$$

Definition: A point p is called an *isolated singularity* of f, if f is not analytic at the point p but f is analytic everywhere in the punctured disk $D_R^*(p)$. For example, $z = 1$ is isolated singularity of $f(z) = \frac{1}{1-z}$, while $z = 0$ is not an isolated singularity of $f(z) = Logz$.

Classification of singularities

Let p be the isolated singularity of f with Laurent series

$$f(z) = \sum_{-\infty}^{\infty} c_n (z-p)^n, \text{ for all } z \in A(p, 0, R),$$

then,

(i) p is a *removable singularity* of f if $c_n = 0$, for $n = -1, -2, -3, ...$

(ii) p is a *pole* of order $k > 0$, if $c_{-k} \neq 0$, but $c_n = 0$ for $n < -k$.

(iii) p is called an *essential singularity* of f, if $c_n \neq 0$ for infinitely many negative integers n.

Definition: A point p is called a *zero* of the function f if and only if f is analytic in $D_R(p)$ and $f^{(n)}(p) = 0$ for each $n = 0, 1, 2, ..., k-1$, but $f^{(k)}(p) \neq 0$. A zero of order 1 is called a simple zero.

Definition: If f has an isolated singularity at the point z_0, then the coefficient c_{-1} of $\frac{1}{(z-z_0)}$ in the Laurent series for f around z_0 is called the *residue* of f at z_0 and is denoted by $\text{Res}(f; z_0)$.

Definition: A function f is called *meromorphic* in a domain D if it is either analytic or has a pole at every point of D.

Definition: A function is called *smooth,* if it has continuous derivatives up to some desired order over some domain.

Definition: A function $f : [0, 1] \times \mathbb{R} \rightarrow \mathbb{R}$ is called *caratheodory* if

(i) $t \rightarrow f(t, x)$ is measurable for each $x \in \mathbb{R}$,

(ii) $x \rightarrow f(t, x)$ is continuous almost everywhere for each $t \in [0, 1]$.

Further f is called L^1-*caratheodory* if

(iii) for each real number $r > 0$, there exist a function $h_r \in L^1(I, \mathbb{R})$ such that $|f(t, x)| \leq h_r(t, x)$ for all $x \in \mathbb{R}$ with $\|x\| \leq r$.

Definition: A solution $x_M(t)$ of an equation is said to be a *maximal* solution if $x(t) \leq x_M(t)$, $t \in I$ for any other solution of that equation. Similarly a solution $x_m(t)$ is said to be a *minimal* solution if $x_m \leq x(t)$, $t \in I$ for any other solution of that equation.

Definition: A subset S of $\mathbb{R}$ is *compact* if and only if it is closed and bounded.

Definition: A set S in a vector space over $\mathbb{R}$ is called *convex,* if the line segment joining any pair of points lies entirely in S.

Definition: Let X be a linear normed space with norm $\|.\|$ and $T : X \to X$ be an operator. The operator T is said to be compact if $\overline{T(S)}$ is compact for any bounded subset S of X. The operator T is said to be *completely continuous,* if it is compact and continuous on X. T is called totally bounded, if for any bounded subset S of X, $T(S)$ is a totally bounded subset of X. Note that every compact operator is totally bounded but the converse may not be true. However, these two notions are equivalent on bounded subsets of Banach space X.

Definition: A family $\{f(x)\}$ of functions defined on some closed interval I is said to be *uniformly bounded***,** if there exist a number $M \geq 0$ such that

$$|f(x)| < M \text{ for all } x \in I \text{ and for all } f \text{ belonging to the given family.}$$

Definition: A family $\{f(x)\}$ of functions is said to be *equi-continuous*, if for given $\epsilon > 0$, there is a $\delta > 0$ such that

$$|f(x_1) - f(x_2)| < \epsilon \text{ whenever } |x_1 - x_2| < \delta.$$

1.2.1 Green's function:

Let L be a linear differential operator and consider a nonhomogeneous problem with an inhomogeneity acting as the source term

$$Lu(t) = f(t, u(t)), t \in I, \tag{i}$$

subject to some two point homogeneous boundary conditions (BCs), where $u(t)$ is the field quantity and $f(t, u(t))$ is the source field. Corresponding homogeneous equation is

$$Lu(t) = 0. \tag{ii}$$

Let the inhomogeneity be delta function δ, then the field generated by the delta function δ is called a Green's function, denoted by $G(t, s)$. If the delta function δ act at a point s, the Green's function $G(t, s)$ satisfies

$$LG(t, s) = \delta(s - t), s, t \in I. \tag{iii}$$

Multiplying equation (iii) by $f(s, u(s))$ and integrating over I, it gives

$$\begin{aligned} \int_I LG(t, s)f(s, u(s))ds &= \int_I \delta(s - t)f(s, u(s))ds \\ L\int_I G(t, s)f(s, u(s))ds &= f(t, u(t)), \text{ by shifting property of } \delta. \end{aligned} \tag{iv}$$

Comparing (i) and (iv), gives

$$u(t) = \int_I G(t, s)f(s, u(s))ds, \tag{v}$$

which is the solution of inhomogeneous equation (i). General solution of (i) is

$$u(t) = \int_I G(t, s)f(s, u(s))ds + u_h(t), \tag{vi}$$

where $u_h(t)$ is solution of homogeneous equation (ii).

The importance of the Green's function G is that, we convert the differential equation into integral equation and then we study integral operator which is easy to handle than to solve differential equation.

In simple words, the effect of a force is called Green's function.

Properties of the Green,s function:

Green's function has the following properties:

(i) Symmetry: $G(t, s) = G(s, t)$, for all $(s, t) \in I \times I$;

(ii) Continuity: $G(s^+, s) = G(s^-, s)$, $s \in I$;

(iii) Jump discontinuity of the derivative: $G_t(s^+, s) - G_t(s^-, s) = -1$.

We state the following theorems without proofs.

Theorem:(1.1)(Arzela-Ascoli) A necessary and sufficient condition that a family of continuous functions defined on a closed interval I be compact in $C(I)$ is that this family be uniformly bounded and equi-continuous. For the proof see [16].

Theorem:(1.2)(Schauder's fixed) Let M be a closed, bounded and convex set in a Banach space X and $f : M \to M$ be compact. Then f has a fixed point. For the proof see [15].

Theorem:(1.3) If f has a pole of order m at z_0, then

$$\text{Res}(f; z_0) = \lim_{z \to z_0} \frac{1}{(m-1)!} \frac{d^{m-1}}{dz^{m-1}} [(z - z_0)^m f(z)].$$

For the proof, see [21].

Theorem:(1.4)(Cauchy's Residue theorem) If Γ is a simple closed positively oriented contour and f is analytic inside and on Γ except at the points $z_1, z_2, ..., z_n$ inside Γ, then

$$\int_\Gamma f(z)dz = 2\pi\imath \sum_{j=1}^{n} \text{Res}(f, z_j).$$

For the proof, see [21].

1.2.2 Winding number:

Theorem:(1.5)(Argument Principle) Suppose that f is meromorphic inside the simply connected domain D and that γ is a simple closed positively oriented contour in D such that f is nonzero and analytic for $z \in \gamma$, then

$$\frac{1}{2\pi\imath} \int_\gamma \frac{f\prime(z)}{f(z)} dz = N_0(f) - N_p(f),$$

where $N_0(f)$ is number of zeros of f that lie inside γ and $N_p(f)$ is the number of poles of f that lie inside γ. For the proof, see [21].

Corollary:(1.6) Let f be analytic in the simply connected domain D. If γ is a simple closed positively oriented contour in D such that $f(z) \neq 0$, for $z \in \gamma$, then

$$\frac{1}{2\pi\iota}\int_{\gamma}\frac{f'(z)}{f(z)}dz = N_0(f).$$

where $N_0(f)$ is number of zeros of f that lie inside γ.

Example: The image of the circle $C_2(0)$ under $f(z) = z^2 + z$ is the curve

$$\{(x, y) = (4\cos 2t + 2\cos t, 4\sin 2t + 2\sin t) : 0 < t < 2\pi\}.$$

Note that the image curve $f(C_2(0))$ winds twice around the origin. Now, we compute, $\frac{1}{2\pi\iota}\int_{c_2^+(0)}\frac{f'(z)}{f(z)}dz = \frac{1}{2\pi\imath}\int_{c_2^+(0)}\frac{2z+1}{z^2+z}dz$, the number of zeros of f that lie inside γ.Clearly, the zeros of $f(z)$ are $0, -1$ lying inside $C_2(0)$ and f has no poles inside $C_2(0)$.The residues of the integrand are at 0 and -1. Thus, by Cauchy residue theorem

$$\begin{aligned}\frac{1}{2\pi\iota}\int_{c_2^+(0)}\frac{f'(z)}{f(z)}dz &= \frac{1}{2\pi\imath}\int_{c_2^+(0)}\frac{2z+1}{z^2+z}dz = \text{Res}\left[\frac{2z+1}{z(z+1)}, 0\right] + \text{Res}\left[\frac{2z+1}{z(z+1)}, -1\right] \\ &= 1 + 1 = 2 = N_0(f),\end{aligned}$$

hence, winding number of the curve $f(C_2(0))$ around the origin is equal to the number of zeros inside the circle $C_2(0)$.■

Winding numbers: Suppose that f is meromorphic in the simply connected domain D. If γ is a simple closed positively oriented contour in D such that $f(z) \neq 0$ and $f(z) \neq \infty$, for all $z \in \gamma$, then

$$W(f(\gamma), z_0) = \frac{1}{2\pi\iota}\int_{\gamma}\frac{f'(z)}{f(z) - z_0}dz, \tag{1.2}$$

known as the *winding number* of $f(\gamma)$ about z_0, counts the number of times the curve $f(\gamma)$ winds around the point z_0. If $z_0 = 0$, the integral counts the number of times the curve $f(\gamma)$ winds around the origin. Hence winding number $W(f(\gamma), z_0)$ counts the number of zeros of f inside D.

Remark: Letting $f(z) = z$ in (1.2), gives

$$W(\gamma, z_0) = \frac{1}{2\pi\iota}\int_{\gamma}\frac{1}{z - z_0}dz = \begin{cases} 1 \text{ if } z_0 \text{ lies inside } \gamma, \\ 0 \text{ if } z_0 \text{ lies outside } \gamma, \end{cases}$$

which counts the number of times γ winds around the point z_0.

If γ is not a simple closed curve, but crosses itself several times, then computation of winding number is not so easy. In this case, the Rouche's theorem will be used to deduce some informations about the location of zeros of a complicated analytic function f by comparing it with an analytic function g whose zeros are known.

Theorem:(1.7)(Rouche's theorem) Suppose f and g are meromorphic functions defined on a simply connected domain D and γ is a simple closed contour in D. Moreover f and g have no zeros or poles for $z \in \gamma$. If the strict inequality

$$|f(z) + g(z)| < |f(z)| + |g(z)|, \tag{1.3}$$

holds for all $z \in \gamma$, then

$$N_0(f) - N_p(f) = N_0(g) - N_p(g),$$

where $N_0(f)$ is the number of zeros of f and $N_p(f)$ is the number of poles of f inside D. Similarly $N_0(g)$ and $N_p(g)$ is number of zeros and poles of g respectively inside D. For the proof, see [21]

Corollary:(1.8) Suppose that f and g are analytic functions defined in the simply connected domain D, that γ is a simple closed contour in D, and that f and g have no zeros for $z \in \gamma$. If the strict inequality

$$|f(z) + g(z)| < |f(z)| + |g(z)|,$$

holds for all $z \in \gamma$, then

$$N_0(f) = N_0(g).$$

Rouche's theorem was improved by Irving Glicksberg [8] by replacing the condition (1.3) by the following weaker condition

$$|f(z) + g(z)| < |g(z)|, \text{ for all } z \in \gamma.$$

Example: Consider the polynomial

$$f(z) = z^4 - 7z - 1,$$

and the disk $D_2(0) = \{z : |z| < 2\}$ with C_2 as its boundary

Take

$$g(z) = -z^4.$$

Then, for all $z \in C_2$,

$$|f(z) + g(z)| \leq |-7z| + |-1| = 7(2) + 1 < 16 = |g(z)|.$$

Hence by Rouche's theorem,

$$N_0(f) = N_0(g),$$

that is, f and g have the same number of zeros inside $D_2(0)$. Consequently, f has four zero inside $D_2(0)$ as g has a zero of order 4 at the origin.■

For a complicated path γ, one can use the concept of homotopy to overcome the difficulties, that is, one needs to find a mapping g for which $w(g(\gamma), 0)$ can be easily computed and a homotopy $H : [0,1] \times \gamma \to C\backslash\{0\}$ between f and g such that

$$\begin{aligned} H(0,z) &= g(z), \\ H(1,z) &= f(z), \forall\ z \in \gamma. \end{aligned}$$

Then, by Rouches theorem, it follows that

$$w(f(\gamma), 0) = w(g(\gamma), 0).$$

To explain it more fully, let us consider the problem of finding the number of zeros of the function $f(z) = \frac{1}{2}z^6 + z - \frac{1}{3}$ inside the unit circle.

In fact, it is required to find a function g such that $w(g(\gamma, 0))$ can be easily computed and a homotopy

$$H : [0,1] \times \gamma \to C\backslash\{0\},$$

such that

$$\begin{aligned} H(0,z) &= g(z), \\ H(1,z) &= f(z), \text{ for } z \in \gamma = \{z \in C : |z| = 1\}. \end{aligned}$$

Let $g(z) = -z$, and H be defined by

$$H(t,z) = (1-t)g(z) + tf(z), \text{ Then}$$

$$|f(z) + g(z)| = \left|\frac{1}{2}z^6 + z - \frac{1}{3} - z\right| = \left|\frac{1}{2}z^6 - \frac{1}{3}\right| < \left|\frac{1}{2}z^6\right| + \frac{1}{3} < 1 = |g(z)| \text{ on } \gamma,$$

which implies that

$$H(t,z) \neq 0 \text{ on } [0,1] \times \gamma.$$

Hence $f(z)$ has one zero inside the circle γ.

For a smooth domain U, which is enclosed by a Jordan curve ∂U, define the Z valued function d by

$$d(f, U, z_0) = w(f(\partial U), z_0) = w(f(\partial U) - z_0, 0) \in Z.$$

The function d measures the number of solutions of the equation

$$f(z) = z_0 \text{ inside } U,$$

and is known as the degree of f at z_0. The invariance of d with respect to certain deformations of f allowed us to compute $\deg(f, U, z_0)$ even in more complicated cases.

1.2.3 Degree theory:

The oldest notion of the degree of a smooth function f from S^1 into S^1, where S^1 is a unit circle, counts "how many times f covers its range taking into account the algebraic multiplicity", that is, the winding number. More generally, a smooth function $f : S^1 \to C$(complex plane) such that $f \neq 0$ on S^1 has a degree given by

$$\deg(f, S^1) = \frac{1}{2\pi i} \int_{S^1} \frac{f'}{f} dz.$$

If instead of S^1, one have a simple curve Γ in $\mathbb{R}^2$ and $f : \Gamma \to S^1$ is a smooth mapping, then

$$\deg(f, \Gamma) = \frac{1}{2\pi} \int_{\Gamma} f \times f_\tau dz,$$

where f_τ is the tangential derivative of f along the curve Γ.

At the end of the 19th century, the notion of the degree was generalized to higher dimensional spaces. For a smooth mapping $f : X \to Y$, where X and Y are n-dimensional manifolds, the degree of f is a measure of the number of solutions of the equation

$$f(x) = y,$$

that is

$$\deg(f) = \sum sign \ \det \ J_f(x_i).$$

In 1912, Brouwer realized that the smoothness assumption on f is not necessary to define a degree, only continuity of f is sufficient assumption.

Let X be an n-dimensional space and $f : D \subset X \to X$ be a mapping, where D is open bounded subset of X. Generally, finding all solutions of the equation

$$f(x) = p, \ x \in D. \tag{1.4}$$

analytically is impossible. Degree theory was found to be a useful tool in such investigations. In applications, not only existence of solutions of an equation is of interest but their number is also significant. This is the case in bifurcation theory, for example, where values of a parameter λ are to be sought such that the number of solutions of an equation of the form

$$f(x, \lambda) = 0,$$

changes. The degree $d(f, D, p)$ of f at a point p relative to D is a measure of the number of solutions of (1.4) in the subset D of X. In particular, (1.4) will have a solution in D whenever $d(f, D, p) \neq 0$. The integer $d(f, D, p)$ has such properties as invariance under small changes in f and p. The Brouwer degree was extended by Leray and Schauder in the thirties to cover certain mappings defined on infinite dimensional normed spaces. Since then, the scope of the theory has been steadily extended. The development of the degree theory is a continuous effort to define a degree in more general situations.

Properties of degree theory:

Let Γ be a collection of open, bounded subsets of X. and $M(D)$ be a set of continuous mappings $\overline{D} \to X$ for all $D \in \Gamma$. For $T \in M(D)$ and $p \notin T(\partial D)$ (p is not an image of the boundary of D), a *topological degree* $d(T, D, p)$ has the following properties.

(**1**) **Normalization:** $d(I, D, p) = 1$, where I is the identity function.

(2) **Additivity:** For disjoint open subsets D_1, D_2 of D with $p \notin T(\overline{D} \backslash (D_1 \cup D_2))$,

$$d(T, D, p) = d(T, D_1, p) + d(T, D_2, p).$$

(3) **Homotopy Invariance:** Suppose H_t is a family of mappings in $M(D)$ depending continuously on $t \in [0, 1]$, and

$$p \notin H_t(\partial D),\ 0 \leq t \leq 1,$$

then $d(H_t, D, p)$ is independent of t.

(4) **Translation:**

$$d(T, D, p) = d(T - p, D, 0).$$

(5) **Boundary dependence:** If f and $g \in M(\overline{D})$ such that $f \mid_{\partial D} = g \mid_{\partial D}$, and $p \notin f(\partial D) \cap g(\partial D)$, then

$$d(f, D, p) = d(g, D, p).$$

(6) **Existence:**

$$d(f, D, p) \neq 0,$$

implies that there exist $x \in D$ such that $f(x) = p$ has a solution in D.

(7) **Excision property:** For every open set $D_1 \subset D$ such that $p \notin T(\overline{D} \backslash D_1)$

$$\deg(T, D, p) = \deg(T, D_1, p).$$

The condition $p \notin T(\partial D)$ is essential, for otherwise the degree would chang by small changes in T or p and the whole theory would collapse.

The Leray-Schauder degree is an extension of the Brouwer's degree to the case of infinite dimensional spaces. Particularly, mapping of the type $I - T$, where T is compact (i.e. $\overline{T(D)}$ is compact whenever D is bounded set). The above properties also hold for Leray-Schauder degree.

1.2.4 Method of lower and upper solutions for BVPs:

The method of lower and upper solutions for BVPs deals with the existence and uniqueness of solutions for boundary value problems. The basic idea is to modify a given problem suitably with respect to upper and lower solutions and then employ known results of the modified problem together with the theory of differential inequalities to establish existence of solution of the original BVP.

It is well known that one of the most important tools for dealing with existence results for nonlinear problems is the method of upper and lower solutions. The method of upper and lower solutions has a long history and some of its ideas can be traced back to Picard [24]. The method of upper and lower solutions was established by Scorza Dragoni [5]. Since then, many authors have developed existence results for nonlinear problems using the method of upper and lower solutions. For an overview of the method of upper and lower solutions of ordinary differential equations, the reader is referred to [4].

Consider the BVP

$$u''(t) = f(t, u(t)), \ t \in [a, b], \tag{1.5}$$

$$a_1 u(a) - a_2 u'(a) = A, \quad b_1 u(b) + b_2 u'(b) = B,$$

where $A,\ B,\ a_1,\ b_1 \in \mathbb{R}, a_2, b_2 \in \mathbb{R}^+, a_1^2 + a_2^2 > 0$ and $b_1^2 + b_2^2 > 0$ and f is continuous. This problem is called separated boundary value problem and it contains the Drichlet problem

$$\begin{aligned} u''(t) &= f(t,u(t)), \\ u(a) &= 0,\ u(b)=0 \end{aligned}$$

and the Neumann problem

$$\begin{aligned} u''(t) &= f(t,u(t)), \\ u'(a) &= 0,\ u'(b)=0, \end{aligned}$$

as special cases.

Definition: A function $\alpha \in C[a,b]$ is a C^2-lower solution of (1.5), if α satisfies the following conditions:

(a) For any $t_0 \in (a,b)$, either $D_-\alpha(t_0) < D^+\alpha(t_0)$ or there exist an open interval $I_0 \subset (a,b)$ with $t_0 \in I_0$ and a function $\alpha_0 \in C^1(I_0)$ such that:

(i) $\alpha(t_0) = \alpha_0(t_0)$ and $\alpha(t) \geq \alpha_0(t)$ for all $t \in I_0$,

(ii) $\alpha_0''(t_0)$ exists and $\alpha_0''(t_0) \geq f(t_0, \alpha_0(t_0))$;

(b)

$$a_1\alpha(a) - a_2 D^+\alpha(a) \leq A,\ b_1\alpha(b) + b_2 D_-\alpha(b) \leq B.$$

Definition: A function $\beta \in C[a,b]$ is a C^2-upper solution of (1.5), if β satisfies the following conditions:

(a) For any $t_0 \in (a,b)$, either $D^-\beta(t_0) > D_+\beta(t_0)$ or there exist an open interval $I_0 \subset (a,b)$ with $t_0 \in I_0$ and a function $\beta_0 \in C^1(I_0)$ such that

(i) $\beta(t_0) = \beta_0(t_0)$ and $\beta(t) \leq \beta_0(t)$ for all $t \in I_0$

(ii) $\beta_0''(t_0)$ exists and $\beta_0''(t_0) \leq f(t_0, \beta_0(t_0))$;

(b)

$$a_1\beta(a) - a_2 D_+\beta(a) \geq A,\ b_1\beta(b) + b_2 D^-\beta(b) \geq B.$$

If the inequalities in (a) or (b) are strict, then α, β are called C^2-strict lower respectively upper solutions of the BVP (1.5).

Geometrically, if the inequalities (ii) are assumed to be strict, graphs of α and β are then curves such that solutions can not be tangent to a lower solution from above or to an upper solution from below. The C^2-lower and C^2-upper solutions can also be constructed from the maximum of lower solutions and minimum of upper solutions.

Remark:(1.9) Let $\alpha_i \in C[a,b]$ $(i = 1, 2, ..., n)$ be lower solutions of (1.5), then the function

$$\alpha(t) = \max_{1\le i\le n} \alpha_i(t),\ t \in [a,b],$$

is a C^2-lower solution of (1.5).

Similarly, if $\beta_j \in C[a,b]$ $(j = 1, 2, ..., m)$ are upper solutions of (1.5), then the function

$$\beta(t) = \min_{1\le j\le m} \beta_j(t),\ t \in [a,b],$$

is a C^2-upper solution of (1.5).

A first existence result for solutions of (1.5) is presented in the following theorem.

Theorem:(1.10) Assume α and $\beta \in C[a,b]$ are C^2-lower solutions and C^2-upper solutions of BVP (1.5) such that $\alpha \le \beta$ on $[a,b]$ and assume $f : [a,b]\times\mathbb{R} \to \mathbb{R}$ is continuous and bounded, then BVP (1.5) has at least one solution $u \in C^2[a,b]$ such that

$$\alpha(t) \le u(t) \le \beta(t),\ t \in [a,b].$$

Proof: Consider the modified boundary value problem

$$\begin{aligned} u'' - u &= f(t, \gamma(t,u)) - \gamma(t,u), \\ u(a) - a_2 u'(a) &= A + (1 - a_1)\gamma(a, u(a)), \\ u(b) + b_2 u'(b) &= B + (1 - b_1)\gamma(b, u(b)), \end{aligned} \tag{1.7}$$

$$\text{where } \gamma(t,u) = \max\{\alpha(t), \min(u, \beta(t))\}. \tag{1.8}$$

Define a linear operator $L : C[a,b] \to C[a,b]$ by

$$(Lu)(t) = u''(t) - u(t), \ t \in [a,b]$$

and a nonlinear operator $N : C[a,b] \to C[a,b]$ by

$$(Nu)(t) = f(t, \gamma(t,u)) - \gamma(t,u) :$$

The BVP (1.7) is equivalent to the operator equation

$$Lu = Nu,$$

$$\text{where } Lu = (u'' - u, u(a) - a_2 u'(a), u(b) + b_2 u'(b))$$

and

$$Nu = (f(t, \gamma(t,u)) - \gamma(t,u), A + (1 - a_1)\gamma(a, u(a)), B + (1 - b_1)\gamma(b, u(b))).$$

The operator L is invertible. Hence the problem (1.7) is equivalent to

$$u = L^{-1}Nu,$$

since f is continuous and bounded, the operator $L^{-1}N$ is compact and by Schauder's fixed point theorem $L^{-1}N$ has a fixed point, which implies (1.7) has a solution.
It is required to show that

$$\alpha(t) \leq u(t) \leq \beta(t), \text{ for all } t \in [a,b]. \tag{1.9}$$

Assume on the contrary that $u(t) - \alpha(t)$ has a negative minimum at some point $t_0 \in [a,b]$. If $t_0 \in (a,b)$, then

$$D^-\alpha(t_0) \geq D_+\alpha(t_0)$$

and using the definition of lower solution, it follows that

$$0 \leq u''(t_0) - \alpha''(t_0) = f(t_0, \alpha_0(t_0)) + u(t_0) - \alpha_0(t_0) - \alpha_0''(t_0) < 0,$$

which is a contradiction.

In case $t_0 = a$ (a similar argument holds for $t_0 = b$), the BCs imply that

$$u'(a) - D^+\alpha(a) \geq 0$$

and

$$\begin{aligned} 0 &= A + (1 - a_1)\gamma(a, u(a)) - u(a) + a_2 u'(a), \\ &> A - a_1\alpha(a) + a_2 D^+\alpha(a), \\ &\geq 0, \end{aligned}$$

a contradiction. Hence $u(t) \geq \alpha(t)$ on $[a, b]$. Similarly, it is not difficult to show that $u(t) \leq \beta(t)$ on $[a, b]$.■

Conclusion: The equation (1.7) has a solution which satisfy (1.9) and therefore is a solution of (1.5).

Example: Consider the two point BVP of the type

$$\begin{aligned} x'' &= -t + xe^x, \ t \in [0, 1] \\ x(0) &= 0, \ x(1) = 1. \end{aligned}$$

Take $\alpha = 0$, then α is a lower solution because

$$f(t, \alpha(t)) = -t \leq 0 = \alpha''(t), t \in [0, 1]$$

and

$$\alpha(0) = 0, \ \alpha(1) = 0.$$

Similarly, $\beta = t$, satisfies

$$\begin{aligned} f(t, \beta) &= -t + te^t, \\ &= t(e^t - 1) \geq 0 = \beta'', \ t \in [0, 1] \end{aligned}$$

and

$$\beta(0) = 0,\ \beta(1) = 1 > 0.$$

Hence β is an upper solution.■

Example: In the following example, consider some three point boundary value problem

$$\begin{aligned} x'' &= x - g(t) = f(t, x), \\ x(0) &= 0, x(1) = \theta x(\eta), \end{aligned}$$

where $\eta \in (0, 1)$ and $g(t)$ is non-negative continuous function such that $g(t) \leq at$, $a > 0$, $t \in [0, 1]$.

Solution: Choose $\alpha = 0$, then

$$\alpha' = 0,\ \alpha'' = 0.$$

Hence,

$$f(t, \alpha) = -g(t) \leq 0 = \alpha'',\ t \in [0, 1]$$

and

$$\alpha(0) = 0,\ \alpha(1) = 0 = \theta\alpha(\eta),$$

which implies that $\alpha = 0$ is lower solution.

Take $\beta = at$, then

$$\beta' = 0, \beta'' = 0,\ t \in [0, 1].$$

Hence,

$$f(t, \beta) = at - g(t) \geq 0 = \beta'',\ t \in [0, 1]$$

and

$$\begin{aligned} \beta(0) &= 0, \\ \beta(1) - \theta\beta(\eta) &= a - \theta a\eta = a(1 - \theta\eta) \geq 0,\ \text{as } \theta\eta < 1, \end{aligned}$$

that is,

$$\beta(1) \geq \theta\beta(\eta),$$

which implies that $\beta = at$ is an upper solution.$\blacksquare$

Modified function:

Let α, $\beta \in C^2(I)$ be the lower and upper solutions of a differential equation

$$u''(t) = f(t, u(t)),$$

such that

$$\alpha(t) \leq \beta(t) \text{ on } I.$$

Define a function F as follows:

$$F(t,u) = \begin{cases} f(t, \beta(t)),\ u > \beta(t) \\ f(t,u),\ \alpha(t) \leq u \leq \beta(t) \\ f(t, \alpha(t)),\ u < \alpha(t). \end{cases}$$

The function $F(t,u)$ is called a modification of $f(t,u)$ associated with α and β. Clearly the modified function $F(t,u)$ is bounded on $I \times \mathbb{R}$ and is continuous on $I \times \mathbb{R}$, provided f is continuous on $I \times \mathbb{R}$. Hence there exists a constant $M > 0$ such that

$$|F(t,u)| < M \text{ on } I \times \mathbb{R},$$

$$\text{where } M = \max\{|f(t,u)| : t \in I, \alpha \leq u \leq \beta\} + \max_{t\in I} |\alpha(t)| + \max_{t \in I} |\beta(t)| + 1.$$

Example: consider the following function

$$f(t,x) = at^2 + bx^3,$$

For simplicity, choose

$$\alpha(t) = -d \text{ and } \beta(t) = d, \text{ where } d \text{ is a constant.}$$

The modification F of f with respect to d and $-d$ is defined as follows

$$F(t,x) = \begin{cases} at^2 + bd^3, \text{ if } x > d, \\ at^2 + bx^3, \text{ if } -d \leq x \leq d, \\ at^2 - bd^3, \text{ if } x < -d. \end{cases}$$

Clearly F is continuous and bounded on $I \times \mathbb{R}$.■

Existence of solution:

In order to discuss the existence of a solution of a given problem by the method of upper and lower solutions, consider a two point BVP of the type

$$\begin{aligned} u''(t) &= f(t, u(t)), t \in [0,1], \\ u(0) &= 0, u(1) = 0, \end{aligned} \tag{1.10}$$

where $f\colon I \times \mathbb{R} \to \mathbb{R}$ is continuous and also the corresponding modified problem

$$\begin{aligned} u''(t) &= F(t, u(t)), t \in [0,1], \\ u(0) &= 0, u(1) = 0. \end{aligned}$$

Recall the concept of lower and upper solutions for the BVP (1.10).

Definition: $\alpha \in C^2(I)$ is a lower solution of (1.10) if

$$\begin{aligned} \alpha''(t) &\geq f(t, \alpha(t)), t \in I; \\ \alpha(0) &\leq 0, \ \alpha(1) \leq 0. \end{aligned}$$

Similarly $\beta \in C^2(I)$ is an upper solution of (1.10) if

$$\begin{aligned} \beta''(t) &\leq f(t, \beta(t)), t \in I; \\ \beta(0) &\geq 0, \ \beta(1) \geq 0. \end{aligned}$$

Theorem:(1.11) Suppose that $\alpha, \beta \in C^2(I)$ are lower and upper solution for (1.10), respectively, such that

$$\alpha(t) \leq \beta(t) \text{ for every } t \in I,$$

then there exist at least one solution u of (1.10) such that $\alpha(t) \leq u(t) \leq \beta(t)$, for every $t \in I$.

Proof: Consider the modified problem

$$\begin{aligned} u''(t) &= F(t, u(t)), t \in I \\ u(0) &= u(1) = 0. \end{aligned} \tag{1.11}$$

$$\text{where } F(t, u(t)) = \left\{ \begin{array}{c} f(t, \beta(t)) + \frac{u-\beta(t)}{1+|u-\beta|}, \ u > \beta(t), \\ f(t, u(t)), \ \alpha(t) \leq u \leq \beta(t), \\ f(t, \alpha(t)) + \frac{u-\alpha(t)}{1+|u-\alpha|}, \ u < \alpha(t). \end{array} \right|$$

Since F is continuous and bounded, hence by theorem (1.2) the modified problem has a solution. Let u be a solution of (1.11). It is required to show that $\alpha(t) \leq u(t) \leq \beta(t)$, $t \in I$. Define $V : I \to \mathbb{R}$ by

$$V(t) = u(t) - \alpha(t),$$

then $V \in C^2(I)$ and the BCs imply

$$V(0) \geq 0, V(1) \geq 0. \tag{1.12}$$

Assume that $V(t) \ngeqq 0$ for every $t \in I$. Then $V(t)$ has a negative minimum at some $t_0 \in I$. From the BCs, it is clear that $t_0 \in (0, 1)$. Hence

$$V(t_0) < 0, V'(t_0) = 0 \text{ and } V''(t_0) \geq 0. \tag{1.13}$$

On the other hand, the definition of F and that of lower solution implies that

$$\begin{aligned} V''(t_0) &= u''(t_0) - \alpha''(t_0) \\ &\leq F(t_0, u(t_0)) - f(t_0, \alpha(t_0)) \\ &= \left[f(t_0, \alpha(t_0)) + \frac{V(t_0)}{1+|V(t_0)|} \right] - f(t_0, \alpha(t_0)), \text{ since } u(t_0) < \alpha(t_0) \\ &= \frac{V(t_0)}{1+|V(t_0)|} < 0, \end{aligned}$$

which contradicts (1.13). Hence $\alpha(t) \leq u(t), t \in I$. Similarly, it can be proved that $u(t) \leq \beta(t)$, for $t \in I$. Hence $\alpha \leq u \leq \beta$ in $[a, b]$ and consequently u is a solution of (1.10).■

In general lower solutions may not always be less than the upper solutions, that is they may not always be well ordered. The following theorem establishes the situation where lower and upper solutions are well ordered.

Theorem:(1.12) Assume that $\alpha, \beta \in C^2(I)$ are lower and upper solutions of (1.10). If $f(t, u)$ is strictly increasing in u for each $t \in I$, then

$$\alpha(t) \leq \beta(t), \ t \in I.$$

Proof: Let $v : I \to \mathbb{R}$ be defined by

$$v(t) = \alpha(t) - \beta(t), \ t \in I,$$

then $v \in C^2(I)$ and the BCs imply that

$$v(0) \leq 0, \ v(1) \leq 0.$$

It is required to show that $v(t) \leq 0$, for every $t \in I$. Suppose on the contrary that $v(t) \nleqq 0$ for every $t \in I$. Then $v(t)$ has a +ve maximum at some $t_0 \in I$, that is,

$$v(t_0) = \max\{v(t) : t \in [0, 1]\} > 0 \text{ for some } t_0 \in [0, 1].$$

From the BCs, it is clear that $t_0 \in (0, 1)$. Hence $v(t_0) > 0$, $v'(t_0) = 0$ and $v''(t_0) \leq 0$. However, the increasing property of $f(t, u)$ in u yields

$$v''(t_0) = \alpha''(t_0) - \beta''(t_0) \geq f(t_0, \alpha(t_0)) - f(t_0, \beta(t_0)) > 0,$$

a contradiction. Hence $\alpha(t) \leq \beta(t)$, $t \in I$.■

Chapter 2

Existence of at least one solution for three-point boundary value problems at resonance

In this chapter, existence of at least one solution of the BVP (1.1) under a resonance condition $\theta(\eta+\epsilon) = 1+\epsilon$ is discussed. Since the homogeneous boundary value problem

$$\begin{aligned} u''(t) &= 0,\ t \in (0,1), \\ u(0) &= \epsilon u'(0),\ u(1) = \theta u(\eta), \end{aligned} \tag{2.1}$$

has non-trivial solutions $u(t) = c(t+\epsilon)$, where $c \in \mathbb{R}$ is an arbitrary constant. Therefore, the problem (1.1) is at resonance. Existence of at least one solution in the presence of lower and upper solutions under the resonance condition $\theta(\eta+\epsilon) = 1+\epsilon$ has been studied in [1]. Almost all of the material of this chapter is taken from the work of II'in [14]. The study of multi-point boundary value problems was initiated by II'in and Moiseev [14]. Recently Gupta [9] studied existence of at least one solution for some three point boundary value problems under some non-resonance conditions. Since then more general non-linear multi-point boundary value problems have been studied by many authors in non-resonance cases [2, 9, 19] and resonance cases [7, 10, 11]. More recently, Ma [19] studied an existence result for the following boundary value problem

$$u''(t) = f(t, u(t)), t \in (0,1), \tag{2.2}$$
$$u(0) = u\prime(0),\ u(1) = \theta u(\eta),$$

under the resonance condition $\theta\eta = 1$ in the presence of well-ordered upper and lower solutions. Note that (2.2) is a special case of (1.1) with $\epsilon = 1$ The methods of lower and upper solution are based on the connectivity properties of the solution sets of parameterized families of compact vector fields.

The following theorem is from Mawhin [22, Lemma 2.3].

Theorem:(2.1) Let E be a Banach space and C be a non-empty, bounded, closed and convex subset of E. Suppose that $T : [a,b] \times C \to C$ is completely continuous, then the set

$$S = \{(\lambda, x) : T(\lambda, x) = x,\ \lambda \in [a,b]\},$$

contains a closed connected subset $\sum$ which connects $\{a\} \times C$ to $\{b\} \times C$.

The following notations are used throughout this work. $X = C[0,1]$ denotes the Banach space with norm $\|x\|_\infty$ and $Y = C^2[0,1]$ denotes the Banach space with norm $\|y\|_y = \max\{\|y\|_\infty, \|y'\|_\infty, \|y''\|_\infty\}$.

Define a *linear differential operator* L by setting

$$D(L) = \{u \in C^2[0,1] : u(0) = \epsilon u'(0), u(1) = \theta u(\eta)\}$$

and for $u \in D(L)$,

$$Lu = u''. \tag{2.3}$$

2.1 Some Basic Lemmas:

Lemma:(2.2) The kernel of L denoted by $Ker(L)$ and image of L denoted by Im(L) are given by

$$Ker(L) = \{c(t + \epsilon) : c \in \mathbb{R}\} \tag{2.4}$$

and

$$\begin{aligned}
\operatorname{Im}(L) &= \{y \in X : \int\limits_0^1 (1-s)y(s)ds = \theta \int_0^\eta (\eta - s)y(s)ds\} \text{ or} \qquad (2.5)\\
\operatorname{Im}(L) &= \{y \in X : \int_0^1 Y(t)dt = \theta\eta \int_0^1 Y(\eta t)dt\},
\end{aligned}$$

$$\text{where } Y(t) = \int_0^t y(s)ds.$$

Proof: By definition of $Ker(L)$, it follows that

$$\begin{aligned}
Ker(L) &= \{u \in D(L) : Lu = 0 \text{ on } I\}\\
&= \{u \in D(L) : u'' = 0 \text{ on } I\}\\
&= \{c(t+\epsilon) : c \in \mathbb{R}\}.
\end{aligned}$$

Let $y \in \operatorname{Im}(L)$, then there exists $u \in D(L)$ such that

$$\begin{aligned}
Lu &= y \text{ on } I, \text{ that is,}\\
y(t) &= u''(t).
\end{aligned}$$

In view of the BCs, it follows that

$$\begin{aligned}
\int_0^1 Y(t)dt &= \int_0^1 dt \int_0^t y(s)ds = \int_0^1 dt \int_0^t u''(s)ds\\
&= \int_0^1 [u'(t) - u'(0)]dt = u(1) - u(0) - u'(0)\\
&= \theta u(\eta) - \epsilon u'(0) - u'(0) = \theta u(\eta) - (1+\epsilon)u'(0)
\end{aligned}$$

and

$$\begin{aligned}
\theta\eta\int_0^1 Y(\eta t)dt &= \theta\eta\int_0^1 dt\int_0^{\eta t} y(s)ds = \theta\eta\int_0^1 dt\int_0^{\eta t} u''(s)ds \\
&= \theta\eta\int_0^1 [u'(\eta t) - u'(0)]dt = \theta\eta[\frac{1}{\eta}\int_0^1 u'(\eta t)d(\eta t) - \int_0^1 u'(0)dt] \\
&= \theta\eta[\frac{1}{\eta}\int_0^{\eta} u'(s)ds - \int_0^1 u'(0)dt] = \theta\eta[\frac{1}{\eta}(u(\eta) - u(0)) - u'(0)] \\
&= \theta u(\eta) - \theta u(0) - \theta\eta u'(0) = \theta u(\eta) - \theta(\epsilon+\eta)u'(0), \\
&= \theta u(\eta) - (1+\epsilon)u'(0).
\end{aligned}$$

Consequently,

$$\int_0^1 Y(t)dt = \theta\eta\int_0^1 Y(\eta t)dt.$$

On the other hand, let $y \in X$ such that

$$\int_0^1 Y(t)dt = \theta\eta\int_0^1 Y(\eta t)dt.$$

It is required to show that $y \in \text{Im}(L)$, that is, there exist $u \in D(L)$ such that

$$Lu = y, \text{ or } u'' = y.$$

Let $u(t) = \int_0^t Y(s)ds$, then

$$u(t) = \int_0^t ds\int_0^s y(t)dt = t\int_0^s y(t)dt.$$

Hence,

$$u'(t) = \int_0^t y(t)dt,$$

which implies that

$$u''(t) = y(t).$$

Consequently, $y(t) \in \text{Im}(L)$.■

Define $Q : C[0,1] \to \mathbb{R}$ by

$$Qy = \Gamma^0 \left(\int_0^1 Y(t)dt - \theta\eta \int_0^1 Y(\eta t)dt \right), \text{ where } \Gamma^0 = \frac{2}{1-\theta\eta^2} > 0.$$

Lemma:(2.3) For every $y \in X$, $y(t) - Qy \in \text{Im}(L)$.

Proof: Take

$$y_1(t) = y(t) - Qy, \text{ then}$$

$$Y_1(t) = \int_0^t y_1(s)ds = \int_0^t (y(s) - Qy)ds = \int_0^t y(s)ds - (Qy)t \tag{2.6}$$

and

$$Y_1(\eta t) = \int_0^{\eta t} y_1(s)ds = \int_0^{\eta t} (y(s) - Qy)ds = \int_0^{\eta t} y(s)ds - (Qy)\eta t. \tag{2.7}$$

Integrating (2.6) and (2.7) from 0 to 1, it follows that

$$\int_0^1 Y_1(t)dt = \int_0^1 [\int_0^t y(s)ds - (Qy)t]dt = \int_0^1 Y(t)dt - \frac{Qy}{2} \tag{2.7*}$$

and

$$\theta\eta \int_0^1 Y_1(\eta t)dt = \theta\eta \int_0^1 [\int_0^{\eta t} y(s)ds - (Qy)\eta t]dt = \theta\eta \int_0^1 Y(\eta t)dt - Qy\frac{\theta\eta^2}{2}. \tag{2.8}$$

Now, from

$$Qy = \Gamma^0 \left(\int_0^1 Y(t)dt - \theta\eta \int_0^1 Y(\eta t)dt \right),$$

it follows that

$$\frac{1-\theta\eta^2}{2} Qy = \int_0^1 Y(t)dt - \theta\eta \int_0^1 Y(\eta t)dt.$$

Hence,

$$\frac{Qy}{2} - Qy\frac{\theta\eta^2}{2} = \int_0^1 Y(t)dt - \theta\eta \int_0^1 Y(\eta t)dt,$$

which leads to

$$\theta\eta \int_0^1 Y(\eta t)dt - Qy\frac{\theta\eta^2}{2} = \int_0^1 Y(t)dt - \frac{Qy}{2}.$$

Substituting (2.7*) and (2.8), in the last equation it follows that

$$\theta\eta \int_0^1 Y_1(\eta t)dt = \int_0^1 Y_1(t)dt.$$

Hence, by lemma (2.2), $y_1 \in \text{Im}(L)$.■

Remark:(2.4) The space $X = C[0,1]$ is the direct sum of $\text{Im}(L)$ and $\mathbb{R}$, that is,

$$X = \text{Im}(L) \oplus \mathbb{R}$$

Proof: By lemma (2.3), for every $y \in X$, $y(t) - Qy \in \text{Im}(L)$. Let

$$y(t) - Qy = z, \text{ where } z \in \text{Im}(L), \text{ then}$$

$$y(t) = z + Qy.$$

This implies that every $y \in X$, can be expressed as $z + Qy$, where $z \in \text{Im}(L)$ and $Qy \in \mathbb{R}$. Hence

$$X = \text{Im}(L) + \mathbb{R}.$$

Now, it remains to show that

$$\text{Im}(L) \cap \mathbb{R} = \{0\}.$$

Let $y \in \text{Im}(L) \cap \mathbb{R}$, then $y \in \text{Im}(L)$ where $y \in \mathbb{R}$. By lemma (2.2), it follows that

$$\int_0^1 (1-s)y\,ds = \theta \int_0^\eta (\eta - s)y\,ds,$$

which implies that

$$y\left[s - \frac{s^2}{2}\right|_0^1 = \theta y \left[\eta s - \frac{s^2}{2}\right|_0^\eta.$$

Consequently,

$$\frac{y}{2}(1 - \theta\eta^2) = 0,$$

which implies $y = 0$ as $1 \neq \theta\eta^2$. Hence,

$$\text{Im}(L) \cap \mathbb{R} = \{0\}.$$

Thus,

$$X = \text{Im}(L) \oplus \mathbb{R}.■$$

Remark:(2.5) Let $P : Y \to Ker(L)$ be defined as

$$(Pu)(t) = u'(0)(t + \epsilon),$$

then,

$$Y = Ker(P) \oplus Ker(L).$$

Proof: For any $y \in Y$, $(Py)(t) \in Ker(L)$, that is,

$$(Py)(t) = y'(0)(t + \epsilon) = c(t + \epsilon),$$

which implies that

$$y'(0) = c.$$

Now, let

$$w(t) = y(t) - c(t + \epsilon),$$

then

$$(Pw)(t) = (y'(0) - c)(t + \epsilon) = 0,$$

which implies $w \in Ker(P)$. Hence $y(t) \in Ker(L) + Ker(P)$. Consequently

$$Y = Ker(L) + Ker(P).$$

Moreover, let $u \in Ker(L) \cap Ker(P)$, then

$$u(t) = c(t + \epsilon) \text{ and } u'(0) = 0,$$

which implies

$$u = 0 \text{ on } [0, 1].$$

Hence,

$$Ker(L) \cap Ker(P) = \{0\}.$$

Thus,

$$Y = Ker(L) \oplus Ker(P).\blacksquare$$

Remark:(2.6) Let $L_p = L \mid_{D(L)\cap Ker(P)}$, then $L_p : D(L) \cap Ker(P) \to \mathrm{Im}(L)$ is one to one operator.

Proof: Let, $y_1, y_2 \in D(L) \cap Ker(P)$ such that

$$(L_P y_1)(t) = (L_P y_2)(t),\ t \in [0,1],\ \text{then}$$

$$(Ly_1)(t) = (Ly_2)(t),\ \ t \in [0,1],$$

that is,

$$\begin{aligned} y_1''(t) &= y_2''(t), \\ (y_1'' - y_2'')(t) &= 0, \\ (y_1 - y_2)''(t) &= 0. \end{aligned}$$

Integrating and using the conditions

$$y_1'(0) = y_2'(0) = 0,$$

it follows that

$$(y_1 - y_2)'(t) = 0, t \in [0,1].$$

Integrating again, and using the BCs,

$$y_1(0) - y_2(0) = \epsilon(y_1'(0) - y_2'(0)) = 0,$$

it follows that

$$y_1(t) = y_2(t).$$

Hence, L_P is one to one.$\blacksquare$

Remark:(**2.7**) Let

$$K_{pQ} = L_P^{-1}(I - Q),$$

and $N : X \to X$, a non-linear operator defined by

$$(Nu)(t) = f(t, u(t)), t \in [0, 1], \tag{2.9}$$

then $K_{pQ}N : X \to X$ is completely continuous and (1.1) is equivalent to the system

$$\begin{aligned} \omega(t) &= K_{pQ}N(\rho(t+\epsilon) + \omega(t)), \qquad (2.10) \\ QN(\rho(t+\epsilon) + \omega(t)) &= 0. \end{aligned}$$

Proof: By remark (2.5), for every $u \in Y$, the unique decomposition of u is

$$u(t) = \rho(t+\epsilon) + \omega(t), \; \rho \in \mathbb{R} \text{ and } \omega \in Ker(P).$$

Since

$$w = K_{pQ}Nu = L_P^{-1}(1 - Q)Nu,$$

which implies that

$$L_p w = (1 - Q)Nu.$$

Hence,

$$\begin{aligned} L_p u &= (1 - Q)Nu = Nu - QNu, \\ &= Nu \text{ iff } QNu = 0, \end{aligned}$$

thus,

$$u'' = f(t, u(t)) \text{ iff } QNu = 0,$$

that is,

$$\left\{ \begin{array}{c} w(t) = K_{pQ}N(\rho(t+\epsilon) + \omega(t)), \\ QN(\rho(t+\epsilon) + \omega(t)) = 0. \end{array} \right\}$$

Hence (1.1) is equivalent to (2.10).■

2.2 Existence of at least one solution:

Consider the BVP (1.1). Let β be a strict upper solution and α be a strict lower solution of (1.1) such that $\beta(t) \geq \alpha(t), t \in [0,1]$. Define $D = \{(t,u) : \alpha(t) \leq u \leq \beta(t),\ t \in [0,1]\}$ and the modification $f^* : [0,1] \times \mathbb{R} \to \mathbb{R}$ by

$$f^*(t,u) = \begin{cases} f(t,\beta(t)), \text{ if } u > \beta(t),\ t \in I, \\ f(t,u), \text{ if } \alpha \leq u \leq \beta,\ t \in I, \\ f(t,\alpha(t)), \text{ if } u < \alpha(t),\ t \in I. \end{cases}$$

Clearly f^* is continuous and bounded. Consider the modified problem

$$\begin{aligned} u''(t) &= f^*(t,u(t)),\ t \in I, \\ u(0) &= \epsilon u'(0),\ u(1) = \theta u(\eta),\ \epsilon > 0. \end{aligned} \tag{2.11}$$

Lemma:(2.8) If the modified problem (2.11) has a solution u, then $\alpha(t) \leq u \leq \beta(t)$, $t \in [0,1]$. In other words u is solution of (1.1).

Proof: Assume that (2.11) has a solution u. It is required to show that

$$\alpha(t) \leq u \leq \beta(t), t \in [0,1].$$

Let

$$v(t) = u(t) - \beta(t),\ t \in [0,1], \tag{2.12}$$

then $v \in C^2[0,1]$ and the BCs imply

$$v(0) \leq \epsilon v'(0), v(1) \leq \theta v(\eta). \tag{2.13}$$

Suppose on the contrary that

$$max\{v(t); t \in [0,1]\} = v(t_0) > 0, \text{for some } t_0 \in [0,1].$$

If $t_0 = 0$, then $v(0) > 0$ and from the BCs (2.13), it follows that $v'(0) > 0$. That is, v is an increasing in the neighborhood $(0, \delta)$ of 0 for small $\delta > 0$. This contradicts $v(0)$ being the maximum value of v on $[0, 1]$. Hence $t_0 \neq 0$. If $t_0 \in (0, 1)$, then

$$v(t_0) > 0, \ v'(t_0) = 0 \text{ and } v''(t_0) \leq 0. \tag{2.14}$$

On the other hand, the definition of f^* and the fact that β is strict upper solution, imply that

$$u''(t_0) = f^*(t_{0,}u(t_0)), \ \ = f(t_{0,}\beta(t_0)) > \beta''(t_0),$$

which implies that

$$v''(t_0) = u''(t_0) - \beta''(t_0) > 0,$$

a contradiction to (2.14). Hence $v(t)$ has no +ve local maximum for $t \in (0, 1)$. If $t_0 = 1$, then $v(1) > 0$. There are two possible cases to be discussed.

Case (I) $v(0) > 0$: From (2.13) and the fact that $v(0) > 0$, it follows that

$$v'(0) > 0.$$

This implies that $v(t)$ is an increasing function on $(0, \delta)$ and since $v(t)$ has no +ve local maximum, hence $v(t) > 0$ on $[0, 1]$, that is,

$$u(t) > \beta(t), \text{for all } t \in [0, 1].$$

Consequently, the definition of f^* and the fact that β is strict upper solution, imply that

$$\begin{aligned} v''(t) &= u''(t) - \beta''(t) > f^*(t, u(t)) - f(t, \beta(t)) \\ &= f(t, \beta(t)) - f(t, \beta(t)) = 0, t \in [0, 1]. \end{aligned} \tag{2.15}$$

Hence the graph of v is convex on $(0, 1)$, that is,

$$\frac{v(\eta) - v(0)}{\eta} < \frac{v(1) - v(0)}{1}. \tag{2.16}$$

However, by the BCs (2.13), it follows that

$$\begin{aligned} v(1) &= u(1) - \beta(1) \leq \theta u(\eta) - \theta\beta(\eta) = \theta[u(\eta) - \beta(\eta)] \leq \theta v(\eta), \\ \frac{v(1)}{1} &\leq \frac{1+\epsilon}{\eta+\epsilon} v(\eta) \text{ or} \end{aligned} \tag{2.17}$$

$$v(1)\eta - v(\eta) + \epsilon(v(1) - v(\eta)) \leq 0. \tag{2.18}$$

By Mean value theorem, there exist $\xi \in (\eta, 1)$ such that

$$\frac{v(1) - v(\eta)}{1 - \eta} = v'(\xi). \tag{2.19}$$

From (2.15), $v''(t) > 0$ on $(0, 1)$, implies $v'(t)$ is strictly increasing on $(0, 1)$. Hence $v'(0) < v'(\xi)$, which in view of (2.19) implies that

$$v'(0) < \frac{v(1) - v(\eta)}{1 - \eta}.$$

Hence,

$$\begin{aligned} \epsilon v'(0)(1 - \eta) &< \epsilon[v(1) - v(\eta)] \text{ or} \\ (1 - \eta)v(0) &< \epsilon[v(1) - v(\eta)], \text{ since } v(0) \leq \epsilon v'(0) \text{ by (2.13).} \end{aligned} \tag{2.20}$$

Combining (2.18) with (2.20), we obtain

$$v(1)\eta - v(\eta) + (1 - \eta)v(0) < 0,$$

which implies that

$$\frac{v(\eta) - v(0)}{\eta} > \frac{v(1) - v(0)}{1}, \tag{2.21}$$

a contradiction to (2.16).

Case (II) $v(0) \leq 0$: From the BCs and the fact that $v(1) > 0$, it follows that

$$v(\eta) = u(\eta) - \beta(\eta) \geq \frac{1}{\theta}[u(1) - \beta(1)] \geq \frac{1}{\theta} v(1) > 0.$$

Hence there exist $\tau \in [0, 1)$ such that

$$v(\tau) = 0 \text{ and } v(t) > 0, \text{ for all } t \in (\tau, 1].$$

If $\tau \in (\eta, 1)$, then there exist $t_1 \in (0, \tau)$ such that

$$\max\{v(t) : t \in [0, \tau]\} = v(t_1) \geq v(\eta) > 0,$$

which implies

$$v'(t_1) = 0 \text{ and } v''(t_1) \leq 0. \tag{2.22}$$

But the definition of f^* and the fact that β is strict upper solution, imply that

$$u''(t_1) = f^*(t_1, u(t_1)) = f(t_1, \beta(t_1)) > \beta''(t_1),$$

that is,

$$v''(t_1) = u''(t_1) - \beta''(t_1) > 0,$$

which is a contradiction to (2.22).

If $\tau \in [0, \eta]$, then for all $t \in (\tau, 1]$, $v(t) > 0$, and by the definition of f^* and the fact that β is strict upper solution, it follows that

$$u''(t) = f^*(t, u(t)) = f(t, \beta(t)) > \beta''(t), \ t \in (\tau, 1],$$

that is,

$$v''(t) = u''(t) - \beta''(t) > 0, \ t \in (\tau, 1].$$

So there exist two numbers t_2, t_3 such that

$$\frac{v(\eta)}{\eta} < \frac{v(\eta) - v(\tau)}{\eta - \tau} = v'(t_2), \text{ for } t_2 \in (\tau, \eta),$$

and

$$\frac{v(1) - v(\eta)}{1 - \eta} = v'(t_3), \text{ for } t_3 \in (\eta, 1).$$

Since $v''(t) > 0$ on $(\tau, 1]$, therefore,

$$\begin{aligned}
\frac{v(\eta)}{\eta} &< v'(t_2) < v'(t_3) = \frac{v(1) - v(\eta)}{1-\eta}, \\
v(\eta)(1-\eta) &< \eta v(1) - \eta v(\eta), \\
v(\eta) &< \eta v(1), \\
\frac{v(\eta)}{\eta} &< \frac{v(1)}{1}. \qquad (2.23)
\end{aligned}$$

However, from (2.17) we obtain

$$\frac{v(1)}{1} \leq \frac{1+\epsilon}{\eta+\epsilon} v(\eta) < \frac{v(\eta)}{\eta},$$

which contradicts (2.23). Hence, the assumption that $v(t_0) > 0$ for some $t_0 \in [0,1]$ is not true. Thus $v(t) \leq 0$ for all $t \in [0,1]$.

Similarly,

$$u(t) \geq \alpha(t), \text{ for } t \in [0,1].\blacksquare$$

Theorem:(2.9) Let $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ be continuous. Assume that β and α are strict upper and lower solutions of (1.1) respectively, such that

$$\alpha(t) \leq \beta(t), \ t \in [0,1],$$

then the BVP (1.1) has a solution $u \in D$.

Proof: Consider the modified problem

$$\begin{aligned}
u''(t) &= f^*(t, u(t)), \ t \in (0,1), \qquad (2.24) \\
u(0) &= \epsilon u'(0), \ u(1) = \theta u(\eta),
\end{aligned}$$

It is equivalent to the system

$$w(t) = K_{PQ} N^*(\rho(t+\epsilon) + w(t)),$$

$$QN^*(\rho(t+\epsilon) + w(t)) = 0, \qquad (2.25)$$

where $K_{PQ}N^* : X \to X$ is completely continuous.

Since f^* is bounded, we know from (2.25) and the Schauder fixed point theorem that for every $\rho \in \mathbb{R}$, the set

$$W(\rho) = \{w \in \widetilde{Y} : (\rho, w) \text{ satisfies } (2.25)\} \neq 0.$$

Moreover, by Theorem (2.1), the set

$$S = \{(\rho, w) \in \mathbb{R} \times \widetilde{Y} : (\rho, w) \text{ satisfies } (2.25)\}, \tag{2.26}$$

contains a connected subset $\sum$ which joins $\{a\} \times W(a)$ and $\{b\} \times W(b)$ for every a,$b \in \mathbb{R}$ with $a < b$. Put

$$W = \{w \in \widetilde{Y} : (\rho, w) \in S\}.$$

Then, by (2.25), there exists a constant $M > 0$ independent of ρ, such that

$$\max\{\|w\|_\infty, \|w'\|_\infty\} \leq M, \text{ for all } w \in W.$$

Choose $\rho_1 \in \mathbb{R}$ so large that for all $w \in W$

$$\rho_1(t + \epsilon) + w(t) > \beta(t), \text{ for } t \in [0, 1],$$

this implies that $f^*(t, \rho_1(t + \epsilon) + w(t)) \equiv f(t, \beta(t))$ and $W(\rho_1)$ reduces to the single-point set $\{K_{PQ}f(t, \beta(t))\}$. Moreover, for every $w \in W(\rho_1)$, we have

$$\begin{aligned} QN^*(\rho_1(t+\epsilon) + w(t)) &= \Gamma_0[\int_0^1 (\int_0^t f^*(s, \rho_1(s+\epsilon) + w(s))ds)dt \\ &\quad -\theta\eta \int_0^1 (\int_0^{\eta t} f^*(s, \rho_1(s+\epsilon) + w(s))ds)dt] \\ &= \Gamma_0[\int_0^1 dt \int_0^t f(s, \beta(s))ds - \theta\eta \int_0^1 dt \int_0^{\eta t} f(s, \beta(s))ds]. \end{aligned} \tag{2.27}$$

Define

$$d = \int_0^1 dt \int_0^t f(s, \beta(s))ds - \theta\eta \int_0^1 dt \int_0^{\eta t} f(s, \beta(s))ds,$$

and

$$e = \int_0^1 dt \int_0^t \beta''(s)ds - \theta\eta \int_0^1 dt \int_0^{\eta t} \beta''(s)ds.$$

From the fact that $\beta''(t) < f(t, \beta(t))$ on $(0, 1)$ and $\theta\eta < 1$, it is claimed that $d > e$. Because

$$\begin{aligned} d - e &= \int_0^1 dt \int_0^t f(s, \beta(s))ds - \theta\eta \int_0^1 dt \int_0^{\eta t} f(s, \beta(s))ds \\ &\quad - \int_0^1 dt \int_0^t \beta''(s)ds + \theta\eta \int_0^1 dt \int_0^{\eta t} \beta''(s)ds, \\ &= \int_0^1 dt \int_0^t (f(s, \beta(s)) - \beta''(s))ds \\ &\quad -\theta\eta \int_0^1 dt[\int_0^{\eta t} (f(s, \beta(s)) - \beta''(s))ds], \\ &> \int_0^1 dt \int_0^t (f(s, \beta(s)) - \beta''(s))ds - \int_0^1 dt[\int_0^{\eta t} (f(s, \beta(s)) - \beta''(s))ds], \\ &= \int_0^1 dt \int_{\eta t}^t (f(s, \beta(s)) - \beta''(s))ds > 0. \end{aligned} \tag{2.28}$$

Combining (2.27) with (2.28), we get

$$\begin{aligned} QN^*(\rho_1(t + \epsilon) + w(t)) &> \Gamma_0[\int_0^1 dt \int_0^t \beta''(s))ds - \theta\eta \int_0^1 dt \int_0^{\eta t} \beta''(s)ds] \\ &= \Gamma_0[\int_0^1 (\beta'(t) - \beta'(0))dt - \theta\eta \int_0^1 (\beta'(\eta t) - \beta'(0)dt] \\ &= \Gamma_0\{(\beta(1) - \beta(0) - \beta'(0) - \theta\eta[\eta^{-1}(\beta(\eta) - \beta(0) - \beta'(0)]\} \\ &= \Gamma_0 \left\{ \beta(1) - \theta\beta(\eta) + \frac{1 - \eta}{\eta + \epsilon}[\beta(0) - \epsilon\beta'(0)] \right\} \\ &\geq 0. \end{aligned}$$

Choose $\rho_2 \in \mathbb{R}$ such that for all $w \in W$

$$\rho_2(t + \epsilon) + w(t) < \alpha(t), \text{ for } t \in [0, 1],$$

this implies that $f^*(t, \rho_2(t + \epsilon) + w(t)) \equiv f(t, \alpha(t))$ and $W(\rho_2)$ reduces to the single-point set

$\{K_{PQ}f(t,\alpha(t))\}$. Moreover, for every $w \in W(\rho_2)$, we have

$$\begin{aligned} QN^*(\rho_2(t+\epsilon)+w(t)) &= \Gamma_0[\int_0^1(\int_0^1 f^*(s,\rho_2(s+\epsilon)+w(s))ds)dt) \\ &\quad -\theta\eta\int_0^1(\int_0^{\eta t} f^*(s,\rho_2(s+\epsilon)+w(s))ds)dt], \\ &= \Gamma_0[\int_0^1 dt\int_0^t f(s,\alpha(s))ds - \theta\eta\int_0^1 dt\int_0^{\eta t} f(s,\alpha(s))ds]. \quad (2.29) \end{aligned}$$

Define

$$d = \int_0^1 dt\int_0^t f(s,\alpha(s))ds - \theta\eta\int_0^1 dt\int_0^{\eta t} f(s,\alpha(s))ds,$$

and

$$e = \int_0^1 dt\int_0^t \alpha''(s)ds - \theta\eta\int_0^1 dt\int_0^{\eta t} \alpha''(s)ds.$$

From the fact that $\alpha''(t) > f(t,\alpha(t)$ on $(0,1)$ and $\theta\eta < 1$, it follows that $d < e$. In fact,

$$\begin{aligned} d-e &= \int_0^1 dt\int_0^t f(s,\alpha(s))ds - \theta\eta\int_0^1 dt\int_0^{\eta t} f(s,\alpha(s))ds \\ &\quad -\int_0^1 dt\int_0^t \alpha''(s)ds + \theta\eta\int_0^1 dt\int_0^{\eta t} a''(s)ds, \\ &= \int_0^1 dt\int_0^t (f(s,\alpha(s)) - \alpha''(s))ds \\ &\quad -\theta\eta\int_0^1 dt[\int_0^{\eta t}(f(s,\alpha(s)) - \alpha''(s))ds], \\ &< \int_0^1 dt\int_0^t (f(s,\alpha(s)) - \alpha''(s))ds - \int_0^1 dt[\int_0^{\eta t}(f(s,\alpha(s)) - \alpha''(s))ds], \\ &= \int_0^1 dt\int_{\eta t}^t (f(s,\alpha(s)) - \alpha''(s))ds < 0. \quad (2.30) \end{aligned}$$

Combining (2.29) with (2.30), we get

$$\begin{aligned} QN^*(\rho_2(t+\epsilon)+w(t)) &< \Gamma_0[\int_0^1 dt\int_0^t \alpha''(s))ds - \theta\eta\int_0^1 dt\int_0^{\eta t}\alpha''(s)ds] \\ &= \Gamma_0[\int_0^1(\alpha'(t)-\alpha'(0))dt - \theta\eta\int_0^1(\alpha'(\eta t)-\alpha'(0)dt] \\ &= \Gamma_0\{(\alpha(1)-\alpha(0)-\alpha'(0)-\theta\eta[\eta^{-1}(\alpha(\eta)-\alpha(0)-\alpha'(0)]\} \\ &= \Gamma_0\left\{\alpha(1)-\theta\alpha(\eta)+\frac{1-\eta}{\eta+\epsilon}[\alpha(0)-\epsilon\alpha'(0)]\right\} \\ &\le 0. \end{aligned}$$

Therefore, by the connectivity of $\sum$, there must exist some $\rho_0 \in (\rho_2, \rho_1)$ and $w(\rho_0) \in W(\rho_0)$ such that $(\rho_0, w(\rho_0)) \in \sum$ and (2.26) holds. Thus $\rho_0(t+\epsilon) + w(\rho_0)$ is a solution of (2.25).■

Example: Consider the following BVP

$$\begin{aligned} u''(t) &= (u(t) - 2t - 3\epsilon)^l + \frac{t^l}{2}\cos^k(u(t)),\ t \in (0,1), \\ u(0) &= \epsilon u'(0),\ u(1) = \frac{1+\epsilon}{\eta+\epsilon}u(\eta), \end{aligned} \tag{2.31}$$

where $k \in N$ and $l = 2n-1$, $n \in N$.

Solution: Take,

$$\alpha(t) = t + \epsilon \text{ and } \beta(t) = 3(t+\epsilon).$$

Clearly,

$$\begin{aligned} \beta''(t) &= 0 \text{ and} \\ f(t,\beta) &= (3t + 3\epsilon - 2t - 3\epsilon)^l + \frac{t^l}{2}\cos^k(3t+3\epsilon), \\ &= t^l + \frac{t^l}{2}\cos^k(3t+3\epsilon) > 0 \text{ on } [0,1]. \end{aligned}$$

Hence,

$$\beta''(t) < f(t,\beta),\ t \in I.$$

Moreover, BCs imply

$$\beta(0) = \epsilon\beta'(0),\ \beta(1) = \frac{1+\epsilon}{\eta+\epsilon}\beta(\eta),$$

thus, β is an upper solution.

Now,

$$\alpha''(t) = 0 \text{ and}$$

$$\begin{aligned}
f(t,\alpha) &= (t+\epsilon-2t-3\epsilon)^l + \frac{t^l}{2}\cos^k(t+\epsilon),\\
&= -(t+2\epsilon)^l + \frac{t^l}{2}\cos^k(t+\epsilon),\\
&= -[t^l + \binom{l}{1}t^{l-1}(2\epsilon) + \binom{l}{2}t^{l-2}(2\epsilon)^2 + \binom{l}{3}t^{l-3}(2\epsilon)^3 + ... + \binom{l}{m}t^{l-m}(2\epsilon)^m + (2\epsilon)^l + \frac{t^l}{2}\cos^k(t+\epsilon)]\\
&= -[t^l + \sum_{m=1}^{l}\binom{l}{m}t^{l-m}(2\epsilon)^m] + \frac{t^l}{2}\cos^k(t+\epsilon),\\
&= -t^l[1-\frac{1}{2}\cos^k(t+\epsilon)] - \sum_{m=1}^{l}\binom{l}{m}t^{l-m}(2\epsilon)^m < 0,\ t\in I,\ \binom{l}{m} = \frac{l!}{m!(l-m)!}.
\end{aligned}$$

Hence,

$$\alpha''(t) > f(t,\alpha),\ t\in I.$$

Moreover, BCs imply

$$\alpha(0) = \epsilon\alpha'(0),\ \alpha(1) = \frac{1+\epsilon}{\eta+\epsilon}\alpha(\eta),$$

thus, α is a lower solution. Clearly, $\alpha(t) \leq \beta(t)$ on $[0,1]$. Hence by Theorem (2.9), the BVP (2.31) has a solution u such that $t+\epsilon \leq u(t) \leq 3(t+\epsilon)$, $t\in[0,1]$.■

Chapter 3

Multiplicity results of three point boundary value problems at non-resonance:

The study of existence of solutions of multi-point boundary value problems for linear second order ordinary differential equations was initiated by II' in and Moiseev [14]. Then Gupta [9] investigated three-point boundary value problems for nonlinear ordinary differential equations at non-resonance. Since then, more general nonlinear multi-point BVPs have been studied by several authors with the theory of fixed point index, coincidence degree theory, Leray-Schauder continuation theorems and fixed point theorems in cones. We refer the readers to [19] for non-resonance cases and to $[7, 10, 11]$ for resonance cases.

Boundary value problems for ordinary differential equations play an important role in a variety of different areas of applied mathematics and physics. Various applications of boundary value problems to physical, biological and chemical process are well documented in the literature; for example, the classic books of love [18], Prescott [24] and Timoshenko [25] on elasticity, the works of Mansfield [20] on deformation of structures, and the monograph of Dulacska [6] on the effects of soil settlement are good sources of such applications. Moreover, multi-point boundary value problems can arise in, for example,

(1) the vibrations of a guy wire of uniform cross-section and composed of N parts of different

densities can be set as a multi-point BVPs (see [23]),

(2) the study of the steady states of a heated bar with a thermostat, where a controller at $t = 1$ adds or dissipates heat according to the temperature detected by a sensor at $t = \eta$, while the left end of the bar is maintained at a fixed temperature,

(3) the bending of a beam where conditions may be imposed at the ends of the beam, as well as at an interior point to improve stability or for other reasons (see [3]).

In this chapter existence of at least three solutions of the BVP (1.1) is investigated by using properties of degree theory. We prove that in the presence of two lower and two upper solutions, the BVP has at least three solutions. The results of this chapter are the original work of the author and his supervisor.

3.1 Basic idea:

In chapter 2 , existence of solution for the second-order three point BVP of the type

$$\begin{aligned} u''(t) &= f(t, u(t)),\ t \in (0,1), \\ u(0) &= \epsilon u'(0),\ u(1) = \theta u(\eta), \end{aligned}$$

is discussed. The function $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ is assumed to be continuous and $\epsilon \in [0,\infty), \theta \in (0,\infty), \eta \in (0,1)$ are given constants such that $\theta(\eta + \epsilon) = 1 + \epsilon$. This problem happens to be at resonance in the sense that the associated linear homogeneous boundary value problem

$$\begin{aligned} u''(t) &= 0,\ t \in (0,1), \\ u(0) &= \epsilon u'(0), u(1) = \theta u(\eta), \end{aligned}$$

has $u(t) = c(t + \epsilon), c \in \mathbb{R}$, as non-trivial solutions.

We study the BVP under the non-resonance condition

$$\theta = \frac{\eta + \epsilon}{1 + \epsilon}.$$

Hence by choosing $\theta = \frac{\eta+\epsilon}{1+\epsilon}$ the BVP problem

$$\begin{aligned} u''(t) &= f(t, u(t)), t \in (0,1), \\ u(0) &= \epsilon u'(0), u(1) = \theta u(\eta), \end{aligned} \tag{3.1}$$

is at non-resonance, because the associated linear homogeneous boundary value problem

$$\begin{aligned} u''(t) &= 0, \ t \in (0,1), \\ u(0) &= \epsilon u'(0), u(1) = \theta u(\eta), \end{aligned} \tag{3.2}$$

has a trivial solution only. In this case, the kernel of the differential operator $L = \frac{d^2}{dt^2}$ is the set $\{0\}$, which implies that L is invertible. Hence the BVP (3.1) is equivalent to integral equation

$$u(t) = \int_0^1 G(t,s) f(s, u(s)) ds, \tag{3.3}$$

where $G(t,s)$ is a Green's function for the associated problem (3.2). Define an integral operator $T : C[0,1] \to C[0,1]$ by

$$(Tu)(t) = \int_0^1 G(t,s) f(s, u(s)) ds.$$

Then solutions of the integral equation (3.3) means solutions of the operator equation

$$(Tu)(t) = u(t), \ t \in [0,1],$$

that is, fixed points of T.

Since our work depends on the nature of the Green's function, we need to get its explicit form.

3.2 Explicit form of Green's function:

We use properties of the Green's function $G(t,s)$ to construct its explicit form. we discuss two cases; that is, $\eta \leq s$ and $\eta \geq s$. Assume $\eta \leq s$ and take

$$G_1(t,s) = \begin{cases} A + Bt, \ 0 \leq t \leq s \leq 1, \\ C + Dt, \ 0 \leq s \leq t \leq 1, \end{cases} \eta \leq s, \Bigg| \tag{3.4}$$

where A, B, C, and D are constant to be determined. The boundary condition

$$G_1(0,s) = \epsilon \frac{\partial}{\partial t} G_1(0,s),$$

implies that

$$A = \epsilon B,$$

and the second BC,

$$G_1(1,s) = \theta G_1(\eta, s),$$

yields

$$C + D = \theta(A + B\eta),$$

which implies

$$C = \theta(A + B\eta) - D.$$

Putting the values of A and C in (3.4), we have

$$G_1(t,s) = \begin{cases} (\epsilon + t)B, \ 0 \leq t \leq s \leq 1, \\ \theta(\epsilon + \eta)B + (t-1)D, \ 0 \leq s \leq t \leq 1, \ \eta \leq s. \end{cases} \Bigg|$$

The continuity of a Green's function at s, that is

$$G_1(s^+, s) = G_1(s^-, s),$$

yields

$$B = \frac{(s-1)D}{[(\epsilon + s) - \theta(\epsilon + \eta)]},$$

which implies

$$G_1(t,s) = \left\{ \begin{array}{ll} (\epsilon+t)\frac{(s-1)}{[(\epsilon+s)-\theta(\epsilon+\eta)]}D, \ 0 \le t \le s \le 1, \\ \frac{\theta(\epsilon+\eta)(s-t)+(t-1)(\epsilon+s)}{(\epsilon+s)-\theta(\epsilon+\eta)}D, \ 0 \le s \le t \le 1, \end{array} \quad \eta \le s. \right|$$

Taking the derivative with respect to t, we get

$$\frac{\partial}{\partial t}G_1(t,s) = \left\{ \begin{array}{ll} \frac{s-1}{(\epsilon+s)-\theta(\epsilon+\eta)}D, \ 0 \le t \le s \le 1, \\ \frac{\epsilon+s-\theta(\epsilon+\eta)}{(\epsilon+s)-\theta(\epsilon+\eta)}D, \ 0 \le s \le t \le 1, \end{array} \quad \eta \le s. \right|$$

Using the jump-discontinuity property of the derivative of G_1, that is,

$$\frac{\partial}{\partial t}G_1(s^+,s) - \frac{\partial}{\partial t}G_1(s^-,s) = -1,$$

we get

$$D = \frac{\theta(\epsilon+\eta)-(s+\epsilon)}{\theta(\epsilon+\eta)-(1+\epsilon)}.$$

Hence,

$$G_1(t,s) = \left\{ \begin{array}{ll} \frac{(s-1)(t+\epsilon)}{\theta(\epsilon+\eta)-(1+\epsilon)}, \ 0 \le t \le s \le 1, \\ \frac{\theta(\epsilon+\eta)(s-t)+(1-t)(\epsilon+s)}{\theta(\epsilon+\eta)-(1+\epsilon)}, \ 0 \le s \le t \le 1, \end{array} \quad \eta \le s, \right|$$

is a Green's function of homogeneous problem (3.2).

In the case , when $\eta \ge s$, let the Green's function be

$$G_2(t,s) = \left\{ \begin{array}{ll} A+Bt, \ 0 \le t \le s \le 1, \\ C+Dt, \ 0 \le s \le t \le 1, \end{array} \quad \eta \ge s. \right| \tag{3.5}$$

Using the boundary condition

$$G_2(0,s) = \epsilon\frac{\partial}{\partial t}G_2(0,s),$$

we get

$$A = \epsilon B$$

and the BC

$$G_2(1,s) = \theta G_2(\eta,s),$$

gives

$$C = \frac{\theta\eta - 1}{1-\theta} D.$$

Putting the values of A and C in (3.5), we have

$$G_2(t,s) = \left\{ \begin{array}{ll} (\epsilon + t)B, \ 0 \le t \le s \le 1, & \\ \frac{\theta\eta - 1 + (1-\theta)t}{1-\theta} D, 0 \le s \le t \le 1 \le & \eta \ge s, \end{array} \right| .$$

Using the continuity of a Green's function at s, that is

$$G_2(s^+, s) = G_2(s^-, s),$$

we get

$$B = \frac{\theta\eta - 1 + (1-\theta)t}{(\epsilon + s)(1-\theta)} D,$$

which gives

$$G_2(t,s) = \left\{ \begin{array}{ll} (\epsilon + t)\frac{\theta\eta - 1 + (1-\theta)t}{(\epsilon+s)(1-\theta)} D, \ 0 \le t \le s \le 1, & \\ \frac{\theta\eta - 1 + (1-\theta)t}{1-\theta} D, \ 0 \le s \le t \le 1, & \eta \ge s. \end{array} \right|$$

Taking the derivative with respect to t, we get

$$\frac{\partial}{\partial t} G_2(t,s) = \left\{ \begin{array}{ll} [\frac{\epsilon+t}{\epsilon+s} + \frac{\theta\eta - 1 + (1-\theta)t}{(\epsilon+s)(1-\theta)}] D, \ 0 \le t \le s \le 1, & \\ D, \ 0 \le s \le t \le 1, & \eta \ge s. \end{array} \right|$$

Using the jump-discontinuity property of the derivative of G_2, that is

$$\frac{\partial}{\partial t} G_2(s^+, s) - \frac{\partial}{\partial t} G_2(s^-, s) = -1,$$

we get

$$D = \frac{(\epsilon + s)(\theta - 1)}{\theta\eta - 1 + (1-\theta)s}.$$

Hence

$$G_2(t,s) = \left\{ \begin{array}{ll} -(\epsilon + t)\frac{\theta\eta - 1 + (1-\theta)t}{\theta\eta - 1 + (1-\theta)s}, \ 0 \le t \le s \le 1, & \\ -(\epsilon + s)\frac{[\theta\eta - 1 + (1-\theta)t]}{\theta\eta - 1 + (1-\theta)s}, \ 0 \le s \le t \le 1, & \eta \ge s, \end{array} \right|$$

is a Green's function of the homogenous three point BVP (3.2).■

Theorem:(**3.1**) Let f be continuous and bounded on $I \times \mathbb{R}$. Then BVP (1.1) has a solution.

Proof: Choose M to be the bound for f on $I \times \mathbb{R}$ and define a mapping $T : C[0,1] \to C[0,1]$ by

$$(Tu)(t) = \int_0^1 G(t,s) f(s, u(s)) ds,$$

where the Banach space is $C[0,1]$ with norm defined by

$$\|u\| = \max_{t \in I} |u(t)|$$

and $G(t,s)$ is a Green's function. Since $G(t,s)$ is continuous and bounded, there exists $N > 0$ such that

$$|G(t,s)| < N \text{ on } I \times I.$$

Hence,

$$|Tu(t)| \leq NM, \ t \in I,$$

which implies that $Tu(t)$ is uniformly bounded and equi-continuous on I. Hence, by Arzela-Ascoli theorem T is compact. Let

$$\Omega = \{u \in C(I) : |u(t)| \leq NM, \ t \in I\}.$$

Clearly, this is closed, bounded and convex subset of $C(I)$. Moreover, for any $u \in \overline{\Omega}$, we have

$$\|(Tu)(t)\| \leq \int_0^1 |G(t,s) f(s, u(s))| \, ds \leq \int_0^1 NM ds = NM,$$

which implies that $(Tu)(t) \in \Omega$ for every $u \in \overline{\Omega}$. Consequently, $T(\overline{\Omega}) \subseteq \Omega$, that is T maps closed, bounded and convex set

$$\Omega = \{u \in C(I) : \|u\| \leq NM\},$$

into itself. Hence by Schauder's fixed point theorem, T has a fixed point, that is, there exist

$u \in \Omega$ such that

$$Tu = u,$$

which implies that the BVP (1.1) has a solution in Ω.■

The following result is known.

Theorem:(3.2) Let Ω be an open bounded subset of a normed space X and let $T : \overline{\Omega} \to X$ be compact. For $y \in X$ such that $y \notin (I - T)(\partial\Omega)$, and the degree $d(I - \lambda T, \Omega, y)$ is defined for all $\lambda \in [0, 1]$, then

$$d(I - T, \Omega, y) = \left\{ \begin{array}{l} 1, \ \text{for } y \in \Omega, \\ 0, \ \text{for } y \notin \Omega. \end{array} \right|$$

For the proof of this result, see [17].

3.3 Existence of at least three solutions:

In this section, we develop the method of upper and lower solutions for the existence of at least three solutions of the BVP (1.1). We use some topology degree theory arguments to get multiplicity results. We show that in the presence of two lower solutions and two upper solutions, the BVP (1.1) has at least three solutions. Moreover, we do not require f to be bounded.

Theorem:(3.3) Assume that the following hold:

(1) α, $\alpha_1 \in C^2(I)$ are two lower solutions and β, $\beta_1 \in C^2(I)$ are two upper solutions of the BVP (1.1) such that

$$\alpha \leq \alpha_1 \leq \beta, \ \alpha \leq \beta_1 \leq \beta \text{ and } \alpha_1 \nleq \beta_1 \text{ on I},$$

(2) α_1 , β_1 are strict lower and upper solutions of BVP (1.1).

Then, the BVP (1.1) has at least three solutions u_i, $i = 1, 2, 3$ such that

$$\alpha \leq u_1 \leq \beta_1, \ \alpha_1 \leq u_2 \leq \beta, \ u_3 \nleq \beta_1 \text{ and } u_3 \ngeq \alpha_1 \text{ on I}.$$

Proof: Define the modification F of f with respect to α, β as follows

$$F\left(t,u(t)\right)=\left\{\begin{array}{c} f(t,\beta(t))+\frac{u-\beta(t)}{1+|u-\beta|},\ u\geq\beta(t), \\ f(t,u(t)),\ \alpha(t)\leq u\leq\beta(t), \\ f(t,\alpha(t))+\frac{\alpha(t)-u(t)}{1+|\alpha-u|},\ u\leq\alpha(t). \end{array}\right|$$

Clearly F is continuous and bounded on $I\times\mathbb{R}$, it follows that the modified BVP

$$\begin{aligned} u''(t) &= F(t,u(t)),\ t\in[0,1], \\ u(0) &= \epsilon u'(0),\ u(1)=\theta u(\eta),\ \theta=\frac{\eta+\epsilon}{1+\epsilon}, \end{aligned} \tag{3.6}$$

has a solution. Using the definition of F and that of lower and upper solutions, we obtain

$$F(t,\alpha(t))=f(t,\alpha(t))\leq\alpha''(t),\ t\in I$$

and

$$F(t,\beta(t))=f(t,\beta(t))\geq\beta''(t),\ t\in I,$$

which implies that α, β are lower and upper solutions of (3.6). Moreover, any solution u of the modified problem (3.6) such that

$$\alpha(t)\leq u(t)\leq\beta(t),\ t\in I,$$

is a solution of problem (1.1). Hence, it is sufficient to show that the modified problem (3.6) has at least three solutions u_i such that

$$\alpha(t)\leq u_i\leq\beta(t),\ i=1,2,3,\ t\in I.$$

Define an integral operator $L_{\alpha\beta}:C^1(I)\to C^1(I)$ by

$$(L_{\alpha\beta}u)(t)=\int_0^1 G(t,s)F_{\alpha\beta}(s,u(s))ds. \tag{3.7}$$

Differentiating(3.7) with respect to t, we get

$$(L_{\alpha\beta}u)'(t)) = \int_0^1 G_t(t,s)F_{\alpha\beta}(s,u(s))ds. \tag{3.8}$$

Since $F_{\alpha\beta}$ is continuous and bounded on $I \times \mathbb{R}$, there exists $M_1 > 0$ such that

$$|F_{\alpha\beta}(t,u(t))| \leq M_1 \text{ on } I \times \mathbb{R}.$$

Also $G(t,s)$ is continuous and bounded on $I \times I$, hence there exists $M_2 > 0$ such that

$$|G(t,s)| \leq M_2 \text{ on } I \times I.$$

$G_t(t,s)$ is discontinuous at $t = s$, but is bounded on $I \times I$. Hence there exists $M_3 > 0$ such that

$$|G_t(t,s)| \leq M_3 \text{ on } I \times I.$$

Now from (3.7) and (3.8), it follows that

$$|(L_{\alpha\beta}u)(t)| \leq \int_0^1 |G(t,s)F_{\alpha\beta}(s,u(s))|\, ds \leq M_2M_1 \tag{3.9}$$

and

$$\left|(L_{\alpha\beta}u)'(t)\right| \leq \int_0^1 |G_t(t,s)F_{\alpha\beta}(s,u(s))|\, ds \leq M_3M_1, \tag{3.10}$$

which imply that the sequence $\{(L_{\alpha\beta}u)(t)\}$ is uniformly bounded and equi-continuous on I. Hence by the Arzela-Ascoli theorem, $L_{\alpha\beta}$ is compact.

Choose,

$$M > \max\{M_2M_1,\ M_3M_1\}.$$

Then, by (3.9) and (3.10), we get

$$|(L_{\alpha\beta}u)(t)| < M \text{ and } \left|(L_{\alpha\beta}u)'(t)\right| < M, \text{ for every } u \in C^1(I).$$

Let

$$\Omega = \{u \in C^1(I) : \|u\| < M\},$$

where $\|u\|$ =max$\{u(t) : t \in I\}$ is the usual norm. Clearly Ω is bounded and open subset of $C^1(I)$. Further, for u, $v \in \Omega$, we have

$$\begin{aligned} \|\alpha u + (1-\alpha)v\| &\leq \|\alpha u\| + (1-\alpha)\|v\| \\ &= \alpha\|u\| + (1-\alpha)\|v\| \\ &= \alpha M + (1-\alpha)M = M, \end{aligned}$$

which implies that Ω is convex. Moreover, for any $u \in \overline{\Omega}$, we have

$$\begin{aligned} \|L_{\alpha\beta}u\| &= \|(L_{\alpha\beta}u)(t)\| \leq \int_0^1 |G(t,s)F_{\alpha\beta}(s,u(s))|\, ds \\ &\leq \int_0^1 M_1M_2 ds = M_1M_2 < M, \end{aligned}$$

which implies that $(L_{\alpha\beta}u)(t) \in \Omega$ for every $u \in \overline{\Omega}$. Consequently, $L_{\alpha\beta}(\overline{\Omega}) \subseteq \Omega$, that is $L_{\alpha\beta} : \overline{\Omega} \to \overline{\Omega}$ maps open, bounded and convex set

$$\Omega = \{u \in C^1(I) : \|u\| < M\},$$

into itself. Hence by Schauder's fixed point theorem, $L_{\alpha\beta}$ has a fixed point, that is, the BVP (1.1) has a solution in Ω.

Now, for $u \in \partial\Omega$ and $\lambda \in [0,1]$, we have $\lambda L_{\alpha\beta}u \neq u$, that is

$$(I - \lambda L_{\alpha\beta})u \neq 0, \text{ for } \lambda \in [0,1] \text{ and for every } u \in \partial\Omega.$$

Hence by theorem (3.2), it follows that

$$\deg(I - \lambda L_{\alpha\beta}, \Omega, 0) = 1, \text{ for } \lambda \in [0,1].$$

By the homotopy invariance property of degree, we get

$$\deg(I - L_{\alpha\beta}, \Omega, 0) = 1.$$

Consider the following open bounded sets of Ω :

$$\Omega_{\alpha_1} = \{u \in \Omega : u > \alpha_1 \text{ on } I\}$$

and

$$\Omega_{\beta_1} = \{u \in \Omega : u < \beta_1 \text{ on } I\}.$$

Clearly, $\overline{\Omega_{\alpha_1}} \cap \overline{\Omega_{\beta_1}} = \emptyset$ and since $\alpha_1 \nleq \beta_1$ on I, therefore the set $\Omega \setminus (\overline{\Omega_{\alpha_1}} \cup \overline{\Omega_{\beta_1}}) \neq \emptyset$. Moreover, there are no solutions of (3.6) on $\partial\Omega_{\alpha_1} \cup \partial\Omega_{\beta_1}$. Because, if u is a solution of (3.6) such that

$$u \in \partial\Omega_{\alpha_1} \cup \partial\Omega_{\beta_1},$$

then $u \in \partial\Omega_{\alpha_1}$ or $u \in, \partial\Omega_{\beta_1}$ which implies that

$$u = \alpha_1 \text{ or } u = \beta_1 \text{ on } I$$

which is impossible as α_1 and β_1 are strict lower and upper solutions.

By the additive property of degree, we have

$$1 = \deg(I-L_{\alpha\beta}, \Omega, 0) = \deg(I-L_{\alpha\beta}, \Omega_{\alpha_1}, 0)+\deg(I-L_{\alpha\beta}, \Omega_{\beta_1}, 0)+\deg(I-L_{\alpha\beta}, \Omega\setminus(\overline{\Omega_{\alpha_1}}\cup\overline{\Omega_{\beta_1}}), 0). \tag{3.11}$$

It is required to show that

$$\deg(I - L_{\alpha\beta}, \Omega_{\alpha_1}, 0) = \deg(I - L_{\alpha\beta}, \Omega_{\beta_1}, 0) = 1,$$

then

$$\deg(I - L_{\alpha\beta}, \Omega\setminus(\overline{\Omega_{\alpha_1}} \cup \overline{\Omega_{\beta_1}}), 0) = -1$$

will follow from (3.11).

Define the modification $F^*_{\alpha_1\beta}$ of f with respect to α_1, β, as follows

$$F^*_{\alpha_1\beta}(t,u) = \left\{ \begin{array}{c} f(t,\beta(t)) + \frac{u-\beta(t)}{1+|u-\beta|},\ u \geq \beta(t) \\ f(t,u(t)),\ \alpha_1 \leq u \leq \beta(t) \\ f(t,\alpha_1(t)) + \frac{\alpha_1-u}{1+|\alpha_1-u|},\ u \leq \alpha_1(t). \end{array} \right|$$

Note that $F^*_{\alpha_1\beta} = F_{\alpha\beta}$ on Ω_{α_1}.

Consider the modified BVP

$$\begin{aligned} u''(t) &= F^*_{\alpha_1\beta}(t,u(t)),\ t \in I, \\ u(0) &= \epsilon u'(0),\ u(1) = \theta u(\eta). \end{aligned} \tag{3.12}$$

This is equivalent to the integral operator equation

$$(I - L_{\alpha_1\beta})u = 0,$$

where

$$(L_{\alpha_1\beta}u)(t) = \int_0^1 G(t,s)F^*_{\alpha_1\beta}(s,u(s))ds,$$

is a compact operator.

By lemma 2.8, any solution u of modified problem (3.12) satisfies $\alpha_1 \leq u$ on I and since α_1 is strict lower solution of (1.1), this implies that $u \neq \alpha_1$ on $(0,1)$ and hence $u \in \Omega_{\alpha_1}$. It follows that the BVP (3.12) has no solution in $\Omega\backslash\Omega_{\alpha_1}$, which implies that

$$\deg(I - L_{\alpha\beta}, \Omega\backslash\overline{\Omega_{\alpha_1}}, 0) = 0. \tag{3.13}$$

Moreover, since $L_{\alpha\beta}(\overline{\Omega}) \subseteq \Omega$, hence

$$\deg(I - L_{\alpha_1\beta}, \Omega, 0) = 1. \tag{3.14}$$

Equations (3.13) and (3.14) implies that

$$\deg(I - L_{\alpha_1\beta}, \Omega_{\alpha_1}, 0) = 1. \tag{3.15}$$

Since $L_{\alpha_1\beta} = L_{\alpha\beta}$ on Ω_{α_1}, so by the property of degree

$$\deg(I - L_{\alpha\beta}, \Omega_{\alpha_1}, 0) = 1. \tag{3.16}$$

Similarly, we can show that

$$\deg(I - L_{\alpha\beta}, \Omega_{\beta_1}, 0) = 1.$$

Thus, from (3.11)

$$\deg(I - L_{\alpha\beta}, \Omega \backslash (\overline{\Omega_{\alpha_1}} \cup \overline{\Omega_{\beta_1}}), 0) = -1$$

Hence there are at least three solutions one in each of the sets $\Omega_{\alpha_1}, \Omega_{\beta_1}$ and $\Omega \backslash (\overline{\Omega_{\alpha_1}} \cup \overline{\Omega_{\beta_1}})$. Thus, if there are two lower and two upper solutions in some specific region , then the BVP (1.1) has at least three solutions.