Double-Null Form for (3+1)-Dimensional Spacetimes

by

Tayyaba Shabbir

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Dr. Tooba Feroze

National University of Sciences and Technology

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Examination Committee Members

1. Name: PROF. AZAD A. SIDDIQUI Signature:
2. Name: DR. MUBASHER JAMIL Signature:
3. Name: ______________ Signature:
4. Name: DR. MUHAMMAD SAJID Signature:

Supervisor's Name: DR. TOOBA SIDDIQUI Signature:

Head of Department

04-09-2015
Date

COUNTERSINGED

Date: 04/09/15

Dean/Principal

Dedicated to

My loving Parents

for their Love,

Endless support &

Encouragement.

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Abstract

Using the coordinate transformations existence of double-null form is discussed in general spacetimes metric in $(3+1)$-dimensions. It is found that a class of $(3+1)$-dimensional spacetimes in which any of coefficients $g_{t\theta}$, $g_{r\theta}$, $g_{t\phi}$, $g_{r\phi}$ or $g_{\theta\phi}$ is non-zero cannot be transformed into double-null form. It has also been obtained that the general class of spacetime metric in (3+1)-dimensional spacetime is

$$ds^2 = g_{tt}dt^2 + g_{rr}dr^2 + 2g_{tr}dtdr + g_{\theta\theta}d\theta^2 + g_{\phi\phi}d\phi^2.$$

Contents

Chapter 1

Introduction to the Theory of Relativity

In this chapter we discuss principles of Special and General Relativity (GR). One of the predications of GR is the existence of black holes. We discuss here Einstein's field equations in vacuum and derive the Schwarzschild metric and describe its features.

Newton's theory of gravitation is a successful theory. For centuries the idea of gravity as a force held away. In Newton's gravitational theory, gravity is described as a force between two particles. There is a gravitational force between the sun and the earth, which provides the centripetal force that keeps the earth in orbit. According to Newton's law, the force between two bodies of masses m_1 and m_2 attracts each other and is inversely proportional to the square of the distance between them. Mathematically,

$$\vec{F} = -G\frac{m_1 m_2}{r^2}\hat{r}, \tag{1.1}$$

where r is the distance between two bodies, G is the gravitational constant, the vector r has direction form m_1 to m_2.

But in the twentieth century the Einstein came up with a completely different approach of gravity. Einstein's theory is different from the Newton's theory and describes gravity as a curvature of spacetime rather than a force. The mass of the sun curves the surrounding spacetime and in the curved spacetime earth moves in a straightest possible path. General Relativity reduces to the Newton's theory under specific circumstances when a particle is moving with slow velocity in a weak and static gravitational field. GR is considered to be one of the greatest achievements in Physics. Major outcomes of GR are the prediction of the precession of perihelion of Mercury, gravitational redshift, and the fact that gravity bends light. The bending of light was observed during the solar eclipse when moon essentially blocks the sun. On the earth, star is observed somewhere surrounding to the sun, which actually is located behind the Sun. Light from the star bends due to the gravitational field of the sun, giving the impression that star is somewhere surrounding the sun. Sun has mass, but photon does not, so the Newton law is inconsistent. Einstein's theory of Relativity is divided into two parts "Special Relativity" and "General Relativity".

In 1905 Einstein introduced the Special Relativity in his paper "On the Electrodynamics of Moving Bodies." Special Relativity is based on two postulates.

- All inertial frames are physically equivalent.

- The speed of light in vacuum (approximately $3 \times 10^8 \ \frac{m}{s}$) is constant for all inertial observers.

An inertial frame is a frame of reference in which the Newton second law of motion holds. The first postulate implies that in an inertial frame there is no, physical difference for any two observers. To both observers physical laws appear the same if they move relative to each other with a constant velocity. For example, suppose you are in a rocket and the rocket is moving at constant speed. An observer in the rocket would say that rocket is not moving, and the rest of the universe is moving around them. An observer outside your rocket would say that you are the one who appears to be moving. In this case, how would we define the coordinates for you in your rocket and the observer outside your rocket? We could say that the outside observer was simply mistaken, and that you were definitely not moving. Thus, his spatial coordinates were changing while you remained stationary. However, the observer could argue that you definitely were moving, and so it is your spatial coordinates that are changing. Hence, there seems to be no absolute coordinate system that could describe every event in the universe for which all observers would agree and we see that each observer has his own way to measure distances relative to the frame of reference. The second postulate implies that the speed of light is independent of the speed of the observer. Special Relativity is limited to the study of uniform, unaccelerated motions of macroscopic objects. For this reason it was called Restricted or Special theory of Relativity. For Unrestricted and General theory of Relativity, Einstein spent 10 more years. The origin of GR is based on the Einstein theory of Special Relativity.

In 1915 Einstein published “The foundation of the General theory of Relativity." GR is the theory of space, time, and gravitation. The two assumptions on which GR is actually based are:

- Principle of equivalence.

  Newtonian mechanics involves two different concepts of mass: Inertial mass, m_i, which describes a particle's resistance to being accelerated by a force. And the gravitational mass, m_g, which determines the force that a given particle experiences due to, or exerts on, another particle as a result of gravity. Einstein concluded that no experiment can distinguish between a gravitational and an accelerating frame of reference. Einstein elevated this concept as the principle of equivalence which is the foundation of the General theory of Relativity.

- Principle general covariance.

  According to the principle of general covariance all frames of references are physically equivalent.

GR is often summarized by the quote of John Wheeler

Spacetime tells matter how to move.

Matter tells spacetime how to curve.

1.1 Metric and Connection Coefficient

In 1908, Hermann Minkwoski introduced the idea of spacetime which is the four dimensional set (ct, x, y, z) with elements labeled by three dimension of space and one dimension of time, the set of all possible events. In four dimensional spacetime the separation is described by the square of distance between two events (ct_1, x_1, y_1, z_1) and (ct_2, x_2, y_2, z_2) as

$$ds^2 = c^2dt^2 - dx^2 - dy^2 - dz^2. \tag{1.2}$$

Two events are separating by the interval

if

$ds^2 > 0$, the interval is timelike,

$ds^2 < 0$, the interval is spacelike,

$ds^2 = 0$, the interval is lightlike or null.

Eq(1.2) can be written as

$$ds^2 = g_{ab}dx^a dx^b, \qquad (a, b = 0, 1, 2, 3) \tag{1.3}$$

where g_{ab} is a symmetric second rank tensor called the metric tensor and the connection related to g_{ab} is

$$\Gamma^a_{bc} = \frac{1}{2}g^{ad}(g_{bd,c} + g_{cd,a} - g_{bc,d}), \tag{1.4}$$

where Γ^a_{bc} is the connection coefficient, it is defined in terms of metric tensor and its partial derivatives. It is symmetric with respect to the interchange of the lower indice

$$\Gamma^a_{bc} = \Gamma^a_{cb}.$$

To describe the geometry of spacetime Riemann curvature tensor is important, which is given as

$$R^a_{bcd} = \Gamma^a_{bd,c} - \Gamma^a_{bc,d} + \Gamma^a_{ce}\Gamma^e_{bd} - \Gamma^a_{de}\Gamma^e_{bc}. \tag{1.5}$$

1.2 Symmetries of the Riemann Tensor

(1) It is antisymmetric with the interchange of the first and second indices

$$R_{abcd} = -R_{bacd}.$$

(2) It is antisymmetric with the interchange of the third and fourth indices

$$R_{abcd} = -R_{abdc}.$$

(3) If the first pair of indices is interchanged with the second pair, of indices it is invariant

$$R_{abcd} = R_{cdab}.$$

(4) It satisfies the first Bianchi identity which is

$$R^a_{bcd} + R^a_{dbc} + R^a_{cdb} = 0.$$

1.3 Contractions of the Ricci Tensor

Ricci tensor, R_{bd}**,** is obtained from the Riemann tensor by contracting the first and third indices i.e., by putting $a = c$

$$R_{bd} = \Gamma^a_{bd,a} - \Gamma^a_{ba,d} + \Gamma^a_{ae}\Gamma^e_{bd} - \Gamma^a_{de}\Gamma^e_{ba}. \tag{1.6}$$

Ricci scalar, R**,** is obtained by contracting the Ricci tensor, as

$$R = g^{ab}R_{ab}. \tag{1.7}$$

1.4 The Einstein Field Equations

The Einstein field equations are

$$R_{ab} - \frac{1}{2}g_{ab}R + \Lambda g_{ab} = \kappa T_{ab}, \tag{1.8}$$

or

$$\epsilon_{ab} = \kappa T_{ab}, \tag{1.9}$$

where

$$\epsilon_{ab} = R_{ab} - \frac{1}{2} g_{ab} R + \Lambda g_{ab}. \tag{1.10}$$

ϵ_{ab} is called the Einstein tensor which is symmetric, $\epsilon_{ab} = \epsilon_{ba}$, T_{ab} is the stress energy tensor, Λ is the cosmological constant and the coupling constant $\kappa = \frac{8\pi G}{c^4}$, where c is the speed of light. The right hand side of the Eq(1.9) describes the mass and energy and the left hand side describes the curvature of spacetime. Eq(1.9) represents 10 non-linear second order partial differential equations in the metric tensor and it is difficult to find exact general solution of Eq(1.9). However, many solutions have been obtained by taking different assumptions.

According to no-hair conjecture, black holes could be identified by three parameters mass, m, angular momentum, J, and charge, Q. The four famous solutions of Einstein's field equations that represent black holes are Schwarzschild, Reissner-Nordström, Kerr, Kerr-Newman solutions.

In 1916, only a year later, after Einstein published the field equations, Karl Schwarzschild obtained the first exact static and spherically symmetric solution of these equations for a point source in vacuum. This solution is known as the *Schwarzschild* black hole solution [1] and given as

$$ds^2 = (1 - \frac{2m}{r})dt^2 - \frac{1}{(1 - \frac{2m}{r})} dr^2 - r^2(d\theta^2 + \sin^2\theta d\phi^2), \tag{1.11}$$

where we took $c,\ G = 1$. As $r \longrightarrow \infty$, metric tends to the Minkwoski metric, therefore, it is asymptotically flat.

Schwarzschild metric was extended by Reissner [2] and Nordström [3]. The

Reissner-Nordström solution is static and spherically symmetric. It depends on mass and charge and is given in polar coordinates as

$$ds^2 = (1 - \frac{2m}{r} + \frac{Q^2}{r^2})dt^2 - \frac{1}{(1 - \frac{2m}{r} + \frac{Q^2}{r^2})}dr^2 - r^2(d\theta^2 + \sin^2\theta d\phi^2), \quad (1.12)$$

The *Kerr* solution [4] is stationary and axially symmetric. It depends on mass and angular momentum and is given by

$$ds^2 = (1 - \frac{2mr}{\rho^2})dt^2 + (\frac{4mra\sin^2\theta}{\rho^2})dtd\phi - (\frac{\rho^2}{\Delta})dr^2 - \rho^2 d\theta^2$$

$$-\big((r^2 + a^2)\sin^2\theta + \frac{2mra^2\sin^4\theta}{\rho^2}\big)d\phi^2, \quad (1.13)$$

where

$\rho^2 = r^2 + a^2\cos^2\theta,$

and

$\Delta = r^2 - 2mr + a^2.$

If $a = 0$ the Kerr metric reduces to the Schwarzschild metric.

The *Kerr-Newman* solution [5] is stationary and axially symmetric. It depends on all three parameters mass, angular momentum and electric charge and is given as

$$ds^2 = (1 - \frac{2mr}{\rho^2})dt^2 + (\frac{4mra\sin^2\theta}{\rho^2})dtd\phi - (\frac{\rho^2}{\Delta})dr^2 - \rho^2 d\theta^2$$

$$-\big((r^2 + a^2)\sin^2\theta + \frac{2mra^2\sin^4\theta}{\rho^2}\big)d\phi^2, \quad (1.14)$$

where

$\rho^2 = r^2 + a^2\cos^2\theta,$

and

$\Delta = r^2 - 2mr + a^2 + Q^2.$

1.5 Stationary and Static Metrics

A metric is said to be stationary if the metric coefficients do not depend on time i.e.,

$$\frac{\partial g_{ab}}{\partial t} = 0. \tag{1.15}$$

A metric is said to be static when the metric is independent of time and the line element ds^2 is unchanged under the transformation $t \to -t$. In static spacetimes the line element do not contain any cross terms like $d\theta dt$, $d\phi dt$. The Schwarzschild metric is both stationary and static. The Kerr metric is the example of stationary spacetime.

1.6 Black Holes

Einstein's theory predictes the existence of black holes, where the gravity is so strong that nothing can escape from it. A black hole is invisible but it pulls things in. In 1784, John Mitchell [6] wrote a paper in the Philosophical Transactions of the Royal Society of London in which he stated that, if a star is sufficiently massive and compact then its gravity would not allow the light to escape. Any light emitted from the surface of the star would be dragged back by the gravitational attraction of the star. A few years later in 1796, Pierre-Simon Laplace gave a similar suggestion and called such objects "dark stars". In 1939, Julius Oppenheimer and Harland Synder [7] showed that a sufficiently massive star must collapse indefinitely. In 1967, John Wheeler called such an object a "black hole"[8] .

Black holes are formed from the collapse of stars. During the lifetime of a star, the gravity acting on the outer edges of the star tries to pull it inwards and is balanced by the pressure created through the nuclear fusion reaction. The pressure reduces due to the exhaustion of nuclear fuel. The balance between pressure and gravity is no longer maintained. The star starts to collapse due to its gravitational attraction. The density of the star increases and the volume decrease which leads to the formation of a black hole.

1.7 Derivation of the Schwarzschild Metric

Taking $\Lambda = 0$, Eq(1.8) becomes

$$R_{uv} - \frac{1}{2} g_{uv} R = \kappa T_{uv}, \tag{1.16}$$

contracting Eq(1.16) with g^{ua}, we get

$$R^a_v - \frac{1}{2} \delta^a_v R = \kappa T^a_v. \tag{1.17}$$

By taking $a = v$, Eq(1.17) leads to $R = -\kappa T$. Then, Eq(1.16) becomes

$$R_{uv} = \kappa (T_{uv} - \frac{1}{2} g_{uv} T). \tag{1.18}$$

In vacuum $T_{uv} = 0$, therefore, Eq(1.18) gives

$$R_{uv} = 0. \tag{1.19}$$

Consider the line element for spherically symmetric and static geometry as [9]

$$ds^2 = e^{2A} dt^2 - e^{2B} dr^2 - r^2 (d\theta^2 + \sin^2\theta d\phi^2), \tag{1.20}$$

where A and B are functions of r only. We choose exponential function because it ensures that the sign of the metric component will be preserved in the desired $(+,-,-,-)$ pattern. In this case the metric is represented by a diagonal matrix. In covariant form the metric tensor is

$$g_{uv} = \begin{pmatrix} e^{2A} & 0 & 0 & 0 \\ 0 & -e^{2B} & 0 & 0 \\ 0 & 0 & -r^2 & 0 \\ 0 & 0 & 0 & -r^2 \sin^2 \theta \end{pmatrix}. \tag{1.21}$$

In diagonal matrices each contravariant component is the reciprocal of the corresponding covariant component and vice versa. In contravariant form the metric tensor becomes

$$g^{uv} = \begin{pmatrix} e^{-2A} & 0 & 0 & 0 \\ 0 & -e^{-2B} & 0 & 0 \\ 0 & 0 & -\frac{1}{r^2} & 0 \\ 0 & 0 & 0 & -\frac{1}{r^2 \sin^2 \theta} \end{pmatrix}. \tag{1.22}$$

Now to get the unknown functions $A(r)$ and $B(r)$, firstly we find the connection coefficients using Eq(1.4). The non zero components are

$\Gamma^r_{tt} = A' e^{2(A-B)}, \quad \Gamma^t_{tr} = A' = \Gamma^t_{rt}, \quad \Gamma^r_{rr} = B',$

$\Gamma^r_{\theta\theta} = -re^{-2B}, \quad \Gamma^r_{\phi\phi} = -e^{-2B} r \sin^2 \theta,$

$\Gamma^\theta_{r\theta} == \Gamma^\theta_{\theta r} = \Gamma^\phi_{r\phi} = \Gamma^\phi_{\phi r} = \frac{1}{r},$

$\Gamma^\theta_{\phi\phi} = -\sin\theta\cos\theta, \quad \Gamma^\phi_{\theta\phi} = \cot\theta = \Gamma^\phi_{\phi\theta}.$

Substituting these connection coefficients in Eq(1.6) to get

$$R_{00} = -e^{2(A-B)}(A'' + A'^{\,2} - A'B' + \frac{2A'}{r}), \tag{1.23}$$

$$R_{11} = A'' + A'^{\,2} - A'B' - \frac{2B'}{r}, \tag{1.24}$$

$$R_{22} = e^{-2B}(1 + A'r - B'r) - 1, \tag{1.25}$$

$$R_{33} = \sin^2\theta R_{22}. \tag{1.26}$$

For a vacuum solution, $R_{uv} = 0$, Eqs(1.23) to (1.26) give

$$-A'' - A'^{\,2} + A'B' - \frac{2A'}{r} = 0, \tag{1.27}$$

$$A'' + A'^{\,2} - A'B' - \frac{2B'}{r} = 0, \tag{1.28}$$

$$e^{-2B}(1 + A'r - B'r) - 1 = 0, \tag{1.29}$$

$$\sin^2\theta R_{22} = 0. \tag{1.30}$$

Solving Eqs(1.27) and (1.28), we have

$$\frac{-2(A' + B')}{r} = 0, \tag{1.31}$$

which implies

$$A' + B' = 0 \qquad \Rightarrow A + B = c_1. \tag{1.32}$$

As $r \to \infty$, the metric reduces to the Minkwoski metric as

$e^{2A} \to 1, \quad e^{2B} \to 1,$

or $\quad A \to 0, \quad B \to 0.$

Then $A + B = 0$, and Eq(1.29) gives

$$e^{2A}(1 + 2A'r) = 1, \tag{1.33}$$

or

$$(re^{2A})' = 1. \tag{1.34}$$

Integrating, we get

$$e^{2A} = 1 + \frac{\alpha}{r}. \tag{1.35}$$

We use weak field limit to find α, i.e.

$$g_{00} = 1 + \frac{2\Phi}{c^2}, \tag{1.36}$$

where $\Phi = -\frac{Gm}{r}$, then Eq(1.35) becomes

$$e^{2A} = 1 - \frac{2Gm}{c^2 r} = g_{00}, \tag{1.37}$$

and the line element in Eq(1.20), takes the form

$$ds^2 = (1 - \frac{2Gm}{c^2 r})dt^2 - \frac{1}{(1 - \frac{2Gm}{c^2 r})}dt^2 - r^2(d\theta^2 + \sin^2\theta d\phi^2). \tag{1.38}$$

In gravitational units $c = G = 1$, we have

$$ds^2 = (1 - \frac{2m}{r})dt^2 - \frac{1}{(1 - \frac{2m}{r})}dt^2 - r^2(d\theta^2 + \sin^2\theta d\phi^2). \tag{1.39}$$

The plan of this thesis is as follows:

In Chapter 2, The singularities of the Schwarzschild metric and different methods to remove the coordinate singularity are discuss and reviewed a paper on "Existence of double-null form for $(2+1)$-dimensional space-time".

In Chapter 3, the existence of $(3+1)$-dimensional spacetimes is discussed.

The brief conclusion of thesis is given in Chapter 4 with the summary and discussion.

Chapter 2

Spacetimes Singularities

Introduction

In Chapter 2, we discuss the singularities of the Schwarzschild metric and different methods to remove the coordinate singularity. The study of GR in lower dimensions specifically (2+1)-dimensions has more recent history. In the mid-1980s several researchers launched investigation into $(2+1)$ gravity. In $(2+1)$-dimension the number of independent components of Riemann tensor and Einstein tensor are the same. The fundamental difference between $(2+1)$ and $(3+1)$-dimensional spacetimes originates in the fact that the curvature tensor in $(2+1)$ dimensions depends linearly on the Ricci tensor [10, 11]. It should be noted that the Newtonian limit for the Einstein field equations cannot be obtained in $(2+1)$ dimensional gravity. In the following section we review a paper "Existence of double-null form for $(2+1)$-dimensional spacetimes" [12].

2.1 Coordinate and Essential Singularities

A singularity is the point at which a given mathematical expression is undefined. For example, the function

$$f(y) = \frac{1}{y}, \tag{2.1}$$

has singularity at $y = 0$. There are two types of singularities.

(1) Coordinate singularity

(2) Essential singularity

Coordinate singularity is due to the bad choice of coordinates and it can be removed by coordinate transformations. An essential singularity cannot be removed by coordinate transformations. In Schwarzschild metric, at $r = 2m$ the factor $(1-\frac{2m}{r})$ causes the metric coefficient $g_{00} \to 0$ and the factor $\frac{1}{(1-\frac{2m}{r})}$ causes the coefficient $g_{11} \to \infty$, and when $r = 0$, the factor $(1 - \frac{2m}{r})$ causes the metric coefficient $g_{00} \to \infty$ and the factor $\frac{1}{(1-\frac{2m}{r})}$ causes the coefficient $g_{11} \to 0$. In both the cases there is singularity but of different nature. To identify singularity the curvature invariants are constructed. If the curvature invariants become infinite at some point, then the singularity is an essential singularity otherwise it is a coordinate singularity. The Ricci scalar is the first curvature invariant and is the simplest one given as

$R_1 = g^{ab}R_{ab} = R.$

In case of the Schwarzschild metric the invariant $R_1 = 0$. Therefore, we consider the second curvature invariant given by

$R_2 = R^{abcd}R_{abcd}.$

For the Schwarzschild metric, the non-zero components of the curvature tensor are

$$R^0_{101} = -\frac{2m}{r^3}(1 - \frac{2m}{r}), \tag{2.2}$$

$$R^0_{202} = -\frac{m}{r}, \tag{2.3}$$

$$R^0_{303} = -\frac{m}{r}\sin^2\theta, \tag{2.4}$$

$$R^1_{212} = -\frac{m}{r}, \tag{2.5}$$

$$R^1_{313} = -\frac{m}{r}\sin^2\theta, \tag{2.6}$$

$$R^2_{323} = \sin^2\theta - (1 - \frac{2m}{r})\sin^2\theta. \tag{2.7}$$

Lowering all the indices, we have

$$R_{0101} = -\frac{2m}{r^3}, \tag{2.8}$$

$$R_{0202} = -\frac{m}{r}(1 - \frac{2m}{r}), \tag{2.9}$$

$$R_{0303} = -\frac{m}{r}(1 - \frac{2m}{r})\sin^2\theta, \tag{2.10}$$

$$R_{1212} = -\frac{m}{r(1 - \frac{2m}{r})}, \tag{2.11}$$

$$R_{1313} = -\frac{m}{r(1 - \frac{2m}{r})}\sin^2\theta, \tag{2.12}$$

$$R_{2323} = 2mr\sin^2\theta. \tag{2.13}$$

Raising all the indices, we have

$$R^{0101} = -\frac{2m}{r^3}, \tag{2.14}$$

$$R^{0202} = -\frac{m}{r^5(1-\frac{2m}{r})}, \tag{2.15}$$

$$R^{0303} = -\frac{m}{r^5(1-\frac{2m}{r})}\sin^2\theta, \tag{2.16}$$

$$R^{1212} = -\frac{m}{r^5}(1-\frac{2m}{r}), \tag{2.17}$$

$$R^{1313} = -\frac{m}{r^5\sin^2\theta}(1-\frac{2m}{r}), \tag{2.18}$$

$$R^{2323} = \frac{2m}{r^7\sin^2\theta}. \tag{2.19}$$

The second curvature invariant is then given as

$$R_2 = R^{abcd}R_{abcd} = \frac{48m^2}{r^6}. \tag{2.20}$$

The curvature invariant is finite at $r = 2m$ and infinite at $r = 0$. Hence at $r = 0$, the singularity is irremovable and is called essential, physical or curvature singularity and the singularity at $r = 2m$ is the coordinate singularity and can be removed by selecting suitable coordinates.

2.2 Double-Null Coordinates

Double-null coordinates demonstrate the ingoing and outgoing null geodesics. One coordinate is constant on each outgoing geodesic and second is constant on each ingoing geodesics.

Before constructing new coordinates to remove coordinate singularity for the Schwarzschild metric we review the construction of null coordinates u and v

in Minkwoski metric [13].

The Minkwoski metric in polar coordinates is

$$ds^2 = dt^2 - dr^2 - r^2(d\theta^2 + \sin^2\theta d\phi^2). \tag{2.21}$$

consider radial null geodesics in Minkwoski spacetime set ($ds^2 = 0$, $d\theta = 0,\ d\phi = 0$).

$$dt^2 - dr^2 = 0, \tag{2.22}$$

therefore

$$t = \pm r + c_i, \qquad (i = 1, 2) \tag{2.23}$$

where c is constant.

$$c_1 = t + r, \qquad c_2 = t - r. \tag{2.24}$$

Choosing

$$c_1 = \sqrt{2}v, \qquad c_2 = \sqrt{2}u. \tag{2.25}$$

Then Eq(2.24) becomes

$$v = \frac{1}{\sqrt{2}}(t + r), \quad u = \frac{1}{\sqrt{2}}(t - r). \tag{2.26}$$

In these coordinates Minkwoski metric takes the form

$$ds^2 = 2dudv - r^2(d\theta^2 + \sin^2\theta d\phi^2). \tag{2.27}$$

The metric tensor components in these coordinates are

$g_{uv} = g_{vu} = 1, \quad g_{\theta\theta} = r^2, \quad g_{\phi\phi} = r^2 sin^2\theta$.

The coordinates are called null coordinates because $g_{uu} = g_{vv} = 0$.

2.3 Eddington-Finkelstein Coordinates

In 1924, Finkelstein [14] discovered transformations in which coordinate singularities disappear. In 1958, Eddington [15] rediscovered the same transformations. To remove the singularity in Schwarzschild metric at $r = 2m$ a new radial coordinate can be defined in which the coordinate singularity disappears. Consider radial null geodesics in Schwarzschild metric set ($ds^2 = 0,\ d\theta = 0,\ d\phi = 0$).

$$(1 - \frac{2m}{r})dt^2 - \frac{1}{(1 - \frac{2m}{r})}dr^2 = 0. \tag{2.28}$$

Introducing a new radial coordinate r^* called the tortoise coordinate as

$$r^* = r + 2m\ln(\frac{r}{2m} - 1). \tag{2.29}$$

Then Eq(2.28) becomes

$$(1 - \frac{2m}{r})dt^2 - (1 - \frac{2m}{r})dr^{*2} = 0, \tag{2.30}$$

$$dt^2 - dr^{*2} = 0, \tag{2.31}$$

therefore

$$t = \pm r^* + c_i, \qquad (i = 1, 2) \tag{2.32}$$

then

$$c_1 = t + r^*, \qquad c_2 = t - r^*. \tag{2.33}$$

Choosing

$$c_1 = v, \qquad c_2 = u. \tag{2.34}$$

The outgoing (retarded), u, and ingoing (advanced), v, null coordinates are

$$u = t - r^*, \qquad v = t + r^*. \tag{2.35}$$

Using the retarded coordinate of Eq(2.35) with Eq(2.29), to have

$$dt = du + dr^* = du + \frac{dr}{1 - \frac{2m}{r}}. \tag{2.36}$$

Squaring both sides of Eq(2.36), we have

$$dt^2 = du^2 + \frac{dr^2}{(1 - \frac{2m}{r})^2} + \frac{2dudr}{1 - \frac{2m}{r}}. \tag{2.37}$$

Using Eq(2.37), Eq(1.11) takes the retarded Eddington-Finkelstein form of the metric

$$ds^2 = (1 - \frac{2m}{r})du^2 + 2dudr - r^2(d\theta^2 + \sin^2\theta d\phi^2), \tag{2.38}$$

where the singularity at $r = 2m$ is removed. Similarly, in advanced coordinate v, Eq(1.11), takes the form

$$ds^2 = (1 - \frac{2m}{r})dv^2 - 2dvdr - r^2(d\theta^2 + \sin^2\theta d\phi^2). \tag{2.39}$$

In coordinates (u, v, θ, ϕ) the Schwarzschild metric, Eq(1.11), takes the form

$$ds^2 = (1 - \frac{2m}{r})dudv - r^2(d\theta^2 + \sin^2\theta d\phi^2). \tag{2.40}$$

2.4 Kruskal-Szekeres Coordinates

In advanced and retarded Eddington-Finkelstein coordinates, we see that, in retarded coordinate the ingoing null rays are discontinuous and in advanced coordinates the outgoing null rays are discontinuous. In 1961, Martin Kruskal [16] and George Szekeres independently introduced a coordinate system in which both ingoing and outgoing null rays are continuous striaght lines. In these coordinates Kruskal exponentiated the retarded and advanced coordinates (u, v) given by Eq(2.35) as

$$\tilde{U} = -e^{-\frac{u}{4m}}, \qquad \tilde{V} = e^{\frac{v}{4m}}, \tag{2.41}$$

$$\tilde{U}\tilde{V} = -e^{\frac{v-u}{4m}} = -e^{\frac{r}{2m}}\Big(\frac{r}{2m} - 1\Big), \tag{2.42}$$

which gives

$$dudv = -\frac{16m^2}{\tilde{U}\tilde{V}} d\tilde{U} d\tilde{V}. \tag{2.43}$$

In Kruskal coordinate $(\tilde{U}, \tilde{V})$ the metric Eq(2.40), takes the form

$$ds^2 = \frac{32m^3}{r} e^{\frac{-r}{2m}} d\tilde{U} d\tilde{V} - r^2(d\theta^2 + \sin^2\theta d\phi^2). \tag{2.44}$$

Kruskal-Szekeres coordinates can be obtained from Kruskal coordinate which has spacelike coordinate, R, and timelike coordinate, T, and defined as

$$T = \frac{1}{2}(\tilde{V} + \tilde{U}), \qquad R = \frac{1}{2}(\tilde{V} - \tilde{U}). \tag{2.45}$$

Substituting Eq(2.41) in Eq(2.45), to get

$$T = \frac{1}{2}(e^{\frac{v}{4m}} - e^{\frac{-u}{4m}}), \qquad R = \frac{1}{2}(e^{\frac{v}{4m}} + e^{\frac{-u}{4m}}). \tag{2.46}$$

Substituting Eq(2.35) in Eq(2.46), to get

$$T = e^{\frac{r}{4m}}(\frac{r}{2m} - 1)^{\frac{1}{2}} \sinh(\frac{t}{4m}), \tag{2.47}$$

$$R = e^{\frac{r}{4m}}(\frac{r}{2m} - 1)^{\frac{1}{2}} \cosh(\frac{t}{4m}), \tag{2.48}$$

then,

$$T^2 - R^2 = -(\frac{r}{2m} - 1)e^{\frac{r}{2m}}, \tag{2.49}$$

$$\frac{T}{R} = tanh(\frac{t}{4m}). \tag{2.50}$$

In coordinates (T, R) the metric Eq(2.44), takes the form

$$ds^2 = \frac{32m^3}{r} e^{\frac{-r}{2m}}(dT^2 - dR^2) - r^2(d\theta^2 + \sin^2\theta d\phi^2). \tag{2.51}$$

Hence, there is no singularity at $r = 2m$. The Kruskal diagram has some important features. The Kruskal Fig (1) shows four regions labelled as I, II, Ì, ÌÌ'. Regions I and Ì correspond to the exterior region of the Schwarzschild black hole in which $r > 2m$. Region ÌÌ is the time-reverse of region II in which the physical singularity $r = 0$ lies and they represent the interior region of Schwarzschild black hole. Region II is the black hole. Region ÌÌ is a part of the spacetime from which things can escape but not enter and is called a white hole. In region I the 45^o line going up is $t = \infty$ and 45^o line going down $t = -\infty$. The regions I and II describe the advance Eddington-Finkelestein solution. The regions Ì and ÌÌ, describe the retarded Eddington-Finkelestein solution. The diagonal lines $r = 2m$, $t = -\infty$ and $r = 2m$, $t = \infty$ describe the event horizon. Lines of constant t is striaght lines passing from the center of the diagram. Lines of constant r are curves of constant $R^2 - T^2$ and hence hyperbolae. The value $r = 2m$ corresponds to either of the straight lines $T = \pm R$ and the value $r = 0$ corresponds a hyperbolae $T = \pm\sqrt{1 + R^2}$. Hence Kruskal-Sezekers coordinates are well behaved everywhere except the physical singularity and these coordinates cover the entire Schwarzschild spacetime.

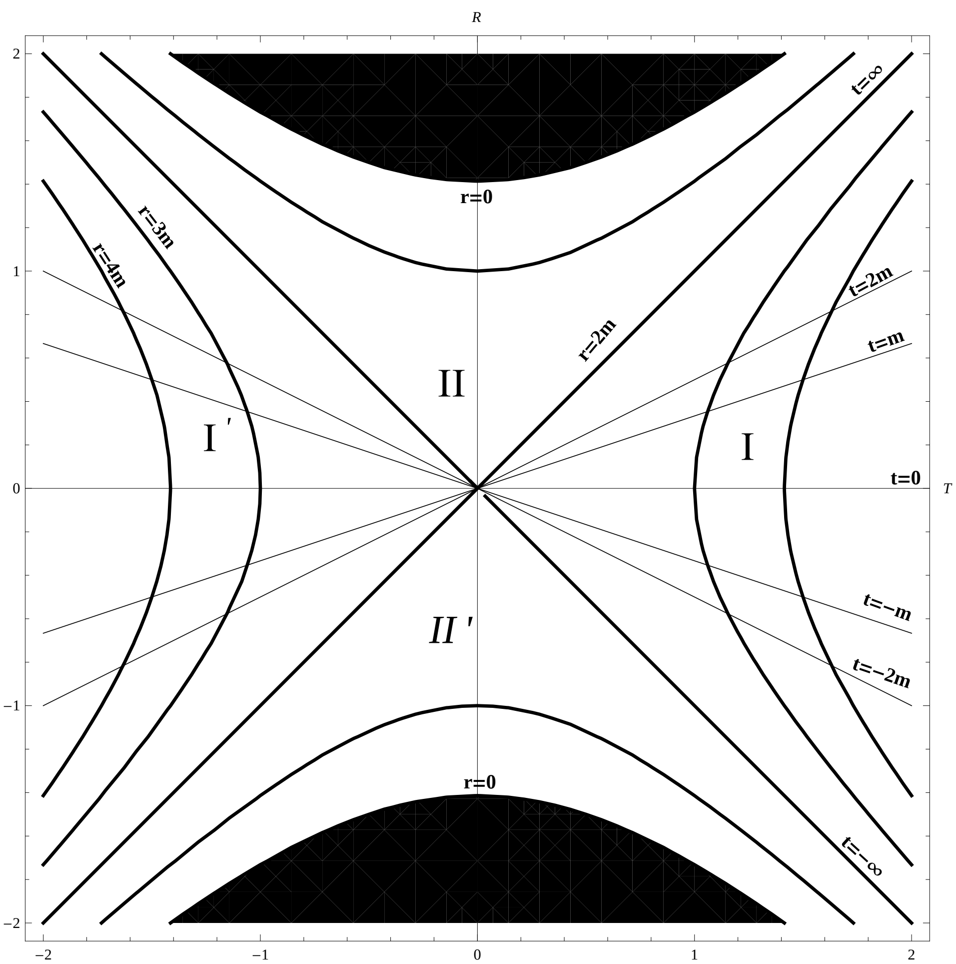

Figure 2.1: Geometrical representation of Schwarzschild metric in Kruskal-Sezekers coordinates.

2.5 Double-Null Form for $(2+1)$-Dimensional Spacetimes

Double-null form is widely used, because of its characteristic to simplify the calculations. These coordinates are useful because the curve along which u or v is constant is lightlike. Hayward widely used double-null form in his investigation of the black hole spacetimes [17] and also in earlier these coordinates were used by Roman and Bergman [18]. It is also useful in simplifications of different calculations in Newman-Penrose formalism which was formed by Ezra T. Newman and Roger Penrose. Double-null form plays an important role in numerical relativity, which is useful in many areas, for instance, perturbed black holes, neutron stars and cosmological models [19, 20, 21, 22].

The double-null form for $(2+1)$-dimensional spacetime is

$$ds^2 = 2\tilde{g}_{uv}dudv + \tilde{g}_{\phi\phi}d\phi^2, \tag{2.52}$$

where $\tilde{g}_{uv}$ and $\tilde{g}_{\phi\phi}$ depend on u, v, and ϕ.

2.6 Existence of Double-Null Form for $(2+1)$-Dimensional Spacetimes

The general spacetime metric for $(2+1)$-dimensional spacetime is

$$ds^2 = g_{tt}dt^2 + g_{rr}dr^2 + g_{\phi\phi}d\phi^2 + 2g_{tr}dtdr + 2g_{t\phi}dtd\phi + 2g_{r\phi}drd\phi, \tag{2.53}$$

where g_{tt}, g_{rr}, $g_{\phi\phi}$, g_{tr}, $g_{t\phi}$ and $g_{r\phi}$ depend on t, r, and ϕ.

To transform metric Eq(2.53) into the double-null form given by Eq(2.52) considering time, t, and radial coordinate, r, as functions of the new coordinates u and v i.e. $t = t(u,v)$ and $r = r(u,v)$. The Jacobian "J" of the transformations is

$$J = \begin{vmatrix} \frac{\partial t}{\partial u} & \frac{\partial t}{\partial v} \\ \frac{\partial r}{\partial u} & \frac{\partial r}{\partial v} \end{vmatrix},$$

or

$$J = \Big(\frac{\partial t}{\partial u}\Big)\Big(\frac{\partial r}{\partial v}\Big) - \Big(\frac{\partial t}{\partial v}\Big)\Big(\frac{\partial r}{\partial u}\Big). \tag{2.54}$$

The transformations are invertible. The metric coefficients obey the transformation law

$$\tilde{g}_{ab} = \frac{\partial x^p}{\partial \tilde{x}^a}\frac{\partial x^q}{\partial \tilde{x}^b} g_{pq}, \tag{2.55}$$

where x^p, x^q and $\tilde{x}^a$, $\tilde{x}^b$ refer to the (t, r, ϕ) and (u, v, ϕ) coordinates respectively with $p, q, a, b = 0, 1, 2$, Eq(2.55) yields the following system of partial differential equations (PDEs):

$$\Big(\frac{\partial t}{\partial u}\Big)^2 g_{tt} + 2\Big(\frac{\partial t}{\partial u}\Big)\Big(\frac{\partial r}{\partial u}\Big) g_{tr} + \Big(\frac{\partial r}{\partial u}\Big)^2 g_{rr} = \tilde{g}_{uu}, \tag{2.56}$$

$$\Big(\frac{\partial t}{\partial v}\Big)^2 g_{tt} + 2\Big(\frac{\partial t}{\partial v}\Big)\Big(\frac{\partial r}{\partial v}\Big) g_{tr} + \Big(\frac{\partial r}{\partial v}\Big)^2 g_{rr} = \tilde{g}_{vv}, \tag{2.57}$$

$$\Big(\frac{\partial t}{\partial u}\Big)\Big(\frac{\partial t}{\partial v}\Big) g_{tt} + \Big(\frac{\partial t}{\partial u}\frac{\partial r}{\partial v} + \frac{\partial t}{\partial v}\frac{\partial r}{\partial u}\Big) g_{tr} + \Big(\frac{\partial r}{\partial u}\Big)\Big(\frac{\partial r}{\partial v}\Big) g_{rr} = \tilde{g}_{uv}, \tag{2.58}$$

$$g_{\phi\phi} = \tilde{g}_{\phi\phi}, \tag{2.59}$$

$$\Big(\frac{\partial t}{\partial u}\Big) g_{t\phi} + \Big(\frac{\partial r}{\partial u}\Big) g_{r\phi} = \tilde{g}_{u\phi}, \tag{2.60}$$

$$\Big(\frac{\partial t}{\partial v}\Big) g_{t\phi} + \Big(\frac{\partial r}{\partial v}\Big) g_{r\phi} = \tilde{g}_{v\phi}. \tag{2.61}$$

For double-null form of metric (2.52), we require $\tilde{g}_{uu} = \tilde{g}_{vv} = \tilde{g}_{u\phi} = \tilde{g}_{v\phi} = 0$, and obtain the following reduced system of partial differential equations:

$$\Big(\frac{\partial t}{\partial u}\Big)^2 g_{tt} + 2\Big(\frac{\partial t}{\partial u}\Big)\Big(\frac{\partial r}{\partial u}\Big)g_{tr} + \Big(\frac{\partial r}{\partial u}\Big)^2 g_{rr} = 0, \tag{2.62}$$

$$\Big(\frac{\partial t}{\partial v}\Big)^2 g_{tt} + 2\Big(\frac{\partial t}{\partial v}\Big)\Big(\frac{\partial r}{\partial v}\Big)g_{tr} + \Big(\frac{\partial r}{\partial v}\Big)^2 g_{rr} = 0, \tag{2.63}$$

$$\Big(\frac{\partial t}{\partial u}\Big)\Big(\frac{\partial t}{\partial v}\Big)g_{tt} + \Big(\frac{\partial t}{\partial u}\frac{\partial r}{\partial v} + \frac{\partial t}{\partial v}\frac{\partial r}{\partial u}\Big)g_{tr} + \Big(\frac{\partial r}{\partial u}\Big)\Big(\frac{\partial r}{\partial v}\Big)g_{rr} = \tilde{g}_{uv}, \tag{2.64}$$

$$g_{\phi\phi} = \tilde{g}_{\phi\phi}, \tag{2.65}$$

$$\Big(\frac{\partial t}{\partial u}\Big)g_{t\phi} + \Big(\frac{\partial r}{\partial u}\Big)g_{r\phi} = 0, \tag{2.66}$$

$$\Big(\frac{\partial t}{\partial v}\Big)g_{t\phi} + \Big(\frac{\partial r}{\partial v}\Big)g_{r\phi} = 0. \tag{2.67}$$

From Eqs(2.66) and (2.67), we get

$$g_{t\phi} = g_{r\phi} = 0.$$

So, a metric cannot be transformed into double-null form given by Eq(2.52) if it contains $g_{t\phi}$ or $g_{r\phi}$. Therefore, the most general form of metric in (2+1)-dimensional spacetime that can be transformed into double-null form is

$$ds^2 = g_{tt}dt^2 + g_{rr}dr^2 + 2g_{tr}dtdr + g_{\phi\phi}d\phi^2. \tag{2.68}$$

To transform metric Eq(2.68) into double null form, we consider the following cases:

(1) $g_{tt},\ \ g_{rr},\ \ g_{\phi\phi}\ \ \neq 0$ and $g_{tr} = 0.$

(2) $g_{rr},\ \ g_{\phi\phi},\ \ g_{tr}\ \ \neq 0$ and $g_{tt} = 0.$

(3) $g_{tt},\ \ g_{\phi\phi},\ \ g_{tr}\ \ \neq 0$ and $g_{rr} = 0.$

(4) $g_{tt},\ \ g_{rr},\ \ g_{\phi\phi},\ \ g_{tr}\ \ \neq 0.$

Case 1: $g_{tt},\ \ g_{rr},\ \ g_{\phi\phi} \neq 0$ and $g_{tr} = 0$

The system of Eqs(2.62) to (2.65) reduces to

$$\left(\frac{\partial t}{\partial u}\right)^2 g_{tt} + \left(\frac{\partial r}{\partial u}\right)^2 g_{rr} = 0, \tag{2.69}$$

$$\left(\frac{\partial t}{\partial v}\right)^2 g_{tt} + \left(\frac{\partial r}{\partial v}\right)^2 g_{rr} = 0, \tag{2.70}$$

$$\left(\frac{\partial t}{\partial u}\right)\left(\frac{\partial t}{\partial v}\right) g_{tt} + \left(\frac{\partial r}{\partial u}\right)\left(\frac{\partial r}{\partial v}\right) g_{rr} = \tilde{g}_{uv}, \tag{2.71}$$

$$g_{\phi\phi} = \tilde{g}_{\phi\phi}. \tag{2.72}$$

Eqs(2.69) and (2.70) imply

$$\left(\frac{\partial t}{\partial v}\frac{\partial r}{\partial u} - \frac{\partial t}{\partial u}\frac{\partial r}{\partial v}\right)\left(\frac{\partial t}{\partial u}\frac{\partial r}{\partial v} + \frac{\partial t}{\partial v}\frac{\partial r}{\partial u}\right) g_{rr} = 0. \tag{2.73}$$

In the present case, g_{rr} and the Jacobian of the transformations to be non-zero, Eqs(2.73) imply

$$\left(\frac{\partial t}{\partial v}\right)\left(\frac{\partial r}{\partial u}\right) + \left(\frac{\partial t}{\partial u}\right)\left(\frac{\partial r}{\partial v}\right) = 0. \tag{2.74}$$

Eq(2.74) gives the following transformations

$$t = \Phi(u \pm v), \qquad r = \Psi(u \mp v), \tag{2.75}$$

where Φ and Ψ are arbitrary functions.

Case 2: $g_{rr}, \quad g_{\phi\phi}, \quad g_{tr} \quad \neq 0$ and $g_{tt} = 0$

The system of Eqs(2.62) to (2.65) reduces to

$$\left(\frac{\partial r}{\partial u}\right)^2 g_{rr} + 2\left(\frac{\partial t}{\partial u}\right)\left(\frac{\partial r}{\partial u}\right) g_{tr} = 0, \tag{2.76}$$

$$\left(\frac{\partial r}{\partial v}\right)^2 g_{rr} + 2\left(\frac{\partial t}{\partial v}\right)\left(\frac{\partial r}{\partial v}\right) g_{tr} = 0, \tag{2.77}$$

$$\left(\frac{\partial r}{\partial u}\right)\left(\frac{\partial r}{\partial v}\right) g_{rr} + \left(\frac{\partial t}{\partial u}\frac{\partial r}{\partial v} + \frac{\partial r}{\partial u}\frac{\partial t}{\partial v}\right) g_{tr} = \tilde{g}_{uv}, \tag{2.78}$$

$$g_{\phi\phi} = \tilde{g}_{\phi\phi}. \tag{2.79}$$

The system of Eqs(2.76) and (2.77) can be written as

$$\frac{\partial r}{\partial u}\Big(\frac{\partial r}{\partial u}g_{rr}+2\frac{\partial t}{\partial u}g_{tr}\Big)=0, \quad \text{and} \quad \frac{\partial r}{\partial v}\Big(\frac{\partial r}{\partial v}g_{rr}+2\frac{\partial t}{\partial v}g_{tr}\Big)=0. \tag{2.80}$$

Eq(3.40) are satisfied under the following cases

(a) $\frac{\partial r}{\partial u}=0=\frac{\partial r}{\partial v}$,

(b) $\frac{\partial r}{\partial u}g_{rr}+2\frac{\partial t}{\partial u}g_{tr}=0=\frac{\partial r}{\partial v}g_{rr}+2\frac{\partial t}{\partial v}g_{tr}$,

(c) $\frac{\partial r}{\partial u}=0=\frac{\partial r}{\partial v}g_{rr}+2\frac{\partial t}{\partial v}g_{tr}$,

(d) $\frac{\partial r}{\partial v}=0=\frac{\partial r}{\partial u}g_{rr}+2\frac{\partial t}{\partial u}g_{tr}$.

The double-null form is not possible in cases (a) and (b). In case (a), the Jacobian of the transformations is zero which is not possible and in case (b) we get contradictions $\tilde{g}_{uv}=0,\ g_{tr}$ and $g_{rr}=0$. In cases (c) and (d) existence of double-null form is possible.

Case 3: $g_{tt},\ \ g_{\phi\phi},\ \ g_{tr}\ \ \neq 0$ and $g_{rr}=0$

The system of Eqs(2.62) to (2.65) reduces to

$$\Big(\frac{\partial t}{\partial u}\Big)^2 g_{tt}+2\Big(\frac{\partial t}{\partial u}\Big)\Big(\frac{\partial r}{\partial u}\Big)g_{tr}=0, \tag{2.81}$$

$$\Big(\frac{\partial t}{\partial v}\Big)^2 g_{tt}+2\Big(\frac{\partial t}{\partial v}\Big)\Big(\frac{\partial r}{\partial v}\Big)g_{tr}=0, \tag{2.82}$$

$$\Big(\frac{\partial t}{\partial u}\Big)\Big(\frac{\partial t}{\partial v}\Big)g_{tt}+\Big(\frac{\partial t}{\partial u}\frac{\partial r}{\partial v}+\frac{\partial t}{\partial v}\frac{\partial r}{\partial u}\Big)g_{tr}=\tilde{g}_{uv}, \tag{2.83}$$

$$g_{\phi\phi}=\tilde{g}_{\phi\phi}. \tag{2.84}$$

Eqs(2.81) and (2.82) can be written as

$$\frac{\partial t}{\partial u}\Big(\frac{\partial t}{\partial u}g_{tt}+2\frac{\partial r}{\partial u}g_{tr}\Big)=0, \quad \text{and} \quad \frac{\partial t}{\partial v}\Big(\frac{\partial t}{\partial v}g_{tt}+2\frac{\partial r}{\partial v}g_{tr}\Big)=0. \tag{2.85}$$

Eq(2.85) are satisfied under the following cases

(a) $\frac{\partial t}{\partial u}=0=\frac{\partial t}{\partial v}$,

(b) $\frac{\partial t}{\partial u}g_{tt} + 2\frac{\partial r}{\partial u}g_{tr} = 0 = \frac{\partial t}{\partial v}g_{tt} + 2\frac{\partial r}{\partial v}g_{tr}$,

(c) $\frac{\partial t}{\partial u} = 0 = \frac{\partial t}{\partial v}g_{tt} + 2\frac{\partial r}{\partial v}g_{tr}$,

(d) $\frac{\partial t}{\partial v} = 0 = \frac{\partial t}{\partial u}g_{tt} + 2\frac{\partial r}{\partial u}g_{tr}$.

Just like case (2) the double-null form is not possible in cases (a) and (b). In case (a), the Jacobian of the transformations is zero which is not possible and in case (b) we get contradictions $\tilde{g}_{uv} = 0,\ g_{tr}$ and $g_{tt} = 0$. Existence of double-null form is possible in cases (c) and (d).

Case 4: $g_{tt},\ g_{rr},\ g_{\phi\phi},\ g_{tr}\ \neq 0$

The system of Eqs(2.62) to (2.65) reduces to

$$\left(\frac{\partial t}{\partial u}\right)^2 g_{tt} + 2\left(\frac{\partial t}{\partial u}\right)\left(\frac{\partial r}{\partial u}\right)g_{tr} + \left(\frac{\partial r}{\partial u}\right)^2 g_{rr} = 0, \tag{2.86}$$

$$\left(\frac{\partial t}{\partial v}\right)^2 g_{tt} + 2\left(\frac{\partial t}{\partial v}\right)\left(\frac{\partial r}{\partial v}\right)g_{tr} + \left(\frac{\partial r}{\partial v}\right)^2 g_{rr} = 0, \tag{2.87}$$

$$\left(\frac{\partial t}{\partial u}\right)\left(\frac{\partial t}{\partial v}\right)g_{tt} + \left(\frac{\partial t}{\partial u}\frac{\partial r}{\partial v} + \frac{\partial t}{\partial v}\frac{\partial r}{\partial u}\right)g_{tr} + \left(\frac{\partial r}{\partial u}\right)\left(\frac{\partial r}{\partial v} = \right)g_{rr} = \tilde{g}_{uv}, \tag{2.88}$$

$$g_{\phi\phi} = \tilde{g}_{\phi\phi}. \tag{2.89}$$

Along with Jacobian to be non-zero this case can be solved and one may obtain the transformations which will give the required double-null form .

Chapter 3

Double-Null Form for $(3+1)$-Dimensional Spacetimes

The double-null form for $(3+1)$-dimensional spacetimes is [23]

$$ds^2 = 2\tilde{g}_{uv}dudv + \tilde{g}_{\theta\theta}d\theta^2 + \tilde{g}_{\phi\phi}d\phi^2, \tag{3.1}$$

where $\tilde{g}_{uv}$, $g_{\theta\theta}$ and $g_{\phi\phi}$ depend on u, v, θ, and ϕ.

Considering the general spacetime metric in $(3+1)$-dimensional spacetime as

$$\begin{aligned} ds^2 =& g_{tt}dt^2 + g_{rr}dr^2 + g_{\theta\theta}d\theta^2 + g_{\phi\phi}d\phi^2 + 2g_{tr}dtdr + 2g_{t\theta}dtd\theta + \\ & 2g_{t\phi}dtd\phi + 2g_{r\theta}drd\theta + 2g_{r\phi}drd\phi + 2g_{\theta\phi}d\theta d\phi, \end{aligned} \tag{3.2}$$

where g_{tt}, g_{rr}, $g_{\theta\theta}$, $g_{\phi\phi}$, g_{tr}, $g_{t\theta}$, $g_{t\phi}$, $g_{r\theta}$, $g_{r\phi}$, and $g_{\theta\phi}$ depend on t, r, θ, and ϕ. To transform metric Eq(3.2) into the double-null form given by Eq(3.1) considering time, t, and radial coordinate, r, as functions of the new coordinates u and v i.e. $t = t(u,v)$ and $r = r(u,v)$. The Jacobian "J" of the transformations is

$$J = \begin{vmatrix} \frac{\partial t}{\partial u} & \frac{\partial t}{\partial v} \\ \frac{\partial r}{\partial u} & \frac{\partial r}{\partial v} \end{vmatrix},$$

or

$$J = \left(\frac{\partial t}{\partial u}\right)\left(\frac{\partial r}{\partial v}\right) - \left(\frac{\partial t}{\partial v}\right)\left(\frac{\partial r}{\partial u}\right). \tag{3.3}$$

The transformations are invertible. In the following, possibility of converting Eq(3.2) into the double-null form is discussed using the transformation given by Eq(2.55)

$$\tilde{g}_{ab} = \frac{\partial x^p}{\partial \tilde{x}^a}\frac{\partial x^q}{\partial \tilde{x}^b} g_{pq}, \tag{3.4}$$

x^p, x^q and $\tilde{x}^a$, $\tilde{x}^b$ refer to the (t, r, θ, ϕ) and (u, v, θ, ϕ) coordinates respectively with $p, q, a, b = 0, 1, 2, 3$. Eq(3.4) yields the following system of partial differential equations:

$$\tilde{g}_{uu} = \left(\frac{\partial t}{\partial u}\right)^2 g_{tt} + 2\left(\frac{\partial t}{\partial u}\right)\left(\frac{\partial r}{\partial u}\right) g_{tr} + \left(\frac{\partial r}{\partial u}\right)^2 g_{rr}, \tag{3.5}$$

$$\tilde{g}_{vv} = \left(\frac{\partial t}{\partial v}\right)^2 g_{tt} + 2\left(\frac{\partial t}{\partial v}\right)\left(\frac{\partial r}{\partial v}\right) g_{tr} + \left(\frac{\partial r}{\partial v}\right)^2 g_{rr}, \tag{3.6}$$

$$\tilde{g}_{uv} = \left(\frac{\partial t}{\partial u}\right)\left(\frac{\partial t}{\partial v}\right) g_{tt} + \left(\frac{\partial t}{\partial u}\frac{\partial r}{\partial v} + \frac{\partial t}{\partial v}\frac{\partial r}{\partial u}\right) g_{tr} + \left(\frac{\partial r}{\partial u}\right)\left(\frac{\partial r}{\partial v}\right) g_{rr}, \tag{3.7}$$

$$\tilde{g}_{\theta\theta} = g_{\theta\theta}, \tag{3.8}$$

$$\tilde{g}_{\phi\phi} = g_{\phi\phi}, \tag{3.9}$$

$$\tilde{g}_{u\theta} = \left(\frac{\partial t}{\partial u}\right) g_{t\theta} + \left(\frac{\partial r}{\partial u}\right) g_{r\theta}, \tag{3.10}$$

$$\tilde{g}_{v\theta} = \left(\frac{\partial t}{\partial v}\right) g_{t\theta} + \left(\frac{\partial r}{\partial v}\right) g_{r\theta}, \tag{3.11}$$

$$\tilde{g}_{u\phi} = \left(\frac{\partial t}{\partial u}\right) g_{t\phi} + \left(\frac{\partial r}{\partial u}\right) g_{r\phi}, \tag{3.12}$$

$$\tilde{g}_{v\phi} = \left(\frac{\partial t}{\partial v}\right) g_{t\phi} + \left(\frac{\partial r}{\partial v}\right) g_{r\phi}, \tag{3.13}$$

$$\tilde{g}_{\theta\phi} = g_{\theta\phi}. \tag{3.14}$$

For double-null form of metric Eq(3.1), we require $\tilde{g}_{uu} = \tilde{g}_{vv} = \tilde{g}_{u\theta} = \tilde{g}_{u\phi} = \tilde{g}_{v\theta} = \tilde{g}_{v\phi} = \tilde{g}_{\theta\phi} = 0$, and obtain the following reduced system of partial differential equations:

$$\left(\frac{\partial t}{\partial u}\right)^2 g_{tt} + 2\left(\frac{\partial t}{\partial u}\right)\left(\frac{\partial r}{\partial u}\right) g_{tr} + \left(\frac{\partial r}{\partial u}\right)^2 g_{rr} = 0, \quad (3.15)$$

$$\left(\frac{\partial t}{\partial v}\right)^2 g_{tt} + 2\left(\frac{\partial t}{\partial v}\right)\left(\frac{\partial r}{\partial v}\right) g_{tr} + \left(\frac{\partial r}{\partial v}\right)^2 g_{rr} = 0, \quad (3.16)$$

$$\left(\frac{\partial t}{\partial u}\right)\left(\frac{\partial t}{\partial v}\right) g_{tt} + \left(\frac{\partial t}{\partial u}\frac{\partial r}{\partial v} + \frac{\partial t}{\partial v}\frac{\partial r}{\partial u}\right) g_{tr} + \left(\frac{\partial r}{\partial u}\right)\left(\frac{\partial r}{\partial v}\right) g_{rr} = \tilde{g}_{uv}, \quad (3.17)$$

$$g_{\theta\theta} = \tilde{g}_{\theta\theta}, \quad (3.18)$$

$$g_{\phi\phi} = \tilde{g}_{\phi\phi}, \quad (3.19)$$

$$\left(\frac{\partial t}{\partial u}\right) g_{t\theta} + \left(\frac{\partial r}{\partial u}\right) g_{r\theta} = 0, \quad (3.20)$$

$$\left(\frac{\partial t}{\partial v}\right) g_{t\theta} + \left(\frac{\partial r}{\partial v}\right) g_{r\theta} = 0, \quad (3.21)$$

$$\left(\frac{\partial t}{\partial u}\right) g_{t\phi} + \left(\frac{\partial r}{\partial u}\right) g_{r\phi} = 0, \quad (3.22)$$

$$\left(\frac{\partial t}{\partial v}\right) g_{t\phi} + \left(\frac{\partial r}{\partial v}\right) g_{r\phi} = 0, \quad (3.23)$$

$$g_{\theta\phi} = 0. \quad (3.24)$$

From Eqs(3.20) to (3.23), we have only trivial solution

$$g_{t\theta} = g_{r\theta} = g_{t\phi} = g_{r\phi} = 0.$$

So, the metric Eq(3.2) cannot be transformed into the double-null form if it contains $g_{t\theta}$, $g_{r\theta}$, $g_{t\phi}$, $g_{r\phi}$ or $g_{\theta\phi}$. Therefore, the most general form of a (3+1)-dimensional spacetime metric that can be transformed into double-null form is

$$ds^2 = g_{tt}dt^2 + g_{rr}dr^2 + 2g_{tr}dtdr + g_{\theta\theta}d\theta^2 + g_{\phi\phi}d\phi^2. \quad (3.25)$$

To transform metric Eq(3.25) into double-null form. We consider the following cases:

(1) $\quad g_{tt}, \quad g_{rr}, \quad g_{\theta\theta}, \quad g_{\phi\phi} \quad \neq 0$ and $g_{tr} = 0$.

(2) $\quad g_{rr},\quad g_{\theta\theta},\quad g_{\phi\phi},\quad g_{tr}\quad \neq 0$ and $g_{tt}=0.$

(3) $\quad g_{\theta\theta},\quad g_{\phi\phi},\quad g_{tr},\quad g_{tt}\quad \neq 0$ and $g_{rr}=0.$

(4) $\quad g_{tt},\quad g_{rr},\quad g_{\theta\theta},\quad g_{\phi\phi},\quad g_{tr}\quad \neq 0.$

Case 1: $g_{tt},\quad g_{rr},\quad g_{\theta\theta},\quad g_{\phi\phi}\quad \neq 0$ and $g_{tr}=0$

The system of Eqs(3.15) to (3.19) reduces to

$$\Big(\frac{\partial t}{\partial u}\Big)^2 g_{tt}+\Big(\frac{\partial r}{\partial u}\Big)^2 g_{rr}=0, \tag{3.26}$$

$$\Big(\frac{\partial t}{\partial v}\Big)^2 g_{tt}+\Big(\frac{\partial r}{\partial v}\Big)^2 g_{rr}=0, \tag{3.27}$$

$$\Big(\frac{\partial t}{\partial u}\Big)\Big(\frac{\partial t}{\partial v}\Big)g_{tt}+\Big(\frac{\partial r}{\partial u}\Big)\Big(\frac{\partial r}{\partial v}\Big)g_{rr}=\tilde{g}_{uv}, \tag{3.28}$$

$$g_{\theta\theta}=\tilde{g}_{\theta\theta}, \tag{3.29}$$

$$g_{\phi\phi}=\tilde{g}_{\phi\phi}. \tag{3.30}$$

Eqs(3.26) and (3.27) implies

$$\Big(\frac{\partial t}{\partial v}\frac{\partial r}{\partial u}-\frac{\partial t}{\partial u}\frac{\partial r}{\partial v}\Big)\Big(\frac{\partial t}{\partial u}\frac{\partial r}{\partial v}+\frac{\partial t}{\partial v}\frac{\partial r}{\partial u}\Big)g_{rr}=0. \tag{3.31}$$

In the present case, g_{rr} and the Jacobian of the transformations to be non-zero, Eq(3.31) implies

$$\Big(\frac{\partial t}{\partial v}\Big)\Big(\frac{\partial r}{\partial u}\Big)+\Big(\frac{\partial t}{\partial u}\Big)\Big(\frac{\partial r}{\partial v}\Big)=0. \tag{3.32}$$

which gives the following transformations

$$t=\Phi(u\pm v),\qquad r=\Psi(u\mp v), \tag{3.33}$$

where Φ and Ψ are arbitrary functions.

Case 2: $g_{rr},\quad g_{\theta\theta},\quad g_{\phi\phi},\quad g_{tr}\quad \neq 0$ and $g_{tt}=0$

The system of Eqs(3.15) to (3.19) reduces to

$$\Big(\frac{\partial r}{\partial u}\Big)^2 g_{rr} + 2\Big(\frac{\partial t}{\partial u}\Big)\Big(\frac{\partial r}{\partial u}\Big) g_{tr} = 0, \tag{3.34}$$

$$\Big(\frac{\partial r}{\partial v}\Big)^2 g_{rr} + 2\Big(\frac{\partial t}{\partial v}\Big)\Big(\frac{\partial r}{\partial v}\Big) g_{tr} = 0, \tag{3.35}$$

$$\Big(\frac{\partial r}{\partial u}\Big)\Big(\frac{\partial r}{\partial v}\Big) g_{rr} + \Big(\frac{\partial t}{\partial u}\frac{\partial r}{\partial v} + \frac{\partial r}{\partial u}\frac{\partial t}{\partial v}\Big) g_{tr} = \tilde{g}_{uv}, \tag{3.36}$$

$$g_{\theta\theta} = \tilde{g}_{\theta\theta}, \tag{3.37}$$

$$g_{\phi\phi} = \tilde{g}_{\phi\phi}. \tag{3.38}$$

The system of Eqs(3.34) and (3.35) can be writtten as

$$\frac{\partial r}{\partial u}\Big(\frac{\partial r}{\partial u} g_{rr} + 2\frac{\partial t}{\partial u} g_{tr}\Big) = 0, \quad \text{and} \quad \frac{\partial r}{\partial v}\Big(\frac{\partial r}{\partial v} g_{rr} + 2\frac{\partial t}{\partial v} g_{tr}\Big) = 0. \tag{3.39}$$

Eq(3.39) are satisfied under the following cases

(a) $\frac{\partial r}{\partial u} = 0 = \frac{\partial r}{\partial v}$,

(b) $\frac{\partial r}{\partial u} g_{rr} + 2\frac{\partial t}{\partial u} g_{tr} = 0 = \frac{\partial r}{\partial v} g_{rr} + 2\frac{\partial t}{\partial v} g_{tr}$,

(c) $\frac{\partial r}{\partial u} = 0 = \frac{\partial r}{\partial v} g_{rr} + 2\frac{\partial t}{\partial v} g_{tr}$,

(d) $\frac{\partial r}{\partial v} = 0 = \frac{\partial r}{\partial u} g_{rr} + 2\frac{\partial t}{\partial u} g_{tr}$.

The double-null form is not possible in cases (a) and (b). In case (a), the Jacobian of the transformations is zero which is not possible and in case (b) we get contradictions $\tilde{g}_{uv} = 0,\ g_{tr}$ and $g_{rr} = 0$. Existence of double-null form is possible in cases (c) and (d).

In case (c)

$$\frac{\partial r}{\partial u} = 0 = \frac{\partial r}{\partial v} g_{rr} + 2\frac{\partial t}{\partial v} g_{tr}. \tag{3.40}$$

Eq(3.40) imply, r depends on v

$$r = r(v), \tag{3.41}$$

requiring the Jacobian (3.3) to be non zero, Eq(3.41) implies

$$J = \frac{\partial t}{\partial u}\frac{\partial r}{\partial v} \neq 0, \tag{3.42}$$

$$\frac{\partial r}{\partial v}g_{rr} + 2\frac{\partial t}{\partial v}g_{tr} = 0, \tag{3.43}$$

$$\frac{\partial t}{\partial u}\frac{\partial r}{\partial v}g_{rr} + 2\frac{\partial t}{\partial v}\frac{\partial t}{\partial u}g_{tr} = 0, \tag{3.44}$$

$$-(\frac{\frac{2\partial t}{\partial v}\frac{\partial t}{\partial u}}{J})g_{tr} = g_{rr}, \tag{3.45}$$

g_{tr} and Jacobian of the transformations to be non-zero so Eq(3.45) implies that $\frac{\partial t}{\partial u}\frac{\partial t}{\partial v} \neq 0$. Therefore from Eq(3.45), we get

$$t = t(u, v).$$

In case (d)

$$\frac{\partial r}{\partial v} = 0 = \frac{\partial r}{\partial u}g_{rr} + 2\frac{\partial t}{\partial u}g_{tr}. \tag{3.46}$$

Eq(3.46) imply, r depends on u

$$r = r(u), \tag{3.47}$$

requiring the Jacobian (3.3) to be non zero, Eq(3.47) implies

$$J = -\frac{\partial t}{\partial v}\frac{\partial r}{\partial u} \neq 0, \tag{3.48}$$

$$\frac{\partial r}{\partial u}g_{rr} + 2\frac{\partial t}{\partial u}g_{tr} = 0, \tag{3.49}$$

$$\frac{\partial t}{\partial v}\frac{\partial r}{\partial u}g_{rr} + 2\frac{\partial t}{\partial v}\frac{\partial t}{\partial u}g_{tr} = 0, \tag{3.50}$$

$$(\frac{\frac{2\partial t}{\partial u}\frac{\partial t}{\partial v}}{J})g_{tr} = g_{rr}, \tag{3.51}$$

g_{tr} and the Jacobian of the transformations to be non-zero so Eq(3.51) implies that $\frac{\partial t}{\partial u}\frac{\partial t}{\partial v} \neq 0$. From Eq(3.51), we get

$$t = t(u, v).$$

Case 3: $g_{tt},\ \ g_{\theta\theta},\ \ g_{\phi\phi},\ \ g_{tr}\ \ \neq 0$ and $g_{rr} = 0$

The system of Eqs(3.15) to (3.19) reduces to

$$\Big(\frac{\partial t}{\partial u}\Big)^2 g_{tt} + 2\Big(\frac{\partial t}{\partial u}\Big)\Big(\frac{\partial r}{\partial u}\Big) g_{tr} = 0, \tag{3.52}$$

$$\Big(\frac{\partial t}{\partial v}\Big)^2 g_{tt} + 2\Big(\frac{\partial t}{\partial v}\Big)\Big(\frac{\partial r}{\partial v}\Big) g_{tr} = 0, \tag{3.53}$$

$$\Big(\frac{\partial t}{\partial u}\Big)\Big(\frac{\partial t}{\partial v}\Big) g_{tt} + \Big(\frac{\partial t}{\partial u}\frac{\partial r}{\partial v} + \frac{\partial t}{\partial v}\frac{\partial r}{\partial u}\Big) g_{tr} = \tilde{g}_{uv}, \tag{3.54}$$

$$g_{22} = \tilde{g}_{\theta\theta}, \tag{3.55}$$

$$g_{33} = \tilde{g}_{\phi\phi}. \tag{3.56}$$

The system of Eqs(3.52) and (3.53) can be writtten as

$$\frac{\partial t}{\partial u}\Big(\frac{\partial t}{\partial u} g_{tt} + 2\frac{\partial r}{\partial u} g_{tr}\Big) = 0, \quad \text{and} \quad \frac{\partial t}{\partial v}\Big(\frac{\partial t}{\partial v} g_{tt} + 2\frac{\partial r}{\partial v} g_{tr}\Big) = 0. \tag{3.57}$$

Eq(3.57) are satisfied for the following cases

(a) $\quad \frac{\partial t}{\partial u} = 0 = \frac{\partial t}{\partial v}$,

(b) $\quad \frac{\partial t}{\partial u} g_{tt} + 2\frac{\partial r}{\partial u} g_{tr} = 0 = \frac{\partial t}{\partial v} g_{tt} + 2\frac{\partial r}{\partial v} g_{tr}$,

(c) $\quad \frac{\partial t}{\partial u} = 0 = \frac{\partial t}{\partial v} g_{tt} + 2\frac{\partial r}{\partial v} g_{tr}$,

(d) $\quad \frac{\partial t}{\partial v} = 0 = \frac{\partial t}{\partial u} g_{tt} + 2\frac{\partial r}{\partial u} g_{tr}$.

The double-null form is not possible in cases (a) and (b). In case (a), the Jacobian of the transformations is zero which is not possible. In case (b) we get contradictions $\tilde{g}_{uv} = 0,\ g_{tr}$ and $g_{tt} = 0$. Existence of double-null form is possible in cases (c) and (d).

In case (c)

$$\frac{\partial r}{\partial u} = 0 = \frac{\partial r}{\partial v} g_{tt} + 2\frac{\partial r}{\partial v} g_{tr}. \tag{3.58}$$

Eq(3.58) gives, t depends on v

$$t = t(v), \tag{3.59}$$

requiring the Jacobian (3.3) to be non zero, Eq(3.59) implies

$$J=-\frac{\partial r}{\partial u}\frac{\partial t}{\partial v}\neq 0, \tag{3.60}$$

$$\frac{\partial t}{\partial v}g_{tt}+2\frac{\partial r}{\partial v}g_{tr}=0, \tag{3.61}$$

$$\frac{\partial r}{\partial u}\frac{\partial t}{\partial v}g_{tt}+2\frac{\partial r}{\partial v}\frac{\partial r}{\partial u}g_{tr}=0, \tag{3.62}$$

$$(\frac{\frac{2\partial r}{\partial u}\frac{\partial r}{\partial v}}{J})g_{tr}=g_{tt}, \tag{3.63}$$

g_{tr} and Jacobian of the transformations to be non-zero so Eq(3.63) implies that $\frac{\partial r}{\partial u}\frac{\partial r}{\partial v}\neq 0$. Therefore from Eq(3.63), we get

$$r=r(u,v).$$

In case (d)

$$\frac{\partial t}{\partial v}=0=\frac{\partial t}{\partial u}g_{tt}+2\frac{\partial r}{\partial u}g_{tr}. \tag{3.64}$$

Eq(3.64), gives t depends on u

$$t=t(u), \tag{3.65}$$

requiring the Jacobian (3.3) to be non zero, Eq(3.65) implies

$$J=\frac{\partial t}{\partial u}\frac{\partial r}{\partial v}\neq 0, \tag{3.66}$$

$$\frac{\partial t}{\partial u}g_{tt}+2\frac{\partial r}{\partial u}g_{tr}=0, \tag{3.67}$$

$$\frac{\partial r}{\partial v}\frac{\partial t}{\partial u}g_{tt}+2\frac{\partial r}{\partial v}\frac{\partial r}{\partial u}g_{tr}=0, \tag{3.68}$$

$$-(\frac{\frac{2\partial r}{\partial u}\frac{\partial r}{\partial v}}{J})g_{tr}=g_{tt}, \tag{3.69}$$

g_{tr} and Jacobian of the transformations to be non-zero so Eq(3.69) implies that $\frac{\partial r}{\partial u}\frac{\partial r}{\partial v}\neq 0$. Therefore from Eq(3.69), we have

$$r=r(u,v).$$

Case 4: $g_{tt}, \quad g_{rr}, \quad g_{\theta\theta}, \quad g_{\phi\phi}, \quad g_{tr} \quad \neq 0$

The system of Eqs(3.15) to (3.19) reduces to

$$\Big(\frac{\partial t}{\partial u}\Big)^2 g_{tt} + 2\Big(\frac{\partial t}{\partial u}\Big)\Big(\frac{\partial r}{\partial u}\Big) g_{tr} + \Big(\frac{\partial r}{\partial u}\Big)^2 g_{rr} = 0, \qquad (3.70)$$

$$\Big(\frac{\partial t}{\partial v}\Big)^2 g_{tt} + 2\Big(\frac{\partial t}{\partial v}\Big)\Big(\frac{\partial r}{\partial v}\Big) g_{tr} + \Big(\frac{\partial r}{\partial v}\Big)^2 g_{rr} = 0, \qquad (3.71)$$

$$\Big(\frac{\partial t}{\partial u}\Big)\Big(\frac{\partial t}{\partial v}\Big) g_{tt} + \Big(\frac{\partial t}{\partial u}\frac{\partial r}{\partial v} + \frac{\partial t}{\partial v}\frac{\partial r}{\partial u}\Big) g_{tr} + \Big(\frac{\partial r}{\partial u}\Big)\Big(\frac{\partial r}{\partial v}\Big) g_{rr} = \tilde{g}_{uv}, \qquad (3.72)$$

$$g_{\theta\theta} = \tilde{g}_{\theta\theta}, \qquad (3.73)$$

$$g_{\phi\phi} = \tilde{g}_{\phi\phi}. \qquad (3.74)$$

Eqs(3.70) and (3.72) imply

$$\frac{\partial r}{\partial u} g_{rr} + \frac{\partial t}{\partial u}\, g_{tr} = \Big(\frac{\frac{\partial t}{\partial u}}{J}\Big)\tilde{g}_{uv}, \qquad (3.75)$$

Eqs(3.71) and (3.72) imply

$$\frac{\partial r}{\partial v} g_{rr} + \frac{\partial t}{\partial v} g_{tr} = -\Big(\frac{\frac{\partial t}{\partial v}}{J}\Big)\tilde{g}_{uv}. \qquad (3.76)$$

From Eqs(3.75) and (3.76), we obtain

$$g_{tr} = \Big(\frac{\frac{\partial r}{\partial v}\frac{\partial t}{\partial u} + \frac{\partial r}{\partial u}\frac{\partial t}{\partial v}}{J^2}\Big)\tilde{g}_{uv}, \qquad (3.77)$$

$$\Longrightarrow r \neq r(u \pm v), \qquad\qquad t \neq t(u \mp v). \qquad (3.78)$$

$\tilde{g}_{uv}$ and Jacobian of the transformations is non-zero so Eq(3.77) implies that

$$\frac{\partial r}{\partial v}\frac{\partial t}{\partial u} + \frac{\partial r}{\partial u}\frac{\partial t}{\partial v} \neq 0. \qquad (3.79)$$

one of the solution of Eq(3.79) is

$$t = \Phi(u \pm v), \qquad r = \Psi(u \pm v), \qquad (3.80)$$

where Φ and Ψ are arbitrary functions. it will be interesting to see other solutions of Eq(3.79).

Chapter 4

Conclusion

In this thesis we have discussed the existence of double-null form of metrics in $(3+1)$-dimensional spacetimes, using the coordinate transformations. It is found that a metric that contains $g_{t\theta}$, $g_{r\theta}$, $g_{t\phi}$, $g_{r\phi}$ or $g_{\theta\phi}$ cannot be transformed into a double-null form. Hence the most general form of metric in $(3+1)$-dimensional spacetimes that can be converted into a double-null form is

$$ds^2 = g_{tt}dt^2 + g_{rr}dr^2 + 2g_{tr}dtdr + g_{\theta\theta}d\theta^2 + g_{\phi\phi}d\phi^2. \tag{4.1}$$

In order to obtain the required transformations, we dealt with the following cases

Case 1: $g_{tt}, \quad g_{rr}, \quad g_{\theta\theta}, \quad g_{\phi\phi} \quad \neq 0$ and $g_{tr} = 0$

In this case the metric can be transformed into double-null form and the required transformation have been obtained.

Case 2: $g_{rr}, \quad g_{\theta\theta}, \quad g_{\phi\phi}, \quad g_{tr} \quad \neq 0$ and $g_{tt} = 0$

$(a) \qquad \frac{\partial r}{\partial u} = 0 = \frac{\partial r}{\partial v}, \quad (b) \qquad \frac{\partial r}{\partial u}g_{rr} + 2\frac{\partial t}{\partial u}g_{tr} = 0 = \frac{\partial r}{\partial v}g_{rr} + 2\frac{\partial t}{\partial v}g_{tr}.$

In cases (a) and (b) the double-null form is not possible.

(c) $\quad \frac{\partial r}{\partial u} = 0 = \frac{\partial r}{\partial v} g_{rr} + 2\frac{\partial t}{\partial v} g_{tr}, \quad (d) \quad \frac{\partial r}{\partial v} = 0 = \frac{\partial r}{\partial u} g_{rr} + 2\frac{\partial t}{\partial u} g_{tr}.$

In cases (c) and (d) the metric can be transformed into double-null form and t depends on both u and v.

Case 3: $g_{\theta\theta}, \quad g_{\phi\phi}, \quad g_{tr}, \quad g_{tt} \quad \neq 0$ and $g_{rr} = 0$

(a) $\quad \frac{\partial t}{\partial u} = 0 = \frac{\partial t}{\partial v}, \quad (b) \quad \frac{\partial t}{\partial u} g_{tt} + 2\frac{\partial r}{\partial u} g_{tr} = 0 = \frac{\partial t}{\partial v} g_{tt} + 2\frac{\partial r}{\partial v} g_{tr}.$

In cases (a) and $b)$ the double-null form is not possible.

(c) $\quad \frac{\partial t}{\partial u} = 0 = \frac{\partial t}{\partial v} g_{tt} + 2\frac{\partial r}{\partial v} g_{tr}, \quad (d) \quad \frac{\partial t}{\partial v} = 0 = \frac{\partial t}{\partial u} g_{tt} + 2\frac{\partial r}{\partial u} g_{tr}.$

In this case the metric can be transformed into double-null form and r depends on both u and v.

Case 4: $g_{tt}, \quad g_{rr}, \quad g_{\theta\theta}, \quad g_{\phi\phi}, \quad g_{tr} \quad \neq 0$

Requiring the Jacobian to be non-zero in this case the metric can be transformed into double-null form. It is also found out that required transformations can not be of the type as obtained in Case 1. i.e.

$$r = r(u \pm v), \qquad t = t(u \mp v). \tag{4.2}$$

Reference

[1] K. Schwarzschild, *Sitzer. Deut Akad. Wiss. Berlin* **189** (1916).

[2] H. Reissner, *Ann. Phys.* **50** (1916) 106.

[3] G. Nordström, *Proc. Koninkl. Ned. A. Akad. Wetenschap*, **20** (1918) 1238.

[4] R. P. Kerr, *Phys. Rev. Lett.* **11** (1963) 237.

[5] E. T. Newman, *J. Math. Phys.* **6** (1965) 918.

[6] C. Montgomery, W. Orshiston and L. Whittinghn, *J. Astron. Hist. Herit.* **12** (2009) 90.

[7] J. R. Oppenheimer and H. Snyder, *Phys. Rev.* **56** (1939) 455.

[8] J. A. Wheeler, *Geon, Black Holes and Quantum Foam, A Life in Physics* Norton (1998).

[9] R. Lewis, *Introduction to General Relativity*, Cambridge University Press, (2009).

[10] S. Carlip, *Quantum Gravity in (2+1)-Dimensions*, Cambridge University Press, (2003).

[11] C.W. Misner, K.S. Thorne and J.A. Wheeler, *Gravitation*, W.H. Freeman, (1973)

[12] A. A. Siddiqui and S. M. J. Riaz, *Proc. of* 13^{th} *Marcel Grossman Meeting*, World Scientific **1** (2015) 1917.

[13] A. Qadir, *Einstein's General Theory of Relativity*, (under preparation).

[14] D. Finkelstein, *Phys. Rev.* **110** (1958) 965.

[15] A. S. Eddington, *Nature* **113** (1924) 192.

[16] M. D. Kruskal, *Phys. Rev.* **119** (1960) 1743.

[17] S. A. Hayward, *Phys. Rev.* D **49** (1994) 831.

[18] A. Roman and P. G. Bergmann, *Phys. Rev.* D **28** (1983) 1265.

[19] G. Ludwig and S. B. Edgar, *Gen. Rel. Grav.* **34** (2002) 807835.

[20] A. Pravdova and V. Pravda, *Class. Quantum Grav.* **25** (2008) 235008.

[21] A. Papapetrou, *Comm. Math. Phys.* **34** (1973) 229.

[22] Y. Sucu and N. Unal, *Class. Quantum Grav.* **21** (2004) 1443.

[23] C. Kiefer, *Quantum Gravity*, Oxford University Press (2012).